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**DO  $3n-5$   
EDGES  
FORCE A  
SUBDIVISION  
OF  $K_5$ ?**

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## REPORT DOCUMENTATION PAGE

Form Approved  
OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION Unclassified		1b. RESTRICTIVE MARKINGS None	
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE		5. MONITORING ORGANIZATION REPORT NUMBER(S)	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) UIIU-ENG-90-2213 (ACT-112)		7a. NAME OF MONITORING ORGANIZATION Office of Naval Research	
6a. NAME OF PERFORMING ORGANIZATION Coordinated Science Lab University of Illinois	6b. OFFICE SYMBOL (if applicable) N/A	7b. ADDRESS (City, State, and ZIP Code) Arlington, VA 22217	
6c. ADDRESS (City, State, and ZIP Code) 1101 W. Springfield Ave. Urbana, IL 61801		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER N00014-85-K-0570 N00014-88-K-0316	
8a. NAME OF FUNDING / SPONSORING ORGANIZATION ONR	8b. OFFICE SYMBOL (if applicable)	10. SOURCE OF FUNDING NUMBERS	
8c. ADDRESS (City, State, and ZIP Code) Arlington, VA 22217		PROGRAM ELEMENT NO.	PROJECT NO.
		TASK NO.	WORK UNIT ACCESSION NO.
11. TITLE (Include Security Classification) Do $3n-5$ edges force a subdivision of $K_5$ ?			
12. PERSONAL AUTHOR(S) Kezdy, Andre E. and McGuinness, Patrick J.			
13a. TYPE OF REPORT Technical	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Year, Month, Day) May 1990	15. PAGE COUNT 25
16. SUPPLEMENTARY NOTATION			
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB-GROUP	
19. ABSTRACT (Continue on reverse if necessary and identify by block number)			
<p style="text-align: center;">Sub 4 + -</p> <p>A conjecture of Dirac states that every simple graph with <math>n</math> vertices and <math>3n - 5</math> edges must contain a subdivision of <math>K_5</math>. We prove that a topologically minimal counterexample is 5-connected, and that no minor-minimal counterexample contains <math>K_4 - e</math>. Consequently, we prove Dirac's conjecture for all graphs that can be imbedded in a surface with Euler characteristic at least <math>-2</math>.</p>			
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS		21. ABSTRACT SECURITY CLASSIFICATION Unclassified	
22a. NAME OF RESPONSIBLE INDIVIDUAL		22b. TELEPHONE (Include Area Code)	22c. OFFICE SYMBOL

# Do $3n-5$ edges force a subdivision of $K_5$ ?

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## Abstract

A conjecture of Dirac states that every simple graph with  $n$  vertices and  $3n - 5$  edges must contain a subdivision of  $K_5$ . We prove that a topologically minimal counterexample is 5-connected, and that no minor-minimal counterexample contains  $K_4 - e$ . Consequently, we prove Dirac's conjecture for all graphs that can be imbedded in a surface with Euler characteristic at least  $-2$ .



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\*Supported by the Office of Naval Research under grant N00014-85K0570.

†Supported by the Office of Naval Research contract N00014-88-K-0316

## 1. Introduction

Let  $H$  be a simple undirected graph. An *elementary subdivision* of  $H$  is a graph obtained from  $H$  by removing some edge  $e = xy$  and adding a new vertex  $z$  together with two new edges  $xz$  and  $zy$ . A *subdivision* of  $H$  is a graph obtained from  $H$  by a succession of elementary subdivisions. If a subdivision of  $H$  is isomorphic to a subgraph of  $G$ , we write  $TH \subset G$ , where  $TH$  represents an arbitrary subdivision of  $H$ . A vertex of  $TK_p$  ( $p \geq 4$ ) with degree at least three is called a *branch vertex*.

A conjecture due to Dirac [2], and reported by Thomassen [8], states that any simple graph with  $n$  vertices and  $3n - 5$  edges contains a subdivision of  $K_5$ . By Kuratowski's Theorem, no planar graph contains a subdivision of  $K_5$ . Thus Dirac's conjecture, if true, would be sharp. Thomassen [7] proved that  $4n - 10$  edges force a  $TK_5$ . In [3], Dirac showed that, if  $\delta(G) \geq 3$ , then  $G$  contains a subdivision of  $K_4$ . A similar result by Pelikán [6] and Thomassen [7] established that  $\delta(G) \geq 4$  forces  $G$  to contain a subdivision of  $K_5 - e$ . More generally, Mader [5] proved that, if  $\delta(G) \geq 3(2)^{p-2} - 2p$  ( $p > 3$ ), then  $TK_p \subset G$ .

A simple graph  $G$  with  $n$  vertices is called a *counterexample* if  $|E(G)| \geq 3n - 5$  and  $TK_5 \not\subset G$ . Let  $\mathcal{D}$  be the set of all counterexamples. A *minor* of  $G$  is a subgraph obtained from  $G$  by a sequence of edge deletions, vertex deletions, and edge contractions. A graph is *minor-minimal* in  $\mathcal{D}$  provided it is a counterexample but no minor is a counterexample. Similarly, a graph is *(topologically) minimal* in  $\mathcal{D}$  provided it is a counterexample and contains no subdivision of a smaller counterexample. Observe that any minor-minimal counterexample is also a (topologically) minimal counterexample.

In section 3 we prove that any minimal counterexample is 5-connected. From this we deduce, in section 4, that no minor-minimal counterexample contains  $K_4 - e$ . Finally, in section 5, we prove Dirac's conjecture for all graphs that can be imbedded in a surface with Euler characteristic at least  $-2$ .

## 2. Menger's Theorem and Extensions

We make use of several fundamental results which we list here. The reader is referred to Bollobás [1] for further details.

A *vertex cut* of  $G$  is a subset of vertices whose removal disconnects  $G$ . A *k-separator* of  $G$  is a vertex cut of  $k$  vertices. The *connectivity* of  $G$  is the least  $k$  such that there exists a  $k$ -separator of  $G$ . If  $k$  is the connectivity of  $G$ , we write  $\kappa(G) = k$  and say that  $G$  is *k-connected*.

**Theorem 1 (Menger).** *A non-trivial graph is k-connected if and only if every pair of vertices is connected by k disjoint paths.*

Let  $S$  be a set of vertices in the graph  $G$  and let  $x$  be a vertex not in  $S$ . An *x-S fan* is a set of  $|S|$  paths from  $x$  to  $S$ , any two of which share only the vertex  $x$ .

**Theorem 2 (Dirac).** *A graph  $G$  is k-connected if and only if  $|G| \geq k + 1$  and for any k-set  $S \subset V(G)$  and vertex  $x \in V(G) - S$  there is an  $x - S$  fan.*

The following two theorems follow as corollaries of the previous one.

**Theorem 3 (Dirac).** *If  $G$  is k-connected and  $k \geq 2$ , then for any set of k vertices there is a cycle containing all of them.*

Suppose  $X, Y \subset V(G)$ . We say that  $X$  is *linked* to  $Y$  if there are  $|X|$  vertex disjoint paths from  $X$  to  $Y$ . Notice that the paths linking  $X$  to  $Y$  cannot share any vertices including initial and terminal vertices.

**Theorem 4 (Dirac).** *Let  $|G| \geq 2k$ .  $G$  is k-connected if and only if whenever  $V_1$  and  $V_2$  are disjoint k-sets of vertices, then  $V_1$  is linked to  $V_2$ .*

### 3. 5-connectivity

Let  $G$  be a (topologically) minimal counterexample as defined in the introduction. In this section we show that  $G$  is 5-connected. We begin by examining the minimum degree. Observe that a minimal counterexample with  $n$  vertices has  $3n - 5$  edges.

**Lemma 1.** *If  $G$  is minimal in  $\mathcal{D}$ , then  $\delta(G) = 5$ .*

**Proof:** The average degree is less than six, so the minimum degree is at most five. If the minimum degree is less than four, then we may delete a vertex of degree at most three from  $G$ , obtaining a smaller graph with  $3(n - 1) - 5$  edges and no subdivision of  $K_5$ , which contradicts minimality of  $G$ . Hence, it suffices to show that the minimum degree is not four.

Suppose, for a contradiction, that  $\delta(G) = 4$ . Let  $v \in V(G)$  have  $d_G(v) = 4$  with neighbors  $a, b, c, d$ . There must be a pair of these neighbors, say  $c$  and  $d$ , that are not adjacent, otherwise the five vertices  $\{v, a, b, c, d\}$  form a  $K_5$ . Deleting the edges  $va$  and  $vb$ , then contracting  $v$  to edge  $cd$  yields a subgraph of  $G$  in  $\mathcal{D}$ , contradicting that  $G$  is a minimal counterexample.  $\square$

From Lemma 1, by counting edges and degrees, it is easy to deduce that a minimal counterexample must have at least ten vertices.

Suppose  $S$  is a set of vertices of  $G$ .  $G[S]$  denotes the subgraph induced by  $S$ , and  $E(S)$  are the edges of  $G[S]$ .

**Lemma 2.** *If  $G$  is minimal in  $\mathcal{D}$ , then  $\kappa(G) \geq 3$ .*

**Proof:** Suppose, for a contradiction, that  $G$  is 2-connected with a 2-separator  $\{x, y\}$ . Let  $C_1$  be one component of  $G - \{x, y\}$ , and  $C_2 = G - (\{x, y\} \cup C_1)$ . Define  $G_i = G[C_i \cup \{x, y\}]$  for  $i = 1, 2$ . Lemma 1 ensures that the number of vertices in each  $G_i$  ( $i = 1, 2$ ) is at least six. Because  $G_1$  and  $G_2$  are sufficiently large subgraphs of  $G$ , the minimality of  $G$  implies that they do not contain a subdivision of  $K_5$ ; thus they each must have at most  $3n_i - 6$  edges, where  $n_i$  represents the number of vertices in  $G_i$ . Observing  $n_1 + n_2 = n + 2$ , we find

$$3n - 5 = |E(G)| \leq |E(G_1)| + |E(G_2)| \leq (3n_1 - 6) + (3n_2 - 6) = 3n - 6$$

a contradiction.  $\square$

Suppose  $G$  is a minimal in  $\mathcal{D}$  with  $S$  a  $\kappa(G)$ -separator of  $G$ . Let  $C_1$  be a component of  $G - S$  and  $C_2 = G - (S \cup C_1)$ . Define  $G_i = G[C_i \cup S]$ , for  $i = 1, 2$ . We say that  $S$  divides  $G$  into  $G_1$  and  $G_2$ . Let  $n_i$  and  $e_i$  represent the number of vertices and edges of  $G_i$ , respectively. Observe that  $n_1 + n_2 = n + \kappa(G)$  and, because  $G$  is a minimal counterexample,  $e_i < 3n_i - 5$ , for  $i = 1, 2$ .

We strengthen the ideas of the previous lemma by augmenting each  $G_i$  with edges corresponding to paths in  $G$ . More precisely, consider a pair of non-adjacent vertices  $x, y \in S$ , and a path  $P$  connecting  $x$  to  $y$  in  $G_2 - (S - \{x, y\})$ . Now  $H = G_1 + \{xy\}$  is a simple graph. Furthermore, if  $TK_5 \subset H$ , then  $TK_5 \subset G$ . Therefore, by the minimality of  $G$ ,  $|E(H)| < 3n_1 - 5$  which implies that  $e_1 < 3n_1 - 6$ . Thus we have used the path  $P$  to reduce the number of edges in  $G_1$ .

In general, suppose  $G$  is minimal  $\mathcal{D}$  with  $S$  a  $\kappa(G)$ -separator that divides  $G$  into  $G_1$  and  $G_2$ . Let  $P$  be a path in  $G_i - (S - \{x, y\})$  connecting two vertices of  $x, y \in S$  with  $xy \notin E(G)$ . We call  $P$  a *substituting path* for  $G_j$  (where  $j = \{1, 2\} - i$ ) and say  $P$  *substitutes* for  $xy$  (see figure 1). Define  $\sigma(G_i)$  to be the maximum number of internally vertex-disjoint substituting paths for  $G_i$  that pairwise do not share the same initial and terminal vertex. Observe that, if some pair of vertices in  $G[S]$  are not adjacent, then  $\sigma(G_i) \geq 1$ , for  $i = 1, 2$ . We make implicit use of this observation throughout the rest of the paper. The following lemma is the essence of this section.

**Lemma 3.** *Suppose  $G$  is minimal in  $\mathcal{D}$ , and  $S$  is a  $\kappa(G)$ -separator dividing  $G$  into  $G_1$  and  $G_2$ . Then*

$$7 + |E(S)| + \sigma(G_1) + \sigma(G_2) \leq 3|S| \quad (1)$$

**Proof:** For each  $i = 1, 2$ , form the simple graph  $H_i$  from  $G_i$  by adding the  $\sigma(G_i)$  edges corresponding to the substituting paths for  $G_i$ . By construction,  $TK_5 \subset H_i$  implies  $TK_5 \subset G$ ; hence  $TK_5 \not\subset H_i$ . Consequently, by the minimality of  $G$ ,  $|E(H_i)| < 3n_i - 5$  and  $e_i < 3n_i - 5 - \sigma(G_i)$ . Now,

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| - |E(G_1) \cap E(G_2)| \\ &\leq 3(n_1 + n_2) - 12 - \sigma(G_1) - \sigma(G_2) - |E(S)| \end{aligned}$$

So the result follows from  $n_1 + n_2 = n + |S|$  and  $|E(G)| = 3n - 5$ .  $\square$

To establish the 5-connectivity of a minimal counterexample, we shall use Lemma 3 repeatedly, forcing contradictions using equation (1).

**Lemma 4.** *If  $G$  is minimal in  $\mathcal{D}$ , then  $\kappa(G) \geq 4$ .*

**Proof:** As in Lemma 2, we argue by contradiction. Suppose that  $S = \{x, y, z\}$  is a 3-separator, dividing  $G$  into  $G_1$  and  $G_2$ . By Lemma 2,  $S$  is a  $\kappa(G)$ -separator of  $G$ .

If  $|E(S)| = 3$ , then equation (1) immediately yields a contradiction. We conclude that there is some pair of non-adjacent vertices in  $S$ , say  $x$  and  $y$ . Because  $S$  is a minimum separator, there is a substituting path for both  $G_1$  and  $G_2$ , substituting for  $xy$ . That is,  $\sigma(G_i) \geq 1$  for  $i = 1, 2$ , implying  $E(S) = \emptyset$  by equation (1).

Because  $G$  is 3-connected, Theorem 3 implies there is a cycle containing  $x, y$  and  $z$ . The cycle segments  $P_{xy}$ ,  $P_{yz}$  and  $P_{zx}$  can be considered as three vertex-disjoint paths. Indeed the three paths  $P_{xy}$ ,  $P_{zx}$ , and  $P_{yz}$ , are three substituting paths substituting for  $xy$ ,  $xz$ , and  $yz$  since  $E(S) = \emptyset$ . Thus,  $\sigma(G_1) + \sigma(G_2) \geq 3$ , and we again obtain a contradiction via equation (1). We conclude that  $\kappa(G) > 3$ .  $\square$

Observe that if  $G$  is a minimal counterexample, then  $G$  may not contain a  $K_4$ . To see this, consider a set  $U \subset V(G)$  with  $G[U]$  isomorphic to  $K_4$ . For any vertex  $x \in V(G) - U$  there exists an  $x - U$  fan by Lemma 4 and Theorem 2. This implies  $TK_5 \subset G$ . We use this observation to prove the following useful lemma. Let  $N_G(x) = \{y \in V(G) : xy \in E(G)\}$  denote the neighborhood of the vertex  $x$  in the graph  $G$ .

**Lemma 5.** *Suppose  $G$  is minimal in  $\mathcal{D}$ , and  $S$  is a 4-separator of  $G$  that divides  $G$  into  $G_1$  and  $G_2$ . For  $i = 1, 2$ ,  $S$  contains at most one vertex  $x$  such that  $|N_{G_i}(x) - S| = 1$ .*

**Proof:** By contradiction. Suppose  $x, y \in S$  such that  $N_{G_1}(x) - S = \{u\}$  and  $N_{G_1}(y) - S = \{v\}$ . Note that  $u \neq v$ , otherwise  $\kappa(G) = 3$  contradicting Lemma 4. Because  $G[S]$  is not isomorphic to  $K_4$ ,  $\sigma(G_i) \geq 1$ , for  $i = 1, 2$ . Hence  $e_i \leq 3n_i - 7$ . Moreover,  $H = G_1 - \{x, y\}$  has at most  $3(n_1 - 2) - 7$

edges, by similar reasoning. Therefore we obtain the following contradiction:

$$\begin{aligned}
3n - 5 = |E(G)| &\leq 2 + |E(H)| + |E(G_2)| \\
&\leq 2 + (3n_1 - 13) + (3n_2 - 7) \\
&\leq 3n - 6
\end{aligned}$$

since  $n_1 + n_2 = n + 4$ .  $\square$

**Theorem 5.** *If  $G$  is minimal in  $\mathcal{D}$ , then  $\kappa(G) = 5$ .*

**Proof:** As in the previous lemmas, we assume that  $G$  is 4-connected and obtain a contradiction. To this end, suppose  $S = \{w, x, y, z\}$  is a 4-separator of  $G$  that divides  $G$  into  $G_1$  and  $G_2$ . Because  $G[S]$  is not isomorphic to  $K_4$ ,  $\sigma(G_i) \geq 1$ , for  $i = 1, 2$ . From equation (1), we conclude that  $|E(S)| \leq 3$ .

Let  $P_j$  and  $E_j$  denote a path and independent set on  $j$  vertices, respectively;  $G_1 \cup G_2$  denotes the disjoint union of  $G_1$  and  $G_2$ . So,  $G[S]$  is isomorphic to one of  $K_3$ ,  $P_4$ ,  $K_{1,3}$ ,  $P_2 \cup P_2$ ,  $P_3 \cup E_1$ ,  $P_2 \cup E_2$ , or  $E_4$ . To prove that  $G$  is 5-connected, it remains to exclude these seven cases.

**Case 1:**  $K_3$ . Suppose  $\{x, y, z\}$  form a triangle. There are four vertex-disjoint paths from any vertex  $u \in G_1 - S$  to  $v \in G_2 - S$  since  $\kappa(G) \geq 4$ . Consequently  $G$  contains a subdivision of  $K_5$  with branch vertices  $\{x, y, z, u, v\}$ .

**Case 2:**  $P_4$ . Suppose  $E(S) = \{wx, xy, yz\}$ . By equation (1), it suffices to show  $\sigma(G_1) + \sigma(G_2) \geq 3$ . Let  $v \in N_{G_1}(z) - S$ . Because  $G$  is 4-connected, Theorem 2 guarantees a fan from  $w$  to  $\{x, y, z, v\}$  consisting of four vertex-disjoint paths  $P_{wx}, P_{wy}, P_{wz}, P_{wv}$ . The paths  $P_{wy}$  and  $P_{wz}$  each lie completely in  $G_1$  or  $G_2$  since  $\{w, x, y, z\}$  is a 4-separator. Similarly,  $P_{wv}$  must lie completely in  $G_1$ . If  $P_{wz} \in G_2$ , then  $P_{wz}$  is a substituting path for  $G_1$  and  $P_{wv} + vz$  is a substituting path for  $G_2$ ; so together with  $P_{wy}$ ,  $\sigma(G_1) + \sigma(G_2) \geq 3$ .

Suppose  $P_{wz}, P_{wv} \in G_1$ . To show  $\sigma(G_1) + \sigma(G_2) \geq 3$ , it suffices to find vertex-disjoint paths  $P_{xz}$  and  $P_{wz}$  in  $G_1$  that avoid  $y$ . Consider a path  $P_{xz}$  connecting  $x$  to  $z$  in  $G_1$  such that  $P_{xz}$  avoids the vertices  $w, y$  (if no such path exists then  $\{w, x, y\}$  is a 3-separator, contradicting Lemma 4). If  $P_{xz}$  avoids either  $P_{wz}$  or  $P_{wv}$ , then we have found the desired paths. Otherwise, let  $u$  be the vertex

closest to  $x$  where  $P_{xz}$  intersects one of these paths. Without loss of generality, we may assume that  $u \in P_{wz}$ . Let  $P_{uz}$  be the segment of  $P_{wz}$  from  $u$  to  $z$ . Then  $P_{xu} + P_{uz}$  and  $P_{wv} + vz$  are the two desired paths.

**Case 3:**  $K_{1,3}$ . Suppose  $E(S) = \{wx, wy, wz\}$ . By Lemma 4 and Theorem 3, there is a cycle in  $G - w$  containing  $x, y, z$ . This cycle determines three substituting paths  $P_{xy}, P_{xz}$ , and  $P_{yz}$ . Hence,  $\sigma(G_1) + \sigma(G_2) \geq 3$  and equation (1) yields a contradiction.

**Case 4:**  $P_2 \cup P_2$ . Suppose  $E(S) = \{wx, yz\}$ . To obtain a contradiction from equation (1), it suffices to show, for  $i = 1, 2$ , that  $\sigma(G_i) \geq 2$ . We show  $\sigma(G_2) \geq 2$ . The other case is symmetric.

Observe that, by Lemma 5, there is at most one vertex of  $S$ , say  $z$ , such that  $|N_{G_1}(z) - S| = 1$ . So there are two vertices  $a, b \in N_{G_1}(w) - S$ ,

Now  $G - \{w, x\}$  is 2-connected. Hence, by Theorem 4 there are two disjoint paths linking  $\{a, b\}$  and  $\{y, z\}$ . Because  $\{y, z\}$  is a 2-cut in  $G - \{w, x\}$ , these two paths must lie entirely in  $G_1$ . These paths substitute for edges  $wy$  and  $wz$ , and so  $\sigma(G_2) \geq 2$ .

**Case 5:**  $P_3 \cup E_1$ . Suppose  $E(S) = \{wx, xy\}$ . In this case, we show that  $\sigma(G_1) + \sigma(G_2) \geq 4$  by showing that, for some  $i \in \{1, 2\}$ ,  $\sigma(G_i) \geq 3$ . Equation (1) provides the contradiction.

Because  $\delta(G) = 5$ , there is some  $j \in \{1, 2\}$  such that there exist three vertices  $a, b, c \in N_{G_j}(z) - S$ . By theorem 4, there exist vertex disjoint paths linking  $\{a, b, c\}$  to  $\{w, x, y\}$  in  $G - z$ . These three paths must all lie in  $G_j$ . Therefore they form three substituting paths  $P_{zw}, P_{zx}$ , and  $P_{zy}$  for  $G_i$ , where  $i = \{1, 2\} - j$ .

**Case 6:**  $P_2 \cup E_2$ . Suppose  $E(S) = \{wx\}$ . By Lemma 5 and  $\delta(G) = 5$ , we may assume, without loss of generality, that there are three vertices  $a, b, c \in N_{G_1}(y) - S$ . Arguing as in the previous case, theorem 4 implies the existence of three substituting paths for  $G_2$ ,  $P_{yw}, P_{yx}$ , and  $P_{yz}$  by linking  $\{a, b, c\}$  with  $\{w, x, z\}$  in  $G - y$ . Hence,  $\sigma(G_2) \geq 3$ .

Furthermore, by Lemma 5 and  $\delta(G) = 5$ , either  $|N_{G_2}(y) - S| \geq 2$  or  $|N_{G_2}(z) - S| \geq 2$ . In either case, linking the neighborhood vertices with  $\{w, x\}$  in  $G - \{y, z\}$  shows that  $\sigma(G_1) \geq 2$ . Thus,  $\sigma(G_1) + \sigma(G_2) \geq 5$ , and equation (1) yields a contradiction.

**Case 7:  $E_4$ .** In this case, it suffices to show that  $\sigma(G_1) + \sigma(G_2) \geq 6$ . Observe that, applying the method in the previous case, if there is a vertex of  $S$ , say  $w$ , such that  $|N_{G_i}(w) - S| \geq 3$ , then  $\sigma(G_j) \geq 3$ , where  $j = \{1, 2\} - i$ . Thus, it is enough to consider the case that, for some  $i \in \{1, 2\}$ , for all  $v \in S$ ,  $|N_{G_i}(v) - S| \leq 2$ . Without loss of generality, suppose  $i = 1$ .

Applying the method of the previous case, it is easy to show  $\sigma(G_2) \geq 2$ . Hence,  $e_2 \leq n_2 - 8$ . Consider  $H = G_1 - S$ . If  $H$  has at least three vertices (i.e.  $n_1 - 4 \geq 3$ ), then  $|E(H)| \leq 3(n_1 - 4) - 6$ , by the minimality of  $G$  (This is clearly true if  $n_1 - 4 \geq 5$ . The remaining cases,  $n_1 - 4 \in \{3, 4\}$ , follow because  $G$  is simple). Therefore,

$$\begin{aligned} 3n - 5 = |E(G)| &\leq |E(H)| + |E(G_2)| + 8 \\ &\leq 3(n_1 - 4) - 6 + 3n_2 - 8 + 8 \\ &= 3n - 6 \end{aligned}$$

This contradiction implies  $H$  has exactly two vertices (the minimum degree prohibits  $H$  having a single vertex).

So,  $H$  consists of two adjacent vertices,  $u$  and  $v$ , each of which is adjacent to every vertex of  $S$ . Suppose  $G - \{u, v\}$  is 3-connected. In this case, theorem 3 guarantees that  $\{x, y, z\}$  lie on a cycle of  $G - \{u, v\}$ . Consequently,  $G$  contains a subdivision of  $K_5$ ; the branch vertices are  $u, v, x, y, z$ . This is a contradiction.

Therefore,  $G - \{u, v\}$  must be 2-connected, with a 2-separator  $S'$ . However, in this case, we may form a 4-separator  $\{u, v\} \cup S'$  of  $G$  with at least one edge. This reduces to a previous case.  $\square$

## 4. Forbidden subgraphs

Recall that, in the previous section,  $K_4$  was forbidden from any minimal graph in  $\mathcal{D}$ . Applying similar arguments and 5-connectivity, we now extend these results and summarize them in the following theorem. Let  $G_1 + G_2$  denote the *join* of  $G_1$  and  $G_2$ ; it is the graph obtained from  $G_1$  and  $G_2$  by joining each vertex of  $G_1$  to each vertex of  $G_2$ .

**Theorem 6.** *No minimal graph in  $\mathcal{D}$  contains  $K_4$ ,  $K_{3,3}$ ,  $K_2 + E_3$  or  $K_{2,4}$ .*

**Proof:** We prove only that  $K_2 + E_3$  is forbidden; the other proofs are similar and are omitted. Suppose that  $G$  is minimal in  $\mathcal{D}$ , and  $K_2 + E_3 \subset G$  such that  $x, y$  are the vertices of the  $K_2$  portion of  $K_2 + E_3$ . By Theorem 3, there is a cycle in  $G - \{x, y\}$  containing the three vertices of  $E_3$ , since  $G - \{x, y\}$  is 3- connected. This implies  $TK_5 \subset G$ .  $\square$

The aim of this section is to forbid  $K_4 - e$  in any minor-minimal graph in  $\mathcal{D}$ . To prove this we require some preliminary definitions and technical lemmas. Graph  $L$  is defined as shown in figure 2. A *branch vertex* of a subdivision is a vertex of degree at least three; and, a *branch path* is a path between branch vertices. In any subdivision of  $L$ , the branch vertices of degree three are called *minor branch vertices*, and the branch vertices of degree four are called *major branch vertices*. The following lemma is presented by Thomassen in [Th74]:

**Lemma 6 (Thomassen).** *Let  $G' = G/_{xy}$ , the graph obtained by contracting edge  $xy$  in  $G$ .*

- (a) *If  $TK_5 \subset G'$  such that  $xy \in V(G')$  is not a branch vertex, then  $TK_5 \subset G$ .*
- (b) *If  $TK_5 \subset G'$  with vertex  $xy \in V(G')$  a branch vertex, then either  $TK_5 \subset G$  such that  $x$  or  $y$  is a branch vertex, or  $TL \subset G$  such that  $x$  and  $y$  are minor branch vertices.*

**Lemma 7.** *If  $G$  is minor-minimal in  $\mathcal{D}$  then, for every  $x, y \in V(G)$  with  $xy \in E(G)$ , there is a subdivision of  $L$  in  $G$  such that  $x$  and  $y$  are minor branch vertices.*

**Proof:** Let  $G$  be minor-minimal in  $\mathcal{D}$ , with  $x, y \in V(G)$  such that  $xy \in E(G)$ . Since the graph  $K_2 + E_3$  is forbidden from  $G$ ,  $G/_{xy}$  has at most three fewer edges than  $G$ . Hence  $|E(G/_{xy})| \geq 3|V(G/_{xy})| - 5$ , and  $G/_{xy}$  contains a  $TK_5$ . By Lemma 6,  $G$  contains a subdivision of  $L$  such that  $x$  and  $y$  are minor branch vertices.  $\square$

From Lemma 7, we may now obtain more detailed structural information about any minor-minimal graph in  $\mathcal{D}$  with a triangle. We introduce a few definitions to refine our view of  $TL$  and describe this structure.

Label the minor branch vertices of  $TL$ ,  $x$  and  $y$ , and the major branch vertices  $a, b, c$ , and  $d$  as in figure 3. The four branch paths between  $\{x, y\}$  and  $\{a, b, c, d\}$  are designated  $P_1, P_2, P_3$ , and  $P_4$  and are called *P-paths*.  $P$  is the set of vertices in  $V(G) - \{x, y\}$  that appear in a  $P$ -path. The

six branch paths between the major branch vertices are labelled  $R_1, \dots, R_6$  and are called *R-paths*.  $R$  is the set of vertices in  $V(G) - \{a, b, c, d\}$  that appear in an R-path.  $R_i$  and  $R_j$  are *adjacent* if they are incident to the same branch vertex, and *parallel* if they are not. For example,  $R_1$  and  $R_2$  are adjacent;  $R_1$  and  $R_6$  are parallel.  $\{R_1, R_2, R_5, R_6\}$  are the *middle R-paths*, and  $\{R_3, R_4\}$  the *outside R-paths*. If  $Q$  is a path with a single endpoint in  $R - \{a, b, c, d\}$ , we define  $\Phi(Q)$  to be the R-path that contains the endpoint of  $Q$  in  $R$ . If  $S$  is a set of paths with endpoints in  $R - \{a, b, c, d\}$ ,  $\Phi(S)$  is defined to be the set of R-paths that contain the endpoints of  $S$  in  $R$ .

**Lemma 8.** *Suppose  $G$  is minor-minimal in  $\mathcal{D}$  with a triangle  $\{x, y, z\}$ . Then,  $G$  contains a subdivision of  $L$  such that  $x$  and  $y$  are minor branch vertices. Furthermore, given  $R$  and  $P$  as defined above,*

- (1)  $z$  is separated from  $P$  by  $R$  in  $G - \{x, y\}$ ,
- (2) If  $z \notin R$ , then there are three disjoint paths in  $G - \{x, y\}$  from  $z$  to  $R$  such that all interior vertices avoid  $V(TL)$ , and all three endpoints are either
  - (a) all in the same R-path, or
  - (b) incident to three different R-paths, which are pairwise adjacent, though not all incident to the same major branch vertex.

**Proof:** Let  $G$  be minor-minimal in  $\mathcal{D}$  with a triangle  $\{x, y, z\}$ . By Lemma 7,  $G$  contains a subdivision of  $L$  with minor branch vertices  $x$  and  $y$ .

If  $z$  is a vertex of a  $P$ -path, then  $TK_5 \subset G$  with branch vertices  $a, b, c, d$  and, either  $x$  or  $y$  depending upon which  $P$ -path contains  $z$ . More generally, if there is a path from  $z$  to  $P$  using only vertices of  $V(G) - V(TL)$ , then  $TK_5 \subset G$ , as shown in figure 4. Thus,  $z \notin P$ , and no path from  $z$  to  $P$  avoids  $V(TL)$ ; that is, the vertices in  $R$  separate  $z$  from  $P$  in  $G - \{x, y\}$ , and statement (1) has been established.

Suppose  $z \notin R$  (if not, statement (2) is vacuous). Because  $G$  is 5-connected, there are three disjoint paths from  $z$  to  $\{a, b, c\}$  in  $G - \{x, y\}$ . Each of these paths must contain a vertex in  $R$ , since  $R$  separates  $z$  from  $\{a, b, c\}$  in  $G - \{x, y\}$ . Let  $Z_1, Z_2$ , and  $Z_3$  be the three disjoint paths from  $z$  to  $R$  defined by these three paths. Call these paths  $Z$ -paths, and let  $Z$  be the set  $Z$ -paths.

Suppose two  $Z$ -paths have endpoints in parallel  $R$ -paths. If the parallel  $R$ -paths are both middle  $R$ -paths, there is a  $TK_5$  in  $G$  with branch vertices  $\{c, d, x, y, z\}$ , as shown in figure 5. Otherwise the endpoints are in  $R_3$  and  $R_4$ , and  $\{a, c, x, y, z\}$  are branch vertices of a  $TK_5$  (see figure 6).

Suppose the endpoints of the  $Z$ -paths lie in three different  $R$ -paths all incident to the same major branch vertex. Without loss of generality, we may assume  $\Phi(Z) = \{R_1, R_2, R_3\}$ ; they are all incident to  $a$ . In this case,  $\{b, c, d, y, z\}$  are branch vertices of a  $TK_5$  as shown in figure 7.

Suppose  $\Phi(Z)$  consists of two adjacent  $R$ -paths. Without loss of generality, we may assume they are incident to  $a$ . In this case,  $\{y, z, b, c, d\}$  are the branch vertices of a  $TK_5$  (figure 8).

For every  $1 \leq i < j \leq 3$ ,  $\Phi(Z_i)$  and  $\Phi(Z_j)$  cannot be parallel, and hence must be equal or mutually adjacent. But if  $\Phi(Z)$  consists of three  $R$ -paths all incident to a single branch vertex, then  $\Phi(Z)$  must consist of a single  $R$ -path. This shows that the endpoints of the  $Z$ -paths are either,

(a) all in the same  $R$ -path, or

(b) incident to three different  $R$ -paths, which are pairwise adjacent, though not all incident to the same major branch vertex.

These are the configurations given in the statement of the lemma.  $\square$

We now can state the main result of this section:

**Theorem 7.** *No minor-minimal graph in  $\mathcal{D}$  contains  $K_4 - e$ .*

**Proof:** We prove the theorem by contradiction. Suppose  $G$  is minor-minimal in  $\mathcal{D}$  such that  $w, x, y$ , and  $z$  induce a  $K_4 - e$ . Let  $x$  and  $y$  be the vertices of degree three in the induced  $K_4 - e$ . By Lemma 7, there is a subdivision of  $L$  in  $G$  with  $x$  and  $y$  as minor branch vertices. Label this  $TL$  as in the previous lemma. Also define the  $P$ -paths and  $R$ -paths as in the previous lemma.

We divide the proof into three cases depending upon whether all, one, or none of  $z$  and  $w$  are in  $R$ . To prove the theorem, it suffices to exclude these three cases.

**Case 1:**  $w, z \in R$ . We consider three subcases according to the placement of  $w$  and  $z$  in  $R$ : the same  $R$ -path, adjacent  $R$ -paths, or parallel  $R$ -paths.

Case 1.1:  $w$  and  $z$  are in the same  $R$ -path. If  $w$  and  $z$  are both in  $R_1$ , there is a  $TK_5$   $\{c, w, x, y, z\}$ , as shown in figure 9. Similar arguments apply for the other  $R$ -paths. (Figure 10 shows the case where  $w$  and  $z$  are in  $R_3$ .)

Case 1.2:  $w$  and  $z$  are in adjacent  $R$ -paths, say  $R_w$  and  $R_z$ . By symmetry, it suffices to consider the case that one of  $R_w, R_z$  is an outside  $R$ -path, and the case that they are both middle  $R$ -paths:  $R_w = R_1, R_z = R_3$ ; and,  $R_w = R_1, R_z = R_2$ . If  $R_w = R_1$  and  $R_z = R_3$ , then  $\{a, b, x, y, z\}$  are the branch vertices of a  $TK_5$ , as shown in figure 11. If  $R_w = R_1$  and  $R_z = R_2$ , then  $\{w, x, y, z, d\}$  are the branch vertices of a  $TK_5$ , as shown in figure 12.

Case 1.3:  $w$  and  $z$  are in parallel  $R$ -paths, say  $R_w$  and  $R_z$ . By symmetry, it suffices to consider when these  $R$ -paths are both middle or both outside  $R$ -paths:  $R_w = R_1, R_z = R_6$ ; and,  $R_w = R_3, R_z = R_4$ . If  $R_w = R_1$  and  $R_z = R_6$ , then  $\{a, b, w, x, y\}$  are the branch vertices of a  $TK_5$ , as shown in figure 13. If  $R_w = R_3$  and  $R_z = R_4$ , a subdivision of  $K_5$  appears as in figure 14.

**Case 2:**  $|R \cap \{z, w\}| = 1$ . Without loss of generality, assume  $w \in R$ . By symmetry, there are only two subcases to consider:  $w \in R_1$  or  $w \in R_3$ . Because  $z \notin R$ , Lemma 8 guarantees three disjoint paths from  $z$  to  $R$ . Call these three paths  $Z$ -paths. By Lemma 8, either  $\Phi(Z)$  is a single  $R$ -path, or  $\Phi(Z)$  consists of three pairwise adjacent  $R$ -paths, not all incident to the same major branch vertex. We may assume that  $\Phi(Z)$  is not a single  $R$ -path because, in this case, one can form a new subdivision of  $L$  in  $G$  such that  $z, w \in R$  and  $x, y$  are the minor branch vertices, by redirecting the  $R$ -path in  $\Phi(Z)$  through  $z$  (this reduces to case 1). We also may assume no  $Z$ -path ends at  $w$  since, in such a case,  $G$  contains a subdivision of  $K_5$  with branch vertices  $\{w, x, y, z\}$  plus one vertex in  $\{a, b, c, d\}$  depending upon the location of  $w$  and  $\Phi(Z)$  in  $R$  (another  $Z$ -path is used to complete a path from  $z$  to the fifth branch vertex).

Case 2.1:  $w \in R_1$ . Because  $\Phi(Z)$  consists of pairwise adjacent  $R$ -paths not all incident to one major branch vertex, some  $Z$ -path ends in an outside  $R$ -path. Therefore  $\{a, c, x, y, w\}$  are the branch vertices of a  $TK_5$ , as in figure 15.

Case 2.2:  $w \in R_3$ .  $\Phi(Z)$  consists of pairwise adjacent  $R$ -paths, not all incident to the same major branch vertex. By symmetry, we may assume, without loss of generality, that  $\Phi(Z)$  contains  $R_2$ ; that is,  $\Phi(Z) = \{R_2, R_3, R_6\}$  or  $\{R_2, R_1, R_4\}$ . In either case,  $\{w, x, y, z, a\}$  are the branch vertices of a  $TK_5$ , as shown in figure 16 (which shows the case where a  $Z$ -path ends in  $R_6$ ).

Case 3:  $R \cap \{w, z\} = \emptyset$ . Because both  $w$  and  $z$  are neighbors to  $x$  and  $y$ , Lemma 8 guarantees three disjoint paths from  $z$  to  $R$ , and three disjoint paths from  $w$  to  $R$ . Let  $Z_1, Z_2$  and  $Z_3$  be the three disjoint paths from  $z$  to  $R$  (the  $Z$ -paths), and  $Z$  the set of  $Z$ -paths. Similarly, let  $W_1, W_2$  and  $W_3$  be the three disjoint paths from  $w$  to  $R$  (the  $W$ -paths), and  $W$  the set of  $W$ -paths. Observe that, by definition, only terminal vertices of  $Z$ -paths or  $W$ -paths are vertices of  $R$ .

By Lemma 8, either  $\Phi(Z)$  is a single  $R$ -path, or  $\Phi(Z)$  consists of three pairwise adjacent  $R$ -paths, not all incident to the same major branch vertex. We may assume that  $\Phi(Z)$  is not a single  $R$ -path because, in this case, one can form a new subdivision of  $L$  in  $G$  such that  $z \in R$  and  $x, y$  are the minor branch vertices, by redirecting the  $R$ -path in  $\Phi(Z)$  through  $z$  (this reduces to case 2). The same argument shows that  $\Phi(W)$  is not a single  $R$ -path.

Because  $\Phi(Z)$  and  $\Phi(W)$  each consist of three pairwise adjacent  $R$ -paths not all incident to the same branch vertex, we may assume, without loss of generality, that  $\Phi(Z_1) = R_1$  and  $\Phi(W_1) = R_3$ . If  $Z_1$  and  $W_1$  do not intersect, then  $G$  contains a subdivision of  $K_5$  with branch vertices  $\{x, y, b, c, d\}$ , as shown in figure 17. Hence,  $Z_1$  and  $W_1$  must intersect.

Reorder the  $W$ -paths so that  $W_1$  is the first  $W$ -path that  $Z_1$  intersects, and  $u$  is a vertex of their intersection closest to  $z$ . Our immediate goal is to construct, from the  $Z$ -paths and  $W$ -paths, three internally disjoint paths: one  $zw$ -path, one  $zR$ -path ( $Q_z$ ), and one  $wR$ -path ( $Q_w$ ). If  $Z_2$  does not meet any  $W$ -path, then we let  $Q_z = Z_2$ ,  $Q_w = W_2$ , and form the  $zw$ -path with the initial segments of  $Z_1$  and  $W_1$  that meet at  $u$ . Otherwise,  $Z_2$  first intersects some  $W$ -path, say  $W_i$ , at some vertex  $v$ . If  $W_i \neq W_1$ , then let  $Q_w = W_j (j = \{2, 3\} - \{i\})$ ,  $Q_z$  the path formed by the initial segment of  $Z_2$  from  $z$  to  $v$  and the final segment of  $W_i$  from  $v$  to  $R$ , and form the  $zw$ -path from the initial segments of  $Z_1$  and  $W_1$ . If  $W_i = W_1$ , we may assume, without loss of generality, that  $u$  is closer to  $v$  along  $W_1$ . In this case, let  $Q_z$  be the path formed by the initial segment of  $Z_2$  and the final segment of  $W_1$ , let  $Q_w = W_2$ , and form the  $zw$ -path from the initial segments of  $Z_1$  and  $W_1$ .

The  $zw$ -path together with the edges in the  $K_4 - e$  form a subdivision of  $K_4$  in  $G$ . To show that  $G$  has a subdivision of  $K_5$ , it suffices to show that some vertex in  $\{a, b, c, d\}$  can be the fifth branch vertex of a  $TK_5$  involving  $\{w, x, y, z\}$ . The branch paths from the fifth branch vertex are constructed using  $Q_w, Q_z, P$ -paths, and  $R$ -paths.

Suppose  $\Phi(Q_w) = \Phi(Q_z)$ . If  $Q_w$  and  $Q_z$  end in the same vertex  $q \in R$ , then  $\{q, w, x, y, z\}$  are the branch vertices of a  $TK_5$ . If  $Q_w$  and  $Q_z$  do not share a common endpoint, but  $\Phi(Q_z) = \Phi(Q_w) = R_1$  say, then  $\{a, w, x, y, z\}$  are the branch vertices of a  $TK_5$  (figure 18). Other cases where  $\Phi(Q_z) = \Phi(Q_w)$  are similar.

Suppose  $\Phi(Q_w) \neq \Phi(Q_z)$ . By symmetry, we may assume that  $\Phi(Q_z)$  is incident to  $a$ , while  $\Phi(Q_w)$  is not. It suffices to find four vertex disjoint paths: one path from each of  $w, x, y, z$  to  $a$ .  $P_2$  connects  $x$  and  $a$ . A segment of  $\Phi(Q_z)$  plus  $Q_z$  connects  $z$  and  $a$ . A path in  $\{R_1, R_2\} - \Phi(Q_z)$  plus a path in  $\{P_3, P_4\}$  connect  $y$  and  $a$ . The remaining  $R$ -paths and  $Q_w$  contain a path connecting  $w$  and  $a$ . Thus,  $\{a, w, x, y, z\}$  are the branch vertices of a  $TK_5$ .  $\square$

## 5. Genus

We assume the reader is familiar with the notation and results found in [4]. Let  $S$  be a closed, connected 2-manifold. We denote the *Euler characteristic* of a cellular imbedding,  $G \rightarrow S$  of a connected graph  $G$  into  $S$  by  $\chi(G \rightarrow S)$ ; its value is  $|V(G)| - |E(G)| + f$ , where  $f$  is the number of faces of the imbedding. The Euler characteristic is an invariant of the surface  $S$ . Let  $\chi(S)$  be the Euler characteristic of  $S$  (so  $\chi(G \rightarrow S) = \chi(S)$  for any cellular imbedding of any  $G$  into  $S$ ).

**Theorem 8.** *Suppose  $G$  is a simple graph on  $n$  vertices that is minor-minimal in  $\mathcal{D}$ , and  $G \rightarrow S$  a cellular imbedding of  $G$  into  $S$ , a closed, connected 2-manifold. Then,*

$$\chi(S) \leq \lfloor 5/3 - n/4 \rfloor.$$

**Proof:** Let  $\chi = \chi(S)$ ,  $\alpha =$  number of triangles in  $G$ , and  $f_i =$  the number of  $i$ -sided faces in the imbedding  $G \rightarrow S$ . Now,  $\chi = n - (3n - 5) + f$ , since  $|E(G)| = 3n - 5$ . On the other hand,

$$3\alpha + 4(f - \alpha) \leq \sum_{i \geq 3} i f_i = 2(3n - 5).$$

Combining these two, we find

$$-4\chi \geq 2n - 10 - \alpha \tag{2}$$

so it suffices to show that  $\alpha \leq (3n - 10)/3$ .

Theorem 7 implies that every edge of  $G$  is in at most one triangle. Furthermore, every vertex of degree five in  $G$  is incident to an edge in no triangle, otherwise  $G$  has a  $K_4 - e$ . Because  $G$  has at least ten vertices of degree five, there are at least five edges of  $G$  that appear in no triangle. Thus, at most  $3n - 10$  edges are in triangles, and  $\alpha \leq (3n - 10)/3$ .  $\square$

We say that Dirac's conjecture holds for a surface  $S$  if every simple graph  $G$  with  $n$  vertices,  $3n - 5$  edges, and a cellular imbedding into  $S$ , contains a subdivision of  $K_5$  (the conjecture holds vacuously for the sphere). In this section, we use Theorem 8 to prove that Dirac's conjecture holds for several surfaces. First we prove a technical lemma.

**Lemma 9.** *Suppose  $G$  is minor-minimal in  $\mathcal{D}$ , and  $F = \{v \in V(G) : d_G(v) = 5\}$ . Then, the girth of  $G[F]$  is at least five.*

**Proof:** We prove that  $G[F]$  does not have a triangle or four-cycle.

Suppose, to the contrary, that  $x_1, x_2, x_3 \in F$  form a triangle of  $G$ . By Theorem 7,  $N_G(x_i) \cap N_G(x_j) = \{x_k\}$  for  $\{i, j, k\} = \{1, 2, 3\}$ . Furthermore, for each  $i = 1, 2, 3$ , there exist a pair of vertices  $y_i, z_i \in N_G(x_i)$  such that  $y_i z_i \notin E(G)$ . Consider  $H = G + \{y_1 z_1, y_2 z_2, y_3 z_3\} - \{x_1, x_2, x_3\}$ .  $H$  has  $n - 3$  vertices and  $3(n - 3) - 5$  edges. By the minimality of  $G$ ,  $TK_5 \subset H$  contradicting  $TK_5 \not\subset G$ . Thus,  $G[F]$  has no triangle.

Suppose  $x_1, x_2, x_3, x_4 \in F$  form a four-cycle. By Theorem 7, we may assume  $N_G(x_i) \cap N_G(x_j) = \emptyset$ , for  $i - j$  odd. Furthermore, one can show that, for each  $i = 1, \dots, 4$ , there exist a pair of vertices  $y_i, z_i \in N_G(x_i)$  such that  $y_i z_i \notin E(G)$  and  $\{y_i, z_i\} \cap \{y_j, z_j\} = \emptyset$  for all  $j \neq i$ . Now consider

$H = G + \{y_i z_i\}_{i=1}^4 - \{x_i\}_{i=1}^4$ .  $H$  has  $n - 4$  vertices and  $3(n - 4) - 5$  edges, so by the minimality of  $G$ ,  $TK_5 \subset H$ . This contradicts  $TK_5 \not\subset G$ .  $\square$

The conclusion of Lemma 9 may be extended in the case that  $G$  has large girth. In particular, if  $G$  has girth at least five, then  $G[F]$  must be acyclic.

**Corollary 1.** *Suppose  $G$  is a simple graph with  $n$  vertices,  $3n - 5$  edges, and a cellular imbedding into a surface  $S$  with  $\chi(S) \geq -2$ . Then  $TK_5 \subset G$ .*

**Proof:** We show that no minor-minimal counterexample can be imbedded into a surface with Euler characteristic greater than  $-3$ . To this end, let  $G$  be a minor-minimal counterexample with an imbedding  $G \rightarrow S$  into a surface  $S$  with  $\chi(S) \geq -2$ . By Theorem 8,  $\chi(S) \leq 5/3 - n/4$ , so  $n \leq 14$ . By remarks following Lemma 1,  $n \geq 10$ .

Observe that  $G$  must contain a triangle  $T$ ; otherwise, by equation (2),  $-4\chi(S) \geq 2n - 10 \geq 10$ . By Lemma 9,  $T$  must contain a vertex of degree six. Counting the neighborhood of  $T$  reveals that  $n \geq 13$  since Theorem 7 implies the neighborhoods of vertices in  $T$  are disjoint.

**Case 1:**  $n = 13$ . Suppose that  $G$  has a vertex with degree at least eight. An edge count reveals that the remaining vertices must then all have degree five. Because every triangle contains the high degree vertex and  $G$  has no  $K_4 - e$ ,  $G$  has at most four triangles so, by equation (2),  $-4\chi(S) \geq 2n - 10 - 4 \geq 12$ .

Thus, the maximum degree of  $G$  is seven, which implies that  $G$  has three vertices of degree six and ten vertices of degree five. If a triangle of  $G$  contains two vertices of degree six, then  $n \geq 14$  because the neighbors of the triangle are all distinct. So, every triangle in  $G$  contains exactly one degree six vertex. Because  $G$  has no  $K_4 - e$ , we conclude that  $G$  has at most seven triangles and  $-4\chi(S) \geq 2n - 10 - 7 \geq 9$ , which is a contradiction.

**Case 2:**  $n = 14$ . By the proof of Theorem 8,  $G$  has at most ten triangles. On the other hand, equation (2) implies that  $G$  has at least ten triangles. Consequently,  $G$  must have exactly ten triangles.

If  $G$  has a vertex  $v$  with degree at least eight, then an edge count reveals that  $G$  must have a vertex  $u$  of degree six. Now every triangle contains either  $u$  or  $v$  by Theorem 9. However  $v$  is in at most four triangles and  $u$  is in at most three triangles; that is,  $G$  has at most seven triangles, a contradiction.

So the maximum degree of  $G$  is seven. If there is a vertex of degree seven, then there are at most three vertices with degree more than five. Hence,  $G$  has at most nine triangles, a contradiction.

The remaining case is when  $G$  has exactly four degree six vertices and exactly ten degree five vertices. Let  $F$  be the set of degree five vertices, and  $S = \{a, b, c, d\}$  the set of degree six vertices. Note that  $|E(F)| = 13 + |E(S)|$ . Also,  $G[F]$  is connected since  $G$  is 5-connected and  $G[F] = G - S$ . In particular,  $G[F]$  does not have isolated vertices.

If there is a vertex  $v \in F$  with  $d_{G[F]}(v) = 5$ , then  $G[F] - \{v\} - N_G(v)$  has four vertices and at least four edges, contradicting that the girth of  $G[F]$  is at least five. Therefore,  $\Delta(G[F]) \leq 4$ .

Suppose there is a vertex  $v \in F$  with  $d_{G[F]}(v) = 4$ . Let  $N_G(v) \cap S = \{a\}$  and  $N_G(v) \cap F = \{x_1, x_2, x_3, x_4\}$ . If  $d_{G[F]}(x_1) = 1$  say, then  $x_j \notin N_G(a)$  ( $2 \leq j \leq 4$ ) since  $K_4 - e \not\subset G$ , so there must be a pair, say  $x_2, x_3$  such that  $|N_G(x_2) \cap N_G(x_3) \cap \{b, c, d\}| \geq 2$ . However,  $G[\{v, b, c, d, x_1, x_2, x_3\}]$  must then contain  $K_{3,3}$  contradicting Theorem 6. On the other hand, if  $d_{G[F]}(x_i) \geq 2$  for  $i = 1, \dots, 4$ , then  $G[\{v, b, c, d\} \cup N_G(v)]$  must contain  $K_{3,3}$ , by similar reasoning.

Therefore,  $\Delta(G[F]) = 3$ . Notice that this implies that  $\delta(G[F]) = 2$ . To see this, consider, for a contradiction, a vertex  $v \in F$  with  $d_{G[F]}(v) = 1$ . Now  $d_G(v) = 5$ , so  $v$  must be adjacent to every vertex of  $S$ . A neighbor of  $v$  in  $G[F]$  must have at least two neighbors in  $S$  (since  $\Delta(G) = 3$ ). Therefore  $S, v$ , and the neighbor of  $v$  in  $G[F]$  must induce  $K_4 - e$ , a contradiction.

**Subcase A:**  $|E[S]| \geq 3$ . In this case,  $G[F]$  has at least 16 edges and so it must contain a vertex of degree four, contradicting  $\Delta(G[F]) \leq 3$ .

**Subcase B:**  $|E(S)| = 2$ . Consider two adjacent vertices  $c, d \in S$ . If  $c$  and  $d$  share no common neighbor, then the edge  $cd$  appears in no triangle; consequently each of  $c$  and  $d$  appear in at most two triangles. However, if  $c$  and  $d$  have a common neighbor  $w \in F$ , then  $(N_G(c) \cup N_G(d)) \cap N_G(w) = \emptyset$  because  $G$  has no  $K_4 - e$ . Therefore, there exists a common neighbor of  $c$  and  $d$ , say  $z \in F - w$ , since

$|E(S)| = 2$  and  $\delta(G[F]) = 3$ . However, this implies that  $G$  contains a  $K_4 - e$ , namely  $\{c, d, w, z\}$ . Hence,  $c$  and  $d$  appear in at most two triangles. Because  $c$  and  $d$  were arbitrary adjacent vertices of  $S$  and  $|E(S)| = 2$ , there must be three vertices of  $S$  that appear in at most two triangles. That is,  $G$  has at most nine triangles, since each triangle of  $G$  must contain a vertex of  $S$ . This is a contradiction.

**Subcase C:**  $|E(S)| = 1$ . In this case,  $|E[F]| = 14$ . Because  $\Delta(G[F]) = 3$  and  $\delta(G[F]) = 2$ ,  $G[F]$  must have exactly two vertices of degree two, say  $u$  and  $v$ . If  $w \in N_G(u) \cap N_G(v) \cap F$ , then  $K_4 - e \subset G[\{u, v, w\} \cup S]$ , a contradiction. Similarly, if  $u$  and  $v$  are adjacent, then  $K_4 - e \subset G[\{u, v\} \cup S]$ . So, we may assume  $N_G(u) \cap N_G(v) \cap F = \emptyset$ , and  $uv \notin E(G)$ .

Suppose, without loss of generality,  $E(S) = \{cd\}$ . If  $\{c, d\} \subset N_G(v)$ , then  $K_4 - e \subset G[v \cup N_G(v) \cup S]$ . Thus, we may assume  $|N_G(v) \cap \{c, d\}| = 1$ . The same argument applies to  $u$ . Thus, there are two cases to consider:  $N_G(v) \cap S \neq N_G(u) \cap S$ , and  $N_G(v) \cap S = N_G(u) \cap S$ . Let  $H = G[\{u, v\} \cup N_G(u) \cup N_G(v) \cup S]$ .

Suppose  $N_G(v) \cap S \neq N_G(u) \cap S$ . Without loss of generality, assume  $c \in N_G(v)$  and  $d \in N_G(u)$ . Figure 19 shows the ten vertices of  $H$ , the edges forced into  $H$  by degree requirements and  $K_4 - e \not\subset G$ , and a new vertex  $z \in N_G(a) \cap N_G(b) - H$ . The vertex  $z$  must exist since  $a$  and  $b$  each have six neighbors in  $G$  while  $a$  has only four neighbors in  $H$ ,  $b$  has only three neighbors in  $H$ , and there are only four vertices in  $G - H$ . Thus  $G$  contains a subdivision of  $K_5$  as shown by the bold lines in the figure.

Similarly, suppose  $N_G(v) \cap S = N_G(u) \cap S$ . Figure 20 shows the ten vertices of  $H$ , the edges forced into  $H$  by degree requirements and  $K_4 - e \not\subset G$ , and a vertex  $z \in N_G(b) \cap N_G(c) - H$  guaranteed by arguing as in the previous paragraph. Thus  $G$  contains a subdivision of  $K_5$  as shown by the bold lines in the figure.

**Subcase D:**  $E(S) = \emptyset$ . In this case,  $|E[F]| = 13$ . Because  $\Delta(G[F]) \leq 3$  and  $\delta(G[F]) = 2$ ,  $G[F]$  has a set  $T$  of four vertices of degree two.

Suppose there are two vertices  $u, v \in T$ , such that  $N_G(u) \cap S = N_G(v) \cap S$ ; without loss of generality,  $N_G(u) \cap S = \{a, b, c\} = N_G(v) \cap S$ . If  $u$  and  $v$  are adjacent, then  $a, b, c, u, v$  form a

$K_2 + E_3$ . Similarly, if  $u$  and  $v$  share a common neighbor  $w \in F$ , then  $w$  must have a neighbor among  $a, b, c$  so a  $K_4 - e$  is formed. Thus  $N_G(v) \cap F = \{x, y\}$  and  $N_G(u) \cap F = \{p, q\}$  such that  $p, q, x, y \in F - T$ . Since  $K_4 - e \not\subset G$ ,  $\{p, q, x, y\} \subset N_G(d)$ . We may assume that  $x \in N_G(a)$  and  $y \in N_G(b)$ . Now there are three cases according to whether  $S - (N_G(p) \cup N_G(q))$  is equal to  $a, b$ , or  $c$ . The three cases are shown in figures 21, 22, and 23. The figures include a vertex  $z \notin \{u, v\} \cup N_G(u) \cup N_G(v) \cup S$  adjacent to two vertices of  $S$  (the existence of  $z$  can be established by considering the neighborhoods of vertices adjacent to  $z$  in  $S$ ). In each case a subdivision of  $K_5$  is indicated by bold lines.

Thus, we may assume that no pair of vertices in  $T$  share the same three neighbors in  $S$ ; that is,  $G[S \cup T]$  is isomorphic to  $K_{4,4}$  minus a one-factor. Because no pair of vertices of  $T$  are adjacent, some pair of vertices  $u, v \in T$  share a common neighbor  $z \in N_G(u) \cap N_G(v) \cap F$ . Let  $w = N_G(u) \cap F - \{z\}$ . Without loss of generality, assume  $N_G(u) \cap S = \{b, c, d\}$  and  $N_G(v) \cap S = \{a, c, d\}$  (so  $N_G(z) \cap S = \{a, b\}$  and  $a \in N_G(w)$ ). However, one can now see that there is a subdivision of  $K_5$  in  $G[S \cup T \cup \{w, z\}]$  with branch vertices  $a, b, u, v, z$ .  $\square$

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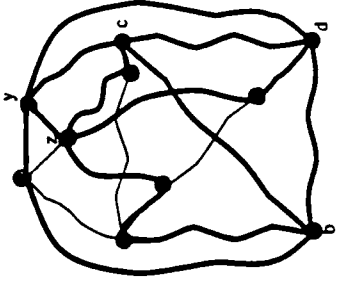


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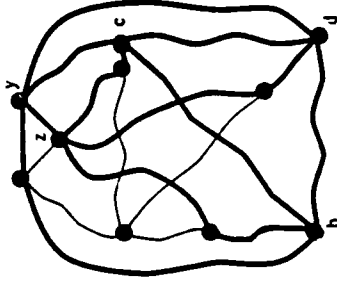


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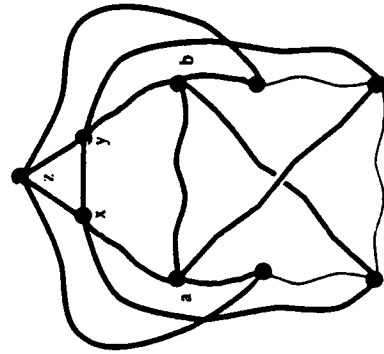


figure 6.

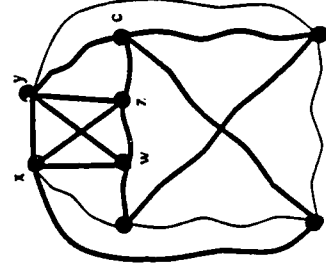


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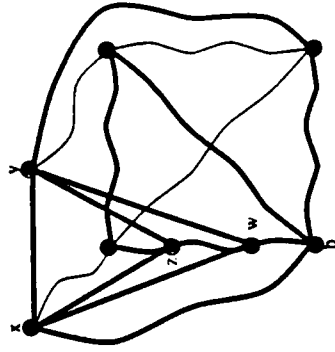


figure 8.

figure 9

figure 10.

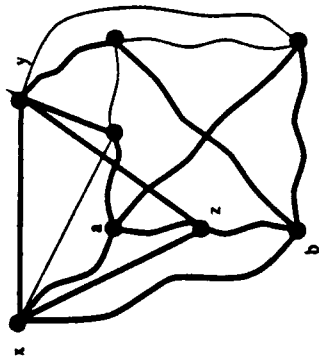


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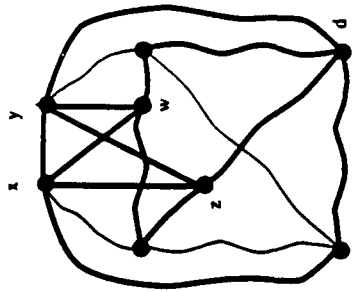


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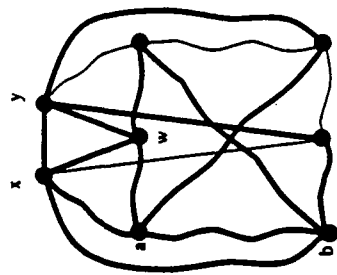


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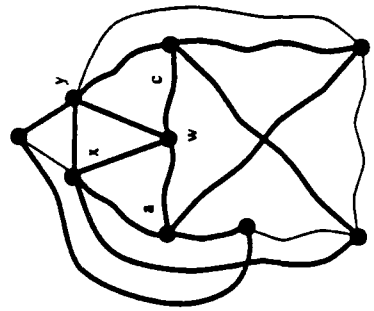


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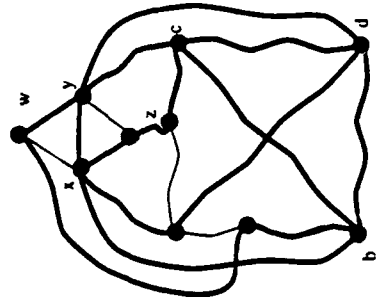


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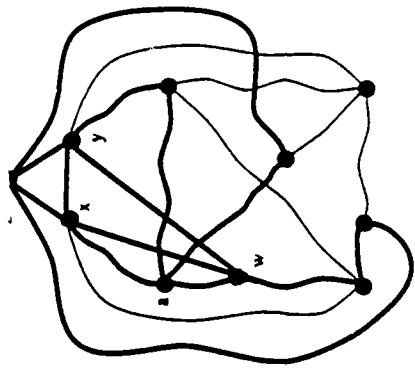


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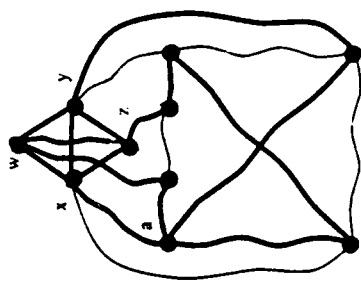


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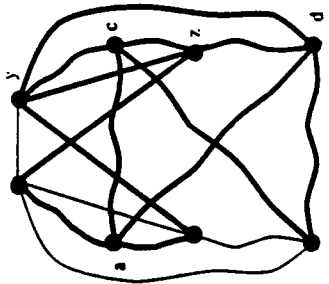


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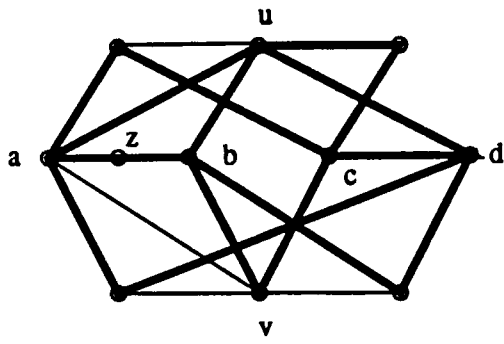


figure 19.

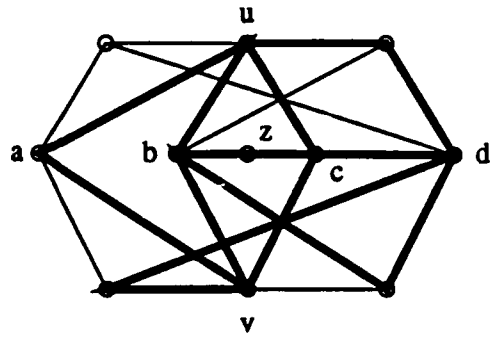


figure 20.

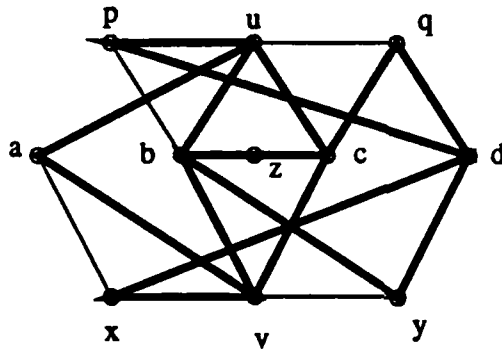


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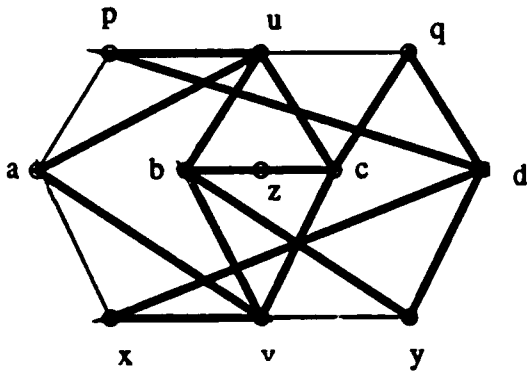


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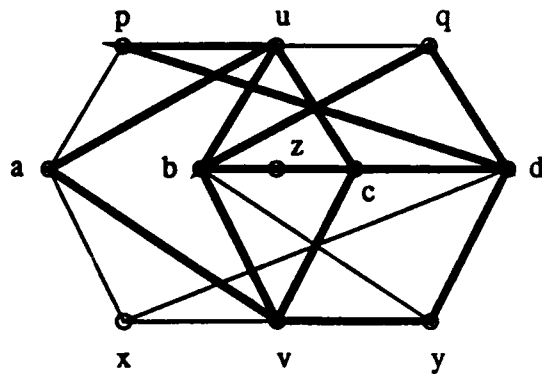


figure 23.