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# ROBUST STABILITY AND PERFORMANCE FOR LINEAR SYSTEMS WITH STRUCTURED AND UNSTRUCTURED UNCERTAINTIES

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This report documents research in the areas of stability and performance robustness. Stability robustness in the presence of structured and unstructured uncertainty is discussed relative to nonminimum phase systems. Simultaneous stabilization under unstructured uncertainties and nonconservative stability robustness bounds are addressed. Performance robustness is discussed using the 'D-Stability' formulation. Keywords:

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# Foreword

This report was prepared by the Department of Aeronautical and Astronautical Engineering, The Ohio State University, Columbus, Ohio under the Air Force Contract F33615-88-C-3603. The work was performed under the direction of Captain Mary Manning and Dr. Siva Banda of the Air Force Flight Dynamics Laboratory, Wright Research and Development Center, Wright Patterson Air Force Base, Ohio.

The technical work was conducted by Dr. R.K. Yedavalli, Principal Investigator and Mr. Y. Liu, a graduate research assistant. The contract was performed during the period July 1988 - April 1990.

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# I. INTRODUCTION AND PERSPECTIVE

The published literature on 'robust stabilization and performance' can be viewed from two viewpoints based on the characterization of uncertainty, namely (i) structured uncertainty and (ii) unstructured uncertainty. Much of the literature addresses only one of these two types of uncertainties while there exist few results dealing with the combined uncertainty case. The 'structured uncertainty' is treated both in the time domain, state space framework (where it manifests itself in the form of perturbations in the entries of the matrices describing the system behavior) and in the frequency domain, transfer function framework (where it manifests itself in the form of perturbations in the coefficients of the numerator and denominator polynomials of the transfer function). However, the 'unstructured uncertainty' (resulting mostly from the high frequency unmodeled dynamics) is treated exclusively in the frequency domain framework.

The robust stabilization problem in the presence of structured uncertainty has been given much attention in recent years. References [1]-[23] provide a representative list of the type of work being carried out in this area. On the other hand, the robust stabilization problem under unstructured uncertainty has been receiving attention much longer and has produced many interesting results, notably the $H_\infty$ theory and the LQG/LTR theory, among others [24]-[33]. Of these, the majority of the results are concerned with stability robustness analysis whereas techniques dealing with the robust synthesis problem are relatively few [31]-[32]. The combined uncertainty problem, being more difficult, produced the least literature [34]-[39].

There are three paths being taken to address the combined uncertainty problem. One strategy is to convert the structured uncertainty into some form of unstructured uncertainty formulation and then use the methods available for the unstructured uncertainty problem. The structured singular value method [17] falls into this category. However, these methods are overly conservative because they do not take advantage of the structure of the uncertainty. The other path has been to consider a 'weakly unstructured perturbation' (in which we assume both phase and magnitude information of the uncertainty are available, whereas in 'highly unstructured uncertainty' we assume that only magnitude information is available) and then convert this into a structured uncertainty formulation and then use the methods available for structured uncertainty. Of course, this strategy obviously is not suitable for the combination of highly unstructured uncertainty and structured uncertainty. Finally, the third path is to derive a new problem formulation in which the method explicitly accommodates both structured uncertainty and unstructured uncertainty. Reference [37] was one of the first papers to consider the design problem for the combined case. In this paper, this problem is cast completely in the frequency domain, transfer function setting. Recently, an interesting new framework was proposed to solve the synthesis problem for the combined uncertainty case [38]-[39].

In this research, some results which consider only one type of uncertainty as well as some results which treat the combined case are presented. The report is organized as follows: Section II considers the robust stabilization aspect. In Subsection 2.1, a robust stabilizing controller design for MIMO (Multiple Input Multiple Output) systems for a class of nonminimum phase systems is presented. Subsection 2.2 presents some interesting results on the simultaneous stabilization of two plants each having a separate unstructured uncertainty profile. In other words, this problem is one form of the combined uncertainty problem wherein structured uncertainty is taken to be simultaneous stabilization (i.e. the uncertain parameter takes on discrete values instead of continuum of values). Subsection 2.3 presents nonconservative stability robustness bounds for linear systems with real parameter variations. In Section III, the aspect of performance robustness is addressed. Finally, Section IV offers some concluding remarks and recommendations for future research.

## II. ROBUST STABILIZATION IN THE PRESENCE OF STRUCTURED AND/OR UNSTRUCTURED UNCERTAINTY

Here, we address the issue of robust stabilization for linear systems. First, in Subsection 2.1, the aspect of robust stabilization under structured uncertainty (real parameter variations) is considered. In previous research [15], the control design problem was solved under some restrictive assumptions (such as minimum phase, etc). In this research, we relax the minimum phase assumption, consider a class of nonminimum phase systems, and present a control design method suitable for MIMO systems. Then in Subsection 2.2 the problem of combined uncertainty is addressed. Since it is difficult to solve the control design problem for very general nonminimum phase systems under the combined uncertainty case, a different problem formulation in which the structured uncertainty is cast as the simultaneous stabilization problem (for two plants) is presented. In this framework, there are no severe assumptions made on the nominal plants. Finally, in Subsection 2.3, the problem of analyzing the stability robustness of a given controller is addressed. This problem is cast as one of obtaining stability robustness measures for a given stable system. Using known results on the characteristic polynomials of a Bialternate sum matrix, nonconservative perturbation bounds are obtained to maintain stability.

### 2.1 Robust Stabilization Under Structured Uncertainty for a Class of Nonminimum Phase Systems:

Robust stabilization of linear uncertain systems with real parametric uncertainty has been an active topic of research in recent years. In fact, the research documented in the two final reports of the contracts F33615-86-K-3611 and F33615-84-K-3606 carried out by this principal investigator [40]-[41] serve as useful reference for this research (along with references therein). In the very few methods available for control design with guaranteed stability robustness, a common assumption is the minimum phase assumption. Attempts to relax this minimum phase assumption and still guarantee robust stability turned out to be quite nontrivial, and only partial inroads have been made in this area. Towards this direction, in [42], a class of nonminimum phase Single Input Single Output (SISO) systems were considered. In this subsection, we present a design method useful for MIMO systems, for a class of nonminimum phase systems, namely those with unstable poles only at the origin.

This problem is solved in an indirect way as described below. We first discuss the robust stabilization of a 'type L' feedback system. We then show that the control design strategy for this problem is directly applicable to the robust stabilization of parameter uncertain systems with unstable poles being only at the origin (with possible right half plane zeros).

### 2.1.1. Robust Stability of Type L Feedback Systems:

In many engineering applications, control system designs utilizing some type of integral feedback are common. For example, it is well known that a single-input/single-output 'type 2' system (with double integrators in the loop transfer function) can track a ramp signal with zero steady-state error. Similarly, 'type L' system design has found use in engineering practice. In [43], the concept of type L system is generalized to multivariable systems. However, it was not until recently that the robust stability of integral feedback control ('type 1' system) has received attention [44].

In this subsection, we consider the robust stability of type L feedback systems. We shall extend the results reported in [44] to general type L systems.

Unless stated otherwise, we shall assume throughout the subsection that the plant is an open loop stable, linear, time-invariant system. For simplicity in notation but without loss of generality, we shall restrict the plant transfer function to be a square matrix relating p inputs to p outputs. Our objective is to propose a simple design technique for type L feedback systems which possess certain stability robustness in the closed-loop. Then this result is directly applied to the problem of robust stabilization of parametric uncertain systems with right half plane zeros, a problem discussed in [42].

### 2.1.2 Preliminaries:

This subsection is concerned with synthesis of type L feedback systems as shown in Figure 1. The plant is assumed to be linear, time invariant and finite dimensional and is described by the following state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^p$ with $m = p$, and the eigenvalues of the matrix A are all in the open left half plane. The transfer function matrix P(s) is represented by

$$P(s) = C(sI - A)^{-1}B. \tag{2.2}$$

The compensator K(s) is assumed to be of the form

$$K(s) = K_L(s) = \frac{K_{L-1} + K_{L-2}s + \dots + K_0 s^{L-1}}{s^L} \tag{2.3}$$

where $K_i$'s $\in \mathbf{R}^{p \times m}$, with $m = p$ and $L \geq 1$. The problem in consideration is to synthesize the $K_i$'s such that the closed-loop system admits certain stability robustness. It will be shown that the aforementioned synthesis task is closely related to singular perturbation theory. The next result will be repeatedly used throughout this section.

**Theorem** 2.1 [45, p. 52]. Consider the singularly perturbed (autonomous) system

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = \begin{bmatrix} F & G \\ H & J \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{2.4}$$

where $F \in R^{n \times n}$, $J \in R^{p \times p}$ and $\epsilon \geq 0$ is a scalar. Suppose that matrices J and $F-GJ^{-1}H$ are both stable (having all eigenvalues in the open left half plane). Let

$$\epsilon^* = \frac{1}{\|J^{-1}\|\left(\sqrt{\|F-GJ^{-1}H\|} + \sqrt{\|G\|}\sqrt{\|J^{-1}H\|}\right)^2}. \tag{2.5}$$

Then for all $\epsilon \in (0, \epsilon^*)$, the system (2.4) is asymptotically stable.

<u>**Remark 1:**</u> The estimate of $\epsilon^*$ in (2.5) is conservative. The exact value of $\epsilon^*$ can be computed from a more complex procedure as discussed in [46].

Using Theorem 2.1, we can easily obtain the following result which was discussed extensively in [44].

**Theorem** 2.2: Let the feedback compensator K(s) be of the form (2.3) with L = 1. Then, there exists $K_0 = \epsilon_0 E_0$ such that the closed-loop system is stable whenever $\epsilon \in (0, \epsilon_0^*)$ for some $\epsilon_0^* > 0$, $E_0 \in R^{p \times p}$ and if and only if the matrix $P(0) = -CA^{-1}B$ is nonsingular.

Proof: Suppose the matrix $CA^{-1}B$ is nonsingular, then $E_0 \in R^{m \times m}$ can be found such that $-E_0CA^{-1}B$ has all eigenvalues in the open left half plane. If the feedback compensator as in (2.3) is used with L = 1, then we have

$$u(t) = \varepsilon_0 \int_0^t E_0 y(\tau)\, d\tau = \varepsilon_0 \int_0^t E_0 C x(\tau)\, d\tau \tag{2.6}$$

as the feedback control law. Differentiating both sides of (2.6) yields

$$\frac{1}{\varepsilon_0} \dot{u}(t) = E_0 C x(t)\,. \tag{2.7}$$

We may write the above equation as a unified matrix equation with (2.1) as below:

$$\begin{bmatrix} \frac{1}{\varepsilon_0} \dot{u} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} O & E_0 C \\ B & A \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix}. \tag{2.8}$$

Define $\epsilon t = \tau$, then $\frac{dx}{dt} = \varepsilon \frac{dx}{d\tau}$ and $\frac{du}{dt} = \varepsilon \frac{du}{d\tau}$ . Hence, (2.8) can be rewritten as

$$\begin{bmatrix} \frac{du}{d\tau} \\ \varepsilon_0 \frac{dx}{d\tau} \end{bmatrix} = \begin{bmatrix} O & E_0 C \\ B & A \end{bmatrix} \begin{bmatrix} u(\tau) \\ x(\tau) \end{bmatrix}. \tag{2.9}$$

It should be clear that the stability of the closed-loop system is equivalent to the stability of the singularly perturbed system (2.8). We can now apply Theorem 2.2 to (2.9) to conclude the existence of $\epsilon_0^* > 0$ such that the closed-loop system is stable for all $\epsilon \in (0,\ \epsilon_0^*)$. Conversely, if the closed-loop system is stable for all $\epsilon \in (0,\ \epsilon_0^*)$ for some $\epsilon_0^* > 0$ and $E_0 \in R^{p \times p}$, then by singular perturbation theory, we must have the stability of matrices A and $-E_0CA^{-1}B$, i.e. the eigenvalues of A and $-E_0CA^{-1}B$ are all in the open left half plane. Note that the stability of matrix A resulting from this theorem is superfluous since we have already assumed it to be stable.

Clearly, $\epsilon_0^*$ represents the gain margin of the closed-loop system. Hence, the larger value of $\epsilon_0^*$ implies a better stability robustness.

<u>Remark 2:</u> The value of $\epsilon_0^*$ can be estimated from (2.5) with $F = 0$, $J = A$, $G = E_0C$ and $H = B$. Further, the matrix $E_0$ can be used to maximize the estimate (2.5). Further, Remark 1 can also be used if a maximum value of $\epsilon_0^*$ is needed.

It is noted that the assumption on equal number of inputs and outputs is not a severe restriction. Indeed, if the transfer function matrix of the plant is not square, then the condition presented in Theorem 2.2 can be termed as the matrix $\begin{bmatrix} -A & B \\ -C & O \end{bmatrix}$ having full rank. In this case, we may in principle employ the technique in [47] to square down the plant by static or dynamic compensation while keeping the full rank condition. The details are omitted here.

2.1.3. <u>Main Result</u>:

In this section, we consider synthesis of a type L compensator as defined in (2.3) for $L > 1$. The next theorem is the main result.

**Theorem 2.3:** Let the feedback compensator K(s) be of the form (2.3) with $L \geq 1$ and $K_i = \epsilon_i E_i$. Then, the closed-loop system can be stabilized by some K(s) of the form (2.3), if the matrix $P(0) = -CA^{-1}B$ is nonsingular.

Proof: The theorem is true for $L = 1$ in light of Theorem 2.2. By induction, it is assumed that the theorem is also true for $L = m-1$. That is, the closed-loop system is stable for $K(s) = K_{m-1}(s)$ which is, in fact, equivalent to the matrix

$$F_{m-1} = \begin{bmatrix} A & -BC_{m-1} \\ B_{m-1}C & A_{m-1} \end{bmatrix} \tag{2.10}$$

having all eigenvalues in the open left half plane where $(A_{m-1}, B_{m-1}, C_{m-1})$ is the realization of $K_{m-1}(s)$ with $K_i = \epsilon_i E_i$ for $i = 0, 1, \ldots, m-2$ and $K_{m-2}$ nonsingular. We shall show that $\epsilon_m^{*-1} > 0$ exists such that the closed-loop system is stable for some $E_{m-1} \in \mathbb{R}^{p \times p}$ whenever $\epsilon_{m-1} \in (0, \epsilon_m^{*-1})$. Indeed, with K(s) defined as in (2.3), we have

$$K_m(s) = \frac{\epsilon_{m-1}E_{m-1} + \epsilon_{m-2}E_{m-2} + \ldots + \epsilon_0 E_0 s^{m-1}}{s^m} = \frac{\epsilon_{m-1}E_{m-1}}{s^m} + K_{m-1}(s), \tag{2.11}$$

and the stability of the closed-loop system in Figure 1 is equivalent to the stability of the closed-loop system in Figure 2 because the characteristic equation of the closed-loop system in Figure 1 can be written as

$$\det(I + K_m(s)P(s)) = \det\{I + \frac{\epsilon_{m-1}}{s^m}P(s)(I + K_{m-1}(s)P(s))^{-1}\}\det(I + K_{m-1}(s)P(s) \tag{2.12}$$

Since $K_{m-1}(s)$ and $\frac{E_{m-1}}{s^{m-1}}$ have the same McMillan degree and have all the poles at the origin, we may always find the realizations of $K_{m-1}(s)$ and $\frac{E_{m-1}}{s^{m-1}}$ such that they have the same A and B matrices. Let $(A_{m-1}, B_{m-1}, C_{m-1})$ be the realization of $K_{m-1}(s)$ and $(A_{m-1}, B_{m-1}, C_c)$ be the realization of $\frac{E_{m-1}}{s^{m-1}}$. Define

$$\tilde{P}(s) = \frac{E_{m-1}}{s^{m-1}}P(s)(I+K_{m-1}(s)P(s))^{-1}, \tag{2.13}$$

then the realization of $\tilde{P}(s)$ can be represented by $(\tilde{A}, \tilde{B}, \tilde{C},)$ with

$$\tilde{A}=\begin{bmatrix} A & -BC_{m-1} & 0 \\ B_{m-1}C & A_{m-1} & 0 \\ B_{m-1}C & 0 & A_{m-1} \end{bmatrix}, \tilde{B}=\begin{bmatrix} B \\ O \\ O \end{bmatrix}, \text{and } \tilde{C}^T=\begin{bmatrix} O \\ O \\ C_c \end{bmatrix}. \tag{2.14}$$

Apply the similarity transformation

$$T=\begin{bmatrix} I & O & O \\ O & I & O \\ O & -I & I \end{bmatrix}$$

to realization (3.14) and eliminate the unobservable states. We then obtain a realization $(\hat{A}, \hat{B}, \hat{C})$ of $\tilde{P}(s)$ with

$$\hat{A}=\begin{bmatrix} A & -BC_{m-1} \\ B_{m-1}C & A \end{bmatrix}, \hat{B}=\begin{bmatrix} B \\ O \end{bmatrix} \text{ and } \hat{C}^T=\begin{bmatrix} O \\ C_c \end{bmatrix}. \tag{2.15}$$

It is noted that $\hat{A}$ is the same as $F_{m-1}$ as in (2.10) which is stable by induction. Hence, $\bar{P}(s)$ is stable. Further, it is noted that $\bar{P}(0) = E_{m-1}E_{m-2}$ from (2.13). By induction, $E_{m-2}$ is nonsingular. This implies that there exists $E_{m-1} \in R^{p \times p}$, such that all the eigenvalues of $\bar{P}(0)$ are on open left half plane. Therefore, if we apply Theorem 2.3 to the feedback system in Figure 2 with $P(s) = \bar{P}(s)$ and $\epsilon_0 = \epsilon_{m-1}$, we can then conclude the existence of $\epsilon_m^{*-1} > 0$, such that the closed-loop system is stable whenever $\epsilon_{m-1} \in (0, \epsilon_m^{*-1})$. Hence, the proof is complete.

The condition of nonsingular $P(0) = -CA^{-1}B$ may not be necessary for Theorem 2.3. However, if we intend to use Theorem 2.2 as a synthesis tool for the design of K(s), then, nonsingularity of $P(0) = -CA^{-1}B$ is also necessary. Next, we present an algorithm for synthesis of K(S) as in (2.3) based on Theorems 2.2 and 2.3:

**Algorithm 1:**

Let $P(s) = C(sI - A)^{-1}B$ be defined as in (2.2) satisfying the condition that $P(0) = -CA^{-1}B$ is nonsingular. Define the type L compensator K(s) as in (2.3) with $K_i = \epsilon_i E_i$ and $L \geq 1$. Then, $K_i$ can be synthesized as follows:

Step 1: Choose $E_0 \in R^{p \times p}$ such that $-E_0CA^{-1}B$ has all eigenvalues in the open left half plane;

Step 2: Compute $\epsilon_0^*$ according to (2.5) with $F = 0$, $J = A$, $G = E_2C$ and $H = B$. Choose $\epsilon_0 \in (0, \epsilon_0^*)$ and set $K_0 = \epsilon_0 E_0$;

Step 3: For $m = 2$ to L, do the following:

(i) Define realizations for $K_m(s)$ and $\frac{E_{m-1}}{s^{m-1}}$ such that they have the same A and B matrices. Denote the realization of $K_m(s)$ as $(A_m, B_m, C_m)$ and hence the realization for $\frac{E_{m-1}}{s^{m-1}}$ can be written as $(A_m, B_m, C_c)$ for some $C_c$.

(ii) Choose $\epsilon_{m-1} \in R^{p \times p}$ such that the matrix $E_{m-1}E_{m-2}$ has all eigenvalues in the open left half plane;

(iii) Compute $\epsilon_m^{*-1}$ according to (2.5) with $F = 0$, $J = \hat{A}$, $G = E_{m-1}\hat{C}$ and $H = \hat{B}$, where $(\hat{A}, \hat{B}, \hat{C})$ is defined as in (2.15). Choose $\epsilon_{m-1} \in (0, \epsilon_m^{*-1})$ and set $K_{m-1} = \epsilon_{m-1}E_{m-1}$.

It should be noted that in synthesizing the compensator K(s) with Algorithm 1, we may use Remark 2 to achieve a better bound for $\epsilon_i^*$'s. However, it may increase the computational burden.

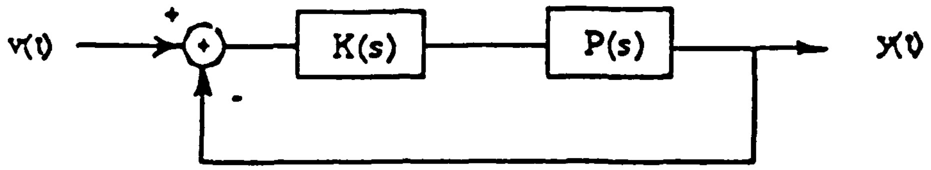



Fig. 1: Standard Single Input Single Output Control System.

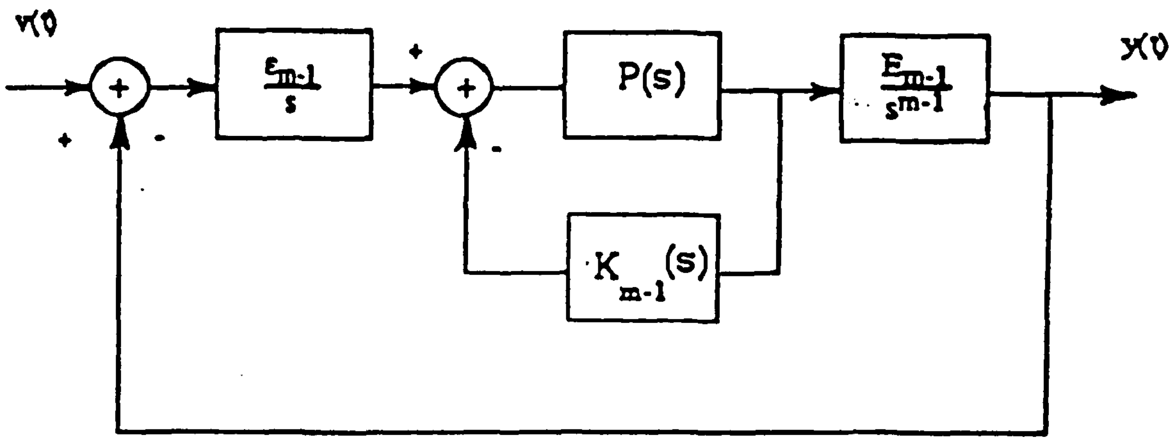



Fig. 2: Integral Feedback Control System

2.1.4. Application to Robust Stabilization of Parametric Uncertain Systems:

Robust stabilization of parametric uncertain systems has been studied extensively in recent years [48]-[49]. A common assumption used in these references is the minimum phase condition. Here, we intend to apply the results in Subsections 2.1.2 and 2.1.3 to a class of parametric uncertain systems without the minimum phase condition. The uncertain system is assumed to have unstable poles only at the origin and have equal number of inputs and outputs which can be represented as

$$P(s,q)=\frac{1}{s^{L}}C(q)\,(sI-A(q))^{-1}B(q), \tag{2.16}$$

where $L \geq 1$, and $q \in Q$ is the uncertain parameter vector. The following assumptions are adopted.

**Assumption A.1** (continuous coefficients): The elements of $A(q)$, $B(q)$ and $C(q)$ are continuous functions of q.

**Assumption A.2** (compact parameter set): Q is a compact set.

**Assumption A.3** (stability): $A(q)$ has all eigenvalues in the open left half plane for all $q \in Q$.

**Assumption A.4** (nonsingular D.C. gain): $-C(q)A(q)^{-1}B(q)$ is nonsingular for all $q \in Q$ and there exists a matrix $E \in R^{p \times p}$ such that the eigenvalues of $-EC(q)A(q)^{-1}B(q)$ are all in the open left half plane for all $q \in Q$.

**Theorem 2.4**: Suppose the transfer function $p(s, q)$ as in (2.16) satisfies Assumptions A1-A4. Then there exists a stable feedback compensator

$$K(s)=N(s)D(s)^{-1}=(N_{L-1} + N_{L-2}s + \ldots + N_0 s^{L-1})(D_m + D_{m-1}s + \ldots + D_0 s^{m-1})^{-1} \tag{2.17}$$

with $m \geq L$ such that the closed-loop is robustly stable for all $q \in Q$.

Proof: Choose $D(s) = D_m + D_{m-1}s + \ldots + D_0 s^{m-1}$ such that $\det(D(s))$ is strictly Hurwitz and $\det(D_0) \neq 0$. Define $\tilde{P}(s, q) := D(s)^{-1}C(q)(sI - A(q))^{-1}B(q)$ with realization $(\hat{A}(q), \hat{B}(q), \hat{C}(q))$. Then for the case $L = 1$, we may apply Theorem 2.2 to $\tilde{P}(s, q)$. Indeed, Assumption A4 implies the existence of $E_0 \in R^{p \times p}$ such that the eigenvalues of $-E_0\hat{C}(q)\hat{A}(q)^{-1}\hat{B}(q)$ are all in the open left half plane for all $q \in Q$. Hence, we may use (2.5) to estimate the value of $\epsilon_0^*$ with $F = 0$, $J = \hat{A}(q)$, $G = E_0\hat{C}(q)$ and $H = \hat{B}(q)$. Since $Q$ is compact and $\hat{A}(q)$, $\hat{B}(q)$, $\hat{C}(q)$ are continuous functions of $q$, $\epsilon_0^* > 0$ exists. Hence, we may set $N(s) = \epsilon_0 E_0$ where $\epsilon_0 \in (0, \epsilon_0^*)$. For the case $L > 1$, we may also follow the proof of Theorem 2.3 with $P(s)$ replaced by $P(s,q)$. The same argument can clearly be carried through in the case $L > 1$ and hence the details are omitted.

### 2.2 Simultaneous Stabilization Under Unstructured Uncertainties:

As indicated in the previous subsection, the current literature on the robust stabilization problem can be attributed to treating two classes of perturbations, namely i) parametric (structured) uncertainty and ii) high frequency unstructured uncertainty. In turn, the literature on the structured uncertainty addresses the problem from two perspectives: namely, the frequency domain (input/output, transfer function) viewpoint, and the time domain (state space) viewpoint. In what follows, these developments are briefly reviewed.

#### 2.2.1. Parametric (Structured) Uncertainty Problem:

One school of thought developed in the area of the robust stabilization problem is the so-called 'simultaneous stabilization' concept. The objective is to design a controller which guarantees stability for a set of plants. This set of plants constitutes the uncertain system in which the parameters of the system are unknown but bounded within given ranges. The simultaneous stabilization problem was introduced by Saeks and Murray [50] and Vidyasagar and Viswanadham [51]. Essentially, the problem formulation is as follows: given a set of k different plants $\{P_1(s), P_2(s), \ldots P_k(s)\}$, does there exist a single compensator C(s) which stabilizes the entire set? Clearly the problem of robust stabilization under parameter variations can be cast in the framework of simultaneous stabilization as follows: Given

$$\{P(s, q) : q \in Q\} \tag{2.18}$$

where Q is a compact index set and q is an uncertain parameter vector, does there exist a single compensator C(s) to stabilize all the plants in the set $\{P(s, q)\}$? If the parameter vector q takes distinct (discrete) values, then the problem is one of stabilizing a discrete, finite number of plants whereas if q takes on a continuum of values within the bounded set Q, then it is one of stabilizing an infinite set of plants (within the bounded set Q).

In their paper, Saeks and Murray [50] develop geometric conditions for simultaneous stabilizability but do not offer any computational criterion for implementing the design. It is mentioned that the computational criterion is given only for the two plant case [52]. In [51] Vidyasagar and Viswanadham generalize the above motion for the multiple input multiple output (MIMO) case. Their result states that the problem of simultaneously stabilizing (k + 1) plants is equivalent to the problem of stabilizing k plants with the added requirement that the compensator itself be stable. In the two plant case, this leads to the requirement that the difference plant ($P_2(s) - P_1(s)$) be stabilizable via a stable compensator. Evidently one can solve the 2 plant case completely using these results since, in that case, the task amounts to that of finding a single plant (namely $P_2(s) - P_1(s)$) for which one can apply the results of Youla, Bongiorno, and Lu [53] which essentially involves the checking of the parity-interlacing property. In [51] it is also shown that given two nxm plants, one can generically stabilize them simultaneously provided either n or m is greater than one. This result is further generalized in Ghosh and Byrnes [54] where it is shown that the generic simultaneous stabilizability of r nxm plants is guaranteed if max $(n, m) \geq r$. Subsequently the problem of determining a computationally feasible criterion for simultaneous stabilizability involving more than 2 plants was addressed in a paper by Emre [55] which presents a solution for the special case with the imposition that all the k + 1 closed loop systems possess the same characteristic equation which clearly limits the application of the method. An important result which addresses the problem of simultaneous stabilization under a <u>continuum of variations</u> in the plant parameters is the one by Wei and Barmish [15]. In this paper, sufficient conditions are given under which a family of single input single output (SISO) plants can be stabilized by a proper (or strictly proper, if desired) stable compensator. Regularity conditions are imposed on the plant family coefficients, and it is assumed that the plant family is minimum phase with one sign high frequency gain. The generalization of these results to the multiple input multiple output (MIMO) case is given in [16]. In the present research, these concepts are used at the problem formulation stage.

2.2.2 <u>Unstructured Uncertainty Problem</u>:

When the system uncertainty is mainly caused by high frequency unmodeled dynamics, it can be characterized in the form of unstructured uncertainty. In this framework, an uncertain system is represented by a class of plants

$$M[P_o(s),\ r(s)] \triangleq \{(I + L(s))P_o(s):\ \|L(j\omega)\|<|r(j\omega)|:\forall\omega\geq o\} \tag{2.19}$$

where $P_o(s)$ is a given nominal plant, $r(s)$ is a prespecified rational function and $L(s)$ is any unknown but stable rational matrix whose norm is bounded by $|r(j\omega)|$ and the uncertainty $L(s)$ is a multiplicative perturbation. A similar model can be given for an additive perturbation also. In this formulation, the design model is characterized by a given fixed nominal plant affected by a norm bounded perturbation.

One of the most popular design methods in the above formulation is the $H^\infty$ method [24]-[30]. In this method, the $H^\infty$ norm of the return difference matrix of the nominal plant is minimized, thereby allowing one to find the maximum tolerable norm bounded (unstructured) perturbation ($|r(j\omega)|$) for guaranteed stability. On the other hand, if the perturbation bound profile is given (i.e. if $r(s)$ is known), then a condition for maintaining closed-loop system stability is given by Doyle and Stein in [28] (see Chen and Desoer [29] and Ridgely and Banda [33] for related work). This condition essentially led to the popular LQG/LTR design technique [30]. In a different direction, combining Nevalinna-Pick theory and Youla's parameterization of controllers [53], robust stability conditions are provided by Kimura [26] for single input single output systems and are further generalized by Vidyasagar and Kimura [32] for multiple input multiple output systems. It is this concept that will be used in this research to address the combined case.

2.2.3. <u>Simultaneous Stabilization Under Unstructured Uncertainty</u>:

Consider the simultaneous stabilization problems for the two-plant case with each nominal plant having different unstructured uncertainties. Let $S_1$ and $S_2$ denote the sets of plants with the nominal systems $P_{10}$ and $P_{20}$, respectively.

$$S_1: = \{ P_1: \|P_1(j\omega)-P_{10}(j\omega)\|_\infty \leq |r_1(j\omega)| \} \tag{2.20a}$$

$$S_2: = \{ P_2: \|P_2(j\omega)-P_{20}(j\omega)\|_\infty \leq |r_2(j\omega)| \} \tag{2.20b}$$

where we always assume $P_1$ and $P_2$ have the same number or fewer unstable poles of $P_{10}$ and $P_{20}$, respectively. Suppose $\Delta P := P_{20}-P_{10}$ is strongly stabilizable, i.e. $\Delta P$ satisfies the so-called *parity interlacing property* (p.i.p.). The objective is to find a single compensator which stabilizes both $S_1$ and $S_2$, simultaneously, and furthermore, makes the closed-loop systems satisfy some specifications of system performance.

At first, we consider the relatively simple case where $P_{10}$ itself is stable. From the assumption commonly made for the stabilization under the unstructured uncertainties, $S_1$ must be stable. In this case, the formulation of the problem can be greatly simplified. The simultaneous stabilizing compensators for $P_{10}$ and $P_{20}$ are now parameterized by a free parameter R, say

$$C[P_{10}] = (I-RP_{10})^{-1}R \tag{2.21}$$

provided R is a stable stabilizing compensator for the difference system $\Delta P=P_{20}-P_{10}$. When the unstructured uncertainties are taken into consideration, we at first give a result for robust stability due to Chen and Desoer [29].

**Lemma** 1: A compensator C is a robust stabilizer of $S_1$ for the unstructured perturbation bound $r_1$, if and only if:

i): C stabilizes $P_{10}$ and

ii): $||C(I+P_{10}C)^{-1}r_1||_\infty<1$

A little modification will give a condition for simultaneous stabilization under the unstructured uncertainties.

**Lemma** 2: a single compensator is a simultaneously robust stabilizer for $S_1$ and $S_2$ if and only if C stabilizes $P_{10}$ and $P_{20}$ simultaneously and

$$||C(I+P_{10}C)^{-1}r_1||_\infty<1 \tag{2.22a}$$

$$||C(I+P_{20}C)^{-1}r_2||_\infty<1 \tag{2.22b}$$

Now the problem has been reduced to finding a compensator $C\in C[P_{10}]$ such that the norm constraints (2.22a) and (2.22b) are satisfied. Note that if C is a simultaneously stabilizing compensator then

$$C=(I-RP_{10})^{-1}R \tag{2.23}$$

Substitute (2.23) into (2.22a) and (2.22b), respectively. The robust stability conditions are now

$$||(I-RP_{10})^{-1}R[I+PP_{10}(I-RP_{10})^{-1}R]^{-1}r_1||_\infty<1 \tag{2.24a}$$

$$||(I-RP_{10})^{-1}R[I+PP_{20}(I-RP_{10})^{-1}R]^{-1}r_2||_\infty<1 \tag{2.24b}$$

with R being the strongly stabilizing compensator for $\Delta P$. This result can be summarized as the following theorem.

**Theorem 2.4:** The two sets of systems $S_1$ and $S_2$ can be simultaneously robustly stabilized by a compensator C if and only if

$$||Rr_1||_\infty<1 \tag{2.25a}$$

$$||R(I+\Delta PR)^{-1}r_2||_\infty<1 \tag{2.25b}$$

where R is a strong stabilizer for $\Delta P$. The simultaneously robust compensator for $S_1$ and $S_2$ is given by (2.23).

The proof of this theorem is available by direct manipulation of (2.22a) and (2.22b) combined with (2.23) and is, therefore, omitted. From Kimura [31] we can see our problem is quite similar to the problem therein, say, an interpolation problem with strictly bounded real (SBR) functions which interpolate the given values at some RHP points. Define

$$G(s):=\begin{bmatrix} Rr_1 & 0 \\ 0 & R(I+\Delta PR)^{-1}r_2 \end{bmatrix} \tag{2.26}$$

The necessary and sufficient conditions for simultaneously robust stabilizability of $S_1$ and $S_2$ under the unstructured uncertainties are

a) G(s) is a SBR function and

b) R is a strongly stabilizing compensator for $\Delta P$.

From now on, we restrict our attention to SISO systems. Let us at first consider the coprime factorization representation of the systems. From [56] we write the right coprime factorization (r.c.f.) representation of the difference system as

$$\Delta P = ND^{-1} \tag{2.27}$$

where N and D are stable right coprime functions. The strongly stabilizing compensator for $\Delta P$ can be now characterized by the units in $H^\infty$ space satisfying some interpolation constraints. More precisely, the strong compensator has the form

$$R = \frac{U-D}{N} \tag{2.28}$$

where U is an arbitrary unit which interpolates D at the RHP zeros of N. By applying (2.28) to G(s), the requirements for G(s) to be an SBR function are now equivalent to

$$\left\| \frac{U-D}{N} r_1 \right\|_\infty < 1 \tag{2.29a}$$

and

$$\left\| \frac{D}{U} \left[\frac{U-D}{N}\right] r_2 \right\|_\infty < 1 \tag{2.29b}$$

Denote the zeros of N in the RHP as $\alpha_1, \alpha_2 \ldots, \alpha_\ell$ and recall the definition of the corresponding Blaschake product B(s)

$$B(s) := \frac{(\alpha_1-s)(\alpha_2-s)\dots(\alpha_l-s)}{(\alpha_1+s)(\alpha_2+s)\dots(\alpha_l+s)} \tag{2.30}$$

Since B(s) is all-pass, (2.29a) and (2.29b) are now equivalent to the new inequalities which read

$$\left\| \frac{U-D}{N} r_1 B(s) \right\|_\infty < 1 \tag{2.31a}$$

$$\left\| \frac{D}{U} \left[\frac{U-D}{N}\right] r_2 B(s) \right\|_\infty < 1 \tag{2.31b}$$

By denoting

$$\tilde{E}_1 := \frac{U-D}{N} r_1 B(s) \tag{2.32a}$$

$$\tilde{E}_2 := \frac{D}{U} \left[\frac{U-D}{N}\right] r_2 B(s) \tag{2.32b}$$

the interpolation constraints are now reduced to finding $\tilde{E}_i \epsilon SBR$, $i=1,2$, which interpolate zero-value at $\alpha_i$, $i=1,2,\dots,l$. Note that the interpolation problem here is quite simple compared with the standard Nevanlinna-Pick interpolation problem and possesses a very simple solution [57]-[58]. In fact, the desired SBR functions can be parameterized by a free parameter, an arbitrary SBR function, as

$$\tilde{E}_i = B(s)E_i, \quad i=1,2. \tag{2.33}$$

**Theorem 2.5:** The closed-loop systems of $S_1$ and $S_2$ can be robustly stabilized by a compensator C if and only if two arbitrary SBR functions $E_1$ and $E_2$ can be found such that

$$D + \frac{E_1 N}{r_1} \epsilon U \tag{2.34a}$$

$$\Delta P = \frac{r_2}{E_2} - \frac{r_1}{E_1} \tag{2.34b}$$

where U denotes the set of units in $H^{\infty}$.

*Proof*: Necessity: It is a straightforward manipulation.

Sufficiency: Suppose $D+E_1Nr_1^{-1}=U^*$ is a unit. From the properties of the SBR function and the assumptions for the uncertainty size $r_1$ and $r_2$ (without loss of generality $r_1$ and $r_2$ can always be assumed to be a stable and minimal phase function [6]), it is obvious that $E_1r_1^{-1}=(U^*-D)N^{-1}$ is stable. Take $R^*=E_1r_1^{-1}$ as the strongly stabilizing compensator for $\Delta P$. By applying $R^*$, (2.25a) is satisfied. It can also be verified that (2.25b) is satisfied by substituting $R^*$ and (2.34b) into it.

Since the solvability of the problem in theorem 2.5 is not generally available at present, we can only handle some simplified cases by making further assumptions on the radius of the unstructured uncertainties and the difference nominal system $\Delta P$. It seems very possible to extend the results to more general cases in future study.

**Assumption A5**: The radii of the unstructured uncertainties of $S_1$ and $S_2$ are constant and equal, i.e.

$$r_1(s) = r_2(s) = r_0$$

**Assumption A6**: $\Delta P$ itself is minimal phase with relative degree one.

It is clear with the above assumption that any first order unit which interpolates at infinity is qualified for the stabilization purpose. The first-order interpolation at infinity can be simply reduced to some requirement for the high frequency gain (HFG) of the desired unit. More specifically, the interpolation constraint is satisfied if and only if the HFG of the unit is equal to the inverse of the HFG of $\Delta P$. Let the HFG of $\Delta P$ be $1/k$. Then $\Delta P$ can be rewritten in the form

$$\Delta P = \frac{1}{k}\frac{n}{d} = \frac{1}{k}\left[\frac{n}{c}\right]\left[\frac{d}{c}\right]^{-1} \tag{2.35}$$

where n and d are monic polynomials and deg(d)=deg(n)+1, c is an arbitrary monic polynomial which serves as the common determinator in forming the coprime factorization representation of ΔP. Then the desired unit should have the form

$$u = k\prod_{i=1}^{v}\frac{(s+a_i)}{(s+b_i)} \tag{2.36}$$

where $a_i$ and $b_i$, i=1,2,...v. are all constants with positive real parts. The simplest unit is zero-order which is a constant k. It is our observation that in the cases of minimal phase ΔP, the order of the employed unit is not a big issue. The following is the comparison of the cases with the unit of order zero and the unit with order v. (2.34) is reduced to

$$\left\|kr_0\left[\frac{c-d}{n}\right]\right\|_\infty < 1 \tag{2.37a}$$

$$\left\|kr_0\left[\frac{c-d}{n}\right]\left[\frac{d}{c}\right]\right\|_\infty < 1 \tag{2.37b}$$

for unit of order zero and

$$\left\|kr_0\left[\frac{(c\prod_{i=1}^{v}(s+a_i)-d\prod_{i=1}^{v}(s+b_i)}{n\prod_{i=1}^{v}(s+b_i)}\right]\right\|_\infty < 1 \tag{2.38a}$$

$$\left\| kr_0 \left[ \frac{c \prod_{i=1}^{v} (s+a_i) - d \prod_{i=1}^{v} (s+b_i)}{n \prod_{i=1}^{v} (s+b_i)} \right] \left[ \frac{d \prod_{i=1}^{v} (s+b_i)}{c \prod_{i=1}^{v} (s+a_i)} \right] \right\|_\infty < 1 \qquad (2.38b)$$

for unit of order V case. By defining

$$\tilde{c}: = c \prod_{i=1}^{v} (s+a_i)$$

$$\tilde{d}: = d \prod_{i=1}^{v} (s+b_i)$$

$$\tilde{n}: = n \prod_{i=1}^{v} (s+b_i)$$

we can see the formulation of the problem with (2.38) is the same as that of the problem with (2.37). Now we concentrate our attention on (2.37). It will be shown the problem can be reduced to a nonlinear programming problem with linear constraints.

If c is chosen as monic (this can always be assumed without loss of generality), it can be easily shown from the definition of the $H^\infty$ norm that

$$\left\| \frac{d(s)}{c(s)} \right\|_\infty = \epsilon \geq 1 \qquad (2.39)$$

From the property of the $H^\infty$ norm [25], we obtain a sufficient condition for simultaneously robust stabilization for $S_1$ and $S_2$ under Assumption A5 and Assumption A6.

**Theorem 2.6**: $S_1$ and $S_2$ can be simultaneously robustly stabilized by a single compensator C under Assumption A5 and Assumption A6 if

$$\left\| kr_0 \left[ \frac{c-d}{n} \right] \right\|_\infty < \frac{1}{\epsilon} \qquad (2.40)$$

We can choose the parameters of c(s), say, the zeros of c(s), such that the $H^\infty$ norm of $d(s)c^{-1}(s)$ is equal to identity. Suppose

$$d(s) = (s+p_1)(s+p_2)\ldots(s+p_m) = \prod_{i=1}^{m}(s+p_i)$$

$$c(s) = (s+c_1)(s+c_2)\ldots(s+c_m) = \prod_{i=1}^{m}(s+c_i)$$

with $c_i$, $i=1,2,\ldots,m$ are all positive. From the multiplicative property of the $H^\infty$ norm, if we choose $c_i>|p_i|$, $i=1,2,\ldots,m$ it always has

$$\left\|\frac{d(s)}{c(s)}\right\|_\infty \le \left\|\frac{(s+p_1)}{(s+c_1)}\right\|_\infty \left\|\frac{(s+p_2)}{(s+c_2)}\right\|_\infty \ldots \left\|\frac{(s+p_m)}{(s+c_m)}\right\|_\infty = \prod_{i=1}^{m}\left\|\frac{(s+p_i)}{(s+c_i)}\right\|_\infty \le 1 \qquad (2.41)$$

Hence, a corollary of Theorem 2.6 follows from the arguments between (2.39) and (2.41).

**Corollary 2.7**: $S_1$ and $S_2$ can be simultaneously robustly stabilized by a single compensator C under Assumption A5 and Assumption A6 if

$$\left\|kr_0\left[\frac{c-d}{n}\right]\right\|_\infty < 1 \qquad (2.42a)$$

under the constraints that

$$c_i > |p_i|, \quad i=1,2,\ldots m. \qquad (2.42b)$$

To verify the condition in Corollary 2.7, it is sufficient to solve a nonlinear programming optimization problem under the constraints (2.42b) with the objective function defined as

$$f(c) = \left\|kr_0\left[\frac{c-d}{n}\right]\right\|_\infty \qquad (2.43)$$

## 2.3. Nonconservative Stability Robustness Bounds Under Structured Uncertainty for Linear State Space Models

The problem of analyzing the stability of a family of matrices arises in many control systems applications. When the system is described by linear state space representation, the plant matrix elements typically depend on some uncertain parameters which vary within a given bounded interval. Traditionally, in the analysis and design of control systems, these parameters are given a specific nominal value and the studies (such as stability, performance, etc.) are carried out using that single value of the parameter set. Naturally, one issue of interest in such a case would be to establish whether the system retains the properties (such as stability) for the entire given range of parameters or not. Since stability is usually the fundamental issue, we concentrate, in this research, on the stability robustness aspect of the problem. Consider the linear state space description

$$\dot{x}(t) = A\,(q)\;x\,(t) \quad q \in Q \tag{2.44}$$

where $x(t) \in R^n$ and q is a p vector of uncertain parameters varying in the prescribed compact set Q. Specifically, let the parameters $q_i$ be given a priori bounds as

$$\underline{q}_i \le q_i \le \bar{q}_i \tag{2.45}$$

We can write the matrix A(q) of (2.44) as

$$A(q) = A_0 + E(q) \tag{2.46}$$

where $A_0$ is an asymptotically stable nominal matrix and E(q) is the perturbation matrix. This nominally asymptotically stable matrix $A_0$ could represent the closed loop (nominal) system matrix for a linear system. In that case, the problem of obtaining bounds on $q_i$ to maintain stability is indirectly related to the assessment of stability robustness of the control gain matrix.

The above issue of determining the stability of an interval parameter matrix has attracted considerable amount of research in the last few years. A very informative account of this research is elegantly summarized in refs. [1]-[2] (and their bibliographies).

There are essentially three paths being followed to answer the question of interval parameter matrix stability.

One framework is based on the polynomial theory, where the matrix stability problem is converted to a polynomial root testing problem and the results available in the uncertain polynomial theory are used. The references [3]-[4] belong to this category. The second viewpoint is the direct matrix theory. That is, trying to solve the problem without converting to polynomials using such time domain techniques as Lyapunov theory and other matrix domain results. The references [5]-[14] fall into this category. Finally the third route follows the frequency domain (transfer function) viewpoint wherein the concepts of multivariable stability margin (MSM) and structured singular value (SSV) are employed (as measures of stability robustness). The research of references [17]-[22] adopts this route. Most of the results to date on this problem are essentially in the form of sufficient conditions except for these references. The results of [3] consider special cases such as $n=2$ with only real spectrum. The recent paper of [20] uses an iterative perturbation domain splitting technique to close the gap between the necessary bound and the sufficient bound. In this research, we present highly nonconservative bounds on the parameters to maintain stability. For some problems, these are almost necessary and sufficient bounds.

In the next subsection, we recall a result that is fundamental to the development of the robustness problem, and in Section III we present the main result.

2.3.1. <u>A Fundamental Stability Result for a Nominal Matrix</u>

In Jury [59], the following necessary and sufficient condition is stated for the stability of an nxn real matrix $A_0$ in terms of the elements of the matrix $A_0$ which was originally due to Fuller [60]. We reproduce the theorem as it is stated in Jury [59].

<u>Theorem 2.8</u> [59]: Let $A_0 = [a_{ij}]$ be a real square matrix of dimension $n > 1$. Let $\tilde{A}_0=[\tilde{a}_{pq,rs}]$ be the 'bialternate sum' of $A_0$ itself. That is, let

$$\tilde{A}_0 = 2A_0 \bullet I_n = A \bullet I_n + I_n \bullet A \tag{2.47}$$

where $\bullet$ denotes bialternate product. Thus, $\tilde{A}_0$ is a square matrix of dimension $m = \frac{1}{2}n(n-1)$ with rows pq ($p=2,3,\ldots n$; $q=1,2\ldots p-1$), columns rs ($r=2,3,\ldots,n$, $s=1,2\ldots r-1$) and elements given by

$$\tilde{a}_{pq,rs} = \begin{cases} -a_{ps} & \text{if } r=q \\ a_{pr} & \text{if } r=p \text{ and } s=q \\ a_{pp}+a_{qq} & \text{if } r=p \text{ and } s=q \\ a_{qs} & \text{if } r=p \text{ and } s=q \\ -a_{qr} & \text{if } s=p \\ o & \text{otherwise} \end{cases} \tag{2.48}$$

For the characteristic roots of $A_0$ to have all their real parts negative (i.e. stable matrix), it is necessary and sufficient that in $(-1)^n$ times the characteristic polynomial of $A_0$, namely

$$(-1)^n \; |A_0-\lambda I_n| \tag{2.49}$$

and in $(-1)^m$ times the characteristic polynomial of $\tilde{A}_0$, namely

$$(-1)^m \; |\tilde{A}_0-\mu I_m| \tag{2.50}$$

the coefficients of $\lambda^i$ ($i=0,1,\ldots n-1$) and $\mu^i = (\lambda_i+\lambda j)^i$ ($i=0, 1,\ldots m-1$) should all be positive.

Before going into the implications of this theorem, we borrow some examples from [59] to illustrate the form of $\tilde{A}_0$ for a few matrices.

For n=2

$$A_0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\tilde{A}_0 = -(a_{11} + a_{22}) \tag{2.51a}$$

For n=3

$$A_0 = \begin{bmatrix} a_{11} & a_{22} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\tilde{A}_0 = \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{33}+a_{11} & a_{12} \\ -a_{31} & a_{21} & a_{33}+a_{22} \end{bmatrix} \tag{2.51b}$$

For n=4

$$A_0 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{2.51c}$$

$$\tilde{A}_0 = \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} & a_{24} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{12} & a_{34} & 0 & -a_{14} \\ -a_{31} & a_{21} & a_{22}+a_{33} & 0 & a_{34} & -a_{24} \\ a_{42} & a_{43} & 0 & a_{11}+a_{44} & a_{12} & a_{13} \\ -a_{41} & 0 & a_{43} & a_{21} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & -a_{42} & a_{31} & a_{32} & a_{33}+a_{44} \end{bmatrix}$$

In [59], a method to form $\tilde{A}_0$ from $A_0$ is presented and for brevity reasons, this method is not discussed here. It suffices to mention that it is computer amenable and easy to build.

Evidently, the above theorem is a very powerful and useful theorem on the stability of the matrix $A_0$. The main reason that this theorem forms the backbone of our further analysis on uncertain matrices is the property of 'convexity' in the individual coefficients (of the characteristic polynomials of $A_0$ & $\tilde{A}_0$) to guarantee stability of $A_0$. By constructing the matrix $\tilde{A}_0$, which henceforth will be called the 'tilde' matrix instead of the long name 'bialternate sum of $A_0$ with itself', we can do away with the interdependence of the coefficients (in the form of Hurwitz conditions). Of course, the price we pay for this simple 'positivity testing' condition is the construction of $\tilde{A}_0$ which is of higher dimension. Recall the well known fact that the Hurwitz region of 2x2 matrices is convex in coefficient space. The above theorem is, in a way, a generalization of this fact for nxn matrices. Clearly this important property of convexity in individual coefficients proves to be extremely useful in our extension to uncertain matrices.

It may be recalled from (2.45) that the perturbation parameter domain is a p dimensional 'hyperrectangle' with $r=2^p$ vertices whose outer (exposed) edges are parallel to the coordinate axes $q_i$. However, in sequel, it proves to be useful to work with a symmetric range in the parameters and hence we will describe the interval parameter matrix (2.44) in an alternate form as follows:

The interval parameter matrix

$$A\ (\underline{q}) = A_0 + \sum_{i=1}^{p} q_i A_i$$

can be written as

$$M\ (\underline{e}) = M^* + \sum_{i=1}^{p} e_i A_i \tag{2.52}$$

where the uncertain parameters $e_i$ have a symmetrical range of $[-\epsilon_i, \epsilon_i]$, $[\epsilon_i>0]$, about the 'center' matrix $M^* = M(0)$. Note that $\epsilon_i$ and $M^*$ can be determined easily from the bounds $\underline{q}_i$, $\bar{q}_i$ and the vertex matrices $M_i$.

Let $\epsilon_i = k_{ij}\, \epsilon_j$. Then $k_{ij}$ determines the shape of the hyperrectangle in the parameter space.

We now state an obvious theorem on interval parameter matrix stability as follows:

Theorem 2.9: The interval parameter matrix (2.52) is stable if and only if, in

$$(-1)^n\, |M(\underline{e}) - \lambda I_n| \tag{2.53a}$$

and $$(1)^m\, |\,\tilde{M}(\underline{e}) - \mu I_m| \tag{2.53b}$$

the coefficients of $\lambda^i$ (which we will call $\beta_i$, $i=0,1, \ldots n-1$) and the coefficients of $\mu^i$ (which we call $\delta_i$, $i=0,1, \ldots m-1$) are all positive.

$$\text{Let } \gamma_i = (\beta_i\ \mu_i) \tag{2.54}$$

However, this theorem is hardly useful from a practical viewpoint. What we need is a finitely computable test or condition to establish the stability of all the matrices in the hyperrectangle. With this in mind, we present a sufficient test for establishing the stability of the uncertain matrix (2.52).

Theorem 2.10: The interval parameter matrix of (2.52) is stable if

$$\text{Max } \epsilon_i = \epsilon < \mu_n \tag{2.55}$$

where $\mu_n$ is obtained as follows:

i) Obtain the coefficients $\gamma_i$ of (2.54) in terms of the parameters $e_i$

ii) Obtain the polynomials $\gamma_i$ (of $e_i$) where the coefficients of the terms in the polynomials are all made negative (except for the constant term which is always positive since it corresponds to the stable matrix M*).

iii) Obtain the single variable polynomials $\bar{\gamma}_i$ from the multivariate polynomials $\gamma_i$ by substituting $\epsilon_i = k_{ij}\,\epsilon_j$

iv) Solve for the positive real roots of all these single variable polynomials.

v) Then $\mu_n$ = minimum of all the positive real roots.

Proof : For the interval parameter matrix M(e) of (2.52), the coefficient $\gamma_i$ of the characteristic polynomials of M(e) and M(e) are polynomic in the variable $e_i$. For example, when there are two uncertain parameters $e_1$ and $e_2$, a typical expression for the coefficient $\gamma_i$ (say $\gamma_3$) is of the form:

$$\gamma_3 = b_{03} + b_{13}e_1 + b_{23}e_2 + b_{33}e_1e_2 + b_{43}e_1^2 + b_{53}e_2^2 + b_{63}e_1^3$$
$$+ b_{73}e_2^3 + b_{83}e_1e_2^2 + b_{93}e_1^2e_2$$

where the constant $b_{03}$ is positive (corresponding to the stable matrix M*)

Now letting $|e_1| = e_{1m}$ and $|e_2| = e_{2m}$, we write

$$\gamma_3 = b_{03} - |b_{13}|e_{1m} - |b_{23}|e_{2m} - |b_{33}|e_{1m}e_{2m} - |b_{43}|e_1^{2m}$$
$$+ |b_{53}|e_{2m}^2 - |b_{63}|e_{1m}^3 - |b_{73}|e_{2m}^3 - |b_{83}|e_{1m}e_{2m}^2$$
$$- |b_{93}|e_{1m}^2\,e_{2m}$$

Then using $e_{1m} = k_{12}\,e_{2m}$ where $k_{12} > 0$ is known, we get

$$\bar{\gamma}_3 = b_{03} - |b_{13}|k_{12}e_{2m} - |b_{23}|e_{2m} - |b_{33}|k_{12}\,e_{2m}^2$$
$$- |b_{43}|k_{12}^2\,e_{1m}^2 - |b_{53}|e_{2m}^2 - |b_{63}|k_1^{32}\,e_{2m}^3$$
$$- |b_{73}|e_{2m}^3 - |b_{83}|k_{12}\,e_{2m}^3 - |b_{93}|k_1^{22}\,e_{2m}^3$$

Clearly, this is a single variable polynomial in $e_{2m}$.

We, then, solve for the positive real roots of $\bar{\gamma}_3 = 0$ and denote the minimum of these as $\alpha_3$. If there are no positive real roots of this equation, it means $\alpha_3 = \infty$. Clearly $\bar{\gamma}_3 > 0$ for all $e_{2m} < \alpha_3$ and $\gamma_3 > 0$ for all $e_{2m} = |e_2| < \alpha_3$ and $e_{1m} = |e_1| < k_{12}\,\alpha_3$.

Clearly $\mu_n = \text{Min}\ \{\ldots\alpha_i\ldots\}$

Note that the proposed sufficient condition is valid for any general type of variation of the parameter vector in the matrix E(q).

# III. PERFORMANCE ROBUSTNESS IN LINEAR UNCERTAIN SYSTEMS

As mentioned earlier, much of the published literature on robustness essentially deals with the stability robustness aspect. However, it is very well recognized that present day control systems are required to not only stabilize the plant but also achieve some prescribed level of performance. Thus, performance robustness is an important feature of any controller design for realistic application problems. Studies discussing the performance robustness aspect have been relatively scarce since it is a much more difficult problem. Here we use the term 'performance' in a qualitative manner, which may include achieving low regulation cost with low control effort in regulation problems or achieving desired transient and steady-state dynamic response with minimum percentage overshoot, damping ratios and steady-state errors, etc. Yet another way of measuring performance is disturbance rejection. It is well recognized in eigenstructure assignment theory that the majority of these performance constraints can be ·dequately modeled in terms of desired closed loop pole regions in the complex plane. Hence, in this research, the performance robustness problem is cast as a 'D-stability' problem where the 'D-regions' are regions in the complex plane (which are symmetric with respect to the real axis and are simply connected) in which the desired closed loop poles are assigned. Thus, assuming the 'nominal' system achieves D-stability, the performance robustness problem is taken to be the problem of achieving D-stability under perturbations. Another reason the D-stability problem formulation is preferred is that it becomes amenable to treat discrete system stability (where the D-region is a unit circle in the complex plane).

In the proposed D-stability problem formulation the uncertainty under consideration is termed 'weakly unstructured.' As mentioned earlier, 'highly unstructured uncertainty' is meant to be the case where only a magnitude bound on the norm of the perturbation is assumed known. In this case, the magnitude bounding function may not be realized by a rational transfer function. On the other hand, in the weakly unstructured case, the bounding function can be realized by a finite dimensional rational transfer function. The D-stability problem under a highly unstructured uncertainty problem is not amenable for analytical formulation. Hence, in this research, the D-stability problem is formulated for weakly unstructured uncertainty.

### 3.1 D-Stability Problem Under Weakly Unstructured Uncertainty

In what follows, we make these assumptions on the D-region:

The D-region is:

i): Symmetric with respect to the real axis.

ii): Simply connected

**Lemma 3.1:** [SISO Systems]:

Suppose g(s) is a rational continuous transfer function. If g(s) has $\gamma$ poles outside $D \cup \partial D$ [i.e. $\rho_i \epsilon \bar{D}$, $i=1\ldots\gamma$. $\bar{D}$ is complementary of D] then g(s)/(1+g(s) has $\rho+\gamma$ poles in $\bar{D}$ if "-1" is not on the Nyquist locus along $\partial D$, and it encircles "-1" $\rho$ times in clockwise sense.

**Lemma 3.2:**

The closed-loop system g(s)/(1+g(s) is D-stable if the Nyquist locus along $\partial D$ encircles "-1" $-\gamma$ times in the clockwise sense.

**Remark:** D-stability of g(s)/(1+g(s) ia assured if and only if

$$|1+g(s)|_{s\epsilon\bar{D}} \neq 0 \quad \text{or} \quad |1+g(s)|_{s\epsilon\bar{D}} > 0 \tag{3.1}$$

**Lemma 3.3:** [Robust D-stability]:

Suppose: i) $g(s) = g_0(s) + \Delta g(s)$

ii) $g_0(s)k(s)/(1+g_0(s)k(s)_\infty)$ is D-stable.

iii) $\Delta g(s)$ is D-stable, and $||\Delta g \cdot R^{-1}||_\infty < 1$

Then the closed-loop system $g(s)k(s)/(1+g(s)k(s))$ is D-stable if:

$$|1+(g_0(s) + \Delta g(s))k(s)|_{s\epsilon\partial D} \neq 0 \quad \text{(or. >0)}$$

Proof: Since the Nyquist locus $\Gamma[1+g_0(s)k(s)]$ encircles the origin $\gamma$ times, and the number of D-unstable poles of $g_0(s)k(s)$ and of $[g_0(s)+\Delta g(s)]k(s)$ are the same (i.e. $\Delta g(s)$ is D-stable), the closed-loop system $g(s)k(s)/(1+g(s)k(s))$ is robust D-stable if $\Gamma(1+(g_0(s)+\Delta g(s))k(s))$ encircles the origin $-\gamma$ times. Now, we need to show that:

Number of encirclements of $\Gamma(1+(g_0+\Delta g)k)$ about "0"

= Number of encirclements of $\Gamma(1+g_0k)$ about "0" *

= $-\gamma$

This is true if and only if

$$|1+(g_0(s) + \epsilon\cdot\Delta g(s))k(s)|_{s\epsilon\partial D} \neq 0 \quad \text{(or positive) for all } 0 \le \epsilon \le 1$$

Sufficiency:

Suppose $|1+(g_0(s^*) + \epsilon\cdot\Delta g(s^*))| = 0$ where $s^*\epsilon\partial D$, the number of encirclements of $\Gamma(1+(g_0(s) + \epsilon\cdot\Delta g(s)k(s))$ about "0" $\neq -\gamma$. Since $\Gamma(\cdot)$ is continuous with $\epsilon$, this contradicts with statement (*).

Necessity:

Since $1+(g_0+\epsilon\cdot\Delta g)k$ is a continuous function of $\epsilon$, if () is not true, there must be a $0\le\epsilon^*\le 1$ such that, $\Gamma(1+(g_0+\epsilon^*\Delta g)k)$ passes through "0". i.e. $|1+(g_0(s^*)+\epsilon^*\Delta g(s^*)| = 0$, $s^*\epsilon\partial D$. Therefore, a contradiction.

This completes the proof.

Remark: We have the following equivalence:

$$|1+(g_0+\epsilon\cdot\Delta g)k|_{s\epsilon\partial D} > 0$$

if and only if

$$|(1+g_0k)[1+\epsilon\Delta g\cdot k(1+g_0k)^{-1}]|_{s\epsilon\partial D} > 0$$

if and only if

$$|1+\epsilon\Delta gk(1+g_0k)^{-1}|_{s\epsilon\partial D} > C$$

if and only if

$$|1+\Delta gk(1+g_0k)^{-1}|_{s\epsilon\partial D} > 0$$

Now we are in a position to state the following theorem:

Theorem: 3.1 [Robust D-stability]:

The closed-loop system $g(s)k(s)/(1+g(s)k(s))$, where $g(s)=g_0(s)+\Delta g(s)$, is D-stable if:

$$|\Delta g(s)k(s)(1+g_0(s)k(s)^{-1})|_{s\epsilon\partial D} < 1 \qquad (3.2)$$

In order to use the above condition for design purposes, we take note of the following result:

Lemma 3.4: [61]:

Every simply connected region D in the complex plane is conformally equivalent to the open unit disc U.

Remark: Obviously, D is also conformally equivalent to the LHP.

Lemma 3.5:

Suppose $\phi$ is the conformal mapping function $\phi^{-1}$: D→LHP. For any real function g(s), we have:

$$\sup_{s=jw} |g(s)| = \sup_{Z\epsilon\partial D} |g(\phi^{-1}(z))|$$

or: $$\sup_{Z\epsilon\partial D} |g(z)| = \sup_{s=jw} |g[\phi(s)]|$$

Define $H_D$ as the following Banach space:

$$H_D \triangleq \{f(s): f(s) \text{ is analytic in } \bar{D} \text{ and } \sup_{s\epsilon D} |f(s) \text{ is bounded}\}.$$

and $$||f(s)||_D \triangleq \sup_{S\epsilon D} |f(s)|$$

Remark: If D LHP and g(s) is D-stable then $||g(s)||_\infty \le ||g(s)||_D$. This is called the Maximum Modulus Theorem [61].

The robust D-stability condition is given by

$$||\Delta g k(1+g_0 k)^{-1}||_D < 1$$

or:

$$||\Delta g(\phi(s))k(\phi(s))[1+g_0(\phi(s)\cdot k(\phi(s))]^{-1}||_\infty < 1$$

Denote $\quad f(\cdot) = k(\cdot)[1+g_0(\cdot)k(\cdot)]^{-1}$

We reduce the condition to:

$$||\Delta g(\phi(s))\cdot f(\phi(s))||_\infty < 1 \tag{3.3}$$

However, in practice we only know the frequency dependent bounding function on the uncertainty, i.e., $|\Delta g(jw)| < |R(jw)|$. The question then is to be able to get a condition for D-stability, knowing only R(jw) (i.e., frequencies only along the imaginary axis).

<u>Theorem 3.2</u>:

Suppose we can make an estimation

$$|\Delta g_0\phi(jw)/\Delta g(jw)| < |\theta(jw)| \tag{3.4}$$

Then: the closed-loop system is D-stable if:

$$|f(\phi(jw))\cdot R(jw)\cdot\theta(jw)| < 1$$

Proof: (3.4) can be rewritten as:

$$\left|\frac{\Delta g(\phi(jw))\cdot f(\phi(jw))\cdot\Delta g(jw)R^{-1}(jw)}{\Delta g(jw)\cdot R^{-1}(jw)}\right| < 1 \tag{3.5}$$

(3.5) can be guaranteed if: $(|\Delta g\cdot R^{-1}|)<1$.

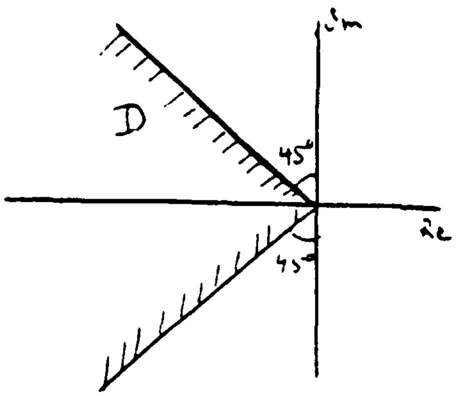
$$\text{i.e. } |f(\phi(jw))\cdot R(jw)\theta(jw)| < 1 \tag{3.6}$$

If $\Delta g$ is strictly unstructured, the estimation of $|\Delta g(\phi(jw))/\Delta g(jw)|$ may not exist. Consider the weakly unstructured case, i.e. when $\Delta g(s)$ can be written as a rational function with parameter variation:

$$\Delta g(s) = E(s,q) \qquad q \epsilon Q$$

In this case, an estimation of $\left|\frac{\Delta g(\phi(jw))}{\Delta g(jw)}\right|$ can be obtained and finally the robust D-stability problem can be reduced to an ordinary robust stability problem which may be solved using the available techniques.

Consider a simple case:




D is given as shown and $\Delta g$ can be written as $\Delta g(s) = a/s+b$ with: $a^- \le a \le a^+ \quad b^- \le b \le b^+$

so:

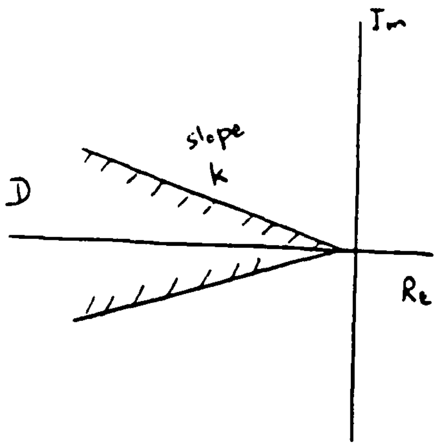
$$\Phi(w) = \left|\frac{\Delta g(\phi(jw))}{\Delta g(jw)}\right| = \left|\frac{\frac{a}{jw-w+b}}{\frac{a}{jw+b}}\right| = \left|\frac{\sqrt{w^2+b^2}}{\sqrt{w^2+(b-w)^2}}\right| \tag{3.7}$$

Straightforward calculation gives:

$$1 \le \Phi(w) \le \sqrt{\frac{3\sqrt{5}-7}{8\sqrt{5}-18}} = 1.618 \tag{3.8}$$

In this case $\theta(jw) = 1.618$.

A more general case is given as follows,




which includes the slope k as a parameter.

$$\Phi_k(w) = \left|\frac{\frac{a}{\boxed{-\frac{w}{k}+jw}+b}}{\frac{a}{jw+b}}\right| = \left[\frac{\sqrt{w^2+b}}{\sqrt{\left(\frac{w}{k}-b\right)^2+w^2}}\right] \tag{3.9}$$

Direct calculation gives:

$$f\max \overset{\Delta}{=} \max_{w} \Phi(w) = \sqrt{\frac{k^2\sqrt{4k^2+1} - 4k^4 - k^2}{(3k^2+1)\sqrt{4k^2+1} - 4k^4 - 5k^2 - 1}} \tag{3.10}$$

The plot fmax (k)~k is given in fig 3 to show the significance of k in estimation of $\Phi(w)$. It can be seen when $k > 0.2$

$$f\max < 5.8.$$

and: $k \to \infty, \quad f\max \to 1.$

Remark: There is one important point to be mentioned here; the final estimation of $\Phi(w)$ is not related to the uncertain parameter bounds. Of course this is only true for first order transfer functions.

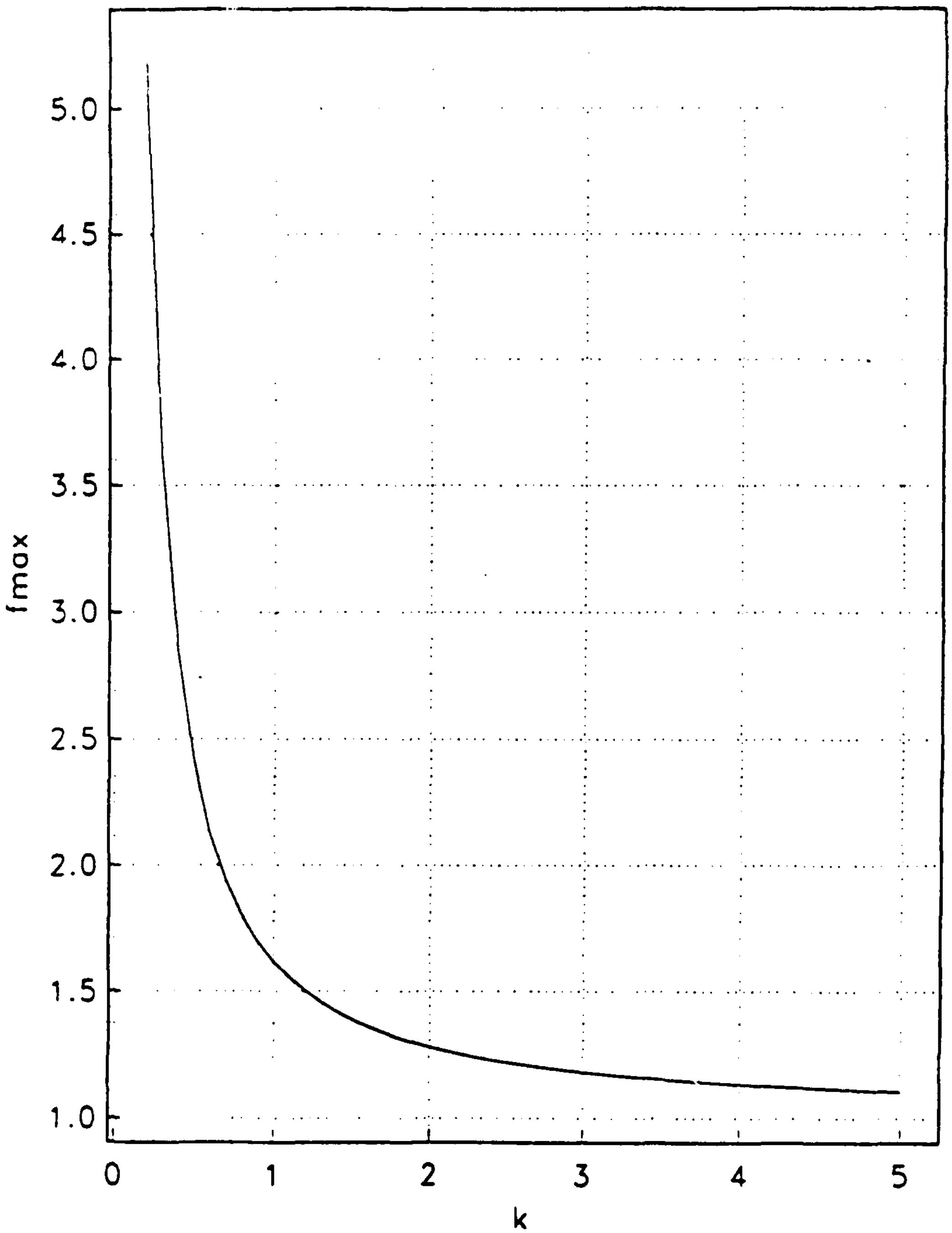



Fig. 3: Variation of k in the estimation of Ø(w)

VI. CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

4.1 Work in Retrospect:

The main theme of the described research under the present contract has been to analyze and synthesize controllers for robust stability and performance under the presence of structured and unstructured uncertainties. First the aspect of robust stabilization under structured uncertainty is considered. A class of nonminimum phase MIMO systems with structured uncertainty is considered and a robust control design scheme is presented using the concept of integral feedback control. Then the aspect of simultaneous stabilization (of two plants) under unstructured uncertainty is considered, and a method for determining a stabilizing controller using $H_\infty$ concepts is presented. These two sections cover the task described as 'task 1' in the original proposal.

Next, the aspect of analyzing the stability robustness of linear systems under structured uncertainty is addressed. In this research, the state space description of the linear system subject to real parameter variations is taken as the uncertain system. Using the theory of characteristic polynomials of some specially constructed matrices, a method for obtaining nonconservative stability robustness measures is presented. This section covers the task defined as 'task 2' of the original research.

Finally the aspect of performance robustness in linear uncertain systems is addressed. The performance robustness problem is cast as a 'D-stabilization' problem, and a method that leads to the design of controllers to keep the closed-loop poles in a designed region under 'weakly unstructured' uncertainty is presented. This result covers the term labeled as 'task 3' of the original proposal.

As it normally occurs in an open research effort such as this one, the time schedule for carrying out these tasks did not adequately match the time schedule originally proposed in the proposal. The research carried out under task 2 consumed much more time than anticipated.

The publications listed as references [62-63-64] are the result of this study.

## Topics for Further Research

1) An interesting area of research is in the lines of the work being carried out in [39], labeled 'mixed $H_2/H_\infty$' problem. This problem formulation seems amenable to carry out robust stability and performance studies under the combined (simultaneous presence) uncertainty case.

2) Another topic for further research would be to obtain necessary and sufficient conditions for robust stability of linear interval parameter systems.

3) One research area that needs attention is simultaneous eigenstructure (eigenvalues as well as eigenvectors) assignment for linear uncertain systems.

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