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ON THE UPPER AND LOWER CLASSES FOR STATIONARY
GAUSSIAN RANDOM FIELDS ON ABELIAN GROUPS WITH A REGULARLY VARYING ENTROPY

by

J.M.P. ALBIN

Technical Report No. 299

June 1990

ON THE UPPER AND LOWER CLASSES FOR STATIONARY GAUSSIAN RANDOM FIELDS ON ABELIAN GROUPS WITH A REGULARLY VARYING ENTROPY[1]

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We give a complete and relatively explicit characterization of the upper and lower classes for a general stationary Gaussian random field.

1. Introduction. We shall assume that our probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is complete and that $\{\xi(t)\}_{t\in T}$ is an $\Re$-valued separable stochastically continuous standardized Gaussian random field on a pseudo-metric unbounded space (T,ρ) equipped with an abelian group-operation $+$ such that the covariance $r(s,t) \equiv \mathbf{E}\{\xi(s)\xi(t)\}$ satisfies $r(s+u,t+u) = r(s,t)$ for $s,t,u \in T$ and whose bounded subsets are totally bounded in the cannonical pseudo-metric $d(s,t)\equiv[\mathbf{E}\{(\xi(t)-\xi(s))^2\}]^{1/2}$. We also define the entropy $N_S(\varepsilon)$ as the minimum number of closed d-balls $\mathcal{O}_\varepsilon$ of radius ε needed to cover $S\subseteq T$ and $M_S(\varepsilon)$ as the largest n for which there exist $t_1,\dots,t_n\in S$ satisfying $d(t_i,t_j)>\varepsilon$ for each $i\neq j$, and we write $\mathbf{P}_\circ\{S\}$ $\equiv \sup\{\mathbf{P}\{B\} : S\supseteq B\in\mathcal{F}\}$, $\mathbf{P}^\circ\{S\}\equiv\inf\{\mathbf{P}\{B\} : S\subseteq B\in\mathcal{F}\}$, Φ for the standard Gaussian d.f., $\underline{\Phi}\equiv 1-\Phi$, $0\cdot\infty\equiv 0$, $S_\rho(t,\varepsilon)\equiv\{s\in T : \rho(s,t)<\varepsilon\}$, $S(t,\varepsilon)\equiv\{s\in T : d(s,t)\le\varepsilon\}$ and $\sigma(t,\varepsilon)\equiv\sup\{0\vee r(s,t) : s\in T - S_\rho(t,\varepsilon)\}$.

In view of recent tight tail-estimates for local suprema (over d-compact sets) of general Gaussian random fields (cf. e.g., [**1**], [**2**], [**3**], [**16**] and [**21**]), it seems motivated to study also the global behaviour of suprema. Here the only tractable approach seems to be upper and lower classes:

Let Ψ be the class of functions $\psi : T\to[-\infty,\infty]$. Provided that $\sigma(t,\Delta)\to 0$ not too slowly as $\Delta\to\infty$ we prove a zero-one law for the sets

$$E(\psi)\equiv\{\omega\in\Omega : \text{the set } \{t\in T : \xi(\omega;t)>\psi(t)\} \text{ is } \rho\text{-unbounded}\},\ \psi\in\Psi.$$

We also give an explicit characterization of when the different values for $\mathbf{P}\{E(\psi)\}$ occur, i.e., we determine the upper and lower classes for $\xi(t)$.

Related work are e.g., [**5**], [**6**], [**10**], [**13**], [**14**], [**15**], [**17**], [**19**] and [**20**].

2. Main result. Our main result is the following theorem.

THEOREM 1. *Assume that there exists an* $R\in(0,\sqrt{2})$ *such that*

$$\text{(2.1)}\qquad \overline{\lim}_{\varepsilon\downarrow 0} N_{\mathcal{O}_R}(x\varepsilon)/N_{\mathcal{O}_R}(\varepsilon)<\infty \quad \textit{for some } x\in(0,1),$$

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and such that to each $C>0$ and $s\in T$ there exists an increasing sequence $\{\varrho_s(n)\}_{n=0}^{\infty}$, with $\varrho_s(0)=0$ and $\lim_{n\to\infty}\varrho_s(n)=\infty$ for $s\in T$, satisfying

$$\text{(2.2)}\quad \sup_{s\in T}\sum_{\{n\geq 0\,:\,\sigma(s,\varrho_s(n))>0\}} N_{S_\rho(s,\varrho_s(n+1))}(R)\exp\{-C/\sigma(s,\varrho_s(n))\}<\infty.$$

Then $E(\psi)\in\mathcal{F}$ with $\mathbf{P}\{E(\psi)\}$ zero or one for each $\psi\in\Psi$ and moreover

$$\text{(2.3)}\quad \mathbf{P}\{E(\psi)\}=0\ \Leftrightarrow\ \sum_{n=1}^{\infty}N_{\mathcal{O}_{r_n}}\Big(\big(1\vee\inf_{t\in S_n}\psi(t)\big)^{-1}\Big)\underline{\Phi}\Big(1\vee\inf_{t\in S_n}\psi(t)\Big)<\infty$$

for some covering $S_n=S(t_n,r_n)$, $n=1,2,\ldots,$ of T with $r_n\leq R$ for all n.

REMARK 1. Note that, by (2.2), given $\varepsilon>0$ and $t_0\in T$, $r(t,t_0)<\varepsilon$ for $\rho(t,t_0)\geq k$ and k large, which yields $S(t_0,\sqrt{2(1-\varepsilon)})\subseteq S_\rho(t_0,k)$. Thus $\mathcal{O}_\delta$ is d-totally bounded for $\delta<\sqrt{2}$ so that (2.1) makes sense and each covering $\{S(t_n,r_n)\}$ of T with $r_n\leq R$ is infinite.

PROOF: $\boxed{\Leftarrow}$ We have, for $\varepsilon\leq\delta\leq R/3$, (since $N_S(\varepsilon)\leq M_S(\varepsilon)\leq N_S(\varepsilon/2)$),

$$M_{\mathcal{O}_\delta}(\varepsilon)\leq N_{\mathcal{O}_\delta}(\varepsilon/2)\leq\frac{N_{\mathcal{O}_{R/3+\delta+\varepsilon}}(\varepsilon/2)}{M_{\mathcal{O}_{R/3}}(2\delta+2\varepsilon)}\leq\frac{N_{\mathcal{O}_R}(\varepsilon/2)}{N_{\mathcal{O}_R}(4\delta)/N_{\mathcal{O}_R}(R/3)},$$

and this inequality trivially extends to $\varepsilon\leq\delta\leq R$. Writing l for the smallest integer having $x^{-l}\geq 8\delta/\varepsilon$ and $K_1\equiv\sup_{\varepsilon>0}N_{\mathcal{O}_R}(x\varepsilon)/N_{\mathcal{O}_R}(\varepsilon)$ ($<\infty$ by (2.1)), we readily get $K_1^l\leq K_1(8\delta/\varepsilon)^{-\log K_1/\log x}$, and thus

$$\text{(2.4)}\quad M_{\mathcal{O}_\delta}(\varepsilon)\leq N_{\mathcal{O}_\delta}(\varepsilon/2)\leq N_{\mathcal{O}_R}(R/3)\textstyle\prod_{k=0}^{l-1}N_{\mathcal{O}_R}(4\delta x^{k+1})/N_{\mathcal{O}_R}(4\delta x^k)$$
$$\leq K_1 N_{\mathcal{O}_R}(R/3)(8\delta/\varepsilon)^{-\log K_1/\log x}\ \text{for}\ \varepsilon\leq\delta\leq R.$$

Now, by (2.4), $\overline{\lim}_{\varepsilon\downarrow 0}\log\log N_{\mathcal{O}_R}(\varepsilon)/\log(1/\varepsilon)=0$ so $\{\xi(t)\}_{t\in\mathcal{O}_R}$ has an a.s. bounded version; cf. [7], [8] and [18]. Since $N_{S_\rho(t_0,\delta)}(R)<\infty$ for $t_0\in T$, $\delta>0$, ρ-separability yields that $\{\xi(t)\}_{t\in S_\rho(t_0,\delta)}$ is a.s. bounded so

$$\mathbf{E}\{\sup_{t\in S_\rho(t_0,\delta)}\xi(t)^2\}\leq 2\mathbf{E}\{(\sup_{t\in S_\rho(t_0,\delta)}\xi(t))^2\}<\infty;$$

cf. [8], [9] and [11]. Since $\xi(t)$ is stochastically continuous we get

$$d(t,t_0)^2\leq\varepsilon^2+\int_{G_\varepsilon}(\xi(t)-\xi(t_0))^2d\mathbf{P}\leq\varepsilon^2+4\int_{G_\varepsilon}\sup_{t\in S_\rho(t_0,\delta)}\xi(t)^2d\mathbf{P}\to\varepsilon^2$$

as $\rho(t,t_0)\to 0$, where $G_\varepsilon\equiv\{\omega\in\Omega:|\xi(\omega;t)-\xi(\omega;t_0)|>\varepsilon\}$, so $d(t,t_0)\to 0$ as $\rho(t,t_0)\to 0$. Hence d-opens are ρ-open and so $\{\xi(t)\}_{t\in T}$ is d-separable. In view of $\xi(t)$:s (trivial) d-stochastic continuity it follows readily that any countable d-dense subset of $\mathcal{O}_\varepsilon$ is a separator for $\{\xi(t)\}_{t\in\mathcal{O}_\varepsilon}$.

Take $a_0=\min\{(1-x^{1/2})^{1/2}/4,R/2\}$ and $t\in T$, let $C_0=\{t\}$ and let C_n be a $(a/u)x^n$-net in $S(t,a/u)$ with $d(s_1,s_2)>(a/u)x^n$ for $C_n\ni s_1\neq s_2\in C_n$, so $\#C_n\leq M_{\mathcal{O}_{a/u}}((a/u)x^n)$. Write $p_n=(1-x^{1/2})x^{(n-1)/2}$ and $C=\cup_{n=0}^{\infty}C_n$ and choose $t_n(s)\in C_n$ with $d(t_n(s),s)\leq(a/u)x^n$ for $s\in C$.

Then $\xi(s)=\xi(t)+\sum_{n=1}^{N}[\xi(t_n(s))-\xi(t_{n-1}(s))]$ for some N for each $s\in C$. Adapting [4, the proof of Theorem 6] to the present context we get

$$\{\xi(s)>u+1/u,\ \xi(t)\le u\}$$
$$\subseteq \bigcup_{n=1}^{N}\{\xi(t_n(s))-\xi(t_{n-1}(s))>p_n/u,\ \xi(t_n(s))>u,\ \xi(t_{n-1}(s))\le u+1/u\}.$$

Thus, since $d(t_n(s),t_{n-1}(s))\le d(t_n(s),s)+d(s,t_{n-1}(s))\le 2(a/u)x^{(n-1)}$,

$$(2.5)\quad \mathbf{P}\{\sup_{s\in S(t,a/u)}\xi(s)>u+1/u,\ \xi(t)\le u\}$$
$$=\mathbf{P}\{\textstyle\bigcup_{s\in C}\{\xi(s)>u+1/u\},\ \xi(t)\le u\}$$
$$\le \sum_{n=1}^{\infty}\ \sum_{s_1\in C_{n-1}}\ \sum_{s_2\in C_n\cap S(s_1,2(a/u)x^{n-1})}$$
$$\mathbf{P}\{\xi(s_2)-\xi(s_1)>p_n/u,\ \xi(s_2)>u,\ \xi(s_1)\le u+1/u\}.$$

Now take $a\in(0,a_0]$ and $u\ge 1$ so that $r(s_1,s_2)=1-d(s_1,s_2)^2/2\ge 1-2(a/u)^2\ge 1/2$ for $d(s_1,s_2)\le 2(a/u)x^{n-1}$, which yields

$$\Big(\frac{1}{r(s_1,s_2)}-1\Big)\xi(s_1)=\frac{d(s_1,s_2)^2}{2r(s_1,s_2)}\xi(s_1)\le 4(a/u)^2x^{2(n-1)}2u\le p_n/(2u)$$

for $\xi(s_1)\le u+1/u$. Hence we have, for $d(s_1,s_2)\le 2(a/u)x^{n-1}$,

$$(2.6)\quad \mathbf{P}\{\xi(s_2)-\xi(s_1)>p_n/u,\ \xi(s_2)\ge u,\ \xi(s_1)\le u+1/u\}$$
$$\le \mathbf{P}\{\xi(s_2)-r(s_1,s_2)^{-1}\xi(s_1)>p_n/(2u),\ \xi(s_2)\ge u\}$$
$$=\underline{\Phi}\bigg(\frac{\sqrt{2}r(s_1,s_2)p_n/(2u)}{\sqrt{1+r(s_1,s_2)}d(s_1,s_2)}\bigg)\underline{\Phi}(u)\le\underline{\Phi}\bigg(\frac{1-x^{1/2}}{8ax^{(n-1)/2}}\bigg)\underline{\Phi}(u).$$

Combining (2.4)-(2.6) we conclude that, uniformly for $u\ge 1$, as $a\downarrow 0$,

$$(2.7)\quad \underline{\Phi}(u)^{-1}\mathbf{P}\{\sup_{s\in S(t,a/u)}\xi(s)>u+1/u,\ \xi(t)\le u\}$$
$$\le\sum_{n=1}^{\infty}M_{\mathcal{O}_{a/u}}((a/u)x^{n-1})M_{\mathcal{O}_{2(a/u)x^{n-1}}}((a/u)x^n)\,\underline{\Phi}\bigg(\frac{1-x^{1/2}}{8ax^{(n-1)/2}}\bigg)$$
$$\le K_1^2N_{\mathcal{O}_R}(R/3)^2\sum_{n=1}^{\infty}(128x^{-n})^{-\log K_1/\log x}\,\underline{\Phi}\bigg(\frac{1-x^{1/2}}{8ax^{(n-1)/2}}\bigg)=\mathrm{o}(a).$$

Arguing as for (2.5) for $\eta_u(s)\equiv 2u+1/u-\xi(s)$ we deduce for future use that, by (2.4), (2.6) and symmetry, uniformly for $u\ge 1$, as $a\downarrow 0$,

$$(2.8)\quad \underline{\Phi}(u)^{-1}\mathbf{P}\{\inf_{s\in S(t,a/u)}\xi(s)<u,\ \xi(t)\ge u+1/u\}$$
$$=\underline{\Phi}(u)^{-1}\mathbf{P}\{\sup_{s\in S(t,a/u)}\eta_u(s)>u+1/u,\ \eta_u(t)\le u\}$$

$$\leq \underline{\Phi}(u)^{-1}\sum_{n=1}^{\infty}\sum_{s_1\in C_{n-1}}\sum_{s_2\in C_n\cap S(s_1,2(a/u)x^{n-1})}$$

$$\mathbf{P}\{\eta_u(s_2)-\eta_u(s_1)>p_n/u,\ \eta_u(s_2)>u,\ \eta_u(s_1)\leq u+1/u\}$$

$$=\underline{\Phi}(u)^{-1}\sum_{n=1}^{\infty}\sum_{s_1\in C_{n-1}}\sum_{s_2\in C_n\cap S(s_1,2(a/u)x^{n-1})}$$

$$\mathbf{P}\{\xi(s_1)-\xi(s_2)>p_n/u,\ \xi(s_1)\geq u,\ \xi(s_2)<u+1/u\}=\mathrm{o}(a).$$

In order to proceed we observe that, by (2.4), for $a\leq 1$ and $\delta\leq R$,

$$\begin{cases} N_{\mathcal{O}_\delta}(a\varepsilon)/N_{\mathcal{O}_\delta}(\varepsilon)\leq N_{\mathcal{O}_\varepsilon}(a\varepsilon)\leq K_1 N_{\mathcal{O}_R}(R/3)(8/a)^{-\log K_1/\log x},\ \varepsilon\leq R,\\ N_{\mathcal{O}_\delta}(a\varepsilon)\leq N_{\mathcal{O}_R}(aR)\leq K_1 N_{\mathcal{O}_R}(R/3)(8/a)^{-\log K_1/\log x}N_{\mathcal{O}_\delta}(\varepsilon),\ \varepsilon>R.\end{cases}$$

Further $u-2/u\equiv\tilde{u}\geq\frac{1}{2}u\geq 1$ for $u\geq 2$, so that $\tilde{u}+1/\tilde{u}\leq u$, and $\underline{\Phi}(\tilde{u})\leq \frac{1}{\tilde{u}}\phi(\tilde{u})\leq\frac{2}{u}e^2\phi(u)\leq\frac{8}{3}e^2\underline{\Phi}(u)$, where $\phi(u)=(2\pi)^{-1/2}\exp\{-u^2/2\}$. Now

$$\mathbf{P}\{\sup_{s\in S(t,a/u)}\xi(s)>u+1/u,\xi(t)\leq u\}\leq\underline{\Phi}(u)\quad\text{for } u\geq 1$$

for some sufficiently small $a\in(0,a_0]$ (cf. (2.7)). Hence we conclude

$$\mathbf{P}\Big\{\sup_{s\in\mathcal{O}_\delta}\xi(s)>u\Big\}\leq N_{\mathcal{O}_\delta}(a/u)\Big[\mathbf{P}\Big\{\sup_{s\in S(t,a/u)}\xi(s)>u,\xi(t)\leq\tilde{u}\Big\}+\mathbf{P}\{\xi(t)>\tilde{u}\}\Big]$$

$$\leq N_{\mathcal{O}_\delta}(a/u)\Big[\mathbf{P}\Big\{\sup_{s\in S(t,a/\tilde{u})}\xi(s)>\tilde{u}+1/\tilde{u},\xi(t)\leq\tilde{u}\Big\}+\underline{\Phi}(\tilde{u})\Big]$$

$$\leq\tfrac{16}{3}e^2K_1N_{\mathcal{O}_R}(R/3)(8/a)^{-\log K_1/\log x}N_{\mathcal{O}_\delta}(1/u)\,\underline{\Phi}(u)$$

for $u\geq 2$, $\delta\leq R$, so, with $K_2=\frac{16}{3}e^2K_1N_{\mathcal{O}_R}(R/3)(8/a)^{-\log K_1/\log x}/\underline{\Phi}(2)$,

(2.9) $\quad\mathbf{P}\Big\{\sup_{s\in\mathcal{O}_\delta}\xi(s)>u\Big\}\leq K_2N_{\mathcal{O}_\delta}(1/(1\vee u))\,\underline{\Phi}(1\vee u)$ for $\delta\leq R$ and all u.

Assume that the sum (2.3) is finite for a covering $\{S_n\}=\{S(t_n,r_n)\}$ of T with $r_n\leq R$. Taking $m=\sup\{\rho(t_1,t_n):1\leq n<J\}$ where

$$\textstyle\sum_{n=J}^{\infty}N_{\mathcal{O}_{r_n}}\big((1\vee\inf_{t\in S_n}\psi(t))^{-1}\big)\,\underline{\Phi}\big(1\vee\inf_{t\in S_n}\psi(t)\big)<\varepsilon/K_2,$$

completeness yields that $E(\psi)\in\mathcal{F}$ with $\mathbf{P}\{E(\psi)\}=0$ since, by (2.9),

$$\mathbf{P}^{\circ}\{E(\psi)\}\leq\mathbf{P}^{\circ}\{\xi(t)>\psi(t),\text{ for some } t\in T\text{ with }\rho(t_1,t)>m+R\}$$

$$\leq\mathbf{P}\{\textstyle\bigcup_{\{n:\rho(t_1,t_n)>m\}}\{\xi(t)>\inf_{s\in S_n}\psi(s),\text{ for some } t\in S_n\}\}$$

$$\leq K_2\sum_{n=J}^{\infty}N_{\mathcal{O}_{r_n}}\Big((1\vee\inf_{t\in S_n}\psi(t))^{-1}\Big)\underline{\Phi}\Big(1\vee\inf_{t\in S_n}\psi(t)\Big)<\varepsilon.$$

$\boxed{\Rightarrow}$ Write $\sum(\{S_n\};\psi)$ for the sum (2.3) and assume that $\sum(\{S_n\};\psi)=\infty$ for each covering $S_n=S(t_n,r_n)$, $n=1,2,\ldots$, of T with $r_n\leq R$.

Taking $t_0 \in T$ and $2 \le u_1 \le u_2 \le \dots$ with $\mathbf{P}\{\sup_{t\in S_\rho(t_0,n)} \xi(t) > u_n\} \le n^{-2}$ (recall that $\{\xi(t)\}_{t\in S_\rho(t_0,n)}$ is a.s. bounded), the function $\psi^\star(t) \equiv u_1$ for $t \in S_\rho(t_0,1)$ and $\psi^\star(t) \equiv u_n$ for $t \in S_\rho(t_0,n) - S_\rho(t_0,n-1)$, $n \ge 2$, has

$$\mathbf{P}^\circ\{E(\psi^\star)\} \le \lim_{n\to\infty} \mathbf{P}^\circ\{\xi(t) > \psi^\star(t), \text{ for some } t \in T - S_\rho(t_0,n)\} = 0.$$

Clearly $\mathbf{P}_\circ\{A \cup B\} \le \mathbf{P}^\circ\{A\} + \mathbf{P}_\circ\{B\}$ so that $\mathbf{P}_\circ\{E(\psi \wedge \psi^\star)\} = \mathbf{P}_\circ\{E(\psi) \cup E(\psi^\star)\} \le \mathbf{P}_\circ\{E(\psi)\}$ and so, by completeness, it suffices to prove that

$$\text{(2.10)} \qquad \varphi(t) \equiv \big(\psi(t) \wedge \psi^\star(t)\big) \vee 2 \quad \text{has} \quad \mathbf{P}_\circ\{E(\varphi)\} = 1.$$

Take a (p/u)-net $\{s_i\}_{i=1}^n$ in $\mathcal{O}_\delta$ with $d(s_i,s_j) > p/u$ for $s_i \ne s_j$. Since

$$\text{(2.11)} \quad \begin{aligned} M_{\mathcal{O}_{\delta\wedge(kp/u)}}(p/u) &= M_{\mathcal{O}_{\delta\wedge(kp/u)}}((2\delta)\wedge(p/u)) \\ &\le K_1 N_{\mathcal{O}_R}(R/3)\big(8\tfrac{\delta\wedge(kp/u)}{(2\delta)\wedge(p/u)}\big)^{-\log K_1/\log x} \\ &\le K_1 N_{\mathcal{O}_R}(R/3)(8k)^{-\log K_1/\log x} \quad \text{for } \delta \le R,\ k \ge 1 \end{aligned}$$

(again using (2.4)), and since, by arguing as for [**5**, Eq. 2.16], for $x, y > 0$,

$$\text{(2.12)} \quad \mathbf{P}\big\{\xi(s) > x, \xi(t) > y\big\} \le \underline{\Phi}\big(\tfrac12 d(s,t)x\big)\,\underline{\Phi}(y) + \underline{\Phi}\big(\tfrac12 d(s,t)y\big)\,\underline{\Phi}(x)$$

for all values of $r(s,t)$ (although [**5**] only treat $0 \le r(s,t) < 1$), we obtain

$$\begin{aligned} &\textstyle\sum_{i\ne j} \mathbf{P}\big\{\xi(s_i) > u,\ \xi(s_j) > u\big\} \\ &\le 2\,\underline{\Phi}(u) \sum_{i=1}^{n} \sum_{k=1}^{[2\delta u/p]} \sum_{\{1\le j\le n:\, kp/u < d(s_i,s_j) \le (k+1)p/u\}} \underline{\Phi}\big(\tfrac12 d(s_i,s_j)u\big) \\ &\le 2n\,\underline{\Phi}(u) K_1 N_{\mathcal{O}_R}(R/3) \sum_{k=1}^{\infty} (8(k+1))^{-\log K_1/\log x}\, \underline{\Phi}\big(\tfrac12 kp\big) \le \tfrac12 n\,\underline{\Phi}(u) \end{aligned}$$

for $u > 0$, $\delta \le R$ and for some $p \ge 1$ (not depending on δ). Since, by (2.4),

$$\begin{aligned} N_{\mathcal{O}_\delta}(1/u) &\le N_{\mathcal{O}_{\delta\wedge(p/u)}}(\delta\wedge(1/u)) N_{\mathcal{O}_\delta}(\delta \wedge (p/u)) \\ &\le K_1 N_{\mathcal{O}_R}(R/3)(8p)^{-\log K_1/\log x} n \quad \text{for } \delta \le R \end{aligned}$$

we conclude, taking $K_3 = \frac12 K_1^{-1} N_{\mathcal{O}_R}(R/3)^{-1}(8p)^{\log K_1/\log x}$,

$$\text{(2.13)} \quad \begin{aligned} \mathbf{P}\big\{\sup_{t\in\mathcal{O}_\delta} \xi(t) > u\big\} &\ge \mathbf{P}\big\{\sup_{1\le i\le n} \xi(s_i) > u\big\} \\ &\ge n\,\underline{\Phi}(u) - \textstyle\sum_{i\ne j} \mathbf{P}\big\{\xi(s_i) > u,\ \xi(s_j) > u\big\} \\ &\ge K_3 N_{\mathcal{O}_\delta}(1/u)\,\underline{\Phi}(u) \quad \text{for } u > 0 \text{ and } \delta \le R. \end{aligned}$$

Now, combining (2.9) and (2.13) we get, for each choice of $\{S_n\}$,

$$\text{(2.14)} \quad K_2 \textstyle\sum(\{S_n\};\varphi) \ge \displaystyle\sum_{n=1}^{\infty} \mathbf{P}\Big\{\sup_{t\in S_n} \xi(t) > \inf_{t\in S_n} \varphi(t)\Big\}$$

$$\geq \sum_{n=1}^{\infty} \mathbf{P}\Big\{\sup_{t\in S_n}\xi(t) > \inf_{t\in S_n}\psi(t)\wedge\psi^\star(t)\Big\}\mathbf{P}\Big\{\sup_{t\in S_n}\xi(t)>2\Big\}$$

$$\geq K_3\,\underline{\Phi}(2)\sum(\{S_n\};\psi)=\infty.$$

Let $r_t \equiv \sup\{r>0 : r\inf_{s\in S(t,r)}\varphi(s)<a\}$ for $a\in(0,1]$, $t\in T$, so that $a/\psi^\star(t)\leq r_t\leq a/2$. Taking $\delta_k\uparrow r_t$ with $\delta_k\inf_{s\in S(t,\delta_k)}\varphi(s)<a$ we get

$$(2.15)\quad \begin{cases} a/(\inf_{s\in S(t,r_t)}\varphi(s))\geq \lim_{k\to\infty} a/(\inf_{s\in S(t,\delta_k)}\varphi(s))\geq \lim_{k\to\infty}\delta_k=r_t, \\ a/(\inf_{s\in S(t,r_t)}\varphi(s))\leq \lim_{\varepsilon\downarrow 0} a/(\inf_{s\in S(t,r_t+\varepsilon)}\varphi(s))\leq \lim_{\varepsilon\downarrow 0} r_t+\varepsilon=r_t. \end{cases}$$

Ordering $\mathcal{S}\equiv\{A\subseteq T : A\ni s\neq t\in A\Rightarrow d(s,t)>r_s\wedge r_t\}$ partially by $A\leq B\Leftrightarrow A\subseteq B$, a chain $\{A_\alpha\}\subseteq\mathcal{S}$ has upper bound $\cup\{A_\alpha\}$ so that, by Zorn's Lemma, $\mathcal{S}$ has a maximal element $\mathcal{C}$. Here $\mathcal{C}$:s maximality readily yields $\cup_{t\in\mathcal{C}}S_t=T$, where $S_t\equiv S(t,r_t)$. Further, since $\#\mathcal{C}\cap S_\rho(t_0,n)\leq M_{S_\rho(t_0,n)}(a/u_n)<\infty$, we have $\#\mathcal{C}\leq\aleph_0$ and, by (2.14), $\sum(\{S_t\};\psi)=\infty$. Writing $\varphi_t=\inf_{s\in S_t}\varphi(s)$ we therefore obtain, by (2.4) and (2.15),

$$(2.16)\qquad \textstyle\sum_{t\in\mathcal{C}}\underline{\Phi}(\varphi_t)\geq \frac{(8/a)^{\log K_1/\log x}}{K_1 N_{\sigma_R}(R/3)}\sum_{t\in\mathcal{C}}N_{S_t}(ar_t)\underline{\Phi}(\varphi_t)=\infty.$$

Now let $\varphi_t^\star\equiv\varphi_t+1/\varphi_t$, $J_t\equiv\{\omega\in\Omega : \xi(\omega;t)>\varphi_t^\star,\ \inf_{s\in S_t}\xi(\omega;s)\geq\varphi_t\}$ and $\mathcal{C}_m^N\equiv\{t\in\mathcal{C} : m\leq\rho(t_0,t)<N\}$. Letting I_t "indicate" J_t we get

$$(2.17)\quad \mathbf{P}_\circ\{E(\varphi)\}=\mathbf{P}_\circ\Big\{\bigcap_{m=1}^{\infty}\bigcup_{N=m}^{\infty}\bigcup_{t\in\mathcal{C}_m^N}\{\xi(\omega;s)>\varphi(s),\text{ for some } s\in S_t\}\Big\}$$

$$\geq\mathbf{P}\Big\{\bigcap_{m=1}^{\infty}\bigcup_{N=m}^{\infty}\{\textstyle\sum_{t\in\mathcal{C}_m^N}I_t>0\}\Big\}$$

$$\geq \overline{\lim_{m\to\infty}}\ \overline{\lim_{N\to\infty}}\Big[1+\mathrm{Var}\Big\{\sum_{t\in\mathcal{C}_m^N}I_t\Big\}\Big/\Big(\mathbf{E}\Big\{\sum_{t\in\mathcal{C}_m^N}I_t\Big\}\Big)^2\Big]^{-1},$$

where we used Hölder's inequality as in [**5**]. Now write

$$\mu_{s,t}=\mathbf{P}\{\xi(s)>\varphi_s^\star,\ \xi(t)>\varphi_t^\star\}-\mathbf{P}\{\xi(s)>\varphi_s^\star\}\mathbf{P}\{\xi(t)>\varphi_t^\star\}\ \text{ for } s,t\in\mathcal{C}$$

and note that, by arguing as above, $\underline{\Phi}(\varphi_t^\star)\geq\frac{3}{8}e^{-2}\underline{\Phi}(\varphi_t)$ so that, by (2.8),

$$\mathbf{E}\{I_t\}=\underline{\Phi}(\varphi_t^\star)-\mathbf{P}\{\xi(t)>\varphi_t^\star,\ \inf_{s\in S_t}\xi(s)<\varphi_t\}\geq\tfrac{3}{16}e^{-2}\underline{\Phi}(\varphi_t)\ \text{ for } t\in\mathcal{C}$$

and $a\leq a_1$, for some $a_1\in(0,R]$ (not depending on t). Since, by (2.8),

$$\mathrm{Var}\{\textstyle\sum_{t\in\mathcal{C}_m^N}I_t\}\leq\displaystyle\sum_{s,t\in\mathcal{C}_m^N}\Big[\mu_{s,t}+2\underline{\Phi}(\varphi_s^\star)\mathbf{P}\{\xi(t)>\varphi_t^\star,\ \inf_{v\in S_t}\xi(v)<\varphi_t\}\Big]$$

$$=\textstyle\sum_{s,t\in\mathcal{C}_m^N}\mu_{s,t}+\mathrm{o}(a)\big(\sum_{t\in\mathcal{C}_m^N}\underline{\Phi}(\varphi_t^\star)\big)^2$$

(see also [**5**]), (2.16) and (2.17) combines to show that it suffices to prove

(2.18) $\underline{\lim}_{m\to\infty}\underline{\lim}_{N\to\infty}\left(\sum_{s,t\in\mathcal{C}_m^N}\mu_{s,t}\right)/\left(\sum_{t\in\mathcal{C}_m^N}\underline{\Phi}(\varphi_t)\right)^2\leq 0$ for $a\leq a_1$.

Now, given an integer $k\geq 1$, partition $\mathcal{C}_m^N\times\mathcal{C}_m^N$ into

$$\begin{cases}\mathcal{C}_{m,N}^{k,1}\equiv\left\{(s,t):d(s,t)>R,\ 0<r(s,t)\leq k^{-1}[(\varphi_s^\star)^2+(\varphi_t^\star)^2]^{-1}\right\},\\ \mathcal{C}_{m,N}^{k,2}\equiv\left\{(s,t):d(s,t)>R,\ r(s,t)>k^{-1}[(\varphi_s^\star)^2+(\varphi_t^\star)^2]^{-1}\right\},\\ \mathcal{C}_{m,N}^{3}\equiv\left\{(s,t):0<d(s,t)\leq R,\ r(s,t)>0,\ \tfrac{1}{2}\varphi_s\leq\varphi_t\leq 2\varphi_s\right\},\\ \mathcal{C}_{m,N}^{4}\equiv\left\{(s,t):0<d(s,t)\leq R,\ r(s,t)>0,\ \varphi_t>2\varphi_s \text{ or } \varphi_s>2\varphi_t\right\},\\ \mathcal{C}_{m,N}^{5}\equiv\left\{(s,t):d(s,t)>0,\ r(s,t)\leq 0\right\},\\ \mathcal{C}_{m,N}^{6}\equiv\left\{(s,t):s=t\right\}.\end{cases}$$

Arguing as for [**5**, Eq. 2.12] we then readily get

$$(2.19)\quad \mu_{s,t}\leq\frac{e^{1/(2k)}\phi(\varphi_s^\star)\phi(\varphi_t^\star)}{\sqrt{2}R\,k\,\varphi_s^\star\varphi_t^\star}\leq\frac{16\,e^{1/(2k)}\,\underline{\Phi}(\varphi_s)\,\underline{\Phi}(\varphi_t)}{9\sqrt{2}R\,k}\quad\text{for } (s,t)\in\mathcal{C}_{m,N}^{k,1}.$$

Further we have, by arguing as for [**5**, Eq. 2.13],

$$\mu_{s,t}\leq\frac{\varphi_s^\star\phi(\varphi_s^\star)}{\sqrt{\pi}R}\exp\left\{-\frac{R^2(\varphi_t^\star)^2}{8}\right\}\quad\text{for } \varphi_t^\star\geq\varphi_s^\star \text{ and } d(s,t)>R.$$

Observing that $x\exp\{-Cx^2\}\leq(2C)^{-1/2}$ and taking ϱ corresponding to $C\equiv R^2/(48k)$ in (2.2) we thus obtain, by (2.4) and (2.15), for $s\in\mathcal{C}_m^N$,

$$\begin{aligned}&\sum_{\{t\in\mathcal{C}_m^N:\,(s,t)\in\mathcal{C}_{m,N}^{k,2},\,\varphi_t^\star\geq\varphi_s^\star\}}\mu_{s,t}\\ &\leq\frac{4\underline{\Phi}(\varphi_s)}{3\sqrt{\pi}R}\sum_{\ell=2}^{\infty}\sum_{n=0}^{\infty}\sum_{\{t\in\mathcal{C}_m^N:\,\ell\leq\varphi_t^\star<\ell+1,\,\varrho_s(n)\leq\rho(s,t)<\varrho_s(n+1),\,r(s,t)>0\}}\\ &\qquad\qquad\varphi_t^\star\exp\left\{-\frac{R^2(\varphi_t^\star)^2}{12}\right\}\exp\left\{-\frac{r^2}{48k\,r(s,t)}\right\}\\ &\leq\frac{8\underline{\Phi}(\varphi_s)}{\sqrt{3\pi}R^2}\sum_{\ell=2}^{\infty}\sum_{\{n\geq 0:\,\sigma(s,\varrho_s(n))\geq 0\}}M_{S_\rho(s,\varrho_s(n+1))}(a/(\ell+1))\\ &\qquad\qquad\times\exp\left\{-\frac{R^2\ell^2}{24}\right\}\exp\left\{-\frac{C}{\sigma(s,\varrho_s(n))}\right\}\\ &\leq\frac{8\underline{\Phi}(\varphi_s)}{\sqrt{3\pi}R^2}\sum_{\{n\geq 0:\,\sigma(s,\varrho_s(n))\geq 0\}}N_{S_\rho(s,\varrho_s(n+1))}(R)\exp\left\{-\frac{C}{\sigma(s,\varrho_s(n))}\right\}\\ &\qquad\times K_1N_{\mathcal{O}_R}(R/3)\sum_{\ell=2}^{\infty}(8R(\ell+1)/a)^{-\log K_1/\log r}\exp\left\{-\frac{R^2\ell^2}{24}\right\}.\end{aligned}$$

Since $\sum_{t\in\mathcal{C}_0^m}\underline{\Phi}(\varphi_t)\le N_{S_\rho(t_0,m)}(a/u_m)\underline{\Phi}(u_m)<\infty$ so that, by (2.16), $\lim_{N\to\infty}\sum_{t\in\mathcal{C}_m^N}\underline{\Phi}(\varphi_t)=\infty$, we readily deduce, by (2.2) and symmetry,

(2.20) $\quad \underline{\lim}_{N\to\infty}\big(\sum_{(s,t)\in\mathcal{C}_{m,N}^{k,2}}\mu_{s,t}\big)/\big(\sum_{t\in\mathcal{C}_m^N}\underline{\Phi}(\varphi_t)\big)^2=0\quad$ for $a\le a_1$.

Clearly we have, by (2.12) and (2.15), for $s\in\mathcal{C}_m^N$,

$$\sum_{\{t\in\mathcal{C}_m^N:\,(s,t)\in\mathcal{C}_{m,N}^3,\,\varphi_t\ge\varphi_s\}}\mu_{s,t}$$

$$\le\sum_{\ell=1}^{\infty}\;\sum_{\{t\in\mathcal{C}_m^N:\,\ell a/(2\varphi_s)<d(s,t)\le R\wedge((\ell+1)a/(2\varphi_s)),\,\varphi_t\le 2\varphi_s\}} 2\underline{\Phi}(\varphi_s)\,\underline{\Phi}\big(\tfrac12 d(s,t)\varphi_s\big)$$

$$\le 2\underline{\Phi}(\varphi_s)\sum_{\ell=1}^{\infty}M_{\mathcal{O}_{R\wedge((\ell+1)a/(2\varphi_s))}}(a/(2\varphi_s))\,\underline{\Phi}\big(\tfrac14\ell a\big),$$

and using (2.11) and symmetry we thus get (since $\sum_{t\in\mathcal{C}_m^N}\underline{\Phi}(\varphi_t)\to\infty$)

(2.21) $\quad \underline{\lim}_{N\to\infty}\big(\sum_{(s,t)\in\mathcal{C}_{m,N}^{3}}\mu_{s,t}\big)/\big(\sum_{t\in\mathcal{C}_m^N}\underline{\Phi}(\varphi_t)\big)^2=0\quad$ for $a\le a_1$.

Further we have, for $s\in\mathcal{C}_m^N$, by [**12**, Theorem 4.2.1] and using (2.4), (2.15) and the facts that $\varphi_s\ge2$ and that $x^\beta\exp\{-Cx^2\}\le(\beta/(2C))^{\beta/2}$,

$$\sum_{\{t\in\mathcal{C}_m^N:\,(s,t)\in\mathcal{C}_{m,N}^4,\,\varphi_t>2\varphi_s\}}\mu_{s,t}$$

$$\le\sum_{\ell=2}^{\infty}\;\sum_{\{t\in\mathcal{C}_m^N:\,\ell\varphi_s<\varphi_t\le(\ell+1)\varphi_s,\,r(s,t)>0,\,0<d(s,t)\le R\}}\frac{r(s,t)}{\sqrt{2}\pi\,d(s,t)\sqrt{1+r(s,t)}}\exp\Big\{-\frac{(\varphi_s^*)^2+(\varphi_t^*)^2}{2(1+r(s,t))}\Big\}$$

$$\le\sum_{\ell=2}^{\infty}\frac{(\ell+1)\varphi_s}{\sqrt{2}\pi\,a}M_{\mathcal{O}_R}(a/((\ell+1)\varphi_s))\exp\Big\{-\frac{(\ell^2+1)\varphi_s^2}{4}\Big\}$$

$$\le\underline{\Phi}(\varphi_s)\sum_{\ell=2}^{\infty}\frac{4K_1N_{\mathcal{O}_R}(R/3)(\ell+1)\varphi_s^2}{3\sqrt{\pi}a\big(8R(\ell+1)\varphi_s/a\big)^{\log K_1/\log x}}\exp\Big\{-\frac{(\ell^2-1)\varphi_s^2}{4}\Big\}$$

$$\le\underline{\Phi}(\varphi_s)\sum_{\ell=2}^{\infty}\frac{N_{\mathcal{O}_R}(R/3)\big(2-\frac{\log K_1}{\log x}\big)^{1-\log K_1/\log x^2}}{6\sqrt{\pi}R\big(8R(\ell+1)/a\big)^{\log K_1/\log x-1}}\exp\{-(\ell^2-3)\},$$

and using symmetry we thus get (again since $\sum_{t\in\mathcal{C}_m^N}\underline{\Phi}(\varphi_t)\to\infty$)

(2.22) $\quad \underline{\lim}_{N\to\infty}\big(\sum_{(s,t)\in C^4_{m,N}}\mu_{s,t}\big)/\big(\sum_{t\in C^N_m}\underline{\Phi}(\varphi_t)\big)^2=0\quad$ for $a\le a_1$.

Finally, by [**12**, Theorem 4.2.1], $\mu_{s,t}\le 0$ for $r(s,t)\le 0$, $s\ne t$, so that

(2.23) $\quad \underline{\lim}_{N\to\infty}\big(\sum_{(s,t)\in C^5_{m,N}}\mu_{s,t}\big)/\big(\sum_{t\in C^N_m}\underline{\Phi}(\varphi_t)\big)^2\le 0\quad$ for $a\le a_1$,

and moreover

(2.24) $$\lim_{N\to\infty}\frac{\sum_{(s,t)\in C^6_{m,N}}\mu_{s,t}}{\big(\sum_{t\in C^N_m}\underline{\Phi}(\varphi_t)\big)^2}=\lim_{N\to\infty}\frac{1}{\sum_{t\in C^N_m}\underline{\Phi}(\varphi_t)}=0\quad\text{for } a\le a_1.$$

Combining (2.19)-(2.24) we see that (given $a<a_1$) the left hand side of (2.18) is at most $\mathcal{O}(1/k)$, and so (2.18) follows from sending $k\uparrow\infty$.

COROLLARY 1. *Assume the hypothesis of Theorem 1 and that d is a metric and (T,d) locally compact. Then there exists an invariant (w.r.t. $+$) Haar-measure μ on (T,d):s Borel-sets with $\mu(\mathcal{O}_\delta)<\infty$ for $\delta\in(0,\sqrt{2})$. If further λ is any version of this Haar-measure, then $\mathbf{P}\{E(\psi)\}=0$ i.f.f.*

(2.25) $$\sum_{n=1}^{\infty}\Big[1+\lambda(\mathcal{O}_{r_n})N_{\mathcal{O}_R}\Big(\big(1\vee\inf_{t\in S_n}\psi(t)\big)^{-1}\Big)\Big]\underline{\Phi}\Big(1\vee\inf_{t\in S_n}\psi(t)\Big)<\infty$$

for some covering $S_n=S(t_n,r_n)$, $n=1,2,\ldots$, of T with $r_n\le R$ for all n.

PROOF: An easy argument yields d-continuity for $(s,t)\to t-s$ so that $(T,d,+)$ is a locally compact (Hausdorff) topological group and μ exists and is finite on compacts, where, by Remark 1 and local compactness, $\mathcal{O}_\delta$ is compact for $\delta<\sqrt{2}$. Now observe that, by arguing as for (2.4),

$$N_{\mathcal{O}_\delta}(\varepsilon)\le 1+\frac{K_1N_{\mathcal{O}_R}(R/3)^2N_{\mathcal{O}_R}(\varepsilon)}{32^{\log K_1/\log x}N_{\mathcal{O}_R}(\delta)}\le 1+\frac{K_1N_{\mathcal{O}_R}(R/3)^2\lambda(\mathcal{O}_\delta)N_{\mathcal{O}_R}(\varepsilon)}{32^{\log K_1/\log x}\lambda(\mathcal{O}_R)}$$

for $\varepsilon>0$ and $\delta\le R$, and so the sum (2.3) is finite when (2.25) holds. Conversly (2.25) holds when the sum (2.3) is finite since, for $\delta\le R$,

$$\begin{aligned}\frac{N_{\mathcal{O}_R}(\varepsilon)}{N_{\mathcal{O}_\delta}(\varepsilon)}&\le N_{\mathcal{O}_R}(R/2)M_{\mathcal{O}_{R/2}}((R/2)\wedge(2\delta))N_{\mathcal{O}_{(R/2)\wedge(2\delta)}}(\delta)\\&\le\frac{K_1N_{\mathcal{O}_R}(R/3)N_{\mathcal{O}_R}(R/2)\lambda(\mathcal{O}_R)}{16^{\log K_1/\log x}\lambda(\mathcal{O}_{(R/4)\wedge\delta})}\le\frac{K_1N_{\mathcal{O}_R}(R/3)^2\lambda(\mathcal{O}_R)^2}{16^{\log K_1/\log x}\lambda(\mathcal{O}_{R/4})\lambda(\mathcal{O}_\delta)}.\end{aligned}$$

The following local result improves on [**16**] and [**21**] (but note that they also treat non-stationary fields); we leave to the reader to find what conditions in Section 1 one can omit without violating it's conclusion.

COROLLARY 2. *Assume that there exists an $R\in(0,\sqrt{2})$ such that (2.1) holds. Then there exist constants $C_1,C_2\in(0,\infty)$ such that*

$$C_1 \le \frac{\mathbf{P}\{\sup_{t\in\mathcal{O}_\delta}\xi(t)>u\}}{N_{\mathcal{O}_\delta}\big((1\vee u)^{-1}\big)\,\underline{\Phi}(u)} \le C_2 \quad \textit{for } u\in\Re \textit{ and } \delta\in[0,R].$$

If in addition d is a metric, (T,d) is locally compact and λ is a version of the Haar-measure, then there exist constants $C_1,C_2\in(0,\infty)$ such that

$$C_1 \le \frac{\mathbf{P}\{\sup_{t\in\mathcal{O}_\delta}\xi(t)>u\}}{\big[1+\lambda(\mathcal{O}_\delta)N_{\mathcal{O}_R}\big((1\vee u)^{-1}\big)\big]\,\underline{\Phi}(u)} \le C_2 \quad \textit{for } u\in\Re \textit{ and } \delta\in[0,R].$$

For homogeneous spaces which in a certain sense are finite-dimensional we have the following very simple criterion for (2.2) to hold.

PROPOSITION 1. *If $\rho(s+u,t+u)=\rho(s,t)$ for $s,t,u\in T$ and if there exists a function $f:\Re\to\Re$ such that, writing $\mathcal{B}_\varepsilon$ for an open ρ-ball of radius ε,*

$$(2.26)\qquad 1<\varliminf_{\Delta\to\infty}\frac{N_{\mathcal{B}_{\Delta+f(\Delta)}}(R)}{N_{\mathcal{B}_\Delta}(R)} \le \varlimsup_{\Delta\to\infty}\frac{N_{\mathcal{B}_{\Delta+f(\Delta)}}(R)}{N_{\mathcal{B}_\Delta}(R)}<\infty,$$

then (2.2) holds if $\sigma(\varepsilon)\equiv\sup\{0\vee r(s,t):\rho(s,t)\ge\varepsilon\}$ satisfies

$$(2.27)\qquad \lim_{\Delta\to\infty}\sigma(\Delta)\log N_{\mathcal{B}_\Delta}(R)=0.$$

PROOF: Take $\varepsilon,y,\Delta>0$ with $1+\varepsilon\le N_{\mathcal{B}_{x+f(x)}}(R)/N_{\mathcal{B}_x}(R)\le y$ for $x\ge\Delta$ and let $\varrho(0)=0$, $\varrho(1)=\Delta$ and $\varrho(n+1)=\varrho(n)+f(\varrho(n))$ for $n\ge1$, so that

$$N_{\mathcal{B}_{\varrho(n+1)}}(R)/N_{\mathcal{B}_{\varrho(1)}}(R)=\textstyle\prod_{k=1}^n N_{\mathcal{B}_{\varrho(k+1)}}(R)/N_{\mathcal{B}_{\varrho(k)}}(R)\ge(1+\varepsilon)^n\to\infty$$

as $n\to\infty$, which yields $\lim_{n\to\infty}\varrho(n)=\infty$. Taking n_0 such that $\sigma(\varrho(n))\times\log N_{\mathcal{B}_{\varrho(n)}}(R)\le C/2$ for $n\ge n_0$ we now readily obtain

$$\begin{aligned}&\sup_{s\in T}\textstyle\sum_{\{n\ge0:\sigma(s,\varrho(n))>0\}}N_{S_\rho(s,\varrho(n+1))}(R)\exp\{-C/\sigma(s,\varrho(n))\}\\&\quad\le\textstyle\sum_{n=1}^{n_0}N_{\mathcal{B}_{\varrho(n)}}(R)+\sum_{n=n_0}^\infty N_{\mathcal{B}_{\varrho(n+1)}}(R)\exp\{-2\log N_{\mathcal{B}_{\varrho(n)}}(R)\}\\&\quad\le\textstyle\sum_{n=1}^{n_0}N_{\mathcal{B}_{\varrho(n)}}(R)+\sum_{n=n_0}^\infty y\,N_{\mathcal{B}_{\varrho(1)}}(R)^{-1}(1+\varepsilon)^{-(n-1)}<\infty.\end{aligned}$$

3. The Euclidian case. Theorem 2 sharpens [**10**] and [**15**]: They need a stronger condition than (3.1) and only treat (and crucially need) $\psi(t)=\hat\psi(|t|)$ with $\hat\psi:[0,\infty)\to[0,\infty)$ increasing (a meaningless notion on general space), which makes (3.3) an integraltest and proofs totally different.

THEOREM 2. *If $\{\xi(t)\}_{t\in\Re^n}$ is separable stationary standard Gaussian, if*

$$(3.1)\qquad \lim_{|t-s|\to\infty}\big(0\vee r(s,t)\big)\log|t-s|=0,$$

and if there exist constants $\alpha,\delta,C_1,C_2\in(0,\infty)$ such that

$$(3.2)\qquad C_1|t-s|^\alpha\le1-r(s,t)\le C_2|t-s|^\alpha\quad\textit{for } 0\le|t-s|\le\delta,$$

then $E(\psi)\in\mathcal{F}$ with $\mathbf{P}\{E(\psi)\}$ zero or one for each $\psi\in\Psi$ and moreover, writing λ for the Lebesgue measure over $\Re^n$, we have $\mathbf{P}\{E(\psi)\}=0$ i.f.f.

$$(3.3)\qquad \sum_{n=1}^{\infty}\Big[1+\lambda(\mathcal{O}_{r_n})\Big(1\vee\inf_{t\in S_n}\psi(t)\Big)^{2n/\alpha}\Big]\,\underline{\Phi}\Big(1\vee\inf_{t\in S_n}\psi(t)\Big)<\infty$$

for some covering $S_n=S(t_n,r_n)$, $n=1,2,\ldots$, of T with $r_n\le 1$ for all n.

Proof: Here $(T,\rho,+)=(\Re^n,|\cdot|,+)$ and $R=1$. Take $\Delta>0$ with $r(0,t)<\frac{1}{2}$ for $|t|\ge\Delta$ and suppose $|t|\not\to 0$ as $d(0,t)\to 0$. Then $\inf\{d(0,t):|t|\ge\varrho\}=0$ for some $\varrho\in(0,\Delta]$, and picking s with $|s|\ge\varrho$ and $d(0,s)<\frac{\varrho}{2\Delta}$ we get $d(0,([\frac{\Delta}{\varrho}]+1)s)<1$ so $r(0,([\frac{\Delta}{\varrho}]+1)s)>\frac{1}{2}$. This is a contradiction since $|\frac{\Delta}{\varrho}s|\ge\Delta$, and so, by homogenity, $|t-s|\to 0$ as $d(s,t)\to 0$. Now pick $\varrho>0$ with $|t-s|\le\delta$ for $d(s,t)\le\varrho$. Then (3.1) readily yields that

$$(3.4)\quad S_{|\cdot|}\big(t,(2C_2)^{-1/\alpha}\varepsilon^{2/\alpha}\big)\subseteq S(t,\varepsilon)\subseteq S_{|\cdot|}\big(t,C_1^{-1/\alpha}\varepsilon^{2/\alpha}\big)\quad\text{for } \varepsilon\le\delta\wedge\varrho.$$

Thus $|\cdot|$- and d-topologies coincide so we have stochastic continuity, $|\cdot|$-boundeds are d-totally bounded, d is a metric, (T,d) is locally compact and the Lebesgue measure is a Haar-measure on $(T,d,+)$. Further it follows easily from (3.4), since $S(t,1)\subseteq S_{|\cdot|}(t,\Delta)$, that $K_1\varepsilon^{2n/\alpha}\le N_{\mathcal{O}_1}(\varepsilon)\le K_2\varepsilon^{2n/\alpha}$ for $\varepsilon\in(0,1]$ and $K_1x^n\le N_{B_x}(1)\le K_2x^n$ for $x\ge x_0$, for some $K_1,K_2,x_0\in(0,\infty)$. This proves (2.1), (2.26) and (using (3.1)) (2.27).

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