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OPTIMAL CONTROL AND MODEL REDUCTION USING A FINITE-INTERVAL H_∞ CRITERION

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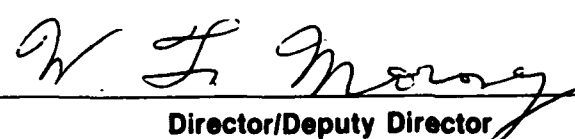
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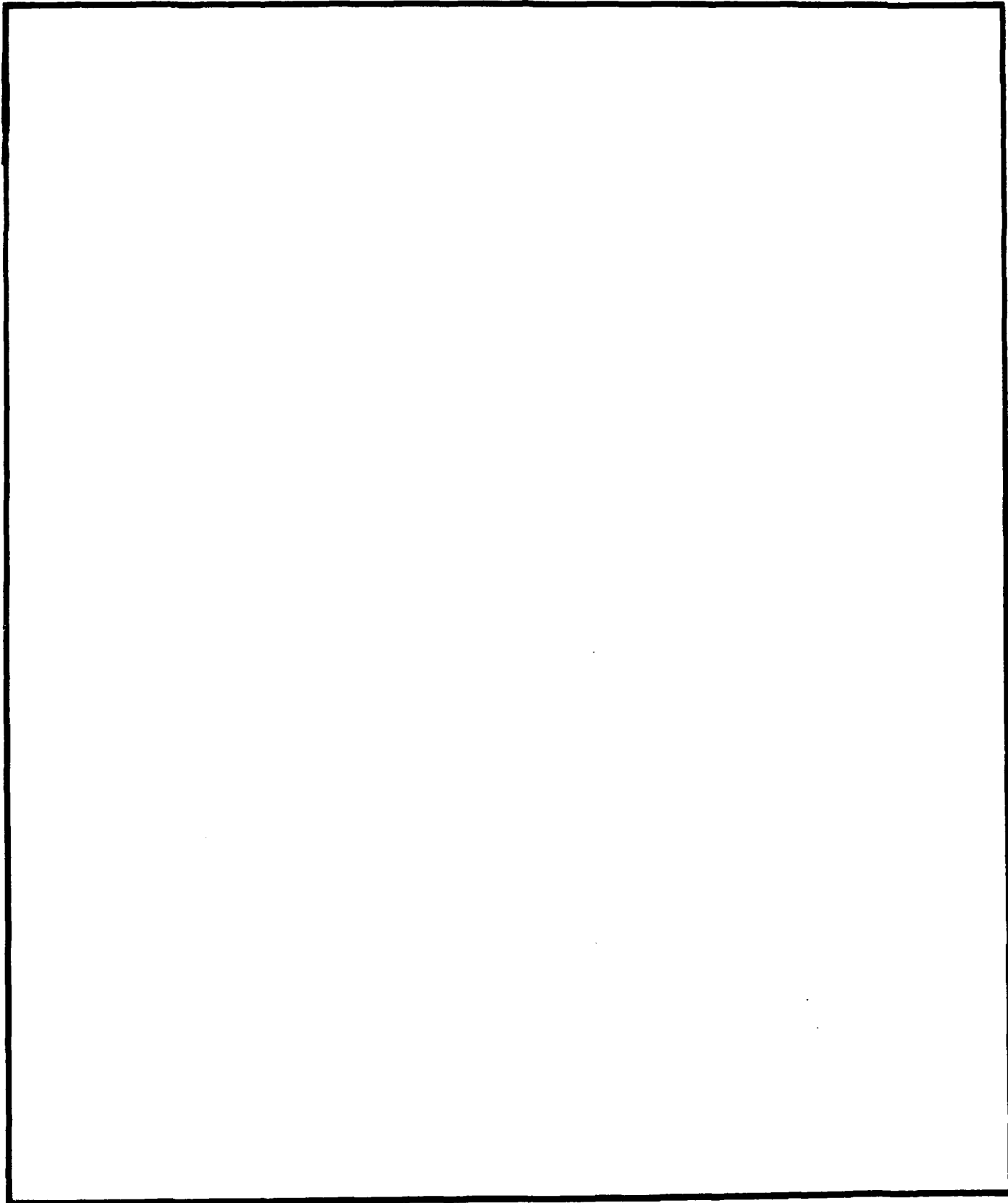
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I. Introduction

In Chapter 2 a state space formulation of the H_∞ optimal control problem is given. Assuming a finite interval of control, the problem of synthesizing a finite-interval H_∞ controller is converted into an optimization problem in which a parameter occurring in a boundary value problem needs to be maximized. An optimality condition for the maximization of this parameter is given. The proposed method makes use of the observer-based parametrization of all stabilizing controllers. An example is worked out.

An important problem in flight control and flying qualities is the approximation of a complex high order system by a low order model. In Chapter 3, for a given reduced order model, we define the correlation measure between the plant and the model outputs to be the minimum of the ratio of weighted signal energy to weighted error energy. We give a criterion for the evaluation of the correlation measure in terms of minimization of a parameter occurring in a two-point boundary value problem. Once the correlation measure for a given reduced order model can be evaluated, a nonlinear programming algorithm can be used to select a model which maximizes the correlation between the plant and model outputs. The correlation index used can be regarded as an extension of the H_∞ performance criterion to the finite-interval time-varying case. However, the usual H_∞ problem seeks an optimal controller, whereas our problem is to select the reduced order model matrices which give the best correlation index. We also give an expression for the variation of the correlation owing to parameter variations and pose a robust model reduction problem. The utilization of the theory is demonstrated by means of some examples. In particular, a problem which involves the reduction of an unstable aircraft model with structural modes is worked out. The computations for Chapter 3 were performed by Marc Steinberg, an engineer with the Flight Control group.

II. Synthesis of Finite-Interval H_∞ Controllers by State Space Methods

1. INTRODUCTION

The H_∞ optimal control theory has been pioneered by Zames [1] and important contributions have been made by Francis and Doyle [2,3]. Recent work [4] indicates that the theory has important applications in the design of flight control systems.

In this chapter a variant of the H_∞ problem is considered in terms of state space formulation. Optimization routines are needed for the synthesis of the final controller. The formulation is based on considering optimal control problems with finite terminal time in which the cost is a quotient of two definite integrals.

Other authors have considered the H_∞ problem from different points of view. In [5] a parametrization of all stabilizing controllers that achieve a specified H_∞ norm bound is given in a specialized case. The computation of the controller involves the solution of two Riccati equations. This result has been extended to the general case in [6]. In [7] the H_∞ problem is solved by introducing a generalized algebraic operation called conjugation. The approach again yields two Riccati equations whose solution leads to the synthesis of a controller. In [8] a certain LQG problem with a side constraint on the H_∞ -norm of the closed loop transfer function is solved. In this approach it is necessary to solve three coupled Riccati equations. In special cases these three equations can be reduced to two Riccati equations.

Our approach results in a two-point boundary value problem. The approach has the advantage of being applicable to time-varying systems with observer-based controllers and dynamic controllers. Ref. 9 contains one such application in which the objective is to maximize the disturbance rejection capacity of a time-varying linear system. Also, given a controller it is important to know the performance measure of the controller. For the

general time-varying system with a given controller, the parameter λ of Section 3 gives a measure of the performance of the controller.

Our time-domain approach has several advantages even in the case of time-invariant systems. First of all, it provides an alternate new approach to the computation of finite-interval H_∞ controllers. The H_∞ algorithms usually cannot handle time domain specifications. In our optimization algorithm it is possible to include time domain constraints. Also time domain approach is convenient for handling parameter uncertainties.

2. STATE SPACE FORMULATION OF THE H_∞ PROBLEM

The standard H_∞ problem can be stated with reference to Fig. 1 (p. 18). In Fig. 1 w, u, z , and y denote the exogenous input (command signals, disturbances, sensor noises etc.), the control input, the output to be controlled, and the measured output, respectively. The plant $G(s)$ and the controller $K(s)$ are assumed to be real-rational and proper. Partition G as

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \quad (1)$$

The equations corresponding to Fig. 1 are

$$z = G_{11}w + G_{12}u, \quad y = G_{21}w + G_{22}u, \quad u = Ky. \quad (2)$$

The standard H_∞ problem is to find a real-rational proper K which minimizes the H_∞ norm of the transfer matrix from w to z under the constraint that K stabilize G .

In terms of state space equations $G(s)$ is written as

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u \\ z &= C_1x + D_{11}w + D_{12}u \\ y &= C_2x + D_{21}w + D_{22}u. \end{aligned} \quad (3)$$

Doyle [10] showed that every stabilization procedure can be realized as an observer-based controller by adding stable dynamics to the plant. The realization of the observer-based controller is shown in Fig. 2 (p. 18) where the stable dynamics added is represented by $Q(s)$, with $Q(s)$ proper and $I - D_{22}Q(\infty)$ invertible. In Fig. 2, F and H are chosen such that $A + B_2F$ and $A + HC_2$ are stable. Assume that $Q(s)$ is described by the minimal representation

$$\dot{q} = \tilde{A}q + \tilde{B}\tilde{y}, \quad u_2 = \tilde{C}q + \tilde{D}\tilde{y}. \quad (4)$$

Following the notation of [11], define the following quantities.

$$\begin{aligned}
 \beta_1 &= -H - (B_2 + HD_{22})(I - \tilde{D}D_{22})^{-1}\tilde{D} \\
 \beta_2 &= \tilde{B} + \tilde{B}D_{22}(I - \tilde{D}D_{22})^{-1}\tilde{D} \\
 \gamma_1 &= F + (I - \tilde{D}D_{22})^{-1}\tilde{D}(C_2 + D_{22}F) \\
 \gamma_2 &= -(I - \tilde{D}D_{22})^{-1}\tilde{C} \\
 \alpha_{11} &= A + HC_2 + (B_2 + HD_{22})\gamma_1 \\
 &= A + B_2F - \beta_1(C_2 + D_{22}F) \\
 \alpha_{12} &= (B_2 + HD_{22})\gamma_2 \\
 \alpha_{21} &= -\beta_2(C_2 + D_{22}F) \\
 \alpha_{22} &= \tilde{A} - \tilde{B}D_{22}\gamma_2 \\
 \kappa &= -(I - \tilde{D}D_{22})^{-1}\tilde{D}.
 \end{aligned} \tag{5}$$

Then the closed loop system is given by

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{x}} \\ \dot{\hat{q}} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \\ q \end{pmatrix} + \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{pmatrix} w, \tag{6}$$

$$y = (I - D_{22}\kappa)^{-1}[C_2x + D_{22}\gamma_1\hat{x} + D_{22}\gamma_2q + D_{21}w], \tag{7}$$

$$z = C_1x + D_{11}w + D_{12}(\gamma_1\hat{x} + \gamma_2q + \kappa y), \tag{8}$$

where

$$\begin{aligned}
 A_{11} &= A + B_2 \kappa (I - D_{22} \kappa)^{-1} C_2 \\
 A_{12} &= B_2 \gamma_1 + B_2 \kappa (I - D_{22} \kappa)^{-1} D_{22} \gamma_1 \\
 A_{13} &= B_2 \gamma_2 + B_2 \kappa (I - D_{22} \kappa)^{-1} D_{22} \gamma_2 \\
 A_{21} &= \beta_1 (I - D_{22} \kappa)^{-1} C_2 \\
 A_{22} &= \alpha_{11} + \beta_1 (I - D_{22} \kappa)^{-1} D_{22} \gamma_1 \\
 A_{23} &= \alpha_{12} + \beta_1 (I - D_{22} \kappa)^{-1} D_{22} \gamma_2 \\
 A_{31} &= \beta_2 (I - D_{22} \kappa)^{-1} C_2 \\
 A_{32} &= \alpha_{21} + \beta_2 (I - D_{22} \kappa)^{-1} D_{22} \gamma_1 \\
 A_{33} &= \alpha_{22} + \beta_2 (I - D_{22} \kappa)^{-1} D_{22} \gamma_2 \\
 B_1 &= B_1 - B_2 \tilde{D} (I - D_{22} \kappa)^{-1} D_{21} \\
 B_2 &= -(H + B_2 \tilde{D}) D_{21} \\
 B_3 &= \tilde{B} D_{21}.
 \end{aligned} \tag{9}$$

Consider equations (4)-(9). Now the H_∞ control problem is to find among all sets of matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ which give a stable transfer matrix from $\tilde{y}$ to u_2 (see Fig. 2) one for which the H_∞ -norm of the transfer matrix from w to z is minimized.

The above problem is equivalent to the following problem. Suppose $\tilde{A}$ is selected to be a stable matrix. For fixed $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, let

$$\lambda = \inf_w \frac{\int_0^\infty w^*(t) w(t) dt}{\int_0^\infty z^*(t) z(t) dt}, \tag{10}$$

where the superscript $*$ denotes matrix or vector transpose. Now find the values of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ which make λ a maximum. The initial conditions for the variables x , $\hat{x}$, and q are of course zero.

It is clear that the H_∞ -norm of the transfer function from w to z is $1/\sqrt{\lambda}$ and the objective is to minimize the H_∞ -norm by choosing a controller.

The input $w(t)$ considered in the above problem is an element of $L_2(0, \infty)$. However, in many physical systems, the control interval is finite. For example, in the case of an advanced fighter, most maneuvers are accomplished in the course of a few seconds. Thus, in the next section we consider an approximate H_∞ problem in the sense that the control interval will be finite. If the integration limit T in, say equation (13), approaches infinity, then $\sqrt{\lambda}$ is the inverse of the H_∞ -norm of the transfer matrix from w to z . For lack of a better term, we call this a finite-interval H_∞ problem. On the other hand, the problem will be more general in the sense that time-varying linear systems and a broader class of performance indices will be considered in Section 3.

To motivate the problem considered in the next section, let $\mathbf{x} = (x^*, \hat{x}^*, q^*)^*$. Equations (6)-(8) are written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w}, \quad \mathbf{x}(0) = 0, \quad \mathbf{w} = w, \quad (11)$$

$$\mathbf{z} = \mathbf{z} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{w}, \quad (12)$$

where the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ depend on $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$. Let the control interval be $[0, T]$. For fixed $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ with $\tilde{A}$ being stable, let

$$\lambda = \inf_{\mathbf{w}} \frac{\int_0^T \mathbf{w}^*(t) \mathbf{w}(t) dt}{\int_0^T \mathbf{z}^*(t) \mathbf{z}(t) dt}. \quad (13)$$

Using an optimization routine, find the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ for which λ is maximized.

3. OPTIMALITY CONDITIONS

In this section we develop conditions for determining λ in a general case which subsumes the problem considered at the end of Section 2. These conditions will be developed for time-varying systems. The system equations are given by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{w}, \quad \mathbf{x}(t_0) = 0. \quad (14)$$

The problem on hand is to select $\mathbf{w}$ which minimizes the performance index given by

$$J(\mathbf{w}) = \frac{\int_{t_0}^T \left\{ \frac{1}{2} \mathbf{x}^* \mathbf{R}_1 \mathbf{x} + \mathbf{x}^* \mathbf{R}_2 \mathbf{w} + \frac{1}{2} \mathbf{w}^* \mathbf{R}_3 \mathbf{w} \right\} dt}{\int_{t_0}^T \left\{ \frac{1}{2} \mathbf{x}^* \mathbf{W}_1 \mathbf{x} + \mathbf{x}^* \mathbf{W}_2 \mathbf{w} + \frac{1}{2} \mathbf{w}^* \mathbf{W}_3 \mathbf{w} \right\} dt}. \quad (15)$$

Note that the performance index given by (13) can be regarded as a special case of (15) since $\mathbf{z} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{w}$. To get the performance index of (13), set $\mathbf{R}_1 = \mathbf{R}_2 = 0$, $\mathbf{W}_1 = \mathbf{C}^* \mathbf{C}$, $\mathbf{W}_2 = \mathbf{C}^* \mathbf{D}$, and $\mathbf{W}_3 = \mathbf{D}^* \mathbf{D}$ in equation (15). In (15) we assume that the weighting matrices $\mathbf{R}_1, \mathbf{R}_3, \mathbf{W}_1$, and $\mathbf{W}_3$ are symmetric and the integrands of both the numerator and the denominator are nonnegative for each $\mathbf{w}(t)$. Further, we assume that there is some $\mathbf{w}(t)$ for which the denominator is positive. Let $\lambda = \inf_{\mathbf{w}} J(\mathbf{w})$. We also assume that $\mathbf{R}_3 - \lambda \mathbf{W}_3$ is nonsingular.

Cost functionals of the form of (15) are the subject matter of this report. For the sake of completeness, we derive the necessary conditions satisfied by an optimal $\mathbf{w}(t)$.

Since the infimum of (15) is λ , we have

$$\begin{aligned} & \int_{t_0}^T \left\{ \frac{1}{2} \mathbf{x}^* \mathbf{R}_1 \mathbf{x} + \mathbf{x}^* \mathbf{R}_2 \mathbf{w} + \frac{1}{2} \mathbf{w}^* \mathbf{R}_3 \mathbf{w} \right\} dt \\ & - \lambda \int_{t_0}^T \left\{ \frac{1}{2} \mathbf{x}^* \mathbf{W}_1 \mathbf{x} + \mathbf{x}^* \mathbf{W}_2 \mathbf{w} + \frac{1}{2} \mathbf{w}^* \mathbf{W}_3 \mathbf{w} \right\} dt \geq 0 \end{aligned} \quad (16)$$

for all $(\mathbf{w}, \mathbf{x})$ which satisfy (14). Thus, if $\mathbf{w}$ minimizes the cost functional in (15), it also minimizes the alternate cost functional

$$J_1(\mathbf{w}) = \int_{t_0}^T \left\{ \frac{1}{2} \mathbf{x}^* (\mathbf{R}_1 - \lambda \mathbf{W}_1) \mathbf{x} + \mathbf{x}^* (\mathbf{R}_2 - \lambda \mathbf{W}_2) \mathbf{w} + \frac{1}{2} \mathbf{w}^* (\mathbf{R}_3 - \lambda \mathbf{W}_3) \mathbf{w} \right\} dt. \quad (17)$$

The necessary conditions for optimal $\mathbf{w}(t)$ can be stated as follows.

THEOREM 3.1. Consider the system given by (14) with the performance index given by (15). If $\mathbf{w}(t)$ minimizes (15), then there exists an adjoint vector $\psi(t)$ such that

$$\frac{d\psi}{dt} = -\mathbf{A}^* \psi + (\mathbf{R}_1 - \lambda \mathbf{W}_1) \mathbf{x} + (\mathbf{R}_2 - \lambda \mathbf{W}_2) \mathbf{w}, \quad \psi(T) = 0, \quad (18)$$

and

$$\mathbf{w}(t) = (\mathbf{R}_3 - \lambda \mathbf{W}_3)^{-1} \{ \mathbf{B}^* \psi - (\mathbf{R}_2 - \lambda \mathbf{W}_2)^* \mathbf{x} \}. \quad (19)$$

Proof. To give a short proof, consider the alternate cost functional given by (17). By the maximal principle [12], the Hamiltonian is given by

$$\begin{aligned} H(\psi, \mathbf{x}, \mathbf{w}) = & \psi^* (\mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{w}) - \left\{ \frac{1}{2} \mathbf{x}^* (\mathbf{R}_1 - \lambda \mathbf{W}_1) \mathbf{x} \right. \\ & \left. + \mathbf{x}^* (\mathbf{R}_2 - \lambda \mathbf{W}_2) \mathbf{w} + \frac{1}{2} \mathbf{w}^* (\mathbf{R}_3 - \lambda \mathbf{W}_3) \mathbf{w} \right\}. \end{aligned} \quad (20)$$

The adjoint vector $\psi(t)$ satisfies

$$\frac{d\psi}{dt} = -\frac{\partial H}{\partial \mathbf{x}} \quad (21)$$

with the transversality condition $\psi(T) = 0$. Equation (18) is obtained from (21). Optimal $\mathbf{w}(t)$ is obtained by setting $\partial H / \partial \mathbf{w} = 0$ and is given by (19). $\square$

Let $\mathbf{V}_i = \mathbf{R}_i - \lambda \mathbf{W}_i$ for $i = 1, 2, 3$. We have a two-point boundary value problem given by

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \mathbf{A} - \mathbf{B} \mathbf{V}_3^{-1} \mathbf{V}_2^* & \mathbf{B} \mathbf{V}_3^{-1} \mathbf{B}^* \\ \mathbf{V}_1 - \mathbf{V}_2 \mathbf{V}_3^{-1} \mathbf{V}_2^* & -\mathbf{A}^* - \mathbf{V}_2 \mathbf{V}_3^{-1} \mathbf{B}^* \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \psi \end{pmatrix}, \quad (22)$$

with

$$\mathbf{x}(t_0) = 0, \quad \psi(T) = 0. \quad (23)$$

We now show that the minimum value of (15) is the least positive λ for which (22)-(23) has a solution with $\int_{t_0}^T \{ \frac{1}{2} \mathbf{x}^* \mathbf{W}_1 \mathbf{x} + \mathbf{x}^* \mathbf{W}_2 \mathbf{w} + \frac{1}{2} \mathbf{w}^* \mathbf{W}_3 \mathbf{w} \} dt > 0$.

THEOREM 3.2. Consider the boundary value problem given by (22) and (23). Let λ be the least positive value for which the boundary value problem has a solution with $\int_{t_0}^T \{ \frac{1}{2} \mathbf{x}^* \mathbf{W}_1 \mathbf{x} + \mathbf{x}^* \mathbf{W}_2 \mathbf{w} + \frac{1}{2} \mathbf{w}^* \mathbf{W}_3 \mathbf{w} \} dt > 0$, where $\mathbf{w}(t) = \mathbf{V}_3^{-1} \{ \mathbf{B}^* \psi - \mathbf{V}_2^* \mathbf{x} \}$. Then λ is the minimum value of (15) and $\mathbf{w}$ is an optimal input.

Proof. From Theorem 3.1 it follows that if $\mathbf{w}(t)$ is optimal, then the boundary value problem (22)-(23) is satisfied for the optimal value of λ . Now suppose the boundary value

problem is satisfied for some λ such that the corresponding solution $(\mathbf{x}, \psi)$ gives the denominator of (15) a positive value (with $\mathbf{w}(t) \triangleq \mathbf{V}_3^{-1}\{\mathbf{B}^*\psi - \mathbf{V}_2^*\mathbf{x}\}$). We show that the performance index corresponding to $(\mathbf{x}, \psi)$ is λ .

Let $(\cdot, \cdot)$ denote the standard inner product in a real Euclidean space. We have

$$(\mathbf{R}_3 - \lambda \mathbf{W}_3)\mathbf{w} = \mathbf{B}^*\psi - (\mathbf{R}_2 - \lambda \mathbf{W}_2)^*\mathbf{x}.$$

Thus

$$\int_{t_0}^T \{(\mathbf{w}, \mathbf{R}_3\mathbf{w}) - \lambda(\mathbf{w}, \mathbf{W}_3\mathbf{w})\} dt = \int_{t_0}^T \{(\mathbf{w}, \mathbf{B}^*\psi) - (\mathbf{w}, \mathbf{R}_2^*\mathbf{x}) + \lambda(\mathbf{w}, \mathbf{W}_2^*\mathbf{x})\} dt. \quad (25)$$

Since $\mathbf{B}\mathbf{w} = \dot{\mathbf{x}} - \mathbf{A}\mathbf{x}$, the first integral on the right side of (25) is

$$\int_{t_0}^T (\mathbf{w}, \mathbf{B}^*\psi) dt = \int_{t_0}^T \{(\dot{\mathbf{x}}, \psi) - (\mathbf{A}\mathbf{x}, \psi)\} dt. \quad (26)$$

After integrating the right side of (26) by parts and utilizing $\mathbf{x}(t_0) = \psi(T) = 0$,

$$\int_{t_0}^T (\mathbf{w}, \mathbf{B}^*\psi) dt = \int_{t_0}^T \{(\mathbf{x}, \mathbf{R}_1\mathbf{x}) - (\mathbf{x}, \mathbf{R}_2\mathbf{w}) + \lambda(\mathbf{x}, \mathbf{W}_1\mathbf{x}) + \lambda(\mathbf{x}, \mathbf{W}_2\mathbf{w})\} dt. \quad (27)$$

Combining equations (25) and (27), we get

$$\begin{aligned} \int_{t_0}^T \{(\mathbf{x}, \mathbf{R}_1\mathbf{x}) + 2(\mathbf{x}, \mathbf{R}_2\mathbf{w}) + (\mathbf{w}, \mathbf{R}_3\mathbf{w})\} dt &= \lambda \int_{t_0}^T \{(\mathbf{x}, \mathbf{W}_1\mathbf{x}) \\ &\quad + 2(\mathbf{x}, \mathbf{W}_2\mathbf{w}) + (\mathbf{w}, \mathbf{W}_3\mathbf{w})\} dt. \end{aligned} \quad (28)$$

Thus λ is the cost associated with $(\mathbf{x}, \psi)$. Thus, if λ is the least positive value for which the boundary value problem (22)-(23) has a solution $(\mathbf{x}, \psi)$ with the corresponding denominator of (15) being positive, then $\mathbf{x}$ must be an optimal trajectory. $\square$

If the system and weighting matrices are functions of a finite number of parameters, these parameters can be varied to maximize λ . In Section 2, since the system matrices and the weighting matrices depend on $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$, an optimization routine needs to be employed with respect to these quantities to maximize λ .

4. OPTIMALITY CONDITIONS FOR THE MAXIMIZATION OF λ

We consider again the time-invariant H_∞ problem. In this section, we derive a condition that needs to be satisfied when λ is maximized. For this, consider equations (14) and (15). Note that for the standard H_∞ problem of Section 2, the system and weighting matrices depend on $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$. These constitute the set of independent variables. The variations in the system and weighting matrices can be explicitly expressed in terms of variations in $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$. However, the optimality conditions are extremely complicated to derive in such a case. The derivation can be simplified a little by assuming that $D_{22} = 0$ (see (3)). However, we only attempt to derive the basic optimality conditions here.

Consider equations (22) and (23). Let $\hat{A} = A - BV_3^{-1}V_2^*$, $\hat{B} = BV_3^{-1}B^*$, and $\hat{C} = V_1 - V_2V_3^{-1}V_2^*$. Suppose $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ maximize λ . Let $\delta\tilde{A}, \delta\tilde{B}, \delta\tilde{C}$, and $\delta\tilde{D}$ denote elemental perturbations in $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ respectively. Also, denote the corresponding perturbations in $\hat{A}, \hat{B}, \hat{C}, x, \psi$, and λ by $\delta\hat{A}, \delta\hat{B}, \delta\hat{C}, x_1, \psi_1$, and μ respectively. Note that if λ is a maximum, $\mu = 0$. Thus, we have the following set of equations.

$$\dot{x} = \hat{A}x + \hat{B}\psi, \quad (29)$$

$$\dot{\psi} = \hat{C}x - \hat{A}^*\psi, \quad (30)$$

$$x(t_0) = \psi(T) = 0, \quad (31)$$

$$\dot{x}_1 = \hat{A}x_1 + \hat{B}\psi_1 + \delta\hat{A}x + \delta\hat{B}\psi, \quad (32)$$

$$\dot{\psi}_1 = \hat{C}x_1 - \hat{A}^*\psi_1 + \delta\hat{C}x - \delta\hat{A}^*\psi, \quad (33)$$

$$x_1(t_0) = \psi_1(T) = 0. \quad (34)$$

From (33), we have

$$\int_{t_0}^T x^* \dot{\psi}_1 dt = \int_{t_0}^T \{x^* \hat{C}x_1 - x^* \hat{A}^*\psi_1 + x^* \delta\hat{C}x - x^* \delta\hat{A}^*\psi\} dt. \quad (35)$$

Also, by an integration by parts

$$\int_{t_0}^T \mathbf{x}^* \dot{\psi}_1 dt = - \int_{t_0}^T \{ \mathbf{x}^* \hat{A}^* \psi_1 + \psi^* \hat{B} \psi_1 \} dt. \quad (36)$$

From (35) and (36),

$$- \int_{t_0}^T \psi^* \hat{B} \psi_1 dt = \int_{t_0}^T \{ \mathbf{x}^* \hat{C} \mathbf{x}_1 + \mathbf{x}^* \delta \hat{C} \mathbf{x} - \mathbf{x}^* \delta \hat{A} \psi \} dt. \quad (37)$$

From equation (30)

$$\hat{C} \mathbf{x} = \dot{\psi} + \hat{A}^* \psi. \quad (38)$$

Note that $\hat{C}^* = \hat{C}$. Substituting (38) in (37) and integrating by parts, we get

$$2 \int_{t_0}^T \mathbf{x}^* \delta \hat{A} \psi dt + \int_{t_0}^T \psi^* \delta \hat{B} \psi dt - \int_{t_0}^T \mathbf{x}^* \delta \hat{C} \mathbf{x} dt = 0. \quad (39)$$

The above equation needs to be satisfied for all elemental perturbations in $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$.

5. A NUMERICAL EXAMPLE

As an example we consider the tracking problem given in [2]. The plant is given by

$$P(s) = \frac{s-1}{s(s-2)}. \quad (40)$$

The tracking error signal is $r - v$. The weighting filter $W(s)$ in Fig. 3 (p. 18) is given by

$$W(s) = \frac{s+1}{10s+1}. \quad (41)$$

The objective in [2] was to choose $K_1(s)$ and $K_2(s)$ such that the H_∞ -norm of the transfer function from w to v is minimized. Our objective in this section is to synthesize u using the theory of this chapter such that the minimum of

$$\frac{\int_0^{10} w^2(t) dt}{\int_0^{10} \{ (r-v)^2 + u^2 \} dt} \quad (42)$$

is maximized.

Converting the plant equations to state space form, we have

$$\begin{aligned}\dot{x}_1 &= -.1x_1 + w \\ \dot{x}_2 &= u \\ \dot{x}_3 &= 2x_3 + u \\ r &= .1w + .09x_1 \\ v &= .5x_2 + .5x_3.\end{aligned}\tag{43}$$

The matrices corresponding to equation (3) are given by

$$\begin{aligned}A &= \begin{pmatrix} -.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ C_1 &= \begin{pmatrix} .09 & -.5 & -.5 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} .09 & 0 & 0 \\ 0 & .5 & .5 \end{pmatrix}, \\ D_{11} &= \begin{pmatrix} .1 \\ 0 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad D_{21} = \begin{pmatrix} .1 \\ 0 \end{pmatrix}, \quad D_{22} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\end{aligned}$$

The matrices F and H are chosen such that $A + B_2F$ and $A + HC_2$ are stable. The choice is the same as that in [2] and is given by

$$F = (0 \quad .5 \quad -4.5), \quad H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -9 \end{pmatrix}.$$

Assume that $Q(s)$ is described by the three dimensional system

$$\begin{aligned}\dot{q} &= \tilde{A}q + \tilde{B}\tilde{y}, \\ u_2 &= \tilde{C}q + \tilde{D}\tilde{y}.\end{aligned}\tag{44}$$

Let $x = (x_1 \quad x_2 \quad x_3)^*$. Then the state equations for the finite-interval H_∞ problem become

$$\dot{\hat{x}} = (A - B_2\tilde{D}C_2)x + B_2(F + \tilde{D}C_2)\hat{x} - B_2\tilde{C}q + (B_1 - B_2\tilde{D}D_{21})w,\tag{45}$$

$$\begin{aligned}\dot{\hat{x}} &= (A + HC_2 + B_2F + B_2\tilde{D}C_2)\hat{x} - (HC_2 + B_2\tilde{D}C_2)x \\ &\quad - B_2\tilde{C}q - (HD_{21} + B_2\tilde{D}D_{21})w,\end{aligned}\tag{46}$$

$$\dot{q} = \tilde{A}q + \tilde{B}C_2(x - \hat{x}) + \tilde{B}D_{21}w,\tag{47}$$

with the initial conditions being zero. The performance index is

$$\frac{\int_0^{10} w^2 dt}{\int_0^{10} \left\{ (.1w + .09x_1 - .5x_2 - .5x_3)^2 + [F\hat{x} - (\tilde{C}q + \tilde{D}C_2x - \tilde{D}C_2\hat{x} + \tilde{D}D_{21}w)]^2 \right\} dt} \quad (48)$$

Assuming values for $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, we can find λ using the theory given in Section 3. Let $\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$ be the transition matrix corresponding to (22). Satisfaction of (23) gives rise to the condition that $\det(\Phi_{22}(10)) = 0$. Thus λ is found by making use of a sign change of $\det(\Phi_{22}(10))$ over a range of values of λ . In our numerical experiments, much of the computer execution time was consumed by the calculation of λ for a given controller. Efforts are under way to make the computation of λ more efficient.

The transition matrix $\Phi(10)$ was found in this case using the following formula [13]. Let $h = 10/2^8$. Represent the system matrix in (22) by $\mathcal{M}$. Then

$$\Phi(10) = \left\{ \left[I - \frac{1}{2}h\mathcal{M} + \frac{1}{12}h^2\mathcal{M}^2 \right]^{-1} \left[I + \frac{1}{2}h\mathcal{M} + \frac{1}{12}h^2\mathcal{M}^2 \right] \right\}^{2^8} \quad (49)$$

Using the above procedure, we can iterate on $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ to maximize λ . Note that once $\Phi(h)$ is calculated, only eight repeated squarings are needed to evaluate $\{\Phi(h)\}^{2^8}$.

Initially the following values were assumed for the control matrices:

$$\tilde{A} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \tilde{C} = (1 \quad 1 \quad 1), \quad \tilde{D} = (1 \quad 1).$$

Using the Rosenbrock hill climbing algorithm [14], the elements of the matrices were varied to maximize λ . The algorithm usually leads to only local maxima. Note that $Q(s)$ is stable if and only if $\tilde{A}$ is stable. This was not introduced as a constraint in the optimization algorithm since the unconstrained run yielded a stable $\tilde{A}$. The Fortran program was run on a Zenith Z-248 personal computer in double precision using the Microsoft Optimizing Compiler Version 4.01. A local maximum of $\lambda \approx 14.8$ was obtained for the following values

of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$:

$$\begin{aligned}\tilde{A} &= \begin{pmatrix} -2.04 & .318 & .023 \\ -.026 & -1.632 & -.028 \\ -.052 & .358 & -2.054 \end{pmatrix}, & \tilde{B} &= \begin{pmatrix} .945 & 9.973 \\ -1.046 & 33.028 \\ .946 & -1.056 \end{pmatrix}, \\ \tilde{C} &= (.944 \quad 1.48 \quad 1.018), & \tilde{D} &= (.986 \quad 41.92).\end{aligned}$$

After several runs with various initial values for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, the value of $\lambda_{\max} = 14.8$ could not be bettered.

The two components of $Q(s)$ are given by

$$\begin{aligned}Q_1(s) &= \frac{.986(s + 3.26)(s + 2.03)(s + .75)}{(s + 1.68)(s^2 + 4.05s + 4.1)}, \\ Q_2(s) &= \frac{41.92(s + 2.04)(s^2 + 5.04s + 6.58)}{(s + 1.68)(s^2 + 4.05s + 4.1)}.\end{aligned}\tag{50}$$

It was reported in [2] that $Q_2(s)$ is unconstrained and may be taken as zero. To simulate this condition, we set the second columns of the optimal $\tilde{B}$ and $\tilde{D}$ equal to zero. The first positive value of λ for which $\det(\Phi_{22}(10))$ changed sign in this case was still observed to be 14.8.

A few comments on the numerical method are in order. Since the computation of λ consumes most of the execution time, further research needs to be done to find an alternate method to evaluate λ more accurately and efficiently. Also, the value of λ is evaluated in the above example by starting with an initial value and incrementing it in steps of 0.2 until a change in the sign of the determinant is observed. Thus the exact value of λ differs from the computed value by at most 0.2. This sort of inaccurate evaluation of λ may prematurely terminate the optimization routine which seeks to maximize λ .

6. CONCLUSIONS

A design methodology for the synthesis of finite-interval H_∞ controllers is presented using state-space methods. Using observer-based controller parametrization, an optimization problem is formulated. A measure of performance for a given controller is defined

in terms of the least value of a parameter occurring in a two-point boundary value problem. Optimality conditions for finding the measure of performance for a given controller are given. The optimization problem seeks to maximize the measure of performance. An example is given.

Note: This chapter is based on a paper which will appear in the AIAA Journal of Guidance, Control, and Dynamics under the same title.

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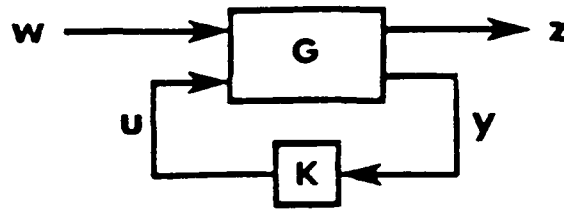


Fig. 1 The Standard Block Diagram

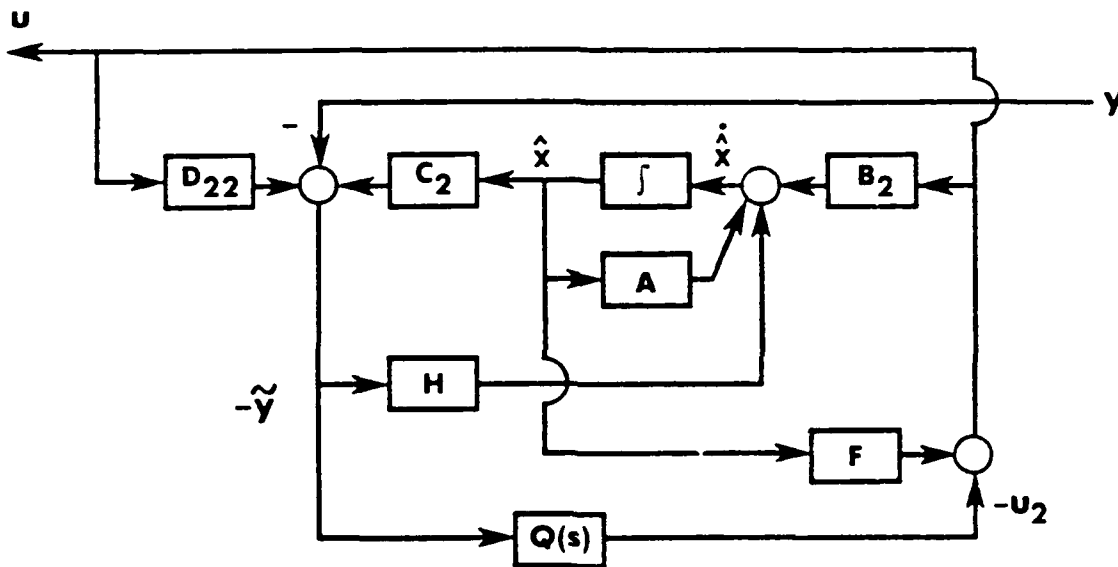


Fig. 2 The Observer-based Controller Parametrization

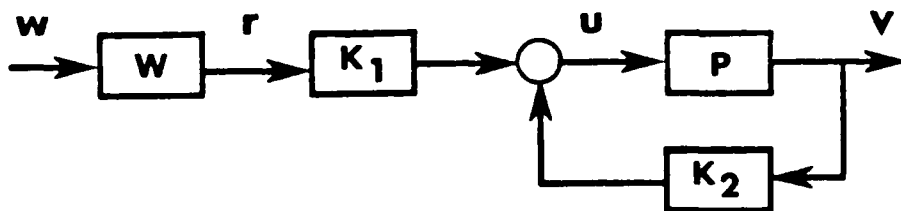


Fig. 3 Block Diagram for the Tracking Problem

III. Model Reduction with a Finite-Interval

H_∞ Criterion

1. Introduction

Model reduction is an important problem in the case of airplanes with significant aeroservoelastic dynamics. The original model in such cases is of high order and thus, the resulting controller will have a complex structure, especially if it uses full state feedback. Also for highly augmented aircraft with flight and propulsive controls, it is useful to develop low order models to analyze flying qualities.

If the aim is to design a low order controller for a high order plant, there are at least three broad approaches to achieve this. A general account of these three approaches is given in [1]. The so called direct design methods assume a stabilizing controller of fixed degree and seek to find the controller that maximizes a quadratic performance index (see [2,3]).

Another approach is to get a high order controller by some design technique, such as LQG or H_∞ , and then to approximate the high order controller by a low order one which possesses certain desirable properties. This approach is the subject matter of [1] and the pertaining literature is referenced in that paper.

The third approach is to approximate the high order plant by a low order one. Then a low order controller is designed and used to control the original plant. In this chapter we concentrate on this approach and consider the problem of approximating the original plant by a low order model in an optimal sense. This problem has been treated recently by several researchers under a variety of approximation criteria and we refer the reader to [4] for the relevant references. Although no computational results are given, [4] gives a sufficient condition which characterizes reduced order models satisfying an optimized L_2 bound as well as a prespecified H_∞ bound. The reduced order model is expressed in terms of solutions of four coupled algebraic Riccati equations.

We now state the main problem. For the sake of generality, we pose it for time-varying systems. Let the plant be described by

$$\dot{x}_p = A_p(t)x_p + B_p(t)u, \quad x_p(t_0) = 0, \quad (1)$$

$$y_p = C_p(t)x_p + D_p(t)u, \quad (2)$$

where $x_p(t)$, $u(t)$, and $y_p(t)$ denote the plant state vector, the control vector, and the plant output vector respectively. Let the reduced order model which approximates the plant be chosen to be

$$\dot{x}_m = A_m(t)x_m + B_m(t)u, \quad x_m(t_0) = 0, \quad (3)$$

$$y_m = C_m(t)x_m + D_m(t)u, \quad (4)$$

where $x_m(t)$ and $y_m(t)$ denote respectively the state vector and the output vector of the reduced order model.

For given $A_m(t)$, $B_m(t)$, $C_m(t)$, and $D_m(t)$, let u be chosen such that the correlation index given by

$$\frac{\int_{t_0}^T \frac{1}{2} u^*(t) R(t) u(t) dt}{\int_{t_0}^T \frac{1}{2} (y_p - y_m)^* Q(t) (y_p - y_m) dt} \quad (5)$$

is minimized. The superscript $*$ denotes matrix or vector transpose. Let this minimum value be denoted by λ . Thus u represents the worst input and λ gives a measure of the worst-case correlation between the plant output and the model output. The problem is to choose $A_m(t)$, $B_m(t)$, $C_m(t)$, and $D_m(t)$ such that λ is maximized.

Since (5) represents the ratio of weighted signal energy to weighted error energy, the above problem may be regarded as a modified H_∞ problem except for a few differences. We consider time-varying systems and in our case the interval of control is finite. There are extensions of the H_∞ results to the finite-interval time-varying case [4]. However, our approach is different and is based on considering the inherent two-point boundary value

problem. Also, the general aim of H_∞ problems is the design of an optimal controller, whereas in this chapter we are interested in the selection of model matrices. It is necessary in our case to use nonlinear programming algorithms in order to select the model matrices which maximize λ . In [5-7], we derived some results which aid in the selection of a controller which maximizes the worst-case performance. The results of [5] are presented in Chapter 2 and these will be utilized in Section 2 of this chapter.

In the case of time-invariant systems, a nonlinear programming algorithm can be used to find at least a local maximum of λ . For the time-varying case, the matrices $A_m(t)$, $B_m(t)$, $C_m(t)$, and $D_m(t)$ need to be expressed in terms of basis functions and a nonlinear programming algorithm needs to be used to maximize λ with respect to the coefficients of the basis functions.

We do not require the plant and the model to be open loop stable. This is significant since many of the modern aircraft have open loop unstable poles. We show in Section 4 by means of examples that the method is indeed applicable to such cases. There is yet another advantage of our method. One of the criticisms in the approach of getting a low order model from a high order plant is that the satisfactory approximation of the plant requires some knowledge in advance of the controller [1]. Since we maximize the correlation between the plant and model outputs for the worst possible input, the correlation in the case of any other controller is bound to be better. Thus, our method furnishes a satisfactory approximation without requiring an *a priori* knowledge of the controller.

We now give a summary of the results of the chapter. In Section 2, conditions that characterize the worst input are derived for a given model. A two-point boundary value problem needs to be solved for the least positive λ to obtain the worst-case correlation between the outputs of the plant and the model. A nonlinear programming algorithm can then be used to find the model matrices which maximize λ .

The stability and control derivatives of aircraft are subject to variations and it is also not possible to determine these exactly from wind tunnel data. There is already some

interest in robust model reduction techniques [8]. In Section 3 we formulate a robust model reduction problem and derive an expression for the variation of correlation between the plant and model outputs as a functional of the variations in system parameters. This value gives an idea of the robustness of the approximate model and can aid in the choice of a reduced order model with a specified level of robustness.

In Section 4 some examples are worked out and details about the computational algorithm utilized are given. Correlation between the plant and the model is shown via time and frequency response plots. In order to keep the examples as simple as possible, we do not consider the robust model reduction problem in the case of these examples.

Finally, certain conclusions are given in Section 5.

2. COMPUTATION OF λ FOR A GIVEN REDUCED ORDER MODEL

Assume that the matrices $A_m(t)$, $B_m(t)$, $C_m(t)$, and $D_m(t)$ are given. In this section, we characterize λ as the minimum positive value for which a certain two-point boundary value problem has a nontrivial solution. Also, we derive a computationally useful characterization of λ .

Letting

$$x^* = (x_p^* \quad x_m^*)^*, \quad (6)$$

$$y = y_p - y_m, \quad (7)$$

$$A(t) = \begin{pmatrix} A_p & 0 \\ 0 & A_m \end{pmatrix}, \quad (8)$$

$$B(t) = \begin{pmatrix} B_p \\ B_m \end{pmatrix}, \quad (9)$$

$$C(t) = (C_p \quad -C_m), \quad (10)$$

and

$$D(t) = D_p - D_m, \quad (11)$$

we can write (1)-(4) as

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = 0, \quad (12)$$

$$y = C(t)x + D(t)u. \quad (13)$$

The correlation index given by (5) can be put in the form

$$J(u) = \frac{\int_{t_0}^T \frac{1}{2} u^*(t) R(t) u(t) dt}{\int_{t_0}^T \left\{ \frac{1}{2} x^* W_1 x + x^* W_2 u + \frac{1}{2} u^* W_3 u \right\} dt}. \quad (14)$$

The problem is to characterize $u(t)$ that minimizes (14).

Let $\lambda = \inf_u J(u)$. We assume that for all t , $R(t) - \lambda W_3(t)$ is invertible, and

$$R(t) \geq 0, \quad (15)$$

$$\begin{pmatrix} W_1(t) & W_2(t) \\ W_2^*(t) & W_3(t) \end{pmatrix} \geq 0. \quad (16)$$

Equations (15) and (16) guarantee that the numerator and denominator of (14) are non-negative for any u . The necessary conditions that characterize the worst input can be stated as follows.

THEOREM 2.1. *Consider the system given by (12)-(14). If $u(t)$ minimizes the correlation index given by (14), then there exists an adjoint vector $\psi(t)$, not identically zero, such that*

$$\frac{d\psi}{dt} = -A^* \psi - \lambda W_1 x - \lambda W_2 u, \quad \psi(T) = 0, \quad (17)$$

with

$$u(t) = (R - \lambda W_3)^{-1} \{ B^* \psi + \lambda W_2^* x \}. \quad (18)$$

Proof. For a proof, see Theorem 3.1 of Chapter 2.

Let

$$\hat{A} = A + \lambda B(R - \lambda W_3)^{-1} W_2^*, \quad (19)$$

$$\hat{B} = B(R - \lambda W_3)^{-1} B^*, \quad (20)$$

and

$$\hat{C} = -\lambda W_1 - \lambda^2 W_2 (R - \lambda W_3)^{-1} W_2^*. \quad (21)$$

Thus, we have a two-point boundary value problem given by

$$\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A}^* \end{pmatrix} \begin{pmatrix} x \\ \psi \end{pmatrix} \quad (22)$$

with

$$x(t_0) = 0, \quad \psi(T) = 0. \quad (23)$$

The following theorem follows from Theorem 3.2 of Chapter 2.

THEOREM 2.2. *Let (x, ψ) satisfy the boundary value problem given by (22) and (23) for the least positive λ such that $\int_{t_0}^T \{ \frac{1}{2} x^* W_1 x + x^* W_2 u + \frac{1}{2} u^* W_3 u \} dt > 0$, where $u = (R - \lambda W_3)^{-1} \{ B^* \psi + \lambda W_2 x \}$. Then λ is the minimum value of the index given by (14) and u is the worst input.*

In [5] and [6], a computational technique which utilizes the transition matrix associated with (22) is given. This technique is also presented in Chapters 2 of this report. In this chapter we use an alternate technique. This technique is more stable numerically. The theory behind the technique is given below.

Let $\Phi(t, \tau)$ be the transition matrix associated with (22). Then we have

$$\begin{pmatrix} x(T) \\ \psi(T) \end{pmatrix} = \Phi(T, \frac{T+t_0}{2}) \Phi(\frac{T+t_0}{2}, t_0) \begin{pmatrix} x(t_0) \\ \psi(t_0) \end{pmatrix}. \quad (24)$$

Let

$$\Phi^{-1}(T, \frac{T+t_0}{2}) = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix} \quad (25)$$

and

$$\Phi(\frac{T+t_0}{2}, t_0) = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}. \quad (26)$$

Multiplying (24) on the left by (25), we get

$$\begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix} \begin{pmatrix} x(T) \\ \psi(T) \end{pmatrix} = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} \begin{pmatrix} x(t_0) \\ \psi(t_0) \end{pmatrix}. \quad (27)$$

Since $x(t_0) = \psi(T) = 0$,

$$\begin{aligned}\zeta_{11}x(T) &= \nu_{12}\psi(t_0), \\ \zeta_{21}x(T) &= \nu_{22}\psi(t_0).\end{aligned}\tag{28}$$

Since the equations in (28) are linearly dependent, (28) has a nontrivial solution for $\psi(t_0)$ and $x(T)$ if and only if

$$\det \begin{pmatrix} \zeta_{11} & \nu_{12} \\ \zeta_{21} & \nu_{22} \end{pmatrix} = 0.\tag{29}$$

Thus, we can characterize λ as the least positive value for which (29) holds.

We can determine λ by doing a search over a range of positive values and picking the first value at which the determinant in (29) changes sign. We give more details on this in Section 4.

3. ROBUST MODEL REDUCTION

In this section we formulate a robust model reduction problem. The aim is to choose the best reduced order model under parameter variations. We derive an expression for the variation in the correlation measure λ in terms of variations in the system matrices. For simplicity of analysis, we assume that $D_p(t)$ and $D_m(t)$ in (11) are zero, which makes $D(t) = 0$.

Consider (1)-(10). The system equations are given by

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = 0,\tag{30}$$

$$y = C(t)x.\tag{31}$$

We can write (5) as

$$\frac{\int_{t_0}^T \frac{1}{2} u^*(t) R(t) u(t) dt}{\int_{t_0}^T \frac{1}{2} x^*(t) C^*(t) Q(t) C(t) x(t) dt}.\tag{32}$$

For given $A_m(t)$, $B_m(t)$, and $C_m(t)$, let λ be the minimum of the correlation index in (32) over $u(t)$. Let the elemental variations in $A_p(t)$, $B_p(t)$, and $C_p(t)$ be denoted by

$\delta A_p(t)$, $\delta B_p(t)$, and $\delta C_p(t)$ respectively. Let $\delta A(t)$, $\delta B(t)$, and $\delta C(t)$ be the variations in the matrices $A(t)$, $B(t)$, and $C(t)$ corresponding to the elemental variations $\delta A_p(t)$, $\delta B_p(t)$, and $\delta C_p(t)$. Notice that

$$\delta A(t) = \begin{pmatrix} \delta A_p(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad (33)$$

$$\delta B(t) = \begin{pmatrix} \delta B_p(t) \\ 0 \end{pmatrix}, \quad (34)$$

and

$$\delta C(t) = (\delta C_p(t) \ 0). \quad (35)$$

Let μ denote the variation in λ caused by δA , δB , and δC . Now the robust model reduction problem can be stated as follows.

Robust model reduction problem. Find $A_m(t)$, $B_m(t)$, and $C_m(t)$ such that

$$\inf_u \frac{\int_{t_0}^T \frac{1}{2} u^* R u \, dt}{\int_{t_0}^T \frac{1}{2} x^* C^* Q C x \, dt} \quad (36)$$

is maximized with the side constraint

$$|\mu/\lambda| \leq \mu_0 \text{ for all } \|\delta A(t)\| \leq a(t), \|\delta B(t)\| \leq b(t), \text{ and } \|\delta C(t)\| \leq c(t), \quad (37)$$

where $a(t)$, $b(t)$, and $c(t)$ are suitably chosen.

We now derive an expression for μ in terms of $\delta A(t)$, $\delta B(t)$, and $\delta C(t)$. For given $A_m(t)$, $B_m(t)$, and $C_m(t)$, let u minimize the index in (32). In the following, we suppress the dependence of the matrices on t for simplicity of notation. From (19)-(23), we get the following boundary value problem which needs to be satisfied by the corresponding pair (x, ψ) .

$$\dot{x} = Ax + BR^{-1}B^*\psi, \quad (38)$$

$$\dot{\psi} = -\lambda C^*QCx - A^*\psi, \quad (39)$$

$$x(t_0) = \psi(T) = 0. \quad (40)$$

Let x_1 and ψ_1 represent the variations in x and ψ due to δA , δB , and δC . From (38)-(40), we have the following equations satisfied by x_1 and ψ_1 .

$$\dot{x}_1 = Ax_1 + \delta A x + BR^{-1}B^*\psi_1 + (BR^{-1}\delta B^* + \delta B R^{-1}B^*)\psi, \quad (41)$$

$$\dot{\psi}_1 = -\mu C^*QCx - \lambda(\delta C^*QC + C^*Q\delta C)x - \lambda C^*QCx_1 - A^*\psi_1 - \delta A^*\psi, \quad (42)$$

$$x_1(t_0) = \psi_1(T) = 0. \quad (43)$$

Theorem 3.1. Consider (38)-(43). Then the variation in λ is given by

$$\mu = \frac{-\int_{t_0}^T \psi^* \delta A x dt - \int_{t_0}^T \psi^* B^* R^{-1} \delta B \psi dt - \lambda \int_{t_0}^T x^* C^* Q \delta C x dt}{\int_{t_0}^T \frac{1}{2} x^* C^* Q C x dt}. \quad (44)$$

Proof. From (42), we get

$$\begin{aligned} \int_{t_0}^T x^* \dot{\psi}_1 dt &= -\mu \int_{t_0}^T x^* C^* Q C x dt - \lambda \int_{t_0}^T x^* (\delta C^* Q C + C^* Q \delta C) x dt \\ &\quad - \lambda \int_{t_0}^T x^* C^* Q C x_1 dt - \int_{t_0}^T x^* A^* \psi_1 dt - \int_{t_0}^T x^* \delta A^* \psi dt. \end{aligned} \quad (45)$$

Also, by an integration by parts and by (38), (40), and (43),

$$\int_{t_0}^T x^* \dot{\psi}_1 dt = - \int_{t_0}^T x^* A \psi_1 dt - \int_{t_0}^T \psi^* B R^{-1} B^* \psi_1 dt. \quad (46)$$

Since

$$\int_{t_0}^T x^* (\delta C^* Q C + C^* Q \delta C) x dt = 2 \int_{t_0}^T x^* C^* Q \delta C x dt, \quad (47)$$

from (45) and (46), we get

$$\begin{aligned} \mu \int_{t_0}^T x^* C^* Q C x dt + 2\lambda \int_{t_0}^T x^* C^* Q \delta C x dt \\ + \lambda \int_{t_0}^T x^* C^* Q C x_1 dt + \int_{t_0}^T x^* \delta A^* \psi dt = \int_{t_0}^T \psi^* B R^{-1} B^* \psi_1 dt. \end{aligned} \quad (48)$$

From (39),

$$\lambda \int_{t_0}^T x^* C^* Q C x_1 dt = - \int_{t_0}^T (\dot{\psi} + A^* \psi)^* x_1 dt. \quad (49)$$

Integrating the first term of the integrand from the right side of (49) by parts, and using (40) and (43), we get

$$\begin{aligned} \lambda \int_{t_0}^T x^* C^* Q C x_1 dt &= \int_{t_0}^T \psi^* \delta A x dt + \int_{t_0}^T \psi^* B R^{-1} B^* \psi_1 dt \\ &\quad + \int_{t_0}^T \psi^* (B R^{-1} \delta B^* + \delta B R^{-1} B^*) \psi dt. \end{aligned} \quad (50)$$

Incorporating (50) in (48) and using the fact that

$$\int_{t_0}^T \psi^* (B R^{-1} \delta B^* + \delta B R^{-1} B^*) \psi dt = 2 \int_{t_0}^T \psi^* B R^{-1} \delta B \psi dt, \quad (51)$$

we get (44). $\square$

Using (44), the variation in the correlation measure λ owing to parameter variations can be computed for any given $A_m(t)$, $B_m(t)$, and $C_m(t)$.

4. NUMERICAL EXAMPLES

In this section we consider only time-invariant examples. The systems in the examples can be put in the form

$$\dot{x} = Ax + Bu, \quad x(0) = 0, \quad x^* = (x_p^* \quad x_m^*)^*, \quad (52)$$

with the correlation index

$$J(u) = \frac{\int_0^T \frac{1}{2} u^* R u dt}{\int_0^T \frac{1}{2} x^* C^* Q C x dt}, \quad (53)$$

where A , B , and C are given by (8)-(10). For given A_m , B_m , and C_m , let $\lambda = \inf_u J(u)$.

To recap the procedure for finding λ , let

$$F = \begin{pmatrix} A & B R^{-1} B^* \\ -\lambda C^* Q C & -A^* \end{pmatrix}, \quad (54)$$

and

$$\exp\left(\frac{FT}{2}\right) = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}, \quad (55)$$

$$\exp\left(\frac{-FT}{2}\right) = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}. \quad (56)$$

From (29), λ is given by the first positive value for which

$$\det \begin{pmatrix} \zeta_{11} & \nu_{12} \\ \zeta_{21} & \nu_{22} \end{pmatrix} = 0. \quad (57)$$

Now we iterate on the matrices A_m , B_m , and C_m using a nonlinear programming algorithm to maximize λ .

There are two primary computational algorithms that are needed to use this method of model reduction. They are a nonlinear global optimization routine to maximize λ and a relatively fast routine to compute λ . Currently λ is determined by finding the smallest positive value for which (57) holds. Due to the oscillatory nature of the value of the determinant as a function of λ , for suitable weighting matrices in (53), λ was originally calculated starting with an initial value of $\lambda = 0.1$ and incrementing by 0.2 until the determinant in (57) changed sign. While this method yielded accurate results, it also used excessive amounts of computational time.

To speed this process up, two modifications were made. First, λ was incremented by large steps until the value of the determinant was less than a percentage of its initial value. At this point, small increments in λ were applied until the determinant changed sign. This technique was successful in this case because the absolute value of the determinant never increased beyond a very small fraction of its initial value after the first zero crossing. The second modification was adaptively scaling down the input weighting matrix R so that the values of λ were consistently in the range of 10 to 20. This modification gives large enough values for accuracy and small enough values to decrease computational time.

The second necessary algorithm is a nonlinear global optimization routine. We are still in the process of developing an algorithm which satisfies both speed and accuracy

requirements. For now, however, two methods were used to test our theory. For Example 1, we used a deterministic tunneling technique [9]. This technique in our case starts with a modified version of the Rosenbrock constrained hill climbing algorithm [10], searches for a better point than the current local maximum, and then restarts the hill climbing algorithm from there. While this method did converge to the global maximum, it also required an excessive number of iterations. The method used for Example 2 was a multi-start hill climbing algorithm with the starting point chosen by truncating the original system as well as by other model reduction techniques. This method will in general not converge to the global maximum without an excellent starting point, but it will find a good local maximum for a decent starting point.

The Rosenbrock hill climbing algorithm and the algorithm to compute λ were written in PC-MATLAB and run on a Zenith Z-248 personal computer.

Example 1. Simple illustrative examples

We will first show the reduction of a stable second order system and an unstable second order system. The system is of the form

$$\dot{x}_p = A_p x_p + B_p u, \quad x_p(0) = 0, \quad (58)$$

$$y_p = C_p x_p, \quad (59)$$

where $A_p = \begin{pmatrix} 0 & 1 \\ -10 & -11 \end{pmatrix}$ for the stable case, and $A_p = \begin{pmatrix} 0 & 1 \\ 10 & -9 \end{pmatrix}$ for the unstable case. The other matrices are $B_p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $C_p = (1 \ 0)$.

Our first order model equations are

$$\dot{x}_m = a_m x_m + b_m u, \quad x_m(0) = 0, \quad y_m = c_m x_m. \quad (60)$$

The weighting matrices in (53) are $R = .01$, $Q = 1$, and the final time T is taken to be 2 seconds. Our optimization technique is the aforementioned tunneling algorithm.

In the stable case, the initial values were $a_m = -1$, $b_m = 1$, and $c_m = 1$. The value of λ increased from an initial value of 3.4 to a maximum of 16.8. The final reduced order model is given by $a_m = -.7485$, $b_m = .0944$, and $c_m = .9$. In the unstable case, the initial values were $a_m = 1$, $b_m = 1$, and $c_m = 1$ and the final values are given by $a_m = .9505$, $b_m = .0812$, and $c_m = 1.1119$. In this case, the value of λ increased from 2.8 to a final maximum of 17.4.

The time responses to a step input are given in Figs. 4 and 5 (p. 37). A comparison of the time responses shows excellent correlation between the plant and model outputs. The frequency responses are also well matched, at least up to 10 rad/sec. These can be seen in Figs. 6 and 7 (p. 38). Divergence in the frequency responses is to be expected, since no low order model can match the frequency response of the high order plant at sufficiently high frequencies.

Example 2. Aircraft with structural modes

In this example, an eighth order plant will be reduced into a fourth order system. The plant is the longitudinal system of the Advanced Supersonic Transport (AST) along with the two lowest frequency structural modes [11]. The structural modes are the first and second fuselage bending modes. The system is of the form

$$\dot{x}_p = A_p x_p + B_p u, \quad x_p(0) = 0, \quad (61)$$

$$y_p = C_p x_p, \quad (62)$$

$$x_p = (v \quad \alpha \quad \theta \quad q \quad x_1 \quad \dot{x}_1 \quad x_2 \quad \dot{x}_2)^* \quad (63)$$

$$u = (\delta_e \quad \delta_t \quad \delta_c \quad \delta_a)^* \quad (64)$$

where the matrices A_p , B_p , and C_p are given in Table 1. In (63), the variables on the right side are, respectively, perturbed speed, angle of attack, pitch angle, pitch rate, first fuselage bending mode, its derivative, second fuselage bending mode, and its derivative.

The quantities on the right side of (64) are the control inputs from the elevator, throttle, canard, and elevon, respectively. The flight condition is supersonic cruise at Mach 2.5. The units for the airspeed are ft/sec, and the angles and the control surface deflections are in degrees.

We attempted to reduce this to a fourth order system given by

$$\dot{x}_m = A_m x_m + B_m u, \quad x_m(0) = 0, \quad (65)$$

$$y_m = C_m x_m, \quad (66)$$

$$x_m = (v \quad \alpha \quad \theta \quad q)^*. \quad (67)$$

Table 1 Plant matrices of the AST longitudinal system

$A_p =$	$\begin{pmatrix} -0.0127 & -0.0136 & -0.0360 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0969 & -0.4010 & 0.0000 & 0.9610 & 19.5900 & -0.1185 & -9.2000 & -0.1326 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.2290 & 1.7260 & 0.0000 & -0.7220 & -12.0200 & -0.3420 & 1.8420 & 0.8810 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.1204 & 0.0000 & 0.0496 & -44.0000 & -1.2740 & -4.0300 & -0.5080 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 0.1473 & 0.0000 & 0.3010 & -7.4900 & -0.1257 & -21.7000 & -0.8030 \end{pmatrix}$
$B_p =$	$\begin{pmatrix} 0.0000 & 0.0194 & 0.0000 & 0.0000 \\ -0.0215 & 0.0000 & -0.0040 & -1.7860 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -1.0970 & 0.0000 & 0.3660 & -0.0569 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.6400 & 0.0000 & 0.1625 & -0.0370 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -1.8820 & 0.0000 & 0.4720 & -0.0145 \end{pmatrix}$
$C_p =$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

Note that in this case, λ is a function of 48 independent variables and the matrix F in (54) is of dimension 24×24 . The exponentials in (55) and (56) were evaluated using the built-in matrix exponential routine of PC-MATLAB. We observed that because of the large numbers involved, it is best not to invert the matrix in (55) to get the matrix in (56), but to compute it directly using the built-in routine. We ran the multi-start hill climbing algorithm with three different starting points, viz., with the truncated system matrices, and with the reduced order matrices from [11], which were obtained by balancing and spectral decomposition. The final time T was taken to be 5 sec, the weighting matrix Q was the identity matrix, and the weighting matrix R was selected as the diagonal matrix with all diagonal entries equal to 0.001. Although all the three runs yielded local maxima for λ , we obtained the best value of λ with the initial matrices obtained by spectral decomposition. In this case, the value of λ increased from 2.6 to 36.3. The resulting reduced order model is characterized by

$$A_m = \begin{pmatrix} -0.0112 & -0.0023 & -0.0258 & -0.0003 \\ -0.1248 & -0.4222 & 0.0139 & 0.8712 \\ -0.2817 & 0.0142 & 0.0152 & 1.0198 \\ -0.2862 & 1.7511 & -0.0032 & -0.6893 \end{pmatrix}, \quad (68)$$

$$B_m = \begin{pmatrix} 0.0027 & 0.0215 & 0.0013 & -0.0005 \\ 0.5117 & 0.0022 & -0.1354 & -1.8368 \\ -0.1422 & 0.0107 & 0.0449 & 0.0035 \\ -1.0341 & 0.0054 & 0.3664 & -0.0521 \end{pmatrix}, \quad (69)$$

$$C_m = \begin{pmatrix} 1.0427 & 0.0099 & 0.0035 & 0.0103 \\ -0.1416 & 1.0082 & 0.0028 & -0.0059 \\ -0.0047 & 0.0068 & 0.9932 & -0.0006 \\ -0.1852 & 0.0197 & 0.0128 & 1.0255 \end{pmatrix}. \quad (70)$$

The original unaugmented plant has the short period eigenvalues at 0.6687 (unstable), and -1.7755 (stable), the phugoid eigenvalues at $-0.0151 \pm i0.0886$, the first fuselage bending mode eigenvalues at $-0.7257 \pm i6.7017$, and the second fuselage bending mode eigenvalues at $-0.3122 \pm i4.4484$. The eigenvalues of A_m are given by -0.0046 , -0.0232 , 0.7113 , and -1.7910 . The correlation between the plant and model outputs is excellent in the

chosen interval, and this can be seen from Figs. 8-11 (p. 39,40), where the responses to an elevator step input are plotted.

Even though this example has weakly coupled flexible and rigid body modes, it revealed another interesting feature. A comparison of the frequency responses in [11] using spectral decomposition and balancing demonstrated the superiority of the spectral decomposition method in this case. As a comparison, in Table 2, we list the initial and final values of λ with starting points obtained from the truncated system matrices, balancing, and spectral decomposition. It can be observed from Table 2 that spectral decomposition gives the best initial and final values for λ .

Table 2 Values of λ with different starting points

	Initial λ	Maximum λ
Truncation	0.1	2.0
Balancing	0.3	3.3
Spectral decomposition	2.6	36.3

While the starting point obtained from the spectral decomposition method gave the best value for the correlation measure, an improvement in the value of λ can be seen for all the starting points. This suggests another use of our method. Using our algorithm, we can fine-tune the reduced order models obtained by some other model reduction methods. We are currently developing an optimization algorithm utilizing certain characteristics of λ as a function of the reduced order model parameters. It is hoped that these characteristics will allow us to overcome the problems associated with the large number of local maxima and the sharp rise in the value of λ near the global maximum. Global optimization algorithms currently being considered include stochastic search methods [9], methods of global increase such as simulated annealing [9,12], and methods of improvement of local maxima such as

tunneling [9].

As is mentioned at the beginning of this section, a vast amount of computation time was used up in the evaluation of λ for given reduced order model matrices. An important topic for further research is to devise an alternate method for the efficient evaluation of λ . Also, our evaluation of λ is only accurate to within the specified incremental step size of λ and this may lead to premature termination of the optimization routine that seeks to maximize λ .

5. CONCLUSIONS

In this chapter, we presented a technique for reduced order modeling with a modified H_∞ optimality criterion. A characterization for the determination of the correlation between plant and model outputs is given. Also the problem of robust model reduction is addressed and an expression for the variation of correlation with plant parameter variations is derived. Nonlinear programming algorithms were utilized to reduce the longitudinal flexible-body model of an Advanced Supersonic Transport. Further work needs to be done to devise a suitable global optimization algorithm.

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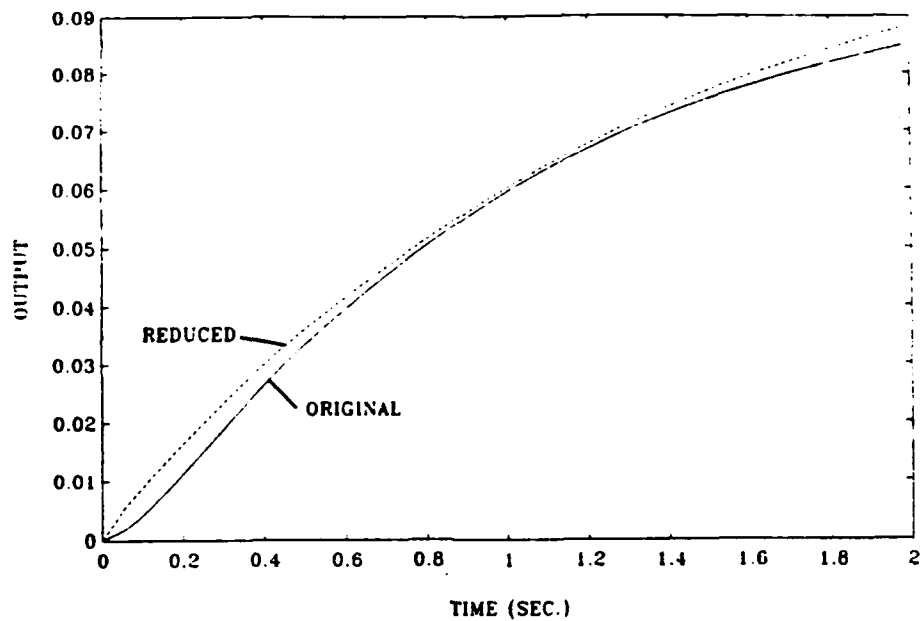


Fig. 4 Step Responses in the Simple Stable Case

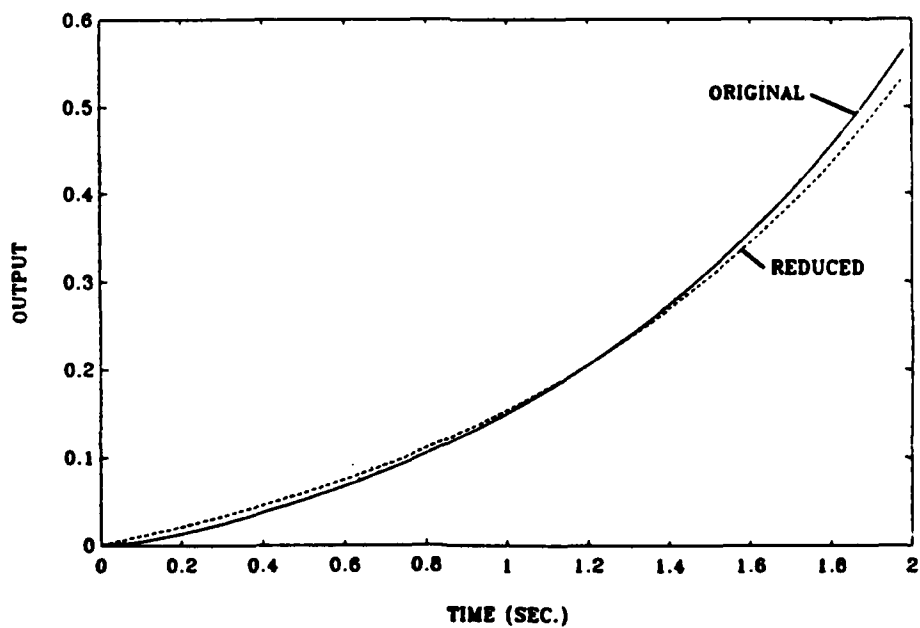


Fig. 5 Step Responses in the Simple Unstable Case

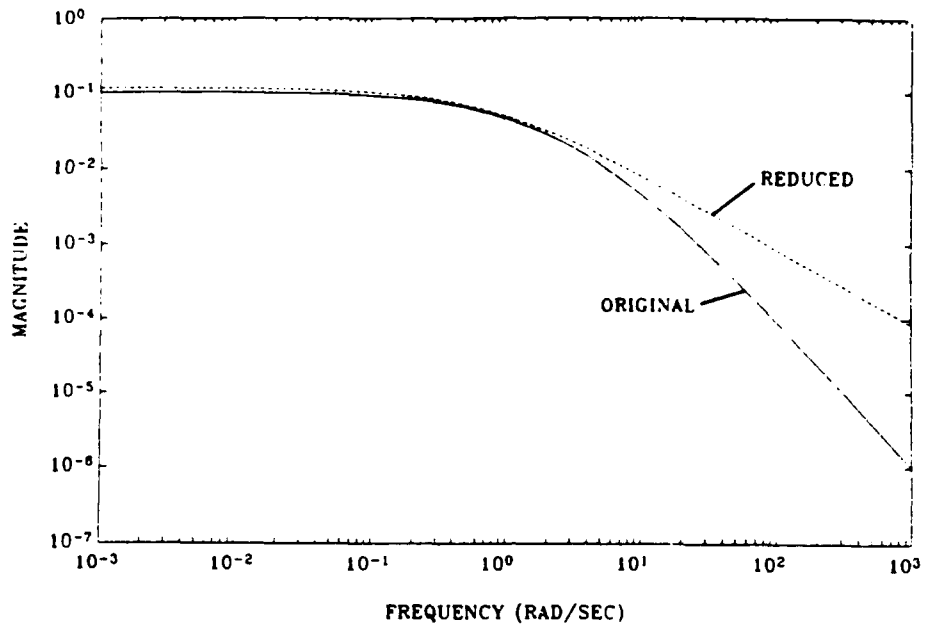


Fig. 6 Frequency Responses in the Simple Stable Case

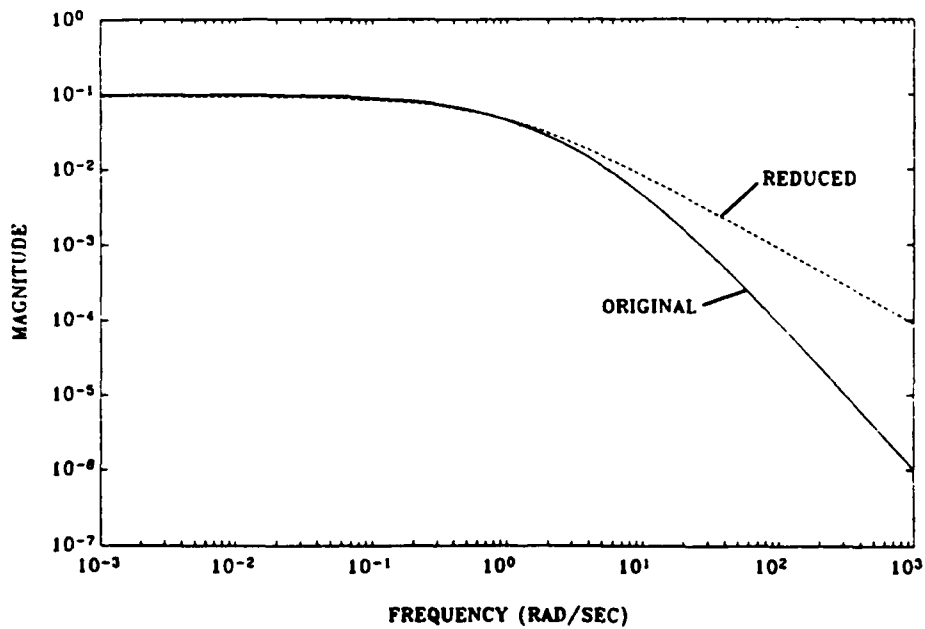


Fig. 7 Frequency Responses in the Simple Unstable Case

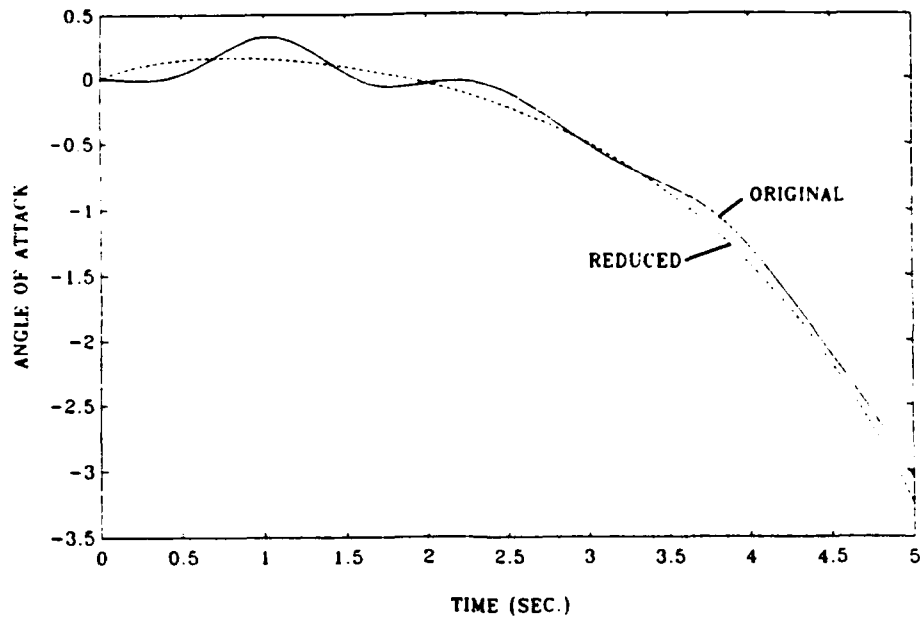


Fig. 8 Angle of Attack due to Application of an Elevator Step

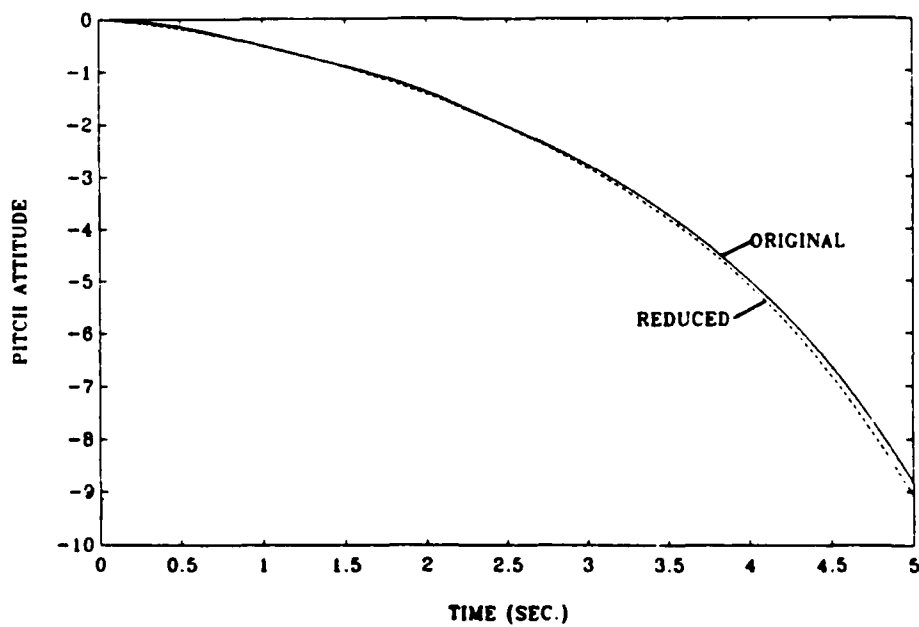


Fig. 9 Pitch Attitude due to Application of an Elevator Step

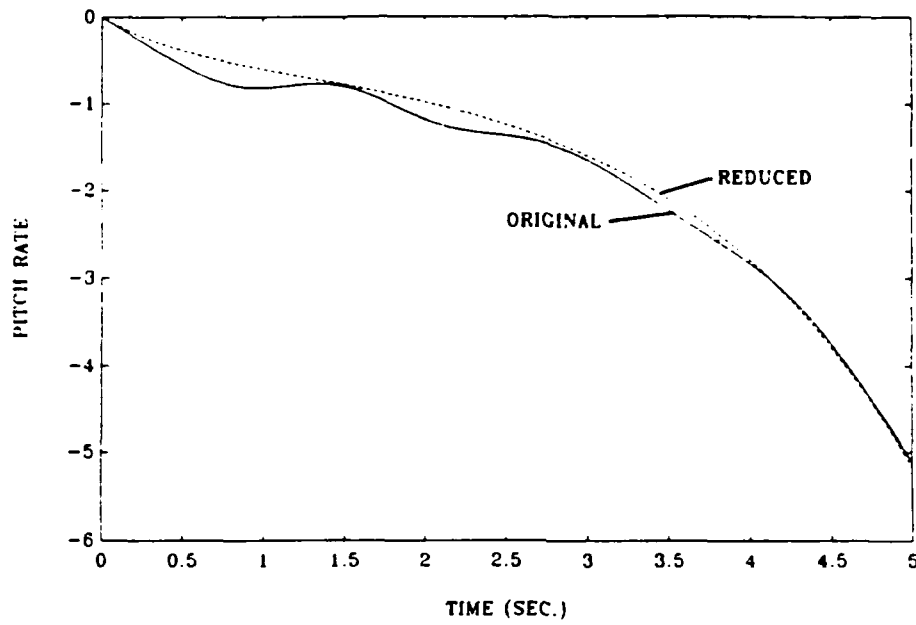


Fig. 10 Pitch Rate due to Application of an Elevator Step

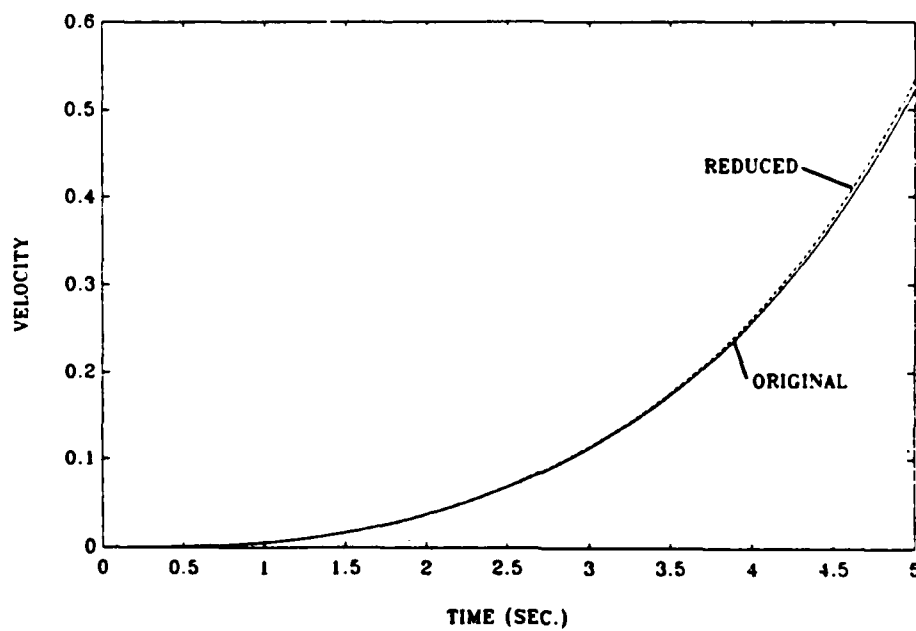


Fig. 11 Velocity due to Application of an Elevator Step

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