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VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
DEPARTMENT OF ENGINEERING SCIENCE AND MECHANICS

Final Scientific Report on

CONTROL OF LARGE SPACE STRUCTURES

Covering the Period
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Principal Investigator: Leonard Meirovitch
University Distinguished Professor

Abstract

Work during this period was concerned with two aspects: 1) development of a new method for the derivation of the state equations of motion for control of flexible spacecraft and 2) development of a method for the control of spacecraft in the form of articulated flexible multi-bodies.

In deriving the equations of motion for the control of flexible spacecraft, it is common practice to express the rigid-body rotational motions in terms of nonorthogonal inertial components, such as Euler's angles. On the other hand, in controlling the same motions of the spacecraft, the control law is generally in terms of angular motions components along the orthogonal body axes. The elastic motions are ordinarily expressed in terms of components along body axes. In implementing the control laws, it is desirable that both the equations of motion and the control laws be in terms of the same type of variables, and in particular in terms of the body axes variables. Reference 1 presents the development of state equations of motion for flexible bodies in terms of quasi-coordinates, which represent this very type of variables. Indeed, the angular velocity components about the orthogonal body axes are time derivatives of quasi-coordinates. The state equations of motion for flexible bodies in terms of quasi-coordinates render the control implementation task considerably easier.

In many space applications, it becomes necessary to reorient the line of sight. If the line of sight is fixed in the spacecraft, such as in the space telescope, this implies reorientation of the whole spacecraft. In other space applications, the spacecraft consists of a rigid platform and a number of flexible appendages, such as antennas. If the mission requires the reorientation of the line of sight of these antennas relative to the inertial space, then the preferred strategy is to stabilize the platform relative to the inertial space and to reorient the flexible antennas relative to the platform. Reference 2 provides a

derivation of the state equations of motion for the task described above, based on the equations of motion in terms of quasi-coordinates derived in Ref. 1. Because the motions defining the maneuvering of the antennas are known a priori, the state equations contain time-dependent coefficients and persistent disturbances.

The problem of maneuvering an articulated flexible spacecraft and controlling its vibration at the same time is a very complex task. Reference 3 shows an approach to the problem, in which the maneuvering of the appendages is carried out open-loop using a bang-bang control law. On the other hand, the vibration suppression is carried out closed-loop, which amounts to controlling a time-varying system subjected to persistent disturbances.

In addition to the above research, work was carried out on several other papers (Refs. 4 - 7) describing research under the preceding AFOSR grant. In particular, we single out Ref. 5, which was presented during this grant period. The paper is concerned with control of the perturbations experienced by a flexible spacecraft during a minimum-time maneuver. The spacecraft is modeled as a flexible appendage attached to a rigid hub. The perturbed model can be divided into a rigid-body part and an elastic part. The model is described by a linear, time-varying set of ordinary differential equations subjected to piecewise-constant disturbances caused by inertial forces resulting from the minimum-time maneuver. The control consists of a reduced-order compensator designed such that the perturbed model is finite-time stable.

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State Equations of Motion for Flexible Bodies in Terms of Quasi-Coordinates

LEONARD MEIROVITCH**

Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061

Summary

This paper is concerned with the general motion of a flexible body in space. Using the extended Hamilton's principle for distributed systems, standard Lagrange's equations for hybrid systems are first derived. Then, the equations for the rigid-body motions are transformed into a symbolic vector form of Lagrange's equations in terms of general quasi-coordinates. The hybrid Lagrange's equations of motion in terms of general quasi-coordinates are subsequently expressed in terms of quasi-coordinates representing rigid-body motions. Finally, the second-order Lagrange's equations for hybrid systems are transformed into a set of state equations suitable for control. An illustrative example is presented.

Introduction

The derivation of the equations of motion has preoccupied dynamicists for many years, as can be concluded from the texts by Whittaker [1], Pars [2] and Meirovitch [3]. References 1-3 consider the motion of systems of particles and rigid bodies, and the equations of motion are presented in a large variety of forms. In this paper, we concentrate on a certain formulation, namely, Lagrange's equations. For an n-degree-of-freedom system, Lagrange's equations consist of n second-order ordinary differential equations for the system displacements.

In the control of dynamical systems, it is often convenient to work with first-order rather than second-order differential equa-

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** University Distinguished Professor.

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tions. Introducing the velocities as auxiliary variables, it is possible to transform the n second-order equations into 2n first-order state equations. The state equations are widely used in modern control theory [4].

With the advent of man-made satellites, there has been a renewed interest in the derivation of the equations of motion. The motion of rigid spacecraft can be defined in terms of translations and rotations of a reference set of axes embedded in the body and known as body axes. The equations of motion for such systems can be obtained with ease by means of Lagrange's equations. It is common practice to define the orientation of the body relative to an inertial space in terms of a set of rotations about nonorthogonal axes [3]. However, the kinetic energy has a simpler form when expressed in terms of angular velocity components about the orthogonal body axes than in terms of angular velocities about nonorthogonal axes. Moreover, for feedback control, it is more convenient to work with angular velocity components about the body axes, as sensors measure angular motions and actuators apply torques in terms of components about the body axes. In such cases, it is often advantageous to work not with standard Lagrange's equations but with Lagrange's equations in terms of quasi-coordinates [1,3]. If the body contains discrete parts, such as lumped masses connected to a main rigid body by massless springs, it is convenient to work with a set of axes embedded in the undeformed body. The equations of motion consist entirely of ordinary differential equations and can be obtained by a variety of approaches, including the standard Lagrange's equations and Lagrange's equations in terms of quasi-coordinates [5]*.

In the more general case, the body can be regarded as being either entirely flexible with distributed mass and stiffness properties or as consisting of a main rigid body with distributed elastic appendages. Unlike the previous case, the equations of motion are hybrid, in the sense that the equations for the rigid-

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* Note that Ref. 5 refers to Lagrange's equations in terms of quasi-coordinates as Boltzmann-Hamel equations.

body motions are ordinary differential equations and those for the elastic motions are partial differential equations. Hybrid equations were obtained for the first time in Ref. 6. Moreover, the formulation of Ref. 6 was obtained by using Lagrange's equations in terms of quasi-coordinates, but some generality was lost in that the body considered was assumed to be symmetric and to undergo antisymmetric elastic motion. As a result, the rigid-body translations were zero.

This paper is concerned with the general motion of a flexible body in space. Using the extended Hamilton's principle for distributed systems [7], standard Lagrange's equations for hybrid systems are first derived. Then, using the approach of Ref. 3, the equations for the rigid-body motions are transformed into a symbolic vector form of Lagrange's equations in terms of general quasi-coordinates. The hybrid Lagrange's equations of motion in terms of general quasi-coordinates are subsequently expressed in terms of quasi-coordinates representing rigid-body motions. This is a very important step, as the latter form permits the derivation of the hybrid equations of motion with relative ease, thus eliminating a great deal of tedious work. These hybrid equations represent an extension to flexible bodies of Lagrange's differential equations in terms of quasi-coordinates derived in Ref. 3 for rigid bodies. The second-order equations are then used to derive the hybrid state equations.

As an illustration, the hybrid equations of motion of a spacecraft consisting of a rigid hub with a flexible appendage simulating an antenna are derived.

Standard Lagrange's Equations for Hybrid Systems

Let us consider a flexible body and assume that the Lagrangian $L = T - V$, in which T is the kinetic energy and V is the potential energy, can be written in the general form $L = L(q_i, \dot{q}_i, u_j, \dot{u}_j, u'_j, u''_j, \ldots, u_j^{(p)})$, where $q_i = q_i(t)$ $(i = 1,2,\ldots,m)$ are generalized coordinates describing rigid-body motions of the body and $u_j(P,t)$ $(j = 1,2,\ldots,n)$ are generalized coordinates describing elastic motions relative to the rigid-body motions of a typical point in the body identified by the spatial position P. Dots designate derivatives with respect to time and primes derivatives with respect to the spatial position. For convenience, we express the Lagrangian in terms of the Lagrangian density $\hat{L}$ in the form $L = \int_D \hat{L}\, dD$, where D is the domain of extension of the body.

We propose to derive Lagrange's equations by means of the extended Hamilton's principle [7], which can be stated as

$$\int_{t_1}^{t_2} \int_D (\delta\hat{L} + \delta\hat{W}) dD dt = 0, \quad \delta q_i = \delta u_j = 0 \quad \text{at} \quad t = t_1, t_2 \qquad (1)$$

where $\delta\hat{W}$ is the nonconservative virtual work density, which is related to the virtual work by $\delta W = \int_D \delta\hat{W}\, dD$. The virtual work can be written in the form

$$\delta W = \sum_{i=1}^{m} Q_i \delta q_i + \sum_{j=1}^{n} \int_D \hat{U}_j \delta u_j \, dD \qquad (2)$$

where Q_i are nonconservative generalized forces associated with the rigid body motions and $\hat{U}_j$ are nonconservative generalized force densities associated with the elastic motions; δq_i and δu_j are associated virtual displacements. Following the usual steps [7], we obtain Lagrange's equations of motion, which can be expressed in the symbolic vector form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underset{\sim}{q}}}\right) - \frac{\partial L}{\partial \underset{\sim}{q}} = \underset{\sim}{Q}, \qquad \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{\underset{\sim}{u}}}\right) - \frac{\partial \hat{T}}{\partial \underset{\sim}{u}} + \mathcal{L}\underset{\sim}{u} = \hat{\underset{\sim}{U}} \qquad (3a,b)$$

where $\underset{\sim}{q}$ and $\underset{\sim}{Q}$ are m-vectors, $\underset{\sim}{u}$ and $\hat{\underset{\sim}{U}}$ are n-vectors and $\mathcal{L}$ is an $n \times n$ operator matrix. Because of the mixed nature of the differential equations, we refer to the set (3) as hybrid. The elastic displacements are subject to given boundary conditions.

<u>Equations in Terms of Quasi-Coordinates for the Rigid-Body Motions</u>

Quite often it is convenient to express the Lagrangian not in terms of the velocities $\dot{q}_i$ but in terms of linear combinations w_ℓ ($\ell=1,2,\ldots,m$) of $\dot{q}_i$. The difference between $\dot{q}_i$ and w_ℓ is that the former represent time derivatives dq_i/dt, which can be integrated with respect to time to obtain the displacements q_i, whereas w_ℓ cannot be integrated to obtain displacements. It is customary to refer to w_ℓ as derivatives of quasi-coordinates [3]. The relation between w_ℓ and $\dot{q}_i$ can be expressed in the compact matrix form $\underline{w} = A^T\dot{\underline{q}}$, where the notation is obvious. Similarly, we express the velocities $\dot{q}_i$ in terms of the variables w_ℓ as $\dot{\underline{q}} = B\underline{w}$, from which it follows that the $m \times m$ matrices A and B are related by $A^TB = B^TA = I$, where I is the identity matrix of order m.

Our object is to derive Lagrange's equations in terms of w_ℓ instead of $\dot{q}_i$. Using the relations indicated above, it can be shown [3] that Eqs. (3a) can be replaced by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{w}}\right) + B^T E \frac{\partial L}{\partial \underline{w}} - B^T \frac{\partial L}{\partial \underline{q}} = \underline{N} \qquad (4)$$

where

$$E = \left[\underline{w}^T B^T \frac{\partial a_{k\ell}}{\partial \underline{q}}\right] - \left[\underline{w}^T B^T \frac{\partial A}{\partial q_k}\right], \quad \underline{N} = B^T \underline{Q} \qquad (5a,b)$$

and we note that the first matrix in E is obtained by first carrying out a triple matrix product for every one of the m^2 entries in A and then arranging the resulting scalars in a square matrix. On the other hand, the second matrix in E is obtained by first generating a row matrix for every generalized coordinate q_k $(k = 1,2,\ldots,m)$ and then arranging the row matrices in a square matrix. Equation (4) represents a symbolic vector form of the Langrange equations for quasi-coordinates. The complete formulation is obtained by adjoining to Eq. (4), the equations for the elastic motion, Eq. (3b), as well as the associated boundary conditions.

General Equations in Terms of Quasi-Coordinates for a Translating and Rotating Flexible Body.

Let us consider the body depicted in Fig. 1. The motion of the body can be described by attaching a set of body axes xyz to the body in undeformed state. The origin of the body axes coincides with an arbitrary point 0. Then, the motion can be defined in terms of the translation of point 0, and the rotation of the body axes xyz relative to the inertial axes XYZ. The position of 0 relative to XYZ is given by the radius vector $\underline{R} = \underline{R}(R_X, R_Y, R_Z)$. The rotation can be defined in terms of a set of angles θ_1, θ_2 and θ_3 (Fig. 2). Hence, the generalized coordinates are $q_1 = R_X$, $q_2 = R_Y$, $q_3 = R_Z$, $q_4 = \theta_1$, $q_5 = \theta_2$, $q_6 = \theta_3$. In addition, there are the elastic displacement components $u_x(P,t)$, $u_y(P,t)$, $u_z(P,t)$. The displacements R_X, R_Y, R_Z are measured relative to the inertial axes XYZ. On the other hand, the displacements u_x, u_y, u_z are measured relative to the body axes xyz. Moreover, the components $\dot{R}_X$, $\dot{R}_Y$, $\dot{R}_Z$ of the velocity vector $\dot{\underline{R}}$ are also measured relative to XYZ. On the other hand, the angular velocity vector $\underline{\omega}$ has components ω_x, ω_y, ω_z, measured relative to the body axes xyz. It will prove convenient to express all motions in terms of components along the body axes. To this end, if we denote the velocity of point 0 in terms of components along the body axes by $\underline{V}$, then it can be shown that $\underline{V} = C\dot{\underline{R}}$, where $C = C(\theta_1,\theta_2,\theta_3)$ is a rotation matrix. Moreover, the angular velocity vector $\underline{\omega}$ can be expressed in terms of the angular velocities $\dot{\theta}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$ in the form $\underline{\omega} = D\dot{\underline{\theta}}$, where $D = D(\theta_1,\theta_3)$ is a transformation matrix. We note that the angular velocity components ω_x, ω_y and ω_z cannot be integrated with respect to time to yield angular displacements α_x, α_y and α_z about axes x, y and z, respectively. Hence, ω_x, ω_y, ω_z can be regarded as time derivatives of quasi-coordinates and treated by the procedure presented in the preceding section. Although it is not very common to regard the velocity components V_x, V_y and V_z as time derivatives of quasi-coordinates, they can still be treated as such. In view of this, if we introduce the generalized velocity vector $\dot{\underline{q}} = [\dot{R}_X\ \dot{R}_Y\ \dot{R}_Z\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3]^T$, as well as the "quasi-velocity" vector $\underline{w} = [V_x\ V_y\ V_z\ \omega_x\ \omega_y\ \omega_z]^T$, we conclude that the coefficient matrices are defined by

$$A^T = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right], \qquad B^T = A^{-1} = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & (D^T)^{-1} \end{array}\right] \tag{6a,b}$$

where we recognized that $C^{-1} = C^T$, because rotation matrices are orthonormal. It can be shown, after lengthy algebraic manipulations, that

$$B^T E = \left[\begin{array}{c|c} \tilde{\omega} & 0 \\ \hline \tilde{V} & \tilde{\omega} \end{array}\right] \tag{7}$$

where $\tilde{\omega}$ and $\tilde{V}$ are skew-symmetric matrices corresponding to $\underline{\omega}$ and $\underline{V}$ [3], respectively.

Using Eqs. (3b) and (4) in conjunction with the above relations, we obtain the hybrid Lagrange's equations in terms of quasi-coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{V}}\right) + \tilde{\omega}\frac{\partial L}{\partial \underline{V}} - C\frac{\partial L}{\partial \underline{R}} = \underline{F} \qquad (8a)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{\omega}}\right) + \tilde{V}\frac{\partial L}{\partial \underline{V}} + \tilde{\omega}\frac{\partial L}{\partial \underline{\omega}} - (D^T)^{-1}\frac{\partial L}{\partial \underline{\theta}} = \underline{M} \qquad (8b)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \underline{v}}\right) - \frac{\partial \hat{T}}{\partial \underline{u}} + \mathcal{L}\underline{u} = \hat{\underline{U}} \qquad (8c)$$

where $\underline{F}$ and $\underline{M}$ are external nonconservative force and torque, respectively, in terms of components about the body axes, $\partial L/\partial \underline{\theta} = [\partial L/\partial \theta_1 \;\; \partial L/\partial \theta_2 \;\; \partial L/\partial \theta_3]^T$ and $\underline{v} = \dot{\underline{u}}$. Note that $\underline{\theta}$ does not really represent a vector and must be interpreted as a mere symbolic notation. We recall that the components of $\underline{u}$ are still subject to given boundary conditions.

It should be pointed out that, in deriving Eqs. (8), no explicit use was made of the angles θ_1, θ_2 and θ_3, so that Eqs. (8) are valid for any set of angles describing the rotation of the body axes, such as Euler's angles, and they are not restricted to the angles used here. Moreover, point 0 is an arbitrary point, not necessarily the mass center of the undeformed body, and axes xyz are not necessarily principal axes of the undeformed body. Clearly, if xyz are chosen as the principal axes with the origin at the mass center, then the equations of motion can be simplified.

State Equations in Terms of Quasi-Coordinates

Equations (8), and in particular Eqs. (8a) and (8b), can be expressed in more detailed form. To this end, we write the velocity vector of a typical point P in the body in terms of components along the body axes as follows:

$$\underline{v}_P = \underline{V} + \underline{\omega} \times (\underline{r} + \underline{u}) + \underline{v} = \underline{V} + (\tilde{r} + \tilde{u})^T\underline{\omega} + \underline{v} \qquad (9)$$

where $\underline{r}$ is the nominal position of P relative to 0. Moreover, $\tilde{r}$ and $\tilde{u}$ represent skew-symmetric matrices associated with the vectors $\underline{r}$ and $\underline{u}$, respectively. Then, denoting by ρ the mass density, the kinetic energy can be shown to have the expression

$$T = \frac{1}{2}\int_D \rho \underline{v}_P^T \underline{v}_P \, dD = \frac{1}{2} m\underline{V}^T\underline{V} + \underline{V}^T\tilde{S}^T\underline{\omega} + \underline{V}^T\int_D \rho \underline{v} \, dD$$

$$+ \underline{\omega}^T \int_D \rho(\tilde{r} + \tilde{u})\underline{v}\, dD + \frac{1}{2}\underline{\omega}^T J \underline{\omega} + \frac{1}{2}\int_D \rho \underline{v}^T \underline{v}\, dD \qquad (10)$$

where $\tilde{S} = \int_D \rho(\tilde{r} + \tilde{u})dD$, $J = \int_D \rho(\tilde{r} + \tilde{u})(\tilde{r} + \tilde{u})^T dD$, in which $\tilde{S}$ is recognized as a skew-symmetric matrix of first moments and J as a symmetric matrix of mass moments of inertia, both corresponding to the deformed body. Moreover, we assume that the potential energy has the functional form $V = V(\underline{R}, \underline{\theta}, \underline{u}, \underline{u}', \ldots, \underline{u}^{(p)})$.

Inserting Eq. (10) into Eqs. (8) and rearranging, we obtain the explicit Lagrange's equations in terms of hybrid coordinates

$$m\dot{\underline{V}} + \tilde{S}^T\dot{\underline{\omega}} + \int_D \rho\dot{\underline{v}}\, dD = (2\tilde{S}_v + m\tilde{V} + \tilde{\omega}\tilde{S})\underline{\omega} - C\frac{\partial V}{\partial \underline{R}} + \underline{F} \qquad (11a)$$

$$\tilde{S}\dot{\underline{V}} + J\dot{\underline{\omega}} + \int_D \rho(\tilde{r} + \tilde{u})\dot{\underline{v}}\, dD = [2\int_D \rho(\tilde{r} + \tilde{u})\tilde{v}\, dD + \tilde{S}\tilde{V} - \tilde{\omega}J]\underline{\omega}$$

$$- (D^T)^{-1}\frac{\partial V}{\partial \underline{\theta}} + \underline{M} \qquad (11b)$$

$$\rho\dot{\underline{V}} + \rho(\tilde{r} + \tilde{u})^T\dot{\underline{\omega}} + \rho\dot{\underline{v}} = -\rho\tilde{V}^T\underline{\omega} - \rho\tilde{\omega}^2(\underline{r} + \underline{u}) - 2\rho\tilde{v}^T\underline{\omega} - \mathcal{L}\underline{u} + \hat{\underline{U}} \qquad (11c)$$

where $\tilde{S}_v = \int_D \rho\tilde{v}\, dD$. The state equations are completed by adjoining the kinematical relations

$$\dot{\underline{R}} = C^T\underline{V}, \quad \dot{\underline{\theta}} = D^{-1}\underline{\omega}, \quad \dot{\underline{u}} = \underline{v} \qquad (11d,e,f)$$

<u>Illustrative Example</u>

As an illustration, we consider a spacecraft consisting of a rigid hub and a flexible appendage, as shown in Fig. 3. From the figure, we can write

$$\underline{r} = x\underline{i}, \quad \underline{u} = u_y\underline{j} + u_z\underline{k}, \quad \underline{v} = v_y\underline{j} + v_z\underline{k} \qquad (12)$$

so that

$$\tilde{S} = \begin{bmatrix} 0 & -\int\rho u_z dx & \int\rho u_y dx \\ \int\rho u_z dx & \int\rho u_z dx & -m\bar{x} \\ -\int\rho u_y dx & m\bar{x} & 0 \end{bmatrix} \qquad (13a)$$

where ρ is the mass density of the appendage, m is the total mass and $\bar{x}$ is the position of the mass center of the appendage. Moreover,

$$J = \begin{bmatrix} J_{xx}+\int\rho(u_y^2+u_z^2)dx & -\int\rho x u_y dx & -\int\rho x u_z dx \\ -\int\rho x u_y dx & J_{yy}+\int\rho u_z^2 dx & -\int\rho u_y u_z dx \\ -\int\rho x u_z dx & -\int\rho u_y u_z dx & J_{zz}+\int\rho u_y^2 dx \end{bmatrix} \quad (13b)$$

where J_{xx}, J_{yy} and J_{zz} are the mass moments of inertia of the spacecraft regarded as rigid.

Using Eqs. (12) and (13), the state equations, Eqs. (11), can be written in the explicit forms

$$\dot{R}_X = (c\theta_2 c\theta_3 + s\theta_1 s\theta_2 s\theta_3)V_x - (c\theta_2 c\theta_3 - s\theta_1 s\theta_2 c\theta_3)V_y + c\theta_1 s\theta_2 V_z \quad (14a)$$

$$\dot{R}_Y = c\theta_1 s\theta_3 V_x + c\theta_1 c\theta_3 V_y - s\theta_1 V_z \quad (14b)$$

$$\dot{R}_Z = -(s\theta_2 c\theta_3 - s\theta_1 c\theta_2 s\theta_3)V_x + (s\theta_2 s\theta_3 + s\theta_1 c\theta_2 c\theta_3)V_y + c\theta_1 s\theta_2 V_z \quad (14c)$$

$$\dot{\theta}_1 = c\theta_3\,\omega_x - s\theta_3\,\omega_y,\quad \dot{\theta}_2 = \frac{s\theta_3}{c\theta_1}\omega_x + \frac{c\theta_3}{c\theta_1}\omega_y \quad (14d,e)$$

$$\dot{\theta}_3 = \frac{s\theta_1 s\theta_3}{c\theta_1}\omega_x + \frac{s\theta_1 c\theta_3}{c\theta_1}\omega_y + \omega_z,\quad \dot{u}_y = v_y,\quad \dot{u}_z = v_z \quad (14f,g,h)$$

$$m\dot{V}_x + \dot{\omega}_y\int\rho u_z\,dx - \dot{\omega}_z\int\rho u_y\,dx = mV_y\omega_z - mV_z\omega_y + m_1\bar{x}(\omega_y^2+\omega_z^2) - \omega_x\omega_y\int\rho u_y\,dx$$
$$- \omega_x\omega_z\int\rho u_z\,dx + 2\omega_z\int\rho v_y\,dx - 2\omega_x\int\rho v_z\,dx - (c\theta_2 c\theta_3 + s\theta_1 s\theta_2 s\theta_3)\frac{\partial V}{\partial R_X}$$
$$- c\theta_2 s\theta_3\frac{\partial V}{\partial R_Y} + (s\theta_2 c\theta_3 - s\theta_1 c\theta_2 s\theta_3)\frac{\partial V}{\partial R_Z} + F_x \quad (14i)$$

$$m\dot{V}_y - \dot{\omega}_x\int\rho u_z\,dx + m_1\bar{x}\omega_z = mV_z\omega_x - mV_x\omega_z - m_1\bar{x}\omega_x\omega_y + (\omega_x^2+\omega_z^2)\int\rho u_y\,dx$$
$$- \omega_y\omega_z\int\rho u_z\,dx + 2\omega_x\int\rho v_z\,dx + (c\theta_2 s\theta_3 - s\theta_1 s\theta_2 c\theta_3)\frac{\partial V}{\partial R_X} - c\theta_2 c\theta_3\frac{\partial V}{\partial R_Y}$$
$$- (s\theta_2 s\theta_3 + s\theta_1 c\theta_2 c\theta_3)\frac{\partial V}{\partial R_Z} + F_y \quad (14j)$$

$$m\dot{V}_z + \dot{\omega}_x\int\rho u_y\,dx - m_1\bar{x}\dot{\omega}_y = mV_x\omega_y - mV_y\omega_x - m_1\bar{x}\omega_x\omega_z + (\omega_x^2+\omega_y^2)\int\rho u_z\,dx$$
$$- \omega_y\omega_z\int\rho u_y\,dx + 2\omega_x\int\rho v_y\,dx - c\theta_1 s\theta_2\frac{\partial V}{\partial R_X} + s\theta_1\frac{\partial V}{\partial R_Y} - c\theta_1 c\theta_2\frac{\partial V}{\partial R_Z} + F_z$$
$$(14k)$$

$$
\begin{aligned}
&- \left(\int \rho u_z \, dx\right)\dot{V}_y + \left(\int \rho u_y \, dx\right)\dot{V}_z + \left[J_{xx} + \int \rho (u_y^2 + u_z^2) dx\right]\dot{\omega}_x - \left(\int \rho x u_y \, dx\right)\dot{\omega}_y \\
&\quad - \left(\int \rho x u_z \, dx\right)\dot{\omega}_z + \int \rho (u_y \dot{v}_z - u_z \dot{v}_y) dx = - \left(V_y \int \rho u_y \, dx + V_z \int \rho u_z \, dx\right) \\
&\quad + V_x \omega_y \int \rho u_y \, dx + V_x \omega_z \int \rho u_z \, dx + \omega_x \omega_y \int \rho x u_z \, dx - \omega_x \omega_z \int \rho x u_y \, dx \\
&\quad + (\omega_y^2 - \omega_z^2) \int \rho u_y u_z \, dx + \omega_y \omega_z \left[J_{yy} - J_{zz} - \int \rho (u_y^2 - u_z^2) dx\right] \\
&\quad - 2\omega_x \int \rho (u_y v_y + u_z v_z) dx - C\theta_3 \frac{\partial V}{\partial \theta_1} - \frac{S\theta_3}{C\theta_1} \frac{\partial V}{\partial \theta_2} - \frac{S\theta_1 S\theta_3}{C\theta_1} \frac{\partial V}{\partial \theta_3} + M_x \qquad (14l)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\int \rho u_z dx\right)\dot{V}_x - m_1 \bar{x} \dot{V}_z - \left(\int \rho x u_z \, dx\right)\dot{\omega}_x + \left(J_{yy} + \int \rho u_z^2 \, dx\right)\dot{\omega}_y - \left(\int \rho u_y u_z \, dx\right)\dot{\omega}_z \\
&\quad + \int \rho (u_z \dot{v}_x - x \dot{v}_z) dx = m_1 \bar{x} V_y \omega_x - \left(V_z \int \rho u_z \, dx + m_1 \bar{x} V_x\right)\omega_y + V_y \omega_z \int \rho u_z \, dx \\
&\quad - \omega_z \omega_x \left(J_{xx} - J_{zz} + \int \rho u_z^2 \, dx\right) - (\omega_x^2 - \omega_z^2) \int \rho x u_z \, dx + \omega_y \omega_z \int \rho x u_y \, dx \\
&\quad - \omega_x \omega_y \int \rho u_y u_z \, dx + 2\omega_x \int \rho x v_y \, dx - 2\omega_y \int \rho u_z v_z \, dx \\
&\quad + 2\omega_z \int \rho u_z v_y \, dx + S\theta_3 \frac{\partial V}{\partial \theta_1} - \frac{C\theta_3}{C\theta_1} \frac{\partial V}{\partial \theta_2} - \frac{S\theta_1 C\theta_3}{C\theta_1} \frac{\partial V}{\partial \theta_3} + M_y \qquad (14m)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\int \rho u_y \, dx\right)\dot{V}_x + m_1 \bar{x} \dot{V}_y - \left(\int \rho x u_z \, dx\right)\dot{\omega}_x - \left(\int \rho u_y u_z \, dx\right)\dot{\omega}_y + \left(J_{zz} + \int \rho u_y \, dx\right)\dot{\omega}_x \\
&\quad + \int \rho (x \dot{v}_y - u_y \dot{v}_x) dx = m_1 \bar{x} V_z \omega_x + V_z \omega_y \int \rho u_y \, dx - \left(V_y \int \rho u_y \, dx - V_x m_1 \bar{x}\right)\omega_z \\
&\quad + \omega_x \omega_y \left(J_{xx} - J_{yy} + \int \rho u_y^2 \, dx\right) + (\omega_x^2 - \omega_y^2) \int \rho x u_y \, dx - \omega_y \omega_z \int \rho x u_z \, dx \\
&\quad + \omega_x \omega_z \int \rho u_y u_z \, dx + 2\omega_x \int \rho x v_z \, dx + 2\omega_y \int \rho u_y v_z \, dx - 2\omega_z \int \rho u_y v_y \, dx - \frac{\partial V}{\partial \theta_3} + M_z \\
&\qquad (14n)
\end{aligned}
$$

$$
\begin{aligned}
&\rho \dot{V}_y - \rho u_z \dot{\omega}_x + \rho x \dot{\omega}_z + \rho \dot{v}_y = \rho V_z \omega_x - \rho V_x \omega_z - \rho x \omega_x \omega_y + \rho (\omega_x^2 + \omega_z^2) u_y \\
&\quad - \rho \omega_y \omega_z u_z + 2\rho v_z \omega_x - \mathcal{L}_y u_y + \hat{U}_y \qquad (14o)
\end{aligned}
$$

$$
\begin{aligned}
&\rho \dot{V}_z + \rho u_y \dot{\omega}_x - \rho x \dot{\omega}_y + \rho \dot{v}_z = - \rho V_y \omega_x + \rho V_x \omega_y - \rho x \omega_x \omega_z - \rho \omega_y \omega_z u_y \\
&\quad + \rho (\omega_x^2 + \omega_y^2) u_z - 2\rho v_y \omega_x - \mathcal{L}_z u_z + \hat{U}_z \qquad (14p)
\end{aligned}
$$

where m_1 is the mass of the appendage, $S\theta_i = \sin \theta_i$, $C\theta_i = \cos \theta_i (i = 1,2,3)$ and

$$\mathcal{L}_y = \frac{\partial^2}{\partial x^2}\left(EI_y\frac{\partial^2}{\partial x^2}\right) - \frac{\partial}{\partial x}\left[\left(\int_x^L \rho\omega_y^2\zeta\, d\zeta\right)\frac{\partial}{\partial x}\right] \quad (15a)$$

$$\mathcal{L}_z = \frac{\partial^2}{\partial x^2}\left(EI_z\frac{\partial^2}{\partial x^2}\right) - \frac{\partial}{\partial x}\left[\left(\int_x^L \rho\omega_z^2\zeta\, d\zeta\right)\frac{\partial}{\partial x}\right] \quad (15b)$$

in which E is the modulus of elasticity and I_y and I_z are area moments of inertia. The operators $\mathcal{L}_y$ and $\mathcal{L}_z$ include the effects of bending and of the axial force on the appendage [7].

Summary and Conclusions

In deriving the equations of motion for flexible bodies by the Lagrangian approach, it is common practice to express the rotational motion in terms of angular velocities about nonorthogonal axes, which tends to complicate the equations. Moreover, this creates difficulties in feedback control, in which the torque actuators apply moments about body axes and the output of sensors measuring angular motion is also expressed in terms of components about the body axes. The same can be said about force actuators and translational motion sensors. It turns out that the equations of motion are appreciably simpler when the rigid-body translations and rotations are expressed in terms of components about the body axes. Such equations can be obtained by introducing the concept of quasi-coordinates. The concept of quasi-coordinates was used earlier by this author to derive equations of motion of rotating bodies with flexible appendages, but never in the general context considered here. Indeed, in this paper, Lagrange's equations in terms of quasi-coordinates are derived for a distributed flexible body undergoing arbitrary rigid-body translations and rotations, in addition to elastic deformations. The second-order differential equations in time for the hybrid system are then transformed into a set of hybrid state equations suitable for control design. The approach is demonstrated by deriving the hybrid state equations of motion for a spacecraft consisting of a rigid body with a flexible appendage in the form of a beam.

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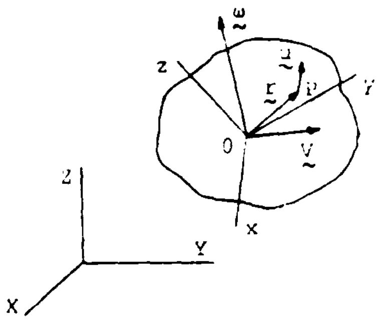

Figure 1. The Flexible Body in Space

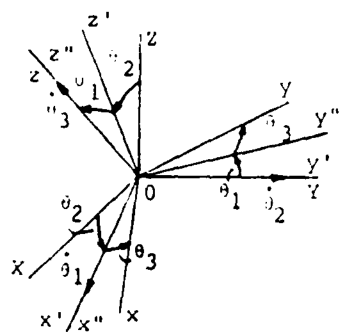

Figure 2. The Angular Motions

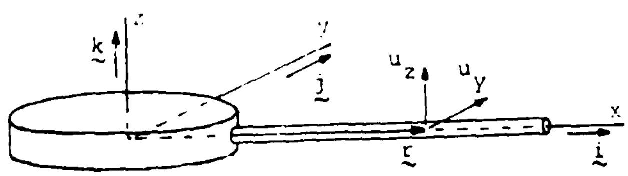

Figure 3. A Rigid Spacecraft with a Flexible Appendage

State equations for a spacecraft with maneuvering flexible appendages in terms of quasi-coordinates

L Meirovitch and M K Kwak

Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061

This paper is concerned with the derivation of the state equations of motion for a spacecraft consisting of a main rigid platform and a given number of flexible appendages changing the orientation relative to the main body. The equations are derived by means of Lagrange's equations in terms of quasi-coordinates. Assuming that the appendages represent distributed-parameter members, the state equations of motion are hybrid. Moreover, they are nonlinear. Following spatial discretization and truncation, the hybrid equations reduce to a system of nonlinear discretized state equations, which are more practical for numerical calculations and control design. To illustrate the effect of nonlinearity on the dynamic response during reorientation, a numerical example involving spacecraft with a membrane-like antenna is presented.

I. INTRODUCTION

In many space applications, it becomes necessary to reorient a certain line of sight in a spacecraft. Examples of this are the reorientation of a space telescope or of an antenna in a spacecraft. In some cases, such as in the space telescope, the line of sight can be regarded as being fixed relative to the undeformed structure, in which cases reorientation of the line of sight implies maneuvering of the whole spacecraft [Turner and Junkins (1980), Breakwell (1981), Baruh and Silverberg (1984), Meirovitch and Quinn (1987), Meirovitch and Sharony (1987) and Quinn and Meirovitch (1988)]. However, many spacecraft can be represented by mathematical models consisting of a rigid platform with one or more flexible appendages, such as flexible antennas, so that the mission involves the maneuvering of a hybrid (lumped and distributed flexible) system. Quite often, the line of sight coincides with an axis fixed in a small component of the spacecraft, such as an antenna, in which case it may be more advisable to retarget only the antenna and not the entire spacecraft. This is particularly true when the inertia of the antenna is much smaller than the inertia of the spacecraft. The argument becomes even stronger when several antennas must be retargeted independently in space. In such cases, it appears more sensible to conceive of a spacecraft consisting of a platform stabilized in an inertial space with several appendages, rigid or flexible, hinged to the platform and capable of pivoting about two orthogonal axes relative to the platform [Meirovitch and Kwak (1988a, 1988b) and Meirovitch and France (1989)]. In this case, reorientation relative to the stabilized platform is equivalent to retargeting in an inertial space. Note that such a maneuvering spacecraft is characterized by the fact that its configuration varies with time. This paper is concerned with the mission of independent retargeting of the line of sight of each antenna relative to the inertial space.

One problem encountered in the dynamics of a complex system is how to derive the equations of motion efficiently. In general, the equations of motion for a flexible spacecraft have very complicated expressions [Grote, McMunn and Gluck (1970) and Likins (1972)], so that new methods for deriving equations of motion have been proposed [Ho (1977) and Kane and Levinson (1980)]. Kane and Levinson compared seven different methods. Lagrange's equations of motion in terms of quasi-coordinates for a hybrid system were derived first by Meirovitch (1966) and then by Williams (1976) and Brown (1981). Recently, Meirovitch (1988) and Meirovitch and Kwak (1989) showed that Lagrange's equations of motion in terms of quasi-coordinates are quite useful for deriving the equations of motion for the maneuvering and control of flexible spacecraft. Because the derived equations of motion are based on body-fixed coordinates, control design based on such equations is very convenient.

The mathematical formulation for a spacecraft including a rigid platform and several flexible antennas consists of a hybrid set of equations of motion, in the sense that there are six ordinary differential equations for the rigid-body translations and rotations of the platform and partial differential equations for the elastic motion of each antenna. The equations of motion are not only hybrid, but the maneuvering of the antennas relative to the platform according to some prescribed function of time introduces time-dependent coefficients into the equations. Moreover, the equations contain terms reflecting persistent disturbances caused by inertial forces. Because both numerical simulation and control design of systems governed by sets of hybrid differential equations are not feasible, it is necessary to discretize the partial differential equations in space, which can be carried out by the classical Rayleigh-Ritz method or the finite element method [Meirovitch (1980)].

II. GENERAL LAGRANGE'S EQUATIONS IN TERMS OF QUASI-COORDINATES

Let us consider a system consisting of a main rigid body, acting as a platform, and a certain number of flexible appendages hinged to the main rigid body. The interest lies in reorienting the flexible appendages independently so as to point in different preselected directions in the inertial space. The object is to derive the equations of motion capable of describing this task.

To describe the motion of the platform, we introduce a set of inertial axes XYZ and a set of body axes xyz attached to the rigid platform. Then, the motion of the platform can be defined in terms of three translations and three rotations of the body axes xyz relative to the inertial axes XYZ. To describe the motion of the flexible appendages, we consider a typical appendage hinged at point e and regard e as the origin of a set of body axes $x_e y_e z_e$ embedded in the appendage in its undeformed state. Then, the motion of a nominal point of the appendage consists of the motion of xyz, the motion of $x_e y_e z_e$ relative to xyz and the elastic motion relative to $x_e y_e z_e$. The system and the various reference frames are shown in Fig. 1.

From Fig. 1, the position vector of a point in the rigid body and in the appendage can be written as

$$\underline{R}_r = \underline{R}_o + \underline{r} \tag{1a}$$

and

$$\underline{R}_e = \underline{R}_o + \underline{r}_{oe} + \underline{r}_e + \underline{u}_e, \quad e = 1,2,\ldots,N \tag{1b}$$

where $\underline{R}_o$ is the radius vector from O to o, $\underline{r}$ is the position vector of a nominal point in the rigid body relative to xyz, $\underline{r}_{oe}$ is the radius vector from o to e, $\underline{r}_e$ is the position vector of a nominal point in undeformed appendage relative to $x_e y_e z_e$ and $\underline{u}_e$ is the elastic displacement of that point. Vector $\underline{R}_o$ is given in terms of components along XYZ, $\underline{r}$ and $\underline{r}_{oe}$ in terms of components along xyz, and $\underline{r}_e$ and $\underline{u}_e$ in terms of components along $x_e y_e z_e$.

The velocity vector of o can be written in terms of components along xyz in the form

$$\underline{V}_o = C\dot{\underline{R}}_o \tag{2a}$$

where C is the matrix of direction cosines between xyz and XYZ and $\dot{\underline{R}}_o$ is the velocity vector of o in terms of components along XYZ. Matrix C depends on the angular displacements θ_i (i = 1,2,3) defining the orientation of axes xyz relative to axes XYZ. Furthermore, the angular velocity vector of axes xyz in terms of components along xyz is given by

$$\underline{\omega} = D\dot{\underline{\theta}} \tag{2b}$$

where $\dot{\underline{\theta}}$ is a vector of angular velocities $\dot{\theta}_i$ and D is a matrix depending on the angular displacements θ_i (i = 1,2,3). Figure 2 shows a set of such angular displacements. For this choice of angles, the matrices C and D are as follows:

$$C = \begin{bmatrix} c\theta_2 c\theta_3 & c\theta_1 s\theta_3 + s\theta_1 s\theta_2 c\theta_3 & s\theta_1 s\theta_3 - c\theta_1 s\theta_2 c\theta_3 \\ -c\theta_2 s\theta_3 & c\theta_1 c\theta_3 - s\theta_1 s\theta_2 s\theta_3 & s\theta_1 c\theta_3 + c\theta_1 s\theta_2 s\theta_3 \\ s\theta_2 & -s\theta_1 c\theta_2 & c\theta_1 c\theta_2 \end{bmatrix} \tag{3a}$$

$$D = \begin{bmatrix} c\theta_2 c\theta_3 & s\theta_3 & 0 \\ -c\theta_2 s\theta_3 & c\theta_3 & 0 \\ s\theta_2 & 0 & 1 \end{bmatrix} \tag{3b}$$

where $c\theta_i = \cos\theta_i$ and $s\theta_i = \sin\theta_i$. Note that this choice of angles helps us avoid singularities at the initial stage of the motion, $\theta_i = 0$.

In view of the above, the velocity vector of a point in the rigid body in terms of components along xyz is simply

$$\underline{V}_r = \underline{V}_o + \underline{\omega} \times \underline{r} \tag{4a}$$

and that of a point in the typical appendage e in terms of components along $x_e y_e z_e$ is

$$\underline{V}_e = E_e(\underline{V}_o + \underline{\omega} \times \underline{r}_{oe}) + (E_e\underline{\omega} + \underline{\omega}_e) \times (\underline{r}_e + \underline{u}_e) + \underline{v}_e$$
$$e = 1,2,\ldots,N \tag{4b}$$

where $\underline{\omega}_e$ is the angular velocity vector of axes $x_e y_e z_e$, E_e is a matrix of direction cosines between the $x_e y_e z_e$ and xyz and $\underline{v}_e$ is the elastic velocity of the point in the appendage relative to $x_e y_e z_e$, $\underline{v}_e = \dot{\underline{u}}_e$. In the maneuver proposed, the angular velocity vectors $\underline{\omega}_e$ of $x_e y_e z_e$ relative to xyz are given, so that the rotational motions of the appendages relative to the platform do not add degrees of freedom. The only degrees of freedom arise from the rigid-body translations and rotations of the platform and the elastic displacements of the appendages.

The equations of motion can be obtained by means of Lagrange's equations in terms of quasi-coordinates [Meirovitch (1988)].

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{V}_o}\right) + \tilde{\omega}\frac{\partial L}{\partial \underline{V}_o} - C\frac{\partial L}{\partial \underline{R}_o} = \underline{F} \tag{5a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{\omega}}\right) + \tilde{V}_o\frac{\partial L}{\partial \underline{V}_o} + \tilde{\omega}\frac{\partial L}{\partial \underline{\omega}} - (D^T)^{-1}\frac{\partial L}{\partial \underline{\theta}} = \underline{M} \tag{5b}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_e}{\partial \underline{v}_e}\right) - \frac{\partial \hat{T}_e}{\partial \underline{u}_e} + \mathscr{L}_e\underline{u}_e = \hat{\underline{U}}_e, \quad e = 1,2,\ldots,N \tag{5c}$$

where

$$L = T - V \tag{6}$$

is the Lagrangian in which T is the kinetic energy and V is the potential energy, $\hat{L}_e$ is the Lagrangian density and $\hat{T}_e$ is the kinetic energy density, both for appendage e, and $\mathscr{L}_e$ is a matrix of homogenous differential stiffness operators. The terms $\underline{F}$ and $\underline{M}$ on the right side of Eqs. (5a) and (5b), respectively, are the force and torque vectors on the platform, both in terms of components along axes xyz, and the term $\hat{\underline{U}}_e$ on the right side of Eq. (5c) is a distributed force vector on appendage e in terms of components along $x_e y_e z_e$. Equations (5) are hybrid in the sense that Eqs. (5a) and (5b) are ordinary differential equations and Eqs. (5c) are partial differential equations. It should be noted that the tilde over a symbol indicates a skew symmetric matrix with entries corresponding to the components of the associated vector [Meirovitch and Kwak (1988a)]. For the system of Fig. 1, we write the kinetic energy

$$T = \frac{1}{2}\int_{D_r}\rho_r\underline{V}_r^T\underline{V}_r dD_r + \sum_{e=1}^{N}\frac{1}{2}\int_{D_e}\rho_e\underline{V}_e^T\underline{V}_e dD_e$$
$$= \frac{1}{2}m_t\underline{V}_o^T\underline{V}_o + \underline{V}_o^T\tilde{S}_t^T\underline{\omega} + \frac{1}{2}\underline{\omega}^T I_t\underline{\omega} + \sum_{e=1}^{N}\Big[\frac{1}{2}\underline{\omega}_e^T I_e\underline{\omega}_e$$
$$+ \frac{1}{2}\int_{D_e}\rho_e\underline{v}_e^T\underline{v}_e dD_e + (\underline{V}_o + \tilde{r}_{oe}^T\underline{\omega})^T E_e^T(\tilde{S}_e^T\underline{\omega}_e + \int_{D_e}\rho_e\underline{v}_e dD_e)$$
$$+ \underline{\omega}^T E_e^T I_e\underline{\omega}_e + (E_e\underline{\omega} + \underline{\omega}_e)^T\int_{D_e}\rho_e(\tilde{r}_e + \tilde{u}_e)\underline{v}_e dD_e\Big] \tag{7}$$

where

$$m_t = m_r + \sum_{e=1}^{N}m_e, \quad m_r = \int_{D_r}\rho_r dD_r, \quad m_e = \int_{D_e}\rho_e dD_e \tag{8a,b,c}$$

$$\tilde{S}_t = \tilde{S}_r + \sum_{e=1}^{N}(m_e\tilde{r}_{oe} + E_e^T\tilde{S}_eE_e) \quad (8d)$$

$$\tilde{S}_r = \int_{D_r}\rho_r\tilde{r}\,dD_r\,,\quad \tilde{S}_e = \int_{D_e}\rho_e(\tilde{r}_e + \tilde{u}_e)dD_e \quad (8e,f)$$

$$I_t = I_r + \sum_{e=1}^{N}(m_e\tilde{r}_{oe}\tilde{r}_{oe}^T + E_e^TI_eE_e - \tilde{r}_{oe}E_e^T\tilde{S}_eE_e - E_e^T\tilde{S}_eE_e\tilde{r}_{oe}) \quad (8g)$$

$$I_r = \int_{D_r}\rho_r\tilde{r}\tilde{r}^TdD_r\,,\quad I_e = \int_{D_e}\rho_e(\tilde{r}_e + \tilde{u}_e)(\tilde{r}_e + \tilde{u}_e)^TdD_e \quad (8i,j)$$

in which ρ_r and ρ_e are mass densities and D_r and D_e are the domain of the rigid platform and of a typical appendage, respectively, m_t is the total mass of the system, $\tilde{S}_t$ is a skew symmetric matrix of first moments of inertia for the system and I_t is the inertia matrix. For simplicity, we assume that the potential energy is due entirely to elastic effects, in which case it can be written in the form

$$V = \frac{1}{2}\sum_{e=1}^{N}[\,\underline{u}_e\,,\underline{u}_e\,] = \sum_{e=1}^{N}\int_{D_e}\underline{u}_e^T\mathscr{L}_e\underline{u}_edD_e \quad (9)$$

where [,] denotes an energy inner product [Meirovitch (1980)]. We note that, in deriving Eq. (9), the boundary conditions were fully considered [Meirovitch (1980)].

III. HYBRID NONLINEAR STATE EQUATIONS OF MOTION

For control purposes, or for mere integration of the equations of motion, it is convenient to work with state equations. As the state vector, we consider

$$\underline{x} = [\,\underline{R}_o^T\ \underline{\theta}^T\ \underline{u}_1^T\ \underline{u}_2^T\ \ldots\ \underline{u}_N^T\ \underline{V}_o^T\ \underline{\omega}^T\ \underline{v}_1^T\ \underline{v}_2^T\ \ldots\ \underline{v}_N^T\,]^T \quad (10)$$

which represents a unique combination of inertial coordinates, angular displacements, elastic deformations, translational velocities, angular velocities and elastic velocities, the last four being in terms of components about the body axes. In view of the definition of the state vector, one half of the state equations consists of the kinematical relations

$$\dot{\underline{R}}_o = C^T\underline{V}_o\,,\quad \dot{\underline{\theta}} = D^{-1}\underline{\omega}\,,\quad \dot{\underline{u}}_e = \underline{v}_e\,,\quad e = 1,2,\ldots,N \quad (11a,b,c)$$

Inserting Eqs. (7) and (9) into Eqs. (5) and considering Eqs. (2) and (3), we obtain the hybrid nonlinear Lagrange's equations for the other half of the state equations

$$m_t\dot{\underline{V}}_o + \tilde{S}_t^T\dot{\underline{\omega}} + \sum_{e=1}^{N}E_e^T\int_{D_e}\rho_e\dot{\underline{v}}_edD_e = -m_t\tilde{\omega}\underline{V}_o + \tilde{\omega}\tilde{S}_t\underline{\omega}$$
$$+ \sum_{e=1}^{N}E_e^T\{2(\widetilde{[\tilde{\omega}_e\underline{S}_e]} + \tilde{S}_{ev})E_e\underline{\omega} - 2\tilde{\omega}_e\int_{D_e}\rho_e\underline{v}_edD_e + \tilde{\omega}_e\tilde{S}_e\underline{\omega}_e$$
$$+ \tilde{S}_e\dot{\underline{\omega}}_e\} + \underline{F} \quad (11d)$$

$$\tilde{S}_t\dot{\underline{V}}_o + I_t\dot{\underline{\omega}} + \sum_{e=1}^{N}\int_{D_e}\rho_e[\tilde{r}_{oe}E_e^T + E_e^T(\tilde{r}_e + \tilde{u}_e)]\dot{\underline{v}}_edD_e = \tilde{S}_t\tilde{V}_o\underline{\omega}$$
$$-\tilde{\omega}I_t\underline{\omega} + \sum_{e=1}^{N}\{[-E_e^T(2\tilde{\omega}_eI_e - \mathrm{tr}\,I_e\tilde{\omega}_e) + 2\tilde{r}_{oe}E_e^T\widetilde{[\tilde{\omega}_e\underline{S}_e]} + \tilde{S}_{ev})$$
$$+ 2E_e^T\int_{D_e}\rho_e(\tilde{r}_e + \tilde{u}_e)\tilde{v}_edD_e]E_e\underline{\omega}$$
$$-\int_{D_e}\rho_e[\tilde{r}_{oe}E_e^T\tilde{\omega}_e + E_e^T\tilde{\omega}_e(\tilde{r}_e + \tilde{u}_e)]\underline{v}_edD_e$$
$$+ \tilde{r}_{oe}E_e^T(\tilde{\omega}_e\tilde{S}_e\underline{\omega}_e + \tilde{S}_e\dot{\underline{\omega}}_e + \tilde{S}_{ev}\underline{\omega}_e)$$
$$- E_e^T(\tilde{\omega}_eI_e\underline{\omega}_e + I_e\dot{\underline{\omega}}_e + I_{ev}\underline{\omega}_e)\} + \underline{M} \quad (11e)$$

$$\rho_e\{E_e\dot{\underline{V}}_o + [E_e\tilde{r}_{oe}^T + (\tilde{r}_e + \tilde{u}_e)^TE_e]\dot{\underline{\omega}} + \dot{\underline{v}}_e\}$$
$$= \rho_e(-\widetilde{[E_e\underline{\omega}]}\,E_e(\underline{V}_o - \tilde{r}_{oe}\underline{\omega})$$
$$+ 2[\tilde{v}_e - (\tilde{r}_e + \tilde{u}_e)\tilde{\omega}_e + \tilde{\omega}_e(\tilde{r}_e + \tilde{u}_e)]E_e\underline{\omega} - 2\tilde{\omega}_e\underline{v}_e$$
$$- (\widetilde{[E_e\underline{\omega}]}^2 + \dot{\tilde{\omega}}_e + \tilde{\omega}_e^2)(\underline{r}_e + \underline{u}_e)) - \mathscr{L}_e\underline{u}_e + \hat{\underline{U}}_e$$
$$e = 1,2,\ldots,N \quad (11f)$$

where tr denotes a trace of a matrix and

$$\tilde{S}_{ev} = \int_{D_e}\rho_e\tilde{v}_edD_e \quad (12a)$$

$$I_{ev} = \int_{D_e}\rho_e[\tilde{v}_e(\tilde{r}_e + \tilde{u}_e)^T + (\tilde{r}_e + \tilde{u}_e)\tilde{v}_e^T]dD_e \quad (12b)$$

IV. THE DISCRETIZED NONLINEAR STATE EQUATIONS OF MOTION

The equations of motion are hybrid, in the sense that the equations for the rigid-body translations and rotations of the platform are ordinary differential equations and those for the elastic motions of the appendages are partial differential equations. Moreover, because of the maneuver angular velocity vector $\underline{\omega}_e$, which is a given function of time, they possess time-dependent coefficients. Control design of systems described by hybrid equations is not feasible, so that we wish to discretize the partial differential equations in space, leaving us with only ordinary differential equations. To this end, we express the elastic displacements as linear combinations of space-dependent admissible functions multiplied by time-dependent generalized coordinates, or

$$\underline{u}_e(\underline{r}_e, t) = \Phi_e(\underline{r}_e)\underline{q}_e(t)\,,\quad e = 1,2,\ldots,N \quad (13)$$

where Φ_e is a matrix of admissible functions and $\underline{q}_e$ is a vector of generalized coordinates.

The Lagrangian equations in terms of quasi-coordinates for the rigid body motions of the platform remain in the form of Eqs. (5a) and (5b). On the other hand, inserting Eq. (13) into Eqs. (5c), we obtain the ordinary differential equations for the discretized elastic motions

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\underline{p}_e}\right) - \frac{\partial L}{\partial\underline{q}_e} = \underline{Q}_e\,,\quad e = 1,2,\ldots,N \quad (14)$$

where

$$\underset{\sim}{p}_e = \dot{\underset{\sim}{q}}_e \tag{15}$$

$$\underset{\sim}{Q}_e = \int_{D_e} \Phi_e^T \hat{\underset{\sim}{U}}_e dD_e \, , \quad e = 1,2,\ldots,N \tag{16}$$

are corresponding vectors of generalized forces.

The Lagrangian remains in the form (6) but the kinetic energy and potential energy change. Indeed, introducing Eq. (13) into Eq. (7), we obtain the discretized kinetic energy

$$T = \frac{1}{2} m_t \underset{\sim}{V}_o^T \underset{\sim}{V}_o + \underset{\sim}{V}_o^T \tilde{S}_t^T \underset{\sim}{\omega} + \frac{1}{2} \underset{\sim}{\omega}^T I_t \underset{\sim}{\omega}$$

$$+ \sum_{e=1}^{N} [\, \frac{1}{2} \underset{\sim}{\omega}_e^T I_e \underset{\sim}{\omega}_e + \frac{1}{2} \underset{\sim}{p}_e^T \int_{D_e} \rho_e \Phi_e^T \Phi_e dD_e \underset{\sim}{p}_e$$

$$+ (\underset{\sim}{V}_o + \tilde{r}_{oe}^T \underset{\sim}{\omega})^T E_e^T (\tilde{S}_e^T \underset{\sim}{\omega}_e + \int_{D_e} \rho_e \Phi_e dD_e \underset{\sim}{p}_e)$$

$$+ \underset{\sim}{\omega}^T E_e^T I_e \underset{\sim}{\omega}_e + (E_e \underset{\sim}{\omega} + \underset{\sim}{\omega}_e)^T \int_{D_e} \rho_e (\tilde{r}_e + \widetilde{[\Phi_e \underset{\sim}{q}_e]}) \Phi_e dD_e \underset{\sim}{p}_e \,] \tag{17}$$

Many of the quantities in Eq. (17) are defined by Eqs. (8), with the exception of

$$\tilde{S}_e = \int_{D_e} \rho_e (\tilde{r}_e + \widetilde{[\Phi_e \underset{\sim}{q}_e]}) dD_e \tag{18a}$$

$$I_e = \int_{D_e} \rho_e (\tilde{r}_e + \widetilde{[\Phi_e \underset{\sim}{q}_e]})(\tilde{r}_e + \widetilde{[\Phi_e \underset{\sim}{q}_e]})^T dD_e \tag{18b}$$

Moreover, inserting Eq. (13) into Eq. (9), the discretized potential energy has the form

$$V = \frac{1}{2} \sum_{e=1}^{N} \underset{\sim}{q}_e^T [\, \Phi_e \, , \Phi_e \,] \underset{\sim}{q}_e \;=\; \frac{1}{2} \sum_{e=1}^{N} \underset{\sim}{q}_e^T K_e \underset{\sim}{q}_e \tag{19}$$

where

$$K_e = [\, \Phi_e \, , \Phi_e \,] \tag{20}$$

Following the same procedure as used earlier, the discretized nonlinear state equations can be written as follows:

$$m_t \dot{\underset{\sim}{V}}_o + \tilde{S}_t^T \dot{\underset{\sim}{\omega}} + 2 \sum_{e=1}^{N} E_e^T (\widetilde{[\tilde{S}_e \underset{\sim}{\omega}_e]} - \tilde{S}_{ev}) E_e \underset{\sim}{\omega} + \sum_{e=1}^{N} E_e^T \overline{\Phi}_e \dot{\underset{\sim}{p}}_e$$

$$+ 2 \sum_{e=1}^{N} E_e^T \tilde{\omega}_e \overline{\Phi}_e \underset{\sim}{p}_e + m_t \tilde{\omega} \underset{\sim}{V}_o - \tilde{\omega} \tilde{S}_t \underset{\sim}{\omega}$$

$$= \underset{\sim}{F} + \sum_{e=1}^{N} E_e^T (\tilde{S}_e \dot{\underset{\sim}{\omega}}_e + \tilde{\omega}_e \tilde{S}_e \underset{\sim}{\omega}_e) \tag{21a}$$

$$\tilde{S}_t \dot{\underset{\sim}{V}}_o + I_t \dot{\underset{\sim}{\omega}} + \sum_{e=1}^{N} \{ E_e^T (2 \tilde{\omega}_e I_e - \mathrm{tr}\, I_e \tilde{\omega}_e) + 2 \tilde{r}_{oe} E_e^T (\widetilde{[\tilde{S}_e \underset{\sim}{\omega}_e]} - \tilde{S}_{ev})$$

$$- 2 E_e^T \int_{D_e} \rho_e (\tilde{r}_e + \widetilde{[\Phi_e \underset{\sim}{q}_e]}) \widetilde{[\Phi_e \underset{\sim}{p}_e]} dD_e \} \, E_e \underset{\sim}{\omega}$$

$$+ \sum_{e=1}^{N} (E_e^T \tilde{\Phi}_e + \tilde{r}_{oe} E_e^T \overline{\Phi}_e + E_e^T \int_{D_e} \rho_e \widetilde{[\Phi_e \underset{\sim}{p}_e]} \Phi_e dD_e) \dot{\underset{\sim}{p}}_e$$

$$+ \sum_{e=1}^{N} [\tilde{r}_{oe} E_e^T \tilde{\omega}_e \overline{\Phi}_e + E_e^T \tilde{\omega}_e \tilde{\Phi}_e + E_e^T \tilde{\omega}_e \int_{D_e} \rho_e \widetilde{[\Phi_e \underset{\sim}{q}_e]} \Phi_e dD_e] \underset{\sim}{p}_e$$

$$- \tilde{S}_t \tilde{V}_o \underset{\sim}{\omega} + \tilde{\omega} I_t \underset{\sim}{\omega} = \underset{\sim}{M} + \sum_{e=1}^{N} [\tilde{r}_{oe} E_e^T (\tilde{S}_e \dot{\underset{\sim}{\omega}}_e + \tilde{\omega}_e \tilde{S}_e \underset{\sim}{\omega}_e + \tilde{S}_{ev} \underset{\sim}{\omega}_e)$$

$$- E_e^T (I_e \dot{\underset{\sim}{\omega}}_e + \tilde{\omega}_e I_e \underset{\sim}{\omega}_e + I_{ev} \underset{\sim}{\omega}_e)] \tag{21b}$$

$$\overline{\Phi}_e^T E_e \dot{\underset{\sim}{V}}_o + \overline{\Phi}_e^T E_e \tilde{\omega} \underset{\sim}{V}_o + (\overline{\Phi}_e^T E_e \tilde{r}_{oe}^T + \tilde{\Phi}_e^T E_e$$

$$+ \int_{D_e} \rho_e \Phi_e^T \widetilde{[\Phi_e \underset{\sim}{q}_e]}^T dD_e E_e) \dot{\underset{\sim}{\omega}}$$

$$+ 2 [\tilde{\Phi}_e^T \tilde{\omega}_e^T + \int_{D_e} \rho_e \Phi_e^T (\widetilde{[\Phi_e \underset{\sim}{q}_e]} \tilde{\omega}_e - \tilde{\omega}_e \tilde{r}_e - \widetilde{[\Phi_e \underset{\sim}{p}_e]}$$

$$- \tilde{\omega}_e \widetilde{[\Phi_e \underset{\sim}{q}_e]}) dD_e] E_e \underset{\sim}{\omega} + M_e \dot{\underset{\sim}{p}}_e + 2 \tilde{H}_e(\underset{\sim}{\omega}_e) \underset{\sim}{p}_e$$

$$+ [K_e + \overline{H}_e(\underset{\sim}{\omega}_e) + \overline{H}_e(E_e \underset{\sim}{\omega}) + \tilde{H}_e(\dot{\underset{\sim}{\omega}}_e)] \underset{\sim}{q}_e$$

$$= \underset{\sim}{Q}_e - \tilde{\Phi}_e^T \dot{\underset{\sim}{\omega}}_e + \int_{D_e} \rho_e \Phi_e^T \tilde{\omega}_e \tilde{r}_e \underset{\sim}{\omega}_e dD_e$$

$$+ \int_{D_e} \rho_e \Phi_e^T \widetilde{[E_e \underset{\sim}{\omega}]}^2 \underset{\sim}{r}_e dD_e + \overline{\Phi}_e^T \widetilde{[E_e \underset{\sim}{\omega}]} E_e \tilde{r}_e \underset{\sim}{\omega} \tag{21c}$$

where

$$M_e = \int_{D_e} \rho_e \Phi_e^T \Phi_e dD_e \tag{22a}$$

$$\overline{\Phi}_e = \int_{D_e} \rho_e \Phi_e dD_e \tag{22b}$$

$$\tilde{\Phi}_e = \int_{D_e} \rho_e \tilde{r}_e \Phi_e dD_e \tag{22c}$$

$$\tilde{H}_e(\underset{\sim}{a}) = \int_{D_e} \rho_e \Phi_e^T \tilde{a} \Phi_e \, dD_e \tag{22d}$$

$$\overline{H}_e(\underset{\sim}{a}) = \int_{D_e} \rho_e \Phi_e^T \tilde{a}^2 \Phi_e \, dD_e \tag{22e}$$

The state vector is redefined as

$$\underset{\sim}{x} = [\, \underset{\sim}{R}_o^T \; \underset{\sim}{\theta}^T \; \underset{\sim}{q}_1^T \; \underset{\sim}{q}_2^T \; \cdots \; \underset{\sim}{q}_N^T \; \underset{\sim}{V}_o^T \; \underset{\sim}{\omega}^T \; \underset{\sim}{p}_1^T \; \underset{\sim}{p}_2^T \; \cdots \; \underset{\sim}{p}_N^T \,]^T \tag{23}$$

In addition, Eq. (11c) is replaced by

$$\dot{\underset{\sim}{q}}_e = \underset{\sim}{p}_e \, , \quad e = 1,2,\ldots,N \tag{24}$$

Equations (11a,b), (24) and (21) represent the discretized nonlinear state equations.

V. NUMERICAL EXAMPLE

The effect of nonlinearity on the system response is illustrated by means of a spacecraft consisting of a rigid platform with a single membrane-type flexible appendage (Fig. 3). The maneuver of the appendage relative to the platform was carried out by means of a smoothed bang-bang [Meirovitch and Quinn (1987)] for the angular acceleration, where the smoothing of the bang-bang was done to reduce the elastic deformations of the appendage. The elastic vibration of the appendage, treated as a circular membrane clamped at $r = a$, was represented by ten degrees of freedom in the z-direction, i.e., by ten admissible functions in the discretization(-in-space) process, so that the matrix Φ_e in Eq. (13) is 3 × 10, or

$$\Phi_e = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \phi_1 & \phi_2 & \cdots & \phi_{10} \end{bmatrix}$$

in which

$$\phi_1 = \kappa_1 \frac{J_0(\beta_{01} r)}{J_1(\beta_{01} a)}, \quad \phi_2 = \kappa_1 \frac{J_0(\beta_{02} r)}{J_1(\beta_{02} a)}$$

$$\begin{matrix}\phi_3 \\ \phi_7\end{matrix} = \kappa_2 \frac{J_1(\beta_{11} r)}{J_2(\beta_{11} a)} \begin{matrix}\cos\theta \\ \sin\theta\end{matrix}, \quad \begin{matrix}\phi_4 \\ \phi_8\end{matrix} = \kappa_2 \frac{J_1(\beta_{12} r)}{J_2(\beta_{12} a)} \begin{matrix}\cos\theta \\ \sin\theta\end{matrix}$$

$$\begin{matrix}\phi_5 \\ \phi_9\end{matrix} = \kappa_2 \frac{J_2(\beta_{21} r)}{J_3(\beta_{21} a)} \begin{matrix}\cos 2\theta \\ \sin 2\theta\end{matrix}, \quad \begin{matrix}\phi_6 \\ \phi_{10}\end{matrix} = \kappa_2 \frac{J_2(\beta_{22} r)}{J_3(\beta_{22} a)} \begin{matrix}\cos 2\theta \\ \sin 2\theta\end{matrix}$$

where

$$\kappa_1 = \frac{1}{\sqrt{\pi \bar{\rho}_e}\, a}, \quad \kappa_2 = \frac{\sqrt{2}}{\sqrt{\pi \bar{\rho}_e}\, a}$$

In the above, ϕ_1 and ϕ_2 are recognized as axisymmetric modes and the other as antisymmetric modes, respectively [Meirovitch (1967)]. The vector q_e is ten-dimensional. The arguments of the Bessel functions of the first kind can be obtained from $\beta_{01}a = 2.405$, $\beta_{02}a = 5.520$, $\beta_{11}a = 3.832$, $\beta_{12}a = 7.016$, $\beta_{21}a = 5.136$ and $\beta_{22}a = 8.417$. The mass matrix, Eq. (22a), and the stiffness matrix, Eq. (20), are 10 × 10 and have the block-diagonal form

$$M_e = I, \quad K_e = \Lambda$$

where I is the 10 × 10 identity matrix and Λ is a diagonal matrix with the diagonal entries

$$\Lambda(1,1) = \frac{c^2}{a^2}(\beta_{01}a)^2, \quad \Lambda(2,2) = \frac{c^2}{a^2}(\beta_{02}a)^2$$

$$\Lambda(3,3) = \Lambda(7,7) = \frac{c^2}{a^2}(\beta_{11}a)^2$$

$$\Lambda(4,4) = \Lambda(8,8) = \frac{c^2}{a^2}(\beta_{12}a)^2$$

$$\Lambda(5,5) = \Lambda(9,9) = \frac{c^2}{a^2}(\beta_{21}a)^2$$

$$\Lambda(6,6) = \Lambda(10,10) = \frac{c^2}{a^2}(\beta_{22}a)^2$$

where $c = \sqrt{T_m/\bar{\rho}_e}$, in which T_m is the membrane tension and $\bar{\rho}_e$ represents the mass per unit area of the membrane. Moreover, the other matrices given by Eqs. (18) and (22) are given in the Appendix.

Two cases with the same numerical data for the elastic appendage but with different inertia terms for the rigid body are tested. The data for the elastic appendage is as follows:

$$m_e = 0.303\ slugs, \quad \underline{S}_e = (0\ 0\ 1.415)^T\ slugs \cdot ft$$

$$I_e = \begin{bmatrix} 7.712 & 0.0 & 0.0 \\ 0.0 & 7.712 & 0.0 \\ 0.0 & 0.0 & 1.294 \end{bmatrix} slugs \cdot ft^2$$

$$\bar{\rho}_e = 0.01\ slugs/ft^2, \quad a = 3\ ft, \quad c = 20\ ft/s$$

$$\underline{r}_{OR} = (0\ 0\ 1.0)^T\ ft$$

For the rigid body, we consider:

Case 1

$$m_r = 134.15\ slugs, \quad \underline{S}_r = \underline{0}\ slugs \cdot ft$$

$$I_r = \begin{bmatrix} 186.021 & 0.0 & 0.0 \\ 0.0 & 186.021 & 0.0 \\ 0.0 & 0.0 & 357.733 \end{bmatrix} slugs \cdot ft^2$$

and Case 2

$$m_r = 21.464\ slugs, \quad \underline{S}_r = \underline{0}\ slugs \cdot ft$$

$$I_r = \begin{bmatrix} 8.943 & 0.0 & 0.0 \\ 0.0 & 8.943 & 0.0 \\ 0.0 & 0.0 & 14.309 \end{bmatrix} slugs \cdot ft^2$$

As seen above, the rigid-body model of Case 1 has large inertias relative to those of the flexible body. On the other hand, the mass moment of inertia of the rigid body of Case 2 is almost the same as that of the flexible body, although the mass of the rigid body is sufficiently large. This is due to the fact that the mass moment of inertia of the rigid body is about the center of mass and the mass moment of inertia of the flexible body is about the hinge, which is far removed from the mass center of the flexible body.

The two cases are compared with results obtained by Meirovitch and Kwak (1988a) using linearized state equations. Figures 4 through 7 show time histories of the rigid-body translations and rotation and the elastic displacements of the membrane at the center and the point defined by $x_e = 0$ and $r = 1.5\,ft$ for Case 1. The figures contain responses for the uncontrolled nonlinear system and linearized system. Figures 8 through 11 show the responses for Case 2.

The elastic displacements at the center and at $r = 0.5\,a, \theta = 90°$ are calculated by using the following formulas:

$$w_{r=0} = \frac{1}{\sqrt{\bar{\rho}_e}}(1.08684\,q_1 - 1.65807\,q_2)$$

$$w_{r=0.5a,\,\theta=90°} = \frac{1}{\sqrt{\bar{\rho}_e}}(0.73006 q_1 + 0.29919 q_2 - 1.06926 q_5 + 0.90574 q_6 + 1.15061 q_7 - 0.35635 q_8)$$

A discretized nonlinear state equation is solved by using IMSL routine DIVPAG.

VI. SUMMARY AND CONCLUSION

State equations of motion for a spacecraft consisting of a rigid body and a given number of flexible appendages are derived. To this end, Lagrange's equations of motion in terms of quasi-coordinates proved to be most effective. In general, the resulting hybrid equations are nonlinear and time-varying.

In addition, the equations contain persistent disturbances due to inertial loading. Because numerical simulation by means of the hybrid state equations is not feasible, discretization and truncation are carried out. The discretized state equation can be used for the maneuver and control of the spacecraft, such as when the object is to change the reorientation of the appendages while suppressing their vibration. A numerical example of a spacecraft with a membrane-like antenna illustrates the differences between the responses obtained by using the nonlinear and linear state equations.

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APPENDIX

For the given configuration, the position vector of a nominal point in the membrane appendage can be expressed as

$$\underline{r}_e = [\, r\sin\theta \;\; r\cos\theta \;\; h \,]^T \qquad (A-1)$$

Inserting the admissible functions given by Eqs. (26), together with Eq. (A-1), into Eqs. (22), we obtain

$$\bar{\Phi}_e^T = 2a\sqrt{\pi\bar{\rho}_e}\begin{bmatrix} 0 & 0 & \frac{1}{\beta_{01}a} \\ 0 & 0 & \frac{1}{\beta_{02}a} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\Phi}_e^T = \sqrt{2\pi\bar{\rho}_e}\,a^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\beta_{11}a} & 0 \\ 0 & -\frac{1}{\beta_{12}a} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\beta_{11}a} & 0 & 0 \\ \frac{1}{\beta_{12}a} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\bar{H}_e(\underline{a}) = -(a_1^2 + a_2^2)\,I$$

where I is 10×10 identity matrix. Moreover,

$$\tilde{H}_e(\underline{a}) = [\,0\,]$$

$$\int_{D_e} \rho_e \widetilde{[\,\Phi_e \underline{q}_e\,]}\,\Phi_e\,dD_e = [\,0\,]$$

where we note that the above null matrices are 10×10 and 3×10, respectively.

In addition,

$$\int_{D_e} \rho_e \tilde{r}_e \widetilde{[\,\Phi_e \underline{p}_e\,]}\,dD_e = \begin{bmatrix} -\alpha_1 & 0 & 0 \\ 0 & -\alpha_1 & 0 \\ \alpha_2 & \alpha_3 & 0 \end{bmatrix}$$

where

$$\alpha_1 = 2ah\sqrt{\pi\bar{\rho}_e}\left[\frac{p_1}{\beta_{01}a} + \frac{p_2}{\beta_{02}a}\right]$$

$$\alpha_2 = a^2\sqrt{2\pi\bar{\rho}_e}\left[\frac{p_3}{\beta_{11}a} + \frac{p_4}{\beta_{12}a}\right]$$

$$\alpha_3 = a^2\sqrt{2\pi\bar{\rho}_e}\left[\frac{p_7}{\beta_{11}a} + \frac{p_8}{\beta_{12}a}\right]$$

Moreover,

$$\int_{D_e} \rho_e \tilde{u}_e \tilde{v}_e\,dD_e = -\sum_{i=1}^{10} q_i p_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\int_{D_e} \rho_e \Phi_e^T \tilde{\omega}_e \tilde{u}_e\,dD_e = \begin{bmatrix} -\omega_1 q_1 & \omega_2 q_1 & 0 \\ -\omega_1 q_2 & \omega_2 q_2 & 0 \\ \vdots & \vdots & \vdots \\ -\omega_1 q_{10} & \omega_2 q_{10} & 0 \end{bmatrix}$$

$$\underline{S}_{ev} = 2a\sqrt{\pi\bar{\rho}_e}\begin{bmatrix} 0 \\ 0 \\ \frac{p_1}{\beta_{01}a} + \frac{p_2}{\beta_{02}a} \end{bmatrix}$$

$$\int_{D_e} \rho_e \Phi_e^T \tilde{a}\,\tilde{r}_e\,dD_e = 2a\sqrt{\pi\bar{\rho}_e}\begin{bmatrix} \frac{ha_1}{\beta_{01}a} & \frac{ha_2}{\beta_{01}a} & 0 \\ \frac{ha_1}{\beta_{02}a} & \frac{ha_2}{\beta_{02}a} & 0 \\ 0 & 0 & \frac{-aa_1}{\sqrt{2}\,\beta_{11}a} \\ 0 & 0 & \frac{-aa_1}{\sqrt{2}\,\beta_{12}a} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-aa_2}{\sqrt{2}\,\beta_{11}a} \\ 0 & 0 & \frac{-aa_2}{\sqrt{2}\,\beta_{12}a} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

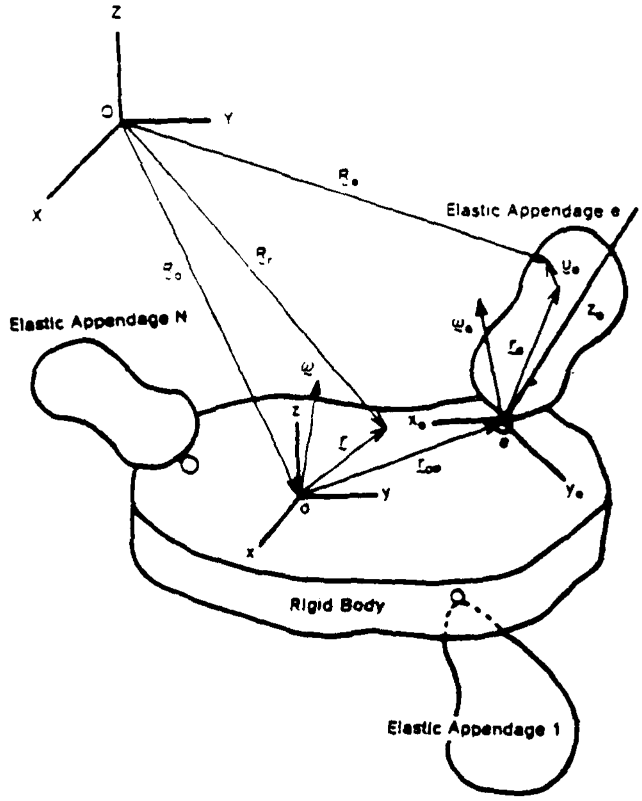

FIG. 1. The Rigid Platform with Flexible Appendages

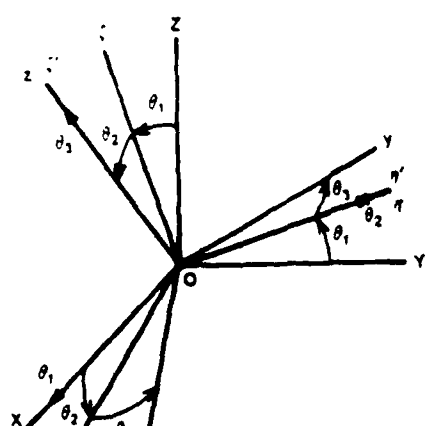

FIG. 2. Angular Displacements and Velocities of the Rigid Platform

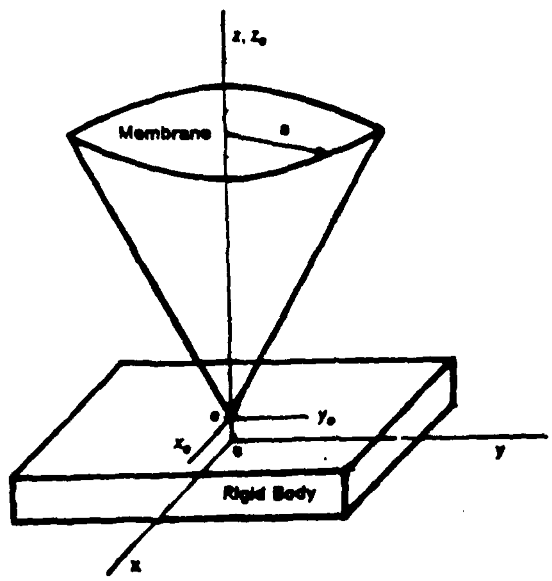

FIG. 3. The Spacecraft with a Membrane Antenna

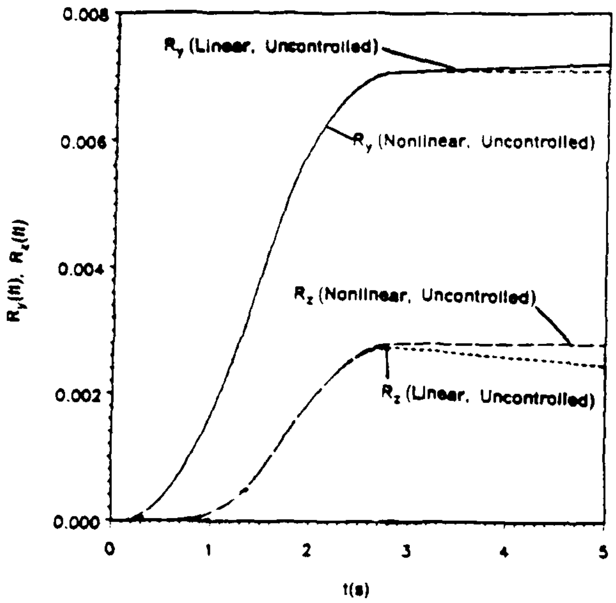

FIG. 4. Time History of the Rigid-Body Translations (Case 1)

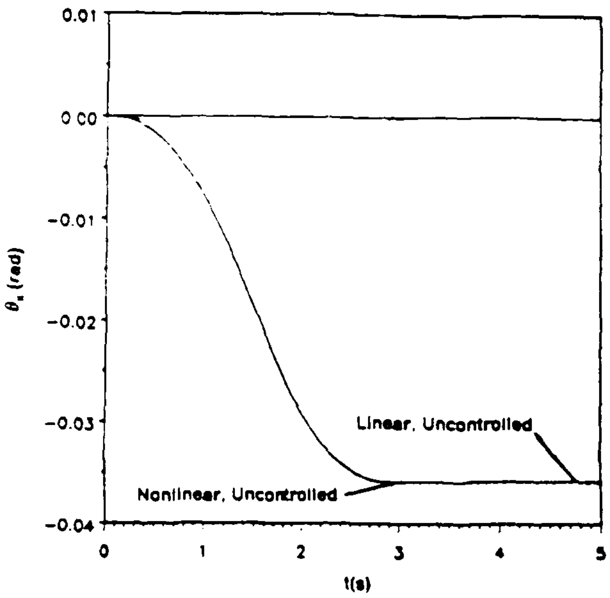

FIG. 5. Time History of the Rigid-Body Rotations (Case 1)

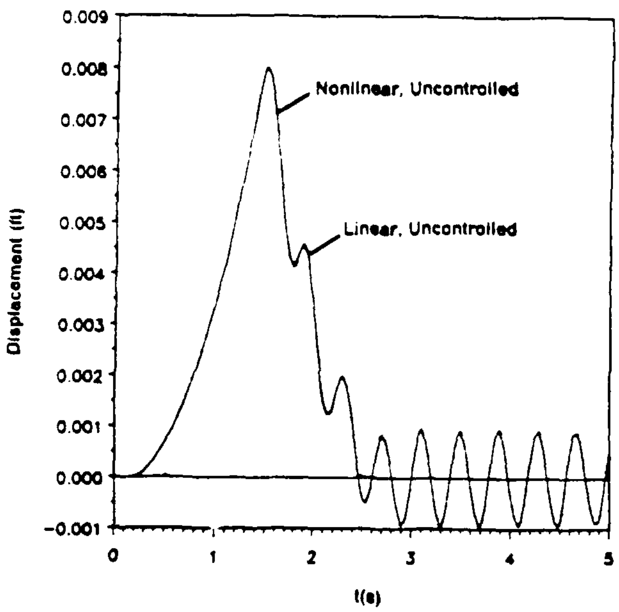

FIG. 6. Elastic Displacements at the Center (Case 1)

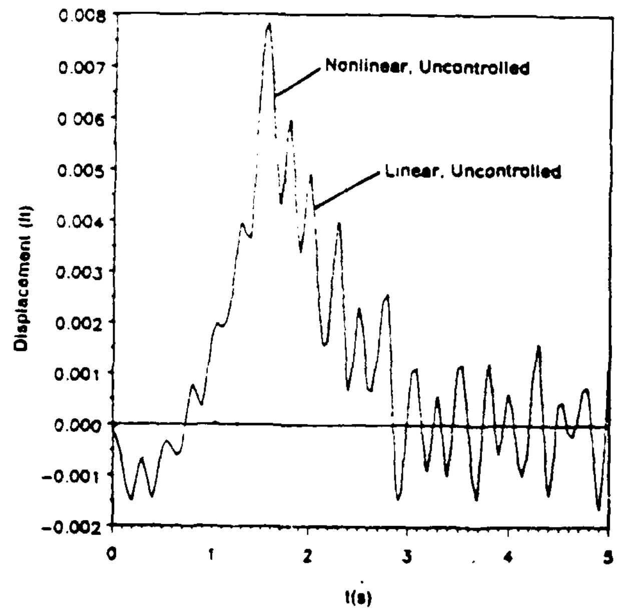

FIG. 7. Elastic Displacements at the $r = 0.5a$ and $\theta = 90°$ (Case 1)

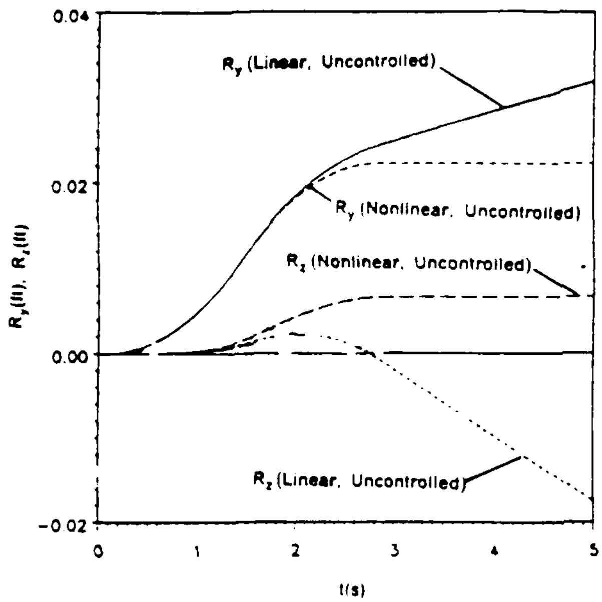

FIG. 8. Time History of the Rigid-Body Translations (Case 2)

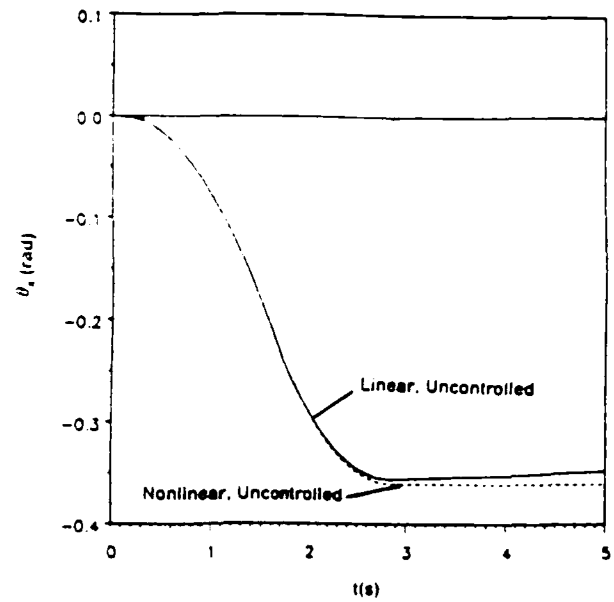

FIG. 9. Time History of the Rigid-Body Rotations (Case 2)

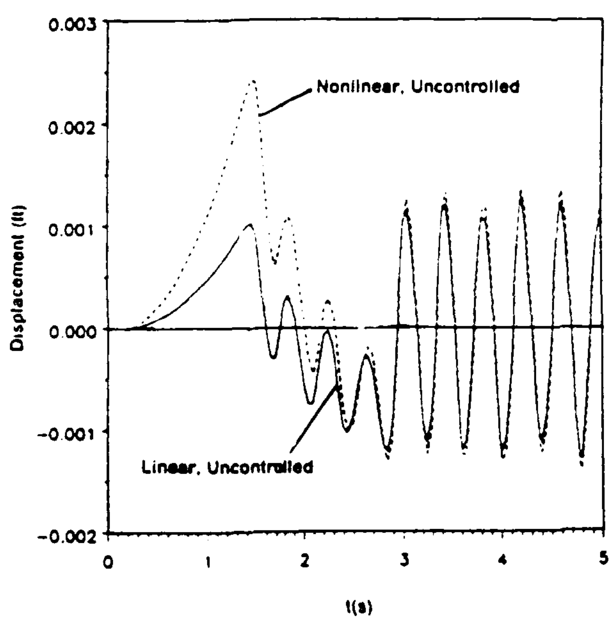

FIG. 10. Elastic Displacements at the Center (Case 2)

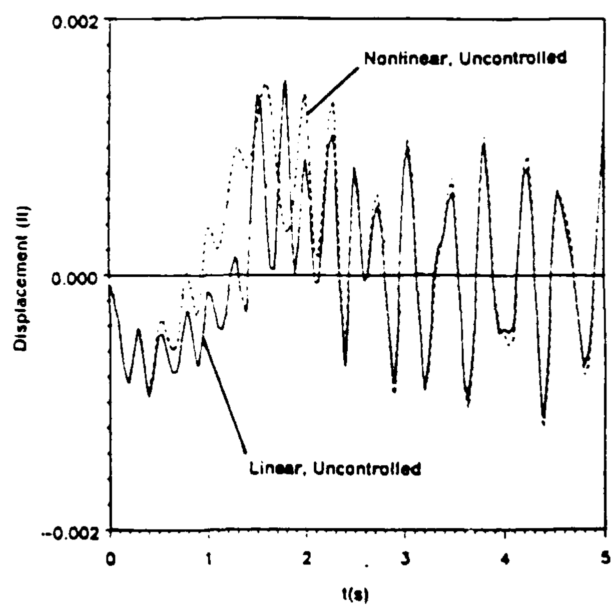

FIG. 11. Elastic Displacements at the $r = 0.5a$ and $\theta = 90°$ (Case 2)

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Control of Spacecraft with Multi-Targeted Flexible Antennas

Leonard Meirovitch and Moon K. Kwak
Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061

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Leonard Meirovitch† and Moon K. Kwak††
Department of Engineering Science and Mechanics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

Abstract

This paper is concerned with the problem of reorienting the line of sight of a given number of flexible antennas in a spacecraft. The maneuver of the antennas is carried out according to a minimum-time policy, which implies bang-bang control. Regarding the maneuver angular motion of the antennas as known, the equations of motion contain time-dependent terms in the form of coefficients and persistent disturbances. The control of the elastic vibration and of the rigid-body motions of the spacecraft caused by the maneuver is implemented by means of a proportional-plus-integral control. The approach is demonstrated by means of a numerical example in which a spacecraft consisting of a rigid platform and two maneuvering flexible antennas is controlled.

1. Introduction

Certain space missions involve the reorientation of the line of sight in a flexible spacecraft. In many cases, the line of sight can be regarded as being fixed relative to the undeformed structure, in which cases the reorientation of the line of sight implies the reorientation of the whole spacecraft (Ref. 1-5). Quite often, however, the line of sight coincides with an axis fixed in a small component of the spacecraft, such as a flexible antenna. In such cases, it may be more advisable to retarget only the flexible antenna and not the entire spacecraft. Retargeting of the antennas can be achieved by attaching the antenna to the platform through a hinge so as to permit pivoting about two orthogonal axes. Retargeting of the antennas, instead of the whole spacecraft, is a virtual necessity when there are several antennas that must be retargeted simultaneously, and the line of sight of each antenna must be reoriented in a different direction.

Reference 6 considers a spacecraft consisting of a rigid platform with a number of flexible antennas (Fig. 1), and the mission is to retarget independently the line of sight of each antenna relative to the inertial space. The maneuvering strategy developed in Ref. 6 consists of stabilizing the platform relative to the inertial space and reorienting the line of sight of each antenna relative to the platform. For given target directions of the antennas, the maneuvers can be designed as if the antennas were rigid. Of course, in actuality the antennas are flexible, so that the maneuvers are likely to cause elastic vibration of the antennas, which in turn will induce perturbations in the platform. Hence, the control task amounts to simultaneous stabilization of the platform relative to the inertial space and vibration suppression in the retargeting antennas.

The mathematical formulation consists of a hybrid set of equations of motion, in the sense that there are six ordinary differential equations for the rigid-body translations and rotations of the platform and partial differential equations for the elastic motion of each antenna. The equations of motion are not only hybrid, but the maneuvering of the antennas relative to the platform introduces time-dependent coefficients into the equations. Moreover, the equations contain terms reflecting persistent disturbances caused by inertial forces. If the mass of the antennas is small relative to the mass of the platform, then the equations of motion can be regarded as linear. Because control of systems governed by sets of hybrid differential equations cannot be readily designed, even when the equations are linear, it is necessary to discretize the partial differential equations in space, which can be carried out by the classical Rayleigh-Ritz method or the finite element method (Ref. 7). Reference 6 presents the mathematical formulation, as well as an example illustrating the maneuvering of a single flexible antenna.

This paper extends the work of Ref. 6 in several respects. In the first place, it treats the problem of retargeting several antennas simultaneously, and not just of a single antenna. In addition, it addresses the problem of persistent disturbances by attempting to mitigate their effect during the maneuver. To cope with known disturbances, disturbance-minimization control is effected. The retargeting is carried out open-loop using a bang-bang control law. This implies that the inertial forces arising from the maneuver angular accelerations are almost constant, except for a sign change at one half of the maneuver period. If the maneuver is not very fast compared to the lowest natural frequency of the antennas, then the control gains can be determined by ignoring the time-dependent terms in the coefficient matrices. This permits the use of proportional-plus-integral feedback control for disturbance accommodation (Ref. 8, 9 and 10). Of course, in the computer simulation of the maneuver and control, the full time-varying system is considered.

This paper contains the procedure for designing the feedback control in the presence of disturbances. The proportional-plus-integral control

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† University Distinguished Professor. Fellow AIAA.

†† Graduate Research Assistant. Student Member AIAA.

procedure described above is demonstrated by means of a numerical example involving a spacecraft consisting of a rigid platform and two flexible beams, each with one end hinged to the platform and the other free (Fig. 2), where the beams are originally parallel to the z-axis of the platform. The maneuver consists of slewing each of the beams through a 45° angle, one about the x-axis and the other about the y-axis of the platform.

2. Equations of Motion

We consider the motion of a spacecraft consisting of a rigid platform with a given number of flexible appendages hinged to the platform as shown in Fig. 1. The rotation of each individual appendage relative to the platform is given, so that the motion of the system consists of the rigid body motions of the platform and the elastic motion of the appendages. To describe the motion, we introduce a set of inertial axes XYZ, a set of body axes xyz attached to the platform. Then, the motion of the rigid body can be defined in terms of three translations and three rotations of axes xyz relative to the inertial axes XYZ. To describe the motion of the flexible appendages, we consider a typical appendage hinged at point e and regard e as the origin of a set of body axes $x_e y_e z_e$ embedded in the appendage in undeformed state. Then, the motion of a nominal point of the appendage consists of the motion of xyz, the motion of $x_e y_e z_e$ relative to xyz and the elastic motion relative to $x_e y_e z_e$. The system and the various reference frames are shown in Fig. 1.

From Fig. 1, the position vector of a point in the rigid body and in the appendage can be written as $\underset{\sim}{R}_r = \underset{\sim}{R}_o + \underset{\sim}{r}$, $\underset{\sim}{R}_e = \underset{\sim}{R}_o + \underset{\sim}{r}_{oe} + \underset{\sim}{r}_e + \underset{\sim}{u}_e$ $(e = 1,2,\ldots,N)$, where $\underset{\sim}{R}_o$ is the radius vector from O to o, $\underset{\sim}{r}$ is the position vector of a point in the rigid body relative to xyz, $\underset{\sim}{r}_{oe}$ is the radius vector from o to e, $\underset{\sim}{r}_e$ is the position vector of a nominal point in undeformed appendage relative to $x_e y_e z_e$ and $\underset{\sim}{u}_e$ is the elastic displacement of that point. Vector $\underset{\sim}{R}_o$ is given in terms of components along XYZ, $\underset{\sim}{r}$ and $\underset{\sim}{r}_{oe}$ in terms of components along xyz, and $\underset{\sim}{r}_e$ and $\underset{\sim}{u}_e$ in terms of components along $x_e y_e z_e$. The velocity vector of o can be written in terms of components along xyz in the form $\underset{\sim}{V}_o = C\dot{\underset{\sim}{R}}_o$, where C is the matrix of direction cosines between xyz and XYZ and $\dot{\underset{\sim}{R}}_o$ is the velocity vector of o in terms of components along XYZ. Furthermore, the angular velocity vector of axes xyz in terms of components along xyz is given by $\underset{\sim}{\omega} = D\dot{\underset{\sim}{\theta}}$, where $\dot{\underset{\sim}{\theta}}$ is a vector of angular velocities $\dot{\theta}_i$ and D is a matrix depending on the angular displacements θ_i $(i = 1,2,3)$. In view of the above, the velocity vector of a point in the rigid body in terms of components along xyz is simply $\underset{\sim}{V}_r = \underset{\sim}{V}_o + \underset{\sim}{\omega} \times \underset{\sim}{r}$ and that of a point in the typical appendage e in terms of components along $x_e y_e z_e$ is $\underset{\sim}{V}_e = E_e(\underset{\sim}{V}_o + \underset{\sim}{\omega} \times \underset{\sim}{r}_{oe}) + (E_e\underset{\sim}{\omega} + \underset{\sim}{\omega}_e) \times (\underset{\sim}{r}_e + \underset{\sim}{u}_e) + \underset{\sim}{v}_e$ $(e = 1,2,\ldots,N)$, where $\underset{\sim}{\omega}_e$ is the angular velocity of axes $x_e y_e z_e$, E_e is a matrix of direction cosines between axes $x_e y_e z_e$ and xyz and $\underset{\sim}{v}_e$ is the elastic velocity of the point in the appendage relative to $x_e y_e z_e$, $\underset{\sim}{v}_e = \dot{\underset{\sim}{u}}_e$. In the maneuver proposed, the angular velocity vectors $\underset{\sim}{\omega}_e$ of $x_e y_e z_e$ relative to xyz are given, so that the rotational motions of the appendages relative to the platform do not add degrees of freedom.

The equations of motion are hybrid, in the sense that the equations for the rigid-body translations and rotations of the platform are ordinary differential equations and those for the elastic motions of the appendages are partial differential equations. Moreover, because of the maneuver angular velocities $\underset{\sim}{\omega}_e$, they possess time-dependent coefficients. Control design of systems described by hybrid equations is not feasible, so that we wish to discretize the system in space. To this end, we express the elastic displacements as linear combinations of space-dependent admissible functions multiplied by time-dependent generalized coordinates, or

$$\underset{\sim}{u}_e(\underset{\sim}{r}_e,t) = \Phi_e(\underset{\sim}{r}_e)\underset{\sim}{q}_e(t) \quad , \quad e = 1,2,\ldots,N \qquad (1)$$

where Φ_e is a matrix of admissible functions and $\underset{\sim}{q}_e$ is a vector of generalized coordinates.

The equations of motion can be obtained by means of Lagrange's equations in terms of quasi-coordinates (Ref. 11). A linearized version of such equations was derived in Ref. 6 for the same mathematical model as the one considered here. From Ref. 6, we obtain the state equations of motion

$$\dot{\underset{\sim}{x}}(t) = A(t)\underset{\sim}{x}(t) + B(t)\underset{\sim}{f}(t) + D(t)\underset{\sim}{d}(t) \qquad (2)$$

where $\underset{\sim}{x}(t) = [\underset{\sim}{R}_o^T \ \underset{\sim}{\theta}^T \ \underset{\sim}{q}_1^T \ \underset{\sim}{q}_2^T \cdots \underset{\sim}{q}_N^T \ \underset{\sim}{V}_o^T \ \underset{\sim}{\omega}^T \ \underset{\sim}{p}_1^T \ \underset{\sim}{p}_2^T \cdots \underset{\sim}{p}_N^T]^T$ is a state vector, in which $\underset{\sim}{\theta}$ is a symbolic vector of angular displacements of the platform, $\underset{\sim}{p}_e = \dot{\underset{\sim}{q}}_e$ $(e = 1,2,\ldots,N)$ and

$$A(t) = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}(t)K(t) & -M^{-1}(t)G(t) \end{array}\right] \qquad (3a)$$

$$B(t) = \left[\begin{array}{c} 0 \\ \hline M^{-1}(t)B^*(t) \end{array}\right], \quad D(t) = \left[\begin{array}{c} 0 \\ \hline M^{-1}(t) \end{array}\right] \qquad (3b,c)$$

are coefficient matrices, in which

$$M(t) = \begin{bmatrix} mI & \tilde{S}_t^T & E_1^T\bar{\Phi}_1 & \cdots & E_N^T\bar{\Phi}_N \\ \tilde{S}_t & J_t & E_1^T\tilde{\Phi}_1 + \tilde{r}_{o1}E_1^T\bar{\Phi}_1 & \cdots & E_N^T\tilde{\Phi}_N + \tilde{r}_{oN}E_N^T\bar{\Phi}_N \\ \bar{\Phi}_1^T E_1 & \tilde{\Phi}_1^T E_1 + \bar{\Phi}_1^T E_1 \tilde{r}_{o1}^T & M_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \bar{\Phi}_N^T E_N & \tilde{\Phi}_N^T E_N + \bar{\Phi}_N^T E_N \tilde{r}_{oN}^T & 0 & \cdots & M_N \end{bmatrix} \tag{4}$$

is a mass matrix,

$$G(t) = \begin{bmatrix} 0 & 2\sum_{e=1}^{N} E_e^T[\widetilde{S_e\omega_e}] & 2E_1^T\tilde{\omega}_1\bar{\Phi}_1 & \cdots & 2E_N^T\tilde{\omega}_N\bar{\Phi}_N \\ 0 & G_{22} & G_{23}^1 & \cdots & G_{23}^N \\ 0 & [\tilde{\Phi}_1^T\tilde{\omega}_1^T - J_1(\omega_1)]E_1 & 2\tilde{H}_1(\omega_1) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & [\tilde{\Phi}_N^T\tilde{\omega}_N^T - J_N(\omega_N)]E_N & 0 & \cdots & 2\tilde{H}_N(\omega_N) \end{bmatrix} \tag{5}$$

is a matrix in which

$$G_{22} = \sum_{e=1}^{N} (E_e^T(2\tilde{\omega}_e J_e - \mathrm{tr}[J_e\tilde{\omega}_e])E_e + 2\tilde{r}_{oe}E_e^T[\widetilde{S_e\omega_e}]E_e) \tag{6a}$$

$$G_{23}^e = 2\tilde{r}_{oe}E_e^T\tilde{\omega}_e\bar{\Phi}_e + E_e^T\tilde{\omega}_e\tilde{\Phi}_e + E_e^T J_e(\omega_e) \tag{6b}$$

and

$$K(t) = \begin{bmatrix} 0 & 0 & E_1^T(\dot{\tilde{\omega}}_1 + \tilde{\omega}_1^2)\bar{\Phi}_1 & \cdots & E_N^T(\dot{\tilde{\omega}}_N + \tilde{\omega}_N^2)\bar{\Phi}_N \\ 0 & 0 & K_{23}^1 & \cdots & K_{23}^N \\ 0 & 0 & K_1 + \bar{H}_1(\omega_1) + \tilde{H}_1(\dot{\omega}_1) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & K_N + \bar{H}_N(\omega_N) + \tilde{H}_N(\dot{\omega}_N) \end{bmatrix} \tag{7}$$

plays the role of a stiffness matrix, where

$$K_{23}^e = \tilde{r}_{oe}E_e^T(\tilde{\omega}_e^2 + \dot{\tilde{\omega}}_e)\bar{\Phi}_e + E_e^T(\tilde{\omega}_e J_e(\omega_e) + J_e(\dot{\omega}_e)) \tag{8}$$

Moreover,

$$B^*(t) = \begin{bmatrix} 1 & b^1 & b^2 & \cdots & b^N \\ 0 & c^1 & 0 & \cdots & 0 \\ 0 & 0 & c^2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & c^N \end{bmatrix} \tag{9}$$

relates the discrete force vectors to the modal force vectors, in which

$$b^e = \begin{bmatrix} E_e^T & E_e^T & \cdots & E_e^T \\ \tilde{r}_{oe}E_e^T + E_e^T\tilde{r}_{e1} & \tilde{r}_{oe}E_e^T + E_e^T\tilde{r}_{e2} & \cdots & \tilde{r}_{oe}E_e^T + E_e^T\tilde{r}_{en_e} \end{bmatrix} \tag{10a}$$

$$c^e = [\Phi_e^T(\zeta_{e1}) \quad \Phi_e^T(\zeta_{e2}) \quad \cdots \quad \Phi_e^T(\zeta_{en_e})] \tag{10b}$$

In addition,

$$\underline{d}(t) = \begin{bmatrix} \sum_{e=1}^{N} E_e^T(\tilde{S}_e \dot{\omega}_e + \tilde{\omega}_e \tilde{S}_e \omega_e) \\ \sum_{e=1}^{N} [\tilde{r}_{oe} E_e^T(\tilde{S}_e \dot{\omega}_e + \tilde{\omega}_e \tilde{S}_e \omega_e) - E_e^T(I_e \dot{\omega}_e + \tilde{\omega}_e I_e \omega_e)] \\ -\tilde{\Phi}_1^T \dot{\omega}_1 + \int_{D_1} \rho_1 \Phi_1^T \tilde{\omega}_1 \tilde{r}_1 \omega_1 dD_1 \\ \cdots \\ -\tilde{\Phi}_N^T \dot{\omega}_N + \int_{D_N} \rho_N \Phi_N^T \tilde{\omega}_N \tilde{r}_N \omega_N dD_N \end{bmatrix} \quad (11)$$

is a vector of disturbances and

$$\underline{f}(t) = [\underline{F}_o^{*T}\ \underline{M}_o^{*T}\ \underline{f}_{11}^T\ \underline{f}_{12}^T \cdots \underline{f}_{1n_1}^T\ \underline{f}_{21}^T\ \underline{f}_{22}^T \cdots \underline{f}_{2n_2}^T\ \underline{f}_{31}^T \cdots \underline{f}_{Nn_N}^T]^T \quad (12)$$

plays the role of a control force vector, where in the latter $\underline{F}_o^*$ and $\underline{M}_o^*$ are actuator force and torque vectors acting on the platform and $\underline{f}_{ei}$ are actuator forces acting on appendages e at points i. Other quantities entering into the above matrices are as follows:

$$m_t = m_r + \sum_{e=1}^{N} m_e\ ,\quad m_r = \int_{D_r} \rho_r dD_r \quad (13a,b)$$

$$m_e = \int_{D_e} \rho_e dD_e \quad (13c)$$

$$\tilde{S}_t = \tilde{S}_r + \sum_{e=1}^{N} (m_e \tilde{r}_{oe} + E_e^T \tilde{S}_e E_e) \quad (13d)$$

$$\tilde{S}_r = \int_{D_r} \rho_r \tilde{r} dD_r\ ,\quad \tilde{S}_e = \int_{D_e} \rho_e \tilde{r}_e dD_e \quad (13e,f)$$

$$I_t = I_r + \sum_{e=1}^{N} (m_e \tilde{r}_{oe} \tilde{r}_{oe}^T + E_e^T I_e E_e - \tilde{r}_{oe} E_e^T \tilde{S}_e E_e - E_e^T \tilde{S}_e E_e \tilde{r}_{oe}) \quad (13g)$$

$$I_r = \int_{D_r} \rho_r \tilde{r}\tilde{r}^T dD_r\ ,\quad I_e = \int_{D_e} \rho_e \tilde{r}_e \tilde{r}_e^T dD_e \quad (13h,i)$$

$$M_e = \int_{D_e} \rho_e \Phi_e^T \Phi_e dD_e \quad (13j)$$

$$\bar{\Phi}_e = \int_{D_e} \rho_e \Phi_e dD_e\ ,\quad \tilde{\Phi}_e = \int_{D_e} \rho_e \tilde{r}_e \Phi_e dD_e \quad (13k,l)$$

$$\tilde{H}_e(\underline{a}) = \int_{D_e} \rho_e \Phi_e^T \tilde{a} \Phi_e dD_e \quad (13m)$$

$$\bar{H}_e(\underline{a}) = \int_{D_e} \rho_e \Phi_e^T \tilde{a}^2 \Phi_e dD_e \quad (13n)$$

$$J_e(\underline{a}) = \int_{D_e} \rho_e (\tilde{r}_e \tilde{a} + \widetilde{[\tilde{r}_e a]}) \Phi_e dD_e \quad (13o)$$

in which $\underline{a}$ is a vector representing ω_e or $\dot{\omega}_e$, ρ_r and ρ_e are mass densities of each body, D_r and D_e are the domain of the rigid platform and of a typical appendage, respectively, m_t is the total mass of the system, S_t is a skew matrix of first moment of inertia for the system and I_t is the inertia matrix. For simplicity, we assumed that the potential energy is due entirely to elastic effects, in which case it could be written in the form

$$V = \frac{1}{2} \sum_{e=1}^{N} \underline{q}_e^T K_e \underline{q}_e \quad (14)$$

where K_e is the stiffness matrix (Ref. 7).

3. Disturbance-Accommodating Control

The maneuver consists of retargeting antennas so as to point in given directions in the inertial space. By stabilizing the platform in an inertial space, the task reduces to reorienting the antennas relative to the platform. For a minimum-time maneuver, the control law is bang-bang, which implies that the angular acceleration of an antenna relative to the platform is constant, with the sign changing at half the maneuver period. Ideally, the maneuver should not cause elastic deformations in the flexible appendages. This is likely to require a long maneuver time, which is in conflict with the minimum-time requirement. To reduce elastic vibration, a smoothed bang-bang can be used. Still, elastic deformations are likely to occur, which in turn implies perturbation of the platform from a fixed position in the inertial space.

The motion of the system is governed by Eq. (2). The system is characterized by two factors that distinguish it from most commonly encountered systems: it is time-varying and it is subjected to persistent disturbances. Both factors arise from the retargeting maneuver angular velocities ω_e, angular accelerations $\dot{\omega}_e$ and the matrices E_e of direction cosines (e = 1,2,...,N), all quantities being known functions of time. Clearly, the disturbance term $D(t)\underline{d}(t)$ in Eq. (2) depends on the maneuver policy. In the case of minimum-time maneuver, the policy is bang-bang, which implies that the maneuver angular acceleration is constant over both halves of the maneuver period. If the maneuver is relatively slow, then the disturbance is constant over both halves of the maneuver period. In this case, we can use proportional-plus-integral feedback control.

Introducing the notation

$$B(t)\underline{u}(t) = B(t)\underline{f}(t) + D(t)\underline{d}(t) \quad (15)$$

Eq. (2) can be rewritten as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (16)$$

Assuming that $D(t)$ and $g(t)$ vary slowly, so that $D(t)g(t)$ is almost constant during the control interval, we can write

$$\dot{u}(t) = \dot{f}(t) = f_d(t) \quad (17)$$

Introducing a new state vector defined by $z = [x^T \ u^T]^T$, Eqs. (16) and (17) can be combined into the expanded state equation

$$\dot{z}(t) = \hat{A}(t)z(t) + \hat{B}(t)f_d(t) \quad (18)$$

where

$$\hat{A}(t) = \begin{bmatrix} A(t) & B(t) \\ 0 & 0 \end{bmatrix}, \quad \hat{B}(t) = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad (19a,b)$$

are coefficient matrices. Note that if $A(t)$ and $B(t)$ are a controllable pair, then $\hat{A}(t)$ and $\hat{B}(t)$ are also a controllable pair (Ref. 8).

We consider an optimal control policy in the sense that $f_d(t)$ minimizes the performance measure

$$J = \frac{1}{2} z^T(t_f)\hat{H}z(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [z^T(t)\hat{Q}(t)z(t) + f_d^T(t)\hat{R}(t)f_d(t)]dt \quad (20)$$

The optimal control law is

$$f_d(t) = -\hat{R}^{-1}(t)\hat{B}^T(t)K(t)z(t) = G(t)z(t) \quad (21)$$

where $G(t)$ represents a control gain matrix in which $K(t)$ satisfies the matrix Ricatti equation

$$\dot{K} = -K\hat{A} - \hat{A}^TK - \hat{Q} + K\hat{B}\hat{R}^{-1}\hat{B}^TK \quad (22)$$

Using Eqs. (16), (17) and (21), it is easy to verify that

$$\frac{df}{dt} = (G_1 - G_2B^{\dagger}A)x + G_2B^{\dagger}\dot{x} \quad (23)$$

where G_1 and G_2 represents submatrices of G corresponding to x and u, correspondingly, and $B^{\dagger} = (B^TB)^{-1}B^T$ is the pseudo-inverse of B. Integrating Eq. (23), we obtain the optimal control law

$$f(t) = f(0) + \int_0^t (G_1 - G_2B^{\dagger}A)x \, d\tau + \int_0^t G_2B^{\dagger}\dot{x} \, d\tau \quad (24)$$

If t_f is sufficiently large and A and B can be assumed to be constant, then the gain matrix is constant. Moreover, if $x(0)$ and $f(0)$ are zero, then Eq. (24) yields

$$f(t) = G_p x + G_i \int_0^t x \, d\tau \quad (25)$$

where

$$G_p = G_2B^{\dagger}, \quad G_i = G_1 - G_2B^{\dagger}A \quad (26a,b)$$

Equation (25) represents the optimal control law for the time-invariant system subjected to unknown constant disturbances, and is known as proportional-plus-integral control.

In general, the above control law cannot be used for the type of problem considered here. When the maneuver is relatively slow, however, so that the matrices A and B are nearly constant, the control law (25) can be used with satisfactory results. In this case, the control must be regarded suboptimal, but close to being optimal. We recall that, to reduce vibration, it is advisable to use a smoothed bang-bang for the rigid-body maneuver of the appendages, instead of an ideal bang-bang.

4. Numerical Example

The mathematical model consists of a rigid platform and two flexible beams, each one with one end hinged to the platform and the other end free (Fig. 2), where the beams are originally parallel to the z-axis of the platform. The maneuver consists of slewing each of the beams through a 45° angle, one about the x-axis and the other about the y-axis of the platform. The beams are discretized in space by using three admissible functions for each component of displacement. Six actuators are used for the rigid platform and three actuators for each displacement component of both beams. The latter actuators are located at 4 ft, 7 ft and 10 ft from the pivot point, the third coinciding with the tip of the beam. Figure 3 shows the maneuver time histories, in which the angular acceleration represents a modified bang-bang. Figures 4a and 4b display both the uncontrolled and controlled translational and angular displacements of the platform, respectively, and Figs. 5a and 5b show the tip displacement of the two beams. As can be verified, the maneuver and control of the spacecraft is quite satisfactory. The disturbance-accommodating control is carried out by the proportional-plus-integral control approach.

In obtaining the numerical results, the following data was used:

$m_r = 134.15$ slugs , $m_e = 0.1873$ slugs

$S_r = [0 \ \ 0 \ \ 0]^T$ slugs·ft ,

$S_e = [0 \ \ 0 \ \ 0.9365]^T$ slugs·ft

$$I_r = \begin{bmatrix} 186.021 & 0 & 0 \\ 0 & 186.021 & 0 \\ 0 & 0 & 357.733 \end{bmatrix} \text{slugs}\cdot\text{ft}^2$$

$$I_e = \begin{bmatrix} 6.243 & 0 & 0 \\ 0 & 6.243 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{slugs}\cdot\text{ft}^2$$

$l_1 = l_2 = 10$ ft, $EI = 3028.9$ lb·ft^2

$r_{01} = [0 \ \ -1.0 \ \ 0.5]^T$ ft

$$\underline{r}_{02} = [0 \quad 1.0 \quad 0.5]^T \text{ ft}$$

Moreover, the weighting matrices appearing in the performance index, Eq. [illegible], are as follows

$$\hat{Q} = \begin{bmatrix} 100\,I & 0 \\ 0 & I \end{bmatrix}, \quad \hat{R} = 0.001\,I$$

where I is the identity matrix. Of course, consistent with a steady-state solution of the matrix Riccati equation, $\hat{H}$ was taken as zero. Finally, for control design purposes, the coefficient matrices A and B were taken as constant and corresponding to the premaneuver configuration of the spacecraft. Of course, in implementing the control, the time-varying matrices A(t) and B(t) were used.

5. Summary and Conclusions

Certain space missions involve the reorientation in an inertial space of the line of sight of certain small flexible components of a spacecraft, such as flexible antennas. In such cases, regarding the main spacecraft as a rigid platform, a sensible strategy is to stabilize the platform relative to the inertial space and reorient the line of sight of the various antennas relative to the platform.

This paper is concerned with the problem of retargeting several antennas simultaneously, while suppressing any rigid-body and elastic perturbations caused by the retargeting maneuver. The retargeting was carried out open-loop using a smoothed bang-bang control law. This implies that the inertial forces arising from the maneuver angular accelerations are almost constant, except for a sign change at one half of the maneuver period. If the maneuver is not very fast compared to the lowest natural frequency of the nonmaneuvering antennas, then the control gains can be determined by ignoring the time-dependent terms in the coefficient matrices. This permitted the use of proportional-plus-integral feedback control for disturbance accommodation. Of course, in the computer simulation of the maneuver and control, the full time-varying system was considered.

A numerical example, in which a spacecraft consisting of a rigid platform and two flexible antennas undergoing reorientation in different planes was controlled, demonstrates the approach.

6. References

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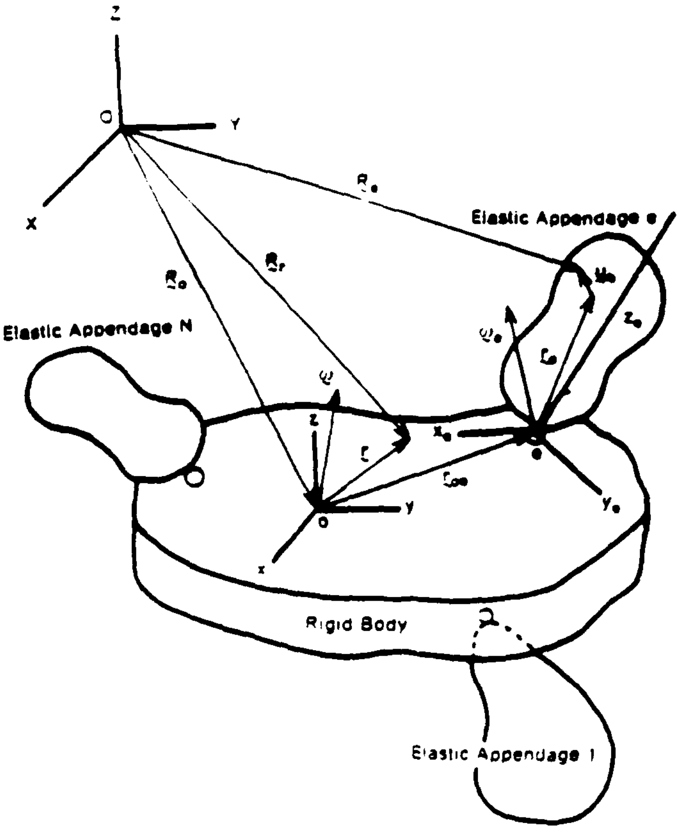

Figure 1. The Rigid Platform with Flexible Appendages

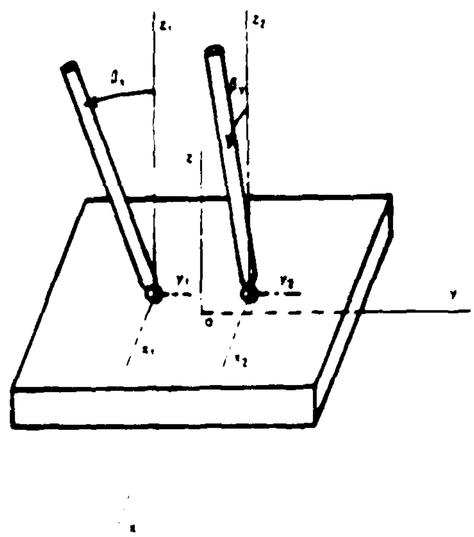

Figure 2. Rigid Platform with Two Flexible Beams

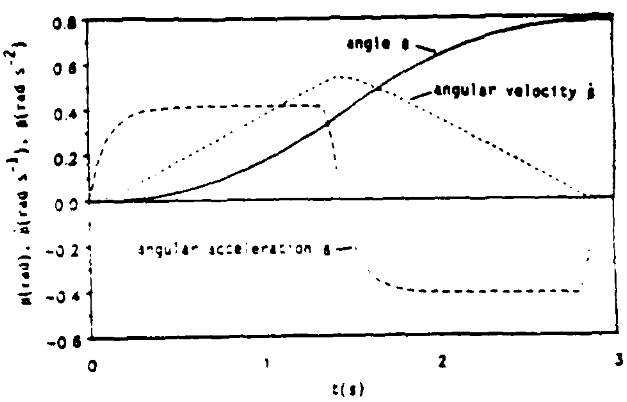

Figure 3. Maneuver Time History

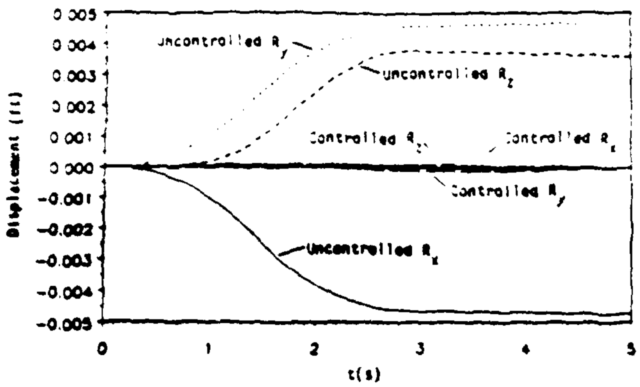

Figure 4a. Time History of the Platform Translational Motion

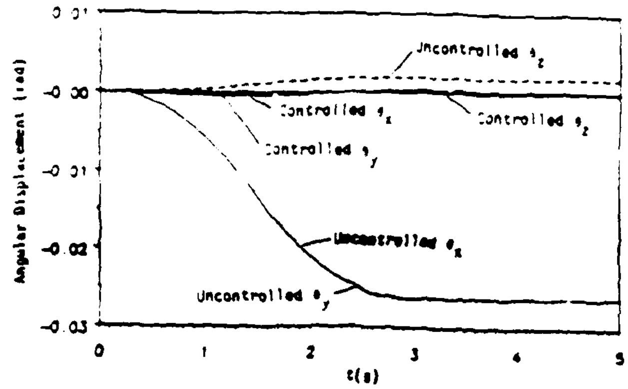

Figure 4b. Time History of the Platform Angular Motion

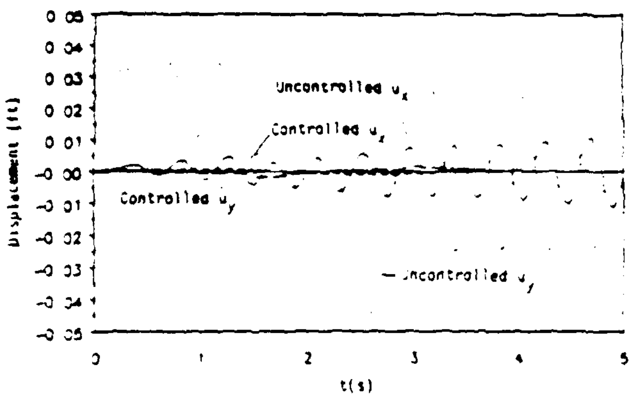

Figure 5a. The Displacement of Beam 1

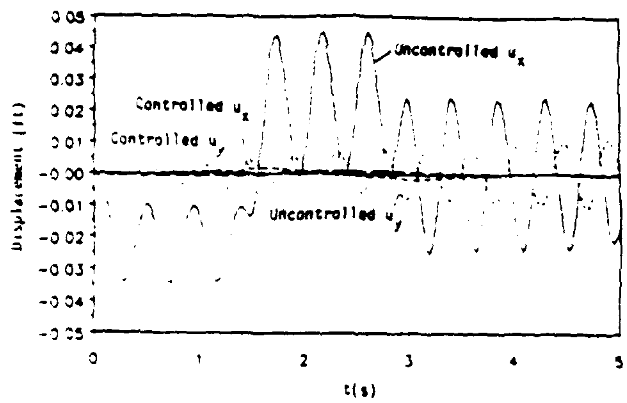

Figure 5b. Tip Displacement of Beam 2