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FACTORIZATION OF THE CHARACTERISTIC EQUATION FOR THREE LAYER DUCTS

by

E. B. Brosius
D. F. McCammon

PENNSTATE

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SYMBOLS

Symbol	Units	Definition
A		Attenuation factor for pressure waves in sediment
A_S		Attenuation factor for shear waves in sediment
C		Reflection factor for improved LLL sediment solutions
c_1	m/s	Sound speed in water
c_2	m/s	Sound speed in sediment
c_{2S}	m/s	Shear sound speed in sediment
c_{3S}	m/s	Shear sound speed in bedrock
E_1		Exponential loss factor for water
E_2		Exponential loss factor for sediment
E_{2S}		Shear exponential loss factor for sediment
E_{12}		Combined exponential loss, phase factor, attenuation and reflection loss for water and sediment
f	Hz	Frequency
h	m	Depth of water layer
K_1		Phase factor for water
K_2		Phase factor for sediment
K_{2S}		Shear phase factor for sediment
k_1	1/m	Wavenumber in water
k_2	1/m	Wavenumber in sediment
k_3	1/m	Wavenumber in bedrock
k_n	1/m	Real part of eigenvalue

M_i		Incoherent summation of modes
n		Number of eigenvalues
P	Pa	Pressure
r	m	Horizontal range
r_2	m	Path distance traveled in sediment
r_{2S}	m	Shear path distance traveled in sediment
S_i		Combined reflection and attenuation factors
S_i^{jj}		Combined shear reflection and attenuation factors
S^{jj}		Combined shear reflection and attenuation factors
t	m	Depth of sediment layer
W		Wronskian
Z_i^j	m	Depth Eigenfunctions
z	m	Depth
α_2	dB/(kHz m)	Attenuation coefficient in sediment
α_{2S}	dB/(kHz m)	Shear attenuation coefficient in sediment
α_3	dB/(kHz m)	Attenuation coefficient in bedrock
ε_n	1/m	Imaginary part of eigenvalue
θ_1	rad	Grazing angle in water
θ_2	rad	Grazing angle in sediment
θ_3	rad	Grazing angle in bedrock
θ_c	rad	Critical angle
κ_i	1/m	Vertical component of wavenumber
κ_i^R	1/m	Real part of the vertical component of wavenumber

ρ_1	kg/m^3	Density of water
ρ_2	kg/m^3	Density of sediment
ρ_3	kg/m^3	Density of bedrock
Φ_1	rad	Phase change for $\Re_1$
Φ_{12}	rad	Phase change for $\Re_{12}$
Φ_{23}	rad	Phase change for $\Re_{23}$
ω	rad/s	Angular frequency
$\Re_1$		Reflection coefficient for water-air surface
$\Re_{12}$		Reflection coefficient for water-sediment surface, liquid/liquid
$\Re_{12}{}^{PP}$		Reflection coefficient for water-sediment surface, liquid/elastic
$\Re_{13}$		Combined reflection coefficient for sound returning from sediment and bedrock
$\Re_{21}{}^{PP}$		Reflection coefficient for sediment-water surface, liquid/elastic
$\Re_{21}{}^{SS}$		Shear reflection coefficient for sediment-water surface, liquid/elastic
$\Re_{23}$		Reflection coefficient for sediment-bedrock surface, liquid/liquid
$\Re_{23}{}^{PP}$		Reflection coefficient for sediment-bedrock surface, elastic/elastic
$\Re_{23}{}^{SS}$		Shear reflection coefficient for sediment-bedrock surface, elastic/elastic
$\Re_{23}{}^{SP}$		Shear to pressure reflection coefficient for sediment-bedrock surface, elastic/elastic
$\Re_{23}{}^{PS}$		Pressure to shear reflection coefficient for sediment-bedrock surface, elastic/elastic

$\mathfrak{T}_{12}^{PP}$	Transmission coefficient for water-sediment surface, liquid/elastic
$\mathfrak{T}_{12}^{PS}$	Pressure to shear transmission coefficient for water-sediment surface, liquid/elastic
$\mathfrak{T}_{21}^{PP}$	Transmission coefficient for sediment - water surface, elastic/liquid
$\mathfrak{T}_{21}^{SP}$	Shear to pressure transmission coefficient for sediment-water surface, elastic/liquid

Chapter 1

INTRODUCTION

Normal mode theory determines exact solutions for ducted sound propagation problems. The theory gives a characteristic equation for a given duct system. Unfortunately, the characteristic equations which determine the eigenvalues are usually transcendental functions that require numerical analysis and the aid of a computer to solve. Using approximations of a characteristic equation can make finding the eigenvalues easier and quicker.

An important duct problem is a three-layer model of the ocean. The characteristic equation determining its eigenvalues is complex and transcendental. It can be solved numerically using Newton's Method for nonlinear systems, but that is a time-consuming computer process. This study describes a factoring technique for breaking the characteristic equation into simpler equations, and compares the eigenvalues produced by the characteristic equation and the simpler equations.

Two types of three layer models will be considered: a liquid/liquid/liquid (LLL) model and a liquid/elastic/elastic (LEE) model. The LLL model is the simpler since only pressure waves can propagate in liquids. The LEE treats the water as a liquid, but treats the sediment and the bedrock as elastic media which support both pressure and shear waves. The use for two models arises from many types of sediments and ocean bottoms.

1.1. Definition of the Three-Layer Model

The model representing the ocean is a three-layer waveguide composed of water, sediment, and bedrock. The water is designated as layer 1 and has constants density ρ_1, sound speed c_1, and wavenumber k_1. Likewise, the sediment is layer 2 with ρ_2, c_2, and k_2, and the bedrock, usually basalt or granite, is layer 3 with ρ_3, c_3, and k_3. The water has depth h and the sediment has depth t. The bedrock is a halfspace. All boundary surfaces are smooth and horizontal, and the water-air surface at z=0 is a pressure release surface.

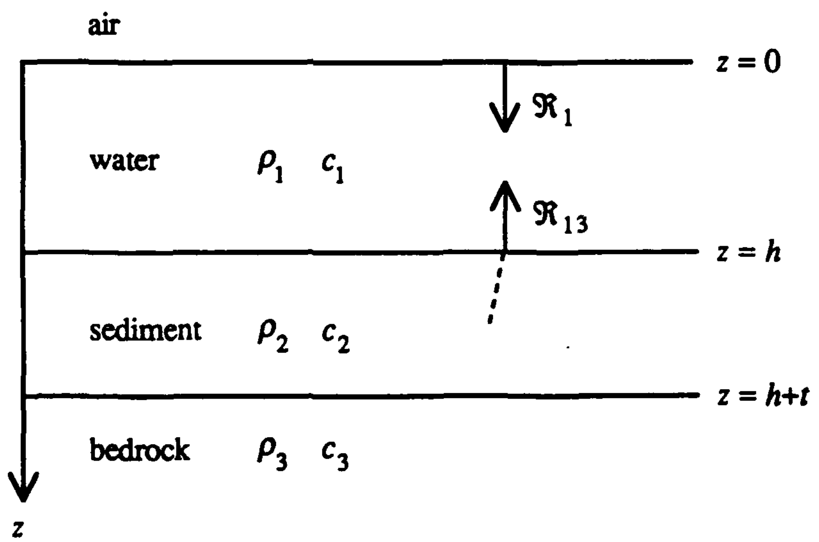

Figure 1.1. The Three-Layer Ocean Model

1.2. Development of the Characteristic Equation

The development of the characteristic equation for a general layered waveguide begins with the wave equation's associated Helmholtz equation for the pressure of plane waves. Pressure P is a function of both range r and depth z. The wavenumber k and sound speed c are generally functions of z, and $k(z)=\omega/c(z)$. Since the interest is in the water layer, the equation is developed for this region.

$$\nabla^2 P(r,z) + k_1(z)^2 P(r,z) = 0 \quad . \tag{1.1}$$

The separation of variables technique can be used to get the differential equation for $Z(z)$ that describes the z-dependence only as in (1.2). The separation constant is denoted k_n, and is the horizontal component of the wavenumber k. The k_n's are the eigenvalues that determine the normal modes for the waveguide. For example, in the water layer,

$$\frac{d^2 Z_1(z)}{dz^2} + (k_1(z)^2 - k_n^2) Z_1(z) = 0 \quad . \tag{1.2}$$

The vertical component of k_i will be denoted as κ_i. The relationship for k and its components is $k_i^{\,2} = \kappa_i^{\,2} + k_n^{\,2}$. For the models developed here, k and κ will not be functions of z, since the sound speeds are assumed to be constants.

$$\frac{d^2 Z_1(z)}{dz^2} + \kappa_1^2 Z_1(z) = 0 \quad . \tag{1.3}$$

The solution to (1.3) is given by (1.4), where $Ae^{i\kappa_1 z}$ represents an upward traveling wave, and $Be^{-i\kappa_1 z}$ represents a downward traveling wave.

$$Z_1(z) = A e^{i\kappa_1 z} + B e^{-i\kappa_1 z} \quad . \tag{1.4}$$

If the water region is divided into a lower and upper region at the depth of a source, then two equations can be written from (1.4): one for a wave that travels upward from the source and strikes the surface, and one that travels downward from the source and strikes the water-sediment interface. The uptraveling wave becomes a downward traveling wave after reflecting off the surface and the down traveling wave becomes an uptraveling wave after hitting the interface. The two equations describing this are

$$Z^U = e^{i\kappa_1 z} + \Re_1 e^{-i\kappa_1 z} \quad , \tag{1.5}$$

and

$$Z^D = \Re_{13} e^{i\kappa_1 (z-h)} + e^{-i\kappa_1 (z-h)} \quad , \tag{1.6}$$

where $\Re_1$ and $\Re_{13}$ are complex reflection coefficients. To relate these two equations to one another, their Wronskian is set equal to zero in (1.7).

$$\mathbf{W}(\kappa_1) = \begin{vmatrix} Z^U & Z^D \\ Z^{U\prime} & Z^{D\prime} \end{vmatrix} = Z^U Z^{D\prime} - Z^{U\prime} Z^D = 0 \quad . \tag{1.7}$$

After completing the necessary math, the characteristic equation is

$$\mathbf{W}(\kappa_1) \;=\; 1 - \Re_1 \Re_{13} e^{-2i\kappa_1 h} \;=\; 0 \quad . \tag{1.8}$$

1.2.1. Making k_n Complex

Now k_n, the horizontal component of wavenumber k, is made complex with small imaginary part ε_n. Allowing the wavenumber to be complex introduces a loss into the characteristic equation.

$$k_n = k_n + i\varepsilon_n \quad . \tag{1.9}$$

The expression for the vertical component of the wavenumber is then

$$\kappa_1^2 = k_1^2 - (k_n + i\varepsilon_n)^2 \quad . \tag{1.10}$$

If it is assumed that ε_n is very small so its squared term may be ignored, after rearranging and using the binomial expansion, the expression for κ_1 becomes

$$\kappa_1 = (k_1^2 - k_n^2)^{1/2} - \frac{ik_n\varepsilon_n}{\kappa_1} \quad . \tag{1.11}$$

The first term on the right side of (1.11) represents the real part of κ_1 and will be denoted as κ_1^R. The κ_1 in the denominator of the second term on the right side of (1.11) can be approximated as κ_1^R so that rewriting (1.11) gives

$$\kappa_1 \equiv \kappa_1^R - \frac{ik_n\varepsilon_n}{\kappa_1^R} \quad . \tag{1.12}$$

Substituting (1.12) into the characteristic equation gives

$$1 - \Re_1\Re_{13}\, e^{-2i\kappa_1^R h} e^{-2k_n\varepsilon_n h/\kappa_1^R} = 0 \quad . \tag{1.13}$$

The exponential with the ε_n in it is the loss or attenuation introduced by the complex k_n. The modes with the smallest ε_n's will propagate best.

1.2.2. The Magnitude and Phase of $\Re_1$

The reflection coefficient $\Re_1$ is complex and may be represented in polar form.

$$\Re_1 = \left|\Re_1\right| e^{-i\Phi_1} \quad . \tag{1.14}$$

The air-water interface is considered to be a near perfect pressure release boundary for all calculations. Therefore the value of $\Re_1$ is set equal to 0.99 and the phase Φ_1 is set equal to pi. The characteristic equation is now

$$1 - \left|\Re_1\right| \Re_{13}\, e^{-i(\kappa_1^R + \Phi_1)} e^{-2k_n \varepsilon_n h/\kappa_1^R} = 0 \quad . \tag{1.15}$$

1.3. Solving the Exact Characteristic Equation

Before the characteristic equation can be solved, $\Re_{13}$ must be known. $\Re_{13}$ will be developed later, and will be a complex function of both k_n and ε_n. The characteristic equation, once $\Re_{13}$ is substituted into it, can be separated into its real and imaginary parts yielding two equations for the two unknowns, k_n and ε_n. Newton's Method for nonlinear systems (Burden and Faires 1985, p. 496) can be used to solve for k_n and ε_n.

Recall that for a linear system, Newton's Method is an iterative process in which x_i's are sought that satisfy

$$f(x_i) = 0 \quad , \tag{1.16}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad , \tag{1.17}$$

(Kunz 1957, pp. 11), where x_i is the initial guess and x_{i+1} becomes the new guess after each iteration. For the two variable case, two equations are needed.

$$f_1(k_n, \varepsilon_n) = 0 \quad . \tag{1.18}$$

$$f_2(k_n, \varepsilon_n) = 0 \quad . \tag{1.19}$$

The iterative equation in matrix notation is

$$\begin{bmatrix} k_{n_{i+1}} \\ \varepsilon_{n_{i+1}} \end{bmatrix} = \begin{bmatrix} k_n \\ \varepsilon_n \end{bmatrix} - \begin{bmatrix} \frac{\partial f_1}{\partial k_n} & \frac{\partial f_1}{\partial \varepsilon_n} \\ \frac{\partial f_2}{\partial k_n} & \frac{\partial f_2}{\partial \varepsilon_n} \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad . \tag{1.20}$$

Defining **x** as in (1.21), (1.20) can be rewritten as (1.22).

$$\mathbf{x} = \begin{bmatrix} k_n \\ \varepsilon_n \end{bmatrix} \quad . \tag{1.21}$$

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} - \mathbf{J}^{-1}(\mathbf{x})\, \mathbf{F}(\mathbf{x}) \quad . \tag{1.22}$$

The matrix $\mathbf{J}(\mathbf{x})$ is the Jacobian matrix, and $\mathbf{J}^{-1}(\mathbf{x})$ is the inverse of the Jacobian. This method usually gives quadratic convergence provided a sufficiently accurate starting value is given and $\mathbf{J}^{-1}(\mathbf{x})$ exists (Burden and Faires 1985, pp. 498).

1.4. The Factoring Technique

In Chapters 2 and 3, $\Re_{13}$ will be shown to be the sum of two or more terms. For simplicity, say $\Re_{13}$ is equal to x plus y, and $\Re_1$ and $e^{(-2i\kappa_1 h)}$ are equal to unity. Then the general form of the characteristic equation is

$$1 - x - y = 0 \quad . \tag{1.23}$$

If we assume that x and y are small, and that the product of the two is also small, then the product xy can be added to the left side of (1.23). The right side is kept at zero, introducing product error xy. The left side can be factored into $(1-x)(1-y)$. Adding xy to the left side and then factoring in this fashion is the factoring technique used on the characteristic equation. Each of the factors $(1-x)$ and $(1-y)$ is then separated into its real and imaginary parts. The equation from taking the imaginary part is always a function of k_n only, as will be shown later, and can be solved using Newton's linear method. The values of k_n can then be substituted into the real part equation, which is a function of the two variables, to find the corresponding values of ε_n.

Chapter 2

THE LIQUID/LIQUID/LIQUID MODEL

In the liquid/liquid/liquid (LLL) model, all three layers in the waveguide are liquids capable of supporting the propagation of compressional waves. The reflection and transmission coefficients needed in $\Re_{13}$ in the characteristic equation must therefore be for waves striking liquid/liquid boundaries. Shear waves cannot propagate in liquids so they are not considered until Chapter 3.

2.1. The LLL Characteristic Equation

In order to solve the characteristic equation developed in Chapter 1 (1.8), $\Re_{13}$ must be obtained. $\Re_{13}$ is a combined reflection coefficient describing the total returned sound from the sediment and basement. The derivation follows from Clay and Medwin (1977, pp. 96).

2.1.1. $\Re_{13}$ for the LLL Model

The reflection coefficient and transmission coefficient for two liquid layers are

$$\Re_{12} = \left|\frac{Z_2 - Z_1}{Z_2 + Z_1}\right| \quad , \tag{2.1}$$

$$\Im_{12} = \left| \frac{2Z_2}{Z_2 + Z_1} \right| \quad , \tag{2.2}$$

where the impedances, Z_1 and Z_2, are giver in Table 2.1. The energy equation that relates these coefficients is also listed in Table 2.1. The phase that arises from reflection off the water-sediment interface Φ_{12} is assumed to be zero. Similarly, the equations for $\Re_{23}$ and $\Im_{23}$ and their energy relationship are in Table 2.1; Φ_{23} is also assumed zero. As shown in Figure 2.1, the angles are measured from the horizontal, and the sediment compressional wave path distance in between reflections or transmissions is r_2.

From Figure 2.1, the total up traveling signal is the sum of an infinite number of reflections and transmissions. Each up and each down path in the sediment has a phase delay of $2k_2 \sin\theta_2 t$ or $2\kappa_2 t$. Assuming the incident wave has unit amplitude, the total reflection $\Re_{13}$ is

$$\Re_{13} = \Re_{12} + \Im_{12}\Re_{23}\Im_{21}\, e^{-2i\kappa_2 t} + \Im_{12}\Re_{23}^2\Re_{21}\Im_{21}\, e^{-4i\kappa_2 t} + \ldots \quad . \tag{2.3}$$

After factoring out $\Im_{12}\Re_{23}\Im_{21}\, e^{-2i\kappa_2 t}$ from all but the first term, the remaining terms in (2.3) have the form of a geometric series (Spiegel 1968, pp. 107).

$$\sum_{j=0}^{\infty} x^{-j} = (1-x)^{-1} \quad . \tag{2.4}$$

The complex coefficients can be replaced by their magnitudes since all of their phases are approximately by zero. Rewriting (2.3) by summing the series gives for $\Re_{13}$

$$\Re_{13} = \Re_{12} + \frac{\Im_{12}\,\Im_{21}\,\Re_{23}\, e^{-2i\kappa_2 t}}{1 - \Re_{21}\,\Re_{23}\, e^{-2i\kappa_2 t}} \quad . \tag{2.5}$$

The reflection coefficient is an oscillating function that depends on $2\kappa_2 t$. It also depends on the frequency and angle of incidence for a given layer (Clay and Medwin 1977, pp. 67).

Table 2.1.		
Reflection and Transmission Coefficients for the LLL Model		
$Z_1 = \dfrac{\rho_1 c_1}{\sin\theta_1}$	$Z_2 = \dfrac{\rho_2 c_2}{\sin\theta_2}$	$Z_3 = \dfrac{\rho_3 c_3}{\sin\theta_3}$
$\Re_{12} = \left\|\dfrac{Z_2 - Z_1}{Z_2 + Z_1}\right\|$		$\Re_{21} = -\,\Re_{12}$
$\Re_{23} = \left\|\dfrac{Z_3 - Z_2}{Z_3 + Z_2}\right\|$		$\Re_{32} = -\,\Re_{23}$
$\Im_{12} = \left\|\dfrac{2Z_2}{Z_2 + Z_1}\right\|$		$\Im_{21} = \left\|\dfrac{2Z_1}{Z_2 + Z_1}\right\|$
$\Im_{23} = \left\|\dfrac{2Z_3}{Z_3 + Z_2}\right\|$		$\Im_{32} = \left\|\dfrac{2Z_2}{Z_3 + Z_2}\right\|$
Energy Relationships:		$\Re_{12}^2 + \Im_{12}\Im_{21} = 1$
		$\Re_{23}^2 + \Im_{23}\Im_{32} = 1$

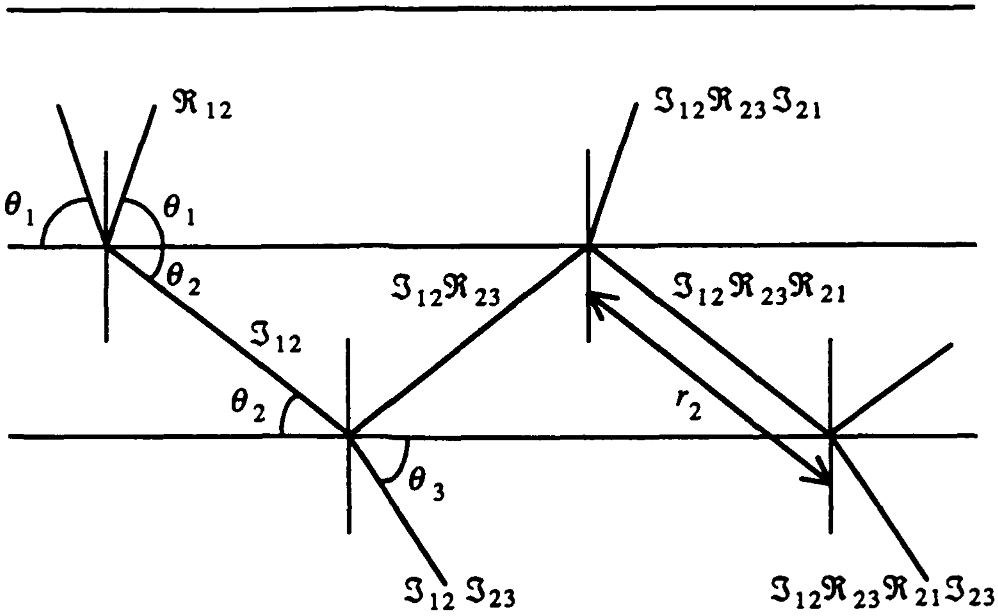

Figure 2.1. Reflection and Transmission Paths for the LLL Model

For the liquid/liquid boundary case, $\Re_{21} = -\Re_{12}$ and $\Im_{12}\Im_{21} = 1-\Re_{12}^2$. Substitution into (2.5) yields

$$\Re_{13} = \Re_{12} + \frac{(1-\Re_{12}^{2})\,\Re_{23}\,e^{-2i\kappa_2 l}}{1+\Re_{12}\,\Re_{23}\,e^{-2i\kappa_2 l}} \quad . \tag{2.6}$$

2.1.2. Incorporation of Attenuation into $\Re_{13}$ for LLL Model

Whenever a signal travels through the sediment it is attenuated. The values for the sediment attenuation coefficient α_2, usually given in dB/(m kHz), vary for different types of sediments. An attenuation factor will be necessary in the same places as the sediment phase factor in the equation for $\Re_{13}$.

$$A = 10^{-(\alpha_2/10)(f/1000)r_2} \quad . \tag{2.7}$$

$$\Re_{13} = \Re_{12} + \frac{(1 - \Re_{12}{}^2)\, \Re_{23} A^2 e^{-2i\kappa_2 t}}{1 + \Re_{12}\, \Re_{23} A^2 e^{-2i\kappa_2 t}} \quad . \tag{2.8}$$

Due to the complex k_r, κ_2 is replaced by

$$\kappa_2 \equiv \kappa_2^R - \frac{ik_n \varepsilon_n}{\kappa_2^R} \quad . \tag{2.9}$$

Making the following simplifications transforms (2.8) into (2.12).

$$S_1 = A^2 (1 - \Re_{12}{}^2) \Re_{23} \quad , \tag{2.10}$$

$$S_2 = A^2\, \Re_{12}\, \Re_{23} \quad , \tag{2.11}$$

$$\Re_{13} = \Re_{12} + \frac{S_1 e^{-2i\kappa_2^R t}\, e^{-2k_n \varepsilon_n t/\kappa_2^R}}{1 + S_2 e^{-2i\kappa_2^R t}\, e^{-2k_n \varepsilon_n t/\kappa_2^R}} \quad . \tag{2.12}$$

2.1.3. Substitution of $\Re_{13}$ into the LLL Characteristic Equation

Before $\Re_{13}$ can be substituted into the characteristic equation, the second term is multiplied by the complex conjugate of its denominator.

$$\Re_{13} = \Re_{12} + S_1 E_2^2\, (e^{-2i\kappa_2^R t} + S_2 E_2^2\,)/\mathrm{Den} \quad , \tag{2.13}$$

where E_2 and Den are

$$E_2 = e^{-k_n \varepsilon_n t/\kappa_2^R} \quad , \tag{2.14}$$

$$\mathrm{Den} = 1 + 2S_2 E_2^2 \cos(2\kappa_2^R t)\, e^{-2i\kappa_2^R t} + S_2^2\, E_2^4 \quad . \tag{2.15}$$

Substituting (2.13) into the characteristic equation gives

$$1 - \Re_1 \Re_{12} E_1^2 e^{-i(2\kappa_1^R h + \Phi_1)} - \frac{\Re_1 S_1 E_{12}}{\text{Den}} = 0 \quad , \tag{2.16}$$

where E_1 and E_{12} are

$$E_1 = e^{-k_n \varepsilon_n h / \kappa_1^R} \quad , \tag{2.17}$$

$$E_{12} = E_1^2 E_2^2 (e^{-i(2\kappa_1^R h + 2\kappa_2^R t + \Phi_1)} + S_2 E_2^2 e^{-i(2\kappa_1^R h + 2\Phi_1)}) \quad . \tag{2.18}$$

2.2. Solving the Characteristic Equation

Equation (2.16) is a complex transcendental equation whose solutions require the Newton's method for two variables described in Chapter 1. The solutions found by this Newton's method will be called the "exact" solutions. To get two equations for $\mathbf{F(x)}$, the characteristic equation is separated into its real and imaginary parts. The real part equation from the characteristic equation is

$$\begin{aligned} \text{F1} = \frac{1}{E_1^2} - \Re_1 \Re_{12} \cos(2K_1 + \Phi_1) \\ - \frac{\Re_1 S_1 E_2^2}{\text{Den}} [\cos(2K_1 + 2K_2 + \Phi_1) + S_2 E_2^2 \cos(2K_1 + \Phi_1)] = 0 \quad , \end{aligned} \tag{2.19}$$

where K_1 and K_2 are

$$K_1 = \kappa_1^R h \quad , \tag{2.20}$$

$$K_2 = \kappa_2^R t \quad , \tag{2.21}$$

and the imaginary part equation is

$$F2 = \Re_{12}\sin(2K_1+\Phi_1) + \frac{S_1E_2^2}{\mathrm{Den}}[\sin(2K_1+2K_2+\Phi_1) + S_2E_2^2\sin(2K_1+\Phi_1)] = 0\ . \quad (2.22)$$

The partial derivatives involved in finding the Jacobian are difficult to obtain, since the κ_i's are functions of k_n and because of the many products. Since the Jacobian is just a focusing mechanism for determining the next guess, the reflection and transmission coefficients are held constant with respect to the k_n's for the derivatives. The consequence of this may be more iterations necessary to focus on a solution.

Newton's Method also requires initial guesses to start the iterations. In the program, the water factored solutions are calculated first, providing the initial guesses for the sediment solutions. Both the water and sediment solutions are used for initial guesses for the exact solutions. See the Appendix for the program listing.

2.3. Factoring the LLL Characteristic Equation

To factor (2.16) as described in section 1.4, a cross term must be added to the left hand side. Then the equation of factors is (2.23).

$$0=(1-\Re_1\Re_{12}E_1^2e^{-i(2K_1+\Phi_1)})(1-\frac{\Re_1S_1E_1^2E_2^2}{\mathrm{Den}}(e^{-i(2K_1+2K_2+\Phi_1)}+S_2E_2^2e^{-i(2K_1+\Phi_1)})). \quad (2.23)$$

Each of the factors can then be independently set equal to zero. The first factor is called the water factor since all the parameters in it are associated with the wave paths in the water. Its solutions will be called the water solutions. The second factor will be called the sediment factor and its solutions, the sediment solutions.

2.3.1. The Water Factor

To solve for the eigenvalues of the water factor ,

$$(1 - \Re_1 \Re_{12} E_1^2 e^{-i(2K_1+\Phi_1)}) = 0 \ , \tag{2.24}$$

the real and imaginary parts are set equal to zero.

$$1 - \Re_1 \Re_{12} \, E_1^2 \cos(2K_1 + \Phi_1) = 0 \ . \tag{2.25}$$

$$\sin(2K_1 + \Phi_1) = 0 \ . \tag{2.26}$$

The equation formed by the imaginary part is a function of k_n only, which is solved for directly.

$$k_n = \left[k_1^2 - \left(\frac{(2n\pi - \Phi_1)}{2h} \right)^2 \right]^{1/2} \ . \tag{2.27}$$

The value of $2n\pi$ is chosen instead of $n\pi$ in order to make $\cos(2n\pi)$ always equal to unity which can be substituted into equation (2.25) along with k_n to get

$$\varepsilon_n = \left| \ln(\Re_1 \Re_{12}) \frac{k_1^R}{2hk_n} \right| \ . \tag{2.28}$$

2.3.2. The LLL Sediment Factor

The sediment factor from (2.23) is

$$(1 - \frac{\Re_1 S_1 E_1^2 E_2^2}{\text{Den}} (e^{-i(2K_1+2K_2+\Phi_1)} + S_2 E_2^2 e^{-i(2K_1+\Phi_1)})) = 0 \ . \tag{2.29}$$

At this point, another approximation is made to make this factor easily solvable. The variable S_2 is set equal to zero. Setting S_2 equal to zero is equivalent to only keeping the first bottom bounce contribution in Figure 2.1.. Equation (2.29) becomes

$$(1 - \Re_1 S_1 E_1^2 E_2^2 \, e^{-i(2K_1+2K_2+\Phi_1)} = 0 \ . \tag{2.30}$$

Taking the real and imaginary parts of (2.30) yields the following two equations.

$$1 - \Re_1 S_1 E_1^2 E_2^2 \cos(2K_1+2K_2+\Phi_1) = 0 \ . \tag{2.31}$$

$$\sin(2K_1+2K_2+\Phi_1) = 0 \ . \tag{2.32}$$

The imaginary part equation is only a function of k_n and can be solved using Newton's method for one variable (Kunz 1957, pp. 11). Following the example of the water factor, the argument of the sine function is set equal to $2n\pi$. Also κ_1^R and κ_2^R are replaced by their k_n forms to get

$$F(k_n) = 2h(k_1^2-k_n^2)^{1/2} + 2t(k_2^2-k_n^2)^{1/2} + \Phi_1 - 2n\pi = 0 \ . \tag{2.33}$$

Newton's method requires the derivative of F and initial guess. The initial guesses are provided by the k_n's found by the water factor. See the Appendix for the program listing.

Once the sediment factor k_n's are found, they can be placed in (2.31), solved for ε_n.

$$\varepsilon_n = \left| \ln(\Re_1 S_1) \ \frac{1}{2k_n \left(h/\kappa_1^R + t/\kappa_2^R \right)} \right| \ . \tag{2.34}$$

2.4. A Qualitative Comparison of the Solutions

The shapes of the three types of mode sets, exact, water, and sediment, varies. Figure 2.2 shows the typical shapes for each set. The number of modes is determined by the frequency f and the water depth h by (2.35) (Clay and Medwin 1977, pp. 306).

$$n \leq \frac{f}{c_1}(2h\sin\theta_c) + \frac{1}{2} \ . \tag{2.35}$$

The modes are always close together for high k_n's and spread out for low k_n's. Also, the harder the sediment, the fewer the modes because the minimum angle θ_c for which there is a mode increases. The discontinuity in the water solutions occurs at the k_n which corresponds to θ_c.

In Figure 2.2 and all the eigenvalue plots, the higher up on the y axis a point is, the better that mode propagates, i.e., the $|\varepsilon_n|$ is smaller so the loss is lower for that mode. The range of the ε_n values is several orders of magnitude, therefore a log scale is used. Note also that for the exact and sediment factor solutions, only eigenvalues above the critical angle are plotted. The critical angle shows the onset of sediment penetration.

2.4.1. Forcing the Water Solutions to the Exact Solutions

If the sediment is very dense and the sediment attenuation is very high, a signal will tend to stay in the water. Therefore the water factor solutions will propagate better than the sediment factored solutions. The high density of the sediment makes $\Re_{12}$ nearly equal to one and thus the transmission into the sediment is small. Whatever waves do get into the sediment will die out quickly due to the high attenuation. Figure 2.3 shows the water, sediment, and exact solutions for these inputs. The matching of the water solutions beyond

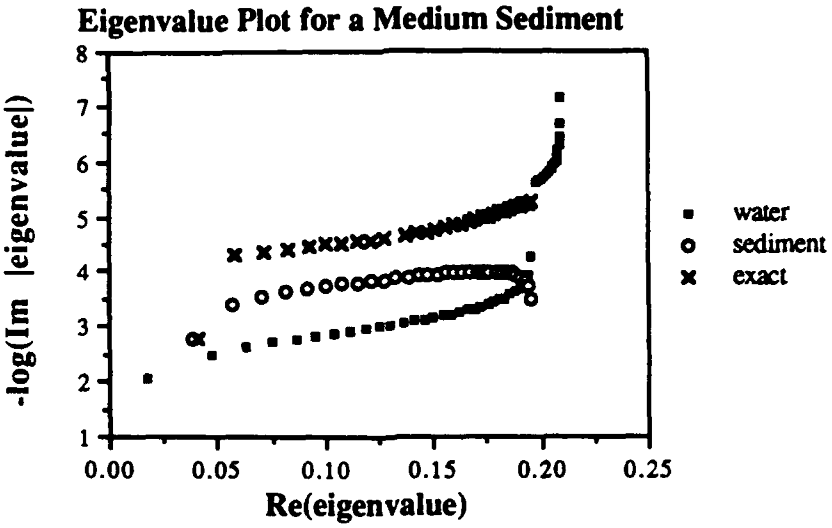

Figure 2.2. Typical LLL Eigenvalue Plot for a Medium Sediment

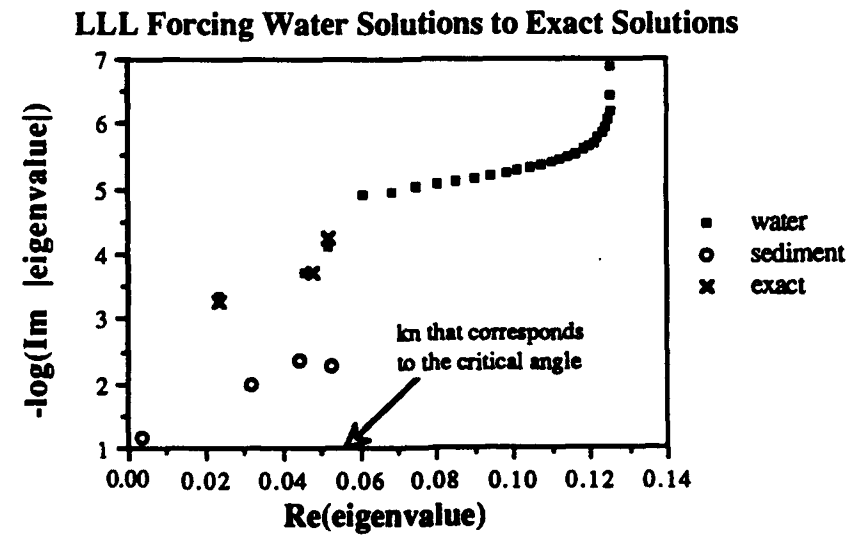

Figure 2.3. Forcing the LLL Water Solutions to the Exact Solutions
(ρ_2/ρ_1=7, c_2/c_1=2.33, α_2=1)

the critical angle with the exact solutions indicates that the factors may be a good approximation of the exact solutions.

2.4.2. Forcing the Sediment Solutions to the Exact Solutions

If the sediment density and sound speed are set approximately equal to the water density and sound speed, and a small sediment attenuation is chosen, the sediment wave path will dominate. Most of the initial wave is transmitted into the sediment because $\Re_{12}$ is small for these conditions. Figure 2.4 shows the sediment and exact solutions to be almost equal for these inputs, again indicating that the factors might be a good model of the exact solutions.

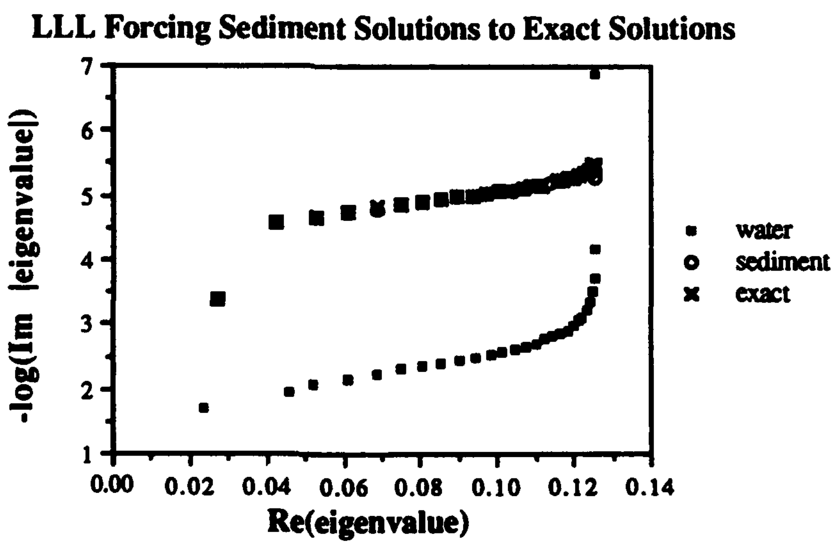

Figure 2.4. Forcing the LLL Sediment Solutions to the Exact Solutions ($\rho_2/\rho_1=1$, $c_2/c_1=1$, $\alpha_2=0.01$)

Chapter 3

THE LIQUID/ELASTIC/ELASTIC MODEL

The LEE model treats both the sediment layer and the basement halfspace as elastic materials capable of supporting shear waves. Four new parameters are thus introduced, the shear speeds of sound in the sediment and basement, c_{2S} and c_{3S}, and the shear attenuation coefficient for the sediment is α_{2S}.

It may seem as though this study is incomplete since it does not address a liquid/ liquid/elastic model. That case was indeed studied but its results proved to be only slightly different than the liquid/liquid/liquid model.

3.1. The LEE Characteristic Equation

The characteristic equation for the LEE model is (1.15). The derivation for $\Re_{13}$ for this case differs from the LLL case because the refection and transmission coefficients are now for liquid/elastic boundaries. Also, new coefficients are needed to account for the shear waves generated from compressional waves at the liquid/elastic and elastic/elastic interfaces.

When a pressure wave strikes a boundary of an elastic medium from a liquid medium, not only are pressure waves reflected from and transmitted through the boundary, but also a shear wave is transmitted (Miklowitz 1978, pp. 156). If the initial medium is also elastic, there will also be a reflected shear wave. These two events are illustrated in Figure 3.1. Pressure waves are drawn as solid lines, SV waves as dashed lines. The notation SV stands for a shear wave with vertical polarization. Shear waves with horizontal polarization are not coupled to the pressure waves and are not considered here (Tolstoy 1973, pp. 188). An incident shear wave generates pressure waves at the boundaries in addition to reflecting and transmitting shear waves (Miklowitz 1978, pp. 156). This is illustrated in Figure 3.2.

3.1.1. Reflection Coefficients for the LEE Model

For the boundary events, new liquid/elastic and elastic/elastic reflection and transmission coefficients are required. The coefficients have been obtained from three sources and are rewritten here with consistent notation and with incident angles measured from the horizontal. The coefficients are listed in Table 3.1 (Brekhovskikh 1980, pp. 43-47) and Table 3.2 (Miklowitz 1978, pp. 160-161). The new coefficients are illustrated in Figure 3.3. Since signals that penetrate the bedrock do not return, the $\Im_{23}$'s are not used in the calculations and therefore are not included in Table 3.2. The energy relationships that govern the behavior of the liquid/elastic and elastic/elastic coefficients are listed in Table 3.3 (Ergin 1952, pp. 350) (Miklowitz 1978, 161-162).

3.1.2. $\Re_{13}$ for the LEE Model

As in the LLL model, the incident pressure wave follows the path shown in Figure 2.1. The total reflection coefficient from the wave is similar to (2.3), assuming again that the phases Φ_{12} and Φ_{23} are zero. The difference from (2.3) is that the coefficients are now for liquid/elastic or elastic/elastic boundaries. The waves which reflect or transmit

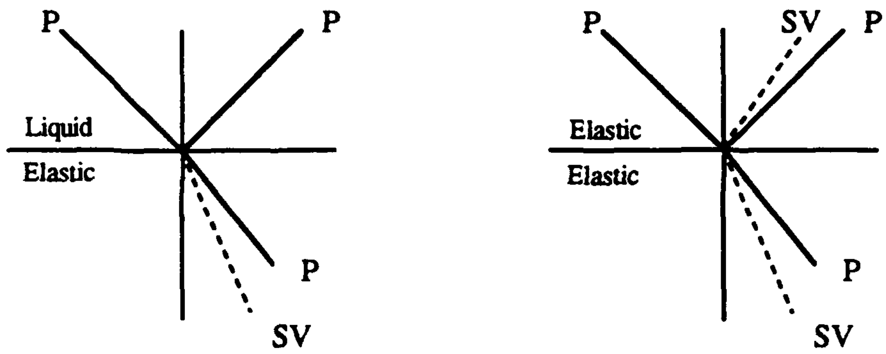

Figure 3.1. Generation of Shear Waves at an Elastic Boundary

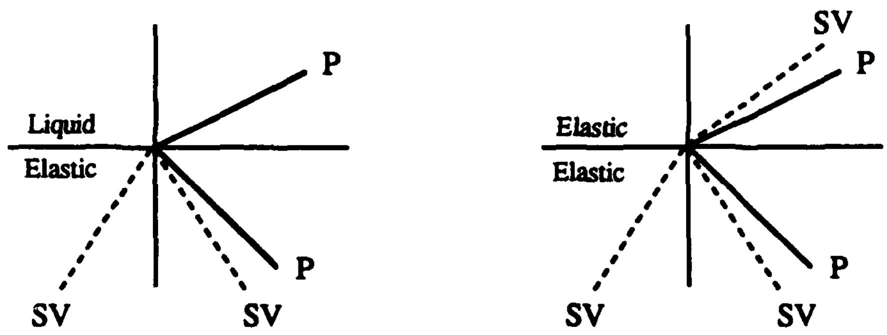

Figure 3.2. Shear Waves Incident at a Boundary

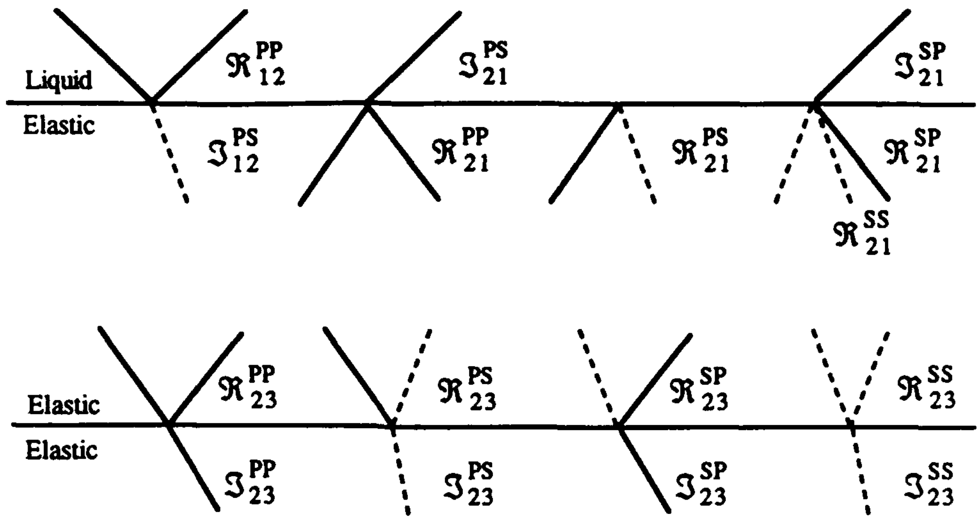

Figure 3.3. Illustration of LEE Reflection and Transmission Coefficients

pressure waves are designated pp. The contribution to $\Re_{13}$ for the incident pressure wave which remains a pressure wave, the pp series, is then

$$\Re_{13}^{PP} = \Re_{12}^{PP} + \frac{\Im_{12}^{PP}\Im_{21}^{PP}\,\Re_{23}^{PP}\, e^{-2i\kappa_2 \ell}}{1-\Re_{21}^{PP}\,\Re_{23}^{PP}\,e^{-2i\kappa_2\ell}} \; . \tag{3.1}$$

In the elastic case, $\Re_{21}$ does not equal $-\Re_{12}$ and no further simplification takes place.

For an incident pressure wave that transmits a shear wave which remains a shear wave in the sediment as shown in Figure 3.4, the ss wave, the contribution to $\Re_{13}$ is

$$\Re_{13}^{SS} = \frac{\Im_{12}^{PS}\Im_{21}^{SP}\,\Re_{23}^{SS}\, e^{-2i\kappa_{2s} \ell}}{1-\Re_{21}^{SS}\,\Re_{23}^{SS}\,e^{-2i\kappa_{2s}\ell}} \; , \tag{3.2}$$

assuming phases $\Phi_{12}{}^{SS}$ and $\Phi_{23}{}^{SS}$ are zero.

Table 3.1.

Reflection and Transmission Coefficients
for
Liquid/Elastic and Elastic/Liquid Boundaries

$Z_1 = \dfrac{\rho_1 c_1}{\sin\,\theta_1}$	$Z_2 = \dfrac{\rho_2 c_2}{\sin\,\theta_2}$	$Z_{2S} = \dfrac{\rho_2 c_{2S}}{\sin\,\theta_{2S}}$

$$\Re_{12}^{PP} = \left[Z_2\cos^2(2\theta_{2S}) + Z_{2S}\sin^2(2\theta_{2S}) - Z_1\right]\Big/D$$

$$\Re_{21}^{PP} = \left[\,Z_1 - Z_2\cos^2(2\theta_{2S}) + Z_{2S}\sin^2(2\theta_{2S})\,\right]\Big/D$$

$$\Re_{21}^{SS} = -\left[\,Z_1 + Z_2\cos^2(2\theta_{2S}) - Z_{2S}\sin^2(2\theta_{2S})\,\right]\Big/D$$

$$\Im_{12}^{PP} = \left[2\rho_1 \tfrac{Z_2}{\rho_2}\cos(2\theta_{2S})\right]\Big/D$$

$$\Im_{12}^{PS} = \left[-2\rho_1 \tfrac{Z_{2S}}{\rho_2}\sin(2\theta_{2S})\right]\Big/D$$

$$\Im_{21}^{PP} = \left[2\rho_2 \tfrac{Z_1}{\rho_1}\cos(2\theta_{2S})\right]\Big/D$$

$$\Im_{21}^{SP} = \left[-2\rho_2 \tfrac{Z_1}{\rho_1}\sin(2\theta_{2S})\right]\Big/D$$

$$D = Z_2\cos^2(2\theta_{2S}) + Z_{2S}\sin^2(2\theta_{2S}) + Z_1$$

Table 3.2.	
Reflection Coefficients for the Elastic/Elastic Boundary	
$\mu_2 = \rho_2 c_{2S}^2$	$e = 2\mu_2 + \mu_3 b_3$
$\mu_3 = \rho_3 c_{3S}^2$	$f = 2(\mu_2 - \mu_3)$
$b_2 = \tan^2\theta_{2S} - 1$	$g = \mu_2 b_2 - \mu_3 b_3$
$b_3 = \tan^2\theta_{3S} - 1$	$h = 2\mu_3 + \mu_2 b_2$
$p_1 = e \tan\theta_2 \tan\theta_{2S}$	$p_2 = g \tan\theta_2$
$p_3 = h \tan\theta_{2S} \tan\theta_3$	$p_4 = -f \tan\theta_2 \tan\theta_{2S} \tan\theta_3$
$q_1 = f \tan\theta_2 \tan\theta_{2S} \tan\theta_{3S}$	$q_2 = h \tan\theta_2 \tan\theta_{3S}$
$q_3 = -g \tan\theta_{2S}$	$q_4 = p_1$
$\Re_{23}^{PP} = [(p_1-p_3)(q_2+q_4) - (p_2+p_4)(q_1-q_3)]/DD$	
$\Re_{23}^{SS} = [-(p_1+p_3)(q_2-q_4) + (p_2-p_4)(q_1+q_3)]/DD$	
$\Re_{23}^{SP} = [2(p_3 q_1 - p_1 q_3)]/DD$	
$\Re_{23}^{PS} = [2(p_4 q_2 - p_2 q_4)]/DD$	
$DD = [(p_1+p_3)(q_2+q_4) - (p_2+p_4)(q_1+q_3)]$	

Table 3.3.

Energy Relationships for LEE Model

Liquid/Elastic

Pressure wave in water against solid

$$1 = \Re_{12}^{PP2} + \frac{\rho_2 \tan\theta_2}{\rho_1 \tan\theta_1} \Im_{12}^{PP2} + \frac{\rho_2 \tan\theta_{2S}}{\rho_1 \tan\theta_1} \Im_{12}^{PS2}$$

Pressure wave in solid against water

$$1 = \Re_{21}^{PP2} + \frac{\tan\theta_{2S}}{\tan\theta_2} \Re_{21}^{PS2} + \frac{\rho_1 \tan\theta_1}{\rho_2 \tan\theta_2} \Im_{21}^{PP2}$$

Shear wave in solid against water

$$1 = \Re_{21}^{SS2} + \frac{\tan\theta_2}{\tan\theta_{2S}} \Re_{21}^{SP2} + \frac{\rho_1 \tan\theta_1}{\rho_2 \tan\theta_{2S}} \Im_{21}^{SP2}$$

Elastic/Elastic

Pressure wave in sediment against basement

$$1 = \Re_{23}^{PP2} + \frac{\tan\theta_{2S}}{\tan\theta_2} \Re_{23}^{PS2} + \frac{\rho_3 \tan\theta_3}{\rho_2 \tan\theta_2} \Im_{23}^{PP2} + \frac{\rho_3 \tan\theta_{3S}}{\rho_2 \tan\theta_2} \Im_{23}^{PS2}$$

Shear wave in sediment against basement

$$1 = \Re_{23}^{SS2} + \frac{\tan\theta_2}{\tan\theta_{2S}} \Re_{23}^{SP2} + \frac{\rho_3 \tan\theta_3}{\rho_2 \tan\theta_{2S}} \Im_{23}^{SP2} + \frac{\rho_3 \tan\theta_{3S}}{\rho_2 \tan\theta_{2S}} \Im_{23}^{SS2}$$

Two combination shear/pressure waves are also included in $\Re_{13}$, the sp wave and the ps wave. (There are many other combinations possible. The pp, ss, sp, and ps are the only ones studied here.) The sp wave travels as indicated in Figure 3.5. Assuming its phase changes at the boundaries $\Phi_{23}{}^{SP}$ and $\Phi_{21}{}^{PP}$ are zero, the sp contribution to $\Re_{13}$ is

$$\Re_{13}^{SP} = \frac{\Im_{12}^{PS}\Im_{21}^{PP}\,\Re_{23}^{SP}\,e^{-i\kappa_{2}l}e^{-i\kappa_{2S}l}}{1-\Re_{21}^{PP}\,\Re_{23}^{PP}\,e^{-2i\kappa_{2}l}} \,. \tag{3.3}$$

Similarly from Figure 3.6 and setting $\Phi_{23}{}^{PS}$ and $\Phi_{21}{}^{SP}$ equal to zero, the fourth component of $\Re_{13}$ is

$$\Re_{13}^{PS} = \frac{\Im_{12}^{PP}\Im_{21}^{SP}\,\Re_{23}^{PS}\,e^{-i\kappa_{2}l}e^{-i\kappa_{2S}l}}{1-\Re_{21}^{SS}\,\Re_{23}^{SS}\,e^{-2i\kappa_{2S}l}} \,. \tag{3.4}$$

Summing the four terms together gives for the LEE case

$$\begin{aligned}\Re_{13} = \Re_{12}^{PP} &+ \frac{\Im_{12}^{PP}\Im_{21}^{PP}\,\Re_{23}^{PP}\,e^{-2i\kappa_{2}l}}{1-\Re_{21}^{PP}\,\Re_{23}^{PP}\,e^{-2i\kappa_{2}l}} + \frac{\Im_{12}^{PS}\Im_{21}^{SP}\,\Re_{23}^{SS}\,e^{-2i\kappa_{2S}l}}{1-\Re_{21}^{SS}\,\Re_{23}^{SS}\,e^{-2i\kappa_{2S}l}} \\ &+ \frac{\Im_{12}^{PS}\Im_{21}^{PP}\,\Re_{23}^{SP}\,e^{-i\kappa_{2}l}e^{-i\kappa_{2S}l}}{1-\Re_{21}^{PP}\,\Re_{23}^{PP}\,e^{-2i\kappa_{2}l}} + \frac{\Im_{12}^{PP}\Im_{21}^{SP}\,\Re_{23}^{PS}\,e^{-i\kappa_{2}l}e^{-i\kappa_{2S}l}}{1-\Re_{21}^{SS}\,\Re_{23}^{SS}\,e^{-2i\kappa_{2S}l}} \,.\end{aligned} \tag{3.5}$$

3.1.3. Incorporation of Attenuation into $\Re_{13}$ for the LEE Model

Just as in the liquid layers, signals in elastic layers are attenuated. The attenuation in the sediment is dependent upon the length of the path traveled and will be included everywhere in $\Re_{13}$ that the phase changes for the paths occur. The path distance traveled is r_{2S}. The attenuation factor for the elastic sediment for α_{2S} in dB/m kHz is

$$A_S = 10^{-(\alpha_{2S}/10)(f/1000)r_{2S}} \,. \tag{3.6}$$

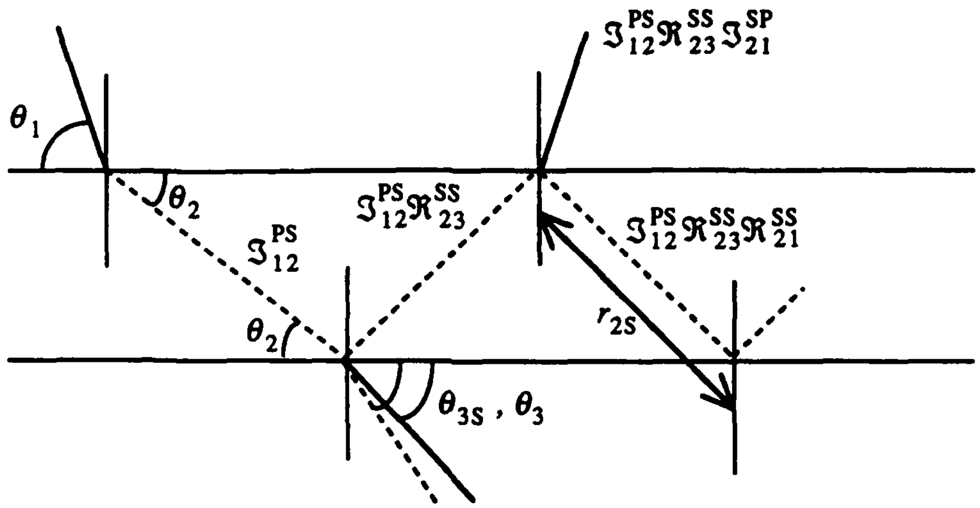

Figure 3.4. Reflection and Transmission Paths for the SS Series

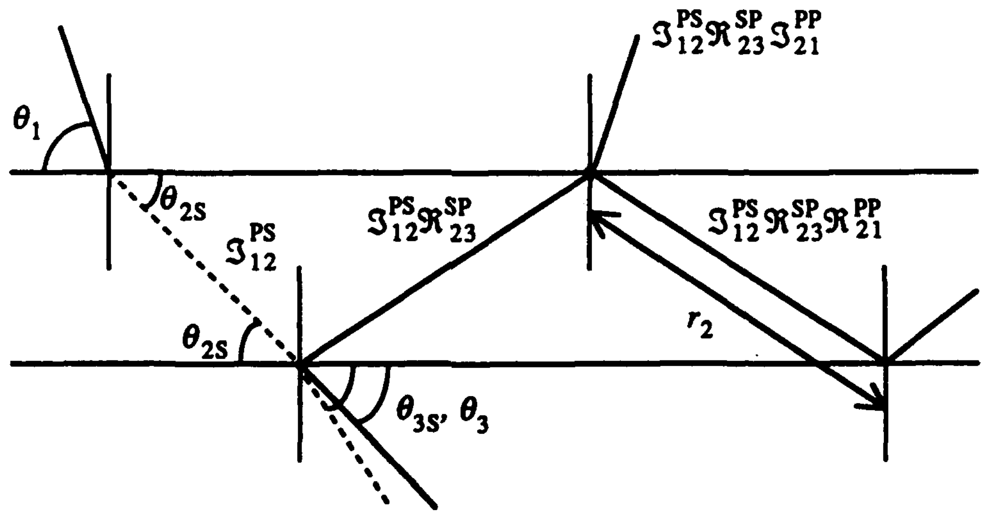

Figure 3.5. Reflection and Transmission Paths for the SP Series

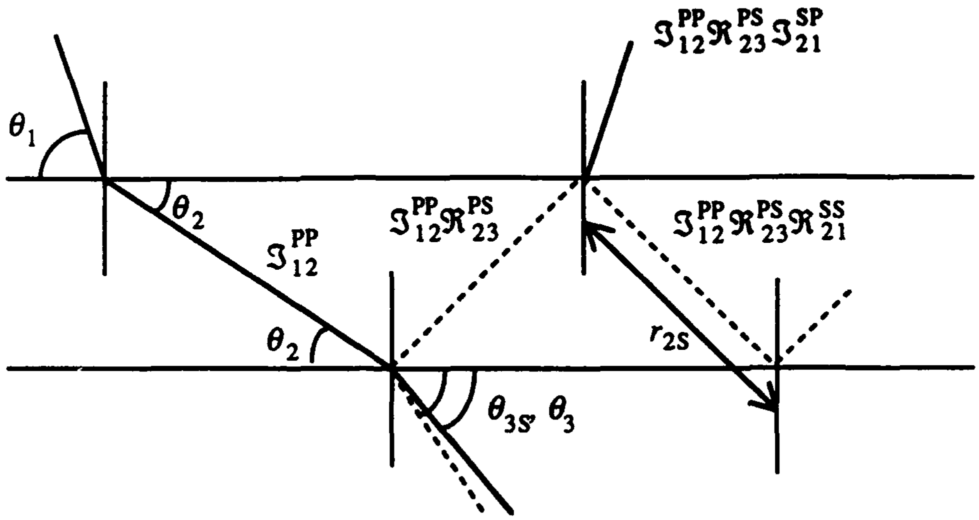

Figure 3.6. Reflection and Transmission Paths for the PS Series

Due to the complex k_n, κ_2 and κ_{2S} are replaced by the following two equations.

$$\kappa_2 = \kappa_2^R - ik_n\varepsilon_n / \kappa_2^R \ . \tag{3.7}$$

$$\kappa_{2S} = \kappa_{2S}^R - ik_n\varepsilon_n / \kappa_{2S}^R \ . \tag{3.8}$$

Making the following definitions and multiplying denominators by their complex conjugates gives for $\mathfrak{R}_{13}$

$$S^{PP} = \mathfrak{I}_{12}^{PP}\mathfrak{I}_{21}^{PP}\mathfrak{R}_{23}^{PP}A^2 \ , \tag{3.9}$$

$$S^{SS} = \mathfrak{I}_{12}^{PS}\mathfrak{I}_{21}^{SP}\mathfrak{R}_{23}^{SS}A_S^2 \ , \tag{3.10}$$

$$S^{SP} = \mathfrak{I}_{12}^{PS}\mathfrak{I}_{21}^{PP}\mathfrak{R}_{23}^{SP}AA_S \ , \tag{3.11}$$

$$S^{\mathrm{PS}} = \mathfrak{I}_{12}^{\mathrm{PP}} \mathfrak{I}_{21}^{\mathrm{SP}} \Re_{23}^{\mathrm{PS}} A A_{\mathrm{S}} \ , \tag{3.12}$$

$$S_2^{\mathrm{PP}} = \Re_{21}^{\mathrm{PP}} \Re_{23}^{\mathrm{PP}} A^2 \ , \tag{3.13}$$

$$S_2^{\mathrm{SS}} = \Re_{21}^{\mathrm{SS}} \Re_{23}^{\mathrm{SS}} A_{\mathrm{S}}^2 \ , \tag{3.14}$$

$$E_{2\mathrm{S}} = e^{-k_a \varepsilon_a t / \kappa_{2\mathrm{S}}^R} \ , \tag{3.15}$$

$$K_2 = \kappa_2^R t \ , \tag{3.16}$$

$$K_{2\mathrm{S}} = \kappa_{2\mathrm{S}}^R t \ , \tag{3.17}$$

$$\begin{aligned} \Re_{13} = \Re_{12}^{\mathrm{PP}} &+ \frac{S^{\mathrm{PP}}(E_2^2 e^{-2iK_2} - S_2^{\mathrm{PP}})}{1 - (S_2^{\mathrm{PP}} E_2^2)\cos(2\kappa_2^R t) + (S_2^{\mathrm{PP}} E_2^2)^2} \\ &+ \frac{S^{\mathrm{SS}} E_{2\mathrm{S}}^2 (e^{-2iK_{2S}} - S_2^{\mathrm{SS}})}{1 - (S_2^{\mathrm{SS}} E_{2\mathrm{S}}^2)\cos(2\kappa_{2\mathrm{S}}^R t) + (S_2^{\mathrm{SS}} E_{2\mathrm{S}}^2)^2} \\ &+ \frac{S^{\mathrm{SP}} E_{2\mathrm{S}} e^{-iK_{2S}} (E_2 e^{-iK_2} - S_2^{\mathrm{PP}} E_2^{-1} e^{iK_2})}{1 - (S_2^{\mathrm{PP}} E_2^2)\cos(2\kappa_2^R t) + (S_2^{\mathrm{PP}} E_2^2)^2} \\ &+ \frac{S^{\mathrm{PS}} E_2 e^{-iK_2} (E_{2\mathrm{S}} e^{-iK_{2S}} - S_2^{\mathrm{SS}} E_{2\mathrm{S}}^{-1} e^{iK_{2S}})}{1 - (S_2^{\mathrm{SS}} E_{2\mathrm{S}}^2)\cos(2\kappa_{2\mathrm{S}}^R t) + (S_2^{\mathrm{SS}} E_{2\mathrm{S}}^2)^2} \ . \end{aligned} \tag{3.18}$$

3.1.4. Substitution of $\Re_{13}$ into the LEE Characteristic Equation

Making a few more definitions and placing (3.18) into (1.15) gives the LEE characteristic equation (3.21).

$$\mathrm{Den1} = 1 - (S_2^{\mathrm{PP}} E_2^2)\cos(2\kappa_2^R t) + (S_2^{\mathrm{PP}} E_2^2)^2 \ . \tag{3.19}$$

$$\text{Den2} = 1 - (S_2^{SS}E_{2S}^2)\cos(2\kappa_{2S}^R t) + (S_2^{SS}E_{2S}^2)^2 \ . \tag{3.20}$$

$$\begin{aligned}
&1 - \Re_1\Re_{12}^{PP}E_1^2 e^{-i(2K_1+\Phi_1)} \\
&-\frac{\Re_1 S^{PP}}{\text{Den1}}\left(E_1^2E_2^2 e^{-i(2K_1+2K_2+\Phi_1)} - S_2^{PP}E_1^2 e^{-i(2K_1+\Phi_1)}\right) \\
&-\frac{\Re_1 S^{SS}}{\text{Den2}}\left(E_1^2E_{2S}^2 e^{-i(2K_1+2K_{2S}+\Phi_1)} - S_2^{SS}E_1^2 e^{-i(2K_1+\Phi_1)}\right) \\
&-\frac{\Re_1 S^{SP}}{\text{Den1}}\left(E_1^2E_2E_{2S} e^{-i(2K_1+K_2+K_{2S}+\Phi_1)} - S_2^{PP}E_1^2E_2^{-1}E_{2S} e^{-i(2K_1-K_2+K_{2S}+\Phi_1)}\right) \\
&-\frac{\Re_1 S^{PS}}{\text{Den2}}\left(E_1^2E_2E_{2S} e^{-i(2K_1+K_2+K_{2S}+\Phi_1)} - S_2^{SS}E_1^2E_2E_{2S}^{-1} e^{-i(2K_1+K_2-K_{2S}+\Phi_1)}\right) = 0 \ .
\end{aligned} \tag{3.21}$$

Two approaches were taken to solve this complex function of both k_n and ε_n. First, Newton's method for nonlinear systems, and second, the factoring technique.

3.2. Solving the LEE Exact Equation with Newton's Nonlinear Method

To solve the LEE characteristic equation using Newton's Method for nonlinear systems as described in section 1.3, the equation must be broken into its real and imaginary parts to give two equations for the two unknowns, k_n and ε_n. The Jacobian for these two equations must also be determined. Since the calculations are done by computer, a discussion of some of the computational pitfalls is also included.

The imaginary part of (3.21) gives for F1:

$$\begin{aligned}
&\text{F1} = 0 = \Re_1\Re_{12}^{PP}E_1^2\sin(2K_1+\Phi_1) \\
&+\frac{\Re_1 S^{PP}}{\text{Den1}}\left[E_1^2E_2^2\sin(2K_1+2K_2+\Phi_1) - S_2^{PP}E_1^2\sin(2K_1+\Phi_1)\right] \\
&+\frac{\Re_1 S^{SS}}{\text{Den2}}\left[E_1^2E_{2S}^2\sin(2K_1+2K_{2S}+\Phi_1) - S_2^{SS}E_1^2\sin(2K_1+\Phi_1)\right] \\
&+\frac{\Re_1 S^{SP}}{\text{Den1}}\left[E_1^2E_2E_{2S}\sin(2K_1+K_2+K_{2S}+\Phi_1) - S_2^{PP}E_1^2E_2^{-1}E_{2S}\sin(2K_1-K_2+K_{2S}+\Phi_1)\right] \\
&+\frac{\Re_1 S^{PS}}{\text{Den2}}\left[E_1^2E_2E_{2S}\sin(2K_1+K_2+K_{2S}+\Phi_1) - S_2^{SS}E_1^2E_2E_{2S}^{-1}\sin(2K_1+K_2-K_{2S}+\Phi_1)\right] ,
\end{aligned} \tag{3.22}$$

and the real part gives F2:

$$
\begin{aligned}
F2 = 0 = 1 &- \Re_1 \Re_{12}^{PP} E_1^2 \cos(2K_1 + \Phi_1) \\
&- \frac{\Re_1 S^{PP}}{\mathrm{Den1}} [E_1^2 E_2^2 \cos(2K_1 + 2K_2 + \Phi_1) - S_2^{PP} E_1^2 \cos(2K_1 + \Phi_1)] \\
&- \frac{\Re_1 S^{SS}}{\mathrm{Den2}} [E_1^2 E_{2S}^2 \cos(2K_1 + 2K_{2S} + \Phi_1) - S_2^{SS} E_1^2 \cos(2K_1 + \Phi_1)] \qquad (3.23) \\
&- \frac{\Re_1 S^{SP}}{\mathrm{Den1}} [E_1^2 E_2 E_{2S} \cos(2K_1 + K_2 + K_{2S} + \Phi_1) - S_2^{PP} E_1^2 E_2^{-1} E_{2S} \cos(2K_1 - K_2 + K_{2S} + \Phi_1)] \\
&- \frac{\Re_1 S^{PS}}{\mathrm{Den2}} [E_1^2 E_2 E_{2S} \cos(2K_1 + K_2 + K_{2S} + \Phi_1) - S_2^{SS} E_1^2 E_2 E_{2S}^{-1} \cos(2K_1 + K_2 - K_{2S} + \Phi_1)] \, .
\end{aligned}
$$

To form the Jacobian matrix for this pair of equations, derivatives of F1 and F2 with respect to k_n and ε_n are needed. These derivatives are quite complicated due to the many products and quotients involving the κ's which depend on both k_n and ε_n. Also the reflection and transmission coefficients depend on κ's. Since the inverse of the Jacobian is only a focusing mechanism to the next guess, the derivatives were simplified by treating the coefficients as constants. The penalty for this is a probable increased number of iterations necessary to find the eigenvalues.

Newton's method works well if the slope of a function is steep enough near a root and the initial guess is close enough. When working with complicated functions such as (3.22) and (3.23), especially when it is not easy to sketch the functions, finding the roots becomes a difficult task. Initial guesses must be very close or roots will be missed.

The roots found by the factoring technique discussed below were used as the initial guesses. Since these were not always good enough to find all the exact roots, the program also used guesses in between the factored roots. Due to slow convergence, round-off error, and insufficient guesses, some roots were still missed. See Appendix for program listing.

To determine if the factored eigenvalues generated by the program were reasonable, the results were compared to those produced by RAYMODE using the same input parameters. RAYMODE, a commly used propagation loss program, was altered to have the same reflection coefficients as used here and the agreement of the two programs was very good.

3.3. Factoring the LEE Characteristic Equation

The factoring technique described in section 1.3 can be used on (3.21) with the introduction of some product errors. Resulting are five factors to be called the water and sediment pp, ss, sp, and ps factors.

3.3.1. The LEE Water Equation

From (3.21), the water equation for the LEE case is

$$\left(1 - \Re_1 \Re_{12}^{PP} E_1^2 e^{-i(2K_1 + \Phi_1)}\right) \ . \tag{3.24}$$

Taking the real and imaginary parts to form two new equations leads to the water solutions.

$$k_n = \left[k_1^2 - \left(\frac{(2n\pi - \Phi_1)}{2h}\right)^2\right]^{1/2} \ . \tag{3.25}$$

$$\varepsilon_n = \left| \ln(\Re_1 \Re_{12}^{PP}) \frac{\kappa_1^R}{2hk_n} \right| \ . \tag{3.26}$$

The LEE water solutions differ from the LLL water solutions only in the $\Re_{12}^{PP}$ factor.

3.3.2. The LEE Sediment Equations

From (3.21), the pp, ss, sp, and ps factors can be written. The method for solving each equation is the same. Take the real and imaginary parts of the equation to form two equations. Solve the imaginary equation for k_n using Newton's method for one variable and substitute the result into the real equation solved for ε_n. The solutions to the sediment factored equations are

$$F^{\mathrm{PP}}(k_n) = 2h(k_1^2-k_n^2)^{1/2} + 2t(k_2^2-k_n^2)^{1/2} + \Phi_1 - 2n\pi = 0 \ , \tag{3.27}$$

$$\varepsilon_n^{\mathrm{PP}} = \left| \ln(\Re_1 S^{\mathrm{PP}}) \ \frac{1}{2k_n\left(h/\kappa_1^R + t/\kappa_2^R\right)} \right| \ . \tag{3.28}$$

$$F^{\mathrm{SS}}(k_n) = 2h(k_1^2-k_n^2)^{1/2} + 2t(k_{2S}^2-k_n^2)^{1/2} + \Phi_1 - 2n\pi = 0 \ , \tag{3.29}$$

$$\varepsilon_n^{\mathrm{SS}} = \left| \ln(\Re_1 S^{\mathrm{SS}}) \ \frac{1}{2k_n\left(h/\kappa_1^R + t/\kappa_{2S}^R\right)} \right| \ . \tag{3.30}$$

$$F^{\mathrm{SP}}(k_n) = F^{\mathrm{PS}}(k_n) = 2h(k_1^2-k_n^2)^{1/2} + t(k_2^2-k_n^2)^{1/2} + t(k_{2S}^2-k_n^2)^{1/2} + \Phi_1 - 2n\pi = 0 \ , \tag{3.31}$$

$$\varepsilon_n^{\mathrm{SP}} = \left| \ln(\Re_1 S^{\mathrm{SP}}) \ \frac{1}{k_n\left(2h/\kappa_1^R + t/\kappa_2^R + t/\kappa_{2S}^R\right)} \right| \ , \tag{3.32}$$

$$\varepsilon_n^{\mathrm{PS}} = \left| \ln(\Re_1 S^{\mathrm{PS}}) \ \frac{1}{k_n\left(2h/\kappa_1^R + t/\kappa_2^R + t/\kappa_{2S}^R\right)} \right| \ . \tag{3.33}$$

3.4. Comparison of Factored to Exact Solutions

As with the LLL model, extreme parameter sets were chosen to force the exact solutions to either the water or various sediment path solutions. For the force to water test, a high density and sound speed are used. Figure 3.7 shows the agreement between the water solutions above the critical angle and the exact solutions. To encourage the sediment paths, similar sound speeds for water and sediment are entered for all three runs. For the pp path, a small sediment attenuation and a high sediment shear attenuation result in the exact solutions being equivalent to the pp solutions as shown in Figure 3.8. Increasing the pressure wave attenuation for the sediment while decreasing the shear wave attenuation enhanced the shear paths as shown in Figure 3.9.

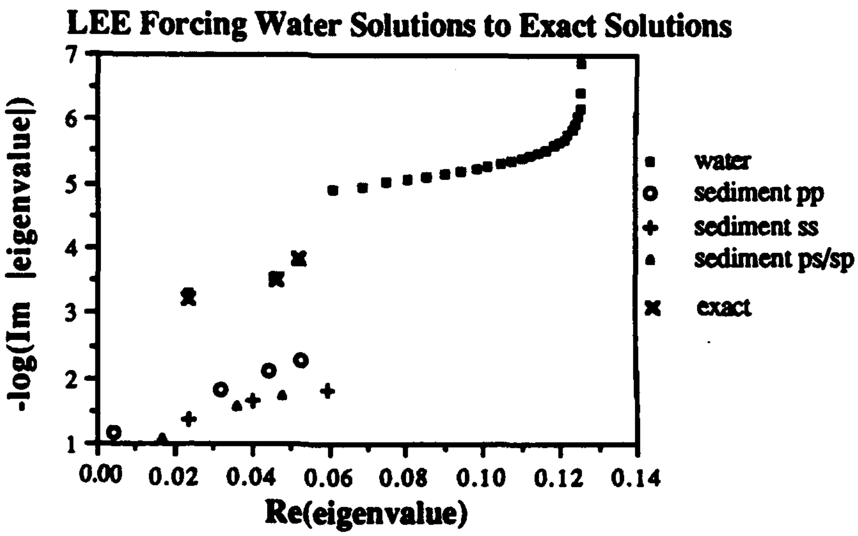

Figure 3.7. LEE Force of Water Solutions to Exact Solutions

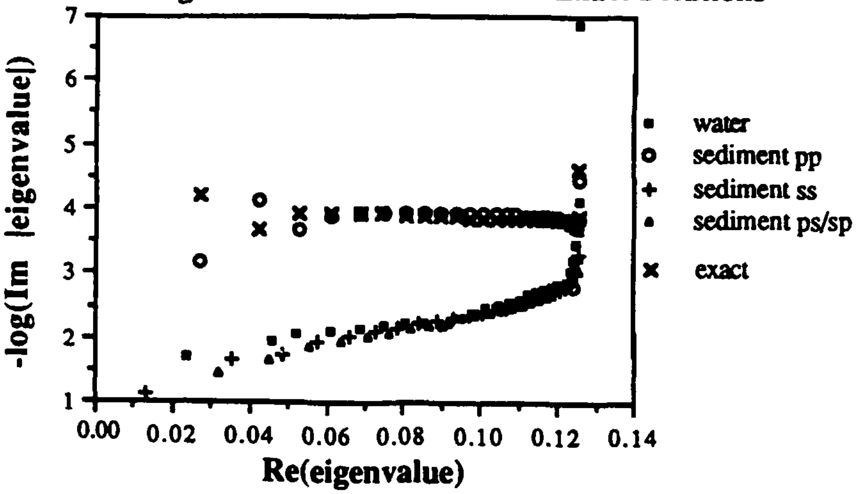

Figure 3.8. LEE Force of Sediment PP solutions to Exact Solutions

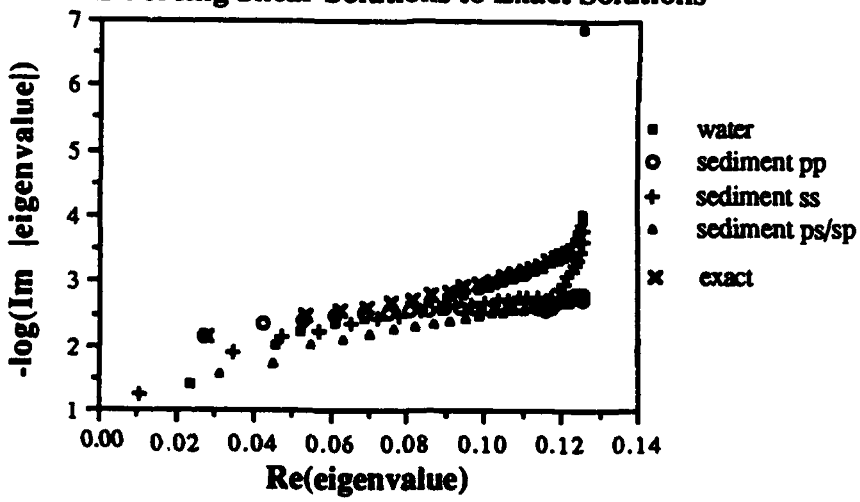

Figure 3.9. LEE Force of Sediment Shear Solutions to Exact Solutions

Chapter 4

RESULTS

A comparison of the eigenvalues generated by the exact solutions and the factored solutions in both the LLL and LEE models was conducted for four types of sediments: a fine fluid, a medium, a hard, and a semiconsolidated sediment. Some typical sediments that fall into these classes are silty clay, till and sand, coarse sand and gravel, and chalk and shale. Table 4.1 lists the numbers used for the sound speeds and attenuation coefficients for the water, sediments and bedrock (Hamilton 1980, pp. 1326) (Stoll 1986, pp. 429) (Clay and Medwin 1977, pp. 258) (Beebe 1981, pp. 78, 94, 115, and 130).

Two criteria are used to evaluate the factored solution eigenvalues: placement on the eigenvalue plot of $-\log(\mathrm{Im}|\varepsilon_n|)$ vs. k_n, and a mode summation comparison between factored solutions and exact solutions. An approximate pressure mode summation equation (4.1), adapted from Clay and Medwin's (9.2.29), is an incoherent sum for a source and receiver located at the same depth. The depth, z, is 100m and the horizontal range, r, is 10km for the calculations. A log ratio of the water (w) plus sediment pp (s) solutions sums divided by the exact (e) solutions multiplied by 10 gives the mode sum comparison in dB as in (4.2). The closer the mode sum is to zero, the better matched the factored solutions are to the exact solutions.

$$M_{w,s,e} = \sum_{n=1}^{\#\text{ of modes}} \left(\frac{\cos(\kappa^{R}_{w,s,e} z)\sin(\kappa^{R}_{w,s,e} z)e^{-|\varepsilon_n r|}}{(k_n^3 r)^{1/2}} \right)^2 . \qquad (4.1)$$

Table 4.1.					
Geoacoustical Parameters for Water, Sediments, and Bedrock					
Layer	ρ (g/cm^3)	c (m/s)	α (dB/m kHz)	c_S (m/s)	α_S (dB/m kHz)
Water	1.0	1500			
Fine Fluid	1.4	1470	0.01	250	15.0
Medium	1.9	1600	0.03	450	13.0
Hard	2.0	1720	0.003	650	10.0
Semiconsolidated	2.1	2400	0.05	1000	1.0
Bedrock	2.7	5500	0.05	2900	(0.07)

$$\textit{mode sum comparison} = 10 \log \frac{M_w + M_s}{M_e} \quad . \tag{4.2}$$

4.1. The LLL and LEE Reflection and Transmissions Coefficients

It is useful to examine the plots of the reflection and transmission coefficients in Figures 4.1-4.5, made for a medium sediment at 30 Hz. Some of the coefficients affect the eigenvalues drastically so it is important to see their behavior.

Figures 4.1 and 4.2 compare the LLL $\Re_{12}$ and $\Re_{23}$ with the LEE $\Re_{12}^{PP}$ and $\Re_{23}^{PP}$. There is little difference between the two $\Re_{12}$'s, the highest difference being for the lowest modes, or highest k_n's. $\Re_{23}$ is equal to one over most of the k_n range

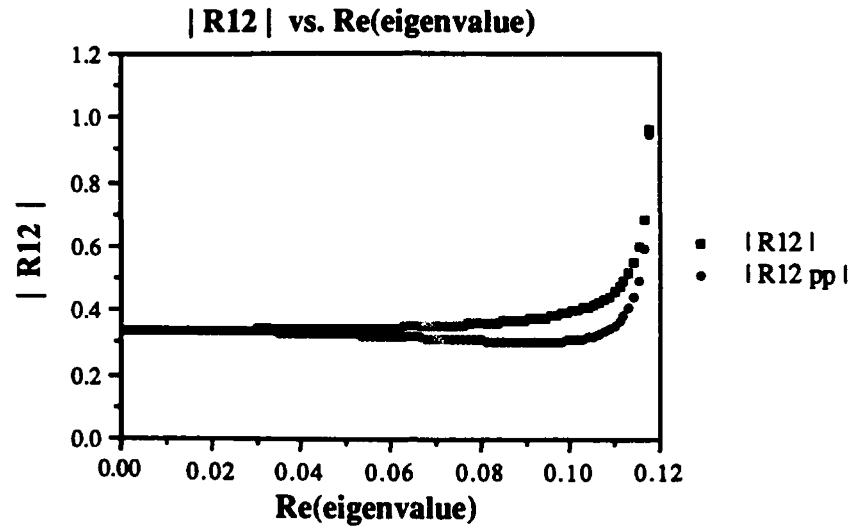

Figure 4.1. Comparison of LLL $\Re_{12}$ and LEE $\Re_{12}^{PP}$

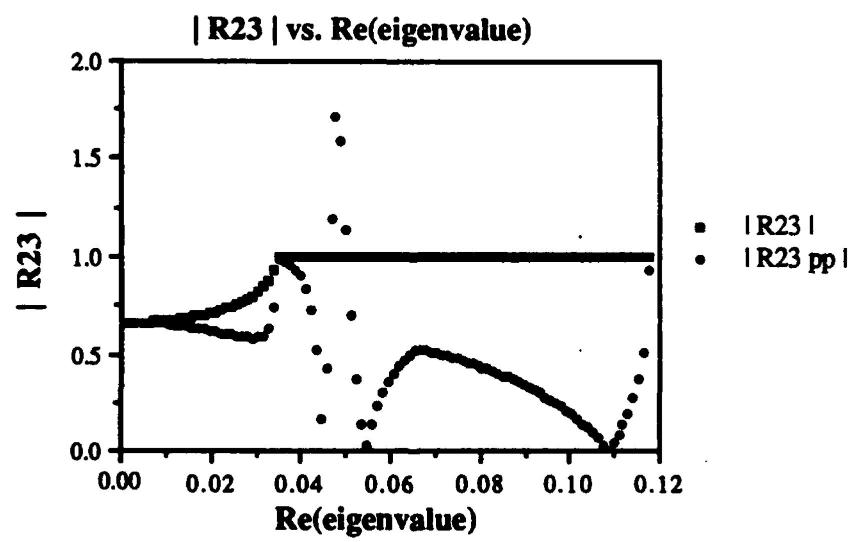

Figure 4.2. Comparison of LLL $\Re_{23}$ and LEE $\Re_{23}^{PP}$

while $\Re_{23}^{PP}$ is not constant or even smooth. At the lowest modes the falloff of $\Re_{23}^{PP}$ is most important. The swings in $\Re_{23}^{PP}$ affect the shape of the LEE sediment pp solutions more than the exact solutions.

Figures 4.3-4.5 display the rest of the LEE coefficients. Figure 4.3 shows the extreme difference between $\Re_{21}^{PP}$ and $\Re_{21}^{SS}$ for the LEE case. From this plot, it is easy to see how the shear waves are attenuated quickly. Figure 4.4 shows the LEE $\Re_{23}^{SS}$, $\Re_{23}^{SP}$ and $\Re_{23}^{PS}$ coefficients. They, like $\Re_{23}^{PP}$, are not smooth, but their peaks and dips fall in the region of low k_n. Figure 4.5 displays the LEE transmission coefficients.

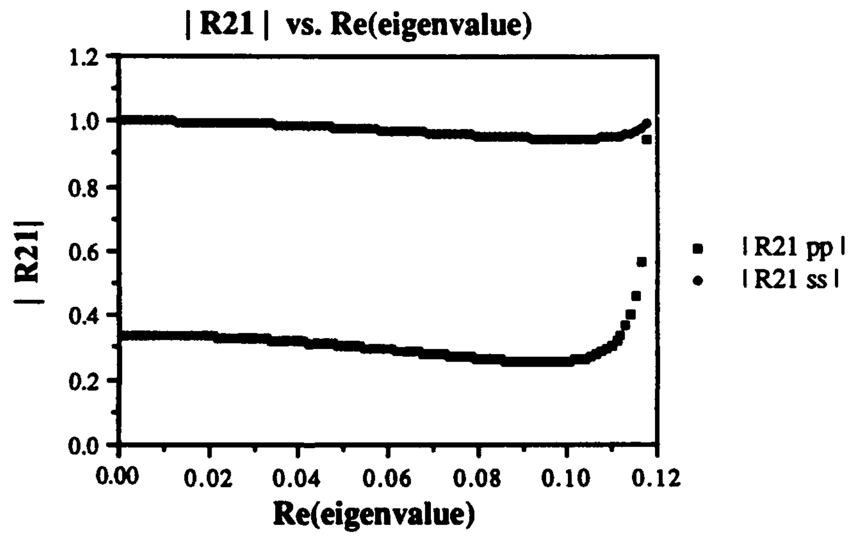

Figure 4.3. Comparison of LEE $\Re_{21}^{PP}$ and $\Re_{21}^{SS}$

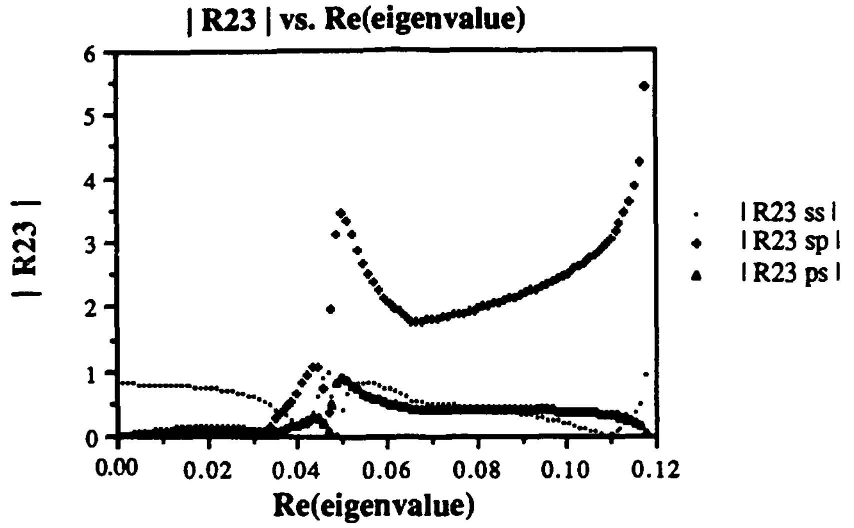

Figure 4.4. Comparison of LEE $\Re_{23}^{SS}$, $\Re_{23}^{SP}$ and $\Re_{23}^{PS}$

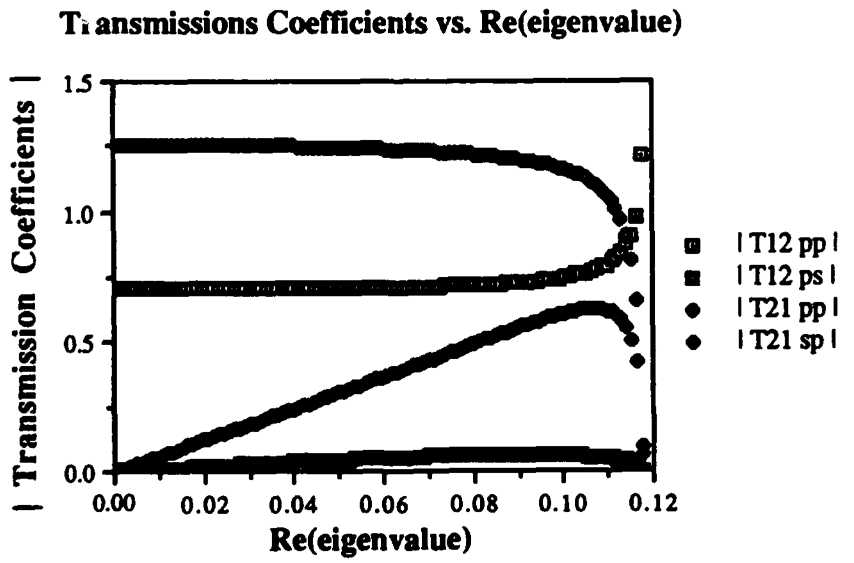

Figure 4.5. Comparison of LEE $\Im_{12}^{PP}$, $\Im_{12}^{PS}$, $\Im_{21}^{PP}$ and $\Im_{21}^{SP}$

4.2. LLL Model Results

The first study compares not only the differences between the four types of sediments, but also tests variations with frequency and sediment thickness. Four combinations of thickness and frequency are examined: low thickness (10m) and low frequency (10Hz), low thickness and high frequency (50Hz), high thickness (200m) and low frequency , high thickness and high frequency.

4.2.1. General Trends

From (2.35), the expression for the number of modes in a duct, it is evident that with increased frequency or water depth there will be an increased number of modes. Consequently frequencies above 50 Hz were not studied to keep the number of modes manageable. The water depth for this study was arbitrarily kept constant at 700m. Comparing Figures 4.6 and 4.7 with 4.8 and 4.9 shows the expected increase in mode number for a frequency increase.

For the larger sediment thickness , the sediment mode attenuation is greater than for the smaller thickness case. The wave travels further in the sediment and is therefore more attenuated. Also, there is more attenuation in the higher frequency runs because the attenuation factor A goes up with frequency.

4.2.2. Comparison of Factored Solutions to Exact Solutions

As can be seen in Figures 4.6-4.9, the mode attenuation, or $|\varepsilon_n|$, is greater for both factored solutions than for the exact solutions. One of the factored solutions should have nearly the

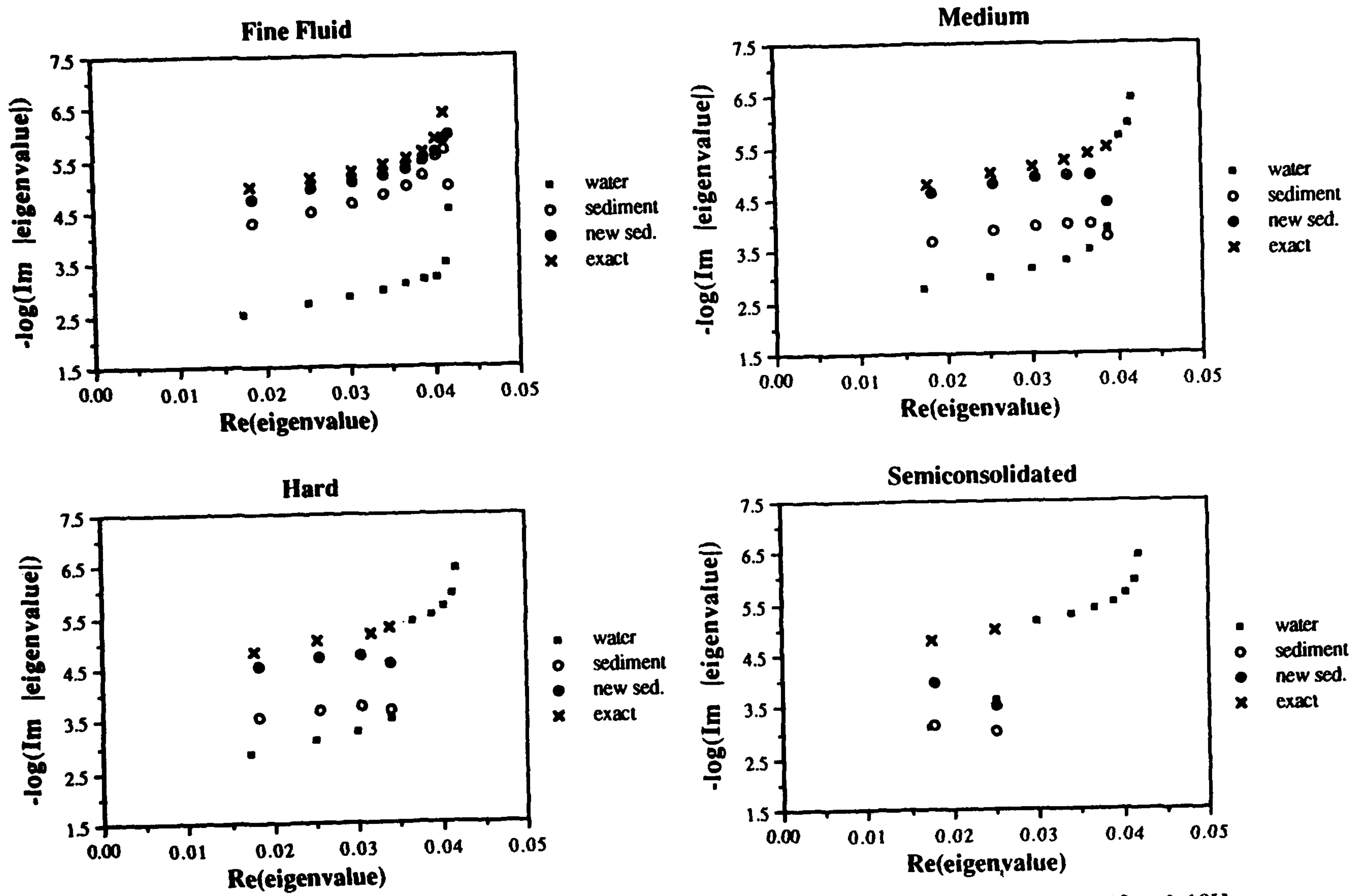

Figure 4.6. LLL Water, Sediment, New Sediment, and Exact solutions for Four Sediments: t=10m, f=10Hz

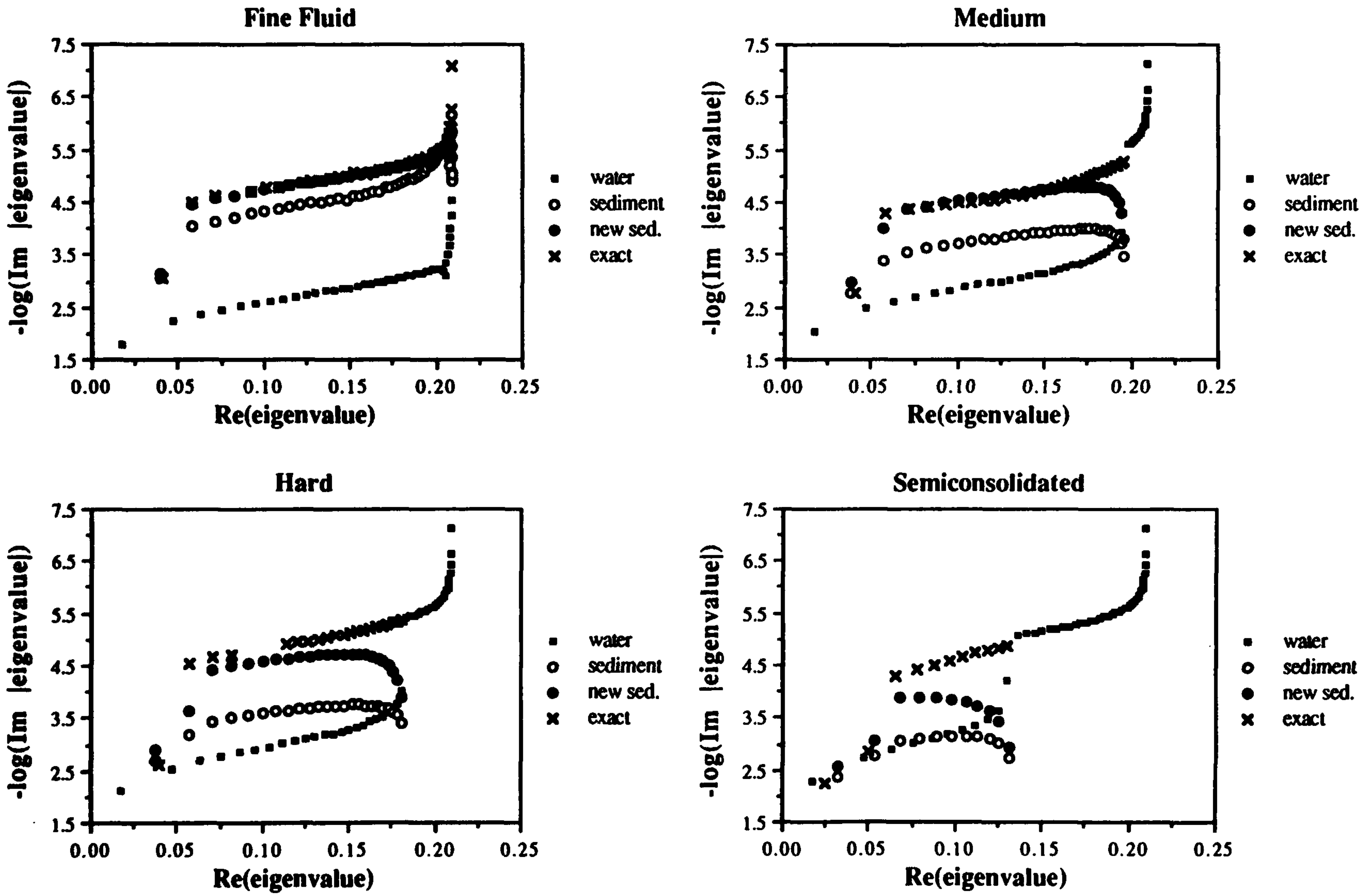

Figure 4.7. LLL Water, Sediment, New Sediment, and Exact Solutions for Four Sediments: t=10m, f=50Hz

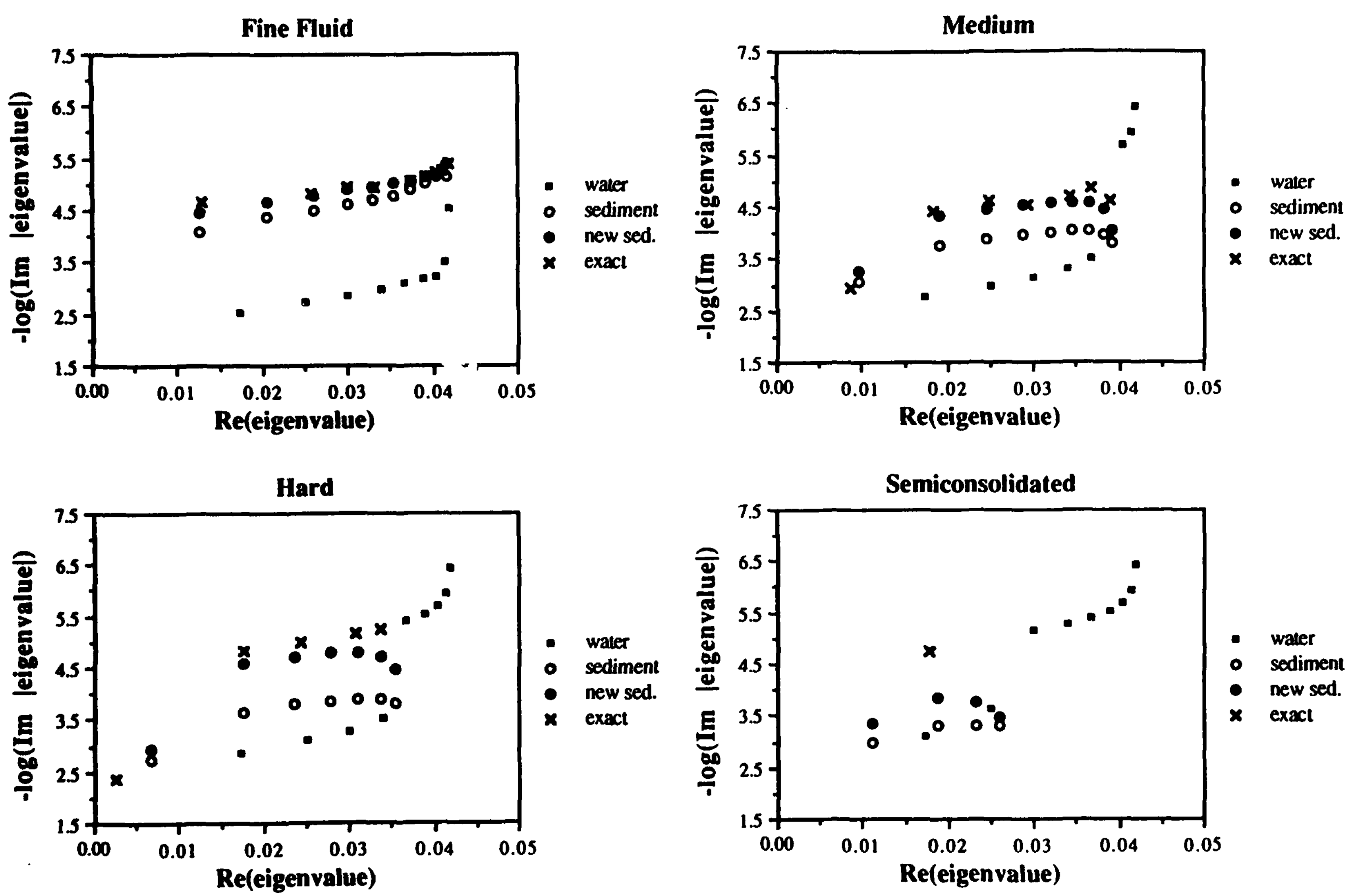

Figure 4.8. LLL Water, Sediment, New Sediment and Exact Solutions for Four Sediments: t=200m, f=10Hz

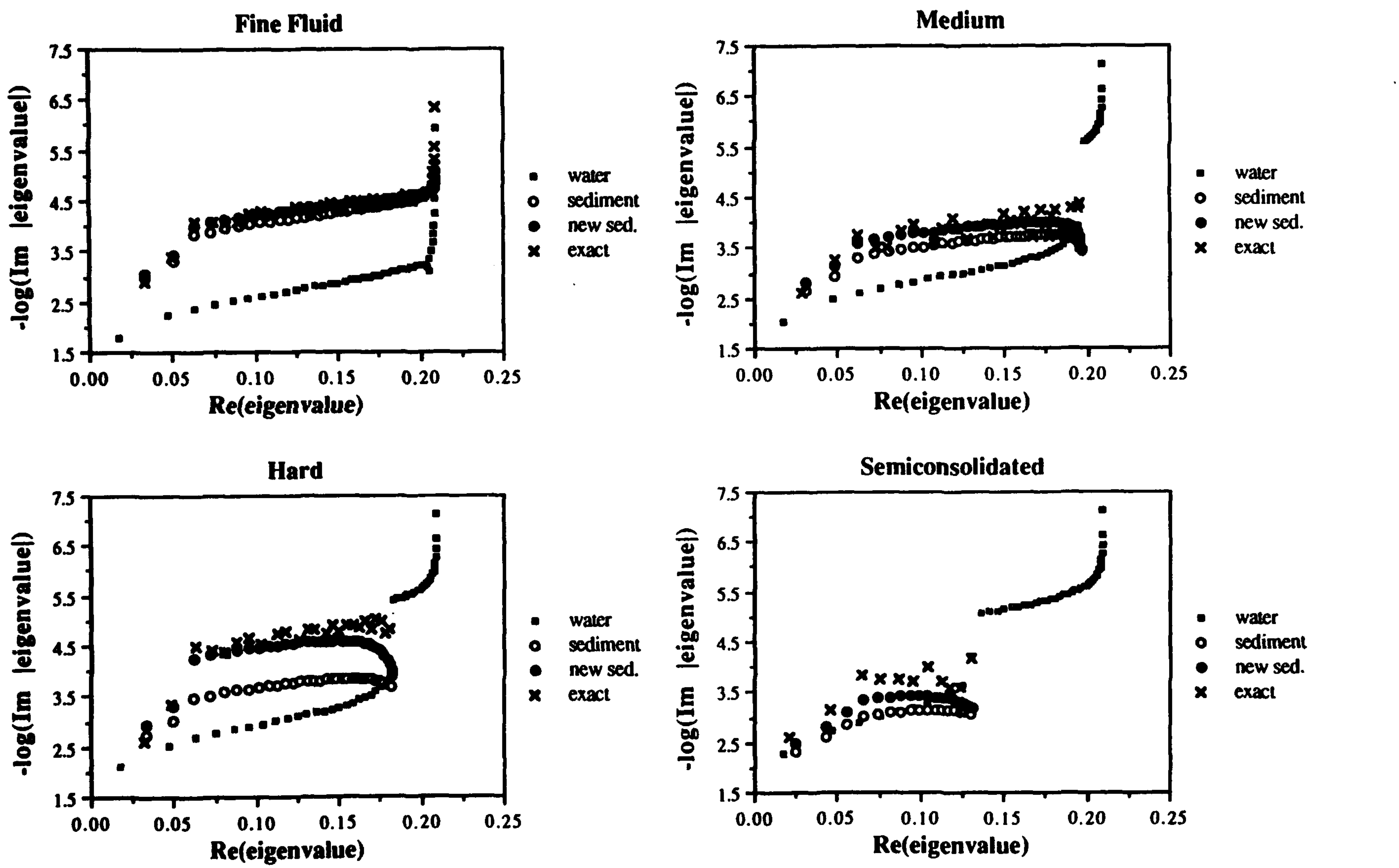

Figure 4.9. LLL Water, Sediment, New Sediment, and Exact Solutions for Four Sediments: t=200m, f=50Hz

equivalent mode attenuation of the exact eigenvalues so it may be used in place of the exact solutions. From Table 4.2, the mode sum comparisons are too high, as high as 30 dB for the semiconsolidated 10Hz, 10m case.

Since the water factor yields solutions that are straight forward to calculate, it will not be changed. An adjusted sediment factor is needed.

4.2.2.1 Derivation of the New LLL Sediment Factor

The sources of error in the sediment factor are the product term added to the left side of the characteristic equation as described in section 1.4 and setting S_2 equal to zero in section 2.3.2. Keeping S_2 in the equation did not improve the results. To change the sediment factor, keep the water factor, and yet remove the error, a new sediment factor was developed by dividing the characteristic equation by the water factor:

$$\frac{(1-\Re_1\Re_{13}e^{-2iK_1})}{(1-\Re_1\Re_{12}e^{-2iK_1})} \, . \tag{4.3}$$

Writing the denominator as a geometric series, expanding that series, and multiplying gives

$$(1-\Re_1\Re_{13}e^{-2iK_1}+\Re_1\Re_{12}e^{-2iK_1}-\Re_1^2\Re_{12}\Re_{13}e^{-4iK_1}+\Re_1^2\Re_{12}^2e^{-4iK_1}-\ldots). \tag{4.4}$$

After substitution for $\Re_{13}$, the positive terms in $\Re_{12}$ drop out leaving

$$1-\frac{\Re_1\Im_{12}\Re_{23}\Im_{21}A^2}{1-\Re_{21}\Re_{23}A^2e^{-2iK_1}}\left[1+\Re_1\Re_{12}e^{-2iK_1}+\Re_1^2\Re_{12}^2e^{-4iK_1}+\ldots\right]e^{-2i(K_1+K_2)}. \tag{4.5}$$

The new sediment factor is to be of the form

$$1-Ce^{-2i(K_1+K_2)}, \tag{4.6}$$

where C, with the denominator expanded, is

$$C = \Re_1 \Im_{12} \Re_{23} \Im_{21} A^2 \left[1 + \Re_1 \Re_{12} e^{-2iK_1} + \Re_1^2 \Re_{12}^2 e^{-4iK_1} + \ldots\right] \times \left[1 - \Re_{12} \Re_{23} A^2 e^{-2iK_2} + \Re_{12}^2 \Re_{23}^2 A^4 e^{-4iK_2} - \ldots\right]. \quad (4.7)$$

In order to obtain a smooth mode attenuation, $2K_1$ is approximated as much larger than $2K_2$, which amounts to h being much larger than t. The eigenvalues of this factor with combined phase K_1+K_2 will be very close to those of the water factor, since the sediment phase is very small. Both exponentials e^{-2iK_1} and e^{-2iK_2} are approximated as unity in the magnitude of C.

$$|C| \cong \Re_1 \Im_{12} \Re_{23} \Im_{21} A^2 \left[1 + \Re_1 \Re_{12} + \Re_1^2 \Re_{12}^2\right] \left[1 - \Re_{12} \Re_{23} A^2 + \Re_{12}^2 \Re_{23}^2 A^4\right]. \quad (4.8)$$

Both series work best truncated to quadratic form as in (4. 8). Higher orders gave mode attenuations of the wrong sign, indicating the sums were diverging. The original sediment factor has the two terms in brackets equal to unity.

4.2.2.2 Comparison of New LLL Sediment Factor with Exact Solutions

From Figures 4.6-4.9 the new sediment factor ε_n's are much closer to the exact ε_n's. Table 4.2 confirms the improvement with much smaller mode sum comparisons. The new sediment factor works very well for the fine fluid, medium, and hard sediments, but is not improved enough for the semiconsolidated sediment.

4.3. LEE Model Results

The tests on the LEE model were done for a constant low frequency (30Hz) since the affects of frequency have already been shown in the LLL study. Two sediment depths, 10m and 300m, were examined for a water depth 900m. The results are shown in Figures 4.10-4.11 and Table 4.3.

Table 4.2.

LLL Mode Sum Comparisons in dB with Sediment and New Sediment Factors

	10m, 10Hz		10m, 50Hz	
	Sediment	New Sediment	Sediment	New Sediment
Fine Fluid	–2	0	–2	0
Medium	–13	–1	–11	0
Hard	–21	–2	–19	–2
Semiconsolidated	–30	–9	–14	–9
	200m, 10Hz		200m, 50Hz	
	Sediment	New Sediment	Sediment	New Sediment
Fine Fluid	–2	0	–1	1
Medium	–9	–1	–7	1
Hard	–15	–1	–14	0
Semiconsolidated	–27	–9	–4	–4

There are four sediment factors, pp, ss, sp, and ps. The pp mode attenuation has a dip and peak corresponding to the dip and peak in the $\Re_{23}^{PP}$ plot in Figure 4.2. The ss, ps, and sp mode attenuations are much lower than the pp. The ps and sp solutions are actually equivalent since their coefficient products, $\Im_{12}^{PS}\Re_{23}^{SP}\Im_{21}$ and $\Im_{12}\Re_{23}^{PS}\Im_{21}^{SP}$ are equivalent.

The water and sediment pp solutions are the best approximation to the exact solutions. The ss and ps/sp solutions are usually highly attenuated since $\Im_{12}^{PS}$ and $\Im_{21}^{SP}$ are small, and can be left out of the LEE model. The dips and peaks in the sediment pp solutions are cause for concern when the using the factors in place of the exact.

The program missed some of the modes for the LEE exact case. The accuracy of mode sum comparisons, which include only the water, sediment pp, and exact solution sums as in the LLL case, is thus somewhat limited.

A new LEE sediment pp was developed analogous to the one for the LLL new sediment factor. With both series linear, quadratic, cubic, fourth order and various combinations of those, ε_n's of the wrong sign were generated. Also the mode attenuation plot was not smooth, so the method of correction was abandoned.

4.4. Comparison of LEE Model to LLL Model

Figures 4.12-4.13 compare the LLL and LEE water and sediment pp solutions. The water factors for the LEE and LLL models are nearly identical for identical input parameters. This is expected since $\Re_1$ is the same for both and $\Re_{12}^{PP}$ for the liquid/elastic boundary is nearly equal to $\Re_{12}$ for the liquid/liquid boundary as shown in Figure 4.1. Consequently, the LLL water factor can be used in a LEE model.

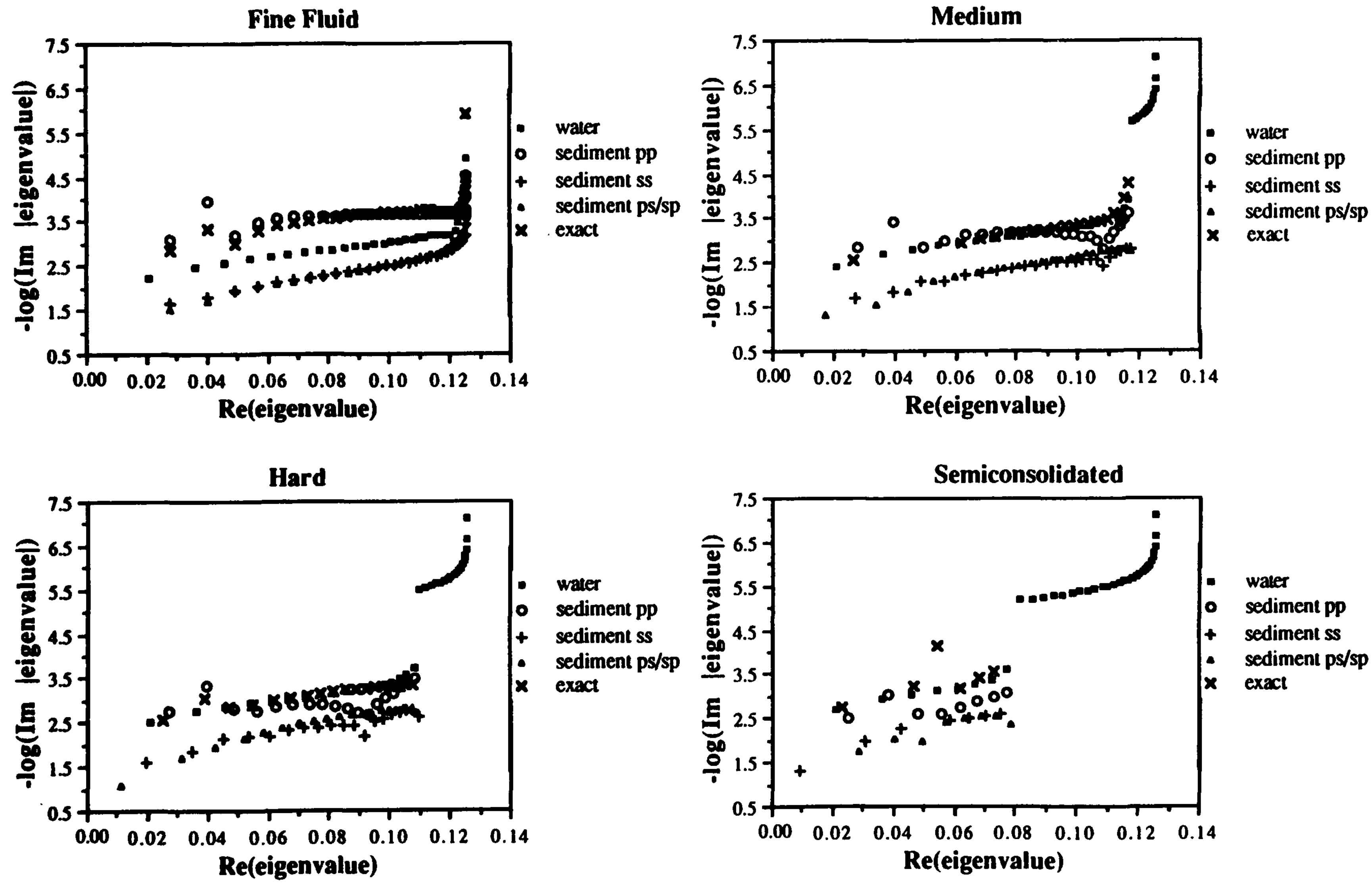

Figure 4.10. LEE Water, Sediment PP, SS, and PS/SP, and Exact Solutions for Four Sediments: t=10m, f=30Hz

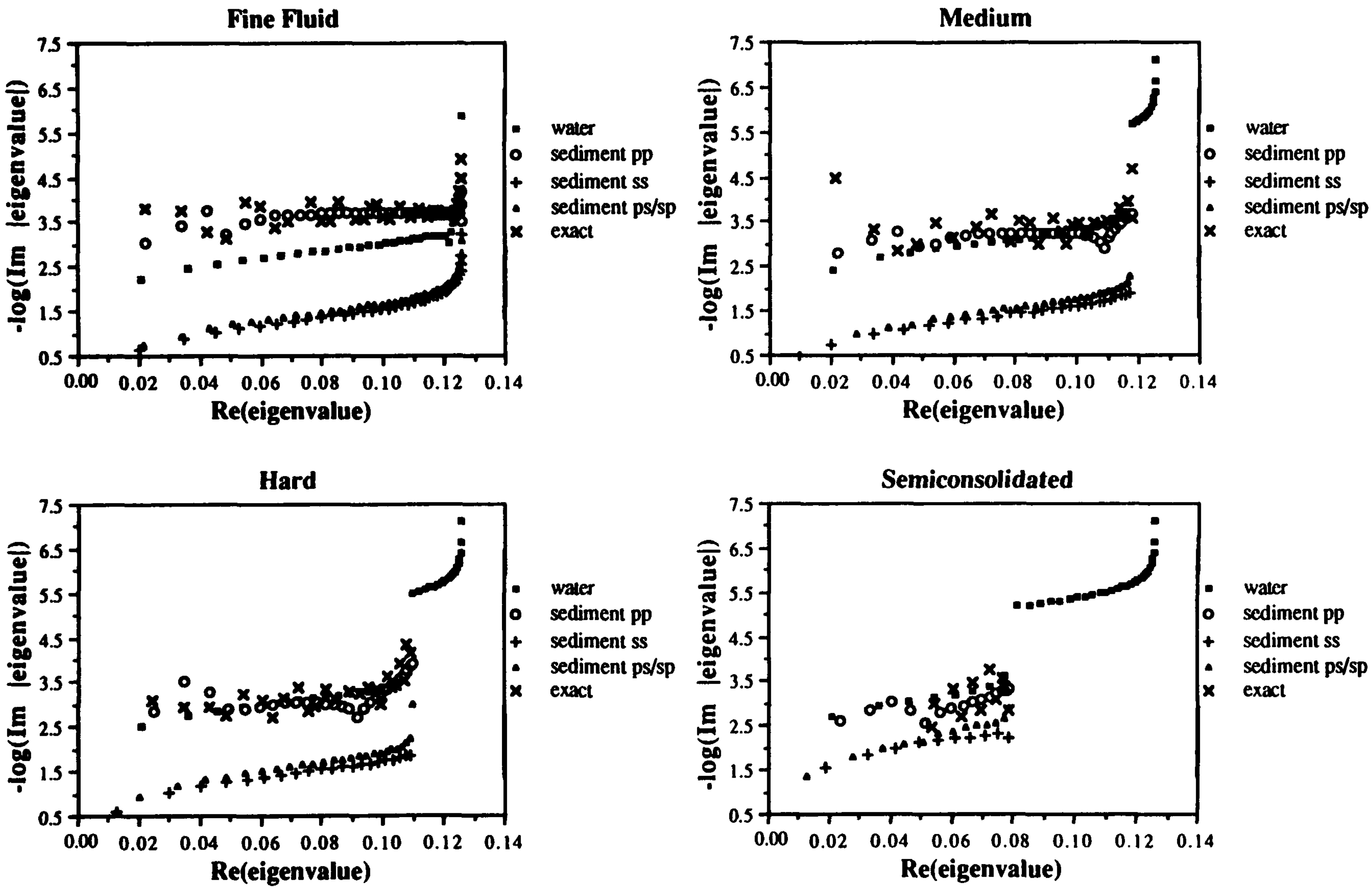

Figure 4.11. LEE Water, Sediment PP, SS, and PS/SP, and Exact Solutions for Four Sediments: t=300m, f=30Hz

Table 4.3.		
Approximate LEE Mode Sum Comparisons in dB		
	10m, 30Hz	300m, 30Hz
Fine Fluid	6	2
Medium	8	10
Hard	8	3
Semiconsolidated	16	7

The sediment solutions do not propagate as well in the LEE environment as they did in the LLL environment due to the added shear loss. $\Re_{23}^{PP}$ is not equal to one for the LEE case as it was for the LLL case as shown in Figure 4.2, but instead has dips and peaks, thus changing the shape of the sediment pp solutions.

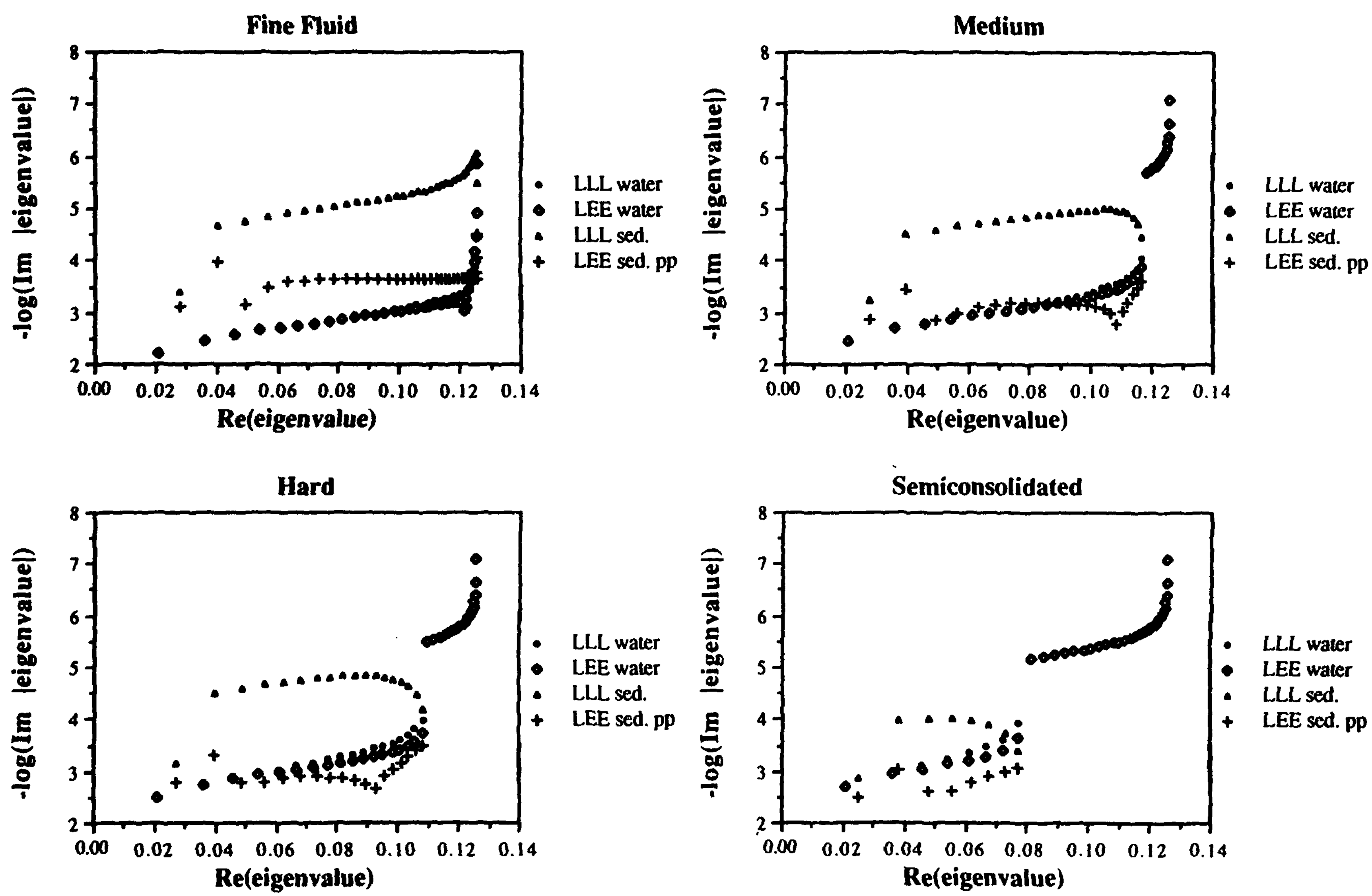

Figure 4.12. LLL and LEE Water and Sediment PP Solutions for Four Sediments: t=10m, f=30Hz

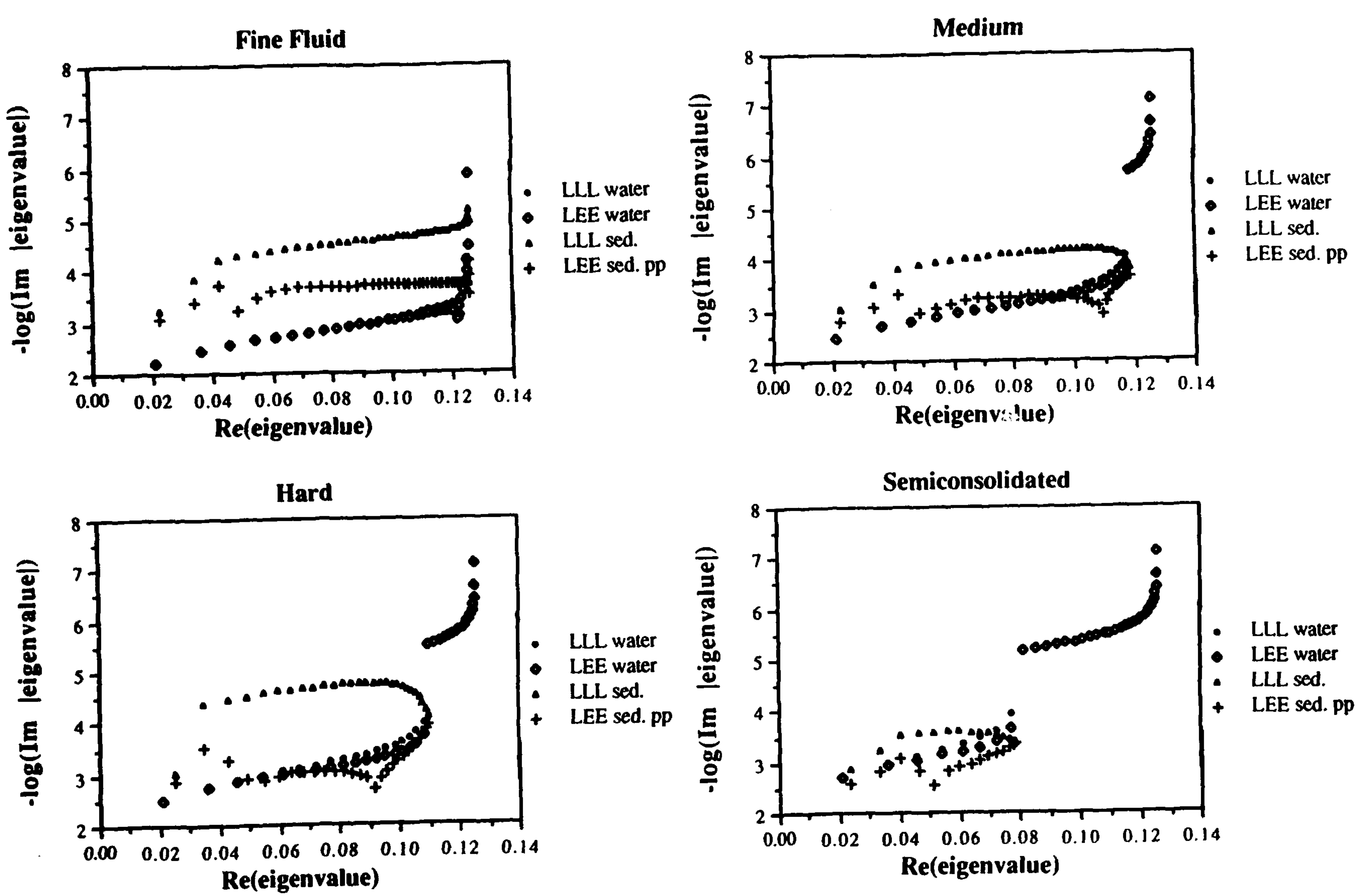

Figure 4.13. LLL and LEE Water and Sediment PP Solutions for Four Sediments: t=300m, f=30Hz

Chapter 5

CONCLUSIONS

Approximate solutions for the three layer duct characteristic equation were developed for quickly obtaining the eigenvalues. The LLL model factors are both quick and a good approximation of the LLL exact solutions. The LEE factored solutions are also easy to calculate, but the loss of accuracy may outweigh the benefits of speed.

The method of factoring described in section 1.4 was applied to both the LLL and LEE characteristic equations. The factored solutions were then compared to the exact solutions for each model. Also the LLL and LEE factored solutions were compared to each other. Variations in sediment, sediment thickness, and frequency were examined.

The LLL factored solutions predicted losses much higher than the LLL exacts. A new sediment factor was then developed, based on the characteristic equation and the water factor. The losses for the new factor are much closer to the losses of the exact solutions with the best results for the fine fluid, medium, and hard sediments. The losses of the new factor for the semiconsolidated sediment, which favors the water path, were improved, but not as much as the others.

The factored solutions for the LEE model also predicted more losses than the LEE exact solutions. Attempts to correct the sediment pp factor proved inadequate. Even though this LEE factoring scheme predicts too much loss, using it may still be better than using a LLL model which would predict far too little loss for an elastic sediment.

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APPENDIX

COMPUTER PROGRAM LISTING

The following computer listing is a program that finds the eigenvalues for the LLL water, sediment, new sediment, and exact solutions. Also it finds the LEE water, sediment pp, ss, sp, and ps, and exact solutions. The program is written in HP BASIC. The symbols and variable names used in the program do not necessarily match the symbols used in the text.

```
RAD
OPTION BASE 1
GINIT
GCLEAR
GRAPHICS ON

DIM     Kw(90), Ew(90), Ks(90), Es(90), Ksm(90), Esm(90), Kn(90), En(90)
DIM     Kw1(90), Ew1(90), Kw2(90), Ew2(90)
DIM     Kw3(90), Ew3(90), Kw4(90), Ew4(90), Kn1(90), En1(90)
COM     /Parameter1/    C1, C2, Rho1, Rho2, H, T, Freq, Alfa2, R1, R1phase
COM     /Parameter2/    C2s, C3, C3s, Rho3, Alfa2s
COM     /Wavenumber/    W, K1, K2, K2s, K3s

C1=1500
C3=2900
Rho1=1.0
Rho3=2.7
H=900
T=10
Freq=30
R1=0.99
R1phase=PI

CALL Label4axes(1)

FOR Diffcase=1 TO 4
SELECT Diffcase
CASE 1
        C2=1470
        C2s=250
        Rho2=1.4
```

```
        Alfa2=0.01
        Alfa2s=15.0
CASE 2
        C2=1600
        C2s=450
        Rho2=1.9
        Alfa2=0.03
        Alfa2s=13.0
CASE 3
        C2=1720
        C2s=650
        Rho2=2.0
        Alfa2=0.03
        Alfa2s=10.0
CASE 4
        C2=2400
        C2s=1000
        Rho2=2.1
        Alfa2=0.05
        Alfa2s=1.0
END SELECT

W=2*PI*Freq
K1=W/C1
K2=W/C2
K3=W/C3
K2s=W/C2s
K3s=W/C3s

ALPHA ON
OUTPUT KBD;" K";

CALL Water(Kw(*),Ew(*),Nmodew,"lll","o",Nofextras)
CALL Water(Kw1(*),Ew1(*),Nmode1w,"lee","o",Nofextras)

CALL Sed(Ks(*),Es(*),Kw(*),Nmodes,Nmodew,"lll"," pp","+",Nofextras,"s ")
CALL Sed(Ksm(*),Esm(*),Kw(*),Nmodems,Nmodew,"llm"," pp","u",Nofextras,"sm")
CALL Sed(Ks1(*),Es1(*),Kw1(*),Nmode1s,Nmode1w,"lee"," pp","+",Nofextras,"s1")
CALL Sed(Ks2(*),Es2(*),Kw1(*),Nmode2s,Nmode1w,"lee"," ss","*",Nofextras,"s2")
CALL Sed(Ks3(*),Es3(*),Kw1(*),Nmode3s,Nmode1w,"lee"," sp","x",Nofextras,"s3")
CALL Sed(Ks4(*),Es4(*),Kw1(*),Nmode4s,Nmode1w,"lee"," ps","-",Nofextras,"s4")

CALL Exact(Kn(*),En(*),Kw(*),Ew(*),Ksm(*),Esm(*),Nmoden,Nmodew,Nmodems,
        "lll","^",Nofextras,Diffcase)
CALL Exact(Kn1(*),En1(*),Kw1(*),Ew1(*),Ks1(*),Es1(*),Nmode1n,Nmode1w,
        Nmode1s,"lee","n",Nofextras,Diffcase)

CALL Sort(Kn(*),En(*),Nmoden)
CALL Sort(Kn1(*),En1(*),Nmode1n)

CALL Nat_log_e(Ew(*),Ln_ew(*),Nmodew)
CALL Nat_log_e(Es(*),Ln_es(*),Nmodes)
CALL Nat_log_e(Esm(*),Ln_esm(*),Nmodems)
```

```
CALL Nat_log_e(En(*),Ln_en(*),Nmoden)
CALL Nat_log_e(Ew1(*),Ln_ew1(*),Nmode1w)
CALL Nat_log_e(Es1(*),Ln_es1(*),Nmode1s)
CALL Nat_log_e(Es2(*),Ln_es2(*),Nmode2s)
CALL Nat_log_e(Es3(*),Ln_es3(*),Nmode3s)
CALL Nat_log_e(Es4(*),Ln_es4(*),Nmode4s)
CALL Nat_log_e(En1(*),Ln_en1(*),Nmode1n)

CALL Fouraxes(5,Diffcase)

CALL Ln_e_vs_k_plot(Kw(*),Ew(*),Ln_ew(*),Nmodew,5,"o")
CALL Ln_e_vs_k_plot(Ks(*),Es(*),Ln_es(*),Nmodes,1,"+")
CALL Ln_e_vs_k_plot(Ksm(*),Esm(*),Ln_esm(*),Nmodems,4,"u")
CALL Ln_e_vs_k_plot(Kn(*),En(*),Ln_en(*),Nmoden,7,"^")
CALL Ln_e_vs_k_plot(Kw1(*),Ew1(*),Ln_ew1(*),Nmode1w,6,"o")
CALL Ln_e_vs_k_plot(Ks1(*),Es1(*),Ln_es1(*),Nmode1s,3,"+")
CALL Ln_e_vs_k_plot(Ks2(*),Es2(*),Ln_es2(*),Nmode2s,4,"*")
CALL Ln_e_vs_k_plot(Ks3(*),Es3(*),Ln_es3(*),Nmode3s,5,"x")
CALL Ln_e_vs_k_plot(Ks4(*),Es4(*),Ln_es4(*),Nmode4s,7,"-")
CALL Ln_e_vs_k_plot(Kn1(*),En1(*),Ln_en1(*),Nmode1n,2,"n")

PRINT "LLL Mode Sum"
CALL Incohrnt(Kw(*),Ew(*),Ks(*),Es(*),Kn(*),En(*),Nmodew,Nmodes,Nmoden,
        Indbex,Nofextras)

PRINT "LLL Mode Sum with New Sediment Factor"
CALL Incohrnt(Kw(*),Ew(*),Ksm(*),Esm(*),Kn(*),En(*),Nmodew,Nmodems,
        Nmoden,Indbex,Nofextras)

PRINT "LEE Mode Sum"
CALL Incohrnt(Kw1(*),Ew1(*),Ks1(*),Es1(*),Kn1(*),En1(*),Nmode1w,Nmode1s,
        Nmode1n,Indbex,Nofextras)

NEXT Diffcase
END

------------------------------------------------------------------------------------------

SUB Water(Kw(*),Ew(*),Nmodew,Type$,Sym$,Nofextras)

COM     /Parameter1/     C1, C2, Rho1, Rho2, H, T, Freq, Alfa2, R1, R1phase
COM     /Wavenumber/  W, K1, K2, K2s, K3s
Frmt1:IMAGE 4X,4A,2D,4A,SD.4DE,5X,4A,2D,4A,SD.4DE,3X,D.2D,5X,2D.D
INTEGER N,M,I

FOR I=1 TO 90
     Kw(I)=0.0
NEXT I
PRINT
PRINT TAB(19);"WATER ";Type$;" SOLNS";Sym$;"                         R    Phi1"
PRINT

Nofextras=0
```

```
Nmodew=0
N=1

WHILE ((2*N*PI-R1phase)/(2*H))<=K1

    Kw(N)=SQR(K1^2-((((2*N*PI-R1phase)/(2*H))^2)
    Kappa1=SQR(K1^2-Kw(N)^2)

    IF (K2^2-Kw(N)^2)<0 THEN
        Kappa2=0
        Noextras=Noextras+1
    ELSE
        Kappa2=SQR(K2^2-Kw(N)^2)
    END IF

    Phi1=ACS(Kw(N)/K1)*180/PI
    Z12=(Rho1*Kappa2)/(Rho2*Kappa1)
    R12ll=(1-Z12)/(1+Z12)

    Nmodew=Nmodew+1
    M=Nmodew
    Kw(M)=Kw(N)

    SELECT Type$
    CASE "lll"
            Ew(M)=LOG(ABS(R1*R12ll))*Kappa1/(Kw(M)*2*H)
            PRINT USING Frmt1;" Kw(",M,") = ",Kw(M)," Ew(",M,") = ",Ew(M),
                    R12ll,Phi1
    CASE "lee"
            IF (K2s^2-Kw(N)^2)<=0 THEN
                R12le=1
            ELSE
                Kappa2s=SQR(K2s^2-Kw(N)^2)
                Phi2s=ATN(Kappa2s/Kw(N))*180/PI
                R12num=(-Z12+1-(SIN(2*Phi2s))^2*(1-Kappa2/Kappa2s))
                R12den=(Z12+1-(SIN(2*Phi2s))^2*(1-Kappa2/Kappa2s))
                R12le=R12num/R12den
            END IF

            Ew(M)=LOG(ABS(R1*R12le))*Kappa1/(Kw(M)*2*H)
            PRINT USING Frmt1;"Kw1(",M,") = ",Kw(M),"Ew1(",M,") = ",Ew(M),
                    R12le,Phi1
    END SELECT
    N=N+1
END WHILE
PRINT
SUBEND

----------------------------------------------------------------------------------------------------

SUB Sed(Ks(*),Es(*),Kw(*),Nmodes,Nmodew,Type$,Kind$,Sym$,Extras,S$)

Frmt1: IMAGE 4X,SD.3DE,6X,2D,SD.3DE,5X,D.3D,3X,D.2D,3X,D.2D,4X,2D.D
```

```
Frmt2: IMAGE 4X,SD.3DE,6X,2D,SD.3DE,5X,D.3D,2X,2D.2D,3X,2D.2D,3X,
       2D.2D,3X,2D.D
Frmt5: IMAGE 8X,A,2A,10X,A,10X,A,2A,9X,5A,3X,3A,4X,3A,4X,4A
Frmt6: IMAGE 8X,A,2A,10X,A,10X,A,2A,9X,5A,3X,3A,5X,3A,5X,3A,3X,4A
Frmt7: IMAGE 8X,A,2A,10X,A,10X,A,2A,9X,5A,2X,5A,2X,5A,2X,5A,3X,4A

COM    /Parameter1/    C1, C2, Rho1, Rho2, H, T, Freq, Alfa2, R1, R1phase
COM    /Parameter2/    C2s, C3, C3s, Rho3, Alfa2s
COM    /Wavenumber/    W, K1, K2, K2s, K3s

INTEGER    Mm, Num_divs, N, M, Stepslow, Nn

PRINT
PRINT TAB(16);"SEDIMENT ";Type$;" SOLNS";Sym$
PRINT

SELECT TYPE$
CASE "lll"
   PRINT USING Frmt5;"K",S$,"N","E",S$,"Atten","R12","R23","Phi1"
CASE "llm"
   PRINT USING Frmt5;"K",S$,"N","E",S$,"Atten","R12","R23","Phi1"
CASE "lee"
  SELECT Kind$
  CASE " pp"
   PRINT USING Frmt6;"K",S$,"N","E",S$,"Atten","R23","T12pp","T21pp","Phi1"
  CASE " ss"
   PRINT USING Frmt7;"K",S$,"N","E",S$,"Atten","R23ss","T12ps","T21sp","Phi1"
  CASE " sp"
   PRINT USING Frmt7;"K",S$,"N","E",S$,"Atten","R23sp","T12ps","T21pp","Phi1"
  CASE " ps"
   PRINT USING Frmt7;"K",S$,"N","E",S$,"Atten","R23ps","T12pp","T21sp","Phi1"
  END SELECT
END SELECT

Onum_divs=Nmodew-Extras
Max_iters=40
Nmodes=0

FOR I=1 TO 90
     Ks(I)=0.0
NEXT I

Nn=1
REPEAT
        X=MIN(0.9999*K1,0.99999*K2)
        Kappa1=SQR(K1^2-X^2)
        Kappa2=SQR(K2^2-X^2)
        Kappa2s=SQR(K2s^2-X^2)
        Phase=2*Nn*PI-R1phase

        IF Kind$=" pp" THEN
                Fs=2*H*Kappa1+2*T*Kappa2-Phase
        ELSE
```

```
            IF Kind$=" ss" THEN
                  Fs=2*H*Kappa1+2*T*Kappa2s-Phase
            ELSE
                  Fs=2*H*Kappa1+T*(Kappa2+Kappa2s)-Phase
            END IF
      END IF

      Nn=Nn+1
UNTIL  Fs<0  OR  Nn>=90

Nstart=Nn-1
N=Nstart
Stepslow=1
Rootsave=10000

IF Kind$=" pp"THEN
      Tst=2*H*K1+2*T*K2
ELSE
      IF Kind$=" ss" THEN
            Tst=2*H*K1+2*T*K2s
      ELSE
            Tst=2*H*K1+T*(K2+K2s)
      END IF
END IF

FOR Stepslow=Stepslow TO Nmodew+1-Extras
   IF Stepslow=1 THEN
      Kstart=Kw(Stepslow+Extras)
      Ks_ip1=MIN(0.9999*K1,0.99999*K2)
   ELSE
      IF Stepslow<Nmodew+1-Extras THEN
            Kstart=Kw(Stepslow+Extras)
            Ks_ip1=Kstart=Kw(Stepslow-1+Extras)
      ELSE
            Kstart=1.E-4
            Ks_ip1=Kw(Stepslow-1+Extras)
      END IF
   END IF

   Num_divs=INT(Onum_divs*(1+Stepslow/(Nmodew+1))/Freq*0.1))
   Kstep=(Ks_ip1-Kstart)/Num_divs

   FOR L=1 TO Num_divs
      Num_iters=0
      Phase=2*N*PI-R1phase
      If Tst<Phase THEN Sub_sed_end
      Ks_ip1=Kstart+(L-1)*Kstep

      REPEAT
            Ks_i=Ks_ip1
            Kappa1=SQR(K1^2-Ks_i^2)
            Kappa2=SQR(K2^2-Ks_i^2)
            Kappa2s=SQR(K2s^2-Ks_i^2)
```

```
            Sign1=SGN(K1^2-Ks_i^2)
            Sign2=SGN(K2^2-Ks_i^2)
            Sign2s=SGN(K2s^2-Ks_i^2)

        IF Kind$=" pp" THEN
            Fs=2*H*Kappa1*Sign1+2*T*Kappa2*Sign2-Phase
            Fsprime=-2*Ks_i*(H/(Kappa1*Sign1)+T/(Kappa2*Sign2))
        ELSE
            IF Kind$=" ss" THEN
                Fs=2*H*Kappa1*Sign1+2*T*Kappa2s*Sign2s-Phase
                Fsprime=-2*Ks_i*(H/(Kappa1*Sign1)+T/(Kappa2s*Sign2s))
            ELSE
                Fs=2*H*Kappa1+T*(Kappa2*Sign2+Kappa2s*Sign2s)-Phase
                Fsprime=-Ks_i*(2*H/(Kappa1*Sign1)+T/(Kappa2*Sign2)
                              +T/(Kappa2s*Sign2s))
            END IF
        END IF
        IF ABS(Fs)>10000 THEN Nxtl

        Ks_ip1=Ks_i-Fs/Fsprime
        IF Ks_ip1>Rootsave THEN Ks_ip1=Rootsave
        Num_iters=Num_iters+1

    UNTIL ABS(Fs)<=1.E-4 OR Num_iters>=Max_iters

    IF Num_iters>=Max_iters THEN
        PRINT "Max iterations"
    ELSE
        IF Ks_ip1>MIN(0.9999*K1,0.99999*K2) THEN Nxtl
        M=N-Nstart+1
        Ks(M)=Ks_ip1
        Rootsave=Ks_ip1

        FOR Mm=1 TO Nmodes
            Testy=ABS(Ks(M)-Ks(Mm))
            IF Testy<=0.00001 THEN Nxtl
        NEXT Mm

        Nmodes=Nmodes+1
        IF Nmodes=1 THEN PRINT

        Phi1=ATN(Kappa1/Ks(M))
        Phi2=ATN(Kappa2/Ks(M))
        Rng=T/SIN(Phi2)
        D=Rho1/Rho2
        P=Kappa2/Kappa1
        Z12=D*P
        Z23=(Rho2/Rho3)*(Kappa3/Kappa2)
        R12ll=ABS((1-Z12)/(1+Z12))
        R23ll=ABS((1-Z23)/(1+Z23))

        SELECT Type$
          CASE "lll"
```

```
    Atten=10^(-2*Rng*Freq*Alfa2/10000)
    S=Atten*(1-R12ll^2)*R23ll
    PRINT USING Frmt1;Ks(M),M,Es(M),Atten,R12ll,R23ll,Phi1
CASE "llm"
    Atten=10^(-2*Rng*Freq*Alfa2/10000)
    S=Atten*(1-R12ll^2)*R23ll
    MagC=R1*S*(1+R1*R12ll+R1^2*R12ll^2)*(1-R12ll*R23ll*Atten+
        R12ll^2*R23ll^2*Atten^2)
    PRINT USING Frmt1;Ks(M),M,Es(M),Atten,R12ll,R23ll,Phi1
CASE "lee"
    SELECT Kind$
    Phi3=ATN(Kappa3/Ks(M))
    Phi2s=ATN(Kappa2s/Ks(M))
    Phi3s=ATN(Kappa3s/Ks(M))
    Rngs=T/SIN(Phi2s)
    B12=1-(SIN(2*Phi2s))^2*(1-Kappa2/Kappa2s)
    R12le=(-Z12+B12)/(Z12+B12)

        T12=2*D*COS(2*Phi2s)/(Z12+B12)
        T21=2*P*COS(2*Phi2s)/(Z12+B12)
        T12ps=-4*D*(COS(Phi2s)^2)*TAN(Phi2)/(Z12+B12)
        T21sp=-4*P*(COS(Phi2s)^2)*TAN(Phi2s)/(Z12+B12)

    R21den=SIN(Phi1)*((C2/C2s)*COS(2*Phi2s)^2+SIN(2*Phi2)*
        SIN(2*Phi2s)*(C2s/C2))+(C2*C1*Rho1*SIN(Phi2))/(C2s*C2*Rho2)
    R21=((SIN(Phi1)*((C2/C2s)*COS(2*Phi2s)^2-SIN(2*Phi2)*
        SIN(2*Phi2s)*(C2s/C2))-(C2*C1*Rho1*SIN(Phi2))/
        (C2s*C2*Rho2)))/R21den
    R21ss=((SIN(Phi1)*((C2/C2s)*COS(2*Phi2s)^2-SIN(2*Phi2)*
        SIN(2*Phi2s)*(C2s/C2))+(C2*C1*Rho1*SIN(Phi2))/
        (C2s*C2*Rho2)))/R21den
    R21ps=-2*SQR(SIN(2*Phi2)*SIN(2*Phi2s))*COS(2*Phi2s)*SIN(Phi1)*
        SIN(2*Phi2)/SIN(2*Phi2s))*(C2s/C2)/R21den
    R21sp=-2*SQR(SIN(2*Phi2)*SIN(2*Phi2s))*COS(2*Phi2s)*SIN(Phi1)*
        SIN(2*Phi2s)/SIN(2*Phi2))*(C2/C2s)/R21den

    Tan2=Kappa2/Ks(M)
    Tan3=Kappa3/Ks(M)
    Tan2s=Kappa2s/Ks(M)
    Tan3s=Kappa3s/Ks(M)
    Mu2s=Rho2*C2s^2
    Mu3s=Rho3*C3s^2

    P1=(2*Mu2s+Mu3s*(Tan3s^2-1))*Tan2*Tan2s
    P2=(Mu2s*(Tan2s^2-1)-Mu3s*(Tan3s^2-1))*Tan2
    P3=(2*Mu3s+Mu2s*(Tan2s^2-1))*Tan3*Tan2s
    P4=2*(Mu3s-Mu2s)*Tan3*Tan2*Tan2s

    Q1=2*(Mu2s-Mu3s)*Tan3*Tan2*Tan2s
    Q2=(2*Mu3s+Mu2s*(Tan2s^2-1))*Tan3s*Tan2
    Q3=((-Mu2s*(Tan2s^2-1))+(Mu3s*(Tan3s^2-1)))*Tan2s
    Q4=P1
```

```
        Denomin=((P1+P3)*(Q2+Q4)-(P2+P4)*(Q1+Q3))
        R23pp=(((P1-P3)*(Q2+Q4)-(P2+P4)*(Q1-Q3))/Denomin)
        R23ss=(((P2-P4)*(Q1+Q3)-(P1+P3)*(Q2-Q4))/Denomin)
        R23sp=(2*(P3*Q1-P1*Q3)/Denomin)
        R23ps=(2*(P4*Q2-P2*Q4)/Denomin)

        CASE " pp"
            Coefs=R1*R23*T12*T21
            Atten=10^(-2*Alfa2*Freq*Rng/10000)
            Es(M)=LOG(ABS(Coefs*Atten))/(2*Ks(M)*(H/Kappa1+T/Kappa2))
            PRINT USING Frmt2;Ks(M),M,Es(M),Atten,R23,T12,T21,Phi1
        CASE " ss"
            Coefs=R1*R23ss*T12ps*T21sp
            Atten=10^(-2*Alfa2s*Freq*Rngs/10000)
            Es(M)=LOG(ABS(Coefs*Atten))/(2*Ks(M)*(H/Kappa1+T/Kappa2s))
            PRINT USING Frmt2;Ks(M),M,Es(M),Atten,R23ss,T12ps,T21sp,Phi1
        CASE " sp"
            Coefs=R1*R23sp*T12ps*T21
            Atten=10^(-(Alfa2*Rng+Alfa2s*Rngs)*Freq/10000)
            Es(M)=LOG(ABS(Coefs*Atten))/(Ks(M)*(H/Kappa1+T/Kappa2+
                T/Kappa2s))
            PRINT USING Frmt2;Ks(M),M,Es(M),Atten,R23sp,T12ps,T21,Phi1
        CASE " ps"
            Coefs=R1*R23ps*T12*T21sp
            Atten=10^(-(Alfa2*Rng+Alfa2s*Rngs)*Freq/10000)
            Es(M)=LOG(ABS(Coefs*Atten))/(Ks(M)*(H/Kappa1+T/Kappa2+
                T/Kappa2s))
            PRINT USING Frmt2;Ks(M),M,Es(M),Atten,R23ps,T12,T21sp,Phi1
        END SELECT
      END SELECT

      N=N+1
      END IF

    Nxtl:!
    NEXT L

  NEXT Stepslow
  PRINT
  Sub_sed_end:!
  SUBEND
```

```
SUB Exact(Kn(*),En(*),Kw(*),Ew(*),Nmoden,Nmodew,Nmodes,Type$,Sym$,Extras,
     S$,OPTIONAL INTEGER Diffcase)

Frmt1:IMAGE 3X,4A,2D,4A,SD.4DE,4X,4A,2D,4A,SD.4DE,2X,2D.D
Frmt2:IMAGE 3X,4A,2D,4A,SD.4DE,4X,4A,2D,4A,SD.4DE,3X,2D.2D,2X,2D.2D,
       3X,3D.D
Frmt3: IMAGE 2X,3D.2D,3X,2D.2D,3X,2D.2D,3X,3D.2D,2X,2D.2D,3X,2D,2D,3X,
       2D.2D,4X,2D.2D,2X,3D.2D
```

```
COM     /Parameter1/    C1, C2, Rho1, Rho2, H, T, Freq, Alfa2, R1, R1phase
COM     /Parameter2/    C2s, C3, C3s, Rho3, Alfa2s
COM     /Wavenumber/    W, K1, K2, K2s, K3s

INTEGER     I, L, M, N, Num_iters, Max_iters, Num_divs, Stepslow

DIM         Wi(2,1), Wip1(2,1), F(2,1), J(2,2), Jinv(2,2)

PRINT
PRINT TAB(19);"EXACT ";Type$;"SOLNS";Sym$
PRINT

Max_iters=100
Wip1(1,1)=0
Wip1(2,1)=0
Nmoden=0
Stepslow=1
Onum_divs=Nmodes/1.2

FOR I=1 TO 90
        Kn(I)=0
        En(I)=0
NEXT I

FOR Stepslow=Stepslow TO Nmodes+1
    SELECT Type$
    CASE "lll"
    IF Stepslow=1 THEN
        Kstart=(Kw(Stepslow+Extras)+Ks(Stepslow+Extras))/2
        Estart=(Ew(Stepslow+Extras)+Es(Stepslow+Extras))/2
        Knext=MIN(0.9999*K1,0.99999*K2)
    ELSE
        IF Stepslow<Nmodes+1 THEN
                Kstart=(Kw(Stepslow+Extras)+Ks(Stepslow+Extras))/2
                Estart=(Ew(Stepslow+Extras)+Es(Stepslow+Extras))/200
                Knext=Ks(Stepslow-1)
        ELSE
                Kstart=1.E-4
                Estart=(Ew(Stepslow+Extras)+Es(Stepslow+Extras))/200
                Knext=Ks(Stepslow-1)
        END IF
    END IF
    CASE "lee"
    IF Stepslow=1 THEN
        Kstart=(Kw(Stepslow+Extras)+Ks(Stepslow+Extras))/2
        Estart=(Ew(Stepslow+Extras)+Es(Stepslow+Extras))/2
        Knext=MIN(0.9999*K1,0.99999*K2)
    ELSE
        IF Stepslow<Nmodes+1 THEN
                Kstart=(Kw(Stepslow+Extras)+Ks(Stepslow+Extras))/2
                Estart=(Ew(Stepslow+Extras)+Es(Stepslow+Extras))/2
                Knext=Ks(Stepslow-1)
        ELSE
```

```
        Kstart=1.E-4
        Estart=(Ew(Stepslow+Extras)+Es(Stepslow+Extras))/2
        Knext=Ks(Stepslow-1)
    END IF
END IF
END SELECT

PRINT
Num_divs=INT(Onum_divs*(1+2*Stepslow/(Nmodes+1)))
Kstep=(Knext-Kstart)/Num_divs

FOR L=1 TO Num_divs
   Num_iters=0
   Wip1(1,1)=Kstart+(L-1)*Kstep
   Wip1(2,1)=Estart
   IF Wip1(1,1)<>0 THEN

     REPEAT
       MAT Wi= Wip1
       GOSUB Function
       IF ABS(F(1,1))>100  OR ABS(F(2,1))>100 THEN Nextl
       GOSUB Jacobian
       MAT Wip1= Jinv*F

       IF Diffcase=1 THEN
          MAT Wip1= Wip1/(1.0)
       ELSE
          MAT Wip1= Wip1/(1.2)
       END IF

       MAT Wip1= Wi-Wip1
       Num_iters=Num_iters+1

       FOR I=1 TO Nmoden
          Test2=ABS(Kn(I)-Wip1(1,1))
          IF Test2<=0.0001 THEN Nextl
       NEXT I

       IF ABS(Wip1(1,1))>1.E+5 THEN Nextl
       IF ABS(Wip1(2,1))>1.E+5 THEN Nextl
       F1=ABS(F(1,1))
       F2=ABS(F(2,1))

     UNTIL F1<=1.E-4 AND F2<1.E-4 OR Num_iters>=Max_iters

     IF Num_iters>=Max_iters THEN
          PRINT "Max Iterations"
     ELSE
          IF Wip1(1,1)<0 THEN Nextl
          IF Phi1<0 Then Nextl
          Nmoden+Nmoden+1
          M=Nmoden
          Kn(M)=Wip1(1,1)
```

```
                En(M)=Wip1(2,1)

                SELECT Type$
                CASE "lll"
                   PRINT USING Frmt2;"Kn(",M,") = ",Kn(M),"En(",M,") = ",En(M),
                      R12ll, R23ll,Phi1*180/PI
                CASE "lee"
                   PRINT USING Frmt1;"Kn1(",M,") = ",Kn(M),"En1(",M,") = ",
                      En(M),Phi1*180/PI
          END SELECT
          IF Nmoden=1 THEN Nextl
          GOTO Nextss
        END IF
      END IF
Nextl:!
      NEXT L
Nextss:!
NEXT Stepslow
GOTO Sub_ex_end

Function:!

    Km=Wi(1,1)
    Em=Wi(2,1)
    IF Km=0 THEN Nextl

    Kappa1=SQR(K1^2-Km^2)
    Kappa2=SQR(K2^2-Km^2)
    Kappa2S=SQR(K2S^2-Km^2)
    Sign1=SGN(K1^2-Km^2)
    Sign2=SGN(K2^2-Km^2)
    Sign2s=SGN(K2s^2-Km^2)

    IF ABS(Km)>=ABS(K3) THEN
        Kappa3=0
    ELSE
        Kappa3=SQR(K3^2-Km^2)
    END IF

    Kappa1=Kappa1*Sign1
    Kappa2=Kappa2*Sign2
    Kappa2s=Kappa2s*Sign2s

    Phi1=ATN(Kappa1/Km)
    Phi2=ATN(Kappa2/Km)
    Rng=T/SIN(Phi2)
    D=Rho1/Rho2
    P=Kappa2/Kappa1
    Z12=D*P
    Z23=(Rho2/Rho3)*(Kappa3/Kappa2)
    R12ll=ABS((1-Z12)/(1+Z12))
    R23ll=ABS((1-Z23)/(1+Z23))
```

```
Aa=2*Kappa1*H+R1phase
Bb=2*Kappa2*T
Ab=Aa+Bb
Saa=SIN(Aa)
Sab=SIN(Ab)
Caa=COS(Aa)
Cab=COS(Ab)

G=Km*Em
G1=H/Kappa1
G2=T/Kappa2
G12=G1+G2
Gf=2*G*G1
Fact1=-2*Km*G1
Fact2=-2*Km*G2
Fact12=Fact1+Fact2
Ex2=-T*G/Kappa2
Ex4=2*Ex2
IF ABS(Ex4)>10  OR  ABS(Gf)>10 THEN Nextl

Y1=Km/Kappa1
Y2=Km/Kappa2
Y=1+Y1^2
Y4=1+Y2^2

B1=-2*Alfa2*Rng*Freq/10000
Fg=G1*EXP(Gf)

SELECT Type$
CASE "lll"
    S=10^(B1)*(1-R12ll^2)*R23ll*EXP(Ex4)
    S2=R12ll*R23ll*10^(B1)*EXP(Ex4)
    Den=1+S2^2+2*S2*COS(Bb)
    F(1,1)=R12ll*Saa+S*(Sab+Saa*S2)/Den
    F(2,1)=EXP(Gf)-R12ll*Caa-R1*S*(Cab+S2*Caa)/Den
CASE "lee"
    IF ABS(Km)>=ABS(K3s) THEN
        Kappa3s=0
    ELSE
        Kappa3s=SQR(K3s^2-Km^2)
    END IF
    Phi3=ATN(Kappa3/Km)
    Phi2s=ATN(Kappa2s/Km)
    Phi3s=ATN(Kappa3s/Km)
    Rngs=T/SIN(Phi2s)

    Cc=2*Kappa2s*T
    Dd=Kappa2*T+Kappa2s*T
    Ac=Aa+Cc
    Sac=SIN(Ac)
    Sad=SIN(Ad)
    Cac=COS(Ac)
    Cad=COS(Ad)
```

```
G2s=T/Kappa2s
G12s=G1+G2s
Gs=2*G1+G2+G2s
Ex2s=-T*G/Kappa2s
Ex4s=2*Ex2s
Ex=Ex2+Ex2s
Y2s=Km/Kappa2s
Y4s=1+Y2s^2

Cc1=-2*Alfa2s*Rngs*Freq/10000
D1=-(Alfa2*Rng+Alfa2s*Rngs)*Freq/10000
B12=1-(SIN(2*Phi2s))^2*(1-Kappa2/Kappa2s)
R12le=(-z12+B12)/(z12+B12)

T12=2*D*COS(2*Phi2s)/(Z12+B12)
T21=2*P*COS(2*Phi2s)/(Z12+B12)
T12ps=-4*D*(COS(Phi2s)^2)*TAN(Phi2)/(Z12+B12)
T21sp=-4*P*(COS(Phi2s)^2)*TAN(Phi2s)/(Z12+B12)

R21den=SIN(Phi1)*((C2/C2s)*COS(2*Phi2s)^2+SIN(2*Phi2)*SIN(2*Phi2s)*
       (C2s/C2))+(C2*C1*Rho1*SIN(Phi2))/(C2s*C2*Rho2)
R21=((SIN(Phi1)*((C2/C2s)*COS(2*Phi2s)^2-SIN(2*Phi2)*SIN(2*Phi2s)*
       (C2s/C2))-(C2*C1*Rho1*SIN(Phi2))/(C2s*C2*Rho2)))/R21den
R21ss=((SIN(Phi1)*((C2/C2s)*COS(2*Phi2s)^2-SIN(2*Phi2)*SIN(2*Phi2s)*
       (C2s/C2))+(C2*C1*Rho1*SIN(Phi2))/(C2s*C2*Rho2)))/R21den

Tan2=Kappa2/Km
Tan3=Kappa3/Km
Tan2s=Kappa2s/Km
Tan3s=Kappa3s/Km
Mu2s=Rho2*C2s^2
Mu3s=Rho3*C3s^2

P1=(2*Mu2s+Mu3s*(Tan3s^2-1))*Tan2*Tan2s
P2=(Mu2s*(Tan2s^2-1)-Mu3s*(Tan3s^2-1))*Tan2
P3=(2*Mu3s+Mu2s*(Tan2s^2-1))*Tan3*Tan2s
P4=2*(Mu3s-Mu2s)*Tan3*Tan2*Tan2s

Q1=2*(Mu2s-Mu3s)*Tan3*Tan2*Tan2s
Q2=(2*Mu3s+Mu2s*(Tan2s^2-1))*Tan3s*Tan2
Q3=((-Mu2s*(Tan2s^2-1))+(Mu3s*(Tan3s^2-1)))*Tan2s
Q4=P1

Denomin=((P1+P3)*(Q2+Q4)-(P2+P4)*(Q1+Q3))
R23pp=(((P1-P3)*(Q2+Q4)-(P2+P4)*(Q1-Q3))/Denomin)
R23ss=(((P2-P4)*(Q1+Q3)-(P1+P3)*(Q2-Q4))/Denomin)
R23sp=(2*(P3*Q1-P1*Q3)/Denomin)
R23ps=(2*(P4*Q2-P2*Q4)/Denomin)

A=ABS(R1*R12le)
B=ABS(R1*R23*T12*T21)
C=ABS(R1*R23ss*T12ps*T21sp)
```

```
        D=ABS(R1*R23sp*T12ps*T21)
        E=ABS(R1*R23ps*T12*T21sp)

        B2=10^(B1)*EXP(Ex4)
        Cc2=10^(Cc1)*EXP(Ex4s)
        D2=10^(D1)*EXP(Ex)

        Pp=R21*R23*10^(B1)*EXP(Ex4)
        Ss=R21ss*R23ss*10^(Cc1)*EXP(Ex4s)

        Deno1=1-2*Pp*COS(Bb)-Pp^2
        Deno2=1-2*Ss*COS(Cc)-Ss^2

        F1wat=A*EXP(-Gf)*Saa
        F1sed=(B*10^(B1)/Deno1)*(EXP(-2*G*G12)*Sab-Pp*EXP(-Gf)*Saa)
        F1sedsp=(D*10^(D1)/Deno1)*(EXP(-G*Gs)*Sad-Pp*EXP(-G*(2*G1+G2s-
              G2))*SIN(Aa+Cc/2-Bb/2))
        F1sedss=(C*10^(Cc1)/Deno2)*(EXP(-2*G*G12s)*Sac-Ss*EXP(-Gf)*SIN(Aa))
        F1sedps=(E*10^(E1)/Deno2)*(EXP(-G*Gs)*Sad-Ss*EXP(-G*(2*G1-
              G2s+G2))*SIN(Aa-Cc/2+Bb/2))

        F2wat=A*Caa
        F2sed=(B*10^(B1)/Deno1)*(EXP(-2*G*G12)*Cab-Pp*EXP(-Gf)*Caa)
        F1sedsp=(D*10^(D1)/Deno1)*(EXP(-G*Gs)*Cad-Pp*EXP(-G*(2*G1+G2s-
              G2))*COS(Aa+Cc/2-Bb/2))
        F1sedss=(C*10^(Cc1)/Deno2)*(EXP(-2*G*G12s)*Cac-Ss*EXP(-Gf)*
              COS(Aa))
        F1sedps=(E*10^(E1)/Deno2)*(EXP(-G*Gs)*Cad-Ss*EXP(-G*(2*G1-
              G2s+G2))*COS(Aa-Cc/2+Bb/2))

        F(1,1)=F1wat+F1sed+F1sedss+F1sedsp+F1sedps
        F(2,1)=1-F2wat*EXP(-Gf)-F1sed-F1sedss-F1sedsp-F1sedps
    END SELECT
RETURN

Jacobian:!

    SELECT Type$
    CASE "lll"
        J(1,1)=Fact1*R12ll*Caa+S*(EXP(EX4)*(Cab*(Fact1+Fact2)-Sab*2*Em*Y4))
        J(1,2)=S*EXP(Ex4)*Sab*Fact2
        J(2,1)=EXP(Gf)*2*Em*G1*Y≠R1*R12ll*Saa*Fact1+R1*S*EXP(Ex4)*
              (Cab*2*Em*G2*Y4+Sab*(Fact1+Fact2))
        J(2,2)=-EXP(Gf)*Fact1-R1*S*EXP(Ex4)*Cab*Fact2
    CASE "lee"
        Fact2s=-2*Km*G2s
        Fact12=Fact1+Fact2
        Fact12s=Fact1+Fact2s
        Dfact=EXP(-G*Gs)*(Fact1+Fact2/2+Fact2s/2)
        J(1,1)=-A*EXP(-Gf)*Caa*Fact1+B*10^(B1)*(EXP(-2*G*G12)*Cab*Fact12-
              Pp*EXP(-Gf)*Caa*Fact1)/Deno1+C*10^(Cc1)*(EXP(-2*G*G12s)*Cac*
              Fact12-Ss*EXP(-Gf)*Caa*Fact1)/Deno2+D*10^(D1)*(Dfact*Cad-Pp*
              EXP(-G*(2*G1+G2s-G2))*COS(Aa+Cc/2-Bb/2)*(Fact1+Fact2s/2-
```

```
            Fact2/2)))/Deno1+E*10^(D1)*(Dfact*Cad-Ss*EXP(-G*(2*G1-G2s+G2))
            *COS(Aa-Cc/2+Bb/2)*(Fact1-Fact2s/2+Fact2/2)))/Deno2
      J(1,2)=A*EXP(-Gf)*Saa*Fact1+B*10^(B1)*(EXP(-2*G*G12)*Sab*Fact12-
            Pp*EXP(-Gf)*Saa*Fact1)/Deno1+C*10^(Cc1)*(EXP(-2*G*G12s)*Sac*
            Fact12-Ss*EXP(-Gf)*Saa*Fact1)/Deno2+D*10^(D1)*(Dfact*Sad-Pp*
            EXP(-G*(2*G1+G2s-G2))*SIN(Aa+Cc/2-Bb/2)*(Fact1+Fact2s/2-
            Fact2/2)))/Deno1+E*10^(D1)*(Dfact*Sad-Ss*EXP(-G*(2*G1-G2s+G2))
            *SIN(Aa-Cc/2+Bb/2)*(Fact1-Fact2s/2+Fact2/2)))/Deno2
      J(2,1)=J(1,2)
      J(2,2)=-J(1,1)
    END SELECT
    MAT Jinv= INV(J)
    RETURN

Sub_ex_end:!
SUBEND

----------------------------------------------------------------------------------------

SUB Nat_log_e(E(*),Ln_e(*),Nmode)

    INTEGER I
    FOR I=1 TO Nmode
        Ln_e(I)=LOG(ABS(E(I)))
    NEXT I
SUBEND

----------------------------------------------------------------------------------------

SUB Ln_e_vs_k_plot(K(*),E(*),Ln_e(*),Nmode,Color,Mark$)

    CSIZE 2,4
    PEN Color
    LORG 5

    FOR I=Nmode TO 1 STEP -1
        MOVE K(I),-Ln_e(I)
        LABEL Mark$
    NEXT I

SUBEND

----------------------------------------------------------------------------------------

SUB Incohrnt(Kw(*),Ew(*),Ks(*),Es(*),Kn(*),En(*),Nmodew,Nmodes,Nmoden,
        Indbex,Extras)

Format1: IMAGE 4X,56A,4D.D,X,2A
COM     /Wavenumber/  W, K1, K2, K2s, K3s
INTEGER     I,R,Z

Z=100
R=10000
```

```
Insumw=0
Insums=0
Insumn=0


FOR I=Extras+1 TO Nmodew
      Kappa1=SQR(K1^2-Kw^2)
      Incow=((COS(Kappa1*Z)*SIN(Kappa1*Z)*EXP(-ABS(Ew(I))*R))/Kw(I)^1.5)^2
      Insumw=Insumw+Incow
NEXT I

FOR I=1 TO Nmodes
      Kappa1=SQR(K1^2-Ks^2)
      Incos=((COS(Kappa1*Z)*SIN(Kappa1*Z)*EXP(-ABS(Es(I))*R))/Ks(I)^1.5)^2
      Insums=Insums+Incos
NEXT I

FOR I=1 TO Nmoden
      Kappa1=SQR(K1^2-Kn^2)
      Incon=((COS(Kappa1*Z)*SIN(Kappa1*Z)*EXP(-ABS(En(I))*R))/Kn(I)^1.5)^2
      Insumn=Insumn+Incon
NEXT I

Insumall=Insumw+Insums
Indball=10*LGT(Insumall)
Indbex=10*LGT(Insumn)

PRINT
PRINT USING Format1;"Incoherent (Water + Sediment) - Incoherent Exacts ",Indball-
        Indbex,"dB"

SUBEND

-----------------------------------------------------------------------------------------------------

SUB Fouraxes(Color, INTEGER Diffcase)

COM    /Parameter1/    C1, C2, Rho1, Rho2, H, T, Freq, Alfa2, R1, R1phase
COM    /Wavenumber/    W, K1, K2, K2s, K3s

DEG
LORG 5
PEN Color

Kmax=INT(Freq/2)*1.E-2
SELECT Diffcase
CASE 1
        VIEWPORT 20,65,47,72
CASE 2
        VIEWPORT 67,112,47,72
CASE 3
        VIEWPORT 67,112,21,46
CASE 4
```

```
        VIEWPORT 20,65,21,46
END SELECT

WINDOW 0,Kmax,0,7
AXES 0,2,0,1,1
MOVE 0.05,1
PEN -1
DRAW Kmax,1
PEN Color
CSIZE 3,.4

FOR J=1 TO 10
        Ang=COS(10*(J-1))*K1
        MOVE Ang,1.25
        DRAW Ang,1
        IF J=1 THEN
                PEN -1
                MOVE Ang+Kmax/35,1
                DRAW Kmax,1
                PEN Color
        END IF
        IF Diffcase=3  OR  Diffcase=4 THEN
                IF J=1 THEN
                        MOVE Ang,.3
                        LABEL "0"
                END IF
                IF J=4 THEN
                        MOVE Ang,.3
                        LABEL "30"
                END IF
                IF J=7 THEN
                        MOVE Ang,.3
                        LABEL "60"
                END IF
                IF J=10 THEN
                        MOVE Ang,.3
                        LABEL "90"
                END IF
        END IF
NEXT J
SUBEND
```

```
SUB Label4axes(Color)

VIEWPORT   0,131,0,100
CSIZE      2.7
LORG       5
DEG

PEN Color
MOVE 31,72
```

```
LABEL "FINE FLUID'
MOVE 75,72
LABEL "MEDIUM"
MOVE 26,46
LABEL "HARD"
MOVE 84,46
LABEL "SEMICONSOLIDATED"

CSIZE 3
MOVE 62,18
LABEL "GRAZING ANGLE"
LDIR 90
MOVE 9,49
LABEL "-ln(ABS(En))"

LDIR 0
CSIZE 3,.4
MOVE 18,72
LABEL "7"
MOVE 18,51
LABEL "1"
MOVE 18,46
LABEL "7"
MOVE 18,25
LABEL "1"

SUBEND
------------------------------------------------------------------------------------------

SUB Sort(K(*),E(*),Nmode)

INTEGER      I, N
FOR N=1 TO Nmode-1
    FOR I=N+1 TO Nmode
        IF K(N)>K(I) THEN
                Junk=K(N)
                K(N)=K(I)
                K(I)=Junk
                Dummy=E(N)
                E(N)=E(I)
                E(I)=Junk
        END IF
    NEXT I
NEXT N
SUBEND
```