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FUZZY MODELING OF ARMOR PLATE BENDING BY BLAST

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AUGUST 1990

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U.S. ARMY LABORATORY COMMAND

BALLISTIC RESEARCH LABORATORY
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13. ABSTRACT (Maximum 200 words) A mathematical model is developed for the deformation of armor plates due to loading by bare explosive. Data inaccuracies are treated by a fuzzy set approach. Uncertainties of the model are expressed by fuzzy model parameters with appropriate spreads and the parameter values are determined by minimizing a least squares objective function. The membership functions of data and parameter vectors are assumed to be conical. The report contains a short discussion of fuzzy vector and function spaces with such membership functions and of corresponding regression techniques. These techniques are used to derive a model for the deformation of a steel armor plate by pressure from a nearby explosion. <i>Keywords:</i>			
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TABLE OF CONTENTS.

	Page
LIST OF ILLUSTRATIONS	v
LIST OF TABLES	vii
1. INTRODUCTION	1
2. FUZZY GEOMETRY	2
2.1. Basic Definitions	2
2.2. Fuzzy Point Space	3
2.3. Fuzzy Functions	4
2.3.1. Linear Functions	5
2.3.2. Nonlinear Functions	9
2.3.3. Fuzzy Function Space	11
2.4. Fuzzy Equations	11
3. FUZZY MODEL FITTING	19
3.1. Fuzzy Models	19
3.2. Model Fitting Principles	20
3.3. Numerical Treatment	24
4. MODEL OF PLATE DEFORMATION	25
4.1. Data and Model	25
4.2. Data and Model Spreads	26
4.3. Numerical Results	28
5. SUMMARY	29
LIST OF REFERENCES	30
DISTRIBUTION LIST	41



Accession For	
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Justification _____	
By _____	
Distribution / _____	
Availability Codes	
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LIST OF ILLUSTRATIONS.

	Page
Figure 1. Conical membership function in R_2	32
Figure 2. Observed plate deflection versus load.	33
Figure 3. Observations with support ellipses.	34
Figure 4. Model curve in Case 1 with prescribed spread of $\tilde{c}$	35
Figure 5. Model curve in Case 2 with undetermined parameter spreads.	36
Figure 6. Local parameter adjustments in Case 1.	37
Figure 7. Local parameter adjustments in Case 2.	38

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LIST OF TABLES.

	Page
Table 1. Input Data.	39
Table 2. Model Parameters.	40

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1. INTRODUCTION.

Mathematical models of terminal ballistics processes contain uncertainties that have two fundamentally different roots. First, one has the usual stochastic errors of observations and responses of materials. These errors can be handled by conventional statistical methods based on estimated probabilities, although in ballistic problems the limits of application for these methods sometimes must be stretched due to missing or scarce data. A second source of inaccuracy is the lack of knowledge about processes of threat/target interaction. Details of these processes are difficult to observe, they are often not well understood and not represented by established theories. For the treatment of this type of inaccuracies one can use the concepts of fuzzy set theory that has been developed to quantify judgement. Fuzzy sets can also represent inaccurate data, in particular if an error distribution of the data is not known or does not exist (for instance, in intelligence data or in future weapon concepts). In the present report we assume such a representation, that is, the data are assumed to be fuzzy numbers and the mathematical model is assumed to be a fuzzy function. We then formulate a fuzzy regression problem based on the minimization of a least squares objective function.

Special types of fuzzy regression problems have been treated by several authors. Tanaka et al., [12] and [13], developed linear programming algorithms for regression with fuzzy linear models and special types of fuzzy data. The algorithm of [12] was applied by Heshmaty and Kandel [9] to economic forecasts. Diamond suggested in [6] and [7] a least squares approach and in [8] a maximum likelihood approach to fuzzy linear regression problems with special fuzzy data. Celmiņš has treated regression of nonlinear models with a slightly more general type of data in the special cases where either only the data [3], or only the model [4] is fuzzy. In the present paper, the results of [3] and [4] are extended to a general regression problem, where the model equations are nonlinear and the model as well as the data are fuzzy. Some restrictions are imposed on the generality in order to obtain a method which is algorithmically simple. The restrictions involve a parametrization of the membership functions which are assumed to be conical functions. Under this restriction, one can solve the model fitting problem with the aid of available regression software for implicitly formulated constraints, supplementing the software only with normalization procedures for some of the input.

In Section 2 we develop some simple concepts and properties of a special fuzzy point space. These results are used in Section 3 to derive a formalism for a fuzzy

regression. In Section 4 the fuzzy regression is applied to the task of developing a model for the bending of armor plates due to loads by pressures from explosions of bare explosives.

2. FUZZY GEOMETRY.

In this section, we provide basic definitions of fuzzy set theory and develop simple concepts of analytic geometry in a fuzzy environment. The purpose of the section is to establish practical tools for model fitting, and to provide a geometrical interpretation of the fitting and its results. Therefore, the exposition is restricted to a special type of fuzzy spaces for which one obtains algorithms that are easily applicable to real life problems.

2.1. Basic Definitions.

In presenting the basic definitions we follow in essence Zimmermann [16]. A classical set is normally defined as a collection of elements of $e \in E$. Each element can either belong or not belong to a set $H \subseteq E$. The set can be described by a characteristic function which equals unity if e is an element of H and is zero if e does not belong to H . We call such a set a *crisp* set. A generalization of the concept of a *crisp* set is obtained by allowing the characteristic function to assume values that are not restricted to zero and unity. To distinguish such a function from a characteristic function it is called *membership function*.

Definition. A fuzzy set $\tilde{H}$ in E is a set of ordered pairs

$$\tilde{H} = \{ e, \mu_H(e) \mid e \in E \} . \quad (2.1)$$

$\mu_H(e)$ is called the membership function or grade of membership of e . The range of $\mu_H(e)$ is a subset of nonnegative numbers whose supremum is finite.

In this paper, we assume that the range of the membership function is the interval $[0,1]$ and that elements with a zero degree of membership are not included in $\tilde{H}$. If the membership function assumes only the values 0 and 1, then the set $\tilde{H}$ is non-fuzzy (*crisp*) and μ_H is its characteristic function.

The concept of a fuzzy set allows one to express uncertainty in a manner which is different from probabilistic concepts and interval analysis. Thus, for instance, the concept "about 5" might be represented by a fuzzy number $\tilde{5}$ whose membership function $\mu_5(x)$ has a positive maximum value at $x=5$ and decreases to zero as $|x-5|$ increases. Zimmermann [16] discusses many examples of such concepts and their

representation as fuzzy sets.

2.2. Fuzzy Point Space.

A fuzzy point in R_n is a fuzzy set of points $X \in R_n$. Let the membership value of any element of the set be given by a membership function $\mu(X)$. In this paper, we only consider a particular type of fuzzy points which are characterized by fuzzy position vectors with *conical membership functions*.

Definition. A regular (not regular) fuzzy vector $\tilde{A}$ in R_n is the pair

$$\tilde{A} = \{ A, P_A \} , \quad (2.2)$$

where $A \in R_n$, and P_A is a positive definite (semidefinite) $(n \times n)$ -matrix. A is called the apex of $\tilde{A}$ and P_A is called the panderance matrix of $\tilde{A}$.

With this definition, a fuzzy vector in an n -dimensional space is specified by $n \cdot (n+3)/2$ real numbers. Let Q_A be the Moore-Penrose generalized inverse of P_A and let $X_A \subseteq R_n$ be the linear space spanned by the columns of P_A . We denote by $\| \cdot \|_A$ the following elliptic norm of the distance between an arbitrary $X \in R_n$ and A :

$$\|X - A\|_A = \begin{cases} [(X - A)^T \cdot Q_A \cdot (X - A)]^{1/2} , & \text{if } X - A \in X_A \\ +\infty & \text{otherwise} . \end{cases} \quad (2.3)$$

Definition. The conical membership function of $\tilde{A}$ is

$$\mu_A(X) = 1 - \min \{ 1, \|X - A\|_A \} . \quad (2.4)$$

The fuzzy point $\tilde{A}$ is the set of all vectors X with the property $\mu_A(X) > 0$. If the panderance matrix P_A is positive definite, then the boundary of the support of μ_A and the level surfaces $\mu_A(X) = \text{constant}$ are hyperellipsoids in R_n and (2.4) defines a cone. The conical membership function μ_A is normalized to $0 \leq \mu_A \leq 1$, and a fuzzy point with a conical membership function is a convex fuzzy set [14]. If the panderance matrix P_A is semidefinite then the cone is degenerate in R_n and is a regular cone in the subspace X_A .

In R_1 , the conical membership function is the triangular function

$$\mu_a(x) = 1 - \min \{ 1, |x - a| / s_a \} , \quad (2.5)$$

where $s_a = \sqrt{P_a}$ is the spread of the function. In R_n , the membership function of each component of $\tilde{A}$ is a triangular function with a spread that equals the square root of the corresponding diagonal element of P_A . The triangular function is, of course, the projection of the cone (2.4) on the corresponding axis. Figure 1 gives a schematic illustration of the conical membership function of a fuzzy point (vector) in R_2 .

The off-diagonal elements of P_A indicate possible interactions [15] between the components of $\tilde{A}$. These elements can be conveniently represented by dimensionless *concordances* which we define for a regular fuzzy point as follows

$$c_{ki} = p_{ki} / \sqrt{p_{kk}p_{ii}} , \quad k, i = 1, \dots, n , \quad (2.6)$$

where c_{ki} and p_{ki} are elements of a concordance matrix and the panderance matrix P_A , respectively.

We introduce a structure in the space of fuzzy points by the following definition of *discord* and *grade of collocation* between fuzzy points.

Definition. The *discord* between two fuzzy points $\tilde{A}$ and $\tilde{B}$ is

$$D(\tilde{A}, \tilde{B}) = \min_{X \in R_n} \max \{ \|X - A\|_A, \|X - B\|_B \} . \quad (2.7)$$

We note that the discord is not a distance in the sense of functional analysis, because it does not satisfy the triangle inequality.

If the panderance matrix of one of the vectors in eq. (2.7), say P_B , approaches a zero-matrix, then $\mu_{\tilde{B}}(X)$ approaches the characteristic function of the apex B . That is, the corresponding $\tilde{B}$ approaches the crisp point B . The corresponding limit of the discord is

$$\lim_{\tilde{B} \rightarrow B} D(\tilde{A}, \tilde{B}) = \|B - A\|_A . \quad (2.8)$$

Definition. The *grade of collocation* between two fuzzy points $\tilde{A}$ and $\tilde{B}$ is

$$\gamma(\tilde{A}, \tilde{B}) = \max_{X \in R_n} \min \{ \mu_A(X), \mu_B(X) \} . \quad (2.9)$$

The grade of collocation and the discord are related by

$$\gamma(\tilde{A}, \tilde{B}) = 1 - \min \{ 1, D(\tilde{A}, \tilde{B}) \} . \quad (2.10)$$

2.3. Fuzzy Functions.

We interpret functions as mappings of a domain in R_n to a range in R_r . Usually, we shall assume that the mapping function is twice continuously differentiable with respect to all arguments of interest, i.e., with respect to $X \in R_n$ and with respect to any free parameters, if present. If some of the arguments are fuzzy numbers or fuzzy vectors, then the image of a crisp or fuzzy point in R_n is a fuzzy point in R_r . For consistency with crisp mappings, we shall always assume that the image of the apex of

the argument is obtained by replacing in the function all fuzzy arguments by their apexes. That is, the apex of a fuzzy point is mapped onto the apex of the image. The pandurance matrix of the image can be computed in terms of the pandurance matrices of the argument and parameters by formulas which are derived in this section.

2.3.1. Linear Functions.

We first consider a mapping of R_n onto itself by an affine coordinate transformation, and derive a pandurance propagation formula for such mappings from the requirement that the membership value of any $X \in R_n$ should be the same in both coordinate systems.

Proposition 1. Let Z, Z_0, X and $A \in R_n$, D be a non-singular $(n \times n)$ -matrix, and

$$Z = L(X) = D \cdot X + Z_0 . \quad (2.11)$$

The image of the fuzzy point which is represented by the fuzzy vector

$$\tilde{A} = \{ A, P_A \} \quad (2.12)$$

is the fuzzy point (vector)

$$\tilde{B} = \{ B, P_B \} , \quad (2.13)$$

with the apex

$$B = D \cdot A + Z_0 \quad (2.14)$$

and the pandurance matrix

$$P_B = D \cdot P_A \cdot D^T . \quad (2.15)$$

Proof. Equating the membership values of an $X \in R_n$ in the new and old coordinate systems, respectively, one obtains

$$\mu_B(L(X)) = \mu_A(X) ,$$

or, from the definition (2.4),

$$\|L(X) - B\|_B^2 = \|X - A\|_A^2 \quad (2.16)$$

in a neighborhood of A . If $X=A$, then the right hand side of (2.16) vanishes and the left hand side produces eq. (2.14). Using this result, we express the left hand side of eq. (2.16) by

$$\begin{aligned}
(D \cdot X + Z_0 - B)^T \cdot Q_B \cdot (D \cdot X + Z_0 - B) &= \\
&= [D \cdot (X - A)]^T \cdot Q_B \cdot [D \cdot (X - A)] = \quad (2.17) \\
&= (X - A)^T \cdot D^T \cdot Q_B \cdot D \cdot (X - A) .
\end{aligned}$$

Hence, from eq. (2.16)

$$D^T \cdot Q_B \cdot D = Q_A , \quad (2.18)$$

or, because D is not singular,

$$P_B = D \cdot P_A \cdot D^T , \quad \text{q.e.d.}$$

The *extension principle for fuzzy functions* [15] states that, if Z is a function of $L(X)$ and $\tilde{B} = L(\tilde{A})$, then one should have the relation

$$\mu_B(Z) = \max_{X: L(X) - Z = 0} \mu_A(X) . \quad (2.19)$$

The formulas (2.14) and (2.15) are componentwise consistent with this principle also in cases where the affine mapping (2.11) is not a coordinate transformation. The consistency is a consequence of the following proposition:

Proposition 2. Let Z and $Z_0 \in R_r$, X and $A \in R_n$, D be a non-zero $(r \times n)$ -matrix of rank r and

$$Z = L(X) = D \cdot X + Z_0 . \quad (2.20)$$

Then the image $\tilde{B}$ of a regular fuzzy vector $\tilde{A}$ as defined by eqs. (2.14) and (2.15) satisfies the extension principle condition (2.19).

Proof. We formulate the right hand side of eq. (2.19) as the following constrained minimization problem:

$$\left. \begin{array}{ll} \text{Minimize} & W = (X - A)^T \cdot P_A^{-1} \cdot (X - A) \\ \text{subject to} & D \cdot (X - A) = Z - B \end{array} \right\} \quad (2.21)$$

We solve (2.21) using a Lagrange multiplier vector k and defining a modified objective function

$$\tilde{W} = \frac{1}{2} (X - A)^T \cdot P_A^{-1} \cdot (X - A) - k^T \cdot (D \cdot (X - A) - Z + B) . \quad (2.22)$$

The normal equations for the minimization of $\tilde{W}$ are

$$\left. \begin{aligned} P_A^{-1} \cdot (X - A) - D^T \cdot k &= 0, \\ D \cdot (X - A) - Z + B &= 0. \end{aligned} \right\} \quad (2.23)$$

Solving the first eq. (2.23) for k , one obtains

$$k = (D \cdot P_A \cdot D^T)^{-1} \cdot D \cdot (X - A). \quad (2.24)$$

We substitute this expression into the first equation (2.23) and multiply the result from left by $(X - A)^T$, obtaining

$$(X - A)^T \cdot P_A^{-1} \cdot (X - A) = (X - A)^T \cdot D^T \cdot (D \cdot P_A \cdot D^T)^{-1} \cdot D \cdot (X - A). \quad (2.25)$$

This equation holds at the solution of the minimization problem. The left hand side of the equation is the sought for minimum value of $\tilde{W}$. The right hand side can be simplified by using the second eq. (2.23) and eq. (2.15) to yield $(Z - B)^T \cdot P_B^{-1} \cdot (Z - B)$. Hence,

$$[(X - A)^T \cdot P_A^{-1} \cdot (X - A)]_{\min} = (Z - B)^T \cdot P_B^{-1} \cdot (Z - B), \quad (2.26)$$

q. e. d.

Corollary 1. If the mapping

$$Z = L(X) = D \cdot X + Z_0 \quad (2.27)$$

maps X into a subspace of R_n and $\mu_B(Z)$ is defined by eqs. (2.14) and (2.15), then (2.19) holds in that subspace.

Proof. Let the dimension of the subspace be s . Then, by a proper coordinate transformation, (2.27) can be rearranged in the form

$$\left. \begin{aligned} Z_1 &= D_1 \cdot X + Z_{10}, \\ Z_2 &= 0, \end{aligned} \right\} \quad (2.28)$$

where D_1 is a $(s \times n)$ -matrix with the rank s . The *Proposition 2* applies to the first s equations (2.28), proving the corollary.

If $\tilde{A}$ is not a regular fuzzy point in R_n then only the subspace X_A (defined in the paragraph following eq. (2.2.)) where X has a positive membership value is relevant for the mapping. This subspace is mapped by (2.20) into another linear subspace $Z_A \subseteq R_n$. *Proposition 2* and *Corollary 1* directly apply to the mapping of X_A into Z_A .

So far we have considered a crisp linear function, eq. (2.20), with a possibly fuzzy argument X . Now we let the function be fuzzy and assume for simplicity that the argument X is crisp. That is, we consider the function

$$Z = \tilde{L}(X) = \tilde{D} \cdot X + \tilde{Z}_0, \quad (2.29)$$

where some of the elements of $\tilde{D}$ and $\tilde{Z}_0$ are fuzzy numbers or vectors. By definition, a fuzzy linear function is a subset in a function space S . That space consists of all linear functions which map R_n into R_r . Its dimension is $r(n+1)$, i.e., the number of elements of D and Z_0 . Each element of S is defined by a particular set of the elements of D and Z_0 , which in turn has a well defined non-negative membership value, given by the apexes and panderances of $\tilde{D}$ and $\tilde{Z}_0$. We assign to the corresponding function $L(X)$ the same membership value thus defining the fuzzy function set $\tilde{L}(X)$. The image of a crisp X by eq. (2.29) is fuzzy due to the fuzziness of the function: a given X produces several images in R_r , each with a corresponding membership value. Also, a given $Z \in R_r$ possibly can be the image of several points $X \in R_n$ whereby each of the corresponding mappings has a specific membership value. According to the extension principle for fuzzy functions we assign to the image $Z = \tilde{L}(X)$ the supremum of these values as its membership value.

To calculate the apex B and the panderance matrix of the image $\tilde{B} = \tilde{L}(A)$ we can invoke the *Proposition 2*, because eq (2.29) can be considered as a crisp linear function (defined by the crisp elements of X) of the fuzzy arguments $\tilde{D}$ and $\tilde{Z}_0$. Let T_1 and $\tilde{T}_2 = \{T_2, P_{T2}\}$ be, respectively, the vector of all crisp and all fuzzy elements in $\tilde{D}$ and $\tilde{Z}_0$, and let eq. (2.29) be expressed by

$$Z = F(X, T_1, \tilde{T}_2). \quad (2.30)$$

Then,

$$\tilde{B} = F(A, T_1, \tilde{T}_2) \quad (2.31)$$

has a panderance matrix which formally is given by

$$P_B = \frac{\partial F}{\partial T_2} \cdot P_{T2} \cdot \left(\frac{\partial F}{\partial T_2} \right)^T. \quad (2.32)$$

We present two examples. First, let z be a scalar function

$$z = f(X; T) = g(X; C, z_0) = X^T \cdot C + z_0 \quad (2.33)$$

and let T be the fuzzy vector

$$\tilde{T} = \left\{ \begin{pmatrix} C \\ z_0 \end{pmatrix}, P_T \right\}. \quad (2.34)$$

Then, for $X=A$, the apex of the function value is

$$z = b = A^T \cdot C + z_0 \quad (2.35)$$

and its pandurance matrix is

$$P_b = (A^T, 1) \cdot P_T \cdot \begin{pmatrix} A \\ 1 \end{pmatrix} \quad (2.36)$$

As a second example, assume that in eq. (2.29) the pandurance matrix P_D of the fuzzy elements of $\tilde{D}$ is diagonal, and that all concordances between elements of $\tilde{D}$ and $\tilde{Z}_0$ are zero. The formula (2.32), evaluated at $X=A$, then is

$$P_B = \begin{pmatrix} \sum_{i=1}^n P_{D1i} A_i^2 & 0 & \dots & 0 \\ 0 & \sum_{i=1}^n P_{D2i} A_i^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sum_{i=1}^n P_{Dri} A_i^2 \end{pmatrix} \quad (2.37)$$

2.3.2. Nonlinear Functions.

We consider twice differentiable functions

$$Z = F(W) \quad (2.38)$$

where $Z \in R_r$ and $W \in R_n$. Let the argument W be a fuzzy vector $\tilde{C}$. Then the value of Z is a fuzzy vector $\tilde{B}$ with the apex

$$B = F(C) \quad (2.39)$$

and a pandurance matrix which may be approximately computed by expanding the function $F(W)$ at $W=C$. The linear part of the expansion is

$$L(W) = F(C) + \frac{\partial F}{\partial W} \cdot (W - C) \quad (2.40)$$

where $\partial F / \partial W$ is evaluated at $W=C$. Applying the pandurance propagation formula (2.15) to the linearized function (2.40), one obtains the approximation

$$P_B \approx \frac{\partial F}{\partial W} \cdot P_C \cdot \left(\frac{\partial F}{\partial W} \right)^T \quad (2.41)$$

For the intended application (mathematical model fitting) it is convenient to split the argument W of a function into two sets: observables X and parameters T , that is, to express model functions $F(W)$ in the form

$$Z = F(X, T) . \quad (2.42)$$

Either X or T , or both, can be fuzzy and the approximate formula (2.41) may be applied accordingly. In particular, if $X = \tilde{A}$ and $T = \tilde{C}$, and the pandurance matrix of the vectors $\tilde{A}$ and $\tilde{C}$ is

$$P_W = \begin{pmatrix} P_A & P_{AC} \\ P_{CA} & P_C \end{pmatrix} , \quad (2.43)$$

then the linearized form (2.41) of the pandurance propagation formula is

$$P_B \approx \frac{\partial F}{\partial X} \cdot P_A \cdot \left(\frac{\partial F}{\partial X} \right)^T + \frac{\partial F}{\partial T} \cdot P_C \cdot \left(\frac{\partial F}{\partial T} \right)^T + \frac{\partial F}{\partial X} \cdot P_{AC} \cdot \left(\frac{\partial F}{\partial T} \right)^T + \frac{\partial F}{\partial T} \cdot P_{CA} \cdot \left(\frac{\partial F}{\partial X} \right)^T . \quad (2.44)$$

We consider as an example the scalar function

$$z = f(X, T, z_0) = X^T \cdot T + z_0 \quad (2.45)$$

with the fuzzy arguments $X = \tilde{A}$, $T = \tilde{C}$ and $z_0 = \tilde{v}$. The function can be expanded at the apex (A, C, v) of its argument to yield

$$f(X, T, z_0) = A^T \cdot C + v + C^T \cdot (X - A) + A^T \cdot (T - C) + (z_0 - v) + (T - C)^T \cdot (X - A) . \quad (2.46)$$

Using the approximate formula (2.44) we obtain

$$P_B \approx C^T \cdot P_A \cdot C + A^T P_C \cdot A + P_v + 2A^T \cdot P_{CA} \cdot C + 2P_{vA} \cdot C + 2A^T \cdot P_{Cv} . \quad (2.47)$$

The approximation comes about by the neglect of the last summand in eq. (2.46), i.e., the nonlinear part of the function. Its contribution generally is small if T is close to C and X is close to A .

The advantages of the approximate calculation of the pandurance of a function by eq. (2.41) are twofold. First, the formula is simple and can be easily included in algorithms using fuzzy arithmetic. Second, it produces function values which have conical membership functions. That is, with this approximation, one stays in conical fuzzy vector spaces (including degenerate vectors with semidefinite pandurance matrices). The disadvantage is, as in any approximation, that in certain cases the approximation may not be sufficiently accurate.

2.3.3. Fuzzy Function Space.

In analogy to the linear functions discussed above we may consider the nonlinear function

$$Z = F(X, \tilde{T}) \quad (2.48)$$

with a fuzzy parameter vector $\tilde{T}$ as a fuzzy function of a crisp argument:

$$Z = \tilde{F}(X) \quad (2.49)$$

To define the fuzzy set of functions, $\tilde{F}$, we specify a corresponding membership function. In the case $\tilde{F}(X) = F(X, \tilde{T})$, each member of the set $\tilde{F}$ is given by a particular parameter value $t \in \tilde{T}$. We assign to the function $F(X, t)$ the membership value of t :

$$\mu_{FT}(F(X, t)) = \mu_T(t) \quad (2.50)$$

In this paper, we only consider fuzzy functions of the described type, that is, we assume that a fuzzy function always is defined in terms of a fuzzy function parameter.

To measure the separation between elements of a fuzzy function space, we define the following norm $\|\cdot\|_{FT}$ for the distance of the crisp function $F(X, t)$ from the crisp "apex" function $F(X, T)$ with the membership value one:

$$\|F(X, t) - F(X, T)\|_{FT} = \|t - T\|_T = [(t - T)^T \cdot P_T^{-1} \cdot (t - T)]^{1/2} \quad (2.51)$$

(If $\tilde{T}$ is not a regular point, then P_T^{-1} is replaced by Q_T and the norm defined in accordance with eq. (2.3)). With this definition, one has in analogy to (2.4)

$$\mu_{FT}(F(X, t)) = 1 - \min \{1, \|F(X, t) - F(X, T)\|_{FT}\} \quad (2.52)$$

The discord and the grade of collocation between two fuzzy elements of the fuzzy function space are defined in analogy to eqs. (2.7) and (2.9), respectively.

2.4. Fuzzy Equations.

A fuzzy equation is a fuzzy set of equations. One obtains a fuzzy equation, for instance, by setting a fuzzy function equal to a crisp or fuzzy zero. Let $X \in R_n$, $T \in R_p$, and the function $F(X, T) \in R_r$. We further assume that F is at least twice continuously differentiable in a proper domain,

$$r \leq n \quad \text{and} \quad r \leq p \quad (2.53)$$

We define a fuzzy function as in Section 2.3.3., and obtain a fuzzy equation by setting the function equal to a crisp vector with r vanishing components:

$$\tilde{F}(X) = F(X, \tilde{T}) = 0 \quad (2.54)$$

We shall assume that the right hand side of a fuzzy equation always is a crisp vector. By permitting it to be a fuzzy vector one does not gain additional generality, but merely increases the number of fuzzy parameters by r . That is, the fuzzy equations

$$\tilde{F}(X) = \tilde{0} \quad (2.55)$$

and

$$\tilde{G}(X) = \tilde{F}(X) - \tilde{0} = 0 \quad (2.56)$$

are equivalent sets of equations in a sense which we shall describe shortly. Any crisp $t \in \tilde{T}$ defines a particular crisp equation of the fuzzy set (2.54). That crisp equation either has no solution or defines a crisp solution $X_F(t)$. We assign to the equation and to its solution the membership value of the function $F(X, t)$, that is, the value $\mu_T(t)$.

The solution $X_F(t)$ of the crisp equation $F(X, t) = 0$ is in R_n a $(n-r)$ -dimensional hypersurface which is the set (locus) of all points X that satisfy the equation. The fuzzy equation (2.54) defines in R_n a fuzzy surface $\tilde{X}_F$, that is, a fuzzy set of $(n-r)$ -dimensional surfaces. The surface corresponding to the parameter value t is assigned the membership value $\mu_T(t)$. The fuzzy surface $\tilde{X}_F$ contains all points X which satisfy eq. (2.54) for some $t \in \tilde{T}$. To an $X \in \tilde{X}_F$ we assign the largest membership value of all solution surfaces that pass through X , that is,

$$\mu_{X_F}(X) = \sup_{t: F(X, t) = 0} \mu_{F_T}(F(X, t)) = \sup_{t: F(X, t) = 0} \mu_T(t) \quad (2.57)$$

If $X \notin \tilde{X}_F$, i.e., if X does not solve the eq. (2.54) for any $t \in \tilde{T}$, then we assign to X the membership value zero.

Now we consider eqs. (2.55) and (2.56). Their equivalency is related to the following proposition.

Proposition 3. Let $X \in R_n$, $C \in R_r$, $\tilde{C} = \{C, P_C\}$, and the function $G(X) \in R_r$. Then the solutions of

$$G(X) = \tilde{C} \quad (2.58)$$

and

$$F(X, \tilde{C}) = G(X) - \tilde{C} = 0 \quad (2.59)$$

are identical fuzzy sets.

Proof. Eq. (2.58) is a fuzzy set of equations. Let its solution be $\tilde{X}_G$. Because one side of the equation is crisp, one may use eq. (2.57) to construct the solution in terms of

its membership value. The result is

$$\mu_{XG}(X) = \sup_{t: G(X)-t=0} \mu_C(t) . \quad (2.60)$$

On the other hand, according to eq. (2.57) the solution of eq. (2.59) is a fuzzy set with the membership function

$$\mu_{XF}(X) = \sup_{t: G(X)-t=0} \mu_C(t) . \quad (2.61)$$

Because the delimiters for the maximization of $\mu_C(t)$ in eqs. (2.60) and (2.61) are equivalent crisp expressions, $\mu_{XG}(X) = \mu_{XF}(X)$, q.e.d.

Corollary 2. The solutions of eqs. (2.55) and (2.56) are identical.

Proof. This is the special case $\tilde{C} = \tilde{0}$ of *Proposition 3*.

Proposition 3 and *Corollary 2* show that certain algebraic manipulations of fuzzy equations are permitted in the sense that they do not affect the solution of the equations. We note, however, that not all manipulations which are permitted for crisp equations can be applied to fuzzy equations. For the present paper, the important consequence of *Proposition 3* is that one can assume on the right hand side of a fuzzy equation a crisp vector with zero-valued components.

The definition (2.57) of the membership function of $\tilde{X}_F$ is not practical for numerical algorithms. We provide, therefore, another equivalent definition in terms of a norm for the distance between X and the solution of $F(X, T) = 0$, i.e., between any $X \in R_n$ and the crisp solution surface $X_F(T)$. We define such a norm by

$$\|X - X_F\|_{XF} = \inf_{t: F(X, t) = 0} \|t - T\|_T \quad (2.62)$$

and assign to $\|X - X_F\|$ an infinite value if $F(X, t) \neq 0$ for all $t \in R_p$. With this definition, eq. (2.57) is equivalent to

$$\mu_{XF}(X) = 1 - \min \{ 1, \|X - X_F\|_{XF} \} . \quad (2.63)$$

This formulation is better suited for numerical calculations than (2.57), because the norm (2.62) can be approximately computed as shown by the following proposition.

Proposition 4. Let the r components of $F(X, t)$ be linearly independent functions of t . Then the following approximation of (2.62) is exact for functions $F(X, t)$ that are linear with respect to t :

$$\|X - X_F\|_{XF} \approx \left[F(X,t)^T \cdot \left(\frac{\partial F}{\partial t} \cdot P_T \cdot \left(\frac{\partial F}{\partial t} \right)^T \right)^{-1} \cdot F(X,t) \right]^{1/2} \quad (2.64)$$

Proof. We use the definition (2.62) and formulate the left hand side of eq. (2.64) as the following constrained minimization problem:

$$\left. \begin{array}{l} \text{Minimize} \\ \\ \text{subject to} \end{array} \right\} \begin{array}{l} W = (t - T)^T \cdot P_T^{-1} \cdot (t - T) \\ \\ F(X,t) = 0 \end{array} \quad (2.65)$$

Using a Lagrange multiplier vector k , one obtains for the problem (2.65) a modified objective function

$$\check{W} = \frac{1}{2}(t - T)^T \cdot P_T^{-1} \cdot (t - T) - k^T \cdot F(X,t) \quad (2.66)$$

The corresponding normal equations are

$$\left. \begin{array}{l} P_T^{-1} \cdot (t - T) - F_t^T \cdot k = 0 \\ \\ F(X,t) = 0 \end{array} \right\} \quad (2.67)$$

where $F_t = \partial F / \partial t$. After elimination of k from the first equation (2.67) and multiplication of the result from left by $(t - T)^T$, one obtains the following formula for $\|t - T\|^2$ at the solution:

$$(t - T)^T \cdot P_T^{-1} \cdot (t - T) = (t - T)^T \cdot F_t^T \cdot (F_t \cdot P_T \cdot F_t^T)^{-1} \cdot F_t \cdot (t - T) \quad (2.68)$$

On the right hand side of eq. (2.68) we approximate

$$\frac{\partial F(X,t)}{\partial t} \cdot (t - T) \approx -F(X,T) \quad (2.69)$$

(because $F(X,t)=0$), and also use the argument (X,T) instead of (X,t) in the product $F_t \cdot P_T \cdot F_t^T$. The result is the right hand side of eq. (2.64).

If F is linear with respect to t , then (2.69) exactly holds, and F_t is independent of t . Therefore, (2.64) is exact in the linear case, q.e.d.

Now we investigate the structure of the membership function $\mu_{XF}(X)$ in the vicinity of the apex of the function. The apex itself is the crisp hypersurface $X_F(T)$, defined by $F(X,T)=0$. The level surfaces $\mu_{XF}=\gamma$, or $\|X - X_F\|_{XF}=1-\gamma$, approximately

are given by the equation (see (2.64))

$$F^T \cdot (F_t \cdot P_T \cdot F_t^T)^{-1} \cdot F = (1 - \gamma)^2 . \quad (2.70)$$

Let $Y \in X_F(T)$, i.e., Y be a point of the apex of $X_F(T)$. Then, $F(Y, T) = 0$ and, expanding F in eq. (2.70) in terms of $X - Y$ and only keeping the first non-zero term, one obtains for the level surface the equation

$$(X - Y)^T \cdot F_X^T \cdot (F_t \cdot P_T \cdot F_t^T)^{-1} \cdot F_X \cdot (X - Y) = (1 - \gamma)^2 , \quad (2.71)$$

where $F_X = \partial F / \partial X$. Within the accuracy provided by the one-term approximation, the structure of the level surface in the vicinity of Y is characterized by the matrix

$$Q_Y = F_X^T \cdot (F_t \cdot P_T \cdot F_t^T)^{-1} \cdot F_X . \quad (2.72)$$

We recall that $F \in R_r$ and $r \leq n$. The rank s of the matrix Q_Y is, therefore, at most equal to $r \leq n$. (It equals r if F_X has a full row rank.) If $s = n$, then the solution $\tilde{X}_F$ is a fuzzy point in R_n , and the surfaces $\|X - X_F\|_{X_F} = 1 - \gamma$ are hyperellipsoids in R_n . If $s < n$, then the matrix Q_Y is only semidefinite and the surfaces are hypercylinders. If γ is close to one, then the boundaries of the support of $\mu_{X_F}(X) - \gamma$ are in case $s < n$ cylindrical surfaces that are approximately parallel to the apex surface $X_F(T)$.

If the apex $X_F(T)$ is a point in R_n then its distance norm $\|\cdot\|_A$ from a fuzzy point $\tilde{A}$ is defined by eqs. (2.2) to (2.3). If the apex $X_F(T)$ is a surface in R_n then we define the distance by

$$\begin{aligned} \|X_F - A\|_A &= \min_{X \in X_F} [(X - A)^T \cdot P_A^{-1} \cdot (X - A)]^{1/2} = \\ &= \min_{X: F(X, T) = 0} [(X - A)^T \cdot P_A^{-1} \cdot (X - A)]^{1/2} . \end{aligned} \quad (2.73)$$

An approximate formula for $\|X_F - A\|_A$ is provided by the following proposition.

Proposition 5. Let the r components of F be linearly independent functions of X . Then the following approximate formula for (2.73) is exact for functions $F(X, t)$ that are linear with respect to X :

$$\|X_F - A\|_A \approx [F^T \cdot (F_X \cdot P_A \cdot F_X^T)^{-1} \cdot F]^{1/2} , \quad (2.74)$$

where F and $\partial F / \partial X$ are evaluated at (A, T) .

Proof. The proof follows the same steps as the proof of Proposition 4, with appropriate changes of the arguments of $F(X, T)$.

The separation between a fuzzy point $\tilde{A}$ and the fuzzy surface $\tilde{X}_F$ may be measured in analogy to (2.7) by the discord

$$D(\tilde{A}, \tilde{X}_F) = \min_{X \in R_n} \max \{ \|X - A\|_A, \|X - X_F\|_{X_F} \} . \quad (2.75)$$

The grade of collocation between $\tilde{A}$ and $\tilde{X}_F$ is (see eq. (2.10))

$$\gamma(\tilde{A}, \tilde{X}_F) = 1 - \min \{ 1, D(\tilde{A}, \tilde{X}_F) \} . \quad (2.76)$$

In some important special cases the discord $D(\tilde{A}, \tilde{X}_F)$ can be explicitly computed. These cases are covered by the following *Propositions 6* and *7*.

Proposition 6. Let F_X and F_t be full rank constant coefficient matrices,

$$F(X, t) = F_0 + F_X \cdot X + F_t \cdot t \quad (2.77)$$

$A \notin X_F(T)$, P_A be non-singular,

$$\|X - A\|_A = [(X - A)^T \cdot P_A^{-1} \cdot (X - A)]^{1/2} , \quad (2.78)$$

$$\|X - X_F\|_{X_F} = [F(X, T)^T \cdot (F_t \cdot P_T \cdot F_t^T)^{-1} \cdot F(X, T)]^{1/2} , \quad (2.79)$$

and $L(X)$ be the locus of points where $\|X - A\|_A = \text{constant}$ is tangent to $\|X - X_F\|_{X_F} = \text{constant}$.

Necessary and sufficient for the locus to be a straight line is that

$$\beta \cdot F_t \cdot P_T \cdot F_t^T = F_X \cdot P_A \cdot F_X^T \quad (2.80)$$

for some positive β .

Proof. We first note that $L(X)$ by definition contains the point A , and that along any straight line through A the planes tangent to $\|X - A\|_A = \text{constant}$ are parallel to each other. The tangent plane which passes through a point $C \neq A$ is spanned by vectors $X - C$ that satisfy the equation

$$(C - A)^T \cdot P_A^{-1} \cdot (X - C) = 0 . \quad (2.81)$$

The plane tangent to $\|X - X_F\|_{X_F} = \text{constant}$ which passes through a point $C \notin X_F(t)$ is spanned by vectors $X - C$ that satisfy the equation

$$F(C, T)^T \cdot (F_t \cdot P_T \cdot F_t^T)^{-1} \cdot F_X \cdot (X - C) = 0 . \quad (2.82)$$

Let $B \in X_F(T)$, i.e., $F(B, T) = 0$. Then, $F(C, T) = F_X \cdot (C - B)$ and (2.82) can be expressed in the form

$$(C - B)^T \cdot F_X^T \cdot (F_t \cdot P_T \cdot F_t^T)^{-1} \cdot F_X \cdot (X - C) = 0 . \quad (2.83)$$

Eq. (2.83) shows that planes through C and tangent to $\|X - X_F\|_{X_F} = \text{constant}$ are parallel to each other for all such C which are located on any straight line that passes through a $B \in X_F$, i.e., through the apex of $\tilde{X}_F$.

We find a point $B \in X_F(T)$ which also is a point of the locus $L(X)$ by solving the following minimization problem:

Minimize

$$W = (B - A)^T \cdot P_A^{-1} \cdot (B - A)$$

subject to

$$F(B, T) = 0$$

Using a Lagrange multiplier vector k , one obtains a modified objective function

$$\hat{W} = \frac{1}{2} (B - A)^T \cdot P_A^{-1} \cdot (B - A) - k^T \cdot F(B, T)$$

and the corresponding normal equations

$$\left. \begin{aligned} P_A^{-1} \cdot (B - A) - F_X^T \cdot k &= 0 \\ F(B, T) &= 0 \end{aligned} \right\} \quad (2.84)$$

Eliminating k from the first equation (2.84) one obtains

$$B - A = P_A \cdot F_X^T \cdot (F_X \cdot P_A \cdot F_X^T)^{-1} \cdot F_X \cdot (B - A),$$

or, because $F(B, T) = F(A, T) + F_X \cdot (B - A) = 0$,

$$B - A = -P_A \cdot F_X^T \cdot (F_X \cdot P_A \cdot F_X^T)^{-1} \cdot F(A, T) \quad (2.85)$$

The plane tangent to $\|X - A\|_A = \text{constant}$ through the point B is spanned by vectors $X - B$ which satisfy the equation (see (2.81))

$$(B - A)^T \cdot P_A^{-1} \cdot (X - B) = 0,$$

or, using eq. (2.85),

$$F(A, T)^T \cdot (F_X \cdot P_A \cdot F_X^T)^{-1} \cdot F_X \cdot (X - B) = 0 \quad (2.86)$$

On the other hand, the surface $\|X - X_F\|_{X_F} = \text{constant}$ which passes through the point A has a tangent plane that is spanned by vectors $X - A$ satisfying the equation (see (2.82))

$$F(A, T)^T \cdot (F_t \cdot P_T \cdot F_t^T)^{-1} \cdot F_X \cdot (X - A) = 0 \quad (2.87)$$

Necessary and sufficient for $L(X)$ to be a straight line is that the planes defined by eqs. (2.86) and (2.87), respectively, are parallel to each other. Inspecting the equations, one sees that necessary and sufficient for the planes to be parallel is that (2.80) holds, q.e.d.

Proposition 7. Let the premises of *Proposition 6* hold, and (2.80) be satisfied. Then,

$$D(\tilde{A}, \tilde{X}_F) = \frac{\|X_F - A\|_A \cdot \|A - X_F\|_{XF}}{\|X_F - A\|_A + \|A - X_F\|_{XF}}, \quad (2.88)$$

where $\|X_F - A\|_A$ is given by eq. (2.74) and $\|A - X_F\|_{XF}$ is given by eq. (2.79).

Proof. The discord $D(\tilde{A}, \tilde{X}_F)$ is defined by eq. (2.75). Because $\|X - A\|_A$ as well as $\|X - X_F\|_{XF}$ are unbounded continuous functions, differentiable everywhere, except at their zeros, the minimum in eq. (2.75) is assumed at a point C where $\|C - A\|_A = \|C - X_F\|_{XF}$, and the planes tangent to $\|X - A\|_A = \text{constant}$ and tangent to $\|X - X_F\|_{XF} = \text{constant}$ coincide. C necessarily is a point of the locus $L(X)$, or, according to *Proposition 6* a point of the straight line through A and B , the latter point being given by eq. (2.85). Therefore, one can compute D by first finding that point C between A and B where both norms are equal, and then computing the value of the norms at $X = C$.

Let

$$C = A + \alpha \cdot (B - A) \quad (2.89)$$

be a point of $L(X)$. Then, by definition

$$\|C - A\|_A = |\alpha| \cdot \|B - A\|_A = |\alpha| \cdot \|X_F - A\|_A$$

and

$$\|C - X_F\|_{XF} = |1 - \alpha| \cdot \|A - X_F\|_{XF}.$$

Both expressions are equal for

$$\alpha = \frac{\|A - X_F\|_{XF}}{\|X_F - A\|_A + \|A - X_F\|_{XF}}.$$

Hence,

$$D(\tilde{A}, \tilde{X}_F) = \|C - A\|_A = \alpha \cdot \|X_F - A\|_A$$

is given by eq. (2.88), q.e.d.

Corollary 3. Let $F(X, t) = F_0 + F_X \cdot X + F_t \cdot t$ be a scalar linear function of X and t . Then the discord $D(\tilde{A}, \tilde{X}_F)$ is given by eq. (2.88).

Proof. If F is a scalar linear function, then eq. (2.80) is satisfied and *Proposition 7* holds.

Corollary 4. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy points in R_n , and $P_A = \beta^2 \cdot P_B$. Then the discord between $\tilde{A}$ and $\tilde{B}$ is given by

$$D(\tilde{A}, \tilde{B}) = \frac{\beta}{1 + \beta} \cdot \|B - A\|_A . \quad (2.90)$$

Proof. We define $\tilde{B}$ as the solution of the fuzzy equation

$$F(X, \tilde{B}) = X - \tilde{B} = 0 .$$

Propositions 6 and 7 hold for this function and, therefore

$$D(\tilde{A}, \tilde{B}) = \frac{\beta \cdot \|B - A\|_A^2}{(1 + \beta) \cdot \|B - A\|_A} = \frac{\beta}{1 + \beta} \cdot \|B - A\|_A , \quad \text{q.e.d.}$$

If $F(X, \tilde{T})$ is not linear in X and T , then the discord formula (2.88) may be applied to a linearized approximation of the equation $F(X, \tilde{T})=0$, e.g., to a fuzzy set of equations with the elements

$$F(X, t) = F_X \cdot (X - Y) + F_t \cdot (t - T) = 0 ,$$

where $Y \in X_F(T)$. For scalar F this yields a reasonable value for the discord if t is sufficiently close to T and X is sufficiently close to $X_F(T)$. If F is not scalar, then also the condition (2.80) needs to be satisfied to have a reasonable approximation of the discord. In particular, in the case $r=n$ the condition (2.80) is important.

We make a final remark about the fuzziness of the surface $F(X, \tilde{T})=0$. Its spread is defined by eqs. (2.62) and (2.63). It is easy to see that the spread depends on the formulation of the equation $F(X, \tilde{T})=0$. Therefore, two forms of the equation which produce the same crisp apex surface $X_F(T)$ (defined by $F(X, T)=0$) in general have different spreads, i.e., they define different fuzzy surfaces $\tilde{X}_F$. This dependence of $\tilde{X}_F$ on the equation formulation is an intrinsic property of fuzzy equations.

3. FUZZY MODEL FITTING.

3.1. Fuzzy Models.

We consider mathematical models of observable events that are formulated as twice differentiable fuzzy equations among components of a fuzzy observable vector $\tilde{X}$. That is, the models are fuzzy equations of the type

$$F(\tilde{X}, \tilde{T}) = 0 , \quad (3.1)$$

where F is a twice differentiable vector function. It is convenient to assume that the model parameter vector $\tilde{T}$ consists of two parts, $\tilde{T}_f$ and $\tilde{T}_p$. The first part, $\tilde{T}_f$, represents free model parameters. The second part, $\tilde{T}_p$, consists of prescribed parameters. (Either of these can be an empty set, of course). The apex T_f of $\tilde{T}_f$ is determined by fitting the model to observations while the apex T_p is predetermined.

The corresponding panderance matrices, P_{Tf} and P_{Tp} , of these vectors may be prescribed or determined concurrently with other unknowns of the regression problem.

We assume throughout the rest of this section that the dimensions of F , X and T are r , n and p , respectively. We will, however, permit the functions F_i and the corresponding dimensions r_i and n_i to be different for different observation vectors $\tilde{X}_i$. For simplicity, we assume that the model parameter dimension p is the same for all observations. This is not a restriction, because not all model functions F_i need to depend explicitly on all parameters.

3.2. Model Fitting Principles.

We determine the unknown model parameter vector $\tilde{T}$ by maximizing the membership values of the corrected observations as well as the membership values of the model surface at the corrected observations. Because the membership functions have a normalized maximum of one, the maximization is equivalent to a minimization of the deviations of the membership values from one. In particular, we minimize the sum of the squares of such deviations.

The constraints of the model fitting problem are the equations

$$F_i(X_i + c_i, T + e_i) = 0, \quad i = 1, \dots, s. \quad (3.2)$$

That is, local adjustments are applied to the apexes X_i of the observations $\tilde{X}_i = \{X_i, P_{X_i}\}$ as well as to the apex $T^T = (T_f^T, T_p^T)$ of the model parameter vector. The local corrections are only applied to the fuzzy components of $\tilde{X}_i$ and $\tilde{T}$. That is, if a component of these vectors is crisp, then the corresponding local adjustment is zero. The non-zero adjustments are obtained by minimizing the objective function

$$W = \sum_{i=1}^s [(1 - \mu_{X_i}(X_i + c_i))^2 + (1 - \mu_T(T + e_i))^2] \quad (3.3)$$

subject to the constraints (3.2).

The solution of the model fitting problem consists of the model parameter vector $\tilde{T}$ and the set of the local adjustments (residuals) c_i and e_i . In the n_i -dimensional space of the i -th observation set, the fitted surface (the model) is the fuzzy solution of the equation $F_i(X, \tilde{T}) = 0$. It is reasonable to require that the observations $\tilde{X}_i$ are close to the fitted surface. In our fuzzy environment we express such a requirement in terms of the grade of collocation between $\tilde{X}_i$ and $\tilde{X}_{Fi}$, that is, by

$$\gamma(\tilde{X}_i, \tilde{X}_{Fi}) > \gamma_i^*, \quad i = 1, \dots, s. \quad (3.4)$$

The choice of γ_i^* depends on the particular application and expresses the modeller's

confidence in the adequacy of the model and in the reliability of the data panderance estimates. Typically one specifies different desired values of γ_i^* in the course of an investigation or sets all desired values equal zero and calculates a posteriori the attained grades of collocation. The attained grades then might be used as outlier indicators.

Using the relation (2.76) between $\gamma(\tilde{X}_i, \tilde{X}_{Fi})$ and the discord $D(\tilde{X}_i, \tilde{X}_{Fi})$, and the discord formula (2.88), this condition can be approximated by

$$\frac{\|X_{Fi} - X_i\|_{X_i} \cdot \|X_i - X_{Fi}\|_{X_{Fi}}}{\|X_{Fi} - X_i\|_{X_i} + \|X_i - X_{Fi}\|_{X_{Fi}}} < 1 - \gamma_i^* , \quad (3.5)$$

or, in case of a scalar linearizable model, by

$$\frac{\|c_i\|_{X_i} \cdot \|e_i\|_T}{\delta_i \cdot \|c_i\|_{X_i} + \epsilon_i \cdot \|e_i\|_T} < 1 - \gamma_i^* , \quad (3.6)$$

where the norms are given by eq. (2.3) in terms of the panderances P_{X_i} and P_T ,

$$\delta_i = 1 - \frac{F_i(X_i, T + e_i)}{F_i(X_i, T)} \quad (3.7)$$

and

$$\epsilon_i = 1 - \frac{F_i(X_i + c_i, T)}{F_i(X_i, T)} . \quad (3.8)$$

The corrector ϵ_i appears in eq. (3.6) because $\|c_i\|_{X_i}$ is not the norm of the whole distance of the observed point X_i from the apex X_{Fi} , but only of a part of it. A linear approximation yields $\|c_i\|_{X_i} = \epsilon_i \cdot \|X_{Fi} - X_i\|_{X_i}$ and, correspondingly, $\|e_i\|_T = \delta_i \cdot \|X_i - X_{Fi}\|_{X_{Fi}}$.

If at the solution all the μ_{X_i} and μ_T in the expression (3.3) are positive then one can use the definition (2.4) and express the objective function (3.3) in terms of the norms $\|c_i\|_{X_i}$ and $\|e_i\|_T$ as follows:

$$W_N = \sum_{i=1}^s [\|c_i\|_{X_i}^2 + \|e_i\|_T^2] = \sum_{i=1}^s [c_i^T \cdot P_{X_i}^{-1} \cdot c_i + e_i^T \cdot P_T^{-1} \cdot e_i] . \quad (3.9)$$

Because W_N is a differentiable function of c_i and e_i for all non-zero values of the residuals, the numerical minimization of W_N subject to eq. (3.2) is much simpler than the minimization of W . After the solution is obtained one may check the membership values μ_{X_i} and μ_T as well as the inequality constraints (3.4) and take appropriate action if the membership values are not positive or (3.4) are not satisfied. Usually they are satisfied if model and data are reasonable. The testing of the conditions is most conveniently done in terms of the discords $D(\tilde{X}_i, \tilde{X}_{Fi})$, i.e., by eq. (3.5), because the

values of D also provide a measure for the identification of outliers. The appropriate action again depends on the particular application and little can be said in general. It may involve a removal of outliers, reexamination of pandurance estimates, change of model, etc.

If the pandurance matrices P_{X_i} and P_T are given, then eqs. (3.2), (3.4) and (3.9) define a constrained minimization problem where the unknowns are T , c_i and e_i . More common are situations where the parameter pandurance matrix P_{T_f} of the free parameters $\tilde{T}_f$ is not known. Then it can be iteratively determined making use of the pandurance propagation formula (2.41). For such an iteration, one assumes an initial estimate $P_{T_f,0}$, numerically solves the minimization problem and computes the ensuing pandurance matrix $P_{T_f,1}$ using the linearized pandurance propagation formula (2.41). Then this matrix is normalized (see below) and taken as initial estimate, etc.

The linearized pandurance propagation formula produces a pandurance matrix with a norm proportional to the norms of the input pandurances. Therefore, an iteration without a normalization of the iterated pandurance matrix between iterations generally does not converge to a non-zero finite pandurance matrix of the model parameters. The iteration with normalization determines on the other hand only the relative fuzziness and concordances of the parameters and thereby allows one to prescribe the relative importance of model and data fuzziness, respectively. Let this relative importance be expressed by a scalar parameter, say α . Let another parameter, β , express the relative importance of the fuzziness of the free and prescribed parameters, respectively. We express the objective function W_N as a function of the two parameters as follows:

$$W_N(\alpha, \beta) = \sum_{i=1}^s q_i^2, \quad (3.10)$$

$$q_i^2 = (c_i^T, e_i^T) \cdot P_{q_i}^{-1} \cdot \begin{pmatrix} c_i \\ e_i \end{pmatrix} \quad (3.11)$$

and

$$P_{q_i}(\alpha, \beta) = \begin{pmatrix} (1 - \alpha)^2 \cdot P_{X_i} & 0 & 0 \\ 0 & \alpha^2 \cdot \beta^2 \cdot P_{T_f} & 0 \\ 0 & 0 & \alpha^2 \cdot (1 - \beta)^2 \cdot P_{T_p} \end{pmatrix}. \quad (3.12)$$

The two parameters α and β may have values between zero and one. If $\alpha=0$ then we have an adjustment problem with fuzzy data and crisp model. If $\alpha=1$ then the data

are crisp and the model is fuzzy. If $\alpha > 0$ and $\beta = 0$ then the data are fuzzy, the free model parameters are crisp and the prescribed model parameters are fuzzy. If $\alpha > 0$ and $\beta = 1$ then the data are fuzzy and the model has only free and fuzzy parameters. By choosing appropriate values for α and β one can obtain a number of different combinations of conditions. We give three examples of such combinations. First, if all the panderance matrices P_{X_i} , P_{T_f} and P_{T_p} are prescribed, then we use (3.12) with $\alpha = 2/3$ and $\beta = 1/2$. Second, if P_{X_i} and P_{T_p} are given, and P_{T_f} is iteratively determined, then we specify α and β by two conditions. First, we require that the relative weights of the contributions $\sum \|c_i\|_{X_i}^2$ and $\sum \|e_{p,i}\|_{T_p}^2$ to W_N are the same. Second, we require that the contribution of $\sum \|e_{f,i}\|_{T_f}^2$ (after the normalization of the iterated parameter panderance matrix P_{T_f}) to W_N have the same weight as the sum of the contributions of $\sum \|c_i\|_{X_i}^2$ and $\sum \|e_{p,i}\|_{T_p}^2$. The first requirement is satisfied if

$$\alpha = \frac{1}{2 - \beta} \quad (3.13)$$

The second requirement we express by the equation

$$\sum_{i=1}^s \|c_i\|_{X_i}^2 + \sum_{i=1}^s \|e_{p,i}\|_{T_p}^2 - \sum_{i=1}^s \|e_{f,i}\|_{T_f}^2 = 0 \quad (3.14)$$

Eqs. (3.13) and (3.14) are a system of equations for α and β , because the residuals depend on α and β . We can obtain the numerical solution of this system, for instance, using a regula falsi algorithm for β .

As a third example for the choice of α and β we assume that the P_{X_i} are known and P_{T_f} is iteratively determined, as before, and that P_{T_p} is known only up to scaling factor. Now, P_{T_f} as well as P_{T_p} contain arbitrary scaling factors. In this case, we postulate the following two conditions for α and β . First, we require that the contribution of $\sum \|e_{f,i}\|_{T_f}^2$ to W_N receives the same weight as the contribution of $\sum \|e_{p,i}\|_{T_p}^2$. Second, we require that the sum of all contributions of the parameter residuals has the same weight as the contribution $\sum \|c_i\|_{X_i}^2$ of the residuals of the observations. The two conditions can be expressed in form of the following two equations.

$$G(\alpha, \beta) = \sum_{i=1}^s \|e_{f,i}\|_{T_f}^2 - \sum_{i=1}^s \|e_{p,i}\|_{T_p}^2 = 0 \quad (3.15)$$

and

$$H(\alpha, \beta) = - \sum_{i=1}^s \|c_i\|_{X_i}^2 + \sum_{i=1}^s \|e_{fi}\|_{Tf}^2 + \sum_{i=1}^s \|e_{pi}\|_{Tp}^2 = 0 \quad (3.16)$$

This system of equations for α and β we solve numerically by a regula falsi algorithm in the α, β -plane. (Note that $G(\alpha, 0) \leq 0$, $G(\alpha, 1) \geq 0$, $H(0, \beta) \leq 0$ and $H(1, \beta) \geq 0$, which assures the existence of a solution within the unit square).

3.3. Numerical Treatment.

The central part of the numerical treatment of the model fitting problem is the solution of the following constrained minimization problem:

Minimize

$$W_N(\alpha, \beta) = \sum_{i=1}^s (c_i^T, e_i^T) \cdot P_{qi}^{-1} \cdot \begin{pmatrix} c_i \\ e_i \end{pmatrix}, \quad (3.17a)$$

subject to

$$F_i(X_i + c_i, T + e_i) = 0, \quad i = 1, \dots, s, \quad (3.17b)$$

where the P_{qi} are defined by eq. (3.12). This problem formally is equal to a usual least squares model fitting problem with vector data and implicit model equations [2]. Software for the solution of such problems is available for instance from [1] and can be used to solve (3.17). The solution of the example presented in Section 4 was computed using the utility routine COLSAC [1]. The result of a COLSAC calculation consists of the optimal values of c_i , e_i and T , and a posterior estimate of the panderance matrix P_{Tf} computed with the panderance propagation formula (2.41) from the input panderance matrices. If no iterations are involved, i.e., if P_T is prescribed, then a single call of COLSAC concludes the calculations.

The inequality constraints (3.4) usually are inactive, except for outliers. If it is determined that data with large discords cannot be discarded or treated in the same manner as other data, (i.e., the inequality constraints ignored), then one has a more complicated programming problem at hand. There is no guarantee that the problem has a solution and, if a solution exists, it generally is not very useful, because it particularly accommodates outliers. The treatment of problems with outliers and too stringent conditions (3.4) very much depends on the application and typically may involve a change of the model functions F_i . We shall not further pursue this type of problem, and assume that the inequality constraints are inactive at the solution.

For certain ranges of α and β the convergence of the procedure for the determination of P_T can be quite slow. In such cases, one may first determine the matrix P_T by iteration for extreme values of α and β and use an interpolated matrix P_T

for subsequent calculations with other α and β . Then, iteration to find the final P_T might be resumed in the vicinity of the final point in the α, β -plane.

4. MODEL OF PLATE DEFORMATION.

4.1 Data and Model.

The vulnerability of armored vehicles depends among other things on the deformation of armor plates by the blast of a nearby explosion. References [10] and [11] describe a series of experiments in which such deformations were measured. We shall use data from these experiments in an example of fuzzy model fitting.

The relevant data consist of the charge energy, the distance of the charge from the armor plate, the thickness of the plate and the maximum deflection of the plate. (The size of the unsupported span of the plate was the same in all experiments and, therefore, its influence on the deformation cannot be obtained from the data. However, in most of the observed cases, the size of the deformation was such that the influence of the plate's support reasonably could be assumed negligible). Some of these data were measured quite precisely (e.g., the plate thickness), while other data were inaccurate and obvious candidates for fuzzy numbers. Such a datum is, e.g., the strength of the plate's material which we chose to characterize by its ultimate yield stress. The yield stress is ill defined and difficult to measure, and it was found to vary considerably between plate specimens [11]. In summary, it was assumed that all data were fuzzy numbers with appropriate spreads.

The regression model of the plate deformation is not derived from a mechanical theory of the event, because such a general theory does not exist. Instead, the model was determined by exploratory data analysis and dimensional arguments. This uncertainty about the form of the model was represented in the model specification by using fuzzy equations as model constraints. Thus, in this example, the model as well as the data are fuzzy.

The following two dimensionless quantities were used as variables:

$$A = \frac{z}{d} , \quad B = \frac{E}{d^2 \cdot T \cdot \sigma_y} , \quad (4.1)$$

where z (m) is the maximum deflection, d (m) is the distance of the charge from the plate, E (J) is the energy released by the explosion, T (m) is the thickness of the plate and σ_y (Pa) is the ultimate yield stress of the plate's material. The maximum deflection was obtained from a detailed analysis of various observed deflections, as described in reference [5], d , E and T were obtained by simple measurements, and for σ_y a nominal

handbook value was used.

In reference [5] the quantity B was called a *ballistic limit indicator*, because the size of B is a predictor for the onset of plate failure: the plate fails (develops a hole) if B exceeds the fuzzy bound $\tilde{B}_0 = \{129., 40.^2\}$. Since B is also useful as an indicator for the size of plate deformation by blast, a better name for B is the *blast damage indicator*.

Table 1 lists the deflection data with their estimated spreads and concordances. The spreads of d , E , T and σ_y were assumed to be constant and estimated by reasonable assumptions about measurement accuracy. The spread of the deflection z was obtained by the data analysis in reference [5]. The spreads of A and B were computed from these estimates using eq. (2.41). The concordances between the observed $\tilde{A}$ and the corresponding $\tilde{B}$ are not zero, because both depend on the distance $\tilde{d}$. Figures 2 and 3 display the data points without and with the corresponding supports of their membership functions, respectively. The different symbols of data points in the figures distinguish different subsets of the experiments but the distinction is not important for the present example.

The model function was chosen to be

$$F(\tilde{A}, \tilde{B}; \tilde{a}, \tilde{b}, \tilde{c}) = \tilde{a} + \tilde{b} \cdot \tilde{B}^{\tilde{c}} - \tilde{A} \quad (4.2)$$

The parameters $\tilde{a}$ and $\tilde{b}$ were assumed to be free and their spreads were not prescribed. The parameter $\tilde{c}$ was assumed to have the apex 1.50 and either a spread of 0.2 or an undetermined spread. Thus, in this example, the pandurance matrix P_{Tf} of the free parameters is a (2×2)-matrix and the pandurance matrix P_{Tp} of the prescribed parameter $\tilde{c}$ is a scalar. The data fitting was carried out as described in Section 3 with the aid of the utility routine COLSAC [1].

4.2 Data and Model Spreads.

The data pandurance matrices P_{Xi} are given in terms of the data in Table 1 by

$$P_{Xi} = \begin{pmatrix} s_{Ai}^2 & s_{Ai} \cdot s_{Bi} \cdot c_{ABi} \\ s_{Ai} \cdot s_{Bi} \cdot c_{ABi} & s_{Bi}^2 \end{pmatrix}, \quad (4.3)$$

where s_{Ai} and s_{Bi} are the spreads of A_i and B_i , respectively and c_{ABi} is the concordance between A_i and B_i . The parameter part of the regression pandurance matrix (3.12) is

$$P_T = \begin{pmatrix} \alpha^2 \cdot \beta^2 \cdot P_{Tf} & 0 \\ 0 & \alpha^2 \cdot (1 - \beta)^2 \cdot 0.2^2 \end{pmatrix}, \quad (4.4)$$

where the (2×2) -matrix P_{Tf} was determined by iteration. We calculated two cases. In one case, P_T was assumed as given by eq. (4.4). In the other case, we assumed that the spread of the exponent $\tilde{c}$ ($=0.2$ in eq. (4.4)) is not known, that is, P_{T_p} was assumed to contain an unknown scaling factor.

In the former case, the normalization factor of P_{Tf} was chosen such that the maximum discord between any of the apexes X_i of the observations $\tilde{X}_i$, $i=1, \dots, s$, and the fuzzy surface $F(X, \tilde{T}_f, T_p)=0$ was equal to five. In the latter case, we first normalized P_{Tf} and P_{T_p} separately such that the maximum discords between the observation apexes and $F(X, \tilde{T}_f, T_p)=0$ and $F(X, T_f, \tilde{T}_p)=0$, respectively, were equal to five. Then the complete matrix P_T (consisting of the two separately normalized submatrices) was normalized such that the maximum discord between the observation apexes and the surface $F(X, \tilde{T}_f, \tilde{T}_p)=0$ was equal to five. We remind that the choice of a normalization procedure (in the present case, also the maximum discord values chosen for the normalization) is arbitrary in the sense that it affects only the numerical values of α and β , which regulate the relative weights of data and parameter residuals (see Section 3.2).

The posterior fuzziness of the fitted model can be estimated using the pandurance propagation formula (2.41) if the pandurances of data and model parameters are all prescribed. If a part of the model parameter pandurance matrix P_T is iteratively determined then the result contains arbitrary parameters and a direct application of (2.41) cannot be justified. Therefore, one should in such cases determine the spread of the model by other means, also taking into account the actually observed deviations between data and fitted model.

In the present example, the fitted function was used in the form

$$\tilde{A}(B) = A(B, \tilde{T}) = \tilde{a} + \tilde{b} \cdot B^{\tilde{c}}, \quad (4.5)$$

and its spread was adapted to observations by using two quadratic modulator functions $M(B)$, one each for the positive and negative spreads of $\tilde{A}$. First, P_T was normalized using a factor ψ such that the largest discord between the fitted curve $A(B, \tilde{T})$ and the X_i was five. This gave a reasonable spread increase pattern for the model outside the observed range of B but a too optimistic spread within the observed range. Therefore, modulator functions $M(B)$ were introduced to define modulated pandurance matrices by

$$P_{T \text{ mod}} = M(B)^2 \cdot \psi^2 \cdot P_T. \quad (4.6)$$

The modulator functions were determined such that the largest discord between the fitted $A(B, \tilde{T})$ and the observations $\tilde{X}_i$ was less than two. The form of the modulator functions was

$$M(B) = \max \{ 1, m_1 + m_2 \cdot (B - B_0)^2 \} \quad (4.7)$$

with proper constants m_1 , m_2 and B_0 , and a different function was determined for positive and negative differences, respectively, between the fitted function and observations. The maximal discord value of two was established by numerical experiments and chosen because it produced a spread pattern in reasonable agreement with the scatter of observations. The final formula for the spread of $\tilde{A}(B)=A(B, \tilde{T})$ is

$$s_A = \left[\frac{\partial A}{\partial T} \cdot P_{T \text{ mod}} \cdot \left(\frac{\partial A}{\partial T} \right)^T \right]^{1/2} \quad (4.8)$$

4.3. Numerical Results.

Table 2 contains the numerical results in the two presented examples. The table lists the optimal values of the model parameters, their spreads and the concordance c_{ab} between a and b . The panderance matrix of the parameter vector is given in terms of the spreads and c_{ab} by eq. (2.6). The quoted spreads correspond to a matrix that is normalized such that in the parameter space the largest discord between the locally corrected parameters and the apex of the parameter vector equals one and, therefore, the listed spread s_c of $\tilde{c}$ in Case 1 is not equal to the prescribed nominal value of 0.2. A comparison of the two cases illustrates the effect of assigning a fixed value to the spread of a model parameter. Thus, in the present example, the spread of $\tilde{c}$ is larger in Case 1 than in Case 2, as are the uncertainties of the other two model parameters $\tilde{a}$ and $\tilde{b}$. The increase of the latter two is a consequence of the condition (3.16). Plots of the final model curve $\tilde{A}(B)$ are shown in Figures 4 and 5. The figures display for each data point its corresponding support ellipse (modified by the factor $1-\alpha$, see eq. (3.12)), the corrected data point $X_i + c_i$, and a short segment of the fitted function that passes through the corrected point and has the highest membership value μ_{FT} . (It is a segment of the curve $F(X, T + e_i) = 0$). The spreads of the model curves shown in Figures 4 and 5 are computed using the factor Ψ and the modulator functions $M(B)$ from Table 2 to modulate the parameter panderance matrix given at the top of the table (see Section 4.2). The differences between the shapes of the spreads of the fitted curves in both figures illustrate how a prescribed model fuzziness may affect the result of the fitting.

An overview of the local parameter adjustments is provided by Figures 6 and 7. They show in the parameter space three views of the locally adjusted parameter sets, i.e., the end points of the vectors $T + e_i = (a + e_{ai}, b + e_{bi}, c + e_{ci})^T$. Also shown are the contours of the support ellipsoid of the fuzzy parameter vector $\tilde{T}$. The ellipsoid corresponds to the spreads s_a , s_b and s_c listed in Table 2.

5. SUMMARY.

Mathematical models of terminal ballistics problems often must be based on incomplete theories and data that contain inaccuracies which are not probabilistic. An example of such a problem is the deformation of an armor plate by a nearby explosion. To treat this problem, we have used concepts of fuzzy set theory and developed algorithms for a special type of fuzzy regression.

The algorithms are based on concepts of analytic geometry and restricted to data and model parameter vectors which are characterized by a particular type of conical membership function. The regression principle is to maximize in a least squares sense membership values of fuzzy observations and of fuzzy fitting functions. The restriction to conical membership functions and the employment of a least squares principle have the advantage that the calculations can be done with available software for least squares regression problems with implicit constraint functions. The model functions can be implicit and nonlinear with respect to data as well as with respect to model parameters, and the algorithms accept fuzzy data as well as fuzzy parameters, whereby the latter may be either free or prescribed. The method is more flexible than previously published methods, which are restricted either to linear models, [6] through [9], [12] and [13], or to crisp models [3], or to crisp data [4]. (The membership functions are restricted to particular types in all of the quoted publications about fuzzy regression methods). We feel that the possibility to choose freely the model functions is a considerable asset of the present method.

The method is applied to a set of data from experiments where armor plates were deformed by bare charge explosions. The result is a fuzzy non-linear function that predicts the maximum of a relative deflection of the plate in terms of a dimensionless blast damage indicator. The model takes into account the inaccuracies of data and the vagueness of the theoretical basis by assigning proper fuzziness measures to the prediction.

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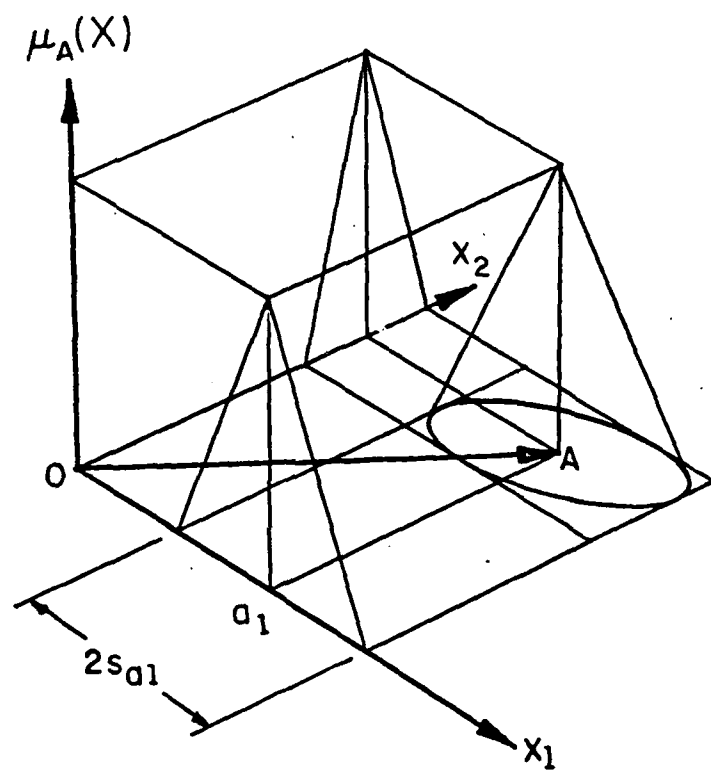


Figure 1. Conical membership function in R_2 .

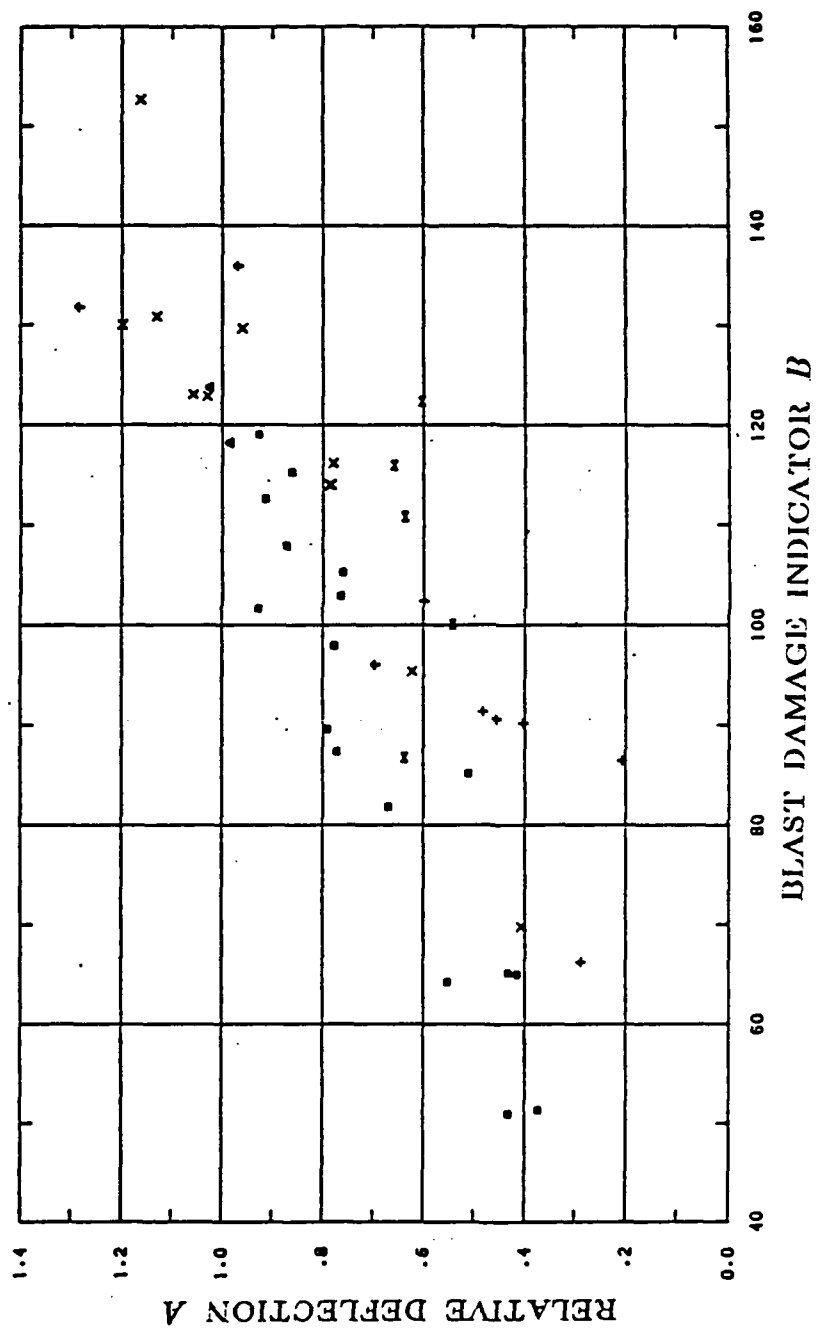


Figure 2. Observed plate deflection versus load.

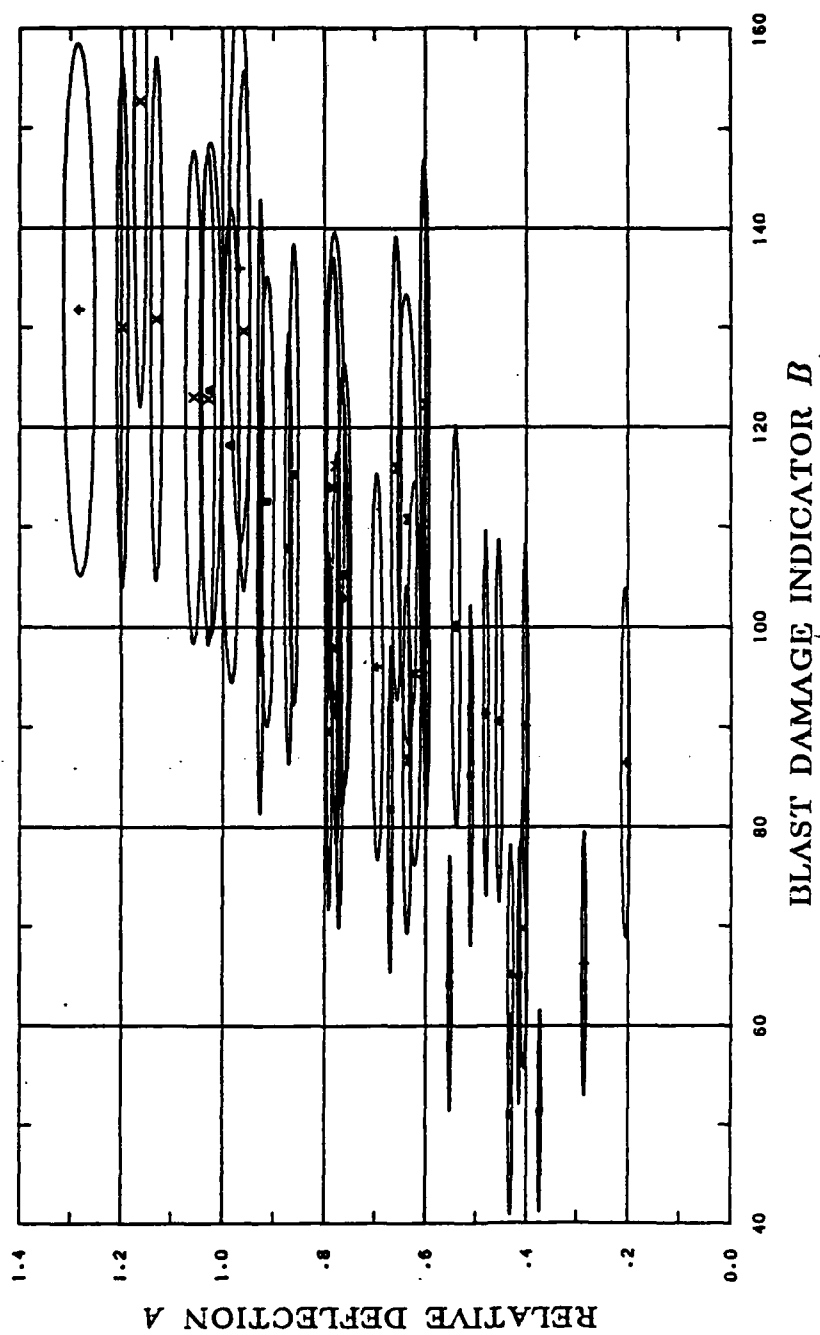


Figure 3. Observations with support ellipses.

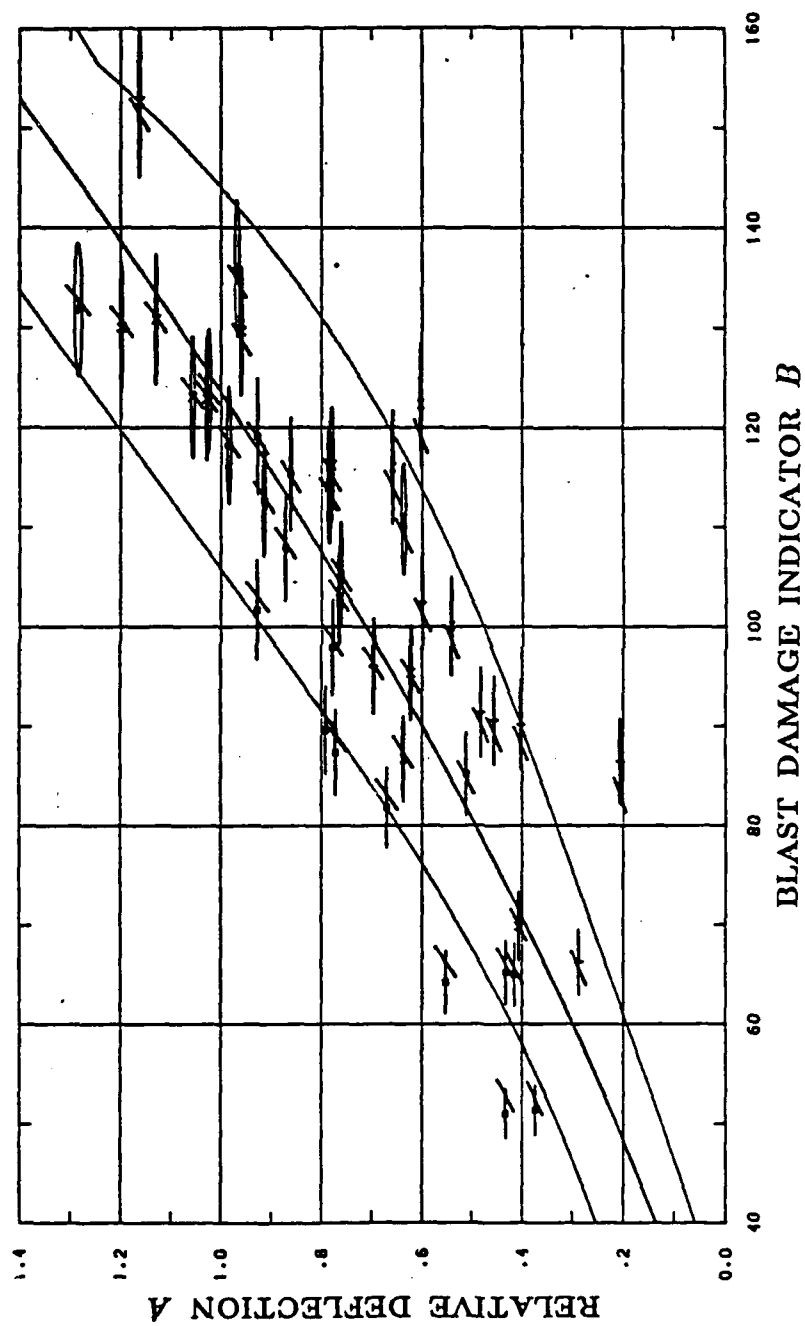


Figure 4. Model curve in Case 1 with prescribed spread of $\tilde{c}$.

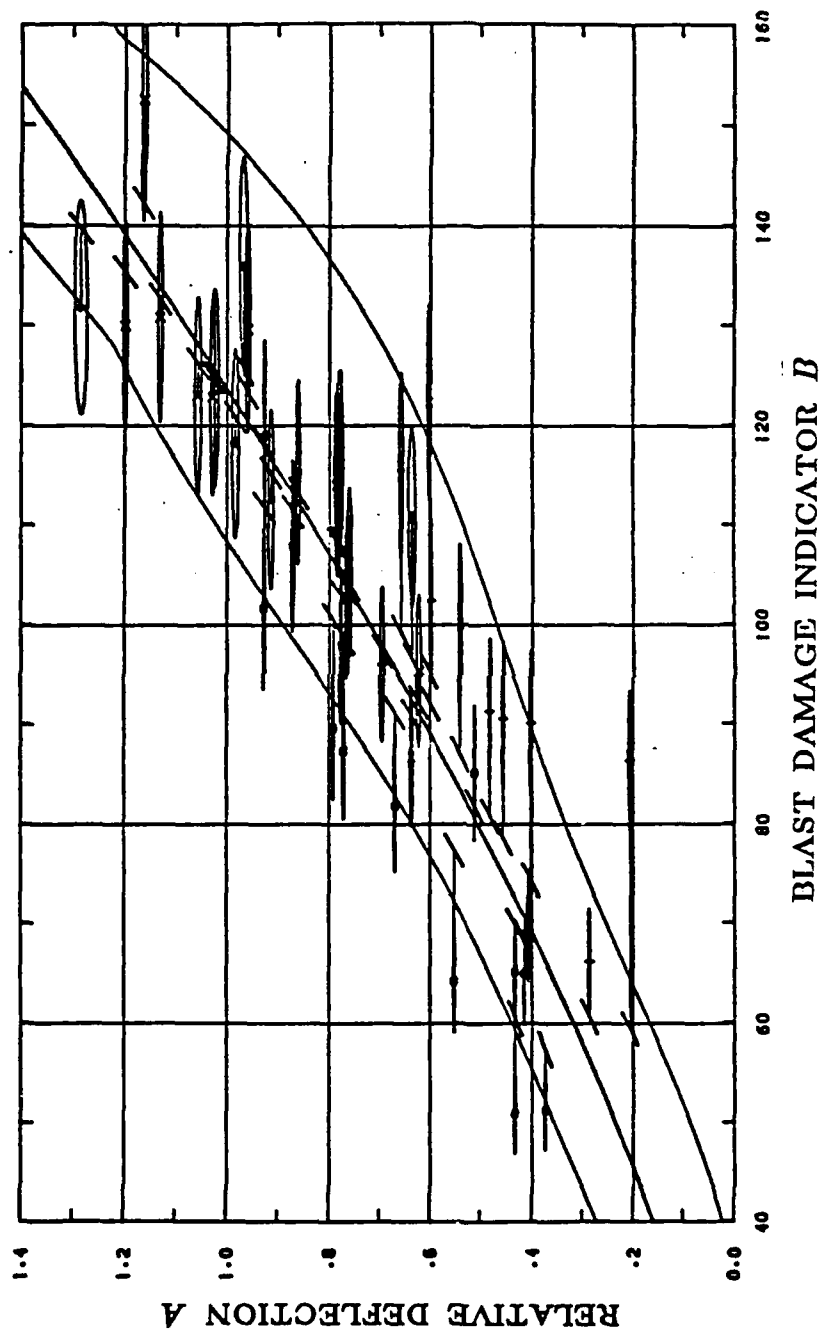


Figure 5. Model curve in Case 2 with undetermined parameter spreads.

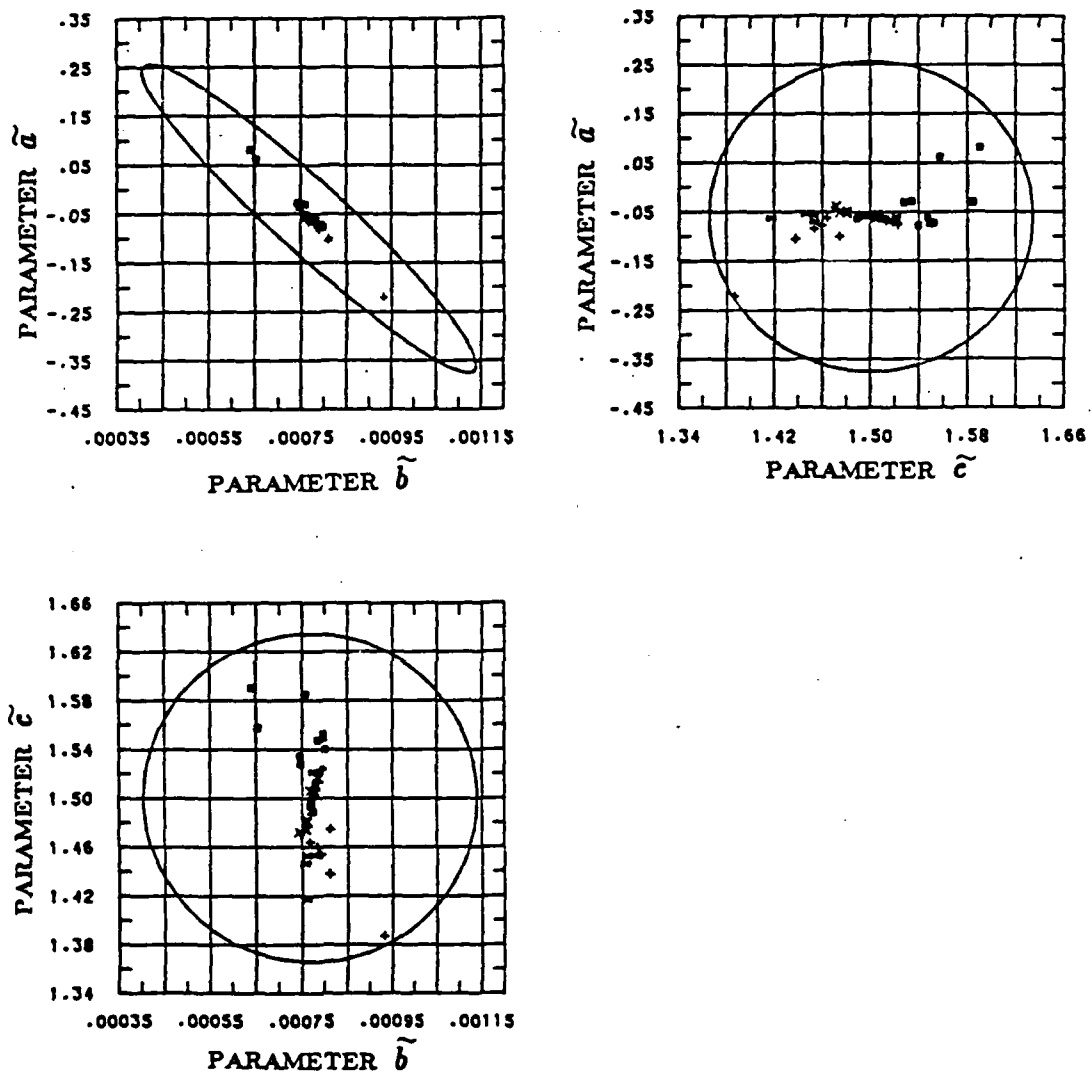


Figure 6. Local parameter adjustments in Case 1.

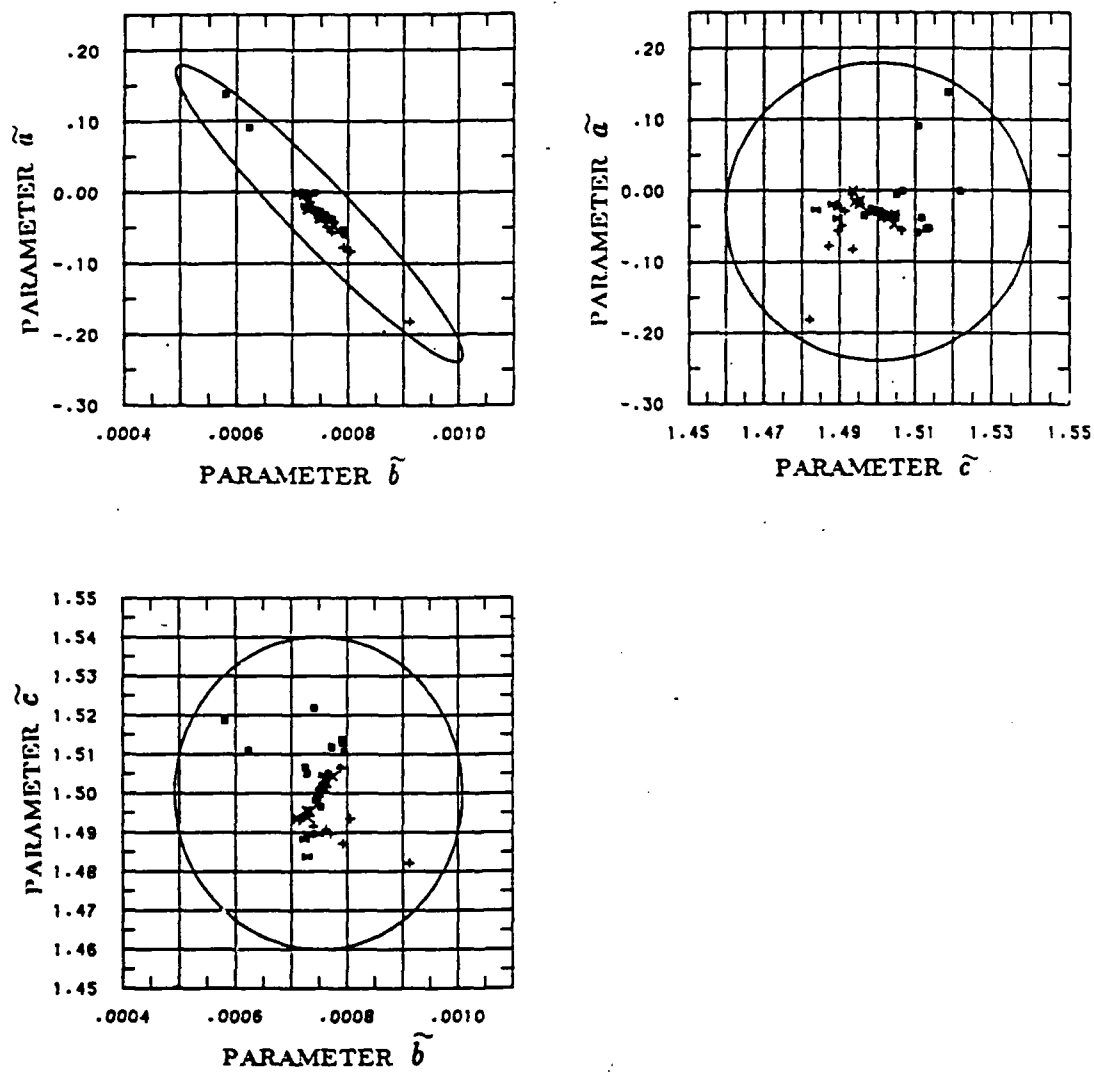


Figure 7. Local parameter adjustments in Case 2.

Table 1. Input Data.

Nr.	Label	B	B-Spread	A	A-Spread	Concordance
1	RD- 2/T# 5	105.26	21.17	0.76056	0.01369	0.053516
2	RD- 5/T# 5	102.93	20.70	0.76508	0.01158	0.063669
3	RD- 6/T# 5	118.21	23.81	0.98405	0.01693	0.063657
4	RD- 8/T# 5	51.00	10.21	0.43281	0.00458	0.018055
5	RD- 9/T# 5	64.24	12.87	0.55264	0.00386	0.034592
6	RD-10/T# 5	51.34	10.28	0.37383	0.00344	0.034592
7	RD-11/T# 5	64.99	43.02	0.41531	0.00366	0.029891
8	RD-12/T# 5	65.18	13.06	0.43272	0.00582	0.001796
9	RD-13/T# 5	85.16	17.07	0.51172	0.00408	0.039626
10	RD-14/T# 5	115.30	23.12	0.86140	0.00971	0.038086
11	RD-16/T# 5	123.73	24.82	1.02472	0.02159	0.022186
12	RD-18/T# 5	97.93	19.63	0.77744	0.00821	0.034652
13	RD-19/T# 5	107.91	21.64	0.87198	0.00772	0.044735
14	RD-21/T# 5	81.80	16.39	0.67031	0.00425	0.030151
15	RD-22/T# 5	87.36	17.50	0.77164	0.00593	0.026309
16	RD-23/T# 5	89.61	17.95	0.79166	0.00471	0.036016
17	RD-24/T# 5	101.66	20.37	0.92772	0.00577	0.038845
18	RD-25/T# 5	112.61	22.56	0.91424	0.01386	0.016990
19	RD-27/T# 5	119.06	23.85	0.92703	0.00606	0.039406
20	RD- 1/T#10	69.83	14.07	0.40709	0.00767	0.066639
21	RD- 2/T#10	135.94	27.54	0.96792	0.02033	0.116586
22	RD- 3/T#10	95.37	19.25	0.62306	0.01206	0.088135
23	RD- 4/T#10	116.19	23.51	0.77957	0.01784	0.088517
24	RD- 7/T#10	96.04	19.39	0.69616	0.01057	0.112396
25	RD- 8/T#10	114.02	23.05	0.78626	0.01431	0.111430
26	RD- 9/T#10	114.04	23.06	0.78291	0.01279	0.124088
27	RD-10/T#10	131.31	26.70	1.28464	0.03112	0.101073
28	RD-11/T#10	129.67	26.07	0.95972	0.01227	0.075380
29	RD-14/T#10	122.87	24.70	1.02897	0.01251	0.079290
30	RD-16/T#10	130.02	26.09	1.19877	0.01199	0.061773
31	RD-18/T#10	130.85	26.26	1.13072	0.01138	0.061376
32	RD-19/T#10	152.65	30.64	1.16320	0.01258	0.063302
33	RD-22/T#10	123.06	24.68	1.05742	0.01652	0.032715
34	RD- 1/T#20	86.41	17.55	0.20707	0.00873	0.071527
35	RD- 2/T#20	110.83	22.60	0.63734	0.01885	0.129906
36	RD- 3/T#20	66.25	13.32	0.28792	0.00438	0.063388
37	RD- 4/T#20	86.75	17.46	0.63775	0.00852	0.094093
38	RD- 6/T#20	100.05	20.16	0.54149	0.00845	0.093388
39	RD- 8/T#20	115.95	23.30	0.65883	0.01016	0.062502
40	RD-12/T#20	90.62	18.19	0.45640	0.00668	0.052095
41	RD-13/T#20	102.40	20.57	0.59983	0.00682	0.075114
42	RD-15/T#20	91.35	18.34	0.48248	0.00518	0.071063
43	RD-16/T#20	90.17	18.15	0.40348	0.00625	0.081118
44	RD-17/T#20	122.35	24.69	0.60291	0.00953	0.107990

Table 2. Model Parameters.

Case 1. Prescribed Apex and Spread of $\tilde{z}$.

$$\tilde{a} = \{ -5.92 \cdot 10^{-2}, 31.60 \cdot 10^{-2} \}, \quad s_a \cdot \Psi = 5.83 \cdot 10^{-2}$$

$$\tilde{b} = \{ 7.52 \cdot 10^{-4}, 3.68 \cdot 10^{-4} \}, \quad s_b \cdot \Psi = 0.68 \cdot 10^{-4}$$

$$\tilde{c} = \{ 1.500, 0.134 \}, \quad s_c \cdot \Psi = 0.025$$

$$c_{ab} = -0.95252, \quad \Psi = 0.18439$$

$$M(B)_+ = 2.588 - 1.19 \cdot 10^{-4} \cdot (B - 48.4)^2$$

$$M(B)_- = -2.617 + 3.75 \cdot 10^{-4} \cdot (B - 90.8)^2$$

$$\alpha = 0.75, \quad \beta = 0.67$$

Case 2. Prescribed Apex and Undetermined Spread of $\tilde{z}$.

$$\tilde{a} = \{ -3.01 \cdot 10^{-2}, 20.88 \cdot 10^{-2} \}, \quad s_a \cdot \Psi = 12.28 \cdot 10^{-2}$$

$$\tilde{b} = \{ 7.50 \cdot 10^{-4}, 2.28 \cdot 10^{-4} \}, \quad s_b \cdot \Psi = 1.52 \cdot 10^{-4}$$

$$\tilde{c} = \{ 1.500, 0.040 \}, \quad s_c \cdot \Psi = 0.024$$

$$c_{ab} = -0.95521, \quad \Psi = 0.58788$$

$$M(B)_+ = 1.952 - 4.18 \cdot 10^{-4} \cdot (B - 80.4)^2$$

$$M(B)_- = -2.641 + 3.80 \cdot 10^{-4} \cdot (B - 93.4)^2$$

$$\alpha = 0.60, \quad \beta = 0.43$$

Note: The fuzzy numbers in this table are given in the form $\{a, s_a\}$, that is, the second number in the braces is the spread (not the panderance).

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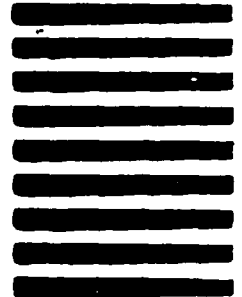


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