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THE MATHEMATICAL BASES FOR QUALITATIVE REASONING

Technical Report AIP - 110

*Yumi Iwasaki, *Jayant Kalagnanam
and Herbert A. Simon

Carnegie Mellon University
Department of Psychology
Pittsburgh, PA 15213

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Much reasoning about quantities takes place without use of mathematical formalisms, but solely in terms of ordinary language. A good deal of such qualitative reasoning makes implicit use of the properties of ordinal variables and monotonic transformations. In this paper, we attempt to provide the formal foundations of qualitative analysis and to show how qualitative reasoning arises naturally and simply out of the structure of systems of algebraic equations and ordinary differential equations. Our goal is to explicate, using familiar mathematical formalisms, the practices of researchers in many fields who use qualitative reasoning, and thereby to gain an understanding of the formal assumptions and mechanisms that underlie such analysis. We sketch out some of the properties of functions, and especially continuous differentiable functions, that are invariant under monotonic transformations of the variables, and show how these properties can be used to analyze phenomena where the variables are only defined ordinally.

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Abstract

Much reasoning about quantities takes place without use of mathematical formalisms, but solely in terms of ordinary language. A good deal of such qualitative reasoning makes implicit use of the properties of ordinal variables and monotonic transformations. In this paper, we attempt to provide the formal foundations of qualitative analysis and to show how qualitative reasoning arises naturally and simply out of the structure of systems of algebraic equations and ordinary differential equations. Our goal is to explicate, using familiar mathematical formalisms, the practices of researchers in many fields who use qualitative reasoning, and thereby to gain an understanding of the formal assumptions and mechanisms that underlie such analysis. We sketch out some of the properties of functions, and especially continuous differentiable functions, that are invariant under monotonic transformations of the variables, and show how these properties can be used to analyze phenomena where the variables are only defined ordinally.

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1. Introduction

In the AI literature, the topics of qualitative reasoning and causality have been closely interwoven, and have sometimes been treated as though there was an essential connection between them. In fact, however, the two topics are quite separable. Qualitative reasoning is mainly used to solve problems in comparative statics: if there is a change in the value of an exogenous (independent) variable of a system of variables, what changes will take place in the endogenous (dependent) variables when the system reaches its new equilibrium? Causal reasoning, on the other hand, is mainly used to explain how a system works: the mechanisms of causal connection among its parts and the way in which these causal connections propagate effects through the system.

Of course the two methods of analysis can be combined. The causal ordering of the variables of a system, together with the algebraic signs associated with the causal links provide sufficient information to carry out some (but not all) qualitative analyses. However, given any set of equations, causal or not, that describe a system, and given the signs of the coefficients in these equations, qualitative analysis can be carried out without concern for the causal ordering of the variables.

In previous papers, two of us have explicated the formal foundations of causal ordering for dynamic systems, static systems, and systems containing a mixture of dynamic and static mechanisms. We have also treated the problem of aggregation and disaggregation of causal systems, and have shown both how to carry out these processes and what their consequences are for the descriptions of the systems [4, 3, 5].

In this paper, we propose to provide comparable foundations for qualitative analysis, and in particular, to show how qualitative reasoning arises naturally and simply out of the structure of systems of algebraic equations and ordinary differential equations. Our goal is to sum up and explicate the practices of researchers in many fields who use qualitative reasoning, and thereby to gain an understanding of the formal assumptions and mechanisms that underlie this kind of analysis. There is no new mathematics in what we have to say. Our reason for saying it is that there does not appear to exist any readily available general and systematic account in the AI literature of the simple mathematics that is implicit in what people are doing when they carry out qualitative analyses of systems of interconnected variables.[1] The extensive discussions of

[1] Perhaps the best treatments of the subject -- all now very old -- are to be found in Lotka [6], Evans [2], and Samuelson [8], the first of which applies the analysis mainly to biology, the last two to economics.

qualitative reasoning to be found in the AI literature are not rooted in the familiar formalisms of the algebra and calculus of real numbers but postulate new, independent formalisms.

Since it has not proved easy to make these new formalisms consistent and rigorous, it would be very convenient if we could simply take advantage, for this purpose, of familiar mathematical systems that are already in place. We will show that this can indeed be done, and that standard formalisms can be used to explicate the practices of qualitative reasoning.

2. Qualitative Reasoning Based on Ordinal Relations

Qualitative reasoning treats quantities that are scaled only ordinally, hence are defined only up to arbitrary strictly monotonic transformations. Greater and less, warmer and cooler, taller and shorter are ordinal terms that can be used to describe the qualitative relations between pairs of values of the corresponding ordinal variables (amount, temperature, height). On the other hand, terms like "twice," "half," "four more than" imply cardinality and cannot be applied to ordinal variables. If ardor is an ordinal variable, then one lover cannot be twice as ardent or half as ardent as another -- only more or less ardent.

Of course, the fact that cardinal properties cannot be attributed to ordinal variables does not prevent us from attributing ordinal properties to cardinal variables. Even in the presence of a Celsius thermometer, one day can simply be warmer than another, or cooler. In particular, it may often be convenient to adopt a particular cardinal scale to represent an ordinal variable, restricting ourselves to just the ordinal properties of the scale and renouncing any interpretation of its cardinal properties. Thus, in the (ordinal) Mohs scale of hardness, cardinal numbers are assigned: for example, 2 for gypsum, 4 for fluorite. One can say that fluorite is harder than gypsum, but not that it is twice as hard. Still, it may be convenient to employ the numerical scale, provided that we remember to attach meaning only to the ordinal, and not to the cardinal properties of the scale.

Ordinal relations are invariant under arbitrary strictly positive monotonic transformations of the scale of measurement, while cardinal relations are invariant only under affine transformations. Although ordinal relations are weaker than cardinal relations (the former are implied by, but do not imply, the latter), they have the advantage that they can often be established empirically when we do not have enough information to establish values on a cardinal scale. Thus, when I step out of doors, I may know that it is warmer than it was yesterday without knowing the exact Fahrenheit or Celsius temperature on either day.

If we are considering an equation connecting two variables, $y=f(x)$, we may know that as x increases y will increase ($dy/dx>0$) without knowing the exact functional form of the relation, much less the values of numerical parameters. Or, even if we can know the form of the function and the parameter values for one particular situation, we may be interested in reasoning about the equation generally, over a whole range of situations of which this one is only a special case, but in all of which y remains some strictly monotonically increasing function of x. In both of these cases, we will want to use the methods of qualitative (ordinal) reasoning. Let us now state these matters a little more formally.

First, a few words about notations: we will sometimes use f_x to denote the derivative of f with respect to x. If f is a function of x alone, f_x is the total derivative of f with respect to f. When f is a function of several variables including x, f_x is a partial derivative. Whenever this notation is used, the context should make it clear which is the case.

2.1. Invariance under Monotonic Transformation

If a variable is meant only to define the ordering of some items along a scale, we style the variable *ordinal*. This variable can be replaced by any other that does not disturb the ordering by transposing items. Thus, if $x(x>0)$ is a real variable, but we are interested only in its ordinal properties, we may replace it by x^2 or e^x or $log x$, without changing these properties, for if $x_i > x_j$ then $x^2_i > x^2_j$, $e^{x_i} > e^{x_j}$ and $log x_i > log x_j$.

More precisely, the ordering of a set of elements by a variable associated with that ordering is invariant under any positive strictly monotonic transformation of the variable. Consider $x \in X$ and $y \in Y$. Then $f(x) \cdot X \rightarrow Y$, is a positive strict monotonic transformation if $y_i = f(x_i) > y_j = f(x_j)$ whenever $x_i > x_j$ and vice versa. If a set of objects has been ordered by the assignment of values of x then replacement of all these values by the corresponding values of y will not change the ordering.[2]

For simplicity, we will confine ourselves to variables that are defined over the reals. Let x and y be a pair of such variables, and let us suppose that they are positively monotonically related. That is, there is a function $f(\bullet)$, $f \cdot R \rightarrow R$, such that $x_i > x_j$ iff $y_i = f(x_i) > y_j = f(x_j)$. Now we

[2]In what follows, we will omit the adjective "strict" from our characterization of monotonic transformations, but for simplicity, we will limit ourselves to the relation, >, of strict inequality instead of using the weaker relation, $\geq$, "greater than or equal to."

subject both x and y to positive monotonic transformations, $g(\bullet)$ and $h(\bullet)$ respectively, so that $z=g(x)$ and $w=h(y)$. Since g and h are monotonic we have $z_i > z_j$ iff $x_i > x_j$, and $w_i > w_j$ iff $y_i > y_j$. But since x and y are positively monotonically related $x_i > x_j$ iff $w_i > w_j$. This is readily generalized to:

> **Theorem 1:** If two variables x and y, are positively (negatively) monotonically related, and if x is transformed to z and y to w as $z = g(x)$ and $w = h(y)$, where g and h are arbitrary positive monotonic transformations, then w and z are positively (negatively) monotonically related.

Proof: Since g is a monotonic transformation

$$x_i > x_j \Longleftrightarrow z_i > z_j.$$

Similarly,

$$y_i > y_j \Longleftrightarrow w_i > w_j,$$

since h is monotonic transformation. Given,

$$x_i > x_j \Longleftrightarrow y_i > y_j,$$

it follows that

$$z_i > z_j, \Longleftrightarrow x_i > x_j, \rightarrow y_j > y_j, \rightarrow w_i > w_j. \cdots \bullet$$

For example, suppose we have observed empirically, using ordinal measures, that pressure grows with temperature. Then if we replace our pressure scale with a new ordinally equivalent scale (e.g., $P\bullet$, where $P\bullet$ is the square of pressure), and our temperature scale with an ordinally equivalent scale (e.g., $T\bullet$, where $T\bullet$ is the logarithm of temperature), it follows that $P\bullet$ grows with $T\bullet$.

Moreover, monotonic relations between ordinal variables are transitive in the following sense:

> **Theorem 2:** If x varies positively with y and y varies positively (negatively) with z, then x will vary positively (negatively) with z. The latter relation will be invariant under arbitrary positive (negative) monotonic transformations of x and z.

Proof: Given $y = f(x), f_x > 0$, and $z = g(y), g_y > 0$,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} > 0$$

Therefore monotonicity is transitive. Note this holds for $\frac{dy}{dx}, \frac{dz}{dy} < 0$.

Let $v=h(x)$, $w=i(z)$ such that $v_x, w_z > 0$.

$$\frac{dw}{dv} = \frac{dw}{dz}\frac{dz}{dv} = \frac{dw}{dz}\frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dv} > 0.$$

Note that $\frac{dx}{dv} > 0$ since $\frac{dv}{dx} > 0$. This is also true for $v_x, w_z < 0. \cdots \bullet$

3. Continuous, Differentiable Functions

If functions of ordinal variables can be represented by functions of cardinals that are continuous and possess appropriate derivatives, then for monotonic transformations that are also continuous and possess appropriate derivatives, we can apply the usual operations of the differential calculus to them. Consider ordinal variables x, y, and z, and suppose we find that z varies positively monotonically with x when y is constant, and z varies negatively monotonically with y when x is constant. Then if $z = f(x, y)$ is everywhere differentiable we have $\partial z/\partial x > 0$ and $\partial z/\partial y < 0$, where $\partial z/\partial x$ and $\partial z/\partial y$ are the partial derivatives of z with respect to x and y, respectively.

In general, it would be convenient if we could always simplify a function $z = f(x,y)$ by taking monotonic transformations of our variables in such a way as to transform the function to $z^* = x^* \times y^*$ or $z^* = x^* + y^*$, where z^*, x^*, and y^* are the transformed variables. Is this possible in general? It is not. The proof is not difficult, but we will omit it here.

But the unavailability of such a transformation is not important, since if we restrict ourselves to qualitative reasoning and ordinal properties, we need not specify any but the ordinal properties of the functions we deal with; the exact form of the function does not matter. If, in the last example, x undergoes a positive monotonic transformation to w, so that now $z = g(w,y)$, then if $f_x > 0$, $g_w > 0$.

3.1. An Example

Consider two equations connecting a pair of variables, x and y, with an exogenous parameter, T, in the first equation:

$$y = f(x,T)$$
$$x = g(y)$$

Suppose the system described by these equations is in equilibrium for a particular value of T, and that T is now displaced by an amount, dT. We now solve for the displacements of x and y that restore the system to equilibrium, dx and dy respectively:

$$dy = f_x\, dx + f_T dT$$
$$dx = g_y dy$$

where the subscripts denote partial derivatives. Then we find that:

$$dy = \frac{f_T\, dT}{(1 - f_x\, g_y)}$$

$$dx = \frac{g_y f_T \, dT}{(1 - f_x g_y)} \tag{1}$$

Without loss of generality, we can take $f_T > 0$, for we can always replace T by $-T$.

IA: Now if f_x and g_y are both positive and $f_x g_y < 1$, then $dy/dT > 0$ and $dx/dT > 0$.

IB: If f_x and g_y are both positive, but $f_x g_y > 1$, then $dy/dT < 0$ and $dx/dT < 0$.

IIA: Now if f_x and g_y are both negative and $f_x g_y < 1$, then $dy/dT > 0$ and $dx/dT < 0$.

IIB: If f_x and g_y are both negative, but $f_x g_y > 1$, then $dy/dT < 0$ and $dx/dT > 0$.

III: If $f_x > 0$ and $g_y < 0$, then $dy/dT > 0$ and $dx/dT < 0$.

IV: If $f_x < 0$ and $g_y > 0$, then $dy/dT > 0$ and $dx/dT > 0$.

Table 3-1 summarizes these results. Since the result in cases IA and IB depends on the cardinal magnitude of $f_x g_y$, we might suppose that it is not invariant under monotonic transformations of the variables. But this is not the case. Suppose we transform x monotonically to $w = p(x)$, and y to $z = q(y)$. Then

$$z_w = \frac{q_y f_x}{p_x}, \text{ and } w_z = \frac{p_x g_y}{q_y},$$

so that

$$z_w w_z = \frac{q_y f_x p_x g_y}{p_x q_y} = f_x g_y.$$

The product of the partial derivatives of the two dependent variables is invariant under monotonic transformations of those variables, hence the sign of $(1 - f_x g_y)$ is an ordinal property of the system.

Notice that the conclusions we reach by solving the simultaneous equations are not always the same as those we would reach simply by propagating dT through the equations, step by step. For consider the cases examined above:

- IA: The increase in T causes an increase in y, which in turn causes an increase in x. If we trace the repercussions further, we see that the increase in x will cause a further increase in y, so that there will surely be a net increase in x and y, in agreement with the algebraic analysis in Case IA. Our curiosity may be aroused as to whether the process will converge to an equilibrium or diverge, a question that did not arise explicitly in our algebraic treatment. We will return to this question later. IB is similar.

- IIA: The increase in T causes an increase in y, which causes a decrease in x. The decrease in x will cause a further increase in y, and so on. There will surely be a net increase in y and a net decrease in x, as in algebraic Case IIA. But again, we do not know whether the process converges. IIB is similar.

- III. The increase in T causes an increase in y, which causes a decrease in x. This time a decrease in x will cause a decrease in y, and this will cause an increase in x. In this case, propagating the change around the feedback loop makes the net effect indeterminate, and again we do not know if the process converges.

- IV. The increase in T causes an increase in y, which causes an increase in x. Now the increase in x causes a decrease in y, and a consequent decrease in x. As in Case III, the final results are indeterminate.

Can we say anything more about convergence? When we propagate the initial impulse around the loop, we have $\Delta x_i = g_y \Delta y_i$ and $\Delta y_{i+1} = f_x \Delta x_i$, so that $\Delta y_{i+1} = g_y f_x \Delta y_i$, or $\Delta y_{i+k} = (g_y f_x)^k \Delta y_i$. Hence, the sum $\sum y_i$ will converge if $|g_y f_x| < 1$, and will diverge otherwise. By introducing an explicit dynamic process of equilibration into our assumptions, we have arrived at conditions for convergence of the process, or what is equivalent, conditions for stability of equilibrium.

Examining Table 1, we see that in the cases of convergence (the right-hand half of the table), the net effects of the change in T are the same in sign as the gross effects of the initial propagation to y, and from y to x. The net change in y will always have the same sign as the change in T, while the change in x will have the same or opposite sign as g_y is positive or negative, respectively. The sign of the feedback link, f_x is irrelevant - at most it magnifies or damps the effect of the initial impulse as $f_x g_y$ is positive or negative, respectively.

Net Effects of Change in dT						
			B		A	
			$f_x g_y > 1$		$f_x g_y < 1$	
	f_x	g_y	dx	dy	dx	dy
I	+	+	-	-	+	+
II	-	-	+	-	-	+
III	+	-	-	+	-	+
IV	-	+	+	+	+	+

Table 3-1: Propagation Effects

3.2. Dynamics and Stability

From the results of the last section, we learn that even in very simple systems with feedback loops, the conclusions we can draw about shifts in equilibrium may depend critically upon assumptions of system stability. But the conditions for stability are not uniquely determined by the equilibrium equations alone. They depend also on the process of adjustment when equilibrium is disturbed.

Consider again the system:

$$y = f(x, T)$$

$$x = g(y)$$

The informal argument made earlier, that $f_y g_x < 1$ was a necessary and sufficient condition for stability, was derived implicitly by embedding our equilibrium equations in the following dynamic structure:

$$y_t = f(x_{t-1}, T_t)$$

$$x_t = g(y_{t-1})$$

for which the previous pair of equations is the (stable or unstable) equilibrium. Since we hold T constant after the initial impulse, dT, we may simply disregard this variable and consider the behavior of the system moving from a non-equilibrium initial position, x_0 and y_0, with T fixed. We obtain the second-order difference equation:

$$y_{t+2} = f(g(y_t))$$

For small disturbances from equilibrium, we can replace this by the linear system:

$$y_{t+2} = Ay_t$$

where $A = f_x g_y$. Let

$$y_t = (Ak)^t y_0$$

so that

$$(Ak)^{t+2} = A(Ak)^t,$$

$$Ak^2 = 1$$

$$\rightarrow k = 1/(A)^{1/2}$$

$$\rightarrow y_t = A^{t/2} y_0.$$

This expression for y_t implies that y will converge iff $A < 1$, which was also our conclusion in Section 3.1. In general, assuming different processes of adjustment may lead to different stability conditions, and as a consequence, to different net effects of a shift in parameters.

3.3. Propagation and Stability

The description of value propagation in Section 3.1 is very similar to the procedure for solving confluence equations to derive the dynamic behavior of a system employed by de Kleer and Brown in ENVISION [1]. Consider again the system in the example.

$$y = f(x, T)$$

$$x = g(y)$$

If we assume that these equations represent components or conduits that transmit signals instantaneously (in other words, that the equilibrium relations represented by these equations are restored instantly after a disturbance,) we can derive a description of the dynamic behavior of the system by differentiating the equations as follows;

$$\frac{dy}{dt} = f_x \frac{dx}{dt} + f_T \frac{dT}{dt} \tag{2}$$

$$\frac{dx}{dt} = g_y \frac{dy}{dt} \tag{3}$$

As in the example in Section 3.1, we assume $\frac{dT}{dt} > 0$ and $f_T > 0$.

Case I: Assume $f_x, g_y > 0$. Equations (2) and (3) become the following confluence equations;

$$\partial y = \partial x + \partial T \tag{4}$$

$$\partial x = \partial y, \tag{5}$$

where $\partial\bullet$ denotes the sign of $\frac{d\bullet}{dt}$. Given the input disturbance,

$$\partial T = +, \tag{6}$$

the values of ∂y and ∂x can be determined by propagating the disturbance through the confluence equations. In this case, we have a choice of propagating it through confluence (4) to ∂x or to ∂y.

Case IA: Assuming that ∂x is initially negligible, we propagate the input to ∂y through (4) to obtain

$$\partial y = +, \tag{7}$$

Propagating this to ∂x through (5), we have

$$\partial x = +. \tag{8}$$

Propagating (8) back to (7) does not produce a contradiction. Therefore, (7) and (8) are a solution. Furthermore, if we now assume that ∂T is negligible and propagate (8) to ∂y through (7), the value obtained for ∂y would be the same, +. Therefore, the feedback must be positive.

Case IB: Likewise, if we assume that ∂y is initially negligible, we obtain $\partial x = -$ and $\partial y = -$.

Other cases lead to the same solutions, as shown in Table 3-1. Notice that in solving the problem by sign propagation, as is done in ENVISION, it is the assumptions about the orders of propagation that lead to different solutions in Case IA and IB above. On the other hand, in Section 3.1, assumptions about the value of $f_x g_y$ lead to the two different solutions. Let's explore these assumptions in terms of stability. Equations (2) and (3) can be rewritten as follows:

$$(1 - f_x g_y)\frac{dy}{dt} = f_T \frac{dT}{dt} \tag{9}$$

$$(1 - f_x g_y)\frac{dx}{dt} = g_y f_T \frac{dT}{dt} \tag{10}$$

$f_x g_y < 1$ — $dy/dt, dx/dt > 0$, since $(1-f_x g_y) > 0, f_T > 0$, and $dT/dt > 0$. In terms of propagation this means that $|dy/dt| > |f_x(dx/dt)|$, hence the effect of dT/dt on

dy/dt dominates the effect on dx/dt. This is equivalent to propagating ∂T to ∂y first, i.e. Case IA above.

$f_x g_y > 1$ $\quad$ $dy/dt,\ dx/dt < 0$. In terms of propagation this means that $|dy/dt| < |f_x(dx/dt)|$, hence the effect of dT/dt on dx/dt dominates the effect on dy/dt. This is equivalent to propagating ∂T to ∂x first, i.e. Case IB above.

As shown in the previous section, A is the stable system, and B is the unstable system, though the question of stability cannot be answered by solving confluence equations. Notice that although the stability assumption is weaker than the assertion that ∂x or ∂y is negligible, nevertheless it is strong enough to determine which term in Equation (2) dominates.

4. Non-linear Dynamic Systems with Ordinal Variables

Thus far we have been concerned with static functional relations among variables. Now we turn to time-dependent relations expressible as differential equations. The discussion will be limited to the simplest case of first-order non-linear differential equations in three variables, including time. Any such system can be written as:

$$\frac{dx}{dt} = f(x,y) \tag{11}$$

$$\frac{dy}{dt} = g(x,y) \tag{12}$$

Taking the ratio of Equations (12) to (11), we also have:

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \tag{13}$$

Here we are on more familiar ground, for textbooks on differential equations do often present the kind of qualitative treatment of non-linear systems that we are about to describe here. Hence, we can be brief. In the x-y plane, we can draw the direction field that describes the paths of the system from any initial conditions. At any point in such a path, the slope of the path will be given by equation (13). Figure 4-1 depicts the direction field for (13) in the case where

$$f(x,y) = -x + b\ log(y) \text{ and}$$
$$g(x,y) = -y + a\ log(x), \tag{14}$$

If we set $\frac{dx}{dt} = 0$, then we obtain the curve, $f(x,y)=0$, along which the intersecting paths of the direction field are vertical. Similarly, if we set $\frac{dy}{dt} = 0$, we obtain the curve, $g(x,y)=0$, along which the intersecting paths of the direction field are horizontal. The points where $\frac{dx}{dt} = \frac{dy}{dt} = 0$ are the (stable and unstable) equilibrium points of the system. In Figure 4-1, at A, where the paths for increasing t converge on an equilibrium point, the equilibrium is stable; at B, where they diverge from one, it is unstable. At saddle points, some paths may converge while others diverge, making the

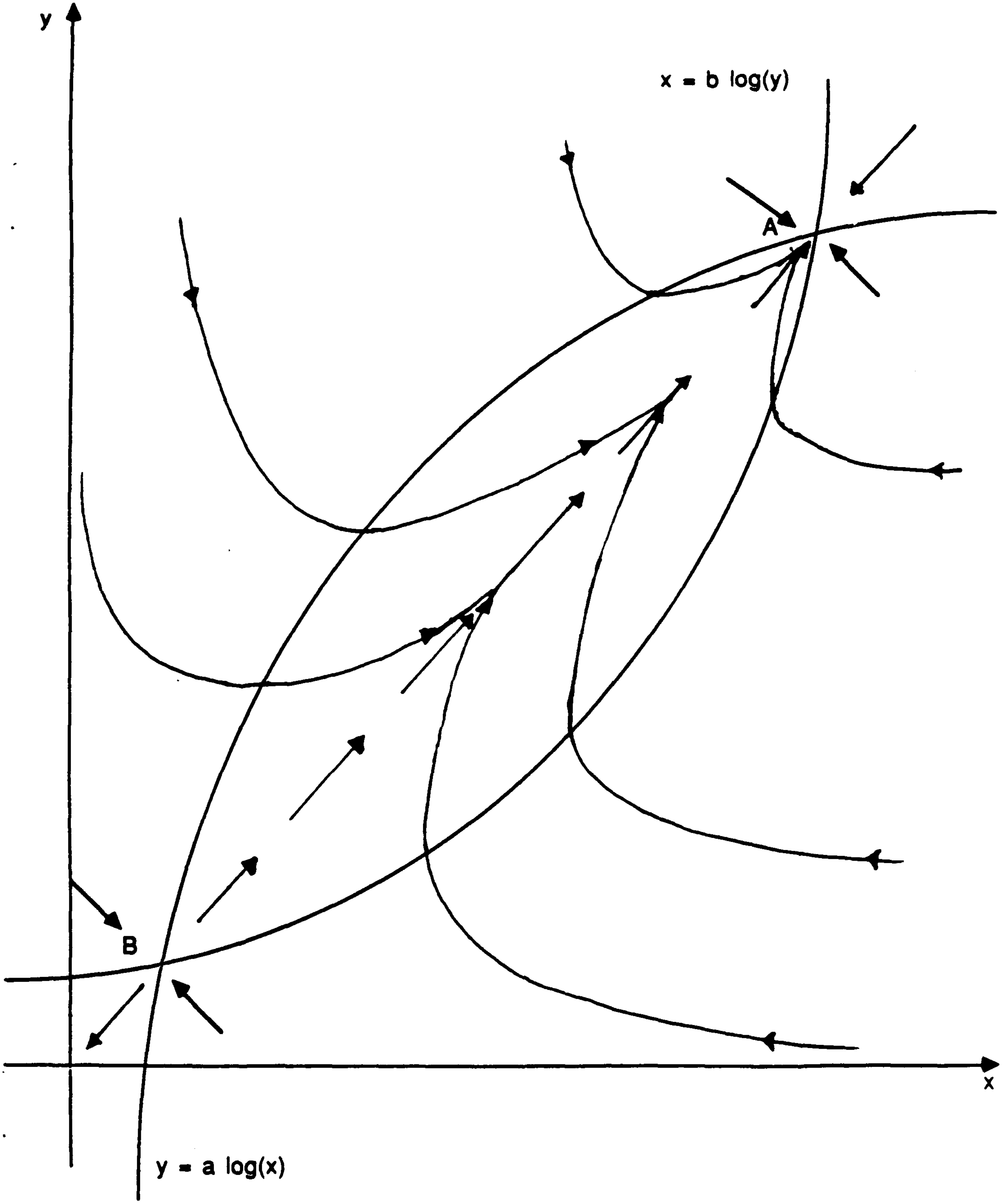

Figure 4-1: Direction Field for Equations (14)

equilibrium unstable.

Sacks developed a program called PLR (Piecewise Linear Reasoner) which examines the properties of the direction field of a system of non-linear differential equations to determine its qualitative behavior [7]. In particular, he divides the space into regions in each of which the system can be approximated by a linear system. In each such region, he examines the direction field to determine the path of the system within and out of the region. The global behavior of the system can be characterized qualitatively in terms of the system's path through these regions.

Weld has proposed a technique, which he calls Comparative Analysis, and which can draw some conclusions about the displacement of the paths and of equilibria with changes in system parameters [9]. In Section 6 we will show how these sorts of conclusions can be drawn directly by our methods of ordinal reasoning.

In non-linear systems of differential equations, the paths of the direction field may also form closed curves in the x-y plane, so that the system describes a periodic motion around the curve. Such paths are called limit cycles. They are stable if neighboring paths all converge to them (from both inside the closed curve and outside), unstable otherwise.

It is obvious that these properties -- the distribution of equilibrium points and limit cycles, of a system of differential equations, and the stability or instability of these equilibria and cycles -- are invariant under ordinal transformations of x and y. For consider such a transformation, $x^* = \phi(x)$, $y^* = \psi(y)$. Since the orderings of *x-coordinates* and of *y-coordinates* are not altered, all direction paths will remain intact. If they converged on a point or limit cycle in the x-y plane, they will converge in the x^*-y^* plane.

Consider the case where the system defined by equations (11) and (12) has an equilibrium point which, by transformations of the coordinates and without loss of generality, we can place at the origin. If we expand $f(\bullet)$ and $g(\bullet)$ in Taylor's series and neglect the higher-order terms, the system is approximated by

$$\frac{dx}{dt} = ax + by \tag{15}$$

$$\frac{dy}{dt} = cx + dy \tag{16}$$

Integrating these equations, we obtain:

$$x = A\, e^{(\lambda_1 t)} + B\, e^{(\lambda_2 t)} \tag{17}$$

$$y = C\,e^{(\lambda_1 t)} + D\,e^{(\lambda_2 t)}$$

where $\lambda = (a+d) \pm [(a+d)^2 - 4(ad-bc)]^{1/2}$. The equilibrium will be stable iff both solutions for λ have negative real parts. But this condition will hold, in turn, iff $(a+d) < 0$ and $(bc-ad) < 0$. >From Equation (15) $(dy/dx)_{dx/dt=0} = -a/b$, while from (16) $(dy/dx)_{dy/dt=0} = -c/d$. So stability depends on which of these two slopes is the greater. If, for example, a and d are negative, while b and c are positive, stability requires that $bc < ad$, whence $-c/d < -a/b$. Then the slope of the curve for $\frac{dx}{dt}$ must be steeper than the slope of the curve for $\frac{dy}{dt}$.

For, by the chain rule for differentiation,

$$\left(\frac{dy^*}{dx^*}\right)_{\frac{dx}{dt}=0} = \frac{dy^*}{dy}\left(\frac{dy}{dx}\right)_{\frac{dx}{dt}=0}\frac{dx}{dx^*}$$

$$\left(\frac{dy^*}{dx^*}\right)_{\frac{dy}{dt}=0} = \frac{dy^*}{dy}\left(\frac{dy}{dx}\right)_{\frac{dy}{dt}=0}\frac{dx}{dx^*}$$

Subtracting these two quantities, we have:

$$\left(\frac{dy^*}{dx^*}\right)_{\frac{dx}{dt}=0} - \left(\frac{dy^*}{dx^*}\right)_{\frac{dy}{dt}=0} = \frac{dy^*}{dy}\frac{dx}{dx^*}\left[\left(\frac{dy}{dx}\right)_{\frac{dx}{dt}=0} - \left(\frac{dy}{dx}\right)_{\frac{dy}{dt}=0}\right]$$

Since the first two factors on the right side are always positive, the sign of the difference of the slopes of the transformed variables is the same as the sign of the difference of the slopes of the original variables. Hence, stability is preserved under positive monotonic transformations of the variables. This can be generalized as follows:

Theorem 3: For a system of differential equations:

$$\frac{dx_i}{dt} = f_i\,(x_1, x_2, \cdots, x_n), \qquad i = 1, \cdots, n,$$

the stability of the equilibrium points is invariant over positive monotonic transformations such as $w_i = g_i\,(x_i)$.

Proof:

The eigenvalues of the transformed equations are the same as those of the original differential equations.

$$\frac{dw_i}{dt} = \frac{dw_i}{dx_i}\frac{dx_i}{dt} = g_i' f_i$$

Linearizing, f_i, g_i' about the equilibrium point, using a Taylor series expansion,

$$f_i = (f_i)_{\bar{x}_0} + \sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)_{\bar{x}_0}(x_j - x_{j0}) + \cdots$$

$$g_i' = g_{i0}' + g_i''(x_i - x_{i0}) + \cdots$$

where $\bar{x}_0 = (x_{10}, x_{20}, \cdots, x_{n0})$ at equilibrium. At equilibrium $f_i = 0$, and with no loss of generality let us assume $\bar{x}_0 = 0$. Therefore, neglecting higher order terms,

$$\frac{dw_i}{dt} = g_i'(x_0)\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)_{\bar{x}_0}(x_j)$$

where, $x_j = g_j^{-1}(w_j)$ and $g_i' = \left(\frac{dw_i}{dx_i}\right)_{x_{i0}}$. Substitute for x_j in terms of w_j by using a linearized relation about the equilibrium point.

$$x_j = g_{w_{i0}} + \left(\frac{dg^{-1}}{dw_i}\right)_{w_{i0}}(w_i - w_{i0}) + \cdots$$

Assume $w_{i0} = 0$, with no loss of generality. Therefore,

$$\frac{dw_i}{dt} = \sum_{j=1}^{n} a_{ij} w_j$$

where, $a_{ij} = \left(\frac{dw_i}{dx_i}\frac{dx_j}{dw_j}\right)$. Notice $a_{ii} = 1$.

In order to determine the eigenvalues of this equation consider the following determinant, say A.

$$\begin{vmatrix} f_{11}-\lambda, & a_{12}f_{12}, & \cdots & a_{1,n-1}f_{1,n} \\ a_{21}f_{21}, & f_{22}-\lambda, & \cdots & a_{1,n-1}f_{2,n} \\ \cdots & & & \\ a_{n1}f_{n1}, & a_{n2}f_{n2}, & \cdots & f_{nn}-\lambda \end{vmatrix} = 0$$

The expansion of this determinant is given by

$$|A| = \sum_{\sigma}(a_{1\alpha}a_{2\beta}\,a_{3\gamma}\cdots a_{n\nu})\,|P_\sigma|$$

where σ is all the permutations of *(1, 2, ..., n)*, and $|P_\sigma| = \pm 1$ is the permutation matrix of A. Therefore,

$$(a_{1\alpha}\,a_{2\beta}\,a_{3\gamma}\cdots a_{n\nu}) = f_{1\alpha}f_{2\beta}f_{3\gamma}\cdots f_{n\nu}$$

where $f_{i\delta} = f_{ii} - \lambda$ for $i = \delta$. Notice that all the terms with non-repeating indices always form a cycle with each index in both positions, i.e. $a_{\bullet\,\alpha}$ *and* $a_{\alpha\,\bullet}$. Therefore $\frac{dw_\alpha}{dx_\alpha}$ is always canceled by $\frac{dx_\alpha}{dw_\alpha}$.

This implies that the characteristic polynomial of the transformed equation is the same as that of the original equation, therefore that the eigenvalues are exactly the same. $\cdots\bullet$

5. Extreme Values

Let $y = f(x)$ be a single-valued function of x, and $x^* = \phi(x)$ and $y^* = \psi(y)$ be positive monotonic transformations of x and y, respectively. Let $f(x)$ have a local maximum at x_0, so that for x in some interval about x_0, $y(x_0) = y_0 > y(x)$. Now $y_i^* > y_j^*$ iff $y_i > y(x)$. Hence, $y^*(x_0^*) > y^*(x^*)$, where $x_0^* = \phi(x_0)$ and $x^* = \phi(x)$ for any other x in the interval. It follows that maxima of the original function correspond to maxima of the transformed function.

This result may seem counterintuitive, since whether a stationary value of a differentiable function is a maximum or a minimum depends on the sign of the second derivative, and this sign is not, in general, invariant under positive monotonic transformations of the variables (concavities can change to convexities, and vice versa). However, it is easy to show that the sign of the *second derivative is invariant* in the neighborhood of a stationary point. The proof, which again makes use of the chain rule for differentiation, is straightforward, and will be omitted here.

6. Comparative Statics

Much qualitative analysis takes the form of comparative statics. The (stable) equilibrium of a system is displaced by a change in one of its parameters, and we wish to know how the values of the system variables are changed when it settles to its new equilibrium. In many cases of interest, *the shapes of the system functions are not known, but only the signs of their derivatives;* and only the sign of the disturbance, and not its magnitude, is given.

To see what kinds of conclusions can be reached under such circumstances, let us consider a simple example from economics. (See Figure 6-1.) For a certain commodity, the quantity, q, will be supplied by producers if the price is $p_S = p_S(q)$, and the same quantity will be purchased by consumers if the price is $p_D = p_D(q)$. More will be supplied if the price is higher ($\frac{dp_s}{dq} > 0$), and less will be purchased if the price is higher ($\frac{dp_D}{dq} > 0$). The market will be in equilibrium when $p_S(q_0) = p_D(q_0) = p_0$.

Now suppose that a sales tax is imposed on the commodity, so that the net supply price for any quantity is increased by the amount of the tax. Call the new supply price $\bar{p}_S = \bar{p}_S(q) > p_S(q)$, and $\bar{p}_S(\bar{q}_0) = p_D(\bar{q}_0) = \bar{p}_0$.

> **Theorem 4:** The equilibrium price will be increased, $\bar{p}_0 > p_0$, and the equilibrium quantity will be decreased, $\bar{q}_0 < q_0$. Moreover, since the tax is such that $\bar{p}_S(q)$ increases with an increase in q, the equilibrium increase in price will be smaller than it would have been if the quantity exchanged had remained constant:
>
> $$\bar{p}_S(q_0) > \bar{p}_S(\bar{q}_0) = \bar{p}_0$$

Proof: These results do not depend on continuity or differentiability of the functions. For all $q > q_0, p_D(q) < p_0$, since the demand price decreases when the quantity increases. But $p_S(q) < p_0$, since the supply price increases under the same conditions.

For all q, $\bar{p}_S(q) > p_S(q)$, the difference being the amount of the tax. Therefore, $p_D(q) < p_0 < p_S(q) < \bar{p}_S(q)$ for all $q > q_0$, and $p_D(q) < \bar{p}_S(q)$ in this range of q. Hence, if a new equilibrium exists, we must have $\bar{q}_0 < q_0$. Now for all $q < q_0$ (including $\bar{q}_0$), $\bar{p}_S(q) < \bar{p}_S(q_0)$, hence $\bar{p}_S(q_0) > \bar{p}_S(\bar{q}_0) = \bar{p}_0$. Similarly, $p_D(q) > p_D(q_0)$, so that $p_D(\bar{q}_0) = \bar{p}_0 > p_0 \cdots \bullet$.

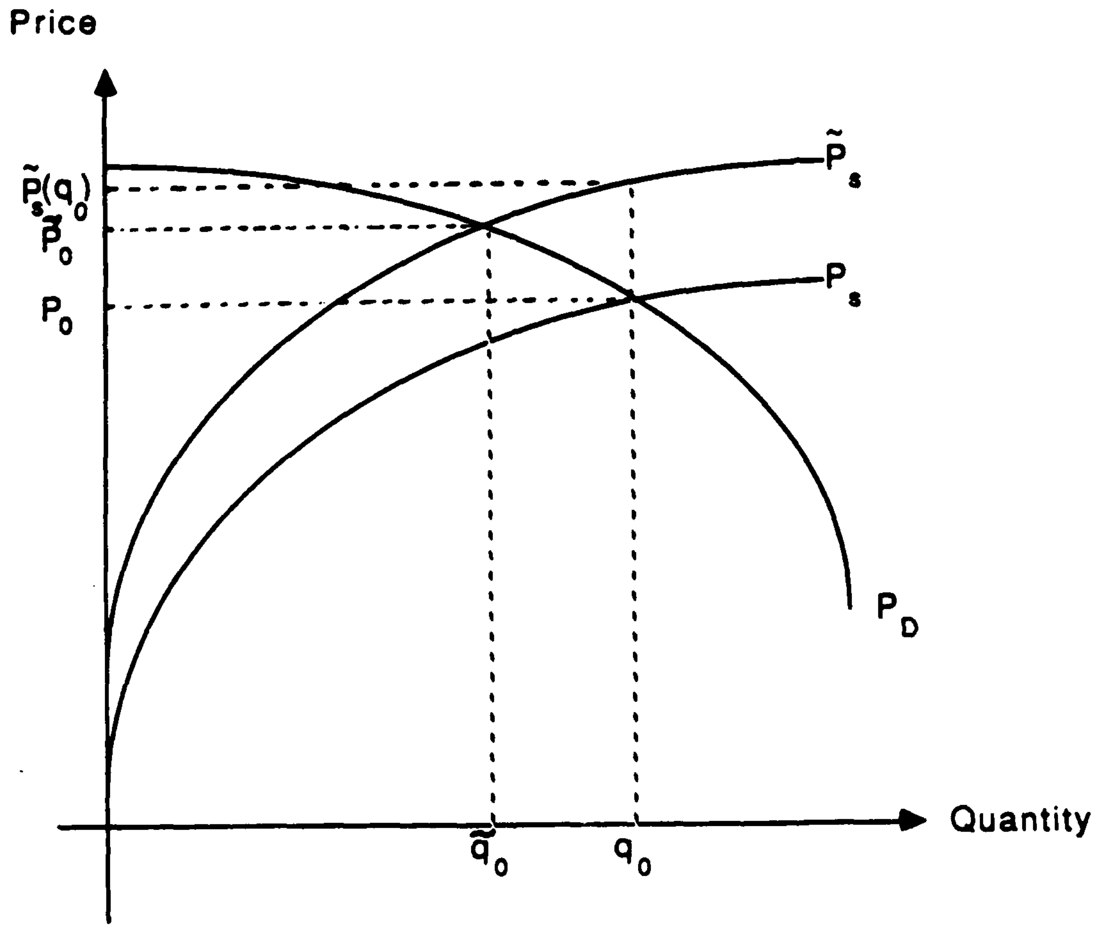

Figure 6-1: Equilibrium for Economic Markets

7. Conclusion

In these pages we have sketched out some of the properties of functions, and especially continuous differentiable functions, that are invariant under monotonic transformations of the variables, and have shown how these properties can be used to analyze phenomena where the variables are only defined ordinally.

The main idea is that, in the case where two variables are positively (negatively) monotonically related, their positive monotonic transforms are positively (negatively) monotonically related. If x is a monotonic function of y and y of z, then x is a monotonic function of z. If we assign the number zero to a monotonic relation if it is positive, and 1 if it is negative, and if $x_1, x_1, x_2, x_3, \ldots x_n$ are a sequence of variables, each of which is a monotonic function of its successor, than the sign of the relation between x_1 and x_n will be positive or negative as the sum, modulo 2, of the numbers assigned to the intervening relations is 0 or 1.

Much reasoning about magnitudes, even in the sciences, takes place without any help from mathematical formalisms, but solely in terms of ordinary language. As the illustrations in this paper suggest, a good deal of this reasoning makes implicit use of the properties of ordinal variables and monotonic transformations. In both the social science and natural science literatures, much reasoning is also done from diagrams, without concern for the exact forms of the functions depicted or the cardinal values of variables. This reasoning also makes use only of the ordinal properties of the magnitudes depicted in the diagrams. We would conjecture that much if not most everyday, "commonsense" reasoning, when it is concerned with magnitudes, is actually implicit reasoning about ordinal relations.

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