SYSTEMS OPTIMIZATION LABORATORY
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305-4022

A Generalized Complementarity Problem in Hilbert Space

by
Jen-Chih Yao

TECHNICAL REPORT SOL 90-5

March 1990

Research and reproduction of this report were partially supported by the National Science Foundation grant DMS-8913089; Office of Naval Research grant N00014-89-J-1659, and the Department of Energy grant DE-FG03-87ER25028.

A Generalized Complementarity Problem in Hilbert Space

Jen-Chih Yao

March 1990

Department of Operations Research
Stanford University
Stanford, CA 94305-4022

Abstract

An existence theorem for a generalized complementarity problem over arbitrary closed convex cone in a Hilbert space is proved.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let K be a nonempty subset of H and F be a point-to-set mapping from K into H. The graph $G(F)$ of F is the subset of $H \times H$ consisting of all (x, y) with $x \in K$ and $y \in F(x)$. Following Berge's definition [2], we say that F is *upper semicontinuous* at $x \in K$ if for each open set O containing $F(x)$ there exists a neighborhood U of x such that $F(u) \subset O$ for each $u \in U$. We say that F is upper semicontinuous in K if it is upper semicontinuous at each point of K and if, also, $F(x)$ is a compact set for each x. For any Hilbert space H, $CC(H)$ denotes the family of all nonempty compact and convex subsets of H. Let K be a closed convex cone in H with the vertex at 0 and K^* be the dual cone of K, that is,

$$K^* = \{u \in H \mid (u, x) \geq 0, \ \forall \, x \in K\}.$$

Let F be a point-to-set mapping from K into H. We consider the following version of the *generalized complementarity problem (GCP)* as formulated by Karamardian [6]: find $x \in K$ and $y \in F(x)$ so that

$$y \in K^*, \quad \langle x, y \rangle = 0. \tag{1}$$

Although GCP has been extensively studied in the literature (see, e.g., [1, 3, 5, 6]), the generalized version (1) of GCP in infinite dimensional spaces seems to have attracted little attention. The purpose of this paper are to establish an existence result for the generalized problem (1) under an appropriate version of a condition of Karamardian [6] and to draw some attention for further investigation on it.

2. The Main Result

Before giving our main result, we first have

Lemma 2.1. *Let K be a nonempty compact and convex subset of a real Hilbert space H. Let F be an upper semicontinuous point-to-set mapping from K into $CC(H)$. Then there exist $x \in K$ and $y \in F(x)$ such that*

$$\langle u - x, y \rangle \geq 0 \qquad \textit{for all } u \in K.$$

Proof. Let C be the convex closure of $F(K) = \bigcup_{x \in K} F(x)$. Since $F(K)$ is compact by [2, Theorem 3, p.110], C is also compact and thus $K \times C$ is compact and convex. Let T be a point-to-set mapping from $K \times C$ into itself defined by $T(x, y) = (V(x, y), F(x))$ where

$$V(x, y) = \{u \in K \mid \langle u - x, y \rangle = \min_{s \in K} \langle s - x, y \rangle\}.$$

For each $(x, y) \in K \times C$, the set $T(x, y)$ is nonempty, closed and convex. Furthermore, since the graph $G(T)$ of T is closed in $H \times H$, the mapping T is upper semicontinuous in $K \times C$ by [2, Corollary, p.112]. Therefore by Ky Fan's generalization of Kakutani's fixed point theorem [4, Theorem 1], there exists $(x, y) \in K \times C$ which is a fixed point of T. Consequently, $x \in K$, $y \in F(x)$ and $\langle u - x, y \rangle \geq 0$ for all $u \in K$.

Now we prove that there exists a solution to the generalized problem (1) under an appropriate modification of a condition of Karamardian [6].

Theorem 2.2. *Let K be a closed convex cone with vertex at 0 in a real Hilbert space H. Let F be an upper semicontinuous point-to-set mapping from K into $CC(H)$. Suppose that there exists a nonempty compact subset D in K with the property that for every $x \in K \backslash D$, there exists $z \in D$ such that*

$$\langle x - z, y \rangle > 0 \qquad \text{for all } y \in F(x). \tag{2}$$

Then there exist $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ such that

$$\bar{y} \in K^*, \ \langle \bar{x}, \bar{y} \rangle = 0. \tag{3}$$

Proof. Let $E = D \times F(D)$. Then E is compact since $F(D)$ is compact. For every $u \in K$, let $D(u)$ be the subset of E defined by

$$D(u) = \{(x, y) \in D \times F(x) \mid \langle u - x, y \rangle \geq 0\}.$$

Then $D(u)$ is nonempty and closed for every $u \in K$. Indeed, if $\{(x_n, y_n)\}$ is a sequence in $D(u)$ converging to (x, y), then since the graph $G(F)$ of F is closed [2, Theorem 6, p.112], it follows that $y \in F(x)$ and $(x, y) \in D(u)$. For an arbitrary finite subset $\{u_i | 1 \leq i \leq n\}$ in K, let $\bar{D}$ be the convex closure of $D \cup \{u_i | 1 \leq i \leq n\}$. Then $\bar{D}$ is compact and convex. Thus by Lemma 2.1, there exist $x \in \bar{D}$ and $y \in F(x)$ so that

$$\langle u - x, y \rangle \geq 0 \qquad \text{for all } u \in \bar{D}. \tag{4}$$

If $x \notin D$, then from (2) there exists $z \in D$ such that $\langle x - z, y \rangle > 0$ which contradicts (4). Hence $x \in D$ and $(x, y) \in \bigcap_{i=1}^{n} D(u_i)$. Conseqnently, the family $\{D(u) \mid u \in K\}$ has the finite intersection property. Since E is compact, there exists $(\bar{x}, \bar{y}) \in \bigcap_{u \in K} D(u)$. This implies that $\bar{y} \in F(\bar{x})$ and

$$\langle u - \bar{x}, \bar{y} \rangle \geq 0 \qquad \text{for all } u \in K. \tag{5}$$

Since K is a convex cone, it follows from (5) that $\bar{y} \in K^*$. By letting $u = 0$ in (5), we have

$$\langle \bar{x}, \bar{y} \rangle \leq 0. \tag{6}$$

Also by letting $u = 2x$ in (5), we get

$$\langle \bar{x}, \bar{y} \rangle \geq 0. \tag{7}$$

From (6) and (7), it follows that $\langle \bar{x}, \bar{y} \rangle = 0$. Hence the result follows.

In finite-dimensional spaces, a result similar to Theorem 2.1 has been obtained by Saigal [8, Theorem 2.1].

References

1. M. S. Bazaraa, J. J. Goode and M. Z. Nashed, A nonlinear complementarity problem in mathematical programming in Banach space, *Proc. Amer. Math. Soc.* **35** (1972), 165-170.

2. C. Berge, *Topological Spaces*, MacMillan Co. New York, 1979.

3. A. T. Dash and S. Nanda, A complementarity problem in mathematical programming in Banach space, *J. Math. Anal. Appl.* **98** (1984), 328-331.

4. K. Fan, Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 121-126.

5. G. Isac and M. Théra, Complementarity problem and the existence of the post-critical equilibrium state of a thin elastic plate, *J. Optim. Theory Appl.* **58** (1988), 241-257.

6. S. Karamardian, Generalized complementarity problem, *J. Optim. Theory Appl.* **8** (1971), 161-168.

7. G. Luna, A remark on the nonlinear complementarity problem, *Proc. Amer. Math. Soc.* **48** (1975), 132-134.

8. R. Saigal, Extension of the generalized complementarity problem, *Math. Oper. Res.* **1** (1976), 260-266.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER SOL 90-5	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Generalized Complementarity Problem in Hilbert Space		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jen-Chih Yao		8. CONTRACT OR GRANT NUMBER(s) N00014-89-J-1659
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research - SOL Stanford University Stanford, CA 94305-4022		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1111MA
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research - Dept. of the Navy 800 N. Quincy Street Arlington, VA 22217		12. REPORT DATE March 1990
		13. NUMBER OF PAGES 5 pp.
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Hilbert space; dual cone; upper semicontinuous point-to-set mapping; generalized complementarity problem.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An existence theorem for a generalized complementarity problem over arbitrary closed convex cone in a Hilbert space is proved.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)