CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

PROPAGATION OF CHAOS AND THE McKEAN-VLASOV EQUATION

IN DUALS OF NUCLEAR SPACES

by

T.S. Chiang

G. Kallianpur

and

P. Sundar

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$$+ (\sigma((t,X_j^n(t)) + \frac{1}{n}\ \Sigma c(t,X_j^n(t),X_i^n(t))dW_t^j \quad j=1,\ldots,n.$$

We prove that under suitable conditions the system has a unique solution and its empirical distributions will converge as $n \longrightarrow \infty$ to the solution of the corresponding McKean-Vlasov equation. An application to a neurophysiological model is also given.

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T.S. Chiang
Institute of Mathematics
Academia Sinica
Taipei, Taiwan

G. Kallianpur
Center for Stochastic Processes
Department of Statistics
University of North Carolina
Chapel Hill, NC 27599-3260

P. Sundar*
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803

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Key words: ~~Nuclear~~ space valued stochastic processes differential equations, McKean-Vlasov equation, propagation of chaos.

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§1. INTRODUCTION

The present paper is concerned with propagation of chaos problems for systems with an infinite number of degrees of freedom such as strings or spatially extended neurons. The investigation of the asymptotic behavior of the voltage (membrane) potentials of large assemblages of interacting neurons leads to precisely such problems and provided the immediate motivation for the work. Another example to which the approach of the present paper could be applied (we believe) is the Ginsburg-Landau model in hydrodynamics recently studied by T. Funaki [4].

Sections 2 and 3 are of an introductory nature. Basic properties of duals of nuclear spaces (denoted throughout by Φ', the strong dual of a countably Hilbertian nuclear space Φ) are briefly discussed and the results of Kallianpur et al. [8] on the existence and uniqueness of the solution to (the martingale problem posed by) a Φ'-valued stochastic differential equation (SDE) is extended to a system of such equations. The principal results in which the infinite dimensionality of our problem call for special arguments are derived in Sections 3, 4, and 5.

In Theorem 4.1, the weak compactness of the sequence of empirical measures $\mu_n(\omega,\cdot) = \frac{1}{n} \sum_{j=1}^{n} \delta_{X_j^n(\cdot,\omega)}(\cdot)$ is established and it is shown in Section 6 (Theorem 6.1) that η_n, the law of $\mu_n(\omega,\cdot)$ converges weakly to the unique solution of the McKean-Vlasov equation.

The infinite dimensional (nucelar space-valued) version of the McKean-Vlasov SDE is introduced in Section 5. The existence and uniqueness of solution of this equation is investigated in detail in Baldwin et al. [1]. In view of the importance of this result for the propagation of chaos, a slightly different proof (with a somewhat stronger conclusion) is given for the special choice of the interaction term for our problem. The main results on the propagation of chaos are given in Theorems 6.2 and 6.3. The existence of a unique solution to the martingale problem posed by the system (3.2.1) and Theorem 5.1 on the McKean-Valsov equation are the key steps that enable Sznitman's technique for finite dimensional SDE's to be used for the nuclear space valued case.

The application, alluded to above, to the voltage potentials of interacting, spatially extended neurons is considered in Section 7. For reasons of space we have limited ourselves to the mathematics of the problem and excluded any discussion of the neurophysiological implications.

In Section 8 we introduce the assumption that the initial measure of the system (3.2.1) is μ_o-chaotic and show that the results of the previous sections hold under this more general condition. This is of importance in application since it is more reasonable to assume (as in the case of the neurons) that the random variables $X_1^n(t),\dots,X_n^n(t)$ are exchangeable than that they are identically distributed.

It is worth remarking that our results contain the finite dimensional results as a particular case and their relationship with other available results (e.g. Sznitman [16]) is also briefly commented upon.

An cutstanding problem, to which we hope to return in a later paper, is that of proving a fluctuation or central limit theorem. The difficulties that lie ahead are foreshadowed in a recent paper by Kallianpur and Mitoma [7] that establishes such a result under restrictive conditions.

§2. PRELIMINARIES ON NUCLEAR SPACES AND Φ'-VALUED SDE's.

In this section we provide the basics on nuclear spaces and on stochastic processes and integrals taking values in duals of nuclear spaces followed by the results of Kallianpur, Mitoma and Wolpert [8] on the existence and uniqueness of solutions of SDE's.

2.1 Nuclear spaces

Let Φ be a real linear space whose topology is given by an increasing sequence $\|\cdot\|_r$, r=1,2,... of Hilbertian norms. Let Φ_r be the completion of Φ with respect to $\|\cdot\|_r$. Then Φ is called a countably Hilbertian nuclear space (CHNS) if the following two conditions are satisfied:

(i) $\Phi = \bigcap_{r=1}^{\infty} \Phi_r$

(ii) For each r, there exists an m>r such that the canonical embedding $\Phi_m \subset \Phi_r$ is Hilbert-Schmidt.

Let Φ' denote the strong dual of Φ whose topology is given by the following family of semi-norms:

$$|f|_B = \sup_{x\in B} |f(x)| \qquad \text{where } B \subseteq \Phi \text{ is a bounded set in } \Phi.$$

It is well known that $\Phi' = \bigcup_{r=1}^{\infty} \Phi_{-r}$ where Φ_{-r} is the dual of Φ_r. Besides, the strong topology on Φ' coincides with the inductive limit topology induced by the canonical embeddings $\Phi_{-r} \subset \Phi'$. Let $\|\cdot\|_{-r}$ denote the norm in Φ_{-r}. If j_r denotes the canonical mapping of Φ_r onto its dual Φ_{-r}, then for $u \in \Phi_{-r}$ and $\phi \in \Phi_r$,

$$u[\phi] = \langle u, j_r\phi\rangle_{-r} = \langle j_{-r}, u, \phi\rangle_r$$

where $\langle \cdot \rangle$ denotes the inner product in the appropriate space.

For any T>0, $C^T_{\Phi'}$ denotes the space of all continuous functions from [0,T] to Φ'. If $\{|\cdot|_\alpha : \alpha \in A\}$ is the set of semi-norms defining the strong topology of Φ', then by defining $|||x|||_\alpha = \sup_{0\le t\le T} |x_t|_\alpha$, $x \in C^T_{\Phi'}$, , the space $C^T_{\Phi'}$ is seen as a completely regular topological space under the projective limit topology of $\{\|\cdot\|_\alpha : \alpha \in A\}$. $C_{\Phi'}$ denotes the space of all continuous functions from $[0,\infty)$ to Φ'. $C^T_{\Phi_{-\ell}}$ is the Banach space with the uniform topology, consisting of all continuous functions from $[0,T]$ to $\Phi_{-\ell}$.

2.2 Φ'-valued processes.

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ be a stochastic basis satisfying the usual hypotheses.

Definition: An adapted Φ'-valued stochastic process $\{M_t\}_{t\geq 0}$ is called a martingale with respect to $(\mathcal{F}_t)$ if for each $\phi \in \Phi$, $\{M_t[\phi]\}_{t\geq 0}$ is a real-valued $(\mathcal{F}_t)$ martingale. $\{M_t\}$ is called an L^2-martingale if $EM_t[\phi]^2<\infty$ for all $t\geq 0$ and $\phi\in\Phi$.

For a detailed discussion of Φ'-valued martingales and their properties we refer the reader to [6] and [12].

Definition: A continuous Φ'-valued process $\{W_t\}_{t\geq 0}$ is called a Wiener process with covariance Q, if the following conditions are satisfied:

(i) $W_0=0$ a.s.

(ii) $\{W_t[\Phi]\}$ is a one-dimensional Wiener process with variance parameter $Q(\phi,\phi)$, where $Q(\cdot,\cdot)$ is a continuous positive definite symmetric bilinear form on Φ.

A result of Mitoma [13] implies that any Φ'-valued Wiener process W has paths that lie in the Banach space $C_{\Phi_{-q}}$ for some $q<\infty$, and which are continuous in the Φ_{-q}-topology P-a.s. The choice of q depends only on the covariance form Q. Let $r \geq q$ be a fixed integer. An important property of the quadratic form Q is that it admits a unique continuous extension to a nuclear form on Φ_r and

$$Q[\phi,\psi] = (\phi,Q_r\psi)_r = (Q_r^{1/2}\phi,Q_r^{1/2}\phi)_r \tag{2.2.1}$$

for a unique non-negative trace-class operator Q_r on Φ_r. The trace norm of Q on Φ_r (or, equivalently, of Q_r^*) is given by

$$|Q|_{-r,-r} = \sum_j Q[h_j^r, h_j^r] \tag{2.2.2}$$

where $\{h_j^r\}$ is a CONS for Φ_r.

2.3 Stochastic integrals in Φ'

Let $\{W_t\}$ be a Φ'-valued Wiener process with covariance form $Q(\cdot,\cdot)$ and let $L(\Phi',\Phi')$ be the space of all continuous linear operators from Φ' to Φ'.

For each T>0 and $\phi \in \Phi$, let L_W^2 denote the space of progressively measurable processes $H: \mathbb{R}_+ \times \Omega \longrightarrow L(\Phi',\Phi')$ for which $E\int_0^T Q[H_s^*\phi, H_s^*\phi]ds < \infty$, where H_s^* is the operator dual to H_s.

<u>Definition</u>: The stochastic integral $I_t^H := \int_0^t H_s dW_s$ $(0 \le s \le T)$ is a Φ'-valued L^2-martingale with the quadratic variation process as $\langle I^H \rangle_t[\phi,\psi] = \int_0^t Q_{H_s}[\phi,\psi]ds$ where $Q_{H_s}[\phi,\psi] = Q[H_s^*\phi, H_s^*\psi]$.

There exists an $\ell > 0$, depending on H and T such that $I_\cdot^H \in C_{\Phi_{-\ell}}^T$ a.s.. If $\{h_j^\ell\} \subset \Phi$ is any CONS in $\Phi_{-\ell}$,

$$I_t^H[\phi] = \int_0^t H_s dW_s[\phi] = \sum_{j=1}^{\infty} \int_0^t (H_s^*\phi, h_j^\ell)_\ell dW_s[h_j^\ell] \tag{2.3.1}$$

where the right hand side is an L^2-convergent series of Itô integrals.

Besides,

$$\langle I^H \rangle_t[\phi,\psi] = \sum_{i,j=1}^{\infty} \int_0^t (H_s^*\phi, h_i^\ell)_\ell (H_s^*\psi, h_j^\ell) ds \; Q[h_i^\ell, h_j^\ell] \tag{2.3.2}$$

2.4 Φ'-valued SDE's.

We give below the result of Kallianpur, Mitoma and Wolpert [8] on the existence and uniqueness of solutions of stochastic differential equations.

For a probability measure μ_0 on Φ' and a pair of functions $A: \mathbb{R}_+ \times \Phi' \longrightarrow \Phi'$ and $B: \mathbb{R}^+ \times \Phi' \dashrightarrow L(\Phi':\Phi')$, consider the following SDE:

$$\left.\begin{aligned} dX_t &= A(t,X_t)dt + B(t,X_t)dW_t \\ X_0 &= X(0) \end{aligned}\right\} \tag{2.4.1}$$

where X_0 is a Φ'-valued random variable with law of X_0 given by μ_0, and W is a Φ'-valued Wiener process with covariance form Q.

Let $\mathcal{D}_b^2(\Phi')$ consist of all functions $f: \Phi' \longrightarrow \mathbb{R}$ with $f(u) = \tilde{f}(u[\phi])$ where $\tilde{f} \in C_b^2(\mathbb{R})$ and $\phi \in \Phi$. The operator L_s is defined as follows: For each $f \in \mathcal{D}_b^2(\Phi')$,

$$L_s f(u) = \tilde{f}'(u[\phi])A(s,u)[\phi] + \frac{1}{2}\tilde{f}''(u[\phi])Q(B^*(s,u)\phi, B^*(s,u)\phi)$$

where $B^*(s,u) : \Phi \longrightarrow \Phi$ is the adjoint of $B(s,u)$, i.e., for all $v \in \Phi'$ and $\phi \in \Phi$,

$$v[B^*(s,u)\phi] = B(s,u)v[\phi].$$

Let $\Omega = C_{\Phi'}$, $\mathcal{F}_t$ = Borel σ-algebra of $C^t_{\Phi'}$, and $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$ and let $\pi_t := C_{\Phi'} \longrightarrow \Phi'$ be the canonical process defined by $\pi_t y = y(t)$ for all $y \in C_{\Phi'}$. If $\mu \in \pi(C^T_{\Phi_{-k}})$, then $\mu\pi_t^{-1}(A) = \mu(y \in C^T_{\Phi_{-k}} : y_t \in A)$, $A \in \mathcal{B}(\Phi_{-k})$.

<u>Definition</u>: A solution to the martingale problem posed by (2.4.1) is a probability measure μ on $C_{\Phi'}$ such that for any $f \in \mathcal{D}_b^2(\Phi')$, the real-valued process $M_t^f = f(x_t) - f(x_0) - \int_0^t L_s f(x_s)ds$ is a $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mu)$ martingale with $\mu \cdot \pi_0^{-1} = \mu_0$.

The following conditions are imposed on the space Φ, measure μ_0 and the coefficients A and B.

Let $\{h_j^m\}$ be a CONS in Φ_m obtained by the Gram-Schmidt process applied to a countable dense set $\{\xi_j\}$ in Φ. For every j, we then have

$$\xi_j = \sum_{k=1}^{n_j} \alpha_{mk} h_k^m + \eta_j \tag{2.4.2}$$

where n_j (depending on m and j) $\leq$ j and $\|\eta_j\|_m = 0$. Our assumption is the following:

(A) For each m and p, $(p \geq m)$, in (2.4.2) $\|n_j\|_p = 0$.

Let $T>0$ be fixed. Then, for each sufficiently large $m \geq r$ where r is introduced in (2.2.1), there exists a number $\theta>0$ and an index $p \geq m$ such that for all $s, t \leq T$,

(IC) Initial Condition: $c_o := \int_\Phi (1+\|u\|_{-m}^2)[\ell n(3+\|u\|_{-m}^2)]^2 \mu_o(du) < \infty$

(CC) Coercivity Condition: For each $u \in j_m\Phi$,

$2A(t,u)[j_{-m}u] + |Q_{B(t,u)}|_{-m,-m} \leq \theta(1+\|u\|_{-m}^2)$

where j_m denotes the canonical map from Φ_m to Φ_{-m}, with j_{-m} as its inverse.

(LG) Linear Growth Condition: If $u \in \Phi_{-m}$, then $A(t,u) \in \Phi_{-p}$ and

$\|A(t,u)\|_{-p}^2 \leq \theta(1+\|u\|_{-m}^2)$

$|Q_{B(t,u)}|_{-m,-m} \leq \theta(1+\|u\|_{-m}^2)$

(JC) Joint Continuity Condition: A and B are each jointly continuous. Further,

(i) $B(s,u)(v) \in \Phi_{-m}$ if $u,v \in \Phi_{-m}$ and

(ii) $Q_{B(s,u)}(\phi,\phi)$ is continuous in u on Φ' for each $\phi \in \Phi$.

The following condition will be needed in the proof of uniqueness.

(MC) Monotonicity Condition: For all $u,v \in \Phi_{-m}$ $(\subset \Phi_{-p})$

$(A(t,u) - A(t,v), u-v)_{-p} + |Q_{B(t,u)-B(t,v)}|_{-p,-p} \leq \theta\|u-v\|_{-p}^2$

We give below the main result of Kallianpur et al. [8].

Theorem 2.4.1. Assuming conditions (A), (IC), (CC), (LG) and (JC), there exists a weak solution to the stochastic differential (2.4.1). Besides, it has the pathwise uniqueness property if ((MC) is satisfied.

Next, we give a moment bound, followed by a tightness result both of which are due to Baldwin et al [1].

Theorem 2.4.2. Let $k \geq 1$ and $E\|X_0\|_{-m}^{2k} \leq c_k < \infty$. Then, under all the conditions of Theorem 2.4.1,

$$E \sup_{0 \leq t \leq T} \|X_s\|_{-m}^{2k} \leq (2C_k + 1)\exp((136k^2 - 4k)\theta T)-1$$

Remark 2.4.1: $E\|X_t\|_{-m}^{2k} \leq (2C_k+1)e^{4k(k-1)\theta t} - 1$ for each $0 \leq t \leq T$.

Theorem 2.4.3: Assume that the coefficients associated with the equations

$$X_t^n = X_0^n + \int_0^t A^n(s,X_s^n)ds + \int_0^t B^n(s,X_s^n)dW_s^n$$

and

$$X_t = X_0 + \int_0^t A(s,X_s)ds + \int_0^t B(s,X_s)dW_s$$

satisfy the conditions:

1) Conditions (IC), (CC), (LG), (JC) and (MC) hold as stated where the constants and indices are independent of n.

2) $X_o^n \xrightarrow{D} X_0$

3) If Q_n and Q denote the covariance forms of $\{W_s^n\}$ and $\{W_s\}$ respectively, then Q_n converges to Q.

4) For each $s \in [0,T]$ and $\phi \in \Phi$, $A^n(s,\cdot)[\phi]$ converges continuously to $A(s,\cdot)[\phi]$.

5) For each $s \in [0,T]$ and $\phi \in \Phi$, $(B^n(s,\cdot))^*\phi$ converges continuously to $B^*(s,\cdot)\phi$.

Then $P\cdot(X^n)^{-1} \Rightarrow P\cdot X^{-1}$ in $C_{\Phi_{-p}}^T$

Remark 2.4.2: If $T>0$ is fixed, the solution of SDE (2.4.1) namely X will have paths in $C^T_{\Phi_{-p}}$ a.s. where p is the index that appears in the conditions.

Remark 2.4.3. Throughout the paper, the notation $\xrightarrow{D}$ is used to denote convergence in distribution of random variables whereas the notation $\Rightarrow$ is used to denote weak convergence of measures. Thus, $X_n \xrightarrow{D} X$ is equivalent to $P\cdot(X_n)^{-1} \Rightarrow P\cdot X^{-1}$, where PX_n^{-1} and PX^{-1} denote the law of X_n and X respectively. We adhere to this notation even when the random variables are measure-valued.

3. SYSTEMS OF Φ'-VALUED SDE's

The aim of this section is to extend the results of the previous section to a system of stochastic differential equations which is done by first introducing the Cartesian product of nuclear spaces.

3.1 Cartesian product of nuclear spaces.

Let Φ denote the nuclear space introduced in Section 2. Consider the linear space $\Phi\times\Phi$ with coordinatewise linear operations. Let

$$\|(\phi_1\times\phi_2)\|_r^2 = \|\phi_1\|_r^2 + \|\phi_2\|_r^2 \quad \text{for } r\geq 1. \tag{3.1.1}$$

An increasing sequence of Hilbertian norms is thus defined on $\Phi\times\Phi$ which preserves nuclearity of $\Phi\times\Phi$. To see this, let $\Phi_{r\times r}$ be the completion of $\Phi\times\Phi$ in the product r-norm given by (3.1.1) for all $r\geq 1$. Clearly, $\Phi_{r\times r} = \Phi_r\times\Phi_r$ and $\Phi\times\Phi = \bigcap_{r\geq 1} (\Phi_r\times\Phi_r)$. Given $n>0$, if $m>n$ such that the canonical injection $i: \Phi_m \subset \Phi_n$ is Hilbert-Schmidt, then the injection $(\Phi\times\Phi)_m \subset (\Phi\times\Phi)_n$ is also Hilbert-Schmidt.

Let $(\Phi\times\Phi)'$ denote the strong dual of $\Phi\times\Phi$ so that $(\Phi\times\Phi)' = \bigcup_{r\geq 1} (\Phi_r\times\Phi_r)'$. If $\ell \in (\Phi_r\times\Phi_r)'$, we can uniquely determine two linear functionals ℓ_1 and ℓ_2 in Φ_{-r} such that

$$\ell(\phi_1,\phi_2) = \ell_1(\phi_1) + \ell_2(\phi_2) \quad \text{for all } \phi_1,\phi_2 \in \Phi_r \tag{3.1.2}$$

Likewise, given ℓ_1 and ℓ_2 in Φ_{-r}, there exists a unique ℓ in $(\Phi_r \times \Phi_r)'$ such that (3.1.2) is satisfied. In short, there exists an isomorphism between $(\Phi_r \times \Phi_r)'$ and $\Phi_{-r} \times \Phi_{-r}$ which is written as $(\Phi_r \times \Phi_r)' \cong \Phi_{-r} \times \Phi_{-r}$.

The above isomorphism is not just algebraic but topological as well, if $\Phi_{-r} \times \Phi_{-r}$ is equipped with the product $-r$ norm. i.e. $\|(u_1 \times u_2)\|^2_{-r} = \|u_1\|^2_{-r} + \|u_2\|^2_{-r}$. We thus get

$$(\Phi \times \Phi)' = \bigcup_{r \geq 1} (\Phi_r \times \Phi_r)' \cong \bigcup_{r \geq 1} (\Phi_{-r} \times \Phi_{-r}) \tag{3.1.3}$$

Besides, $\bigcup_{r \geq 1} (\Phi_{-r} \times \Phi_{-r}) = \Phi' \times \Phi'$ set-theoretically. To see the topological equivalence, consider a neighbourhood of zero in $\Phi' \times \Phi'$ i.e. Let A_1 and A_2 be two bounded sets in Φ, and $\epsilon > 0$ be given. Consider the set

$$\Lambda = ((\ell_1,\ell_2) : \sup_{\phi \in A_1} |\ell_1(\phi)| < \epsilon, \ \sup_{\psi \in A_2} |\ell_2(\psi) < \epsilon) \subseteq \Phi' \times \Phi'.$$

For any fixed $r \geq 1$, $\exists\, a_r \to A_i \subseteq \{\phi : \|\phi\|_r < a_r\}$ $i=1,2$ so that

$$\Lambda \supseteq ((\ell_1,\ell_2) : \|\ell_1\|_{-r} < \frac{\epsilon}{a_1}, \ \|\ell_2\|_{-r} < \frac{\epsilon}{a_1})$$

Thus

$$\Omega \supseteq ((\ell_1,\ell_2) : \|\ell_1\|^2_{-r} + \|\ell_2\|^2_{-r} < \epsilon^2/a_1^2) \tag{3.1.4}$$

On the other hand,

$$\Lambda \subseteq \{(\ell_1,\ell_2) : \sup_{(\phi_1,\phi_2) \in A_1 \times A_2} |\ell_1(\phi_1) + \ell_2(\phi_2)| < 2\epsilon\} \tag{3.1.5}$$

and $A_1 \times A_2$ is a bounded set in $\Phi \times \Phi$. (3.1.4) and (3.1.5) give us the topological equivalence of $\bigcup_{r \geq 1} (\Phi_{-r} \times \Phi_{-r})$ and $\Phi' \times \Phi'$. Therefore (3.1.3) implies that

$$(\Phi \times \Phi)' = \bigcup_{r\geq 1} (\Phi_r \times \Phi_r)' \cong \bigcup_{r\geq 1} (\Phi_{-r} \times \Phi_{-r}) = \Phi' \times \Phi'. \qquad (3.1.6)$$

Equation (3.1.6) carries over for any finite number of Cartesian products.

3.2 System of SDE's

Let X_j^n be Φ'-valued processes, $1 \leq j \leq n$ governed by the SDE

$$dX_j^n(t) = (a(t,X_j^n(t)) + \frac{1}{n} \sum_{i=1}^{n} b(t,X_j^n(t),X_i^n(t)))dt$$

$$+ (\sigma(t,X_j^n(t)) + \frac{1}{n} \sum_{i=1}^{n} c(t,X_j^n(t),X_i^n(t)))dW_t^j \qquad (3.2.1)$$

and $X_j^n(0)$ $1 \leq j \leq n$ being iid Φ'-valued r.v.'s with law of $X_j^n(0)$ given by the probability measure μ_0. $\{W^j\}$, $1 \leq j \leq n$, are independent copies of a Brownian motion with Q as the covariance form. Besides,

a: $\mathbb{R}^+ \times \Phi' \longrightarrow \Phi'$

b: $\mathbb{R}^+ \times \Phi' \times \Phi' \longrightarrow \Phi'$

σ: $\mathbb{R}^+ \times \Phi' \longrightarrow L(\Phi' : \Phi')$

c: $\mathbb{R}^+ \times \Phi' \times \Phi' \longrightarrow L(\Phi' : \Phi')$

Let $(X_1^n(t),\ldots,X_n^n(t)) \in \Phi' \times \ldots \times \Phi'$ be a solution of (3.2.1). Then the isomorphism given by (3.1.6) between $\Phi' \times \ldots \times \Phi'$ and $(\Phi \times \ldots \times \Phi)'$ we can write the system of SDE's (3.2.1) as follows:

$$dX_t^n = (\bar{a}(t,X_t^n) + \bar{b}(t,X_t^n))dt + (\bar{\sigma}(t,X_t^n) + \bar{c}(t,X_t^n))dW_t \qquad (3.2.2)$$

where initial value X_0^n is a $(\Phi \times \ldots \times \Phi)'$-valued r.v. such that X_0^n is isomorphic to $(X_1^n(0),\ldots,X_n^n(0)) \in (\Phi' \times \ldots \times \Phi')$, and W_t is the $(\Phi \times \ldots \times \Phi)'$-valued Wiener process described below.

$$W_t(\underset{\sim}{\varphi}) = \sum_{j=1}^{n} W_t^j(\varphi_j).$$

By the independence of $\{W^j\}_{1\leq j\leq n}$

$$\langle W\rangle[\underset{\sim}{\varphi},\underset{\sim}{\psi}] = t \sum_{j=1}^{n} Q(\varphi_j,\psi_j).$$

Besides, $\{X_t^n\}$ is a $(\Phi \times\ldots\times \Phi)'$-valued process. The coefficients appearing in (3.2.2) are given as follows:

$\bar{a}$: $\mathbb{R}^+ \times (\Phi \times\ldots\times \Phi') \longrightarrow (\Phi \times\ldots\times \Phi)'$

$\bar{b}$: $\mathbb{R}^+ \times (\Phi \times\ldots\times \Phi)' \longrightarrow (\Phi \times\ldots\times \Phi)'$

$\bar{\sigma}$: $\mathbb{R}^+ \times (\Phi \times\ldots\times \Phi)' \longrightarrow L((\Phi \times\ldots\times \Phi)' : (\Phi \times\ldots\times \Phi)')$

$\bar{c}$: $\mathbb{R}^+ \times (\Phi \times\ldots\times \Phi)' \longrightarrow L((\Phi \times\ldots\times \Phi)' : (\Phi \times\ldots\times \Phi)')$

For $\underset{\sim}{\varphi} = (\varphi_1,\ldots,\varphi_n)$ and $\underset{\sim}{\phi}=(\phi_1,\ldots,\phi_n) \in \phi \times\ldots\times \Phi$ and $\underset{\sim}{u} \cong (u_1,\ldots,u_n)$ and $\underset{\sim}{v} \cong (v_1,\ldots,v_n) \in (\Phi \times\ldots\times \Phi)'$, we have

$$\bar{a}(t,\underset{\sim}{u})[\underset{\sim}{\varphi}] = \sum_{j=1}^{n} a(t,u_j)[\varphi_j]$$

$$\bar{b}(t,\underset{\sim}{u})[\underset{\sim}{\varphi}] = \sum_{j=1}^{n} \frac{1}{n} \sum_{i=1}^{n} b(t,u_j,u_i)[\varphi_j]$$

$$\bar{\sigma}(t,\underset{\sim}{u})(\underset{\sim}{v})[\underset{\sim}{\varphi}] = \sum_{j=1}^{n} \sigma(t,u_j)(v_j)[\varphi_j]$$

$$\bar{c}(t,\underset{\sim}{u})(\underset{\sim}{v})[\underset{\sim}{\varphi}] = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} c(t,u_j,u_i)(v_j)[\varphi_j]$$

so that

$$\int_0^t \bar{\sigma}(s,\underset{\sim}{u})dW_s[\underset{\sim}{\varphi}] = \sum_{j=1}^{n} \int_0^t \sigma(s,u_j)dW_s^j[\varphi_j]$$

$$\int_0^t \bar{c}(s,\underset{\sim}{u})dW_s[\underset{\sim}{\varphi}] = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_0^t c(s,u_j,u_i)dW_s^j[\varphi_j]$$

3.3 <u>Existence and uniqueness of solutions</u>.

Under the conditions given below, there exists a weak solution X^n of (3.2.2). Such a solution will also be shown to be pathwise unique and thus ensures a unique strong solution $\{X_t^n\} \in (\Phi \times \ldots \times \Phi)'$ to the equation (3.2.2) (see [5]).

Analogous to (3.1.6), the isomorphism $C_{(\Phi\times\ldots\times\Phi)'} \cong C_{\Phi'}\times\ldots\times C_{\Phi'}$ is easily established. Towards this, fix $T>0$ and consider $y \in C^T_{(\Phi\times\ldots\times\Phi)'}$ so that for each $0 \le t \le T$, $y(t) \in (\Phi \times \ldots \times \Phi)'$. By using (3.1.6), $y(t)$ is isomorphic to, say, $(y_1(t),\ldots,y_n(t)) \in \Phi' \times \ldots \times \Phi'$ for $0 \le t \le T$ and so, $\lim_{t\to t_0} y(t) = y(t_0)$ is equivalent to $\lim_{t\to t_0} y_k(t) = y_k(t_0)$ for each $1 \le k \le n$. Let $\{|\cdot|_\alpha : \alpha \in A\}$ be the set of semi-norms defining the strong topology of Φ'. Set $|||y_k|||_\alpha = \sup_{0\le t\le T} |y_k(t)|_\alpha$ for $y_k \in C^T_{\Phi'}$, $1 \le k \le n$ and $\alpha \in A$. Define $|||y|||^2_\alpha = \sum_{k=1}^n |||y_k|||^2_\alpha$ for each $\alpha \in A$. By replacing $\|\cdot\|_r$ by $|||\cdot|||_\alpha$ in (3.1.4), the arguments used in deriving (3.1.6) hold in the present context as well and thus $C^T_{(\Phi\times\ldots\times\Phi)'}$ is isomorphic to $C^T_{\Phi'}\times\ldots\times C^T_{\Phi'}$. $C_{\Phi'}\times\ldots\times C_{\Phi'}$ and $C_{(\Phi\times\ldots\times\Phi)'}$ equipped with the projective limit topologies of $\{C^T_{\Phi'}\times\ldots\times C^T_{\Phi'} : T \in \mathbb{N}\}$ and $\{C^T_{\Phi'}\times\ldots\times C^T_{\Phi'} : T \in \mathbb{N}\}$ respectively, are therefore isomorphic.

Let $X_j^n(t)$, $1 \le j \le n$ and $t>0$, be such that for each $\underset{\sim}{\varphi} \cong (\varphi_1,\ldots,\varphi_n) \in \Phi \times \ldots \times \Phi$, $X_t^n(\underset{\sim}{\varphi}) \cong \Sigma X_j^n[\varphi_j]$. Then $X_j^n(\cdot) \in C_{\Phi'}$ for $1 \le j \le n$ and solve (3.2.1). The conditions for the existence and uniqueness of solutions of SDE (3.2.2) are as follows: Conditions (A) and (IC) of Section 2 are assumed to hold. It is easy to note that condition (A) on Φ implies that on $\Phi \times \ldots \times \Phi$. Likewise condition (IC) of Section 2 implies

$$\int_{(\Phi\times\ldots\times\Phi)'} (1+\|\underset{\sim}{u}\|^2_{-m})[\ell n(3+\|\underset{\sim}{u}\|^2_{-m})]^2\, \bar{\mu}_0(d\underset{\sim}{u}) < \infty$$

where $\underset{\sim}{u}\in(\Phi\times\ldots\times\Phi)'$ and $\bar{\mu}_0 = \mathcal{L}(X_0^n)$. We will call (A) and (IC) as (SA) and (SIC) where S stands for system of stochastic differential equations. The coefficients a,b,c and σ are assumed to satisfy the following conditions: For any $T>0$, $\exists\, L_T \geq r$ such that for each $m \geq L_T$, $\exists$ a number θ and an index p (note that θ and p depend on m) such that:

(SCC) For $u,v \in j_m\Phi$, and $0 \leq t \leq T$,

$$2a(t,u)[j_{-m}u] + 2b(t,u,v)[j_{-m}u]+$$
$$|Q_{\sigma(t,u)+c(t,u,v)}|_{-m,-m} \leq \theta(1+\|u\|_{-m}^2+\beta(t,u,v)\|v\|_{-m}^2)$$

where

$$\beta(t,u,v) = \begin{cases} 0 & \text{if } b(t,u,v) = c(t,u,v) = 0 \\ 1 & \text{otherwise} \end{cases}$$

(SLG) Let $u,v \in \Phi_{-m}$, and $0 \leq t \leq T$. Then $a(t,u) \in \Phi_{-p}$ and $b(t,u,v) \in \Phi_{-p}$. Besides,

$$\|a(t,u)\|_{-p}^2 \leq \theta(1+\|u\|_{-m}^2)$$
$$\|b(t,u,v)\|_{-p}^2 \leq \theta(1+\|u\|_{-m}^2 + \|v\|_{-m}^2)$$
$$|Q_{\sigma(t,u)}|_{-m,-m} \leq \theta(1+\|u\|_{-m}^2)$$
$$|Q_{c(t,u,v)}|_{-m,-m} \leq \theta(1+\|u\|_{-m}^2+\|v\|_{-m}^2)$$

(SJC) a,b,c and σ are *jointly continuous functions*. Further,

(i) For $u,v,w \in \Phi_{-m}$,

$$\sigma(t,u)(w) \in \Phi_{-m}$$
$$c(t,u,v)(w) \in \Phi_{-m}$$

(ii) $Q_{\sigma(t,u)}(\phi,\phi)$ is continuous in u on Φ' and $Q_{c(t,u,v)}(\phi,\phi)$ is continuous in u on Φ' for each $\phi \in \Phi$.

The following condition is needed to prove the uniqueness of solutions.

(SMC) For $u_1,v_1,u_2,v_2 \in \Phi_{-m}$ $(\subset \Phi_{-p})$

$$(a(t,u_1)-a(t,v_1),u_1-v_1)_{-p} + |Q_{\sigma(t,u_1)-\sigma(t,v_1)}|_{-p,-p} \le \theta \, \|u_1-v_1\|^2_{-p}$$

and

$$(b(t,u_1,u_2)-b(t,v_1,v_2),u_1-v_1)_{-p} + |Q_{c(t,u_1,u_2)-c(t,v_1,v_2)}|_{-p,-p}$$

$$\le \theta \, \{\|u_1-v_1\|^2_{-p} + \|u_2-v_2\|^2_{-p}\}$$

It is easily verified that the above conditions will imply (CC), (LG), (JC) and (MC) for the coefficients $\bar{a},\bar{b},\bar{c}$ and $\bar{\sigma}$ for equations (3.2.2), so that by Theorem 2.4.1 we get existence and uniqueness of solutions for the SDE (3.2.2). The moment bound given in Theorem 2.4.2 becomes

$$E \sup_{0\le t\le T} \|X^n_j(t)\|^{2k}_{-m} \le (2C_k+1)\exp((136k^2-4k)\theta T)-1 \tag{3.3.1}$$

where $C_k = E\|X^n_1(0)\|^{2k}_{-m} < \infty$ and is independent of n. In fact, the following bound can also be derived by an obvious modification of the proof of (3.3.1): Under the Condition (IC)

$$E \sup_{\substack{|t-s|<\delta \\ 0\le t,s\le T}} \|X^n_j(s)-X^n_j(t)\|^2_{-p} \le c\delta^{1/2} \tag{3.3.2}$$

where c is a constant independent of n. To see this, note that if $T \ge t \ge s \ge 0$ with $t-s \le \delta$, then

$$X^n_j(t) = X^n_j(s) + \int_s^t\{(a(s),X^n_j(s)) + \frac{1}{n}\sum_{i=1}^n b(s,X^n_j(s),X^n_i(s))\}ds$$

$$+ \int_s^t \{\sigma(s,X^n_j(s)) + \frac{1}{n}\sum_{i=1}^n c(s,X^n_j(s),X^n_i(s))\}dW^j_s.$$

We get (3.3.2) by the familiar route namely, via. Doob's inequality, Jensen's inequality wherein the condition (SLG) is used crucially.

<u>Note</u>: $0<T<\infty$ will be kept fixed till the last paragraph in Section 6. Thus X^n_j

$1 \leq j \leq n$, the solution of 3.2.1 will each have paths in $C^T_{\Phi_{-p}}$ a.s. where p is the index that appears in the conditions.

§4. WEAK COMPACTNESS OF EMPIRICAL MEASURES.

Let $X^n_{\bullet}$ denote the solution of the SDE (3.2.2) so that $X^n_{\bullet} \cong (X^n_1(\cdot),\ldots,X^n_n(\cdot)) \in C^T_{\Phi_{-p}} \times\ldots\times C^T_{\Phi_{-p}}$ where $X^n_j(\cdot)$, $1 \leq j \leq n$ solve the SDE's (3.2.1).

Let $\mathcal{B}(C^T_{\Phi_{-p}})$ be the Borel σ-algebra of $C^T_{\Phi_{-p}}$. For $\omega \in \Omega$, $B \in \mathcal{B}(C^T_{\Phi_{-p}})$, define the empirical measure

$$\mu_n(\omega,B) = \frac{1}{n} \sum_{j=1}^{n} \delta_{X^n_j(\cdot,\omega)}(B). \tag{4.1}$$

For any $k \geq 1$, let $\pi(C^T_{\Phi_{-k}})$ be the space of all probability measures on $C^T_{\Phi_{-k}}$ equipped with the topology of weak convergence of measures. Likewise $\pi(C^T_{\Phi'})$ will be the space of all probability measures equipped with the topology of weak convergence of measures. Note that the canonical injection $i:\pi(C^T_{\Phi_{-k}}) \subset \pi(C^T_{\Phi_{-\ell}})$ is continuous if $k \leq \ell$. To see this, let $\lambda_n \in \pi(C^T_{\Phi_{-k}})$ such that $\lambda_n \Rightarrow \lambda$ in $\pi(C^T_{\Phi_{-k}})$. Therefore, for all $f \in C_b(C^T_{\Phi_{-k}})$, $\int_{C^T_{\Phi_{-k}}} f(y)\lambda_n(dy) \longrightarrow \int_{C^T_{\Phi_{-k}}} f(y)\lambda(dy)$ as $n \longrightarrow \infty$. If $g \in C_b(C^T_{\Phi_{-\ell}})$, let $j: \Phi_{-k} \subset \Phi_{-\ell}$ be the continuous canonical injection of Φ_{-k} into $\Phi_{-\ell}$ so that the composition $g \cdot j \in C_b(C^T_{\Phi_{-k}})$. Besides, $\int_{C^T_{\Phi_{-k}}} (g \cdot j)(y)\lambda_n(dy) = \int_{C^T_{\Phi_{-\ell}}} g(y)\lambda_n(dy)$ and $\int_{C^T_{\Phi_{-k}}} (g \cdot j)(y)\lambda(dy) = \int_{C^T_{\Phi_{-\ell}}} g(y)\lambda(dy)$, so that $\lambda_n \Rightarrow \lambda$ in $\pi(C^T_{\Phi_{-\ell}})$ as $n \longrightarrow \infty$.

μ_n is a random measure with $\mu_n(\omega,\cdot) \in \pi(C^T_{\Phi_{-p}})$ for each $\omega \in \Omega$ and $n \geq 1$. Let $\upsilon_n(B) = E\mu_n(B)$ for all $B \in \mathcal{B}(C^T_{\Phi_{-p}})$. Let $\eta_n := \mathcal{L}(\mu_n)$, the law of μ_n, i.e., the probability measure on $\pi(C^T_{\Phi_{-p}})$ induced by the random measure μ_n. Thus, $\eta_n \in \pi(\pi(C^T_{\Phi_{-p}}))$

Theorem 4.1: Under the conditions (SIC), (SA), (SCC), (SLG), (SJC) and (SMC), we have

(a) the sequence $\{\upsilon_n\}$ is tight in $\pi(C^T_{\Phi_{-q}})$ for some $q \geq p$.

(b) the sequence $\{\eta_n\}$ is weakly compact in $\pi(\pi(C^T_{\Phi_{-q}}))$.

Proof: (a) For any set $B \in \mathcal{B}(C^T_{\Phi'})$,

$\upsilon_n(B^c) = E\mu_n(B^c) = \frac{1}{n}\sum_{j=1}^{n} P(X^n_j \in B^c) = P(X^n_1 \in B^c)$ since X^n_j, $1 \leq j \leq n$, are identically distributed random variables. Therefore, the tightness of $\{\upsilon_n\}$ in $\pi(C^T_{\Phi'})$ is equivalent to the tightness of $\{X^n_1\}_{n\geq 1}$, i.e., of the probability measures $\{P^n\}$ on $\pi(C^T_{\Phi'})$ where P^n = law of X^n_1.

By a result of Mitoma [14], the tightness of $\{P^n\}$ is equivalent to the tightness of $\{P^n\pi_\phi^{-1}\}$ on $\pi(C^T_{\mathbb{R}})$ for each $\phi \in \Phi$, where $\pi_\phi: C^T_{\Phi'} \longrightarrow C^T_{\mathbb{R}}$ with $\pi_\phi(u) = u[\phi]$. The tightness of $\{P^n\pi_\phi^{-1}\}$ follows by verifying the following two conditions:

(i) Given $\epsilon > 0$, $\exists$ a > 0 such that

$$\sup_n P^n(y \in C^T_{\Phi'}: \sup_{0\leq t\leq T} |y_t[\phi]| > a) < \epsilon$$

(ii) Given $\epsilon > 0$ and $\rho > 0$, $\exists$ $\delta > 0$ such that

$$\sup_n P^n(y \in C^T_{\Phi'}: \sup_{|t-s|<\delta} |y_t[\phi]| - y_s[\phi]| \geq \epsilon) \leq \rho.$$

Condition (i) is verified by noting that $E \sup_{0\le t\le T} \|X_1^n(t)\|_{-p}^2$ is finite and independent of n. To verify (ii), note that

$$P^n(y: \sup_{|t-s|<\delta} |y_t[\phi]-y_s[\phi]| \ge \epsilon) \le P^n(y: \sup_{|t-s|<\delta} \|y_t-y_s\|_{-p}\|\phi\|_p \ge \epsilon)$$

$$\le \frac{\|\phi\|_m^2}{\epsilon^2} E \sup_{|t-s|<\delta} \|X_1^n(t) - X_1^n(s)\|_{-p}^2 \le \rho$$

if δ is such that $E \sup_{|t-s|<\delta} \|X_1^n(t) - X_1^n(s)\|_{-p}^2 \le \dfrac{\epsilon^2\rho}{\|\phi\|_p^2}$.

By the moment bound given in (3.3.2) we have that

$$E \sup_{|t-s|<\delta} \|X_1^n(t) - X_1^n(s)\|_{-p}^2 \le c\delta^{1/2}$$

so that a δ as desired does indeed exist once we are given ϵ, ρ and ϕ.

The tightness of $\{\upsilon_n\}$ is thus established on the space $\pi(C_{\Phi'}^T)$. For any $\phi \in \Phi$, and any given $\epsilon > 0$,

$$P(\sup_{0\le t\le T} |X_j^n(t)[\phi]| > \epsilon) \le P(\sup_{0\le t\le T} \|X_j^n(t)\|_{-m}\|\phi\|_m > \epsilon)$$

$$\le \frac{\|\phi\|_m^2}{\epsilon^2} E \sup_{0\le t\le T} \|X_j^n(t)\|_{-m}^2$$

$$\le \frac{\|\phi\|_m^2}{\epsilon^2} (2C_1+1)e^{132\theta T} \le \rho$$

if $\|\phi\|_m^2 \le \dfrac{\rho\epsilon^2}{(2C_1+1)e^{132\theta T}}$. Thus by Mitoma ([14]; Remark (R.1)), $\{\upsilon_n\}$ are uniformly m-continuous and hence, are uniformly p-continuous as well. Therefore, there exists an index $q \ge p$ such that $\{\upsilon_n\}$ is tight in $\pi(C_{\Phi_{-q}}^T)$.

(b) For the second part of the theorem, let us look upon $\mu_n(\omega,\cdot)$ as

$\pi(C^T_{\Phi_{-q}})$-valued random variables. Then $\upsilon_n \in \pi(C^T_{\Phi_{-q}})$ and $\eta_n \in \pi(\pi(C^T_{\Phi_{-q}}))$ for all $n \geq 1$.

Note that $$\upsilon_n(B) = \int_{\pi(C^T_{\Phi_{-q}})} \lambda(B)\eta_n(d\lambda) \ \forall\ B \in \mathcal{B}\ \pi(C^T_{\Phi_{-q}}). \tag{4.2}$$

Using part (a) of the theorem, for each $j \geq 1$, there exists a compact set K_j in $C^T_{\Phi_{-q}}$ such that $\upsilon_n(K^c_j) = \int \lambda(K^c_j)d\eta_n(\lambda) \leq \epsilon/j^3$ where $\epsilon > 0$ is given. Let $K=\{\lambda\colon \lambda(K_j) \geq 1 - \frac{1}{j} \ \forall\ j\}$ where K_j's can, WLOG, be taken to be increasing sets. $K \subset \pi(C^T_{\Phi_{-q}})$ is compact since closed tight subsets of probability measures on a complete separable metric space (in our case, on $C^T_{\Phi_{-q}}$) are compact (see Chapter II, Theorem 6.7 in [15]).

$$\begin{aligned}
\eta_n(K^c) = P(\mu_n \in K^c) &\leq \sum_{j=1}^{\infty} \eta_n(\lambda:\lambda(K_j) < 1 - \tfrac{1}{j}) \\
&= \sum_{j=1}^{\infty} \eta_n(\lambda:\lambda(K^c_j) \geq \tfrac{1}{j}) \\
&\leq \sum_{j=1}^{\infty} \frac{\int \lambda(K^c_j)\, d\eta_n}{(1/j))} \\
&\leq \epsilon \sum_{j=1}^{\infty} \frac{1}{j^2} < 2\epsilon.
\end{aligned}$$

Thus tightness of $\{\eta_n\}$ in $\pi(\pi(C^T_{\Phi_{-q}}))$ ensues.

Note that $\pi(C^T_{\Phi_{-q}})$ equipped with the topology of weak convergence is a complete separable metric space (see Ch. II, Theorems 6.2 and 6.5 in [15]). Tight subsets of probability measures on a complete separable metric space are relatively compact. The proof is thus complete.

Remark 4.1. Let $\{\eta_{n_k}\}$ be the subsequence given by the above theorem so that $\eta_{n_k} \Rightarrow \eta$ (say). Let μ denote the $\pi(C^T_{\Phi_{-q}})$-valued random variable whose law is given by η. Thus $\eta_{n_k} \Rightarrow \eta$ is equivalent to saying that $\mu_{n_k} \xrightarrow{D} \mu$.

Remark 4.2. By a well-known theorem of Skorohod, there exists a probability space on which are defined $\pi(C^T_{\Phi_{-q}})$-valued random variables, say $\{Z_{n_k}\}$ and Z with law of $Z_{n_k} = \eta_{n_k}$ and law of $Z = \eta$ such that $Z_{n_k} \longrightarrow Z$ a.s.. Using this representation and applying Fatou's lemma, we get

$$\begin{aligned}
E \sup_{0\le s\le T} \int_{C^T_{\Phi_{-q}}} \|y_s\|^2_{-q} Z(dy) &= \int_{\pi(C^T_{\Phi_{-q}})} \{\sup_{0\le s\le T} \int_{C^T_{\Phi_{-q}}} \|y_s\|^2_{-q} \lambda(dy)\}\eta(d\lambda) \\
&\le \varliminf_{k\to\infty} E \sup_{0\le s\le T} \int_{C^T_{\Phi_{-q}}} \|y_s\|^2_{-q} Z_{n_k}(dy) \\
&= \varliminf_{k\to\infty} \int_{\pi(C^T_{\Phi_{-p}})} \{\sup_{0\le s\le T} \int_{C^T_{\Phi_{-p}}} \|y_s\|^2_{-m} \lambda(dy)\}\eta_{n_k}(d\lambda) \\
&\le \lim_{k\to\infty} E \sup_{0\le s\le T} \frac{1}{n_k} \sum_{j=1}^{n_k} \|X_j^{n_k}(s)\|^2_{-m} \\
&\le (2C_1+1)\exp(132\theta T) \qquad (4.3)
\end{aligned}$$

by using (3.3.1). The inequality (4.3) holds with C_1 and θ remaining the same if T is replaced by any t on both sides of the inequality (4.3), $0\le t\le T$.

Remark 4.3: In case μ is a degenerate random variable, and Y is a $C^T_{\Phi_{-q}}$-valued random variable with $\mathcal{L}(Y) = \mu$, then Remark (4.2) implies that for each $0\le t\le T$,

$$E \sup_{0\le s\le t} \|Y_s\|^2_{-q} \le (2C_1+1)\exp(132\theta t). \qquad (4.4)$$

§5. THE McKEAN-VLASOV EQUATION

Let $\{Y_t : 0 \le t \le T\}$ be a Φ'-valued stochastic process that solves the following SDE known as the Global McKean-Vlasov equation: For $0 < t \le T$,

$$Y_t = Y_0 + \int_0^t A(s, Y_s, \mathcal{L}(Y))ds + \int_0^t B(s, Y_s, \mathcal{L}(Y))dW_s \tag{5.1}$$

where

$$A(s,u,\lambda): [0,T] \times \Phi' \times \pi(C^T_{\Phi'}) \longrightarrow \Phi'$$

$$B(s,u,\lambda): [0,T] \times \Phi' \times \pi(C^T_{\Phi'}) \longrightarrow L(\Phi' : \Phi')$$

and W is a Φ'-valued Wiener process. $\mathcal{L}(Y)$ denotes the law of Y, and Y_0 is a Φ'-valued random variable.

The local McKean-Vlasov equation is of the form:

$$Y_t = Y_0 + \int_0^t A(s, Y_s, \mathcal{L}(Y_s))ds + \int_0^t B(s, Y_s, \mathcal{L}(Y_s))dW_s \tag{5.2}$$

for $0 \le t \le T$.

By uniqueness of solutions of the SDE (5.1), we mean the following: For each $\lambda \in \pi(C^T_{\Phi'})$, let, for $0 \le t \le T$,

$$Y^\lambda_t = Y_0 + \int_0^t A(s, Y^\lambda_s, \lambda)ds + \int_0^t B(s, Y^\lambda_s, \lambda)dW_s. \tag{5.3}$$

Suppose there are λ_1 and $\lambda_2 \in \pi(C^T_{\Phi'})$ such that $\lambda_1 = \mathcal{L}(Y^{\lambda_1})$ and $\lambda_2 = \mathcal{L}(Y^{\lambda_2})$. Then $\lambda_1 = \lambda_2$.

Existence and uniqueness of solutions of equation (5.1) in full generality will appear in Baldwin et al. [1]. Here, we content ourselves with the following choice of A and B: For each $\lambda \in \pi(C^T_{\Phi'})$, let

$$A(s,u,\lambda) = a(s,u) + \tilde{b}(s,u,\lambda)$$

where $\tilde{b}(s,u,\lambda) = \int_{C^T_{\Phi'}} b(s,u,y_s)\lambda(dy)$ and

$$B(s,u,\lambda) = \sigma(s,u) + \tilde{c}(s,u,\lambda) \tag{5.5}$$

where $\tilde{c}(s,u,\lambda) = \int_{C^T_{\Phi'}} c(s,u,y_s)\lambda(dy)$. Besides, we assume that there exists a sufficiently large number $M(m)$, possibly depending on m, where m is as in subsection 3.3, such that for each $u,v \in \Phi_{-m}$, and $0 \leq s \leq T$,

$$b(s,u,v) = \begin{cases} b_s(u,v) & \text{if } \|u-v\|_{-m} \leq M(m) \\ \hat{b}_s(u,v) & \text{otherwise} \end{cases} \tag{5.6}$$

where $b_s(u,v)$ and $\hat{b}_s(u,v)$ are functions of u,v with $\|\hat{b}_s(u,v)\|_{-m} \leq C(m)$ for each $s \in [0,T]$. Likewise

$$c(s,u,v) = \begin{cases} c_s(u,v) & \text{if } \|u-v\|_{-m} \leq M(m) \\ \hat{c}_s(u,v) & \text{otherwise.} \end{cases} \tag{5.7}$$

with $c_s(u,v)$ and $\hat{c}_s(u,v)$ are functions of u,v with $|Q_{\hat{c}_s(u,v)}|_{-m,-m} \leq C(m)$ for each $s \in [0,T]$.

With b and c as above we first note that $\tilde{b}$ and $\tilde{c}$ exist and are finite. To see this, consider

$$\tilde{b}(s,u,\lambda) = \int_{C^T_{\Phi'}} b(s,u,v_s)d\lambda(v) = \int_{\Phi'} b(s,u,v)d\lambda\pi^{-1}(v).$$

Since $u \in \Phi'$, there exists an index k such that $u \in \Phi_{-k}$, and such a k can be chosen to be sufficiently large.

Using such an index k in the place of m in our conditions given in subsection (3.3) as well as (5.6) and (5.7), we get for each $\varphi \in \Phi$ that

$$|\tilde{b}(s,u,\lambda)[\varphi]| \leq \int_{\Phi'} |b(s,u,v)[\varphi]|d\lambda\pi_s^{-1}(v)$$

$$= \int_A |b(s,u,v)[\varphi]|d\lambda\pi_s^{-1}(v)$$

$$+ \int_{A^c} |b(s,u,v)[\varphi]|\, d\lambda\pi_s^{-1}(v)$$

where $A = \{v\colon \|u-v\|_{-k} \leq M(k)\}$. Continuing,

$$\leq \int_A \|\varphi\|_{p_k} (1+\|u\|_{-k}+\|v\|_{-k})\, d\lambda\pi_s^{-1}(v) + C(k)$$

by using (SLG) with $p_k \geq k$ as the index that corresponds to k. Continuing,

$$\leq \|\varphi\|_{p_k} (M(k) + 1 + 2\|u\|_{-k} + C(k) < \infty.$$

Likewise one can establish the finiteness of $\tilde{c}(s,u,\lambda)$ by showing that $|Q_{\tilde{c}(s,u,\lambda)}|_{-k,-k} < \infty$ whenever $u \in \Phi_{-k}$.

Such a choice of b and c makes physical sense in it that a pair of particles far apart interact boundedly. This choice includes in particular the case where b(s,u,v) and c(s,u,v) are both bounded in the sense that

$$\|b(s,u,v)\|_{-m} \leq C$$

$$|Q_{\hat{c}_s(u,v)}|_{-m,-m} \leq C$$

for all $0 \leq s \leq T$, $u \in \Phi'$ and $v \in \Phi'$. We assume that the functions a,b,c and σ satisfy the conditions (SCC), (SLG), (SJC) and (SMC). It is then a routine matter to check that A and B, as defined above, satisfy (CC), (LG), (JC) and (MC) as listed in Section 2 with the same indices and with constants independent of the measures $\lambda \in \pi(C^T_{\Phi'})$. We need and hence introduce the following additional condition:

(SJC)' (i) For each $\lambda \in \pi(C^T_{\Phi'})$, $\tilde{b}(s,u,\lambda)$ and $\tilde{c}(s,u,\lambda)$ are jointly continuous in s and u. $Q_{\tilde{c}(s,u,\lambda)}(\phi,\phi)$ is continuous in u on Φ' for each $\phi \in \Phi$ and $s \in [0,T]$.

(ii) For each $\phi \in \Phi$, $u \in \Phi_{-m}$ and $s \in [0,T]$, $\tilde{b}(s,u,\lambda_n)[\phi]$ converges continuously to $\tilde{b}(s,u,\lambda)[\phi]$ and $\tilde{c}^*(s,u,\lambda_n)\phi$ converges continuously to $\tilde{c}^*(s,u,\lambda)\phi$ as $\lambda_n \Rightarrow \lambda$ in $\pi(C^T_{\Phi'})$.

<u>Theorem 5.1:</u> Consider the SDE (5.3) with A and B specified by equations (5.4) to (5.7). Assume conditions (SA), (SIC), (SCC), (SLG), (SJC), (SMC) and (SJC)' and that $E\|Y_0\|^4_{-m}$ where C is a positive constant. Then,

(i) the McKean-Vlasov equation defined by the equation (5.3) admits a solution. The solution Y^λ lies in $C^T_{\Phi_{-p}}$ a.s.

(ii) The solution is unique if the following additional condition holds (MCπ): For all $u,v \in \Phi_{-m}$, and $\zeta_1,\zeta_2 \in \pi(C^T_{\Phi'})$, and $0 \le s \le T$,

$$\langle \tilde{b}(s,u,\zeta_1) - \tilde{b}(s,v,\zeta_2), u-v\rangle_{-p} + \|Q_{\tilde{c}(s,u,\zeta_1)-\tilde{c}(s,v,\zeta_2)}\|_{-p,-p}$$

$$\le C_T\{K_s(\zeta_1,\zeta_2)\|u-v\|_{-p} + \|u-v\|^2_{-p}\} \tag{5.8}$$

where C_T is a constant and

$$K_s(\zeta_1,\zeta_2) = \begin{cases} \inf\limits_{\lambda \in \mathcal{A}(\zeta_1,\zeta_2)} \displaystyle\int_{\Phi'}\int_{\Phi'} \|u_1-u_2\|_{-p}\, \lambda_s(du_1du_2) & \text{if } \zeta_1 \ne \zeta_2 \\ 0 & \text{if } \zeta_1 = \zeta_2 \end{cases}.$$

Here, $\mathcal{A}(\zeta_1,\zeta_2)$ = the set of all probability measures λ on $C^T_{\Phi'} \times C^T_{\Phi'}$ with the prescribed marginals ζ_1 and ζ_2. Besides, $\lambda_t = \lambda\pi_t^{-1}$.

<u>Proof:</u> A complete and detailed proof of this result will appear in Baldwin, et al. [1]. Here we will briefly outline the basic ideas of their proof with modifications to suit our needs.

Let $A = \{\mathcal{L}(Y^\lambda): \lambda \in \pi(C^T_{\Phi'})\}$. Then A is a tight subset of $\pi(C^T_{\Phi'})$ by Theorem (2.4.3). Define the map $\psi \cdot \pi(C^T_{\Phi'}) \longrightarrow A$ by $\psi(\mu) = \mathcal{L}(Y^\mu)$. Again by theorem (2.4.3), ψ is sequentially continuous.

Note that A is a subset of $\pi(C^T_{\Phi_{-p}})$. By using Theorem (2.4.2) for k=1 and a result of Mitoma [14], we know that there exists an index $\bar{p} \geq p$ such that A is tight as a subset of $\pi(C^T_{\Phi_{-\bar{p}}})$. Let $c\ell(\text{coA})$ denote the closure of the convex hull of A in $\pi(C^T_{\Phi_{-\bar{p}}})$. Then $c\ell(\text{coA})$ is tight in $\pi(C^T_{\Phi_{-\bar{p}}})$ and is therefore compact in $\pi(C^T_{\Phi_{-\bar{p}}})$. The canonical inclusion $j:\pi(C^T_{\Phi_{-\bar{p}}}) \subset \pi(C^T_{\Phi'})$ is continuous as can be seen from the proof of part (c) of Theorem 6.3. Therefore, $c\ell(\text{coA})$ can be viewed as a subset of $\pi(C^T_{\Phi'})$ and is a compact and tight subset of $\pi(C^T_{\Phi'})$.

It can be shown that the topology of weak convergence in $\pi(C^T_{\Phi'})$ when relativized to a compact tight subset of $\pi(C^T_{\Phi'})$ is metrizable so that $c\ell(\text{coA})$ is a Polish space under this topology.

Let $\bar{\psi}$: $c\ell(\text{coA}) \longrightarrow c\ell(\text{coA})$ be the restriction of ψ to $c\ell(\text{coA})$.

$c\ell(\text{coA})$ is metrizable and so sequential continuity of $\bar{\psi}$ is equivalent to continuity of $\bar{\psi}$. An application of the Schauder-Tychonoff fixed point theorem (see [3]) gives us the existence of the McKean-Vlasov equation. If $\lambda_o \in \pi(C^T_{\Phi'})$ such that $\lambda_o = \mathcal{L}(Y^{\lambda_o})$, then, for this choice of measure λ_o, the coefficients A and B satisfy the conditions of existence and uniqueness of solutions as listed in Section 2. Therefore $Y^\lambda \in C^T_{\Phi_{-p}}$, since $A \subset \pi(C^T_{\Phi_{-p}})$.

For part (ii) of the theorem, let λ^1 and λ^2 be two measures in $\pi(C^T_{\Phi'})$ such that $\lambda^1 = \mathcal{L}(Y^{\lambda^1})$ and $\lambda^2 = \mathcal{L}(Y^{\lambda^2})$. Then Theorem 2.4.2 implies that $E \sup_{0\leq t\leq T} \|Y^{\lambda^1}_t\|^2_{-p} < \infty$ since $E \sup_{0\leq s\leq T} \|Y^{\lambda^2}_t\|^2_{-p} < \infty$. Likewise, $E \sup_{0\leq t\leq T} \|Y^{\lambda^2}_t\|^2_{-p} < \infty$. Therefore if $Y_t = Y^{\lambda^1}_t - Y^{\lambda^2}_t$, $E \sup_{0\leq s\leq T} \|Y_t\|^2_{-p} < \infty$. Applying the Itô lemma to $\|Y_t\|^2_{-p}\cdot\exp(-2C_T t)$ where $C_T = C_T(\lambda^1,\lambda^2)$ is the constant that appears in (5.8), and then using the condition in part (ii) of this theorem, we get that

$$E\|Y_t^2\|_{-p}\exp(-2C_T t) \le E\int_0^t K_s(\lambda^1,\lambda^2)\|Y_s\|_{-p} C_T e^{-C_T s} ds.$$

Since $K_s(\lambda^1,\lambda^2) \le E\|Y_s\|_{-p}$, the above inequality yields

$$E\|Y_t^2\|_{-p}\exp(-2C_T) \le C_T\int_0^t e^{-C_T s} E\|Y_s\|_{-p}^2 ds.$$

Gronwall's lemma now yields $E\|Y_t^2\|_{-p} = 0$ for all $t \in [0,T]$. Since Y_t is sample continuous, $\sup_{0\le t\le T} \|Y_t^{\lambda^1} - Y_t^{\lambda^2}\|_{-p}^2 = 0$ a.s.

Remark 5.1: Since the conditions of Theorem 5.1 hold for all sufficiently large m with p being the index determined by each such m, the conclusion of the theorem holds in particular when m is replaced by a larger index. Therefore, the measure λ_o obtained in Theorem 5.1 is the unique solution of the McKean-Vlasov equation defined by equation (5.3) even among the measures in the larger space viz. $\pi(C_{\Phi_{-k}}^T)$ for any $k \ge p$. This fact will be used in Section 6 for the particular choice of k=q where q is the index that appears in Theorem 4.1.

§6. PROPAGATION OF CHAOS

Let P_n denote the unique probability measure on $(C_{\Phi_{-q}}^T)^{\otimes n}$ that solves the martingale problem posed by the system of equations (3.2.1) subject to the conditions listed in subsection 3.3, and conditions (SJC)' and (MCπ). Besides, we assume that b and c satisfy conditions (5.6) and (5.7). $\eta_n \in \pi(\pi(C_{\Phi_{-q}}^T))$ is given by

$$\eta_n(B) = P_n(\underset{\sim}{y}^n : \frac{1}{n}\sum_{i=1}^n I_{y_i^n} \in B) \quad \forall\, B \in \mathcal{B}(\pi(C_{\Phi_{-q}}^T))$$

where $\underset{\sim}{y}^n$ denotes a generic point in $(C_{\Phi_{-q}}^T)^{\otimes n}$ so that $\underset{\sim}{y}^n = (y_1^n, y_2^n, \ldots, y_n^n)$ where each component belongs to $C^T_{\Phi_{-q}}$. The method employed by Sznitman [12] is used in proving the following theorem.

<u>Theorem 6.1:</u> Under the conditions specified in Theorem 5.1, let $\lambda_o \in \pi(C^T_{\Phi_{-q}})$ be the unique probability measure that solves the McKean-Vlasov equation (5.3). Then, the subsequence $\{\eta_{n_k}\}$ obtained by Theorem 4.1 is such that $\eta_{n_k} \Rightarrow \delta_{\lambda_o}$ where δ refers to the Dirac δ measure provided that there exists $\alpha > 0$ such that $E\|X^n_1(0)\|^{4+\alpha}_{-m} \leq C$, where C is independent of n.

<u>Proof:</u> Let $f \in \mathcal{D}^2_b(\Phi')$ (see subsection 2.4) so that $f(u) = \tilde{f}(u[\varphi])$ for some $\varphi \in \Phi$ and $\tilde{f} \in C^2_b(\mathbb{R})$.

$$L_{i,j}(f,\underset{\sim}{y}^n,s) = \tilde{f}'(y^n_j(s)[\varphi])\{a(s,y^n_j(s))[\varphi] + b(s,y^n_j(s),y^n_i(s))[\varphi]\}$$

$$+ \frac{1}{2}\tilde{f}''(y^n_j(s)[\varphi])$$

$$\cdot Q\Big(\frac{1}{n}\sum_{k=1}^{n}(\sigma(s,y^n_j(s)) + c(s,y^n_j(s),y^n_k(s)))^*[\varphi],$$

$$\frac{1}{n}\sum_{k=1}^{n}(\sigma(s,y^n_j(s)) + c(s,y^n_j(s),y^n_k(s)))^*[\varphi]\Big)$$

where $\underset{\sim}{y}^n = (y^n_1,\ldots,y^n_n) \in \Phi'\times\ldots\times\Phi'$. Let

$$L(f,y(s),s,\lambda) = \int_{C^T_{\Phi_{-q}}} \Big[\tilde{f}'(y(s)[\varphi])$$

$$\{a(s,y(s))[\varphi] + b(s,y(s),z(s))[\varphi]\} + \frac{1}{2}\tilde{f}''(y(s)[\varphi])$$

$$Q\Big(\Big(\int_{C^T_{\Phi_{-q}}} (\sigma(s,y(s)) + c(s,y(s),z_1(s))dz(z_1)\Big)^*[\varphi]\Big),$$

$$\left(\int_{C^T_{\Phi_{-q}}} (\sigma(s,y(s)) + c(s,y(s),z_2(s))d\lambda(z_2))^*[\varphi])\right]d\lambda(z)$$

By the conditions listed in subsection 3.3, the existence of a unique solution to the martingale problem posed by (3.2.1) is guaranteed. We have called such a solution as P_n. Therefore

$$\tilde{f}(y^n_j(t)[\varphi]) - \tilde{f}(y^n_j(r)[\varphi]) - \int_r^t \{\frac{1}{n} \sum_{i=1}^n L_{i,j}(f,\underset{\sim}{y}^n,s)ds$$

is a P_n-martingale with $0\leq r\leq t\leq T$. By the conditions listed in the statement of Theorem 5.1, a unique solution of the McKean-Vlasov equation posed by (5.3) exists and is denoted by λ_o. Therefore the following is a λ_o-martingale:

$$\tilde{f}(y(t)[\varphi]) - \tilde{f}((y(r)[\varphi]) - \int_r^t L(f,y(s),s,\lambda_o)ds$$

where $0\leq r\leq t\leq T$. Consider the function $F\colon \pi(C^T_{\Phi_{-q}}) \longrightarrow \mathbb{R}$ defined by

$$F(\lambda) = \int_{C^T_{\Phi_{-q}}} \{\tilde{f}(y(t)[\varphi]) - \tilde{f}(y(r)[\varphi]) - \int_r^t L(f,y(s),s,\lambda)ds\} \tag{6.1}$$

$$g_1(y(r_1))\dots g_p(y(r_p))d\lambda(y)$$

where $0\leq r_1\leq r_2\dots\leq r_p\leq t\leq T$ and $g_1,\dots,g_p$ are bounded functions from $\Phi_{-q} \longrightarrow \mathbb{R}$. Hence $F(\lambda_o)=0$. Now we will show that $\int_{\pi(C^T_{\Phi_{-q}})} F^2(\lambda)\eta(d\lambda) = 0$ by direct evaluation. From this, it follows that the support of η is contained in the set of solutions to the L-martingale problem. Corresponding to each solution of the L-martingale problem, we can construct a weak solution of the McKean-Vlasov equation, by the method employed by Kallianpur et al. [8]. From the previous section we know that the McKean-Vlasov equation has a unique strong solution namely Y^{λ_0}. Therefore the set of solutions to the L-martingale problem is the singleton set $\{\lambda_0\}$. From the fact that λ_o is the unique solution to the martingale problem it will then follow that $\eta=\delta_{\lambda_o}$.

<u>Claim</u>: $$\lim_{k\to\infty} \int_{\pi(C^T_{\Phi_{-q}})} F^2(\lambda)\eta_{n_k}(d\lambda) = \int_{\pi(C^T_{\Phi_{-q}})} F^2(\lambda)\eta(d\lambda)$$

<u>Proof of Claim</u>. Let λ_s denote $\lambda\pi_s^{-1}$. If λ is in the support of η_{n_k} recall that λ_s has support in Φ_{-m} for each $0\le s\le T$. For each $u,v \in \Phi'$, $\varphi\in\Phi$, $\lambda \in \pi(C^T_{\Phi_{-q}})$ and $0\le s\le T$, let for $R > 0$,

$$a^R(s,u)[\varphi] = (-R\vee a(s,u)[\varphi])\wedge R$$

$$b^R(s,u)[\varphi] = (-R\vee b(s,u,v))[\varphi])\wedge R$$

$$Q^R_{h(s,u,\lambda_s)}(\varphi,\varphi) = R\wedge Q_{h(s,u,\lambda_s)}(\varphi,\varphi)$$

where $h(s,u,\lambda_s) = \sigma(s,u) + \tilde{c}(s,u,\lambda_s)$. Replace a,b and Q by a^R, b^R and Q^R respectively in the definition of F, and call the resulting function as F_R.

$$\lim_{k\to\infty} \int_{\pi(C^T_{\Phi_{-q}})} F_R^2(\lambda)\eta_{n_k}(d\lambda) = \int_{\pi(C^T_{\Phi_{-q}})} F_R^2(\lambda)\eta(d\lambda)$$

since $F_R \in C_b(\pi(C^T_{\Phi_{-q}}))$. The claim will be proved if we show that $\int_{\pi(C^T_{\Phi_{-q}})} (F^2(\lambda) - F_R^2(\lambda))\eta_{n_k}(d\lambda))$ and $\int_{\pi(C^T_{\Phi_{-q}})} (F^2(\lambda) - F_R^2(\lambda))\eta(d\lambda)$ can both be made arbitrarily small when R is sufficiently large.

Using Fubini's theorem and Jensen's inequality,

$$\int_{\pi(C^T_{\Phi_{-q}})} (F^2(\lambda)-F_R^2(\lambda))\eta_{n_k}(d\lambda)\le 3(t-r)(\max_{1\le i\le p} \|g_i\|^2_\infty)^p$$

$$\times \int_r^t E_{\eta_{n_k}} \Big[\int_{\{y:\, |a(s,y)[\varphi]|>R\}} (\tilde{f}(y[\varphi]a(s,y)[\varphi])^2 \lambda_s(dy) \tag{6.2}$$

$$+ \int_{\{y:\, |\tilde{b}(s,y,\lambda_s)[\varphi]|>R\}} (\tilde{f}'(y[\varphi])\tilde{b}(s,y,\lambda_s)[\varphi])^2 \lambda_s(dy)$$

$$+ \int_{\{y:Q_{h(s,y,\lambda_s)}(\varphi,\varphi)>R\}} (\tfrac{1}{2}\tilde{f}''(y[\varphi])Q_{h(s,y,\lambda_s)}(\varphi,\varphi))^2 \lambda_s(dy)\Big]ds$$

It remains to show that each of the three terms on the right side of (6.2) can be made arbitrarily small, uniformly in k, when R is large. Since the method for each term is essentially the same, we shall consider only the third term.

$$Q_{h(s,y,\lambda_s)}(\varphi,\varphi) \le \|\varphi\|_m^2 \; |Q_{h(s,y,\lambda_s)}|_{-m,-m} \le \|\varphi\|_m^2[3\theta(1+4\|y\|_{-m}^2+M^2(m))+3C^2(m)]$$

by using (SLG) and equation (5.7). Thus

$$\{y:\ Q_{h(s,y,\lambda_s)}(\varphi,\varphi) > R\} \subseteq \{y:\ \|y\|_{-m}^2 > R/k\}$$

where k is a suitable positive constant and R is sufficiently large.

Therefore, the third term on the right side of (6.2) is

$$\le K_1 \int_r^t E_{\eta_{n_k}} \int_{\{y:\|y\|_{-m}^2>R/k\}} (1+\|y\|_{-m}^4)\lambda_s(dy)ds$$

by using (SLG) again with K_1 as a suitable constant independent of η_k. Continuing:

$$\le K_1 \int_r^t \frac{1}{n_k} \sum_{j=1}^{n_k} E[(1+\|x_j^{n_k}(s)\|_{-m}^4) I_{\{\|x_j^{n_k}(s)\|_{-m}^2 > R/k\}}]ds$$

$$\le K_1 2^{4+\alpha} \int_r^t \frac{1}{n_k} \sum_{j=1}^{n_k} \{E(1+\|x_j^{n_k}(s)\|_{-m}^{4+\alpha})\}^{4/(4+\alpha)} \{P(\|x_j^{n_k}(s)\|_{-m}^2 > R/k)\}^{\alpha/(4+\alpha)}ds$$

by Hölder's inequality.

$$\lesssim k_3/R^{\alpha/(4+\alpha)} \longrightarrow 0 \qquad \text{as } R \longrightarrow \infty,$$

wherein the last inequality uses Chebyshev's inequality and the moment bound given by (3.3.1). Besides, k_3 is a constant independent of n_k. The fact that $\int_{\pi(C^T_{\Phi_{-q}})} (F^2(\lambda) - F^2_R(\lambda))\eta(d\lambda)$ can be made small for large R, follows along the same lines as above and its proof is hence omitted.

<u>Continuation of the proof of the theorem</u>: Using the above claim,

$$\int_{\pi(C^T_{\Phi_{-q}})} F(\lambda)^2\eta(d\lambda) = \lim_{k\to\infty} \int_{\pi(C^T_{\Phi_{-q}})^{\otimes n_k}} [\{\frac{1}{n_k} \sum_{j=1}^{n_k}(\tilde{f}(y_j^{n_k}(t)[\varphi])$$

$$- \tilde{f}(y_j^{n_k}(r))[\varphi] - \int_r^t \frac{1}{n_k} \sum_{i=1}^{n_k} L_{i,j}(f,\underset{\sim}{y}^{n_k},s)ds\}g_1(y_j^{n_k}(r_1))\ldots g_p(y_j^{n_k}(r_p))]^2 dP_{n_k}(\underset{\sim}{y}^{n_k})$$

since $\eta_{n_k}(B) = P_{n_k}(\underset{\sim}{y}^{n_k}: \frac{1}{n_k} \sum_{i=1}^{n_k} \delta_{y_i^{n_k}} \in B)$ so that

$$\int_{\pi(C^T_{\Phi_{-q}})} \frac{1}{n_k^2} ((\tilde{f}(y_j^{n_k}(t)[\varphi]) - \tilde{f}(y_j^{n_k}(r)[\varphi]) - \int_r^t \frac{1}{n_k} \sum_{i=1}^{n_k} L^{\varphi}_{i,j}(f,\underset{\sim}{y}^{n_k},s)ds)$$

$$\cdot\, g_1(y_j^{n_k}(r_1))\ldots g_p(y_j^{n_k}(r_p)))^2 d\eta_{n_k}(\frac{1}{n_k} \sum_{i=1}^{n_k} \delta_{y_i^{n_k}})$$

$$= \int_{\pi(C^T_{\Phi_{-q}})^{\otimes n_k}} \frac{1}{n_k^2} ((\tilde{f}(y_j^{n_k}(t)[\varphi]) - \tilde{f}(y_j^{n_k}(r)[\varphi]) - \int_r^t \frac{1}{n_k} \sum_{i=1}^{n_k} L^{\varphi}_{i,j}(f,\underset{\sim}{y}^{n_k},s)ds)$$

$$\cdot\, g_1(y_j^{n_k}(r_1))\ldots g_p(y_j^{n_k}(r_p)))^2 dP_{n_k}(\underset{\sim}{y}^{n_k}) \qquad (6.2)$$

Then, the right side of (6.2) is equal to

$$\int_{\pi(C^T_{\Phi_{-q}})^{\otimes n_k}} \int_r^t \frac{1}{n_k^2} ((g_1(y_j^{n_k}(r_1))\ldots g_p(y_j^{n_k}(r_p)))^2((\tilde{f}'(y_j^{n_k}(s)[\varphi]))^2$$

$$\cdot\ Q\Big(\Big(\frac{1}{n_k}\sum_{i=1}^{n_k}\{\sigma(s,y_j^{n_k}(s)) + c(s,y_j^{n_k}(s),y_i^{n_k}(s))\}\Big)^*[\varphi],$$

$$\Big(\frac{1}{n_k}\sum_{i=1}^{n_k}\{\sigma(s,y_j^{n_k}(s)) + c(s,y_j^{n_k}(s),y_i^{n_k}(s))\}\Big)^*[\varphi]\Big)ds\,dP_{n_k}(\underset{\sim}{y}^{n_k}) = o\Big(\frac{1}{n_k^2}\Big)$$

by using the condition (SLG) and the bound given by (3.3.1). Thus

$$\int_{\pi(C^T_{\Phi_{-q}})} F(\lambda)^2\eta(d\lambda) = \lim_{k\to\infty}\Big\{o\Big(\frac{1}{n_k}\Big) + \frac{n_k-1}{n_k}\int_{(C^T_{\Phi_{-q}})^{\otimes n_k}}\Big(\tilde{f}(y_1^{n_k}(t)[\varphi])$$

$$- \tilde{f}(y_1^{n_k}(r)[\varphi]) - \int_r^t \frac{1}{n_k}\sum_{i=1}^{n_k} L^{\varphi}_{i,1}(f,\underset{\sim}{y}^{n_k},s)ds\Big)$$

$$\cdot\ \Big(\tilde{f}(y_2^{n_k}(t)[\varphi]) - \tilde{f}(y_2^{n_k}(r)[\varphi]) - \int_r^t \frac{1}{n_k}\sum_{i=1}^{n_k} L^{\varphi}_{i,2}(f,\underset{\sim}{y}^{n_k},s)ds\Big)$$

$$g_1(y_1^{n_k}(r_1))\dots g_p(y_1^{n_k}(r_p))g_1(y_2^{n_k}(r_1))\dots g_p(y_2^{n_k}(r_p))$$

$$dP_{n_k}(\underset{\sim}{y}^{n_k})\Big\} = 0$$

since the independence of the Wiener processes W^1 and W^2 implies that

$$\langle M_r^1(t),\ M_r^2(t)\rangle = 0$$

where

$$M_r^1(t) = \tilde{f}(y_1^{n_k}(t)[\varphi]) - \tilde{f}(y_1^{n_k}(r)[\varphi]) - \int_r^t \frac{1}{n_k}\sum_{i=1}^{n_k} L^{\varphi}_{i,1}(f,\underset{\sim}{y}^{n_k},s)ds$$

which is a P_{n_k}-martingale.

Thus $\int_{\pi(C^T_{\Phi_{-q}})} F(\lambda)^2\eta(d\lambda) = 0$ for all F defined by (6.1) with $f \in \mathfrak{D}_b^2(\Phi')$, $p \in \mathbb{N}$ and $g_1,\dots,g_p$ continuous and bounded mapping $\Phi' \longrightarrow \mathbb{R}$, and $0\le r_1\le\dots\le r_p\le t\le T$. Since λ_0 is the unique member of $\pi(C^T_{\Phi_{-q}})$ such that $F(\lambda_0) = 0$, we get that $\eta = \delta_{\lambda_0}$.

<u>Remark 6.1.</u> By Theorem 6.1, the possibly random measure μ such that $\mu_{n_k} \xrightarrow{D} \mu \in \pi(C_{\Phi'}^T)$ has been shown to be a non-random measure and in fact, $\mu=\lambda_0$.

Our result on propagation of chaos is presented in the next theorem.

<u>Theorem 6.2.</u> Under the conditions (SA), (SIC), (SCC), (SLG), (SJC), (SMC), (SJC)' and (MCπ) and with the coefficients b and c satisfying (5.6) and (5.7), and $\sup_n E\|X_1^n(0)\|_{-m}^{4+\alpha} \leq C$ for some $\alpha > 0$ we get

(i) $\eta_n \Rightarrow \delta_{\lambda_0}$ in $\pi(\pi(C_{\Phi_{-q}}^T))$.

(ii) If $\eta_n^t = \mathcal{L}(\mu_n^t)$ where $\mu_n^t(w,\underset{\sim}{X}^n) = \frac{1}{n}\sum_{i=1}^{n} I_{X_i^n(t,w)}$

and $\{X_i^n(\cdot),\ 1\leq i\leq n\}$ solves the system (3.2.1), then $\eta_n^t \Rightarrow \delta_{\lambda_0}\pi_t^{-1}$ for $0\leq t\leq T$. That is to say $\mu_n^t \longrightarrow \lambda_0\pi_t^{-1}$ in distribution and hence, in probability as well.

<u>Proof.:</u> We have shown in Theorem 6.1 that $\eta_{n_k} \Rightarrow \delta_{\lambda_0}$. In fact, for any convergent subsequence $\{\mu_{n_j'}\}$ of the sequence of empirical measures $\{\mu_n\}$, we get from Theorem 6.1 that $\eta_{n_j'} \Rightarrow \delta_{\lambda_0}$. Therefore, the whole sequence η_n weakly converges to δ_{λ_0}.

To prove (ii), note that for all real-valued continuous, bounded functions f on $\pi(C_{\Phi'}^T)$,

$$\int_{\pi(C_{\Phi_{-q}}^T)} f(\lambda)\eta_n(d\lambda) \longrightarrow \int_{\pi(C_{\Phi_{-q}}^T)} f(\lambda)\delta_{\lambda_0}(d\lambda). \tag{6.2}$$

In particular, if $f(\lambda) = \int_{(C^T_{\Phi_{-q}})} g(y)d\lambda(y)$ where for all $y \in C^T_{\Phi_{-q}}$, $g(y) = \tilde{g}(y_t)$ for t fixed in [0,T], and $\tilde{g}$ a continuous bounded function from Φ_{-q} to $\mathbb{R}$, then $f(\lambda)$ is indeed a real-valued, continuous, bounded function on $\pi(C^T_{\Phi'})$. Therefore, (6.2) for this choice of f implies that $\eta^t_n \Rightarrow \delta_{\lambda_0}\pi_t^{-1}$. Thus $\mu^t_n \longrightarrow \lambda_0\pi_t^{-1}$ as $n \longrightarrow \infty$ in probability since the limit is non-random.

Theorem 6.3: (a) For each T>0, let the conditions (SA), (SIC), (SCC), (SLG), (SJC) and (SMC) hold. Then the system of SDE's (3.2.1) admits a weak solution that is pathwise unique. That is, (3.2.1) has a unique strong solution in $(C_{\Phi'})^{\otimes n}$.

(b) Assume the additional conditions (SJC)' and (MCπ) for each T>0. If b and c are as specified by (5.6) and (5.7), the McKean-Vlasov equation posed by (5.3) has a unique strong solution in $C_{\Phi'}$.

(c) In view of (a) and (b) above, define $\mu_n(\cdot,w) = \frac{1}{n}\sum_{i=1}^{n} I_{X_i^n(\cdot,w)}$ so that its law $\bar{\eta}_n \in \pi(\pi(C_{\Phi'}))$ $n \geq 1$. Let $\bar{\lambda}_0$ be the probability measure on $C_{\Phi'}$ that solves the McKean-Vlasov equation posed by (5.3). If $\sup_n E\|X_1^n(0)\|_{-m}^{4+\alpha} \leq C$ for some $\alpha > 0$, then

$$\bar{\eta}_n \Rightarrow \delta_{\bar{\lambda}_0} \quad \text{in } \pi(\pi(C_{\Phi'})).$$

Proof: Part (a) follows by reading off the corresponding result in Kallianpur, Mitoma and Wolpert [8].

(b) Let the conditions of Theorem 6.2 hold for each fixed T>0. Then, the results in Section 5 and 6 hold in the interval $[0,\infty)$. To see this, suppose $\lambda_0 = \mathcal{L}(Y^{\lambda_0})$ solves the McKean-Vlasov equation in the interval $[0,T_0]$, and $\lambda_1 = \mathcal{L}(\bar{Y}^{\lambda_1})$ solves the McKean-Vlasov equation in the interval [0,T], where

$T_1 > T_0$, then, by the uniqueness of solutions to the McKean-Vlasov equation, we get that the projection of λ_1 on the interval $[0, T_0]$ must coincide with λ_0. Thus λ_1 is an extension of λ_0 in the above sense. Such an argument shows the existence and uniqueness of solutions to the McKean-Vlasov equation (5.3) in the interval $[0, T]$ for any $T>0$.

Choose $T_n = n$, and the corresponding measures $\lambda_n \in \pi(C_{\Phi'})$. Solve the McKean-Vlasov equation (5.3). Then, the projective limit of $\{\lambda_n\}$ is a measure $\bar{\lambda}_0 \in \pi(C_{\Phi'})$ that solves (5.3) for all $t \geq 0$.

To prove part (c), we make the following observations: For any positive index k, i: $C^T_{\Phi_{-k}} \subset C^T_{\Phi'}$ is continuous. In fact, the topology on $C^T_{\Phi'}$ as given in Section 2 is equivalent to the weakest topology with respect to which the above canonical inclusions are continuous. Therefore, we have

<u>Claim</u>: If $\Gamma_\alpha \Rightarrow \Gamma$ in $\pi(\pi(C^T_{\Phi_{-k}}))$, then $\Gamma_\alpha \Rightarrow \Gamma$ in $\pi(\pi(C^T_{\Phi'}))$.

<u>Proof of the claim:</u> First, note that the inclusion j: $\pi(C^T_{\Phi_{-k}}) \subset \pi(C^T_{\Phi'})$ defined by $j(\lambda) = \lambda i^{-1}$ is continuous. To see this, let $\{\lambda_\alpha\}$ be a net in $\pi(C^T_{\Phi_{-k}})$ such that $\lambda_\alpha \Rightarrow \lambda$ in $\pi(C^T_{\Phi_{-k}})$. Therefore

$$\int_{C^T_{\Phi_{-k}}} f d\lambda_\alpha \longrightarrow \int_{C^T_{\Phi_{-k}}} f d\lambda \ \forall\, f \in C_b(C^T_{\Phi_{-k}}).$$

Let $g \in C_b(C^T_{\Phi'})$. The composition $g \cdot i$ is then in $C_b(C^T_{\Phi_{-k}})$. Also

$$\int_{C^T_{\Phi'}} g d\lambda_\alpha i^{-1} = \int_{C^T_{\Phi_{-k}}} g \cdot i d\lambda_\alpha \quad \text{for all } \alpha$$

and

$$\int_{C^T_{\Phi'}} g d\lambda = \int_{C^T_{\Phi'}} g \cdot i d\lambda_\alpha$$

so that

$$\int_{C^T_{\Phi'}} g d\lambda_\alpha \longrightarrow \int_{C^T_{\Phi'}} g d\lambda \text{ for all } g \in C_b(C^T_{\Phi'}).$$

Therefore j is continuous.

Now let k be the inclusion from $\pi(\pi(C^T_{\Phi_{-k}})) \subset \pi(\pi(C^T_{\Phi'}))$. Continuity of k can be proved by following step by step the proof of continuity of j. The claim is thus shown.

Now part (c) is shown by observing that Theorem 6.2 part (i) implies that

$$\eta_n \Rightarrow \delta_{\lambda_0} \quad \text{in } \pi(\pi(C^T_{\Phi_{-q}})$$

and hence

$$\eta_n \Rightarrow \delta_{\lambda_0} \quad \text{in } \pi(\pi(C^T_{\Phi'})$$

by the claim shown above.

$$\eta_n \Rightarrow \delta_{\lambda_0} \quad \text{in } \pi(\pi(C_{\Phi'})$$

since the inclusion $\pi(\pi(C^T_{\Phi'})) \subset \pi(\pi(C_{\Phi'}))$ is also continuous. Note that η_n and δ_{λ_0} are the projections of $\bar{\eta}_n$ and $\delta_{\bar{\lambda}_0}$ on $\pi(\pi(C^T_{\Phi'}))$. Thus $\bar{\eta}_n \Rightarrow \delta_{\bar{\lambda}_0}$ in $\pi(\pi(C^T_{\Phi'}))$.

Remark 6.3: The unique strong solutions mentioned in parts (a) and (b) of the above theorem are in general Φ'-valued processes and cannot be guaranteed to lie in a single Hilbert space Φ_{-j}. This is so since the indices m and p vary with T in the conditions.

§7. APPLICATION TO INTERACTING SYSTEMS OF NEURONS.

The random behavior of the voltage potential of a spatially distributed neuron has attracted considerable attention in neurophysiology and can be modeled in the following set-up:

Let H be a separable Hilbert space and T_t be a strongly continuous contraction semigroup on H with a densely defined, closed, negative-definite generator A. In practice, H is usually taken to be $L^2(\mathfrak{X},\mu)$ where $\mathfrak{X}$ is the membrane of a neuron and μ is a suitable measure on $\mathfrak{X}$. If there exists $r_1>0$ so that $(I-A)^{-r_1}$ is Hilbert-Schmidt, then there exists $\{\varphi_j\}_{j=1}^{\infty}$, a CONS for H such that $-A\varphi_j = \lambda_j\varphi_j$, $j=1,2,\ldots$ with $\sum_{j=1}^{\infty}(1+\lambda_j)^{-2r_1} < \infty$. Let $\Phi = \{\varphi\in H: \sum_{j=1}^{\infty}(1+\lambda_j)^{2r}(\varphi,\varphi_j)_H^2 < \infty \text{ for any } r>0\}$. Define on Φ a family of increasing Hilbertian norms $\|\cdot\|_r$ with $\|\varphi\|_r^2 = \sum_{j=1}^{\infty}(1+\lambda_j)^{2r}(\varphi,\varphi_j)^2$ and let Φ_r denote the completion of Φ w.r.t. $\|\cdot\|_r$. Since $\Phi_{r+r_1} \subset \Phi_r$ is Hilbert-Schmidt, it is easy to see that Φ is a nuclear space. The semigroup $\{T_t\}_{t>0}$ can be written as follows. For any $\varphi \in \Phi$

$$T_t\varphi = \sum_{j=1}^{\infty}\exp(-t\lambda_j)(\varphi,\varphi_j)_0\varphi_j \in \Phi.$$

The voltage potential is identified as the solution of the Φ'-valued SDE.

$$dX_t = A'X_t dt + dW_t$$

where A' is the adjoint operator of A and W_t is a Φ'-Brownian motion with a certain covariance function $E(W_t,\varphi)(W_j,\psi) = (t \wedge s)Q(\varphi,\psi)$.

More generally, suppose A_t generates a strongly continuous contraction evolution operators $T(s,t)$, $s\leq t$ on Φ. Assume the following conditions on A_t: For any T and large enough m, there exists a $p>m$ such that $|A_t'u|_{-p}\leq K|u|_{-m}$ for all $t\leq T$, and $u\in\Phi_m'$, i.e., as continuous linear operators from Φ_m' to Φ_p', $\{A_t\}_{t\leq T}$ are uniformly bounded.

Then the following Φ'-valued SDE modeling the voltage potential of a neuron has a unique solution:

$$dX_t = A_t'X_t dt + dW_t$$
$$X_0 = \xi_o.$$

Moreover, the solution can be explicitly written as:

$$X_t = T'(0,t)\xi_0 + W_t + \int_0^t A_s'T_{s,t}'W_s\, ds.$$

Here, $T_{s,t}'$ denotes the adjoint operator of $T_{s,t}$. Now, consider the system of n-interacting neurons whose voltage potentials are governed by the following SDE:

$$dX_i^{(n)}(t) = (A_t X_i^{(n)}(t) + \frac{1}{n}\sum_{j=1}^{n} b_t(X_i^n(t),\ X_j^n(t))dt + dW_i(t),\ i=1,2,\dots,n \quad (7.1)$$

$X_i^n(0) = \xi \in \Phi'$ where $b_t\colon \Phi'\times\Phi' \longrightarrow \Phi'$ represents the interaction between neurons and $\{W_i(t)\}_{i=1}$ are independent copies of a Φ'-valued Brownian motion.

We require that the interaction $b_t\colon \Phi'\times\Phi' \longrightarrow \Phi'$ satisfy the conditions (SCC), (SLG), (SJC), (SMC), (SJC)' and (MCπ) as in Theorem 6.2.

The existence and uniqueness theorem in Section 3 thus guarantees that system (7.1) has a unique solution. The propagation of chaos in Section 6 asserts that the empirical distribution $\frac{1}{n}\sum_{i=1}^{n}\delta_{X_i^{(n)}} \in \pi(C^T_{\Phi_{-q}})$ converges in probability to a deterministic probability measure $\lambda_0 \in \pi(C^T_{\Phi_{-q}})$ which is the law of the solution of the McKean-Vlasov equation corresponding to (7.1):

$$dX_t = (A_tX_t + b_t[X_t,\lambda_0^t])dt + dW_t,$$

where

$$b[X,\lambda_0^t] = \int_{\Phi'} b(x,y)d\lambda_0^t(y)$$

and λ_0^t is the law of X_t.

Thus the asymptotic behavior of a large system of neurons through mean-interactions becomes asymptotically independent with the distribution governed by the McKean-Vlasov equation (7.2).

8. CHAOTIC SYSTEMS

8.1 Exchangeable systems.

Till now, the initial random variables $X_j^n(0)$, $1\le j\le n$ have been assumed to be i.i.d. random variables. We now relax this condition and assume that $X_j^n(0)$, $1\le j\le n$ are exchangeable random variables for each $n\ge 1$. That is, the law of $X_j^n(0)$ $1\le j\le n$, denoted by $\mu_0^n \in \pi(\Phi')^{\otimes n}$ is a symmetric probability measure on $(\Phi')^{\otimes n}$.

We call the symmetric measures μ_0^n μ_0-chaotic if the following condition holds: For every integer $k\ge 1$ and $f_1,\dots,f_k \in C_b(\Phi')$,

$$\lim_{n\to\infty} \int_{(\Phi')^{\otimes n}} f_1(u_1),\dots,f_k(u_k)d\mu_0^n(\underset{\sim}{u}) = \prod_{i=1}^{k} \int_{\Phi} f_i(u_i)d\mu_0(u_i) \tag{8.1}$$

where $\underset{\sim}{u} =(u_1,\dots,u_n) \in (\Phi')^{\otimes n}$, and μ_0 is a probability measure on Φ'. We assume that the measures μ_0^n are μ_0-chaotic.

In the context of the neuorphysiological model described in section 7, the assumption of exchangeability of the law of $(X_1^n(0),\dots,X_n^n(0))$ for each $n\ge 1$ is equivalent to saying that the particular order in which the neuronal membranes are taken, is immaterial. This is so since the random variables $X_1^n(t),\dots,X_n^n(t)$ for each $t\ge 0$ and $n\ge 1$ turn out to be exchangeable random variables. The μ_0-chaoticity assumption is needed in showing the propagation of chaos result. To see this, consider the simplest case where the drift and diffusion coefficients are identically zero so that (8.1) itself becomes the propagation of chaos statement.

The results of the previous sections hold for the exchangeable model as well if we assume μ_0-chaoticity and that $E\|X_1^n(0)\|_{-m}^{4+\delta} \leq C$ where C is a constant independent of n, and m is the index that appears in the conditions listed in subsection 3.3.

8.2 Finite-dimensional systems

By setting $\Phi_n \equiv \mathbb{R}^d$ with $\|x\|_n^2 = \sum_{i=1}^{d} x_i^2$ for each $n\geq 1$, $\Phi = \mathbb{R}^d = \bigcap_{n\geq 1} \Phi_n$ is seen to be a nuclear space with its strong dual Φ' being isomorphic to $\mathbb{R}^d$. In this case, all the norms $\|\cdot\|_k$, $-\infty < k < \infty$ are one and the same, namely, the Euclidean norm on $\mathbb{R}^d$ denoted by $\|\cdot\|$. Therefore, the indices m,p,q etc. in our conditions can and will be taken to be 1. The canonical maps j_m will not appear in the conditions in this case. Besides, expressions such as $|Q_{\sigma(t,u)}|_{-m,-m}$ will simply read as trace$(\sigma\sigma^*(t,u))$. Also, the condition (SA) is trivially seen to hold for the choice of $\Phi = \mathbb{R}^d$. The propagation of chaos result for the finite-dimensional exchangeable system is given in the next theorem:

<u>Theorem 8.2.1</u>. For each T > 0, let the conditions (SIC), (SCC), (SLG), (SJC) and (SMC) hold. Let $X_j^n(0)$ $1\leq j\leq n$ be exchangeable random variables and let μ_o^n = law of $(X_1^n(0),\dots,X_n^n(0))$ be μ_o-chaotic. Then,

a) The system of SDE's (3.2.1) has a unique strong solution in $(C_{\mathbb{R}^d})^{\otimes n}$.

b) In addition, assume (SJC)' and (MCπ) for each T > 0. If the coefficients b and c are as specified by (5.6) and (5.7), then the McKean-Vlasov equation (5.3) has a unique strong solution in $C_{\mathbb{R}^d}$.

c) Assume the conditions in part (b). Further, assume that there exists $\delta>0$ such that $E\|X_1^n(0)\|^{4+\delta} \leq C$ where C is independent of n. Then, in the notation of section 6

$$\bar{\eta}_n \Rightarrow \delta_{\bar{\lambda}_o} \quad \text{in } \pi(\pi(C_{\mathbb{R}^d})).$$

The above result enables us to compare our results with those of Sznitman [16].

The conditions made by Sznitman are the following:

(i) The initial random variables $X_1^n(0),\dots,X_n^n(0)$ are $\mathbb{R}^d$-valued exchangeable random variables and are bounded.

(ii) μ_o^n = law of $(X_1^n(0),\dots,X_n^n(0))$ on $(\mathbb{R}^d)^{\otimes n}$ are μ_o-chaotic, where μ_o is a probability measure on $\mathbb{R}^d$

(iii) The drift and diffusion coefficients are uniformly bounded and satisfy uniform Lipschitz conditions in the space and time variables.

(iv) The covariance form Q is the identity matrix.

In the next paragraph the conditions (i) through (iv) are compared with those that appear in Theorem 8.2.1.

First (SIC) and the moment condition introduced in part (c) of Theorem 8.2.1 are satisfied since (i) says that the initial variables are bounded. (SCC) is verified as follows:

For $u,v \in \mathbb{R}^d$, $0 \le t \le T$, and $h(t,u,v) = \sigma(t,u) + c(t,u,v)$,

$$\begin{aligned}
&|2a(t,u)\cdot u + 2b(t,u,v)\cdot u + \mathrm{tr}(hh^*(t,u,v))| \\
&\qquad \le 2\|a\|_\infty\|u\| + 2\|b\|_\infty\|u\| + \mathrm{trace}(hh^*(t,u,v)) \\
&\qquad \le \theta(1+\|u\|^2)
\end{aligned}$$

by using the uniform boundedness of the coefficients and condition (iv).

The verifications of (SLG) and (SJC) given the Conditions (i) to (iv) are simple and hence left to the reader. (SMC) can be verified by using Lipschitz continuity of the coefficients. (SJC) and (MCπ) are obtained by continuity and boundedness of the coefficients. Thus, our set of conditions for the propagation of chaos is weaker than that imposed in the finite-dimensional set-up by Sznitman [16]. The finite-dimensional result of Leonard [11] is close in spirit to Theorem 8.2.1 and hence, a comparison of the two is left to the reader.

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