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Pseudo-monotone Complementarity Problems
in Hilbert Space

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TECHNICAL REPORT SOL 90-10

July 1990

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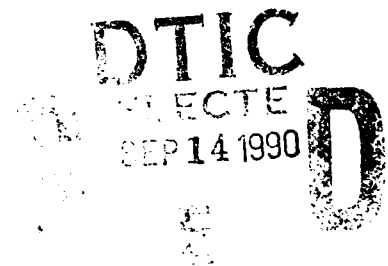Stanford University
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Abstract

In this paper, some existence results for a nonlinear complementarity problem involving a pseudo-monotone mapping over an arbitrary closed convex cone in a real Hilbert space are established. In particular, some known existence results for a nonlinear complementarity problem in a finite-dimensional Hilbert space are generalized to an infinite-dimensional real Hilbert space. Applications to a class of nonlinear complementarity problems and the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions are given.

Key Words: Nonlinear complementarity problems, variational inequality problems, pseudo-monotone mappings, monotone mappings, weakly coercive mappings.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A nonempty subset K of H is said to be a *cone* if $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$. Let K be a closed convex cone in H with dual cone K^*, that is,

$$K^* = \{u \in H \mid \langle u, x \rangle \geq 0, \, \forall \, x \in K\}.$$

Let f be a mapping from K into H. We consider the following *nonlinear complementarity problem* (NCP): find $x \in K$ such that

$$f(x) \in K^* \quad \text{and} \quad \langle x, f(x) \rangle = 0.$$

Such problems were introduced by Karamardian (Ref. 1) and have been extensively studied in the literature. See, e.g., (Refs. 1-12) and the references therein. Another problem that is closely related to NCP is the following *variational inequality problem* (VIP): find $\bar{x} \in K$ such that

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0 \qquad \text{for all } x \in K.$$

In (Ref. 1, Lemma 3.1), Karamardian has shown that if K is a closed convex cone, then both VIP and NCP have the same solution set. Therefore, one approach to studying NCP is by studying VIP over closed convex cones. The purpose of this paper is to use this approach to prove some existence results for a nonlinear complementarity problem involving a pseudo-monotone mapping over an arbitrary closed convex cone in a real Hilbert space. Nonlinear complementarity problems involving pseudo-monotone mappings in Hilbert spaces have not yet been investigated except for those in the finite-dimensional case. See (Ref. 10). In Section 2, we shall prove some necessary and sufficient conditions for existence of solutions to variational inequality problems. Then in Section 3, we give some existence results for solutions to nonlinear complementarity problems by combining existence results given in

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Section 2 with the aforementioned lemma of Karamardian. Finally, in Section 4, we shall consider some applications to a class of nonlinear complementarity problems studied in (Ref. 9) and the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions.

2. Existence Results for Variational Inequality Problems

Let K be a nonempty subset of H. A mapping $f : K \longrightarrow H$ is said to be *continuous on finite-dimensional subspaces* if it is continuous on $K \cap U$ for every finite-dimensional subspace U of H with $K \cap U \neq \emptyset$. The mapping f is said to be *pseudo-monotone* (in the sense of Karamardian (Ref. 10)) if for any x and y in K,

$$\langle x - y, f(y) \rangle \geq 0 \quad \text{implies} \quad \langle x - y, f(x) \rangle \geq 0,$$

and f is said to be *monotone* if

$$\langle x - y, f(x) - f(y) \rangle \geq 0 \qquad \text{for all } x, y \in K.$$

It is easy to see that if f is monotone, then it is pseudo-monotone, but not conversely. The mapping f is said to be *strictly monotone* if the above inequality is strict whenever x and y are distinct. For other types of generalized monotone mappings we refer readers to (Refs. 13, 14). In (Ref. 13), Karamardian and Schaible introduce varies types of generalized monotone mappings starting with pseudo-monotone mappings in (Ref. 10), whereas in (Ref. 14), they present characterizations of differentiable generalized monotone mappings in (Ref. 13) and, as a special case, of affine-linear generalized monotone mappings.

For any real number x, $|x|$ denotes the absolute value of x. For $K \subset H$, int(K), $\partial(K)$ and K^c denote the interior, boundary and complement of K, respectively. For $K, B \subset H$, $\text{int}_K(B)$ and $\partial_K(B)$ denote the relative interior and relative boundary of B in K, respectively. The set $K \backslash B$ denotes the complement of B in K. A subset of a Hilbert space is said to be

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solid if it has a nonempty interior. A cone K is said to be *pointed* if $K \cap -K = \{0\}$. Unless specified otherwise, the topology of a Hilbert space mentioned in this paper refers to the norm topology.

Lemma 2.1. *Let K be a closed convex subset in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then $x \in K$ is a solution of*

$$\langle u - x, f(x) \rangle \geq 0 \qquad \text{for all } u \in K \tag{1}$$

if and only if

$$\langle u - x, f(u) \rangle \geq 0 \qquad \text{for all } u \in K. \tag{2}$$

Proof. Suppose $x \in K$ is a solution of (2). Let $u \in K$ be arbitrary and for $0 < \lambda \leq 1$ let $x_\lambda = \lambda u + (1 - \lambda)x$. Then $x_\lambda \in K$, and from (2) we have

$$\lambda \langle u - x, f(x_\lambda) \rangle \geq 0.$$

Hence

$$\langle u - x, f(x_\lambda) \rangle \geq 0. \tag{3}$$

Let $\lambda \longrightarrow 0$. Then $x_\lambda \longrightarrow x$ along a line segment. By the continuity on finite-dimensional subspaces, $f(x_\lambda)$ converges to $f(x)$ as λ goes to 0. It follows from (3) that

$$\langle u - x, f(x) \rangle \geq 0.$$

Therefore x is a solution of (1).

Conversely, suppose $x \in K$ is a solution of (1). Since f is pseudo-monotone, we have

$$\langle u - x, f(u) \rangle \geq 0 \qquad \text{for all } u \in K.$$

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Thus x is a solution of (2).

The following existence theorem is a key result for the remainder of this paper.

Theorem 2.1. *Let K be a closed, bounded, and convex subset in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then there exists $\bar{x} \in K$ such that $\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0$ for all $x \in K$.*

Proof. Let U be any finite-dimensional subspace of H with $K \cap U \neq \emptyset$ and let P_U be the orthogonal projection of H onto U. Let $f_U = P_U \circ f$, the composition of P_U and f. By a result of Hartman and Stampacchia (Ref. 15, Lemma 3.1), there exists $x_U \in K \cap U$ such that

$$\langle u - x_U, f(x_U) \rangle \geq 0 \qquad \text{for all } u \in K \cap U. \tag{4}$$

Let Λ be the family of all finite-dimensional subspaces U of H with $K \cap U \neq \emptyset$ and let $K_U = \{x_V \mid U \subset V \in \Lambda\}$. For $U \in \Lambda$, let $\overline{K_U}^\omega$ be the weak closure of K_U. Then the family $\{\overline{K_U}^\omega \mid U \in \Lambda\}$ has the finite intersection property. Indeed, for $U, V \in \Lambda$, let $W \in \Lambda$ be such that $U \cup V \subset W$. Then $\emptyset \neq K_W \subset K_U \cap K_V$. Since K is closed, bounded, and convex, it is weakly compact. Also, since $\overline{K_U}^\omega \subset K$ for all $U \in \Lambda$, it follows that $\bigcap_{U \in \Lambda} \overline{K_U}^\omega \neq \emptyset$.

Let $\bar{x} \in \bigcap_{U \in \Lambda} \overline{K_U}^\omega$. Suppose $u \in K$ is arbitrary and let $U \in \Lambda$ contain u. Since K_U is bounded and $\bar{x} \in \overline{K_U}^\omega$, there exists a sequence $\{x_n\} \subset K_U$ which converges to $\bar{x}$ weakly. By Lemma 2.1 and (4) we have $\langle u - x_n, f(u) \rangle \geq 0$ for all n. The function $\langle u - x, f(u) \rangle$ is weakly continuous in x. Letting n go to infinity, we have

$$\langle u - \bar{x}, f(u) \rangle \geq 0 \qquad \text{for all } u \in K.$$

Hence by Lemma 2.1 again we have

$$\langle u - \bar{x}, f(\bar{x}) \rangle \geq 0 \qquad \text{for all } u \in K.$$

Next, we give some necessary and sufficient conditions for the existence of solutions to the variational inequality problem for unbounded sets.

Theorem 2.2. *Let K be a closed convex subset of the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then the following statements are equivalent:*

(i) *There exists $\bar{x} \in K$ such that*

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0 \qquad \text{for all } x \in K. \tag{5}$$

(ii) *There exist $u \in K$ and constant $r > \|u\|$ such that $\langle x - u, f(x) \rangle \geq 0$ for all $x \in K$ with $\|x\| = r$.*

(iii) *There exists $r > 0$ such that the set $\{x \in K \mid \|x\| \leq r\}$ is nonempty and such that for each $x \in K$ with $\|x\| = r$ there exists $u \in K$ with $\|u\| < r$ and $\langle x - u, f(x) \rangle \geq 0$.*

(iv) *There exists a closed, solid, convex set E in H such that $\emptyset \neq K \cap E$ is bounded, and for each $x \in K \cap \partial(E)$ there exists $u \in K \cap \text{int}(E)$ such that $\langle x - u, f(x) \rangle \geq 0$.*

(v) *There exists a nonempty closed, bounded, convex subset B of K with $\text{int}_K(B) \neq \emptyset$ that satisfies the following condition: for each $x \in \partial_K(B)$ there exists $u \in \text{int}_K(B)$ such that $\langle x - u, f(x) \rangle \geq 0$.*

Proof. (i) implies (ii). Let $\bar{x}$ be a solution of (5). Then by choosing $r > 0$ such that $\|\bar{x}\| < r$ and letting $u = \bar{x}$, (ii) follows from the pseudo-monotonicity of the mapping f.

(ii) implies (iii). This is obvious.

(iii) implies (iv). Let $E = \{x \in H \mid \|x\| \leq r\}$. Then (iv) follows from (iii) immediately.

(iv) implies (v). Let $B = K \cap E$. Then B is a nonempty closed, bounded, and convex subset of K. First we claim that $K \cap \text{int}(E) \subset \text{int}_K(B)$ and $\partial_K(B) \subset K \cap \partial(E)$.

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Suppose $x \in K \cap \text{int}(E)$. Then there exists an open set O such that $x \in O \subset E$. Then $A = K \cap O$ is open in K and $x \in A$. Since $A \subset B$, we have $A \subset \text{int}_K(B)$. Therefore $x \in \text{int}_K(B)$. Hence $K \cap \text{int}(E) \subset \text{int}_K(B)$.

Next, suppose $x \in \partial_K(B)$. Let A be any neighborhood of x in K. Then $A \cap B \neq \emptyset$ and $A \cap (K \backslash B) \neq \emptyset$. Then $\emptyset \neq (A \cap K \cap E) \subset (A \cap E)$. Also

$$
\begin{aligned}
A \cap (K \backslash B) &= A \cap (K \cap (E^c \cup K^c)) \\
&= A \cap ((K \cap E^c) \cup (K \cap K^c)) \\
&= A \cap K \cap E^c.
\end{aligned}
$$

So $A \cap E^c \neq \emptyset$. If A is any neighborhood of x, then A is also a neighborhood of x in K. Thus $A \cap E \neq \emptyset$ and $A \cap E^c \neq \emptyset$. Therefore $x \in K \cap \partial(E)$. Hence $\partial_K(B) \subset K \cap \partial(E)$.

Now, let $x \in \partial_K(B)$. Since $\partial_K(B) \subset K \cap \partial(E)$, $x \in K \cap \partial(E)$. Then by (iv), there exists $u \in K \cap \text{int}(E) \subset \text{int}_K(B)$ such that $\langle x - u, f(x) \rangle \geq 0$. Hence (v) follows.

(v) implies (i). By Theorem 2.1 there exists $\bar{x} \in B$ such that

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0 \qquad \text{for all } x \in B. \tag{6}$$

For arbitrary $x \in K$, there are two possibilities.

(a) $\bar{x} \in \text{int}_K(B)$. There exists $0 < \lambda < 1$ such that $\lambda x + (1 - \lambda)\bar{x} \in B$. Then by (6), we have $\lambda \langle x - \bar{x}, f(\bar{x}) \rangle \geq 0$. Thus $\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0$.

(b) $\bar{x} \in \partial_K(B)$. By the condition in (v), there exists $u \in \text{int}_K(B)$ such that

$$\langle \bar{x} - u, f(\bar{x}) \rangle \geq 0.$$

Therefore by (6) we have $\langle \bar{x} - u, f(\bar{x}) \rangle = 0$. Now choose $0 < \lambda < 1$ such that $\lambda x + (1 - \lambda)u \in B$. By (6) we have

$$
\begin{aligned}
0 &\leq \langle \lambda(x - u) + u - \bar{x}, f(\bar{x}) \rangle \\
&= \lambda \langle x - u, f(\bar{x}) \rangle \\
&= \lambda \langle x - \bar{x}, f(\bar{x}) \rangle.
\end{aligned}
$$

So again $\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0$. Hence $\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0$ for all $x \in K$. Therefore $\bar{x}$ is a solution to (5).

In finite-dimensional spaces, the result that (ii) or (iii) implies (i) in Theorem 2.2 without the assumption of pseudo-monotonicity of f is given in (Ref. 11, Theorem 2.3, 2.4).

3. Existence Results for Pseudo-monotone Complementarity Problems

In this section, we obtain some existence results on solutions to nonlinear complementarity problems by combining Theorem 2.2 and the following lemma.

Lemma 3.1. *Let K be a closed convex cone in the Hilbert space H, and let f be a mapping from K into H. Then $\bar{x} \in K$ satisfies*

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0 \qquad for\ all\ x \in K$$

if and only if

$$f(\bar{x}) \in K^* \quad and \quad \langle \bar{x}, f(\bar{x}) \rangle = 0.$$

Proof. This is a special case of (Ref. 1, Lemma 3.1).

By Theorem 2.2 and Lemma 3.1, we have the following necessary and sufficient conditions for the existence of a solution to a nonlinear complementarity problem.

Theorem 3.1. *Let K be a closed convex cone in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Then the following statements are equivalent:*

(i) *There exists $\bar{x} \in K$ such that $f(\bar{x}) \in K^*$ and $\langle \bar{x}, f(\bar{x}) \rangle = 0$.*

(ii) *There exist $u \in K$ and constant $r > \|u\|$ such that $\langle x - u, f(x) \rangle \geq 0$ for all $x \in K$ with $\|x\| = r$.*

(iii) *There exists $r > 0$ such that for each $x \in K$ with $\|x\| = r$ there exists $u \in K$ with $\|u\| < r$ and $\langle x - u, f(x) \rangle \geq 0$.*

(iv) *There exists a closed, solid, convex set E in H such that the set $K \cap E$ is nonempty and bounded, and for each $x \in K \cap \partial(E)$ there exists $u \in K \cap int(E)$ such that $\langle x - u, f(x) \rangle \geq 0$.*

(v) *There exists a nonempty closed, bounded, and convex subset B of K with nonempty $int_K(B)$ satisfying the following condition: for each $x \in \partial_K(B)$ there exists $u \in int_K(B)$ such that $\langle x - u, f(x) \rangle \geq 0$.*

Before we give some other important sufficient conditions for the existence of a solution to NCP implied by Theorem 3.1, we recall some definitions.

Let K be a subset of a real Hilbert space H. Let f be a mapping from K into H. Then f is said to be *coercive* if

$$\frac{\langle x, f(x) \rangle}{\|x\|} \longrightarrow \infty \qquad \text{as } \|x\| \longrightarrow \infty \text{ and } x \in K,$$

and f is said to be *weakly coercive* if

$$\langle x, f(x) \rangle \longrightarrow \infty \qquad \text{as } \|x\| \longrightarrow \infty \text{ and } x \in K.$$

The mapping f is said to be *α-monotone* if there exists an increasing function $\alpha : [0, \infty) \longrightarrow [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ such that

$$\langle x - y, f(x) - f(y) \rangle \geq \|x - y\| \alpha(\|x - y\|) \qquad \text{for all } x, y \in K.$$

If $\alpha(r) = kr$ for some $k > 0$, then f is said to be *strongly monotone*.

Theorem 3.2. *Let K be a closed convex cone in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional sub-*

spaces. Then there exists $\bar{x} \in K$ such that

$$f(\bar{x}) \in K^* \quad and \quad \langle \bar{x}, f(\bar{x}) \rangle = 0$$

under each of the following conditions:

(i) $\lim_{\|x\| \to \infty, \ x \in K} \langle x, f(x) \rangle > 0,$

(ii) *f is weakly coercive,*

(iii) *f is coercive.*

Proof. (i). By the condition, there exists an $r > 0$ such that $\langle x, f(x) \rangle \geq 0$ for all $x \in K$ with $\|x\| = r$. Therefore the result follows from Theorem 3.1 (ii).

Parts (ii) and (iii) follow directly from (i).

Corollary 3.1. *Let K be a closed convex cone in the real Hilbert space H. Let f be a mapping from K into H which is continuous on finite-dimensional subspaces. Then there exists a unique solution to NCP under each of the following conditions:*

(i) *f is strictly monotone and weakly coercive,*

(ii) *f is strongly monotone,*

(iii) *f is α-monotone.*

Proof. The existence of a solution to NCP follows from Theorem 3.2 directly. The uniqueness of the solution can be proved by the same argument as that in (Ref. 11, Corollary 3.2).

We note that Corollary 3.1 (ii) extends a result of Nanda and Nanda (Ref. 12, Theorem) where f is assumed to be strongly monotone and lipschitzian.

Corollary 3.2. *Let K be a closed convex cone in the real Hilbert space H. Let f be a monotone and coercive mapping from K into H which is continuous on finite-dimensional subspaces. Then for each $u \in H$, there exists $\bar{x} \in K$ such that*

$$f(\bar{x}) - u \in K^* \quad and \quad \langle \bar{x}, f(\bar{x}) - u \rangle = 0.$$

Proof. Let $g : K \longrightarrow H$ be defined by $g(x) = f(x) - u$ for all $x \in K$. Since f is monotone and coercive, g is also monotone and coercive. Therefore by Theorem 3.2(ii), there exists $x \in K$ such that $g(\bar{x}) \in K^*$ and $\langle \bar{x}, g(\bar{x}) \rangle = 0$. Hence the result follows.

We note that the conclusion of Corollary 3.2 may not hold if f is simply assumed to be weakly coercive. For example, let $H = \mathbf{R}$, the set of real numbers, $K = \{x \in \mathbf{R} \mid x \geq 0\}$ and f be a constant mapping from K into H with $f(x) = 1$ for all $x \in K$. Then f is weakly coercive. But it is clear that for each $u > 1$, there exists no $\bar{x} \in K$ such that

$$f(\bar{x}) - u \in K^* \quad and \quad \langle \bar{x}, f(\bar{x}) - u \rangle = 0.$$

We also note that when H is finite-dimensional, the monotonicity of f in Corollary 3.2 is not needed. See, e.g., (Ref. 11, Theorem 3.1).

Lemma 3.2. *Let K be a pointed solid closed convex cone in the real Hilbert space H and let $v \in int(K^*)$. Then for any $\alpha > 0$, the set $S = \{x \in K \mid \langle x, v \rangle \leq \alpha\}$ is weakly compact.*

Proof. It is clear that the set S is weakly closed. Let $S^\circ = \{y \in H \mid \langle x, y \rangle \leq 1, \ \forall \, x \in S\}$ be the polar of S. Since K^* has nonempty interior in the norm topology, S° has nonempty interior in the norm topology. Then by (Ref. 16, Proposition 2.4, p.IV.5), S° has nonempty interior in the Mackey topology. Since for each $x \in S$ we have $-\alpha \leq \langle x, v \rangle \leq 0$, therefore S is weakly compact by (Ref. 17, Theorem 1).

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The following result is a generalization of a result of Karamardian (Ref. 10, Theorem 4.1) to infinite-dimensional spaces.

Theorem 3.3. *Let K be a pointed solid closed convex cone in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Suppose that there exists $u \in K$ such that $f(u) \in int(K^*)$. Then there exists $\bar{x} \in K$ such that $f(\bar{x}) \in K^*$ and $\langle \bar{x}, f(\bar{x}) \rangle = 0$.*

Proof. If $\langle u, f(u) \rangle = 0$, then we are done. So we may assume that $\langle u, f(u) \rangle > 0$. Let U be any finite-dimensional subspace of H with $u \in U$ and let P_U be the orthogonal projection of H onto U. Let $f_U = P_U \circ f$. Since $f(u) \in \text{int}(K^*)$, for each $x \in K \cap U$ with $x \neq 0$ we have

$$\langle x, f_U(u) \rangle = \langle x, f(u) \rangle > 0.$$

Therefore $f_U(u) \in \text{int}((K \cap U)^*)$ by (Ref. 10, Lemma 2.1 (i)). Then by (Ref. 10, Theorem 4.1), there exists $x_U \in K \cap U$ such that

$$f_U(x_U) \in (K \cap U)^* \quad \text{and} \quad \langle x_U, f_U(x_U) \rangle = 0.$$

By Lemma 3.1, it follows that

$$\langle x - x_U, f(x_U) \rangle \geq 0 \qquad \text{for all } x \in K \cap U.$$

Let $\Lambda(u)$ be the family of all finite-dimensional subspaces U of H with $u \in U$ and let $K_U = \{x_V \mid U \subset V \in \Lambda(u)\}$. Let $S = \{x \in K \mid \langle x, f(u) \rangle \leq \langle u, f(u) \rangle\}$. If $x \in K \backslash S$, then $\langle x - u, f(u) \rangle > 0$. Therefore $\langle x - u, f(x) \rangle > 0$ and consequently $K_U \subset S$ for all $U \in \Lambda(u)$. For $U \in \Lambda(u)$, let $\overline{K_U}^\omega$ be the weak closure of K_U. Then the family $\{\overline{K_U}^\omega \mid U \in \Lambda(u)\}$ has the finite intersection property. Since S is weakly compact by Lemma 3.2 and $\overline{K_U}^\omega \subset S$ for all $U \in \Lambda(u)$, it follows that $\bigcap_{U \in \Lambda(u)} \overline{K_U}^\omega \neq \emptyset$. Let $\bar{x} \in \bigcap_{U \in \Lambda(u)} \overline{K_U}^\omega$. By the same argument

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as that in the proof of Theorem 2.1, we have

$$\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0 \qquad \text{for all } x \in K.$$

Hence by Lemma 3.1, we have $f(\bar{x}) \in K^*$ and $\langle \bar{x}, f(\bar{x}) \rangle = 0$.

We note that by the same argument as that in the proof of Theorem 3.3 a similar existence result for variational inequality problems can be obtained as follows (cf. (Ref. 7, Theorem 3.4)):

Let K be a closed convex set in the real Hilbert space H. Let f be a pseudo-monotone mapping from K into H which is continuous on finite-dimensional subspaces. Suppose that there exists $u \in K$ such that $f(u) \in int(K^*)$. Then there exists a vector $\bar{x} \in K$ such that $\langle x - \bar{x}, f(\bar{x}) \rangle \geq 0$ for all $x \in K$, where K^* is defined exactly as in the case where K is a convex cone.

A result similar to Theorem 3.3 is proved in (Ref. 2, Theorem 4) with a different approach.

4. Some Applications

In this section, we shall consider some applications of existence results established in Section 3 to a class of nonlinear complementarity problems studied in (Ref. 9), and in particular, to the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions.

Let $K \subset H$ be a closed convex cone and let $L_1, L_2 : K \longrightarrow H$ be two mappings. The nonlinear complementarity problem (NCP)(T, K) is to find $x \in K$ such that $T(x) \in K^*$ and $\langle x, T(x) \rangle = 0$ where $T(x) = x - L_1(x) + L_2(x)$ for each $x \in K$. Such problems were studied in (Ref. 9) and were used as mathematical models for mechanical problems, especially for the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions. By employing Theorem 3.2(i), we have the following existence reuslt

for (NCP)(T, K):

Theorem 4.1. *Suppose the mapping T is pseudo-monotone and is continuous on finite-dimensional subspaces. If $\lim_{\|x\| \to \infty, \ x \in K} \langle x, T(x) \rangle > 0$, then there exists a solution to (NCP)(T, K).*

Remark 4.1. In Theorem 4.1, the closed convex cone K need not be necessarily pointed and the mapping T need not be the one-sided Gâteaux directional derivative of a functional defined on K (cf. (Ref. 9, Theorem 3.1)).

Remark 4.2. Problem (NCP)(T, K) will have a unique solution if both the mappings $-L_1$ and L_2 are monotone since in this case the mapping T is strongly monotone. The uniqueness of the solution of problem (NCP)(T, K) was not discussed in (Ref. 9).

Next, we consider another application to the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions. We adopt the notations used in (Ref. 9).

Let Ω be a thin elastic plate whose thickness is supposed to be constant and which rests without friction on a flat rigid support. The material is also supposed to be homogeneous and isotropic. Mathematically, Ω may be identified as a bounded open connected subset of $\mathbf{R}^2$. The plate Ω is assumed to be clamped on $\gamma_1 \subset \Omega$ and simply supported on $\gamma_2 = \gamma \backslash \gamma_1$, where γ is the boundary of Ω which is supposed to be sufficiently regular.

Suppose that a lateral variable load λL is applied to the boundary of Ω where λ is a fixed positive real number representing the magnitude of lateral loading. Consider the Sobolev space

$$H^2(\Omega) = \{u \in L^2(\Omega) \mid \partial u / \partial x_i, \partial^2 u / \partial x_i \partial x_j \in L^2(\Omega), \ \forall \ i, j = 1, 2\}$$

equipped with the norm $\| \cdot \|_{H^2(\Omega)}$, and let E be the closed subspace of $H^2(\Omega)$ defined by

$$E = \{z \in H^2(\Omega) \mid z_{|_\gamma} = 0, \text{ and } \partial z/\partial n_{|_{\gamma_1}} = 0, \text{a.e.}\},$$

where n denotes the normal to γ exterior to Ω and $\partial(\cdot)/\partial n$ denotes the normal exterior derivative. We may equip E with the inner product defined by a continuous bilinear form on $E \times E$ such that the associated norm is equivalent to the initial norm $\| \cdot \|_{H^2(\Omega)}$ (See Ref. 9). The post-critical equilibrium state of the plate subjected to unilateral conditions is governed by the following variational inequality: Find $x \in K$ such that

$$\langle z - w, w - \lambda L(w) + C(w) \rangle \geq 0 \qquad \text{for all } z \in K, \tag{7}$$

where $K = \{z \in E \mid z \geq 0 \text{ a.e. on } \Omega\}$ represents the admissible vertical displacements of the plate; L is a self-adjoint linear compact operator defined by the nature of the load applied on the boundary of Ω; and C is a bounded nonlinear continuous operator connected with the expansive properties of the plate. In practice, the mapping C is also assumed to be a Gâteaux derivative of a nonlinear functional Φ such that $\Phi(0) = 0$ (See, e.g., (Ref. 18)). In this case, $w = 0$ is a trivial solution to problem (7). The existence of a nonzero solution depends on the forces of magnitude λ. See (Ref. 9) for a detailed discussion on this issue. For more detailed discussion on the problem (7), we refer readers to (Refs. 9, 18) and the references therein.

Since K is a closed convex cone, by Lemma 3.1, problem (7) is equivalent to problem (NCP)(T, K) where $T(x) = x - \lambda L(x) + C(x)$ for all $x \in K$. Therefore, by Theorem 4.1, we have the following existence result for problem (7), and hence the existence of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions:

Theorem 4.2. *Suppose that the mapping T is pseudo-monotone and also suppose that $\lim_{\|x\| \to \infty, \, x \in K} \langle x, T(x) \rangle > 0$. Then there exists a solution to problem (7).*

Finally, by Remark 4.2, we have the following result concerning with the uniqueness of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions:

Theorem 4.3. *Suppose that both the mappings $-L$ and C are monotone. Then there exists a unique solution to problem (7).*

Remark 4.3. Other results in Section 3 can also be employed to obtain existence results for problem $(NCP)(T, K)$ and problem (7).

Acknowledgement

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REPORT DOCUMENTATION PAGE

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1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	July 1990	Technical Report

4. TITLE AND SUBTITLE	5. FUNDING NUMBERS
Pseudo-monotone Complementarity Problems in Hilbert Space	N00014-89-J-1659 DE-FG03-87ER25028

6. AUTHOR(S)

Richard W. Cottle and Jen-Chih Yao

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
Department of Operations Research - SOL Stanford University Stanford, CA 94305-4022	1111MA

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSORING/MONITORING AGENCY REPORT NUMBER
Office of Naval Research - Department of the Navy 800 N. Quincy Street Arlington, VA 22217 Office of Energy Research U.S. Department of Energy Washington, DC 20585	SOL 90-10

12a. DISTRIBUTION/AVAILABILITY STATEMENT	12b. DISTRIBUTION CODE
UNLIMITED	UL

13. ABSTRACT (Maximum 200 words)

In this paper, some existence results for a nonliner complementarity problem involving a pseudo-monotone mapping over an arbitrary closed convex cone in a real Hilbert space are established. In particular, some known existence results for a nonlinear complementarity problem in a finite-dimensional Hilbert speace are generalized to an infinite-dimensional real Hilbert space. Applications to a class of nonlinear complementarity problems and the study of the post-critical equilibrium state of a thin elastic plate subjected to unilateral conditions are given.

14. SUBJECT TERMS	15. NUMBER OF PAGES
Nonlinear complementarity problems; variational inequality problems; pseudo-monotone mappings; monotone mappings; weakly coercive mappings.	17 pp.
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED			SAR