Minimum Covers for Grids and Orthogonal Polygons by Periscope Guards

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Abstract: The problem of finding minimum guard covers is NP-hard for simple polygons [LL84,A84] and open for simple orthogonal polygons. Alternative definitions of visibility have been considered for orthogonal polygons. In this paper we try to determine the complexity of finding guard covers in orthogonal polygons by considering periscope visibility. Under periscope visibility two points in an orthogonal polygon are visible if there is an orthogonal path with at most one bend that connects them without intersecting the exterior of the polygon. Periscope visibility is the closest to standard visibility among various alternatives that have been proposed. We show that finding minimum periscope guard (as well as k-bend and s-guard) covers is NP-hard for 3-d grids. ~~We present~~ an $O(n^3)$ algorithm for finding minimum periscope guard covers in a simple grid with n segments. ~~We also show that~~ this result can be used to obtain minimum periscope guard covers for a class of simple orthogonal polygon in $O(n^3)$.

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The problem of finding minimum guard covers is NP-hard for simple polygons [LL84,A84] and open for simple orthogonal polygons. Alternative definitions of visibility have been considered for orthogonal polygons. In this paper we try to determine the complexity of finding guard covers in orthogonal polygons by considering periscope visibility. Under periscope visibility two points in an orthogonal polygon are visible if there is an orthogonal path with at most one bend that connects them without intersecting the exterior of the polygon. Periscope visibility is the closest to standard visibility among various alternatives that have been proposed. We show that finding minimum periscope guard (as well as k-bend and s-guard) covers is NP-hard for 3-d grids. We present an $O(n^3)$ algorithm for finding minimum periscope guard covers in a simple grid with n segments. We also show that this result can be used to obtain minimum periscope guard covers for a class of simple orthogonal polygon in $O(n^3)$.

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I. Introduction

The problem of covering a polygon with simpler polygons has attracted the interest of many researchers [A84,O'R87,K86,C'R87]. One such problem is finding the minimum number of star polygons needed to cover a given polygon (a star polygon is such that there exists a point in the polygon from which all points in the polygon are visible). Covering a polygon with the minimum number of star polygons (minimum star cover) is equivalent to the placement of the minimum number of point guards (minimum guard cover) so that each point inside the polygon is visible to some guard. This problem was shown to be intractable (NP-hard [GJ79]) for polygons with holes in [O'RS83]. The problem was shown to remain intractable for simple polygons in [LL84, A84]. The complexity of finding minimum guard covers for simple orthogonal polygons remains an open question. Many of the results on guard covers and the related Art Gallery problem can be found in [O'R87].

Because of the intractability of most minimum guard cover problems, restricted classes of polygons and different definitions of visibility have been considered [K86,O'R87,MRS88]. In the standard definition of visibility two points are said to be visible if the straight line segment joining them does not intersect the exterior of the polygon. An important class of polygons is that of orthogonal polygons. A polygon is an **orthogonal polygon** if all its edges are parallel to the major axes. Alternative definitions of visibility have been proposed in connection to orthogonal polygons. Two points inside an orthogonal polygon are said to be **s-visible** [MRS88] if they can be joined by an orthogonally convex staircase path that does not intersect the exterior of the polygon. Two points are said to be **r-visible** [K86] if they can be placed inside an orthogonal rectangle that is completely contained in the polygon. The notions of s-visibility and r-visibility directly lead to **s-star** and **r-star polygons.**

Polynomial time algorithms for solving restricted versions of the guard cover problem in simple orthogonal polygons have been reported in [K86,MRS88]. Keil [K86] gave an $O(n^2)$ time algorithm for minimally covering a horizontally convex orthogonal polygon with the minimum number of r-star polygons. A faster implementation of the same approach that runs in optimal $O(n)$ time is reported in [GN88]. In [MRS88], an $O(n^3)$ time algorithm for covering an orthogonal polygon that has only three (out of the possible four) dent orientations with the minimum number of s-star polygons is presented. They also give an $O(n^{10})$ time algorithm for the case when the polygon has dents in all four directions.

A structure that is often associated with polygons and has found many applications in Computational Geometry is a grid (e.g., the grid induced by the polygon's edges/vertices). If we think of the grid edges as corridors, the star cover problem in a grid is to find the minimum number of guards that need to be stationed in the grid so that each point in the grid is visible to some guard. Finding a minimum guard cover in a three dimensional grid is NP-hard but a minimum cover for a two dimensional grid can be found in $O(n^{2.5})$ time [N86] (where n is the number of segments in the grid).

In this paper we address the problem of finding minimum star covers under periscope visibility. Two points are visible under **periscope visibility** if there is an orthogonal path with at most one bend connecting them without intersecting the exterior of the polygon. Generalizing, **k-bend visibility** allows staircase paths with at most k bends (periscope visibility is the same as 1-bend visibility). If k can have any value but the paths are restricted to be orthogonally convex, we have s-visibility. In the next section we show that finding a minimum periscope guard cover for a three dimensional grid is NP-hard. Also, finding minimum k-bend guard covers and s-guard covers are NP-hard problems. In section III we present an $O(n^3)$ time algorithm for finding the minimum number of periscope guards needed to cover a simple 2-d grid (simple grids are closely related to orthogonal polygons). In section IV we adapt this result to the problem of finding the minimum number of periscope guards needed to cover a simple orthogonal polygon and obtain $O(n^3)$ algorithms for finding minimum periscope guard covers for a class of orthogonal polygons that includes monotone and orthogonally convex polygons.

II. Periscope Guard Covers for 3-d Grids.

The complete two-dimensional grid of size n is the graph with vertex set $V = \{1,2,...,n\} \times \{1,2,...,n\}$ and edge set $E = \{\{(i,j), (k,m)\} : |i-k| + |j-m| = 1\}$. The complete 3-d grid is defined similarly. A (partial) grid is any subgraph of the complete grid. In a geometric setting we think of the grid edges as corridors and the grid vertices as intersections of corridors. We also assume that the grid edges are parallel to the major axes. Finding a minimum set of guards needed to cover (under normal visibility) a 3-d grid is NP-hard [N86]. The reduction is from the vertex cover problem for graphs with maximum degree three [GJ79,O'R87]. We use a similar approach to show that the minimum cover problems for periscope guards, k-bend guards and s-guards are NP-hard for 3-d grids.

Vertex Cover:
Instance: Graph $G = (V,E)$ with all vertices having degree three or less, positive integer $k < |V|$.

Question: Is there a vertex cover of size $\leq k$ for G ? (I.e., a set of vertices such that each edge in G is incident on at least one vertex in the set).

Periscope Guard Cover for 3-d Grid:
Instance: A three dimensional grid with n segments, integer m.
Question: Is there a positioning of m periscope guards in the grid so that every point in the grid is visible to at least one guard?

Given an instance of vertex cover we construct an instance of guard cover as follows. Index the vertices of G arbitrarily from 1 to $|V|$. We construct a three dimensional grid Q such that Q can be covered by less than or equal to m guards (m to be determined later) if and only if there is a vertex cover with at most k vertices in G. We start with a full 3-d grid of size $6|V|$ and assign the vertices of G to grid vertices so that the vertex v_i is assigned to the grid vertex $(6i, 6i, 6i)$. An edge (v_i, v_j) of G is represented by a grid path connecting the corresponding grid points. Grid paths for the edges need to be disjoint, consist of $4x + 2$ segments (for some integer $x \geq 1$) and the three paths corresponding to the three edges incident on a vertex of G need to be orthogonal to each other in the immediate neighborhood of the grid vertex where they meet. Once a path that satisfies these requirements is constructed for each edge, all grid edges and vertices that are not used in these paths are removed. The resulting grid is Q. The two types of grid paths shown in Figure 1 are sufficient to make all the connections. We refer to them as **short** (six segments) and **long** (fourteen segments) paths respectively. Short paths are used as much as possible: long paths are needed to bend around a previous connection so that all connections to a grid vertex that represents one of the original graph vertices will occupy a distinct direction in the immediate vicinity of the vertex.

Theorem 1: There is a vertex cover of size k in G if and only if there is a solution to the corresponding grid cover problem of size $k + \sum s_i$, where $s_i = 1$ or 3 depending on whether the i^{th} edge corresponds to a short or a long path, respectively.

Proof: Assume there is a vertex cover of size k. Position k guards at grid vertices that correspond to the vertices in the vertex cover. Then position the remaining guards at every fourth corner in the six or fourteen bend paths (Figure 1) starting from the position of the guards corresponding to the vertex cover (if both ends of a path correspond to vertices in the vertex cover, arbitrarily select one of them). One additional guard is needed in every short path and three additional guards are needed in every long path.

Conversely, assume that a solution of size $k + \sum s_i$ for the grid cover problem exists. We can obtain a solution to the vertex cover problem from the solution of the grid cover problem as follows. Note that at least one guard per short path and three guards per long path are needed in their interior to cover them (in addition to guards at the endpoints of paths). If we spread these guards as shown in Figure 1, the remaining k guards can be shifted to the grid vertices corresponding to the vertices of G without disturbing the cover. These vertices must constitute a vertex cover in G. If not, there must be an edge in G such that the corresponding short (long) grid path is guarded by just one (three) guard which is impossible since the path consists of six (fourteen) segments. **Q.E.D.**

Corollary 1: Finding the minimum number of periscope guards needed to cover a 3-d grid is NP-hard.

Consider now k-bend guards. We can use the same approach but use grid paths consisting of $2(k+1)x + k+1$ ($x \geq 1$) segments for each edge in G. For s-guards, there is no upper bound on the size of the staircase connecting two points (so that they are visible). However, the staircase is restricted to be orthogonally convex. We can use this requirement to construct grid paths for the edges in G that will consist of $4x + 2$ orthogonally convex staircases connected so that no sequence of two or more staircases is orthogonally convex. Then the same arguments as in Theorem 1 can be used to show that the minimum cover problem for s-guards in 3-d grids is NP-hard.

Corollary 2: The minimum cover problems for k-bend guards and s-guards in 3-d grids are NP-hard.

III. Minimum Periscope Guard Covers for Simple 2-D Grids.

In this section we consider the periscope guard cover problem for a class of 2-d grids (simple grids) which can be used to model the periscope guard problem in simple orthogonal polygons.

A **grid segment** is any maximal straight line sequence of successive grid edges. A grid is called a **simple grid** if all the endpoints of its segments lie on the outer face of the planar subdivision formed by the grid (as in Figure 2a); otherwise, the grid is called a **general grid** (a general grid may have holes as in Figure 2b). The **crossing set** C_i of a segment s_i is the set of segments that intersect s_i. A segment s_1 is said to be **dominated** by a segment s_2 if C_1 is a subset of C_2. In Figure 2a, segments c and d are dominated by segment e. A segment s is called a **cross** if there exists a segment s_1 such that the crossing set of s_1 contains only s. A segment is called a **pseudo cross** if it becomes a cross by removing zero or more segments dominated by it (note that every cross is also a pseudo cross). A segment is called **prime** if it is neither a pseudo cross nor a dominated segment. Two segments are **equivalent** if their crossing sets are the same. In figure 2a, segments 1, 4, 5 are crosses, segment e is a pseudo cross, segment 3 is prime and segments b, g are equivalent. Note that a periscope guard that can see a segment s can see all segments equivalent to s. Therefore we keep only one segment from each set of equivalent segments.

The importance of domination is illustrated in Figure 2a. A guard located on a dominated segment can be moved to the dominating segment and still see all segments visible from its original position (as well as some additional ones). For example, a guard at point x can be moved to point y in Figure 2a and still see all the segments visible from x. This indicates that certain segments are more important than others. We capture this idea by defining a reduced grid. Let G be a simple grid. Mark all segments that are dominated in G. The grid obtained by removing all marked segments is called the **reduced grid**. Figure 3 shows a grid with the dominated segments marked and the reduced grid obtained by removing them.

Lemma 1: The reduced grid of any simple and connected grid is simple and connected.

Proof: All end points of the original grid are in the outer face. After removing dominated segments, all the remaining end points still lie in the outer face and the reduced grid is simple. All segments connected by a dominated segment are also connected by the dominating segment. Therefore removal of a dominated segment does not disconnect the grid and the reduced grid is connected. **Q.E.D.**

Two points on a grid are visible under periscope visibility if they lie on the same segment or the segments on which they lie intersect. Points lying on the same segment are defined to be **directly visible** while points on intersecting segments are said to be **indirectly visible.** Given a simple grid we are interested in finding the minimum number of periscope guards that need to be placed in the grid so that each point in the grid is visible to at least one guard. A set of guards that can see the whole grid is called a **guard cover** for the grid.

Our approach for finding a minimum guard cover for a simple grid is to identify places where any optimum solution should have a guard, place a guard, remove a portion of the grid and repeat until all of the grid is visible to a guard. Let R, R' be two guard covers for a simple grid. We say that a guard g_i in R is **equivalent** to a guard g_j in R' if the two guards see exactly the same set of grid segments. A guard g_i **covers** a guard g_j if the set of segments visible to g_i contains the set of segments visible to g_j. To obtain a minimum guard cover we locate each guard so that any minimum guard cover will contain a guard equivalent to it or covered by it.

Lemma 2: There exists an optimum guard cover in which all guards are located at grid vertices on segments that are not dominated.

Proof: We can adjust any guard cover to obtain one that has an equal (or smaller) number of guards and satisfies the conditions. If guard g_i is located in the interior of a grid edge, a guard g_j located at either endpoint of that edge covers g_i and we can replace g_i with g_j in the guard cover without increasing its size. Similarly, if g_i is on a dominated segment, there is a guard g_j on a parallel dominating segment (that is not itself dominated) that covers g_i. **Q.E.D.**

Lemma 3: There is an optimum guard cover in which every pseudo cross has a guard along it.

Proof: From the definition of a pseudo cross we have that there exists a segment that intersects only the pseudo cross or the pseudo cross and segments dominated by it. To see this segment any solution must have a guard either along it or on one of the segments that intersect it. In the first case we move the guard to the intersection of the segment with the pseudo cross. In the second case the guard is on a dominated segment: if we move the guard to the pseudo cross (dominating segment), the guard still sees all segments seen before (and some additional ones). **Q.E.D.**

Let $C = \{s_1, s_2, \ldots, s_k\}$ be the crossing set of a segment s in the reduced grid. The crossing set C is said to form a **group** if there exists a segment $s' \in C$ such that s' is dominated by all segments in C. Then s' is called a **junior segment** in C. In Figure 3, the crossing set for segment d forms a group and segment 6 is the junior segment in this group. Another group is formed by the crossing set for segment 3 with segment i or segment f as the junior segment (junior segments are not unique).

Lemma 4: Let s_l (respectively s_r) be the left most (right most) segment in the crossing set of a horizontal segment s in the reduced grid. Let s' be a segment of minimum rank (i.e., smallest crossing set) in the crossing set of s. Let s_t (s_b) be the top most (bottom most) segment in the crossing set of s'. Then the crossing set of s forms a group if and only if segments s_l, s_r, s_t, s_b intersect to form a rectangle (as in Figure 4). A similar property holds for any vertical segment. Note that the rectangle may be degenerate, i.e., it may be a segment or a point.

Proof: A junior segment in a group must have the smallest crossing set. Since it is dominated by all other segments in the crossing set of s we must have that s_l, s_k (and all other segments intersecting s') must intersect all segments in the crossing set of s; thus s_l, s_r, s_t, s_b intersect to form a rectangle. The rectangle degenerates to a segment when s_t, s_b are the same and to a point when the crossing set of s contains only s'. **Q.E.D.**

Lemma 5: The crossing set of a prime segment s in the reduced grid can not form a group.

Proof: Assume to the contrary that the crossing set of s forms a group. Let the junior segment in the crossing set of s be s'. If s' intersects only s, s cannot be prime (it will be a pseudo cross), a contradiction. If s' intersects s and some other segment s_1, the crossing set of s_1 must contain the crossing set of s (otherwise s' will not be the junior segment). But then s_1 will dominate s implying that s is not a prime segment, a contradiction. **Q.E.D.**

Lemma 6: Let G_r be the reduced grid of a simple grid G. Let s be a pseudo cross segment in G_r such that its crossing set $C = \{s_1, s_2, \ldots, s_k\}$ forms a group. Let $s_i \in C$ be a junior segment in the group. Then there exists an optimum guard cover for G that contains a guard equivalent to a guard placed at the intersection of s and s_i.

Proof: Let r be the minimum number of guards needed to cover the grid and let Q be a placement of r guards that covers the grid. We prove the lemma by showing that Q can be rearranged so that a guard is placed at the intersection of s and s_i and the resulting guard set still covers the grid. First note that s_i must be a pseudo cross in the original grid (dominated segments do not appear in G_r and prime segments can not be junior segments). Now consider the crossing set of s_i.

Case 1: (The crossing set of s_i contains only s). Since s_i is a pseudo cross, there is a guard along it. We can move this guard to the intersection of s and s_i (see Figure 5a) without affecting the coverage.

Case 2: (The crossing set of s_i contains more than one segment). Since both s and s_i are pseudo crosses, Q must have at least one guard along each of them or on the segments dominated by them. If a guard is on a dominated segment then we can safely move it to the pseudo cross that dominates it. If a guard is already at the point of intersection of s and s_i we are done. Otherwise, let guards g_1, g_2 be at the intersection of s and s'', s_i and s', respectively. Now observe that g_1 and g_2 must be at opposite corners of the rectangle formed by the segments s, s_i, s' and s'' (Figure 5b). If such a rectangle does not exist then s_i can not be a junior segment (contradicting Lemma 4). We can move g_1 and g_2 to the other two corners of the rectangle without affecting the set of segments visible to them (collectively). **Q.E.D.**

From G_r we can construct two trees, the horizontal segment tree $T_h = (V_h, E_h)$ and the vertical segment tree $T_v = (V_v, E_v)$. T_h is constructed as follows (the construction for T_v is similar): If two or more horizontal grid segments are equivalent then we treat them as a single horizontal segment. The set of vertices V_h is, $V_h = \{v |\ v$ is a horizontal segment in G_R $\}$. Two vertices v_1,v_2 are connected by an edge (v_1, v_2) if the corresponding horizontal segments are neighbors, i.e., there is a vertical segment that intersects both of them without intersecting any other horizontal segments between them. Note that T_h, T_v are trees because G_r is a simple grid.

We use T_h (or T_v) to find a segment whose crossing set forms a group. We modify T_h as follows: A branching node in T_h is a node that has degree three or higher. A leaf of T_h may be connected to the nearest branching node directly or through a path that visits vertices of degree two. If, along such a path, the father f and grandfather g of a leaf segment s are such that g dominates f and f dominates s we make both f and s children of g and repeat this process as needed. The resulting tree is the **modified horizontal segment tree** T_{hm}. Figure 6b shows T_h for the reduced grid of Figure 6a and Figure 6c shows the modified tree T_{hm}.

Lemma 7: The reduced grid G_r of any simple grid G contains a segment whose crossing set forms a group.

Proof: Every leaf segment of T_h (or T_v) is a pseudo cross. Consider a leaf l in T_{hm}. If l is neither dominated nor equivalent to another segment there must be a segment s such that s intersects with l but not with the father of l. This means that the crossing set of l forms a group with s as the junior segment. If l is not dominated but is equivalent to another segment, again there is a segment s that intersects l and segments equivalent to it only. Then l forms a group with s as the junior segment.

If all the leaves in T_{hm} are dominated, let T' be the tree obtained by removing the leaf nodes of T_{hm}. Consider now a node f in T_{hm} that appears as a leaf in T'. Node f in T_{hm} may have many leaves as its sons. A leaf segment s corresponding to a son of f is called an **upper leaf segment** (respectively **lower leaf segment**) if the y-coordinate of s is greater (less) than the y-coordinate of f. If we trace the portions of the upper leaf segments of f that lie on the outer face of the planar subdivision induced by the grid (disregarding vertices of degree one) we will get a Manhattan sky line pattern as shown in Figure 7b. Similarly, an upside down Manhattan sky line will be formed if the lower leaf segments of f are traced. (Observe that if they do not form a Manhattan sky line, then some son of f will not be dominated by f, a contradiction). Now the leaf segment corresponding to the left most or right most peak of the sky line is a pseudo cross and its crossing set forms a group. **Q.E.D.**

We are now ready to describe the algorithm for finding a minimum periscope guard cover for a simple 2-d grid. The dominated segments in the given grid G are marked and a reduced grid G_r is obtained from G by removing dominated segments. Lemma 7 guarantees that at least one segment of G_r is such that its crossing set forms a group. Once such a segment is found, the location of a guard g that corresponds to an optimum solution is determined by using Lemma 6 and visible segments in G are marked. We could remove from G the segments that are visible to the current guard set and repeat the process on the resulting smaller grid until the grid becomes empty. However a problem arises when we remove a visible segment that may be used by a guard (positioned later) to see some other segments indirectly. For example, when segments visible to g in Figure 8a are removed, the grid shown in Figure 8b results. Two additional guards are needed for a total of three whereas two guards are enough to cover the original grid. The problem arises due to the removal of segments that are indirectly visible to the current guard set but may have a guard (positioned later) along them. There are cases when it is safe to remove indirectly visible segments. The following observations list the kinds of visible segments that can be safely removed.

Lemma 8: It is safe to remove a segment s from the grid when: (a) s is directly visible (to a guard in the current guard set), (b) s is indirectly visible to a guard placed on segment s' and s is dominated by s', (c) s is visible and all segments in the crossing set of s are visible, (d) s is visible and intersects with only one segment.

The algorithm therefore removes from G those segments that are visible and are safe to remove (making use of the above observations). Note that the resulting grid G' may contain some segments that may be indirectly visible. When a grid has some segments marked visible then the definition of domination has to be adjusted accordingly. We say that segment s_1 **dominates** segment s_2 if the set of invisible segments in C_2 is a subset of the set of invisible segments in C_1. In Figure 9a, segment s_1 is dominated by segment s_2.

Construct a reduced grid G'_r from G' by removing dominated segments. Now observe that cross (or pseudo cross) segments are of two kinds as shown in figure 10 (dotted segments are visible to the current guard set; note that segment s_6 is a cross but segment s_4 is not). Both kinds of crosses always need a guard along them. From this it follows that Lemmas 6, 7 hold even when the reduced grid contains some visible segments.

The algorithm therefore finds a segment that forms a group in the reduced grid of G' and places the next guard using Lemma 6. This process is repeated until the remaining grid is fully visible. A formal description of the algorithm is given below:

Algorithm GRID-COVER:

step1: Find all dominated segments in the simple grid G and remove them. Let G_r be the reduced grid.

step2: Find a segment s in G_r such that its crossing set forms a group. Let s' denote the junior segment in the crossing set of s.

step3: Place a guard at the intersection of s and s' and remove all segments that are safe to remove (Lemma 8) from the reduced grid.

step4: Repeat step1 through step3 until all segments of the grid are visible to a guard.

The execution of the above algorithm is illustrated by an example in Figure 11. Consider the simple 2-d grid shown in Figure 11a. All dominated segments are marked by $\surd$. Segment 2 forms a group and a guard g_1 is placed at the point of intersection of a and 2. The segments that are visible to g_1 are drawn in grey. Visible segments a,b,1 and 2 are safe to remove. In the grid obtained by removing these segments the algorithm finds that it is safe to remove visible segments 3,4,5,7 and 8 (each of them intersects with only one segment). The location of the next four guards is found similarly (Figure 11c). Note that segment k is not safe to remove (it connects invisible segments 15 and 16). When the guard g_8 is placed all the grid segments are visible and the algorithm terminates. Figure 11g shows the final guard placement.

Theorem 2: A minimum periscope guard cover for a simple grid can be found in $O(n^3)$ time.

Proof: We can construct the grid graph G formed by the intersection of grid segments in $O(n^2)$ time using the arrangement of lines algorithm in [CGL84 or EOS83]. This graph is represented by using the doubly connected edge list data structure (as described in [GS83]) to facilitate traversal of the faces of the graph. We can find the crossing set of one segment in $O(n)$ time by simply traversing the graph along the segment. Then the crossing sets of all segments can be determined in time $O(n^2)$. A segment s dominates a segment s' if the crossing set of s contains the crossing set of s'. Note that if a segment is dominated then it is dominated by one of its neighbors. Therefore dominated segments can be identified by comparing the crossing sets of neighboring pairs in $O(n^2)$ time. We can form the reduced graph G_r from G by removing the dominated segments in $O(n^2)$ time by traversing the graph on faces containing the segments, deleting the dominated segments and updating the doubly connected edge list data structure. Whether or not the crossing set of a segment s in G_r forms a group can be determined by using Lemma 4 in $O(n)$ time. Therefore we can find in $O(n^2)$ time a segment whose crossing set form a group (by applying Lemma 4 at most $O(n)$ times). Visible segments that are safe to remove can be removed by traversing the graph in $O(n^2)$ time. Since the minimum guard cover has size at most $O(n)$, all of the above steps will be repeated at most $O(n)$ times and the overall time complexity of algorithm GRID COVER is $O(n^3)$. **Q.E.D.**

The algorithm we have described works on simple grids. It is interesting to consider what happens in general 2-d grids. It is easy to construct 2-d grids in which no segment forms a group (e.g., consider the grid in Figure 12). If all the segments in the reduced grid are pseudo crosses, we can find a minimum guard cover using the matching approach in [N86] (since each pseudo-cross must have a guard along it). However, the reduced grid may also contain prime segments and it is not clear that the approach we use for simple grids can be extended to general grids.

IV. Covering Orthogonal Polygons with Periscope Guards

In this section we consider the problem of finding the minimum number of periscope guards needed to cover an orthogonal polygon. We show that the polygon cover problem can be converted into an appropriate grid cover problem for a class of orthogonal polygons.

Consider the subdivision formed by extending the edges of an orthogonal polygon into its interior. The polygon now consists of rows and columns of rectangles. Without loss of generality, we assume that no two polygon edges that face in opposite directions are collinear (if such a pair exists, we can shift one of the edges slightly so that we create a new row/column of rectangles where a degenerate row/column was before). Each row/column consists of a number of sequences of rectangles separated by portions of the exterior of the polygon. We construct a grid G to represent an orthogonal polygon P by associating a grid segment with each sequence of rectangles. Then the internal grid vertices represent individual rectangles in the polygon. Figure 13a shows an orthogonal polygon and Figure 13b shows the grid corresponding to it.

Lemma 9: The grid G is a simple and connected grid.

Lemma 10: Let X be a guard cover for the grid G. Then X is a guard cover for the underlying polygon P.

Proof: Suppose that there is a point in P that is not visible to any guard in X. Consider the rectangle that contains this point. There must be two grid segments that go through this rectangle and both of them must be visible to a guard in X. Then any guard that sees one of these edges in G sees all points in the rectangle in P. **Q.E.D.**

The difficulty with the grid G is that the reverse of Lemma 8 is not true, i.e., a guard cover for the polygon does not always correspond to a guard cover in the grid. For example, consider the polygon of Figure 14a and the corresponding grid (Figure 14b). Two guards are needed to cover the grid but one guard is sufficient to cover the polygon. The problem here results from the two rectangles at the top of the polygon. Note that the whole area in these rectangles is indirectly visible to a guard located at point x but a distinct guard for each rectangle is needed in the grid (the two horizontal segments in the top rectangles are not visible to any single guard). Suppose that we place a grid guard at point x in both the polygon and the grid. Then the grid segment corresponding to the left side of the polygon is indirectly visible to the guard. Consider then the infinite sequence of vertical segments obtained by sliding segment a along the boundary of P so that it sweeps through the top rectangle. All these segments are indirectly visible from point x; also, all the points in the horizontal segment b (that is not visible to the grid guard at x) are included in this sweep. That is, the segment b need not be included in the grid because any guard that sees the left edge will automatically cover b as well. In effect, the horizontal segment b can be replaced by its intersection with the sweeping segment without affecting Lemma 10.

We say that a grid segment is a **swept segment** if there is a grid segment that intersects it and it can be moved through the entire span of the swept segment without intersecting the exterior of the polygon and while both its ends remain in contact with the boundary of the polygon. This definition can be applied recursively by removing swept rectangles from P each time. All but the bottom horizontal segment in Figure 14d are swept (recursively). Replacing corresponding segments in the grid with their intersections with the sweeping segment results in the grids of Figure 14c,e. Note that a single guard can cover each of the grids. We call the grid resulting from sweeping the **swept grid** G_s for the orthogonal polygon.

There is one more problem with both grids G and G_s. It arises when the polygon contains corners like the one shown in Figure 15. There are two orthogonal grid segments that enter the corner and a guard placed on either one of them can see the whole corner. This is similar to what happens when we have swept segments but here we have a choice of two ways to sweep. The problem is that it is not clear locally which of the two choices is best. We refer to this type of corner as a **swept corner.** Note that choosing to sweep with a vertical (respectively horizontal) segment is equivalent to adding a vertical (horizontal) notch into the corner so that the notch is not visible from the vertical (horizontal) edge that enters the corner. Also, note that addition of such a notch eliminates one of the choices, i.e., the grid cover corresponds to a polygon cover.

Lemma 9: A grid cover in the grid G obtained from a simple polygon without swept corners after replacing swept segments with points is equivalent to a polygon guard cover.

Proof: A guard cover for the grid is clearly a guard cover for the polygon. Given a guard cover for the polygon, we first obtain an equivalent cover in which all guards are in the interior of some rectangle. To do this, we need to shift guards that lie at the border between two or more rectangles (i.e., they are co-linear with some polygon edge(s)) without affecting their visibility. If the guard is at the border between two rectangles, we note that all the polygon edges that are co-linear with this border are facing in the same direction. Then moving the guard to the interior of the rectangle that lies in that same direction does not affect the visibility of the guard. Guards that lie at the (point) border between three or four rectangles are handled similarly.

From the polygon guard cover with all guards in the interior of rectangles, we obtain an equivalent grid cover by shifting each guard to the nearest grid vertex (i.e., the intersection of the grid segments that intersect the rectangle containing the guard). Suppose that the resulting grid guard set does not cover the grid. Then there is a segment in G that is not visible to any of the guards. That means than none of the rectangles intersected by this segment and none of the rectangles intersected by segments in its crossing set contained a guard in the original polygon. But then the set of polygon guards did not cover the polygon, a contradiction. **Q.E.D.**

Algorithm POLYGON-COVER:

step1 Construct G, the grid of the polygon and remove swept segments from it.

step2 Find a leaf segment s in the reduced grid of G such that s is a pseudo cross and the crossing set of s forms a group. Let s' denote a junior segment in the crossing set of s.

step3 Place a guard g at the point of intersection of s and s' and remove all segments that are either directly visible to g or are swept by a segment directly visible to g. Let G the remaining grid.

step4 If $G \neq null$ goto step2.

Theorem 3: A minimum periscope guard cover for a simple orthogonal polygon with a fixed number of swept corners can be constructed in $O(n^3)$.

Proof: Each corner represents two choices. Making the choice is equivalent to removing the corner. As long as the number of corners is fixed, there are O(1) possible choices to consider. For each choice we can apply the algorithm of the previous section to obtain a minimum grid cover. Then we select the best of these grid covers and that will be an optimum polygon cover. **Q.E.D.**

Corollary 2: We can find minimum polygon covers in $O(n^3)$ for simple orthogonally convex polygons, orthogonal monotone polygons and orthogonal spiral polygons.

V. Concluding Remarks:

We presented $O(n^3)$ algorithms for finding optimum periscope guard covers for simple grids and orthogonal polygons with a constant number of corners. There are many interesting open problems. These include the periscope cover problem for general 2-d grids, the k-bend guard cover problem for grids and orthogonal polygons. Our motivation for considering periscope guards is to help determine the complexity of the guard cover problem for orthogonal polygons which remains a well known open problem.

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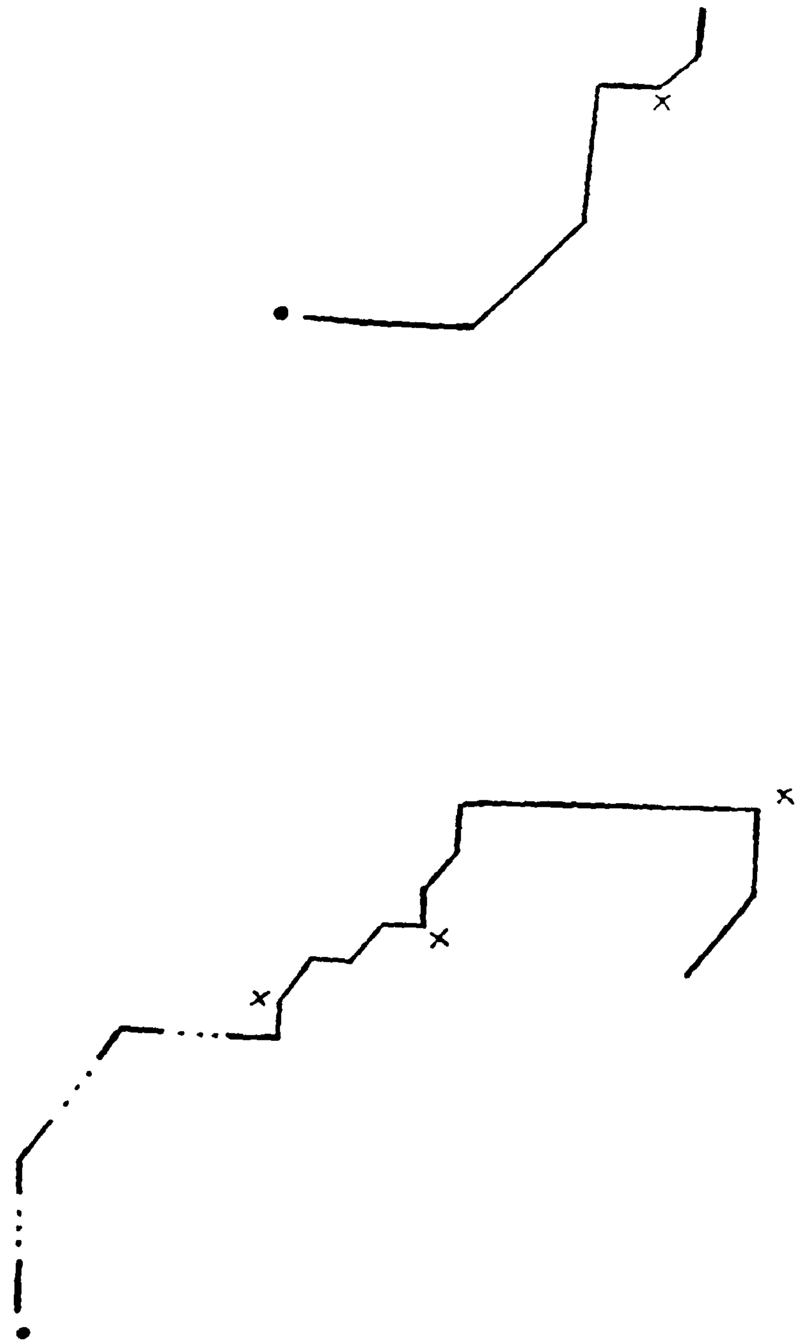
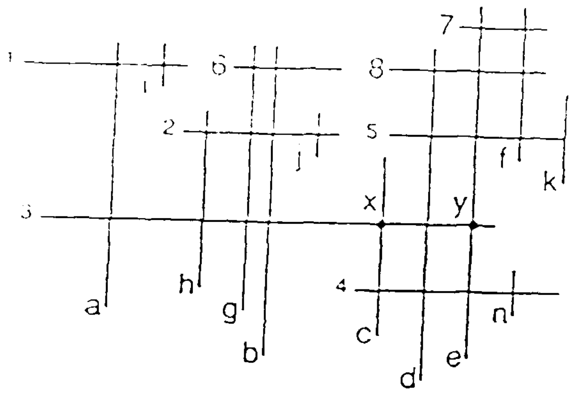
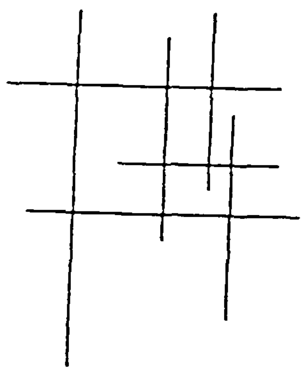


Figure 1: Short (six segment) and long (14 segment) grid paths.



(a)



(b)

Figure 2: Simple Grids and Grids with Holes.

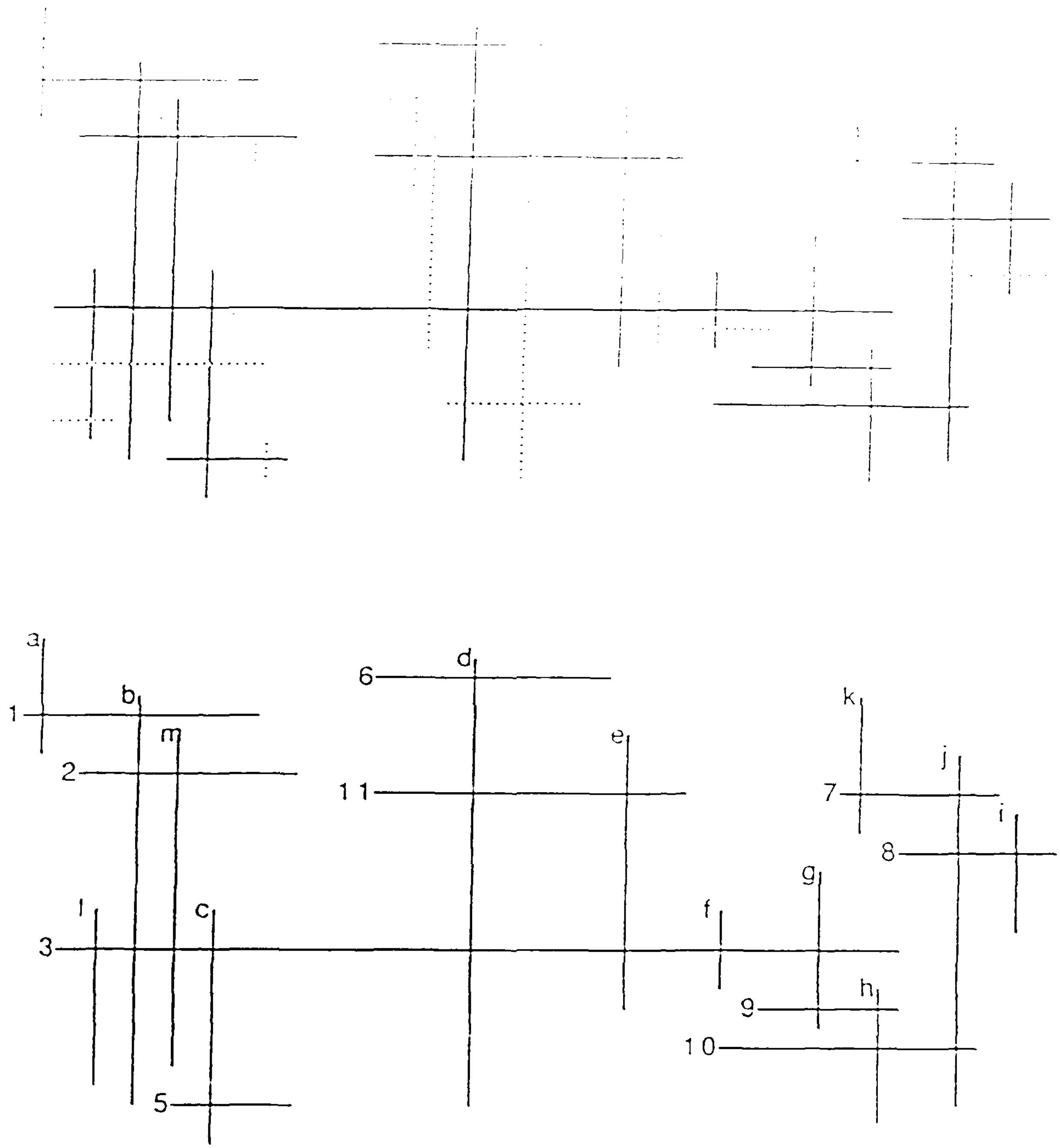


Figure 3. Dominated segments and the reduced grid.

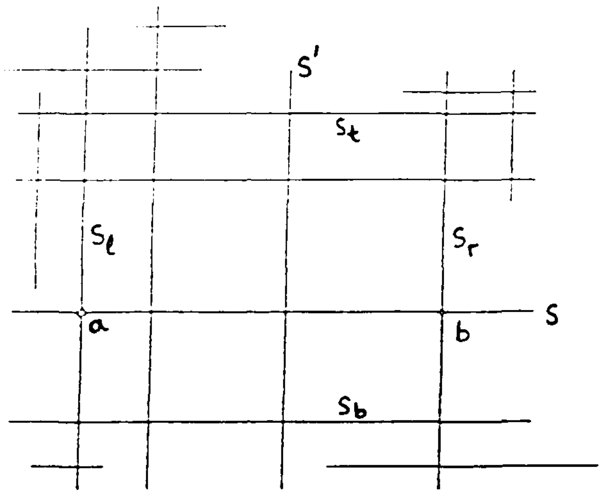


Figure 4: Lemma 4

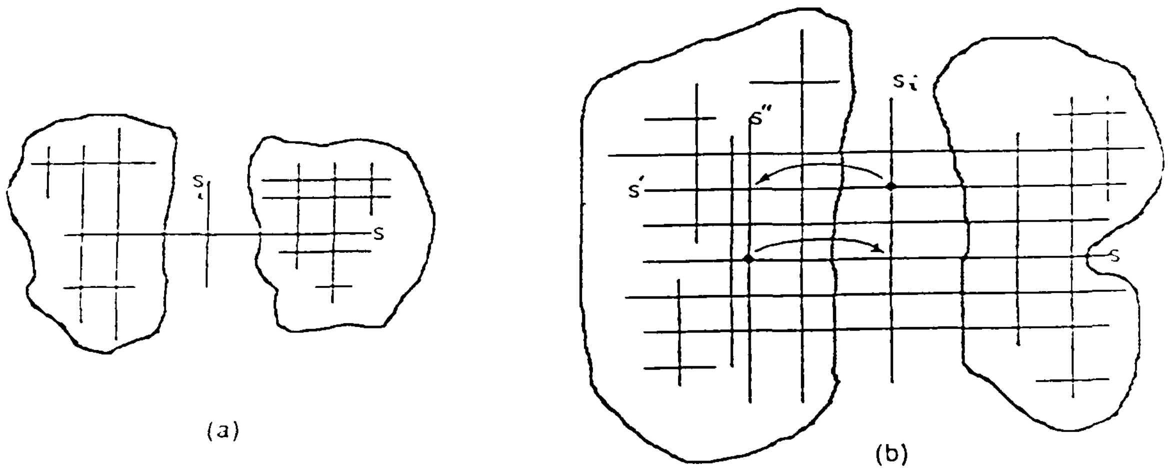


Figure 5: Proof of Lemma 6.

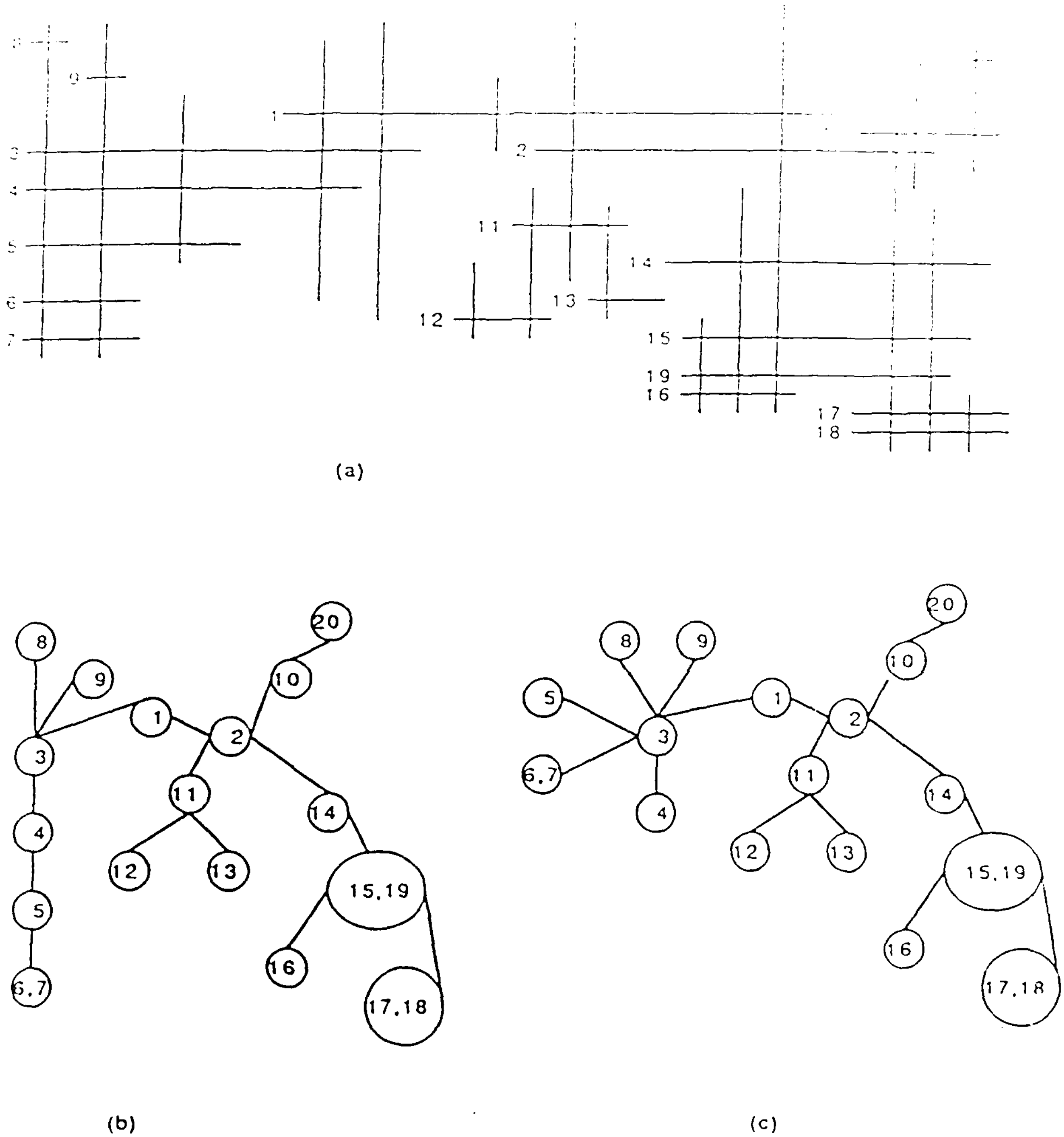


Figure 6. Construction of the horizontal segment tree.

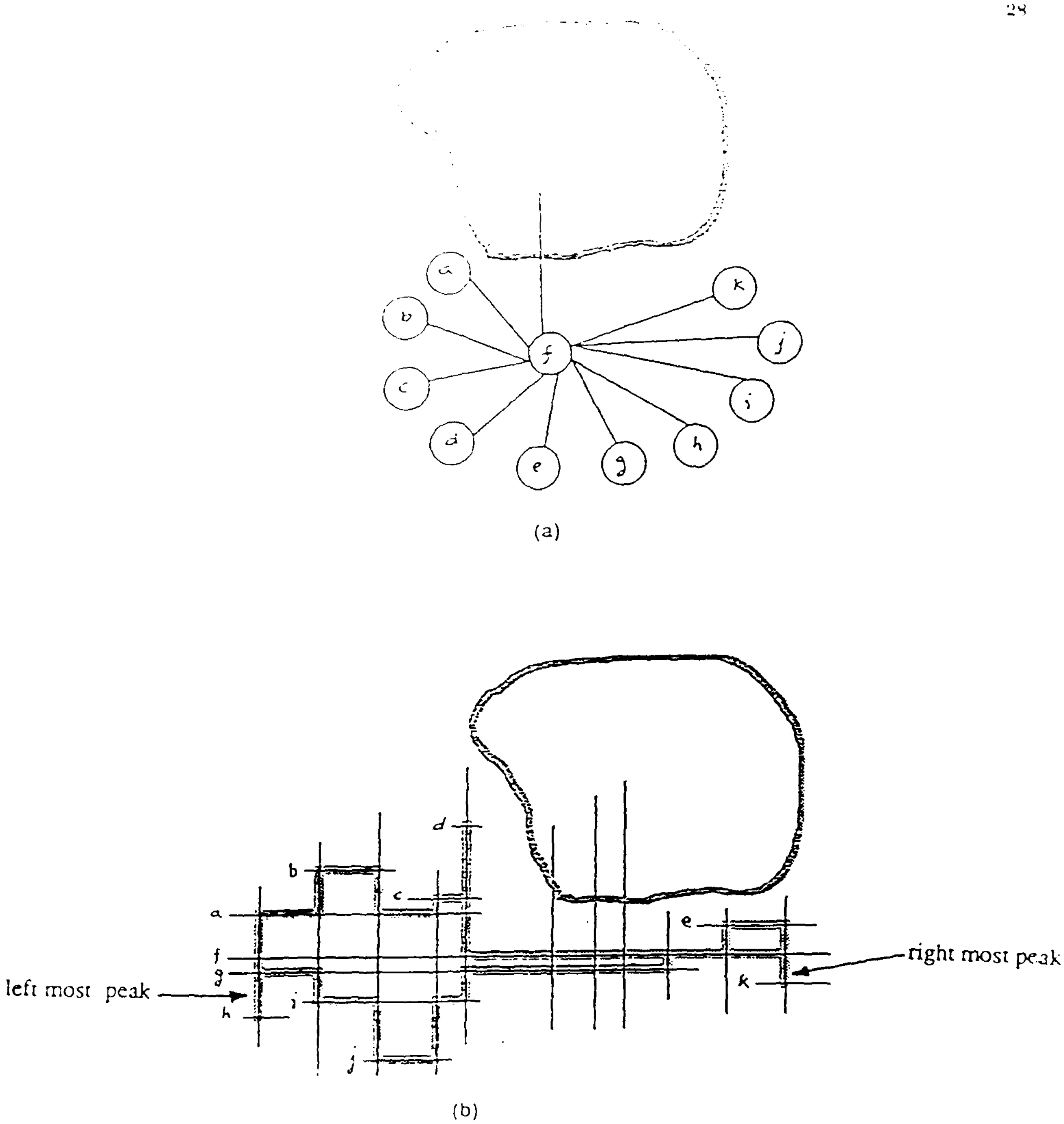


Figure 7. Formation of Manhattan skyline.

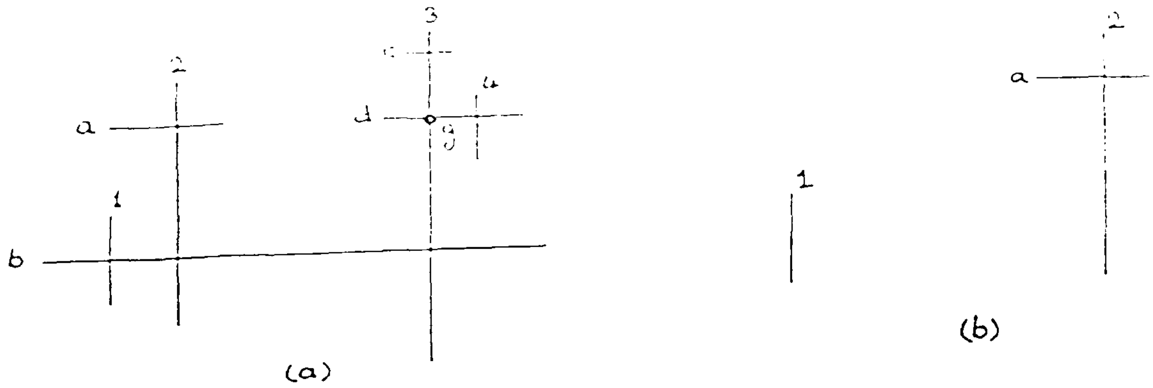


Figure 8. It is not safe to remove visible segments from grid.

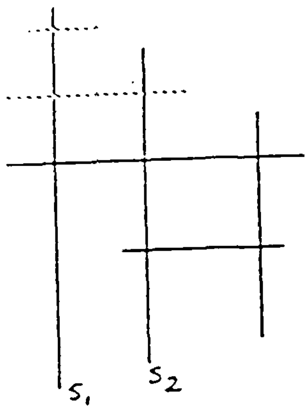


Figure 9. Definition of domination when some segments are visible.

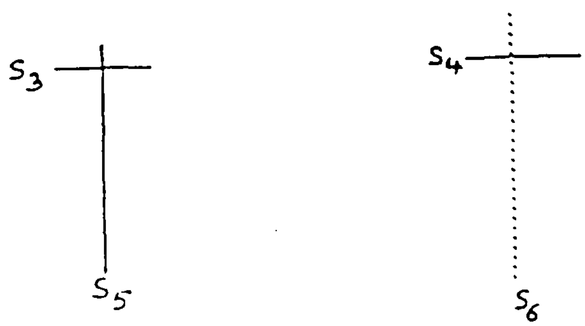


Figure 10. Two kinds of crosses.

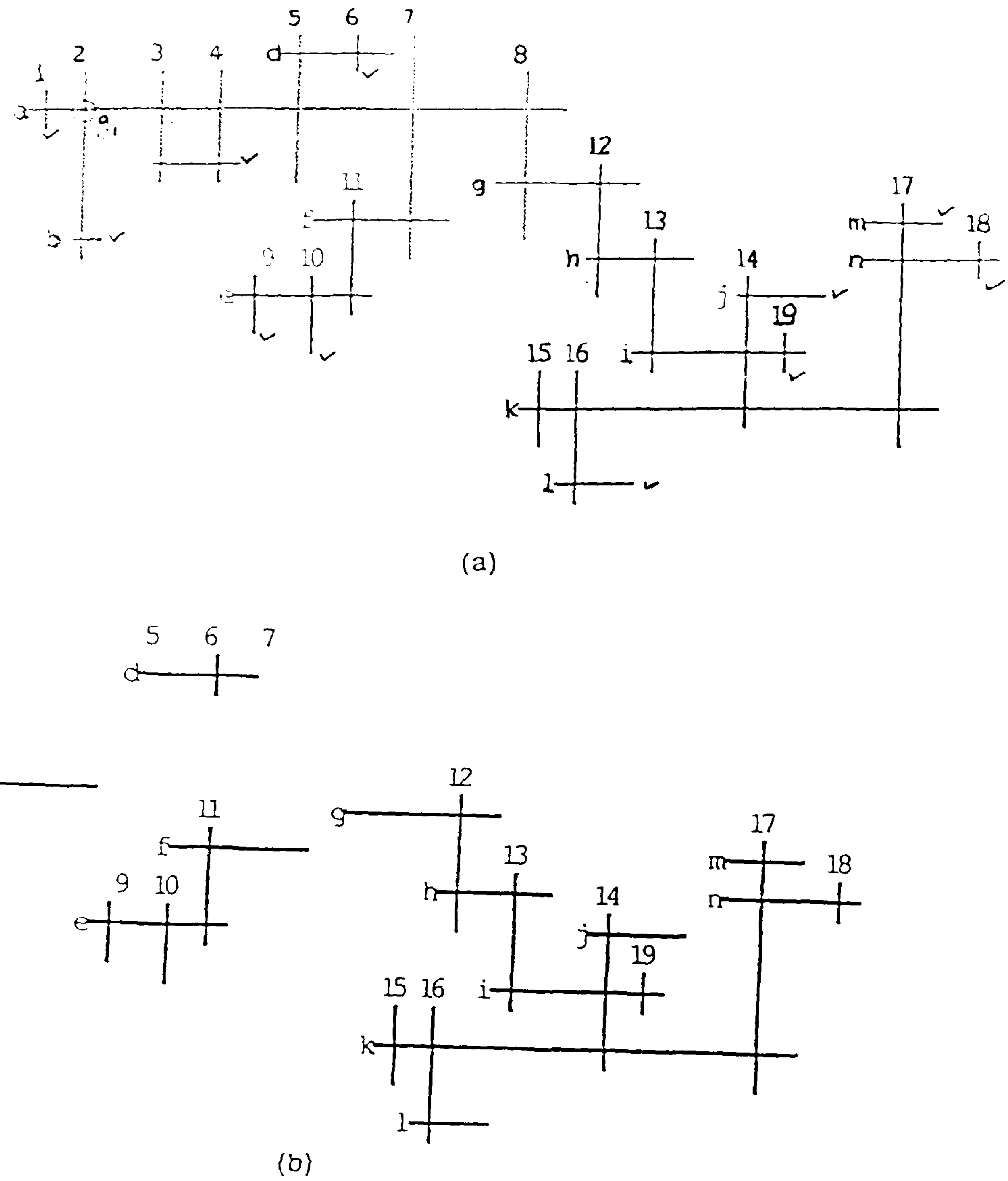


Figure 11. Example showing the execution of algorithm GRID-COVER.

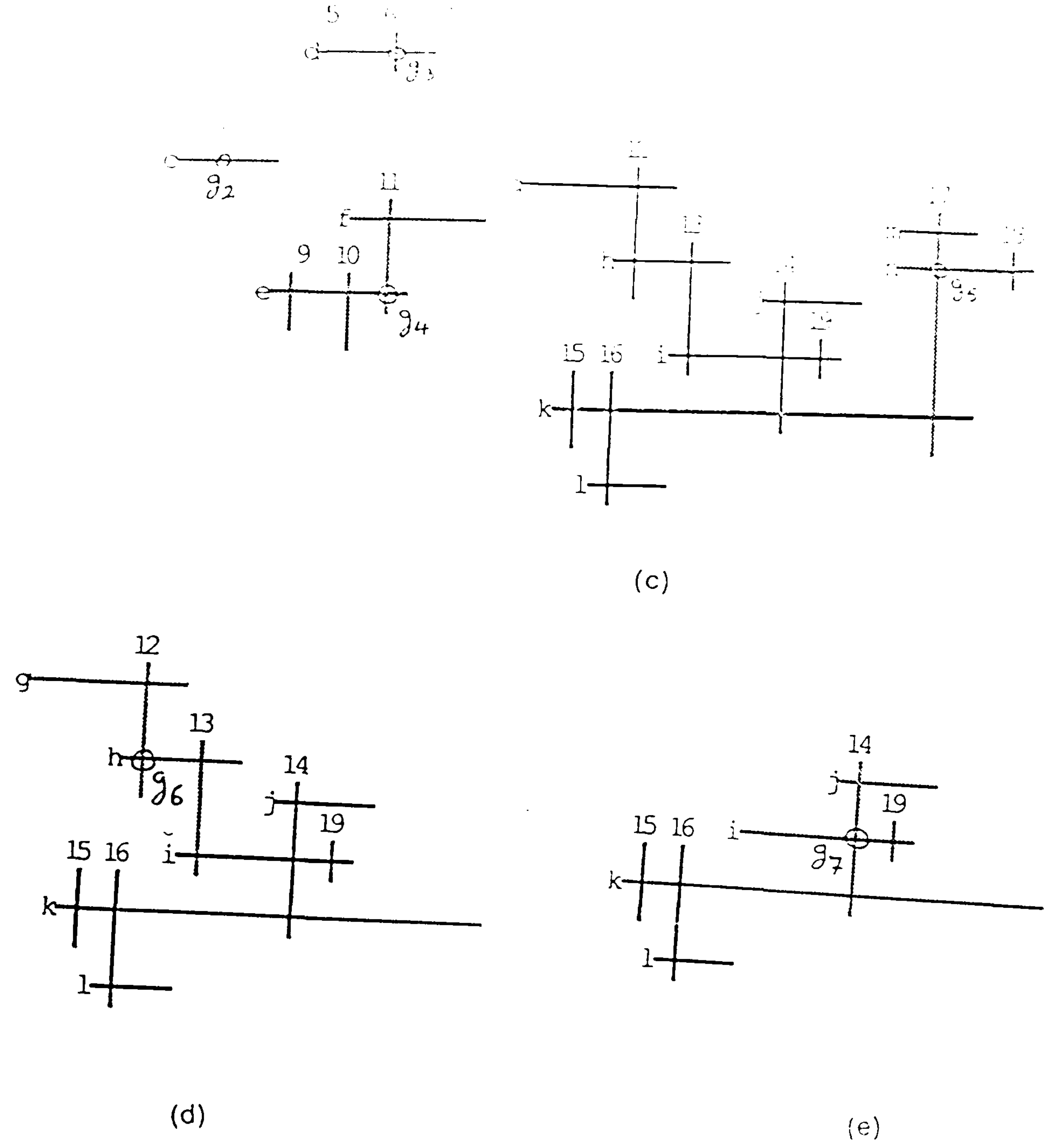
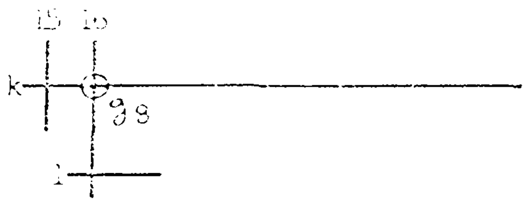
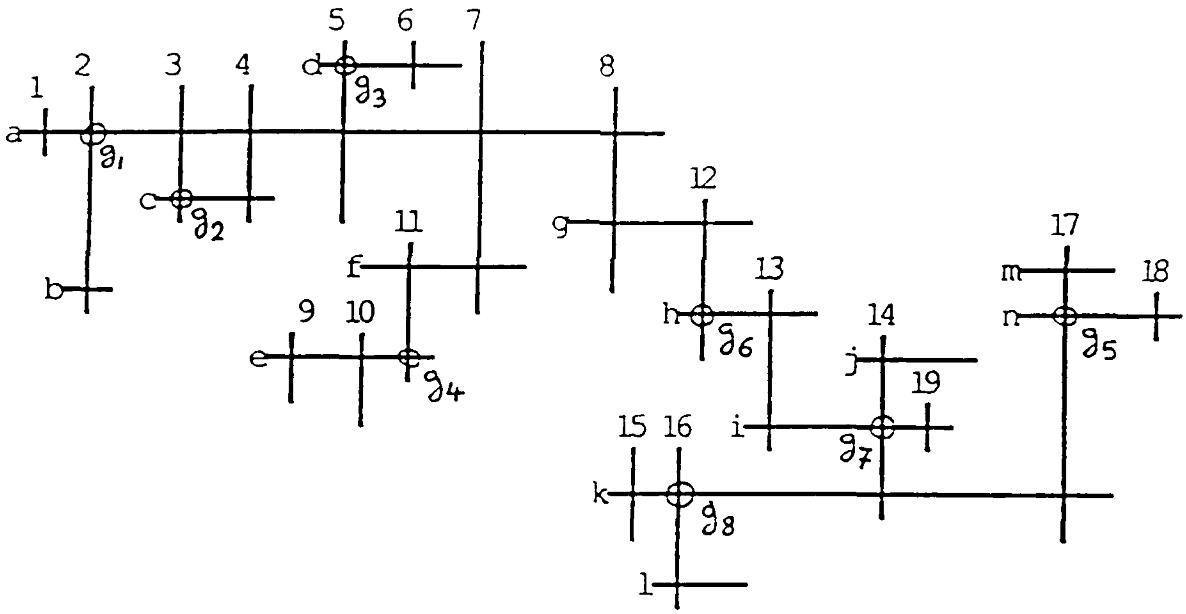


Figure 11. Continued.



(f)



(g)

Figure 11. Continued.

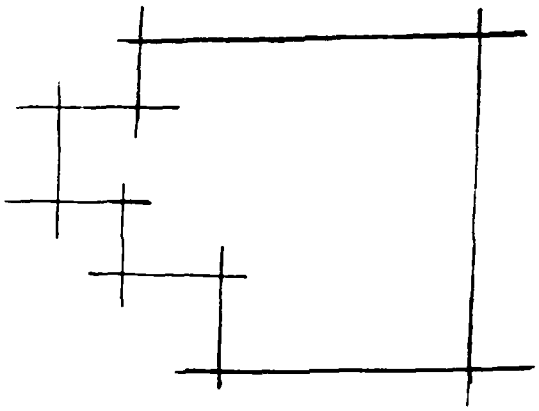
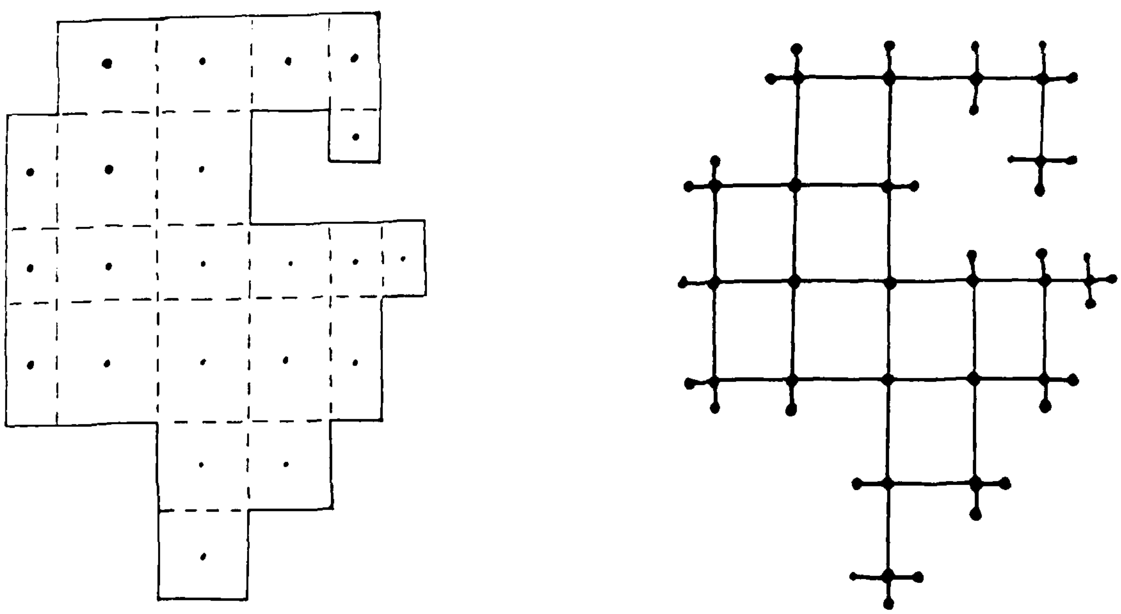


Figure 12. A general grid where no crossing set forms a group.



(a) (b)

Figure 13. The simple grid of an orthogonal polygon.

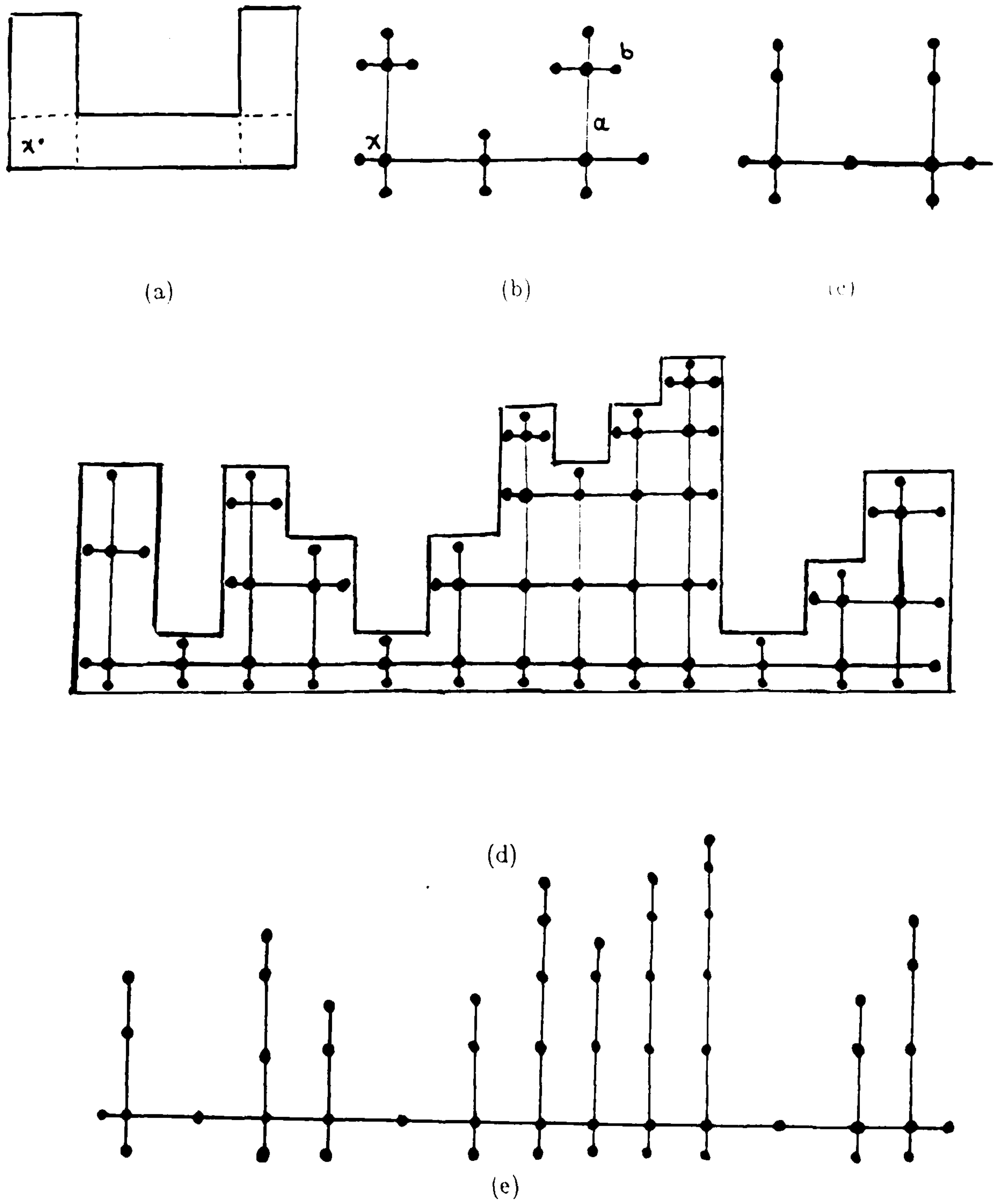


Figure 14. Swept segments and the swept grid G_s.

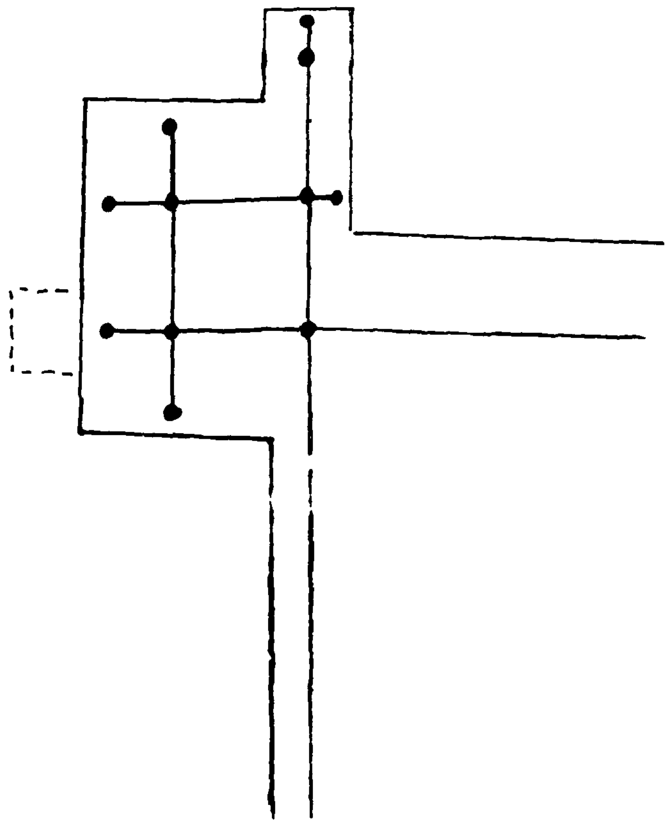


Figure 15. A swept corner in a polygon.