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Optimal maintenance policies are designed to reduce the number of system failures and minimize the cost of repair by scheduling planned replacements. In this area the problem of updating the maintenance policy using the past maintenance history has not been adequately solved. In this thesis we study a sequential estimation procedure in a nonparametric setting to estimate the age replacement policy that minimizes long run expected maintenance costs.

This thesis begins with the discussion of the concepts of preventive maintenance, age replacement policies, the settings of our simulation model, and a detailed description of the sequential estimation procedure. We include examples using actual replacement data which demonstrate the usefulness of the sequential estimation procedure. Monte-Carlo methods are used to study the behavior of estimated optimal age replacement policy for different sample sizes, costs and underlying system life distributions. We also make comparison with Frees and Ruppert's (1985) sequential procedure for estimating optimal age replacement policies. These comparisons show that our sequential estimation procedure is competitive and for large sample sizes performs better than the Frees and Ruppert's procedure. Finally, we will introduce a graphical method to estimate the optimal age replacement policy.

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Sequential Estimation of Age Replacement Policies

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ABSTRACT

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This thesis begins with the discussion of the concepts of preventive maintenance, age replacement policies, the settings of our simulation model, and a detailed description of the sequential estimation procedure. We include examples using actual replacement data which demonstrate the usefulness of the sequential estimation procedure. Monte-Carlo methods are used to study the behavior of estimated optimal age replacement policy for different sample sizes, costs and underlying system life distributions. We also make comparison with Frees and Ruppert's (1985) sequential procedure for estimating optimal age replacement policies. These comparisons show that our sequential estimation procedure is competitive and for large sample sizes performs better than the Frees and Ruppert's procedure. Finally, we will introduce a graphical method to estimate the optimal age replacement policy.

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I. INTRODUCTION

A. BACKGROUND

Optimal maintenance policies are designed to reduce the number of system failures and minimize the cost of repair by scheduling times for replacements. By far, most of the research in this area has been from the modeling standpoint. Even in the most basic scenario, where the underlying system lifetimes are assumed to be independent and identically distributed, the problem of updating the maintenance policy using the past maintenance history has not been adequately solved. In this thesis Monte-Carlo methods are used to study a particular nonparametric procedure which updates estimates of an optimal age replacement policy after each replacement. In particular, we will focus on finding conditions under which this sequential procedure does well.

B. MAINTENANCE POLICIES

The effectiveness of a working system depends not only on the innate properties built into it in the design and production stages, but also on the quality of its operation, maintenance and repair. Maintenance policies are designed to reduce the incidence of system failure by scheduling maintenance actions. For example, a maintenance policy is warranted when the failure of a system during actual operation is costly or dangerous (i.e., when the cost C_1 of unscheduled maintenance due to system failure is more than the cost C_2 of scheduled maintenance before system failure). If the system is characterized by a failure rate that increases with age, it may be wise to replace it before it has aged too greatly. This is the concept of **preventive maintenance**. Broadly speaking, preventive maintenance is the total of all service functions aimed at maintaining and improving reliability performance characteristics and concerns itself with such activities as the replacement and repair of systems, inspections, testing and checking of working parts during their operation.

In the preventive maintenance model developed here, it will be assumed that the maintenance action returns the equipment to the "as good as new" condition, thus providing the same services as equipment that has been "replaced". This assumption implies that the times between failure are independent and have the same distributions. If this is not the case, the model needs to be modified. In addition, the model assumes that the scheduled and unscheduled replacement costs C_1, C_2 remain constant over time.

The two most common replacement policies are the policy based on age (age replacement) and the policy based on time (block replacement). Under age replacement a system is always replaced at the time of failure or at age T, whichever occurs first. Under a block replacement policy the system is replaced upon failure and at times T , 2T , 3T , ... , etc. By its nature, age replacement is administratively more difficult to implement, since the age of the unit must be recorded. On the other hand, block replacement, although simpler to administer since the age of the unit need not be recorded, leads to more frequent replacement of relatively new items [Ref. 1: pp. 178-182]. Thus, in this thesis we only consider age replacement policies.

We define an optimal age replacement policy as the age replacement policy which yields the smallest long run expected replacement costs per unit time. To determine an optimal age replacement policy we require explicit knowledge of the system's life distribution. But in a real world scenario, we may not know the life distribution explicitly, thus the optimal age replacement policy needs to be estimated. Estimation when a fixed sample of independent and identically distributed (*i.i.d.*) system life times were available has been examined in detail [Refs. 2,3,4,5]. To gather such *i.i.d.* data, experimental systems must be left in service until failure. Therefore, the experimenter can not implement an age replacement policy to achieve cost savings while collecting data for estimation. When the luxury of observing system lifetimes until failure is not available, a more cost effective approach is to estimate sequentially. By this we mean that the estimator of the optimal age of replacement is updated after each system replacement. In addition, a system under observation is subject to a replacement policy that is close to the best estimated policy so far. The nonparametric procedure of this type which we will study is described in detail in Chapter 2. The difficulty with such a sequential procedure is that implementing an age replacement policy while collecting data results in censored observations. Thus, there is a trade-off between the goals of controlling cost and gathering data for estimation. This conflict is particularly acute in the nonparametric setting where information about the right tail of the distribution needed to estimate the optimal replacement age can not be obtained under heavy censoring.

This thesis will include small, moderate and large sample studies by Monte-Carlo simulation to determine the behavior of the sequential procedure under various conditions. In Chapter 2, description of the age replacement policy and the sequential estimation procedure are given. Included in this chapter is an example of the sequential estimation procedure applied to actual data. Chapter 3 establishes the setting for the simulation study, with a quick overview of the results of that study. A detailed analysis of the simulation results is given in Chapter 4. In that chapter, we also present results comparing our sequential estimation procedure with another sequential estimation procedure. In Chapter 5, we describe a graphical method for estimating the optimal age replacement policy. Conclusions and recommendations are given in Chapter 6.

II. THE SEQUENTIAL ESTIMATION PROCEDURE

A. THE OPTIMAL REPLACEMENT AGE

Let $\{X_n\}$ be an independent and identically distributed (*i.i.d.*) sequence of positive random variables with distribution function F. The sequence $X_1, X_2, \ldots$ represents the sequence of system lifetimes that would be observable if the systems were replaced at failure. It is intuitively clear that it is not advantageous to use an age replacement policy for a system whose lifetime has a decreasing failure rate or constant failure rate since this would result in a replacement by a new system whose properties are worse than or just the same as the original one. Let C_1 be the cost of an unscheduled replacement (at system failure) and C_2 be the cost of a scheduled replacement (before system failure), where $C_1 > C_2$. Under an age replacement policy with scheduled replacement at age t, standard results from renewal theory [Ref. 6: p. 87] can be used to show that the long run expected cost per unit time is given by

$$R(t) = \frac{C_1 \times F(t) + C_2 \times S(t)}{\int_0^t S(x)\,dx},$$

where $S(x) = 1 - F(x)$ is the survival function. The numerator is the expected cost of one replacement under the age replacement policy, and the denominator is the expected time between replacement. Under some fairly general conditions, there exists unique and finite time $\phi^\star$ where $R(t)$ attains a global minimum [Ref. 7: pp. 161-168]. For example, a sufficient condition for the existence of $\phi^\star$ is that F have failure rate $\lambda(x)$ that strictly increases to infinity. We will assume through out the sequel that $\phi^\star$ exists.

When such a ϕ^* exists, it is the optimal age of replacement, in the sense that a policy with age replacement at ϕ^* will have minimum long run expected costs per unit time. When ϕ^* is unknown then it needs to be estimated from system lifetime data.

B. THE SEQUENTIAL ESTIMATION PROCEDURE

Let $\{\phi_n^*\}$ be the sequence of estimators of ϕ^* where the estimator ϕ_n^* is based on data from the first n replacements. If our goal was just to estimate ϕ^*, then we would construct ϕ_n^* based on observing n complete system lifetimes. In this case all replacements would be unscheduled replacements, at system failure, with cost C_1. We wish, however, to control replacement costs by implementing age replacement policies while collecting data to estimate ϕ^*. There is clearly a trade-off between gathering data for estimation (i.e., observing systems as long as possible) and controlling costs by implementing the age replacement policy with the best estimated scheduled replacement age so far. Therefore, we compromise and choose the scheduled replacement age for the nth system to be ξ_n where $\phi_{n-1}^* < \xi_n < \infty$. Intuitively, it seems reasonable to take $\xi_n = \phi_{n-1}^* + a_n$ where $\{a_n\}$ is a sequence of constants that decrease to zero as $n \to \infty$. How to choose the sequence $\{a_n\}$ is an open question that we study in detail in the Monte-Carlo simulations that follow.

Assuming that the $\{a_n\}$ sequence and ξ_1, the replacement age for the first system under observation, are given, the procedure to compute the estimators $\{ \phi_n^* \}$ developed in [Ref. 8] follows:

1. Determine ith system's scheduled replacement time by

$$\xi_i = \phi_{i-1}^* + a_i .$$

2. At the ith replacement observe the system lifetime X_i or ξ_i whichever comes first. Let $Z_i = \min(X_i, \xi_i)$ and $\delta_i = I(X_i \leq \xi_i)$ where $I(A)$ is indicator function of the set A. In other words, if system is replaced before failure, then the replacement time $Z_i = \xi_i$ and $\delta_i = 1$, otherwise $Z_i = X_i$ and $\delta_i = 0$.

3. Sort $Z_1, Z_2, ..., Z_i$ into an ascending sequence $Z_{(1)}, Z_{(2)}, ..., Z_{(i)}$ with $\delta_{(1)}, \delta_{(2)}, ..., \delta_{(i)}$ ordered according to the ordering of $Z_{(1)}, Z_{(2)}, ..., Z_{(i)}$.

4. Use the data $(Z_{(1)}, \delta_{(1)}), (Z_{(2)}, \delta_{(2)}), ..., (Z_{(i)}, \delta_{(i)})$ to estimate $S \equiv 1 - F$. The estimator $\hat{S}$ of S is defined as

$$\hat{S}(t) = \prod_{\{i : Z_{(i)} \leq t\}} \left(\frac{n-i}{n-i+1} \right)^{\delta_{(i)}} .$$

   Note that although $\hat{S}$ is formally identical to the Kaplan-Meier estimator [Ref. 9: pp. 34-35] based on randomly right censored data, it's properties are substantially different because the underlying data pairs (Z_1, δ_1) , (Z_2, δ_2) , ... , (Z_i, δ_i) are not independent.

5. Use the estimator in step 4 to estimate $R(t)$ as

$$\hat{R}(t) = \frac{C_1 \times \hat{F}(t) + C_2 \times \hat{S}(t)}{\int_0^t \hat{S}(x)\, dx} .$$

6. The minimum of the estimator $\hat{R}(t)$ does not exist. In addition if $\delta_{(i)} = 0$, i.e., if the largest replacement age observed so far occurred before system failure, then $\hat{R}(t) \rightarrow 0$ as $n \rightarrow \infty$. To take care of these two problems, after i replacements, we estimate $\phi^{\star}$ by $\phi_i^{\star}$, where $\phi_i^{\star}$ attains the infimum of $\hat{R}(t)$ over $o \leq t \leq Z_{(i)}$. In fact $\phi_i^{\star}$ is easy to calculate. It can be shown [Ref. 8], that $\phi_i^{\star}$ is equal to the Z_j among $1 \leq j \leq i$ such that

$$\hat{R}(Z_j) = \min \{ R(\hat{Z}_{(1)}^{-}), R(\hat{Z}_{(2)}^{-}), ..., R(\hat{Z}_{(i)}^{-}) \}.$$

The procedure is then repeated.

This sequential estimation scheme means that after $(i-1)$ replacements the experimenter has available a scheduled replacement age ξ_i. The replacement cost for the ith system is C_1 if $X_i < \xi_i$, otherwise the replacement cost is C_2. Thus with this sequential estimation procedure the actual total replacement cost for the first n systems that are observed is

$$c_n = \sum_{i=1}^{n} \{C_1 \times \delta_i + C_2 \times (1 - \delta_i)\},$$

and the total operating time for the n systems is

$$t_n = \sum_{i=1}^{n} \min(X_i, \xi_i) = \sum_{i=1}^{n} Z_i .$$

Let $\varepsilon > 0$, and $a_i > \varepsilon$ for $i = 1, 2, \ldots$.. Then in [Ref. 8] it is shown, that as $n \to \infty$, $\phi_n^\star$ converges to $\phi^\star$ with probability 1.0, and the actual cost per unit time $\frac{c_n}{t_n}$ converges with probability 1.0 to $R(\phi^\star)$. Since $\varepsilon > 0$, can be arbitrarily small, for practical purposes we can choose a sequence $\{a_n\}$ such that $a_n \to 0$. It is clear that choice of $\{a_n\}$ can have considerable impact on how well $\phi_n^\star$ estimates $\phi^\star$ and on how large the actual costs $\frac{c_n}{t_n}$ are. An important feature of this thesis is to use simulation to provide practical guidelines for choice of $\{a_n\}$.

C. EXAMPLE: REPLACEMENT COST ANALYSIS OF THE TRACTOR AT ENGINE FAILURE

To show how the sequential procedure works, we consider the data [Ref. 10: p. 560], summarized in Table 1, which includes the age at engine failure of 22 brand new D9G-66A Caterpillar tractors and the calendar date of the failure.

Assume the unscheduled replacement cost C_1 = \$200.00 and scheduled replacement cost C_2 = \$100.00. If engines are replaced only at failure, with no age replacement policy (i.e., $\xi_i = \infty$ for $i = 1, 2, \ldots, 22$), then at each replace... $Z_i = X_i$, and $\hat{F}$ becomes the empirical distribution based on the system lifetimes observed so far. Table 2 on page 9 gives the estimates of ϕ^* calculated after each replacement.

Table 1. THE AGE OF TRACTOR FAILURE

Sequence of failure	Tractor's number	Date of failure	Age in hours when failure occurred
1	16	4-15-65	5161
2	12	6-28-65	5278
3	13	8-1-66	6378
4	15	8-3-66	6578
5	14	9-14-66	6385
6	17	10-26-66	6717
7	19	4-3-67	5556
8	10	5-8-67	2690
9	18	11-1-67	6869
10	11	3-26-68	6259
11	8	8-8-69	4394
12	22	10-31-69	6150
13	2	4-16-70	5085
14	5	6-1-70	6052
15	7	8-10-70	7774
16	9	9-21-70	10517
17	6	6-7-71	6367
18	1	6-16-71	8230
19	20	7-30-71	3286
20	3	10-11-71	3826
21	21	1-21-72	4815
22	4	5-8-72	10950

From Table 2, we have

$$\text{total replacement cost} = 22 \times 200.00 = 4400.00\,,$$

$$\text{total operating hours} = \sum_{i=1}^{22} Z_i = 135299, \text{ and}$$

$$\text{cost per unit time} = \frac{4400.00}{135299} = 0.03252 \ \frac{\text{dollars}}{\text{hour}}\,.$$

Table 2. ESTIMATION WITH NO AGE REPLACEMENT POLICY

i	ξ_i	X_i	δ_i	Z_i	ϕ_i^*
1	∞	5161	1	5161	5161
2	∞	5278	1	5278	5161
3	∞	6378	1	6378	5161
4	∞	6578	1	6578	5161
5	∞	6385	1	6385	5161
6	∞	6717	1	6717	5161
7	∞	5556	1	5556	5161
8	∞	2690	1	2690	5161
9	∞	6869	1	6869	5161
10	∞	6259	1	6259	5161
11	∞	4394	1	4394	5161
12	∞	6150	1	6150	5161
13	∞	5085	1	5085	5085
14	∞	6052	1	6052	5085
15	∞	7774	1	7774	5085
16	∞	10517	1	10517	5085
17	∞	6367	1	6367	5085
18	∞	8230	1	8230	5085
19	∞	3268	1	3268	5085
20	∞	3826	1	3826	5085
21	∞	4815	1	4815	5085
22	∞	10950	1	10950	5085

We now use the sequential estimation procedure described in the last section on the same set of data, with the same costs $C_1 = 200.00$, $C_2 = 100.00$. For this example, let $\xi_1 = \infty$, so that first replacement will occur at system failure and let $a_i = \frac{500}{i-1}$ hours for $i = 2, 3, ..., 22$. The sequence $\{a_n\}$ used in this example was chosen for convenience. No special attempt was made to find the $\{a_n\}$ sequence that gave the best results. Applying the sequential estimation procedure introduced in last section, the sequential estimates of ϕ^* and the data are summarized in Table 3.

Table 3. ESTIMATION WITH AGE REPLACEMENT POLICY

i	a_i	ξ_i	X_i	δ_i	Z_i	ϕ_i^*
1	0	∞	5161	1	5161	5161
2	500.00	5661.00	5278	1	5278.00	5161
3	250.00	5411.00	6378	0	5411.00	5161
4	166.67	5327.67	6578	0	5327.67	5161
5	125.00	5286.00	6385	0	5286.00	5161
6	100.00	5261.00	6717	0	5261.00	5161
7	83.33	5244.33	5556	0	5244.33	5161
8	71.43	5232.43	2690	1	2690.00	5161
9	62.45	5223.50	6869	0	5223.50	5161
10	55.56	5216.56	6259	0	5216.56	5161
11	50.00	5211.00	4394	1	4394.00	5161
12	45.45	5206.45	6150	0	5206.45	5161
13	41.67	5202.67	5085	1	5085.00	5085
14	38.46	5123.46	6052	0	5123.46	5085
15	35.71	5120.71	7774	0	5120.71	5085
16	33.33	5118.33	10517	0	5118.33	5085
17	31.25	5116.25	6367	0	5116.25	5085
18	29.41	5114.41	8230	0	5114.41	5085
19	27.78	5112.78	3268	1	3268.00	5085
20	26.32	5111.32	3826	1	3826.00	5085
21	25.00	5110.00	4815	1	4815.00	5085
22	23.81	5108.81	10950	0	5108.81	5085

This procedure yields

$$\text{total replacement cost} = \sum_{i=1}^{22} (200.00 \times \delta_i + 100.00 \times (1-\delta_i)) = 3000.00\,,$$

$$\text{total operating hours} = \sum_{i=1}^{22} Z_i = 107395.48, \text{ and}$$

$$\text{cost per unit time} = \frac{3000.00}{107395.48} = 0.02793 \ \frac{\text{dollars}}{\text{hour}}\,.$$

At each updating, both procedures give identical estimates $\phi_i^\star$, $i = 1, 2, ..., 22$. But the final cost per unit time when no age replacement policy was implemented is larger. By varying unscheduled maintenance cost, it can be seen in Table 4 that as the ratio of $\frac{C_1}{C_2}$ increases, the benefit of applying an age replacement policy while gathering data from estimation becomes greater.

Table 4. COMPARISON OF MAINTENANCE COSTS

Unscheduled replacement cost	Scheduled replacement cost	System operation time	Total maintenance cost	Estimated optimal replacement time	Average cost with Age Replacement Policy	Average cost with Failure Replacement Policy
200	100	107395.48	3000	5085	0.02793	0.03252
300	100	106358.44	3800	4815	0.03573	0.04878
400	100	102600.13	4300	4394	0.04191	0.06504
500	100	99185.13	4600	3826	0.04638	0.08130
600	100	98029.13	5200	2690	0.05305	0.09756
700	100	96919.44	5200	2690	0.05365	0.11382
800	100	91040.44	5700	2690	0.06261	0.13008
900	100	86865.44	6200	2690	0.07137	0.14634
1000	100	95194.75	6700	2690	0.07038	0.16260

III. SIMULATION SETTING

A. UNDERLYING LIFE DISTRIBUTION

Guidelines for choosing a sequence $\{a_n\}$ for estimation will depend on three factors, the underlying distribution F, the costs C_1, C_2 and the sample size N where we intend to stop sampling. The two factors that we assume are known, are the sample size N and the costs C_1 and C_2. In much of the simulation, we concentrate on fixed costs $C_1 = 5.0$ and $C_2 = 1.0$. Other costs are also considered, but in much less detail. We choose sample sizes $N = 10$, 50 and 250 to reflect small, moderate and large sample sizes. The third factor is the underlying system life distribution which in general will be unknown. For simulation, we choose this distribution to be Weibull with shape parameter α and scale parameter λ, where the density is given by

$$f(t) = \alpha \lambda (\lambda t)^{\alpha-1} e^{-(\lambda t)^{\alpha}} \qquad for\ t > 0.$$

The Weibull distribution has failure rate

$$\lambda(t) = \alpha \lambda (\lambda t)^{\alpha-1}.$$

When $\alpha > 1.0$, the failure rate is strictly increasing to infinity. Thus, for Weibull distributions, with $\alpha > 1.0$, a unique and finite optimal replacement age ϕ^* exists. The ten different Weibull distributions used in the simulation have α values 1.1, 1.2, ..., 1.9, 2.0. This selection of α values gives us a range of distributions which become more like the exponential distribution as α decreases from 2.0 to 1.1. To make fair comparisons between Weibull distributions, the scale parameter λ was chosen so that the expected system lifetime $E(X_i) = 2.0$. See Figures 1 and 2, for plots of the Weibull densities and corresponding failure rates.

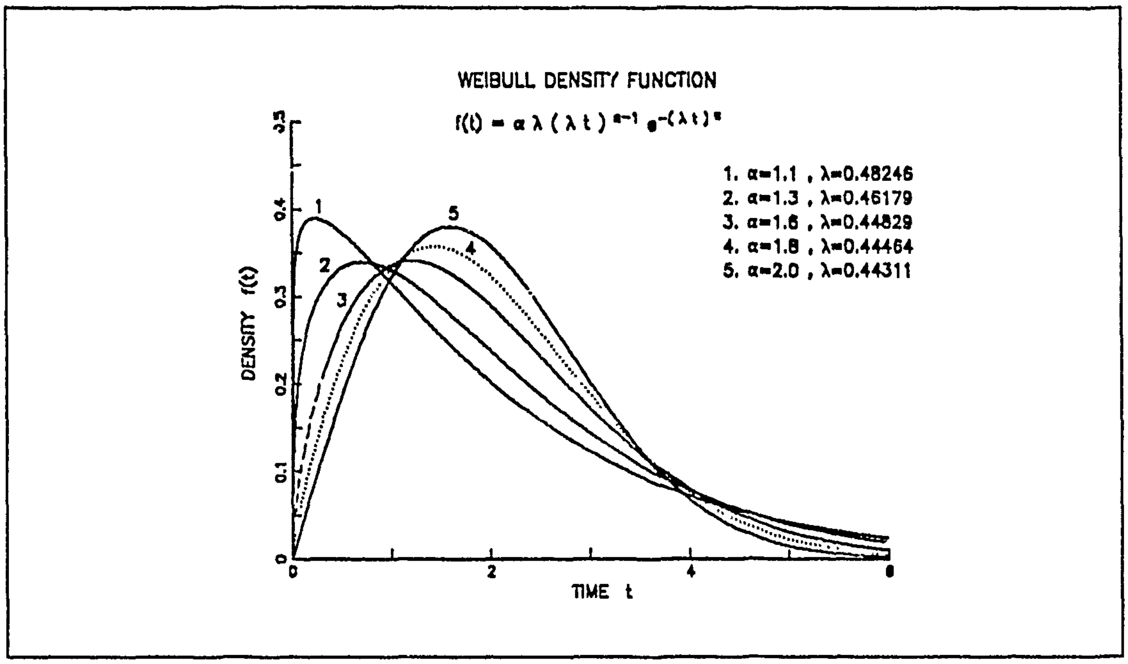

Figure 1. The Weibull density function f(t\ with $E(X_i) = 2.0$

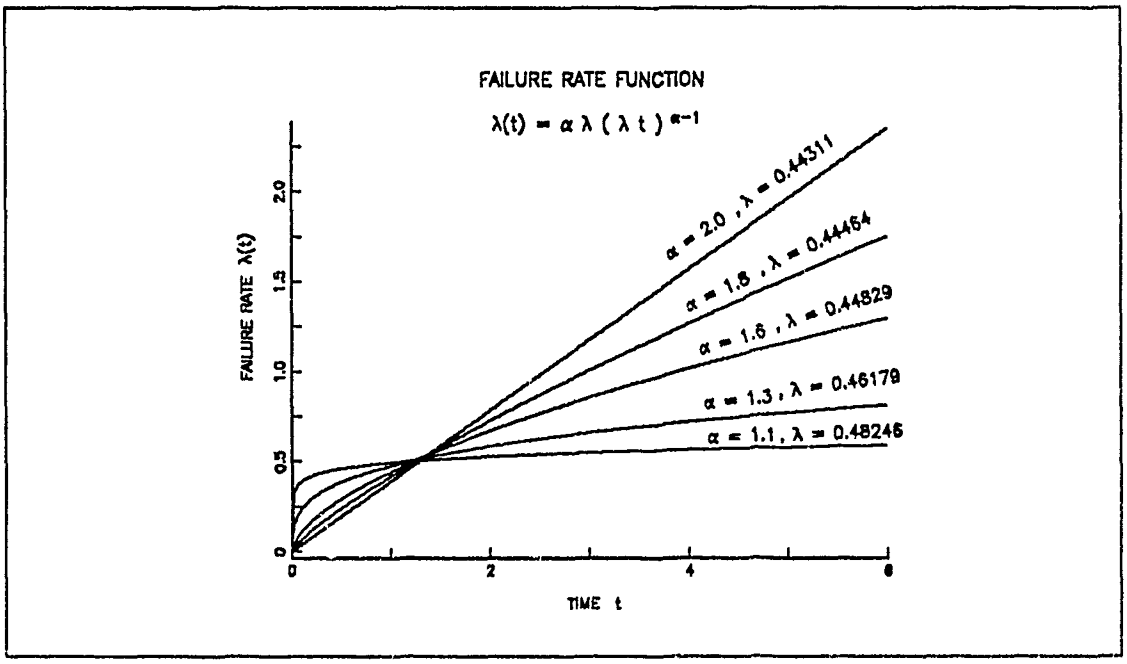

Figure 2. The failure rate of Weibull distribution with $E(X_i) = 2.0$

B. OPTIMAL REPLACEMENT TIME

When the distribution of system lifetime is Weibull, the expected long run average cost per unit time under a simple age replacement policy with scheduled replacement at age t, is given by

$$R(t) = \frac{C_1\left(1 - e^{-(\lambda t)^\alpha}\right) + C_2\, e^{-(\lambda t)^\alpha}}{\int_0^t e^{-(\lambda x)^\alpha}\, dx}.$$

See Figure 3 for a plot of $R(t)$ when the underlying life distribution is Weibull with shape parameter α varying from 1.1 to 2.0. For each curve on Figure 3 the optimal replacement time $\phi^\star$ can be located on the x-axis at the minimum point.

Table 5 on page 16 gives the optimal replacement times $\phi^\star$ for different values of the shape parameter α when the unscheduled replacement cost $C_1 = 5.0$, and the scheduled replacement cost $C_2 = 1.0$. The optimal replacement times $\phi^\star$ in Table 5 are computed from simulation results.[1] Each $\phi^\star$ was generated using 60,000 pseudo random Weibull lifetimes. Included in Table 5, is the probability that a system will be replaced before failure under the optimal age replacement policy,

$$P(X_i < \phi^\star) = 1 - e^{-(\lambda\phi^\star)^\alpha}.$$

From Table 5 on page 16, we observe that the optimal replacement time $\phi^\star$ increases as the shape parameter α decreases from 2.0. For values of α close to 1.0, the performance of a new system will not differ greatly from the old one which is still in use. In this case, very little is gained by replacing the system before failure at the higher cost C_1. The larger values of $\phi^\star$, insure that a small percentage of replacements will be made

1 Since the optimal replacement time $\phi^\star$ comes from a simulation result, it varies slightly with the number of pseudo random variables used and the seed numbers used to generate them.

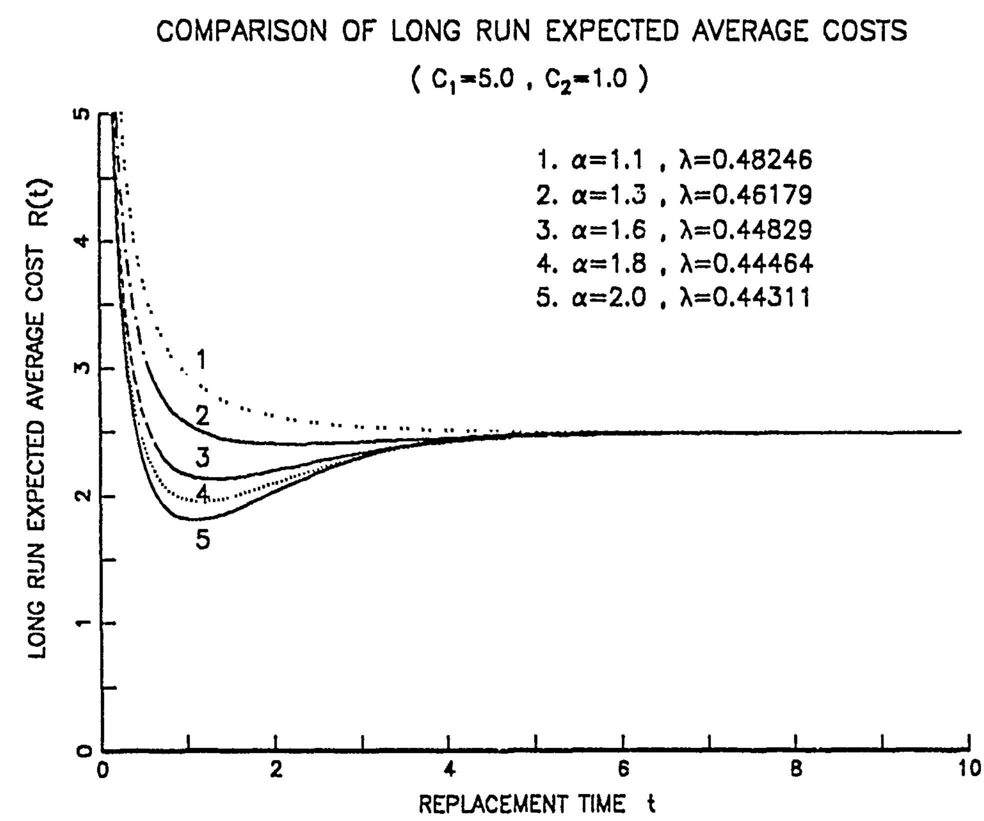

Figure 3. Long run expected average cost curves with $E(X_i) = 2.0$

before failure, which is what we desire if the system's life distribution is close to exponential.

C. OVERVIEW OF SIMULATION RESULTS

A wide range of $\{a_n\}$ sequences have been selected for study. Their functional forms are given in Table 6 on page 18. Recall that ξ_i (i.e., the scheduled replacement time or scheduled censoring time for the ith system lifetime) is

$$\xi_i = \phi^{\star}_{i-1} + a_i , \qquad i = 2, 3, \ldots .$$

Table 5. ESTIMATED OPTIMAL REPLACEMENT TIMES OF WEIBULL MODEL WITH $E(X_i) = 2.0$

Shape parameter α	Scale parameter λ	Optimal replacement time ϕ^*	Long run expected optimal replacement cost $R(\phi^*)$	$P(X_i < \phi^*)$
2.0	0.443113	1.1792297	1.8024244	0.238937
1.9	0.443682	1.1846266	1.8709278	0.255175
1.8	0.444643	1.2031918	1.9470497	0.277008
1.7	0.446122	1.2661085	2.0291557	0.315235
1.6	0.448287	1.3114500	2.1227818	0.347835
1.5	0.451373	1.4643221	2.2216024	0.415706
1.4	0.455712	1.6853428	2.3185654	0.498967
1.3	0.461788	2.3491287	2.4112816	0.670973
1.2	0.470328	3.7472544	2.4730082	0.861093
1.1	0.482456	12.9815130	2.4966393	0.999460

We have arbitrarily selected $\xi_1 = 1.0$ and let $a_1 = 0.0$ for all $\{a_n\}$ sequences. For each α , sample size N and sequence $\{a_n\}$, we have simulated the results of sequentially estimating ϕ^*. Each simulation is based on generating 1000 sequences of system lifetimes.

Let $R_{j,N}$, $j = 1, 2, ..., 1000$ be the actual replacement cost per unit time for the first N replacements of the jth repetition of a simulation. For each simulation, the performance of the sequential estimation procedure is first evaluated by computing

$$\overline{\text{MSE}} = \sum_{j=1}^{1000} \frac{(R_{j,N} - R(\phi^*))^2}{1000} ,$$

where $\overline{MSE}$ is the average squared difference of the actual replacement cost per unit time from the minimum long run expected replacement cost per unit time. Because the ultimate goal of estimating ϕ^* is to reduce costs, we have chosen to evaluate performance by comparing at actual costs per unit time to $R(\phi^*)$, rather than looking at the mean square error of the estimators of ϕ^*. Tables 7, 8 and 9 summarize the simulation results of the $\{a_n\}$ sequences that performed best for different values of α and sample sizes N, with fixed costs $C_1 = 5.0$, $C_2 = 1.0$. The criteria for selecting the best performing $\{a_n\}$ sequence is to choose the sequence which gives the smallest $\overline{MSE}$. In cases where $\{a_n\}$ sequences had nearly equal $\overline{MSE}$, we select the sequence with the lowest actual cost per unit time averaged over the 1000 repetitions.

$$\overline{R}_N = \sum_{j=1}^{1000} \frac{R_{jN}}{1000} .$$

We have also plotted the best performing $\{a_n\}$ sequences using the three sample sizes of 10, 50 and 250 with various values of the shape parameter α in Figure 4 on page 22. From these plots and the results from Tables 7, 8 and 9, we conclude that in general when α decreases from 2.0 (i.e., the underlying life distribution is becomes more exponential), the best performing $\{a_n\}$ sequence tends to have larger values. In the next chapter we provide a detailed analysis of these simulation results, that reveals how the best performing $\{a_n\}$ sequences are effected by sample size, underlying distribution and costs.

Table 6. FUNCTIONAL FORM OF $\{a_n\}$ SEQUENCES

Functional form	A_1	A_2
$\frac{1}{A_1 + A_2 \times (I-1)}$	2.525	0.15
		0.09
		0.06
		0.03
	2.225	0.15
		0.09
		0.06
		0.03
	1.725	0.15
		0.09
		0.06
		0.03
		0.01
	1.400	0.03
$\frac{1}{(A_1 + A_2 \times (I-1))^{0.5}}$	10.00	0.09
		0.07
		0.05
		0.03
	9.00	0.09
		0.07
		0.05
		0.03
	8.00	0.09
		0.07
		0.05
		0.03
$\frac{1}{A_1 + (A_2 \times (I-1))^{0.5}}$	0.50	1.10
	0.50	0.70
	0.35	0.85
	0.20	1.00

Table 7. BEST SIMULATION RESULTS FOR SMALL SAMPLE SIZE (N = 10)

Shape parameter α	$\{a_n\}$ functional form	Actual average cost $\overline{R}_N$	$\overline{\text{MSE}}$ of actual average cost	Estimated optimal replacement time $\hat{\phi}^*$	MSE of $\hat{\phi}^*$
2.0	$\frac{1}{1.725+0.06\times(i-1)}$	2.01557	0.36262	1.26406	0.27861
1.9	$\frac{1}{1.725+0.06\times(i-1)}$	2.06978	0.38469	1.25512	0.28013
1.8	$\frac{1}{1.725+0.01\times(i-1)}$	2.13750	0.41103	1.27617	0.32854
1.7	$\frac{1}{1.400+0.03\times(i-1)}$	2.21857	0.44410	1.33542	0.38985
1.6	$\frac{1}{1.400+0.03\times(i-1)}$	2.29646	0.49618	1.36779	0.41863
1.5	$\frac{1}{1.400+0.03\times(i-1)}$	2.37696	0.54815	1.39615	0.45544
1.4	$\frac{1}{1.400+0.03\times(i-1)}$	2.47533	0.64276	1.39989	0.57325
1.3	$\frac{1}{1.400+0.03\times(i-1)}$	2.58649	0.75869	1.44236	1.34749
1.2	$\frac{1}{1.400+0.03\times(i-1)}$	2.72798	0.95200	1.50447	5.56615
1.1	$\frac{1}{1.400+0.03\times(i-1)}$	2.90896	1.26615	1.53330	131.64487

Table 8. BEST SIMULATION RESULTS FOR MODERATE SAMPLE SIZE (N = 50)

Shape parameter α	$\{a_n\}$ functional form	Actual average cost $\overline{R}_N$	$\overline{\text{MSE}}$ of actual average cost	Estimated optimal replacement time $\hat{\phi}^*$	MSE of $\hat{\phi}^*$
2.0	$\frac{1}{2.225 + 0.06 \times (i-1)}$	1.93167	0.08290	1.26858	0.17087
1.9	$\frac{1}{2.525 + 0.03 \times (i-1)}$	1.99418	0.08626	1.30783	0.19786
1.8	$\frac{1}{(9.00 + 0.03 \times (i-1))^{0.5}}$	2.05713	0.08936	1.33833	0.22572
1.7	$\frac{1}{2.225 + 0.03 \times (i-1)}$	2.13354	0.09605	1.38420	0.25644
1.6	$\frac{1}{(8.00 + 0.03 \times (i-1))^{0.5}}$	2.20758	0.10296	1.45589	0.34657
1.5	$\frac{1}{1.400 + 0.03 \times (i-1)}$	2.30572	0.10254	1.59010	0.41487
1.4	$\frac{1}{1.400 + 0.03 \times (i-1)}$	2.38234	0.11150	1.74888	0.55554
1.3	$\frac{1}{1.400 + 0.03 \times (i-1)}$	2.47548	0.12823	1.91380	0.83251
1.2	$\frac{1}{1.400 + 0.03 \times (i-1)}$	2.56903	0.15299	2.16169	3.34231
1.1	$\frac{1}{1.400 + 0.03 \times (i-1)}$	2.68135	0.20829	2.35770	113.78398

Table 9. BEST SIMULATION RESULTS FOR LARGE SAMPLE SIZE (N=250)

Shape parameter α	$\{a_n\}$ functional form	Actual average cost $\overline{R}_V$	$\overline{\text{MSE}}$ of actual average cost	Estimated optimal replacement time $\hat{\phi}^*$	MSE of $\hat{\phi}^*$
2.0	$\frac{1}{2.525+0.06\times(i-1)}$	1.86401	0.01816	1.17291	0.04966
1.9	$\frac{1}{2.525+0.06\times(i-1)}$	1.93160	0.01831	1.20933	0.05909
1.8	$\frac{1}{2.525+0.03\times(i-1)}$	2.01019	0.01897	1.26157	0.08345
1.7	$\frac{1}{2.525+0.03\times(i-1)}$	2.08492	0.01918	1.30963	0.08491
1.6	$\frac{1}{(10.0+0.09\times(i-1))^{0.5}}$	2.17148	0.01916	1.41962	0.14584
1.5	$\frac{1}{1.725+0.03\times(i-1)}$	2.25858	0.01997	1.51001	0.20026
1.4	$\frac{1}{1.725+0.01\times(i-1)}$	2.35111	0.02039	1.77728	0.37160
1.3	$\frac{1}{1.725+0.01\times(i-1)}$	2.43630	0.02189	2.09759	2.63463
1.2	$\frac{1}{1.725+0.01\times(i-1)}$	2.51301	0.02519	2.60572	2.22276
1.1	$\frac{1}{1.725+0.01\times(i-1)}$	2.58071	0.03489	3.26706	95.67296

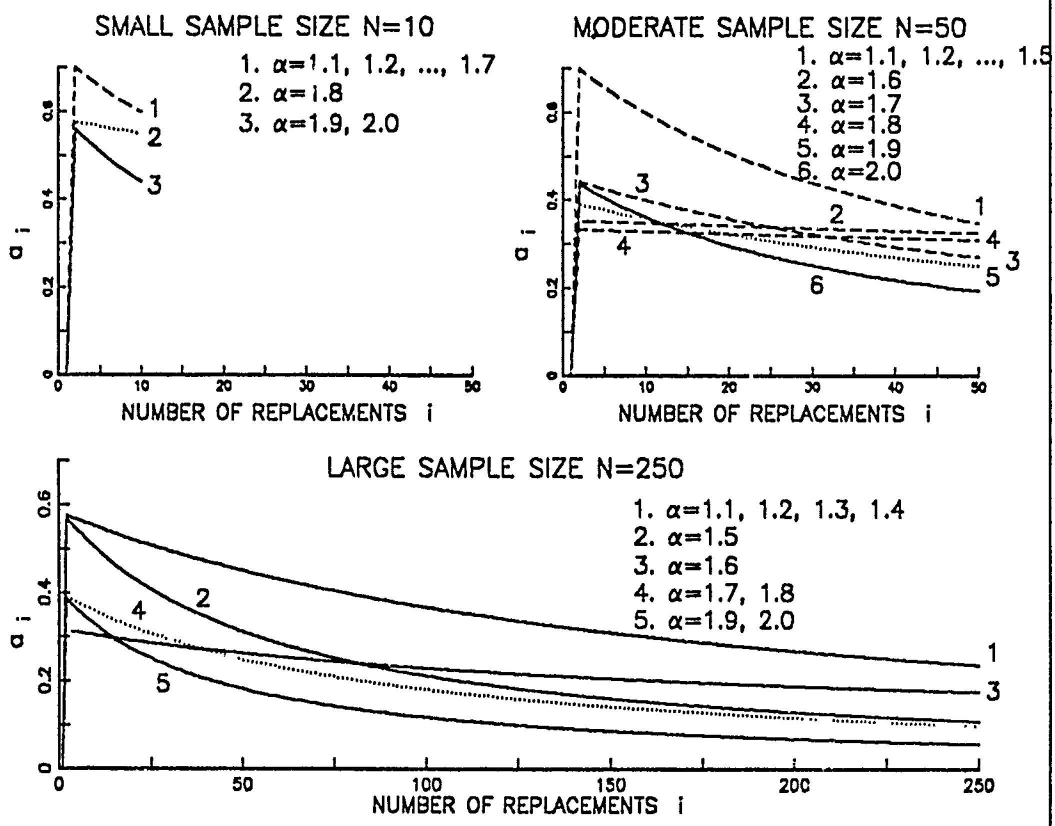

Figure 4. The curves of best performance $\{a_n\}$ sequences

IV. ANALYSIS OF SIMULATION RESULTS

It is clear from Tables 7, 8 and 9 that $\{a_n\}$ sequences with the smallest $\overline{\text{MSE}}$ when the shape parameter α equals 2.0, perform significantly worse when α is equal to 1.1. It is also clear that for a given α , the $\{a_n\}$ sequence performing well for small sample sizes will not be the best $\{a_n\}$ for large sample sizes. In this chapter we analyze in detail how the best performing $\{a_n\}$ sequences differ for different sample sizes, values of α for the underlying system life distribution, and for different costs C_1, C_2. To make these detailed comparisons for $i = 1, 2, ..., N$, we will examine:

- $\overline{\xi}_i$, the ith scheduled replacement times (censoring ages) ξ_i averaged over 1000 repetitions,
- $\overline{P}(X_i < \xi_i)$, the proportion of unscheduled replacements from 1000 repetitions at the ith replacement, and
- $\overline{R}_i$, the average actual cost per unit time for the first i replacements averaged over 1000 repetitions.

We begin by studying simulation results for small, moderate and large sample sizes separately.

A. SMALL SAMPLE SIZE N = 10

From Table 7 on page 19, there are only three different best performing $\{a_n\}$ sequences for $N = 10$. We plot the performance of all three $\{a_n\}$ sequences for $\alpha = 2.0$, 1.8, 1.5 and 1.3 in Figures 5, 6, 7 and 8 respectively. Figures 5(a)~8(a) are the same; they plot a_i versus i, to give a visual representation of the three $\{a_n\}$ sequences. Figures 5(b)~8(b) plot $\overline{\xi}_i$ versus i, and the dotted line is the optimal age of replacement $\phi^\star$. Figures 5(c)~8(c) plot $\overline{P}(X_i < \xi_i)$ versus i, and the dotted line is $P(X_i < \phi^\star)$, the probability of unscheduled replacement under a policy with optimal age of replacement $\phi^\star$.

Figures 5(d)~8(d) plot $\overline{R}_i$ versus i and the dotted line is $R(\phi^*)$. Recall that the first scheduled replacement time $\xi_1 = 1.0$ and $a_1 = 0.0$, so that for each simulation $\overline{\xi}_1 = 1.0$. In Figures 5, 6, 7 and 8:

- the curves marked "1" correspond to the $\{a_n\}$ sequence of functional form $\frac{1}{1.400 + 0.03 \times (i-1)}$ which performs best for $\alpha = 1.1, 1.2, ..., 1.7$,
- the curves marked "2" correspond to the $\{a_n\}$ sequence of functional form $\frac{1}{(1.725 + 0.01 \times (i-1))}$ which performs best for $\alpha = 1.8$, and
- the curves marked "3" correspond to the $\{a_n\}$ sequence of functional form $\frac{1}{1.725 + 0.06 \times (i-1)}$ which performs best for $\alpha = 1.9, 2.0$.

1. Shape Parameter α Equals 2.0

The average censoring time $\overline{\xi}_i$ for all three $\{a_n\}$ sequences overestimates the optimal replacement time $\phi^* = 1.1792$, for $i = 2, 3, ..., 10$ (Figure 5(b)). Because $\xi_i = \phi^*_{i-1} + a_i$, when a_i is large, the scheduled replacement time for the ith replacement ξ_i is also large. Thus we would expect that the $\{a_n\}$ sequence with larger values to yield larger $\overline{\xi}_n$, which in turn would mean that a higher proportion of replacements would be unscheduled. Figures 5(a), 5(b) and 5(c) are consistent with this expectation.

In Figure 5(d) $R(\phi^*) = 1.8024$. This figure shows that for all three sequences the $\overline{R}_i$ the average cost per unit time up to the ith replacement decreases with i, the number of replacements. This result is promising because the goal of the sequential estimation procedure is to decrease costs while sampling. Even though as $n \to \infty$, $\overline{R}_n$ will approach the optimal replacement cost $R(\phi^*)$ with probability 1.0 [Ref. 8], there is no guarantee that $\overline{R}_n$ will decrease for the first few observations.

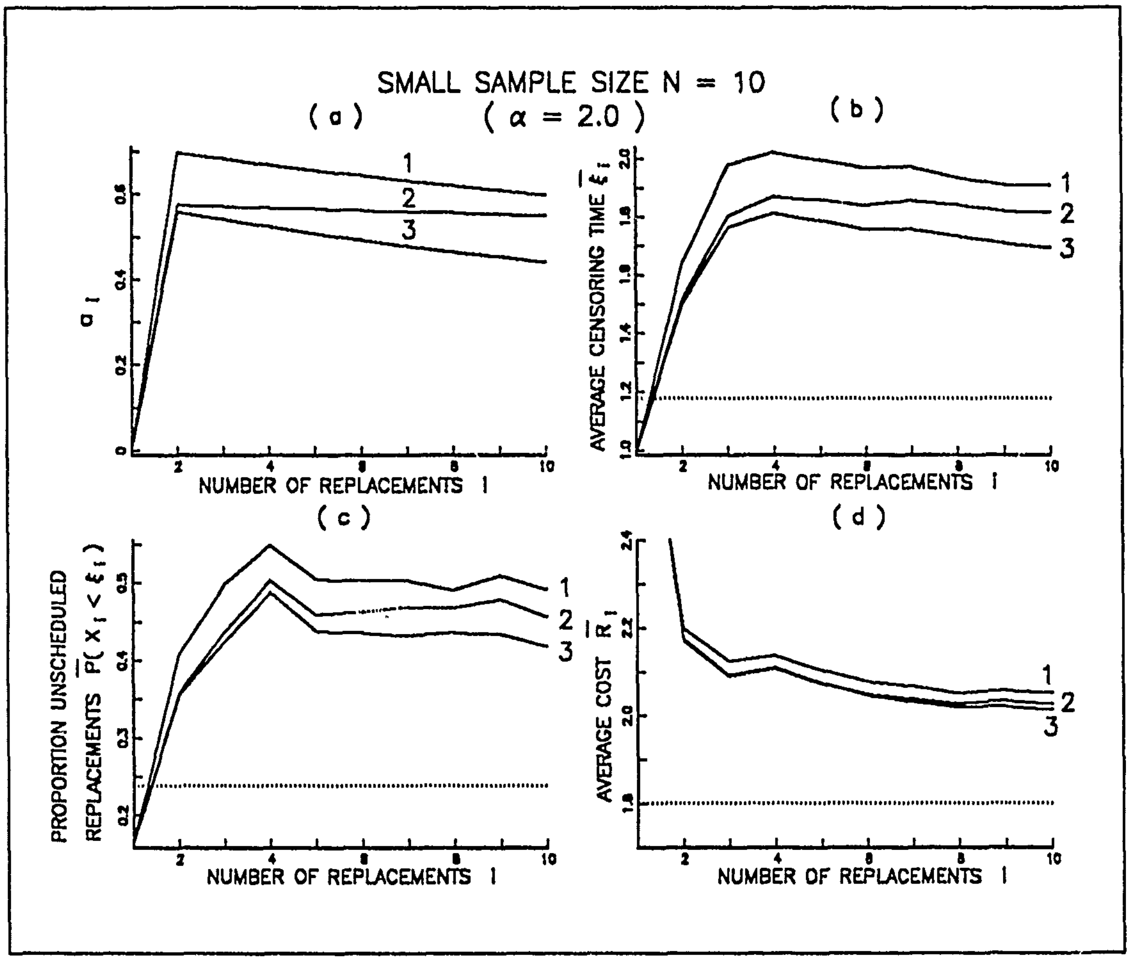

Figure 5. Plot of $\{a_n\}$ performance when N = 10 and $\alpha = 2.0$

The ordering of the curves 1, 2 and 3 in Figure 5(d) supports the intuition that the $\{a_n\}$ sequence which yields $\overline{\xi}_i$'s closest to $\phi^\star$ will have the smallest average costs per unit time $\overline{R}_i$. Thus, curve 3 which is closest to $\phi^\star$ in Figure 5(b) is also closest to $R(\phi^\star)$ in Figure 5(d).

2. Shape Parameter α Equals 1.8

Here $\phi^* = 1.2032$ and $R(\phi^*) = 1.9470$. Again all three $\{a_n\}$ sequences yield $\bar{\xi}_i > \phi^*$, $i = 2, 3, ..., 10$ (see Figure 6(b)). However, comparing curve 3 in Figure 6(b) with curve 3 in Figure 5(b), we see that the third $\{a_n\}$ sequence yields $\bar{\xi}_i$ values closer to ϕ^*: for $\alpha = 1.8$ than for $\alpha = 2.0$. Although the third $\{a_n\}$ sequence clearly yields $\bar{\xi}_i$ values closest to $\phi^*$for $\alpha = 1.8$ (curve 3, Figure 6(b)), both the second and third $\{a_n\}$ sequence have nearly identical average actual cost per unit time (curves 2 and 3, Figure 6(d)). In fact, for $\alpha = 1.8$ the second $\{a_n\}$ sequence performs better than the third $\{a_n\}$ sequence in that the $\overline{\text{MSE}}$ for the second $\{a_n\}$ sequence is smaller.

To explain this, recall that at each replacement, we can only observe the first i replacements. During early stages of sampling (i.e., for small i), the information about the system's life distribution will not be enough to get a very good estimates of ϕ^*. Thus the estimator ϕ_i^* and the scheduled censoring time ξ_i will have large variances in early stages of sampling. Also notice that in Figure 3 on page 15, $R(t)$ to the left-hand side of the global minimum ϕ^* is steeper than on the right-hand side of ϕ^*. Thus slight underestimates of ϕ^* can increase costs much more than corresponding overestimates of ϕ^*. This explains the fact that when $\bar{\xi}_i > \phi^*$ the $\{a_n\}$ sequences with $\bar{\xi}_i$ is closer to ϕ^*, may not do as well as $\{a_n\}$ sequences with slightly large $\bar{\xi}_i$.

We conclude that when the $\{a_n\}$ sequence with average censoring times $\bar{\xi}_i$'s closest to ϕ^* has $\bar{\xi}_i > \phi^*$, and the differences the $\bar{\xi}_i$'s and ϕ^* are small, then this sequence might not yield the lowest average cost, and we need to check the variation in the ξ_i 's carefully before jumping to conclusions.

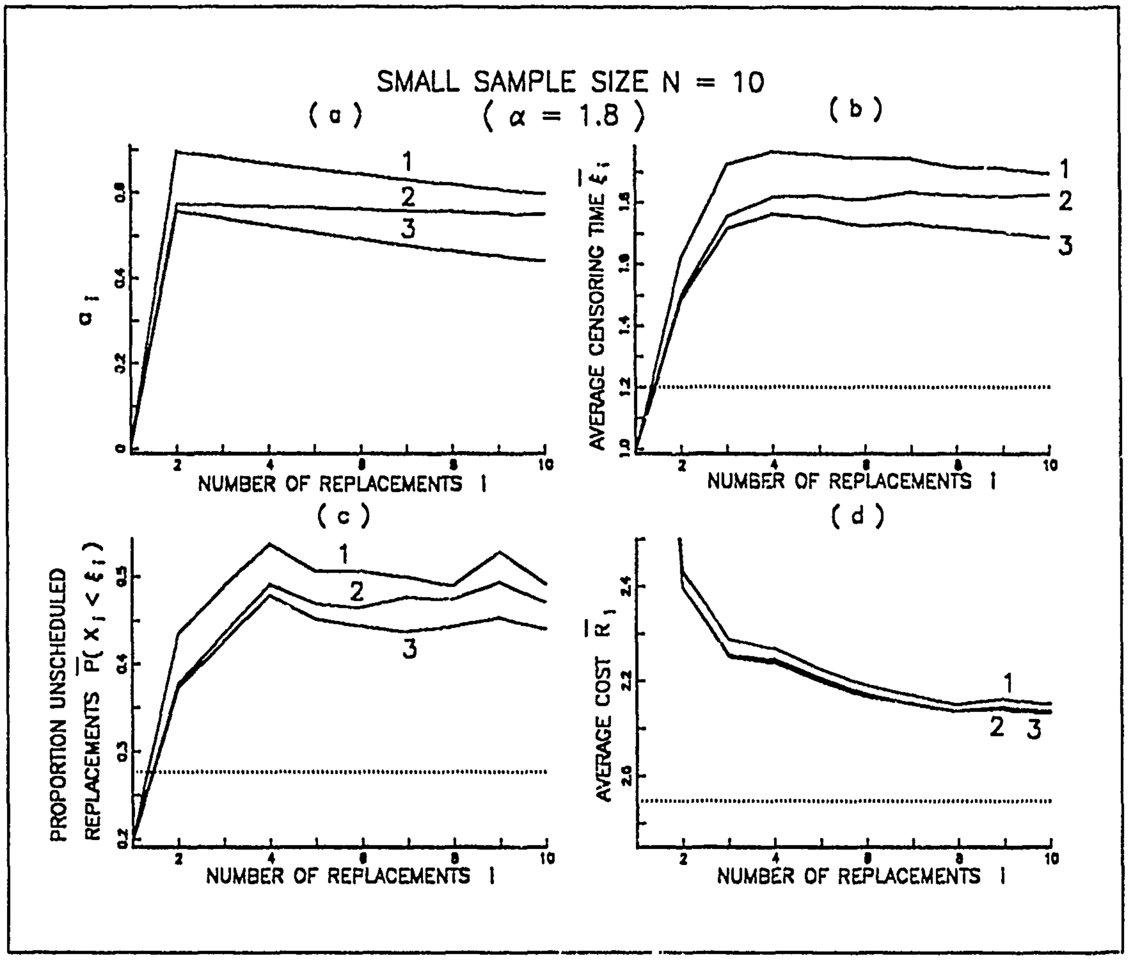

Figure 6. Plot of $\{a_n\}$ performance when N = 10 and $\alpha = 1.8$

3. Shape Parameter α Equals 1.5

The $\{a_n\}$ sequences in Figure 7 perform similarly to those in Figure 6. We see that the $\bar{\xi}_i > \phi^*$ for $i = 2, 3, ..., 10$ for all three sequences, and that the three curves of the average censoring time are even closer to ϕ^* than when $\alpha = 1.8$ and $\alpha = 2.0$ (Figures 6(b) and 5(b)). The same line of reasoning used to explain Figure 6 when $\alpha = 1.8$, can be used here to explain why the average actual costs from the three $\{a_n\}$ sequences are virtually identical (Figure 7(d)) even though the $\bar{\xi}_i$'s for the three sequences are clearly separated (Figure 7(b)). In fact, the first $\{a_n\}$ sequence, with the largest $\bar{\xi}_i$ values (curve 1, Figure 7(b)) has the smallest $\overline{MSE}$.

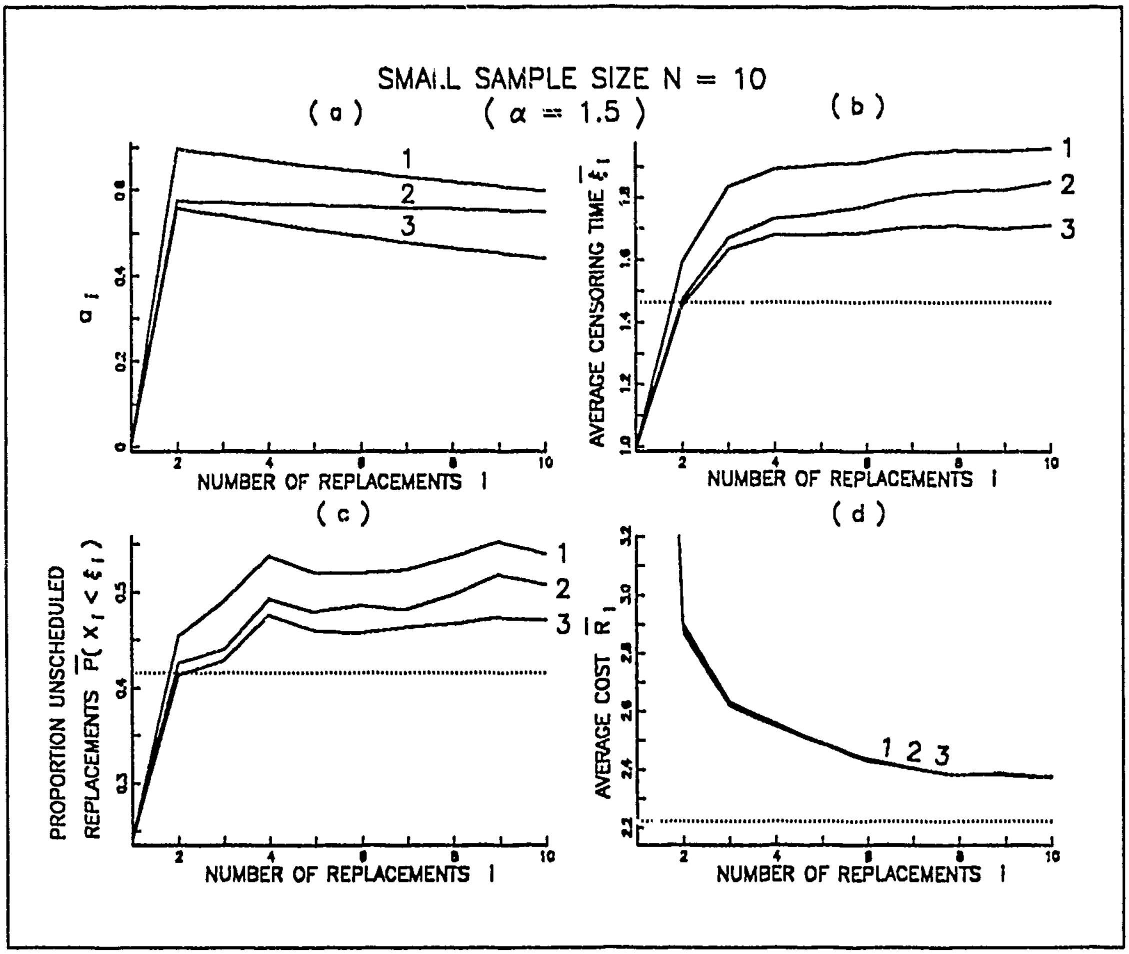

Figure 7. Plot of $\{a_n\}$ performance when N = 10 and $\alpha = 1.5$

4. Shape Parameter α Equals 1.3

Figure 8(b) shows that for all three $\{a_n\}$ sequences $\bar{\xi}_i < \phi^*$, $i = 1, 2, ..., 10$ and that the first $\{a_n\}$ sequence (curve 1) yields $\bar{\xi}_i$ values closest to ϕ^*. Again because $R(t)$ to the left of ϕ^* is steeper than $R(t)$ to the right (Figure 3, curve 2), when several $\{a_n\}$ sequence yield $\bar{\xi}_i < \phi^*$ for $i = 1, 2, ..., 10$, then the $\{a_n\}$ sequence with $\bar{\xi}_i$'s closest to ϕ^*

should yield the lowest average actual costs per unit time $\overline{R}_i$. Thus it is not surprising that the first $\{a_n\}$ sequence has the smallest $\overline{R}_i$, i = 1, 2, ..., 10 and the smallest $\overline{MSE}$.

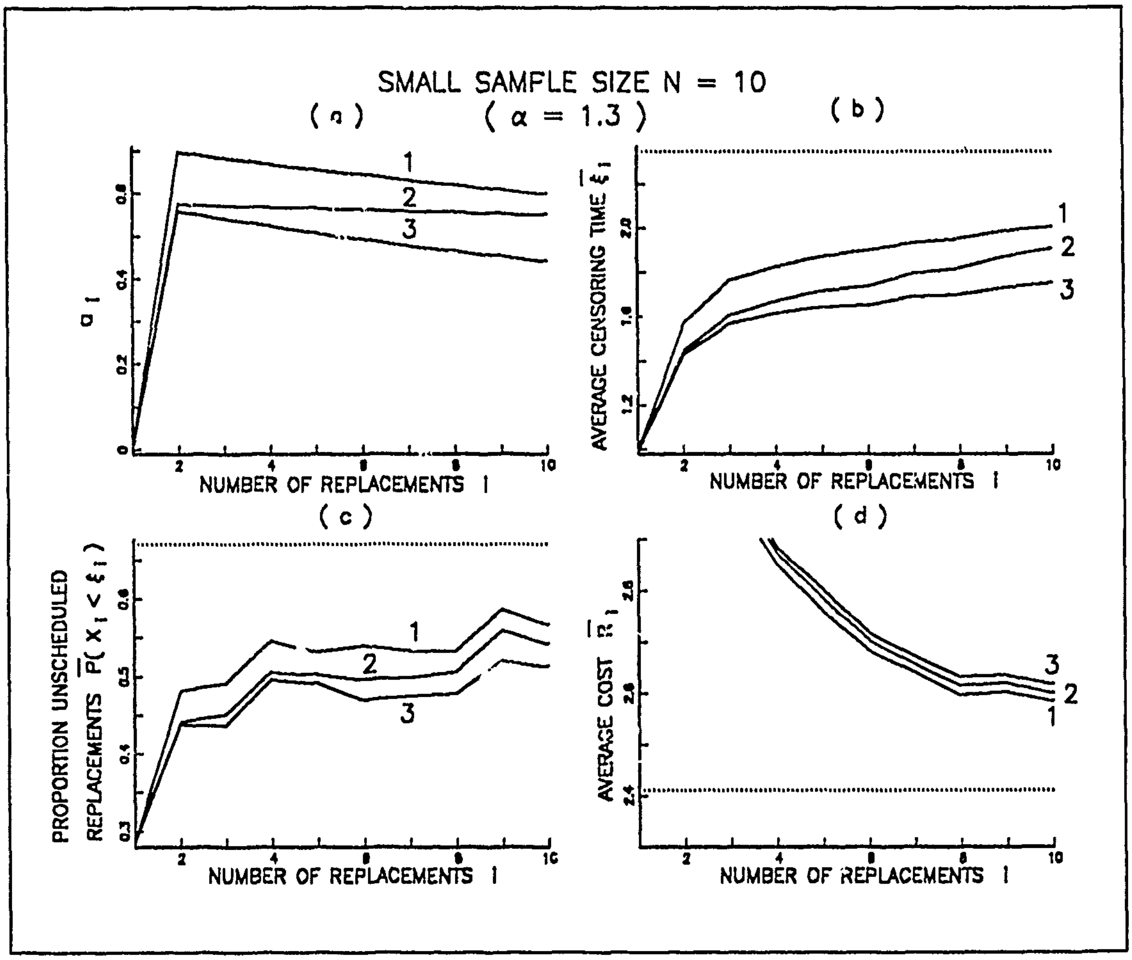

Figure 8. Plot of $\{a_n\}$ performance when N = 10 and α = 1.3

B. MODERATE SAMPLE SIZE N = 50

We use the same approach to analyze the performance of $\{a_n\}$ sequence with moderate sample size as we used for small sample size. In figures 9, 10 and 11:

- the curves marked "1" correspond to the $\{a_n\}$ sequence with functional form $\frac{1}{1.400 + 0.03 \times (i-1)}$ which performs best for α = 1.1, 1.2, ..., 1.5,
- the curves marked "2" correspond to the $\{a_n\}$ sequence with functional form $\frac{1}{(8.00 + 0.03 \times (i-1))^{0.5}}$ which performs best for α = 1.6, and

- the curves marked "3" correspond to the $\{a_n\}$ sequence with functional form $\frac{1}{2.225 + 0.06 \times (i-1)}$ which performs best for $\alpha = 2.0$.

1. Shape Parameter α Equals 2.0

Again in Figures 9(a), 9(b) and 9(c) we see the same trends that were evident for $N = 10$ when $\alpha = 2.0$ (Figures 5(a), 5(b) and 5(c) on page 25). For instance, curve 1 in Figure 9(a) has the largest a_i values, $i = 1, 2, \ldots, 50$, in Figure 9(b) it has the largest $\bar{\xi}_i$ values and in Figure 9(c) it has the largest proportion of unscheduled replacements. Curve 3 has the lowest average cost and it has the smallest $\overline{\text{MSE}}$. Thus the $\{a_n\}$ sequence of functional form $\frac{1}{2.225 + 0.06 \times (i-1)}$, has the best performance for $N = 50$ and $\alpha = 2.0$.

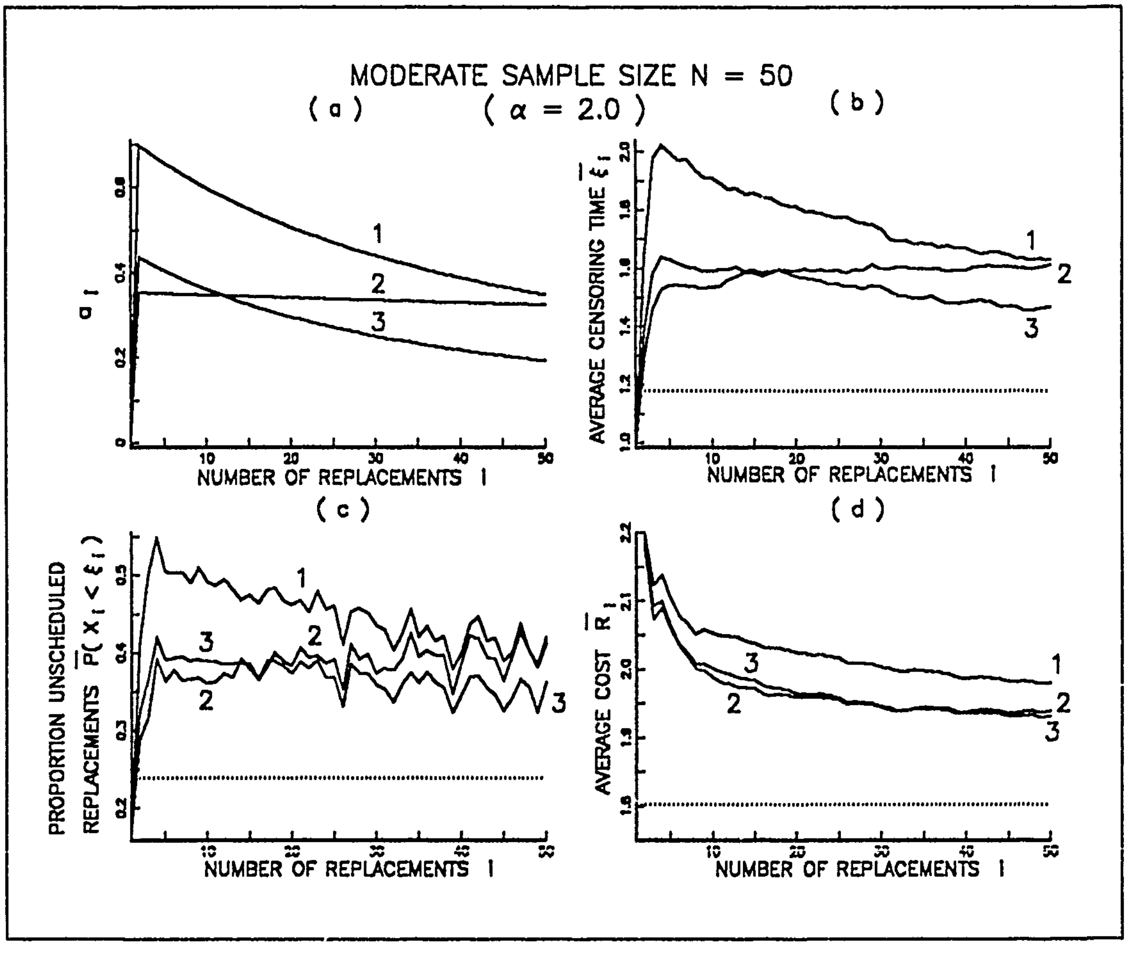

Figure 9. Plot of $\{a_n\}$ performance when N = 50 and $\alpha = 2.0$

2. Shape Parameter α Equals 1.6

From Figure 10(d), we see the same phenomena that occurs for $N = 10$ with $\alpha = 1.8$ (Figure 6 on page 27), i.e., that although the average scheduled replacement times $\bar{\xi}_i$ for curve 3 are closer to ϕ^* than for curve 2, curve 2 has the lower average cost. Again this can be explained by the fact that when $\bar{\xi}_i > \phi^*$ the variance of the ξ_i's has a more dramatic effect on costs when $\bar{\xi}_i$ is close to ϕ^* than when $\bar{\xi}_i$ is further away. It is interesting to note that this effect of the variation of ξ_i's on cost shows up to same extent in Figure 9 when $\alpha = 2.0$.

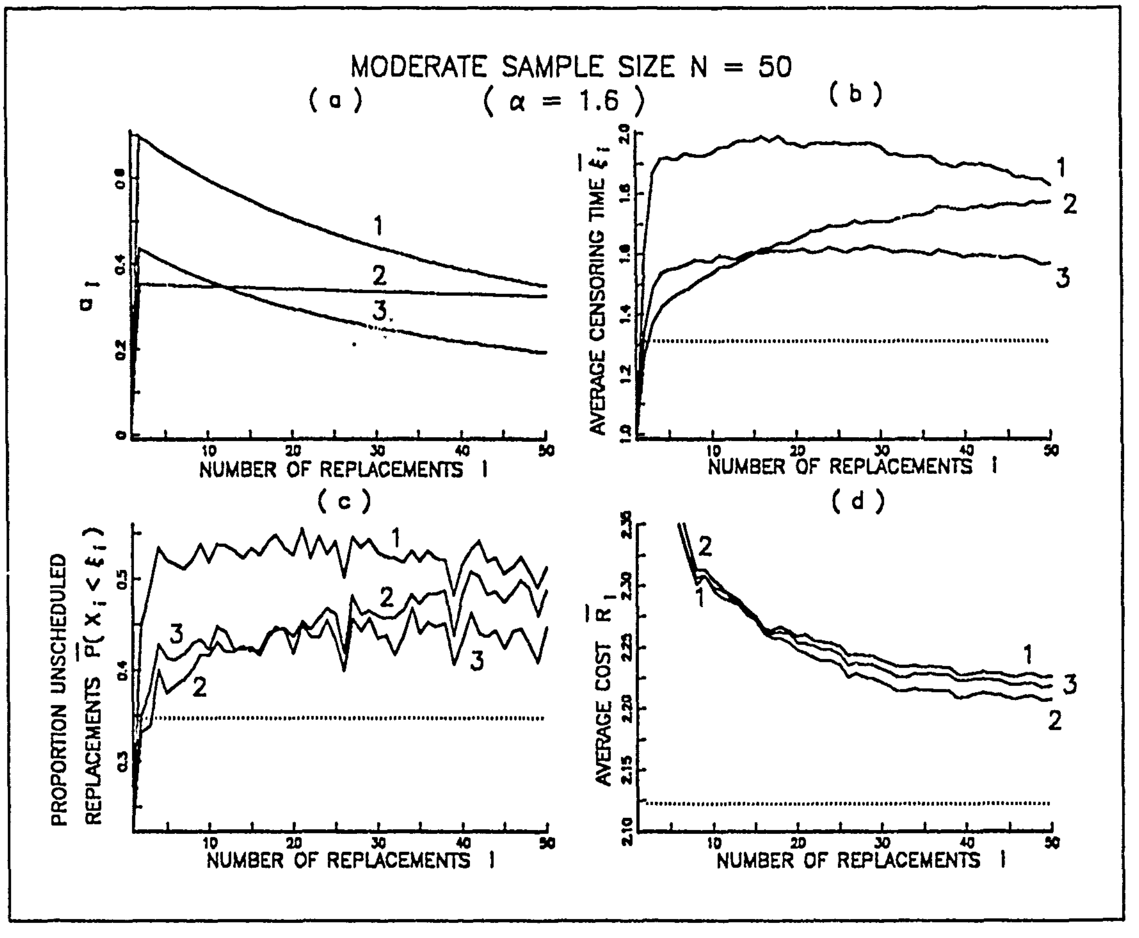

Figure 10. Plot of $\{a_n\}$ performance when N = 50 and $\alpha = 1.6$

3. Shape Parameter α Equals 1.3

All three $\{a_n\}$ sequences yield $\bar{\xi}_i < \phi^*$ for $i = 1, 2, ..., 50$ (Figure 11(b)) consistent with the observations made when $N = 10$ and $\alpha = 1.3$ (Figure 8), the first $\{a_n\}$ sequence (curve 1 in Figure 11(b)) which yields $\bar{\xi}_i$ values closest to ϕ^*, also has the smallest $\bar{R}_i$'s (curve 1 in Figure 11(d)) and has smallest $\overline{\text{MSE}}$.

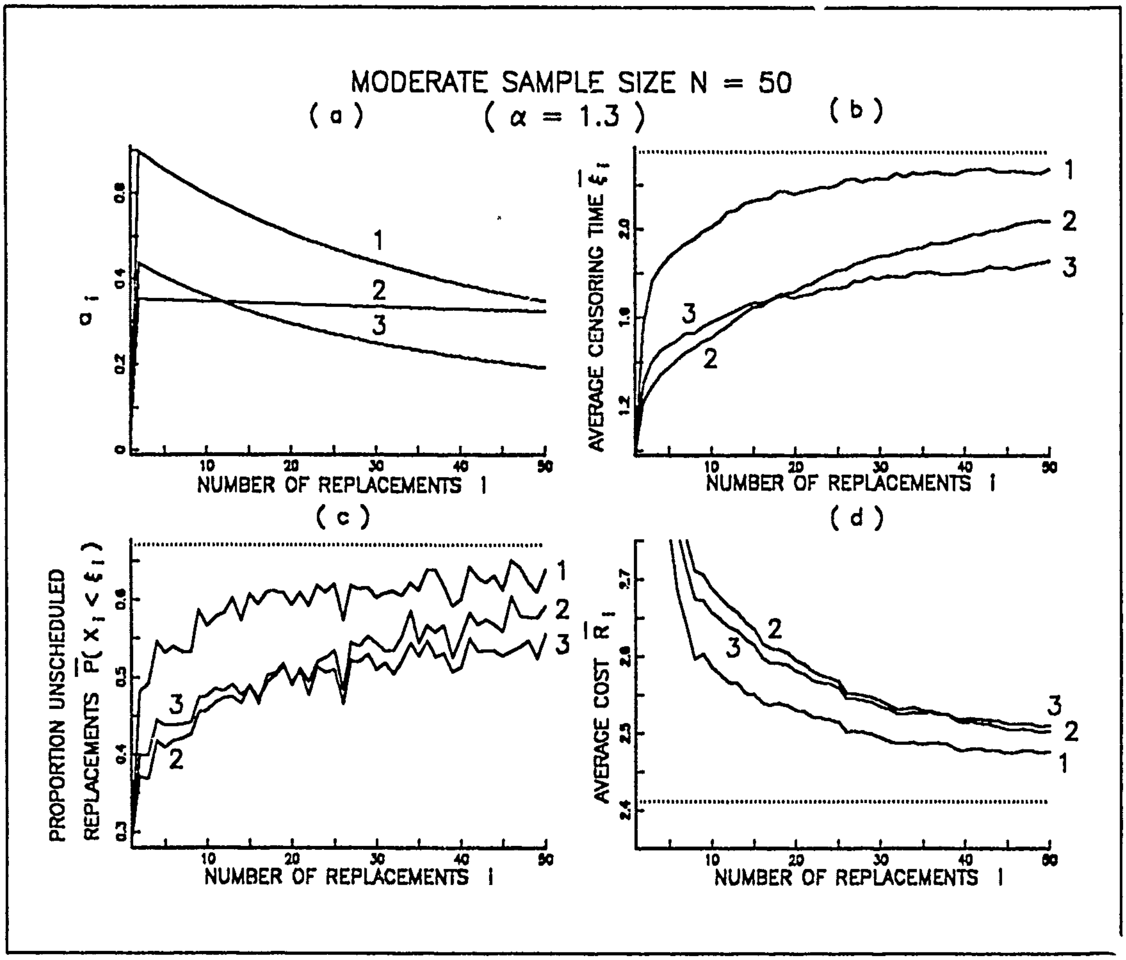

Figure 11. Plot of $\{a_n\}$ performance when N = 50 and $\alpha = 1.3$

C. LARGE SAMPLE SIZE N = 250

Figures 12, 13 and 14 show the performance of $\{a_n\}$ sequences as follows:

- the curves marked "1" correspond to the $\{a_n\}$ sequence of functional form $\frac{1}{1.725 + 0.01 \times (i-1)}$ which performs best for $\alpha = 1.3$,
- the curves marked "2" correspond to the $\{a_n\}$ sequence of functional form $\frac{1}{(10.00 + 0.09 \times (i-1))^{0.5}}$ which performs best for $\alpha = 1.6$, and
- the curves marked "3" correspond to the $\{a_n\}$ sequence of functional form $\frac{1}{2.525 + 0.06 \times (i-1)}$ which performs best for $\alpha = 2.0$.

1. Shape Parameter α Equals 2.0

As for the other sample sizes when $\alpha = 2.0$, all three $\{a_n\}$ sequences yield $\bar{\xi}_i > \phi^*$ for $i = 2, 3, \ldots, 250$. In early stages of sampling, the average costs for the second $\{a_n\}$ sequence (curve 2, Figure 12(d)) are lower than the average costs for the third $\{a_n\}$ sequence (curve 3, Figure 12(d)). But after about 60 replacements the situation reverses and curve 3 falls below curve 2 in Figure 12(d). From Figure 12(b), we observe that the average censoring times of curve 3 are large when i is small. After about 30 replacements, the average censoring times of curve 3 are the closest to ϕ^*. Because the variance of the estimators of ϕ^* should be smaller, after moderate number of replacements, $\bar{\xi}_i$ of curve 3 (Figure 12(d)) decreases faster than the other two curves and finishes lower than curve 2 (Figure 12(d)) in the end.

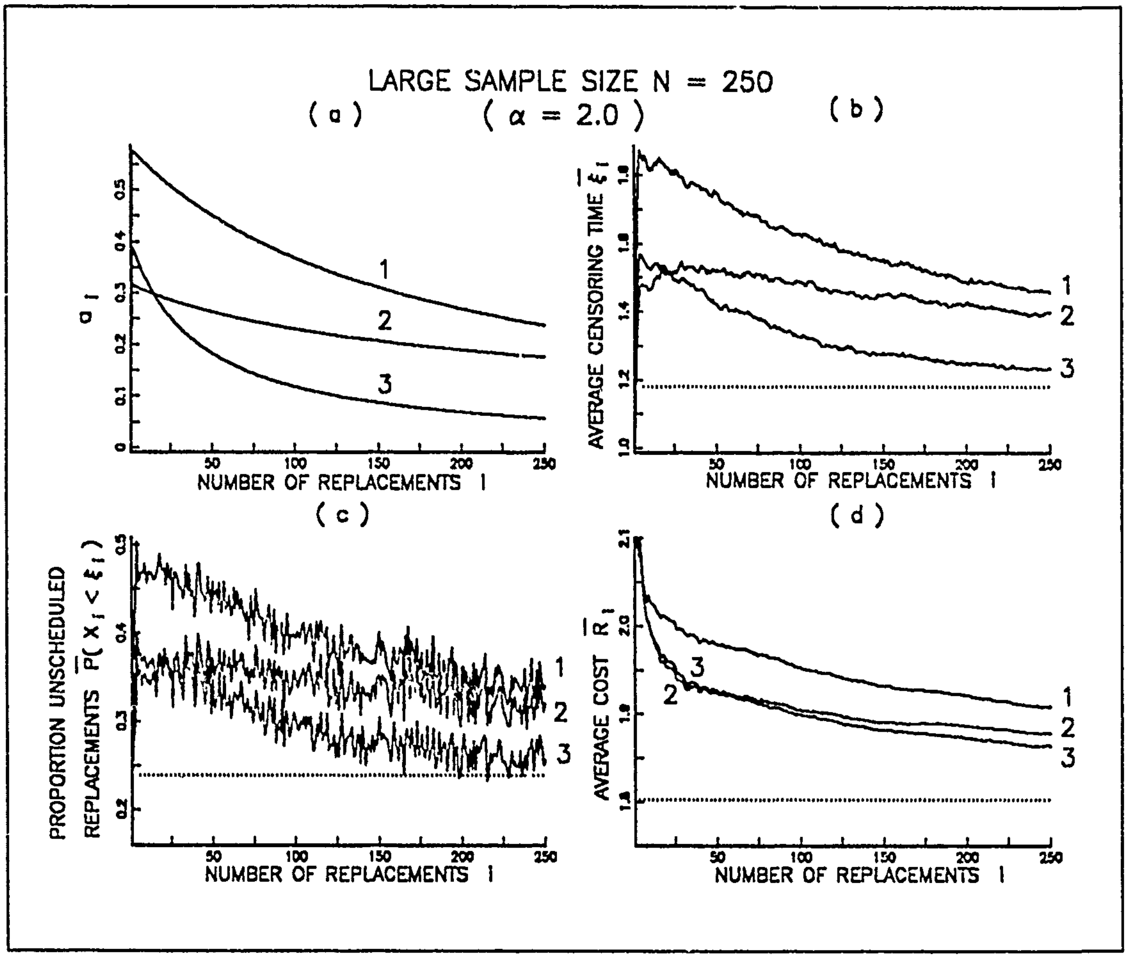

Figure 12. Plot of $\{a_n\}$ performance when N = 250 and $\alpha = 2.0$

2. Shape Parameter α Equals 1.6

Although the second and third $\{a_n\}$ sequences are fairly different (curves 2, 3 in Figure 13(a)), their average costs are about the same. Before these two $\{a_n\}$ sequences intersect, the proportion of uncensored observations (i.e., unscheduled replacements) is higher for the second $\{a_n\}$ sequence than the third (curves 2 and 3, Figure 13(c)). After intersection the second $\{a_n\}$ sequence has a higher proportion of unscheduled replacement. Thus after a moderate number of replacements, it would seem that about the same amount of information about the system's life distribution and is gathered using either $\{a_n\}$ sequence, so that we could expect the average costs to be about the same. However, after a large number of replacements, the variance of censoring times become small and the $\{a_n\}$ sequence yielding $\bar{\xi}_i$'s the closest to the optimal replacement time (curve 3) has the lowest average costs.

Although the third $\{a_n\}$ sequence (curve 3 in figure 12(d)) gives the lowest $\bar{R}_{250}$, our simulation results reveal that the variance of actual costs per unit time are larger than for the second $\{a_n\}$ sequence. Our policy is to choose the sequence with the smallest $\overline{MSE}$ as the best performer. Thus we conclude that the second $\{a_n\}$ sequence (curve 2), is the best performer.

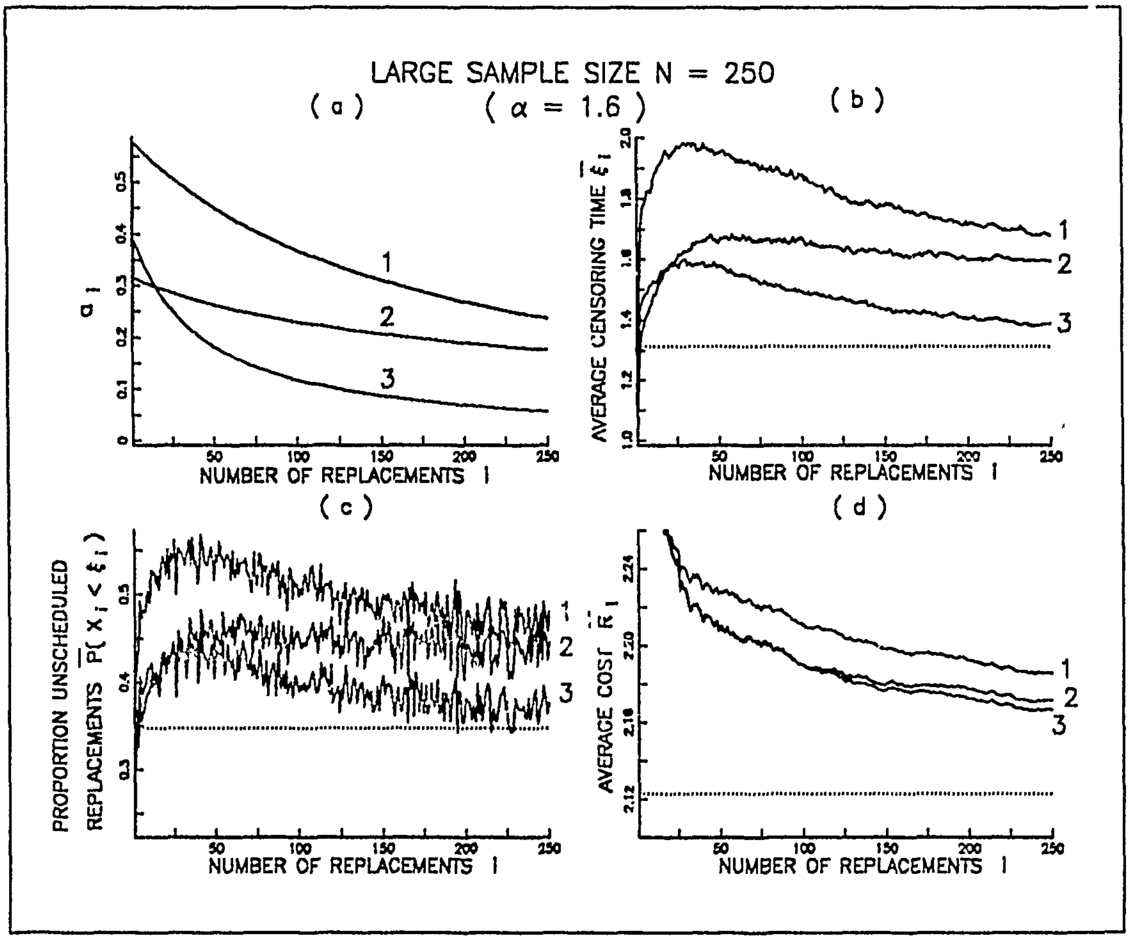

Figure 13. Plot of $\{a_n\}$ performance when N = 250 and α = 1.6

3. Shape Parameter α Equals 1.3

Since the second and third $\{a_n\}$ sequences (curves 2 and 3 in Figure 14(b)) yield $\bar{\xi}_i < \phi^*$ for $i = 1, 2, ..., 250$, we expected the sequences with $\bar{\xi}_i$ closer to ϕ^* to have the lowest cost. This can be seen with curves 2 and 3, we see in Figure 14(b). The first sequence (Curve 1 in Figure 14(b)) yields $\bar{\xi}_i$'s closest to ϕ^*. Thus it is not surprising to find out that the first sequence (curve 1 in Figure 14(d)) has the lowest average cost and the smallest $\overline{\text{MSE}}$.

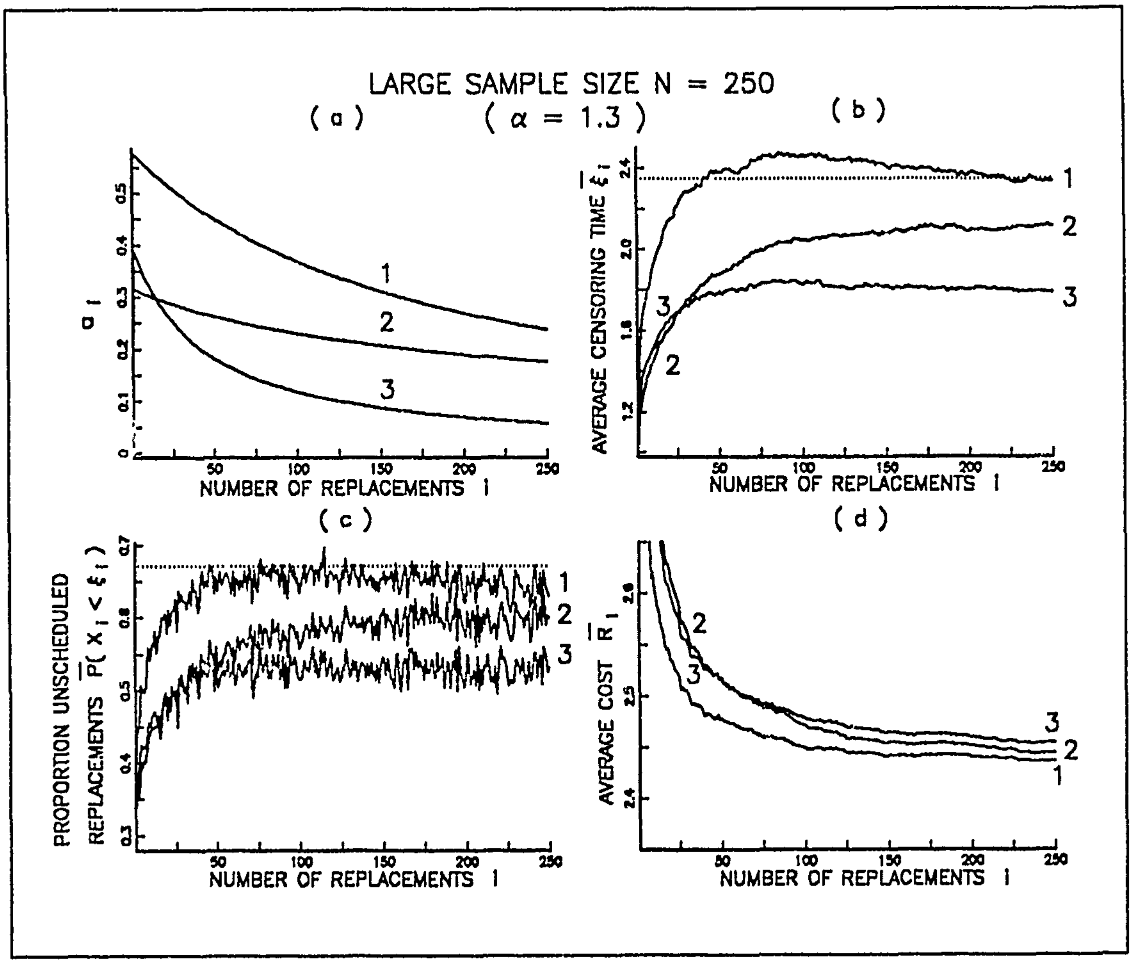

Figure 14. Plot of $\{a_n\}$ performance when N=250 and $\alpha = 1.3$

D. SAMPLE SIZE CHANGES

In the last section for each fixed sample size small, moderate and large, we notice that as the shape parameter α decreases from 2.0 to 1.1, the best performing $\{a_n\}$ sequences tend to have larger values. This means that when the system's life distribution is close to the exponential distribution, the sequential estimation procedure needs to use $\{a_n\}$ sequences with larger values to get smaller actual costs per unit time. Now we examine how the $\{a_n\}$ sequences perform as sample size varies for each fix α. We picked some of the best performing $\{a_n\}$ sequences from Tables 7, 8 and 9, and plotted these $\{a_n\}$ sequences in Figure 15. The short curves are the best performing $\{a_n\}$ sequences for small sample size $N = 10$, the moderate curves are the best performing $\{a_n\}$ sequences for moderate sample size $N = 50$ and the long curves are the best performing $\{a_n\}$ sequences for large sample size $N = 250$.

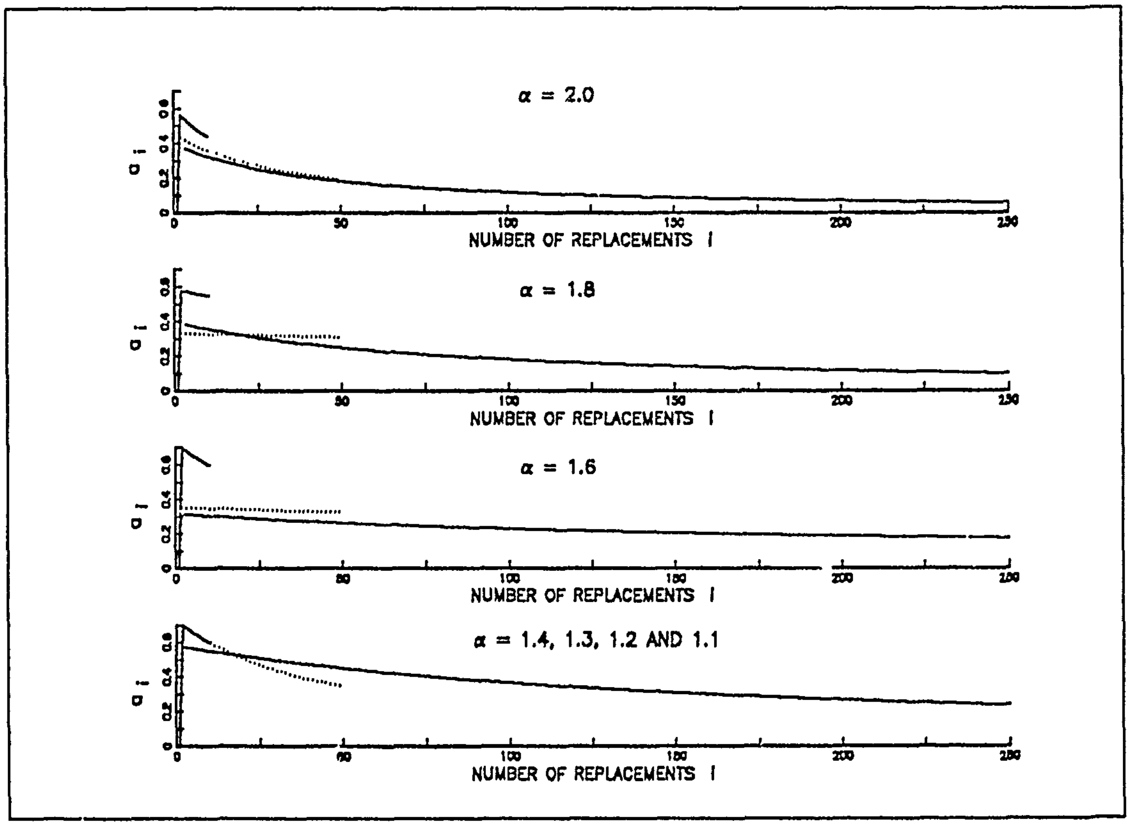

Figure 15. Comparison of best performing $\{a_n\}$ when α is fixed

In the bottom plot of Figure 15, the $\{a_n\}$ sequence were the same for $\alpha = 1.4, 1.3,$ 1.2 and 1.1. We already know that as the system's life distribution approaches the exponential distribution (i.e., as the shape parameter α decreases to 1.0) the sequential estimation procedure performs best with $\{a_n\}$ sequences with larger values. Thus the condition that four different α values have the same best performing sequences, is due to the fact that $\{a_n\}$ sequences with large enough values were not tried in the simulation.

From Figure 15, for each α, the best $\{a_n\}$ sequences for $N = 10$ are larger than the best sequences for $N = 250$. However best $\{a_n\}$ sequences for $N = 50$ sometimes intersect with the best $\{a_n\}$ sequences for $N = 250$. Generally speaking, the relationship between three best $\{a_n\}$ sequences is not consistent for different α values. In early stages of sampling before much data been collected, the variation of censoring ages is very large. From Figure 3 on page 15, if the censoring time is less than the optimal replacement time $\phi^\star$, then the censoring time will be on the steeper side of the curve. This means the censoring time will have a large expected average cost. Thus we would rather have an $\{a_n\}$ sequence with large values so that the ξ_i 's tend to be larger than $\phi^\star$ to reduce the risk of higher costs when $\xi_i < \phi^\star$.

When i becomes large, we have many observations with more information about the system's life distribution which can be used to better estimate $\phi^\star$, thereby reducing the variance of the ξ_i's. Thus the $\{a_n\}$ sequences giving average censoring times $\bar{\xi}_i$ closest to the optimal replacement time $\phi^\star$ performed best. In general $\{a_n\}$ needs to decrease as n increases. However for distributions that are closer to exponential, $\phi^\star$ is larger and the estimators $\phi_i^\star$ tend to be less than $\phi^\star$, we need $\{a_n\}$ sequences with large values that decrease slowly. For large α values, $\phi_i^\star$ tends to overestimate the optimal replacement time, thus the $\{a_n\}$ sequence which drop to zero faster or have light tails, achieves lower cost.

E. COST RATIO CHANGES

1. Optimal Replacement Time

We recall that the expected long run costs per unit time $R(t)$ is dependent on the unscheduled and scheduled maintenance cost, the system's life distribution and the scheduled censoring time t:

$$R(t) = \frac{C_1 \times F(t) + C_2 \times S(t)}{\int_0^t S(x)\,dx} ,$$

where C_1 and C_2 are unscheduled and scheduled replacement cost respectively. By expressing $R(t)$ as

$$R(t) = C_2 \times \left(\frac{\frac{C_1}{C_2} \cdot F(t) + S(T)}{\int_0^t S(x)\,dx} \right),$$

we see that the optimal replacement time $\phi^{\star}$ is a function of cost ratio $\frac{C_1}{C_2}$. Thus for a given system life distribution, as long as the ratio is constant, then $\phi^{\star}$ is the same no matter how large C_1 and C_2 are. Thus we can study the effect of different costs on the sequential estimation procedure by changing the cost ratio. Table 10 shows optimal replacement times for several cost ratios. These were found using the same method used to construct Table 5 on page 16.

Table 10. COMPARISON OF OPTIMAL REPLACEMENT TIMES BY DIFFERENT COST RATIO WITH $E(X_i) = 2.0$

Shape Parameter α	Optimal Replacement Time ϕ^*			
	$C_1 = 2.0$ $C_2 = 1.0$	$C_1 = 5.0$ $C_2 = 1.0$	$C_1 = 8.0$ $C_2 = 1.0$	$C_1 = 10.0$ $C_2 = 1.0$
2.0	2.4638243	1.1792297	0.8876661	0.7378923
1.9	2.7862625	1.1846266	0.9107760	0.6948394
1.8	2.9159756	1.2031918	0.8641901	0.7974927
1.7	3.3440571	1.2661085	0.9595973	0.8142039
1.6	3.6384935	1.3114500	0.9868178	0.7605762
1.5	4.8073025	1.4643221	1.0388746	0.9282054
1.4	7.6937876	1.6853428	1.0966682	0.9166338
1.3	10.2272406	2.3491287	1.2848225	1.0260086
1.2	18.5055695	3.7472544	1.7184162	1.4291582
1.1	21.9620056	12.9815130	4.2228003	3.8461828

From Table 10 for each fixed ratio costs, the optimal replacement time ϕ^* increases as the shape parameter α decreases from 2.0.[2] From Table 10 we also see that for fixed α as the cost ratio $\frac{C_1}{C_2}$ increases, the optimal replacement time ϕ^* decreases. This means that when the unscheduled replacement cost is too large, we don't want to risk unscheduled replacement.

2 Note that in Table 10, when $C_1 = 8.0$, the optimal replacement time ϕ^* value for $\alpha = 1.8$ is less than the ϕ^* value for $\alpha = 1.9$. The same is true when $C_1 = 10.0$, $\alpha = 1.6$ and 1.4. This is caused by the specific series of Pseudo Random Numbers (PRN) used to simulate ϕ^*. When the seed for the PRN generator changes, the simulation results for these situations should change also.

2. Effect on Performance

Let $\alpha = 1.8$, the scheduled replacement cost $C_2 = 1.0$, and let the unscheduled replacement cost C_1 have different values 2.0, 5.0, 8.0 and 10.0. To examine the performance of the $\{a_n\}$ sequence for different $\frac{C_1}{C_2}$ ratios, we simulated the sequential estimation procedure with each of the $\{a_n\}$ sequences from Table 6 on page 18. Again, the best performing $\{a_n\}$ sequences were chosen. They are summarized in Table 11.

Table 11. BEST PERFORMING $\{a_n\}$ FOR SHAPE PARAMETER $\alpha = 1.8$

Unscheduled and Scheduled Replacement Cost	$\{a_n\}$ Functional Form		
	$N = 10$	$N = 50$	$N = 250$
$C_1 = 2.0$ $C_2 = 1.0$	$\frac{1}{1.400 + 0.03 \times (i-1)}$	$\frac{1}{1.400 + 0.03 \times (i-1)}$	$\frac{1}{1.725 + 0.01 \times (i-1)}$
$C_1 = 5.0$ $C_2 = 1.0$	$\frac{1}{1.725 + 0.01 \times (i-1)}$	$\frac{1}{(9.0 + 0.03 \times (i-1))^{0.5}}$	$\frac{1}{2.525 + 0.03 \times (i-1)}$
$C_1 = 8.0$ $C_2 = 1.0$	$\frac{1}{2.525 + 0.03 \times (i-1)}$	$\frac{1}{2.525 + 0.15 \times (i-1)}$	$\frac{1}{2.525 + 0.15 \times (i-1)}$
$C_1 = 10.0$ $C_2 = 1.0$	$\frac{1}{2.525 + 0.03 \times (i-1)}$	$\frac{1}{2.525 + 0.15 \times (i-1)}$	$\frac{1}{2.525 + 0.15 \times (i-1)}$

From Table 11, we observe that for $C_1 = 8$ and 10, the same $\{a_n\}$ sequences perform best for sample sizes $N = 10$, 50 and 250. This could be because the cost ratios $\frac{C_1}{C_2} = 8$ and 10 are close, or it might be caused by the fact that we did not test enough $\{a_n\}$ sequences.

We have also plotted the $\{a_n\}$ sequences from Table 11 in Figure 16, so that we can make a comparison. By comparing Table 11 and Figure 16, we observe that for fixed sample size as the cost ratio $\frac{C_1}{C_2}$ increases, the best performing $\{a_n\}$ sequence tends to become small. Thus the ith replacement's scheduled replacement time will tend to be small and the proportion of unscheduled replacements will decrease. This limits the probability of an unscheduled failure. This conclusion is consistent with the conclusions of the previous section.

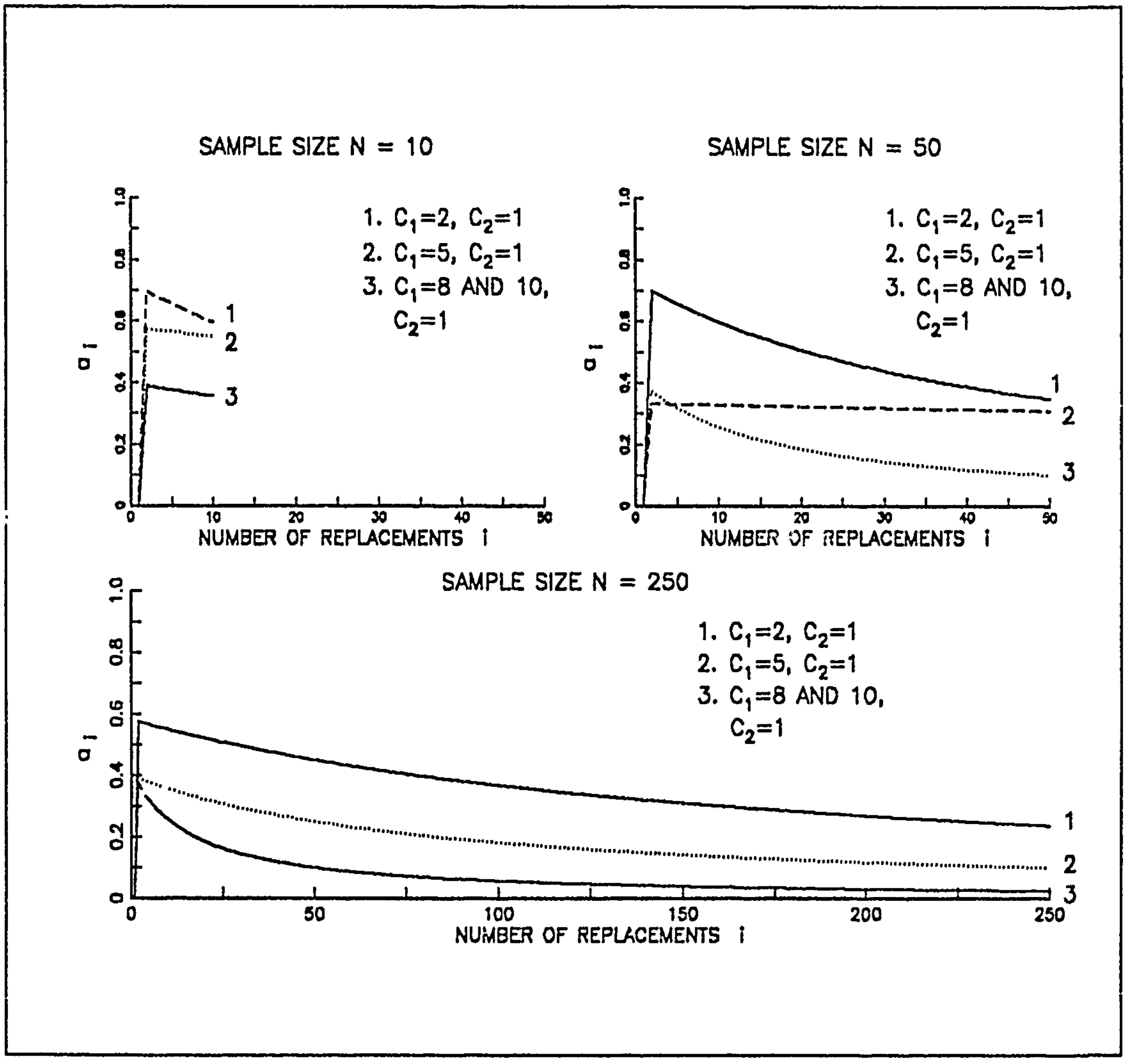

Figure 16. The $\{a_n\}$ sequence performances for $\alpha = 1.8$

F. COMPARISON WITH FREES & RUPPERT'S PROCEDURE

There have been two previous attempts to construct nonparametric sequential estimators of $\phi^{\star}$: Bather [Ref. 11: pp. 253-266] and Frees & Ruppert [Ref. 12: pp. 650-661]. Bather's procedure was not fully sequential in that periodically, the system was allowed to operate until failure regardless of how long it took. Frees and Ruppert use stochastic approximation techniques to construct a fully sequential estimator of $\phi^{\star}$ (i.e., $\xi_i < \infty$ for $i = 1, 2, ...$). In this section we compare the performance of the sequential procedure described in Chapter 3 with the performance of Frees and Ruppert's procedure. Because Frees and Ruppert do not estimate $\phi^{\star}$ directly, we can not compare the procedures using mean square error (MSE) of the estimator of $\phi^{\star}$. We can however compare $\overline{\text{MSE}}$, the squared difference of actual costs per unit time from $R(\phi^{\star})$ averaged over the repetitions of each simulation, for both procedures. Because the ultimate goal is to reduce costs, this is, in fact, the more meaningful comparison.

In their Monte-Carlo study, Frees and Ruppert took the unscheduled replacement cost $C_1 = 5.0$ and the scheduled replacement cost $C_2 = 1.0$. The system's life distribution is Weibull with shape parameter $\alpha = 2.2$ and scale parameter $\lambda = 0.5$. This distribution has expected value 1.7712 and standard deviation 0.8499. The optimal replacement time $\phi^{\star} = 0.99505$ and optimal cost $R(\phi^{\star}) = 1.904$. [Ref. 12: p. 656]

Frees and Ruppert update their estimator of $\phi^{\star}$ after every two observations. They chose to look at the performance of their estimator after 10, 50 and 250 updates to reflect small, moderate and large sample sizes respectively. Thus after 10, 50 and 250 updates, they used 20, 100 and 500 observations respectively. Table 12 on page 44 shows the performance of cost estimator from Frees and Ruppert's sequential estimation procedure.

In order to compare methods, we use the same simulation setting as Frees and Ruppert's, $C_1 = 5.0$, $C_2 = 1.0$, the system's life distribution is Weibull with shape parameter $\alpha = 2.2$ and scale parameter $\lambda = 0.5$, and we select a $\{a_n\}$ sequence of the functional form $\frac{1.5}{(n+50)^{0.7}}$. This $\{a_n\}$ sequence roughly corresponds to the sequence used by Frees and Ruppert. The performance of our cost estimator is shown in the Table 13.

Table 12. THE PERFORMANCE OF COST ESTIMATOR FROM FREES AND RUPPERT'S PROCEDURE

Number of updates	10	50	250
Number of observations	20	100	500
Average cost	2.268	2.159	2.053
Variance of average cost	0.1851	0.0408	0.0096
MSE of average cost	0.3176	0.1058	0.0318

Table 13. THE PERFORMANCE OF COST ESTIMATOR FROM OUR SEQUENTIAL ESTIMATION PROCEDURE

Number of updates	10	50	250
Number of observations	10	50	250
Average cost	2.168	2.005	1.952
Variance of average cost	0.6325	0.0971	0.0158
MSE of average cost	0.7022	0.1073	0.0181
Number of updates	**20**	**100**	**500**
Number of observations	20	100	500
Average cost	2.086	1.983	1.940
Variance of average cost	0.2874	0.0415	0.0087
MSE of average cost	0.3205	0.0477	0.0100

Recalling that Frees and Ruppert's need two observations for each update of their estimator of ϕ and we only need one observation to update our estimator of ϕ^*. The results of our simulation after 20, 100 and 500 observations are comparable to Frees and Ruppert's simulation with 10, 50 and 250 updates of ϕ^* respectively. Comparing Table 13 to Table 12 with the same number of updates, the variance of our simulated average costs are slightly higher than Frees and Ruppert's simulation results. However:

- Our average costs are much lower than Frees and Ruppert's average costs.
- For small sample size (20 observations), for this $\{a_n\}$ sequence , the $\overline{\text{MSE}}$ for both procedures are about the same. For large sample size the $\overline{\text{MSE}}$ of our procedure is smaller than that of Frees and Ruppert, even using half the number of observations.
- This choice of $\{a_n\}$ gives the best performance for Frees and Ruppert's procedure and was chosen knowing that the underlying distribution is Weibull ($\alpha = 2.2$, $\lambda = 0.5$). It is probable that a different choice of $\{a_n\}$ would yield better results for our sequential procedure.
- The intermediate estimators of Frees and Ruppert's sequential estimation procedure are complicated to calculate.

Thus we may conclude that our sequential estimation procedure for age replacement policies is competitive to Frees and Ruppert's procedure, and is in many ways better.

V. GRAPHICAL DETERMINATION OF THE SCHEDULED REPLACEMENT TIME

A graphical method can be used to estimate the optimal age replacement policy based on observing a sample of *i.i.d.* system lifetimes [Ref. 3: pp. 113-115]. This method obtains the estimate of $\phi^{\star}$ from a scaled total time on test plot. The advantage of this method is that you can see immediately how sensitive the estimate of $\phi^{\star}$ is to change in the cost ratio. This method can be extended to estimate $\phi^{\star}$ based on data from our sequential estimation procedure. In this chapter, we describe the graphic determination method for *i.i.d.* data and apply it to the 22 D9G-66A Caterpillar Tractor Engines (this data was used as an example in Chapter 2). We also indicate how this method can be extended to our sequential estimation procedure.

A. SCALED TOTAL TIME ON TEST PLOT

Plotting the data is often the first step to unlocking information contained in data about the underlying model. For example, a plot of the empirical distribution function contains information about the probability density. If the empirical distribution looks concave, then the density may be decreasing. However, this plot is scale dependent so that the perceived shape of the plot depends on the choice of plotting scale.

A total time on test plot provides information about the failure rate. Analyzing failure data, it is often the failure rate which is of chief interest. For example, if the failure rate is constant or decreasing, we know that we should not adopt a scheduled replacement policy, since an old system in this case is actually "better" than a new system. Consider an experiment where n systems system lifetimes $X_1, X_2, ..., X_n$ are observed. Let $n(u)$ be the number of systems that survive to age u, then the total time on test to age x is defined to be

$$T(x) = \int_0^x n(u)\, du\,.$$

Let $X_{(1)} \leq X_{(2)} \leq .... \leq X_{(n)}$, be the order statistics of $X_1, X_2, ..., X_n$, then

$$T(X_{(i)}) = \int_0^{X_{(i)}} n(u)\, du$$

$$= nX_{(1)} + (n-1)(X_{(2)} - X_{(1)}) + \;....\; + (n-i+1)(X_{(i)} - X_{(i-1)})$$

is the total time on test to age $X_{(i)}$. We call

$$\frac{T(x)}{T(X_{(n)})}$$

the scaled total time on test at age x. A plot of $\frac{T(X_{(i)})}{T(X_{(n)})}$ versus the empirical distribution $F_n(X_{(i)}) = \frac{i}{n}$ for $i = 1, 2, ..., n$ is called the scaled total time on test plot. This plot provides information about the failure rate. If the total time on test plot is strongly concave, then there is evidence that the underlying distribution is IFR and a scheduled age replacement policy makes sense.

B. OBTAINING THE AGE REPLACEMENT TIME FROM THE PLOT

Let C_1 be the cost (in dollars) of an in-service tractor engine failure replacement and C_2 be the cost (in dollars) of a scheduled tractor engine replacement, where $C_1 > C_2$. After 22 observations, ϕ^* can be estimated from the tractor engine failure ages using the procedure described in Chapter 2 with $a_i = \xi_i = \infty$, $i = 1, 2, ..., 22$. It can also be estimated using the scaled total time on test plot. Table 14 on page 48 shows the ordered actual failure age of 22 D9G-66A Caterpillar Tractor Engines, and the empirical distribution of tractor engine failures is plotted in Figure 17 on page 49.

To estimate ϕ^* plot the point $\frac{-C_2}{(C_1 - C_2)}$ on the x-axis and draw a tangent line to the scaled total time on test plot through this point. Then drop down to the x-axis from

Table 14. THE ORDERED AGE OF FAILURE TRACTOR ENGINES

Order sequence of failure age	Tractor's number	Date of failure	Age in hours when failure occurred
1	10	5-8-67	2690
2	20	7-30-71	3286
3	3	10-11-71	3826
4	8	8-8-69	4394
5	21	1-21-72	4815
6	2	4-16-70	5085
7	16	4-15-65	5161
8	12	6-28-65	5278
9	19	4-3-67	5556
10	5	6-1-70	6052
11	22	10-31-69	6150
12	11	3-26-68	6259
13	6	6-7-71	6367
14	13	8-1-66	6378
15	14	9-14-66	6385
16	15	8-3-66	6578
17	17	10-26-66	6717
18	18	11-1-67	6869
19	7	8-10-70	7774
20	1	6-16-71	8230
21	9	9-21-70	10517
22	4	5-8-72	10950

the point of tangency closest to the value $\frac{i_o}{n}$ when $i_o = 1, 2, ..., n$. Then $X_{(i_o)}$ will be the estimated optimal replacement age. A straight forward geometric argument shows that estimating ϕ^* from the scaled total time on test plot is identical to the estimate obtained by minimizing $\hat{R}(T)$ [Ref. 3: p. 114].

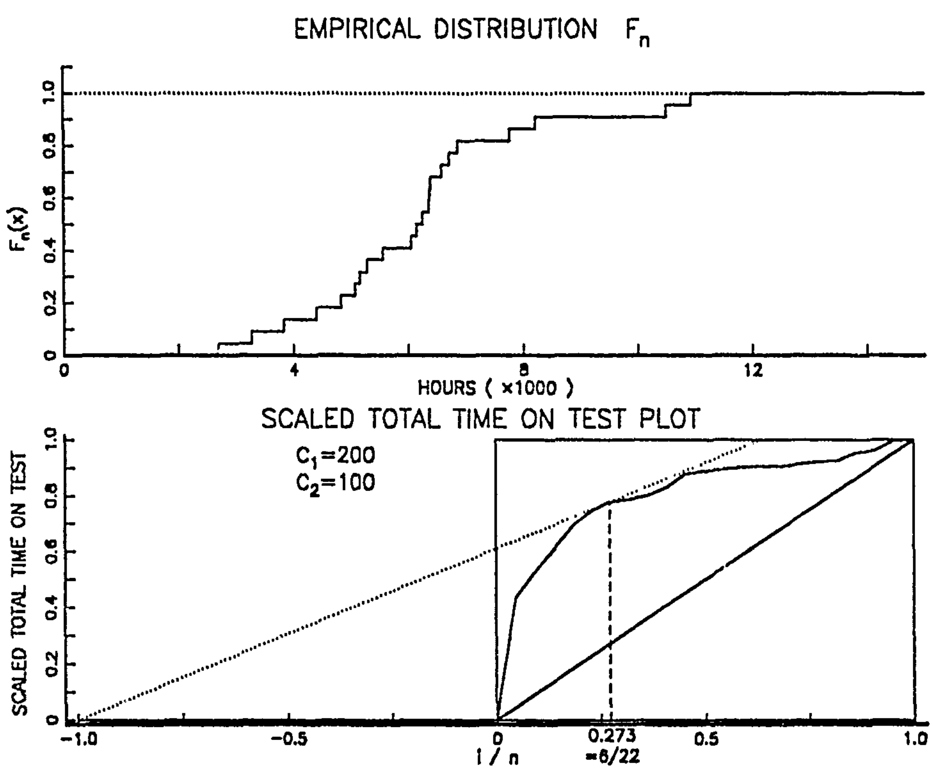

Figure 17. The empirical distribution and the total time on test plot for Caterpillar Tractor Engine data

C. THE TOTAL TIME ON TEST PLOT OF TRACTOR FAILURE ENGINES

The bottom plot in Figure 17 is the scaled total time on test plot when the unscheduled replacement cost $C_1 = 200$ and the scheduled replacement cost $C_2 = 100$. On the horizontal axis we can find the point $\frac{-C_2}{(C_1 - C_2)} = -1.0$. We draw a tangent line to the scaled total time on test plot through this point and drop down to the x-axis from the point of tangency, we find the value $\frac{i_0}{n} \simeq 0.273 \simeq \frac{6}{22}$. Then the order statistic $X_{(6)}$ (i.e., 5085 hours), is the estimated optimal replacement time. Changing the unscheduled replacement cost C_1 to 400 and 700, and holding the scheduled replacement cost C_2 at

100, from the scaled total time on test plots in Figure 18 on page 50, we see that the estimated optimal replacement age is 4394 hours and 2690 hours respectively. The results obtained from this figure are consistent with the optimal replacement ages estimated using the sequential estimation procedures in Table 4 on page 11. The value of the estimator using the scaled total time on test plot depends upon how closely the estimated distribution from the sample approximates the true underlying life distribution.

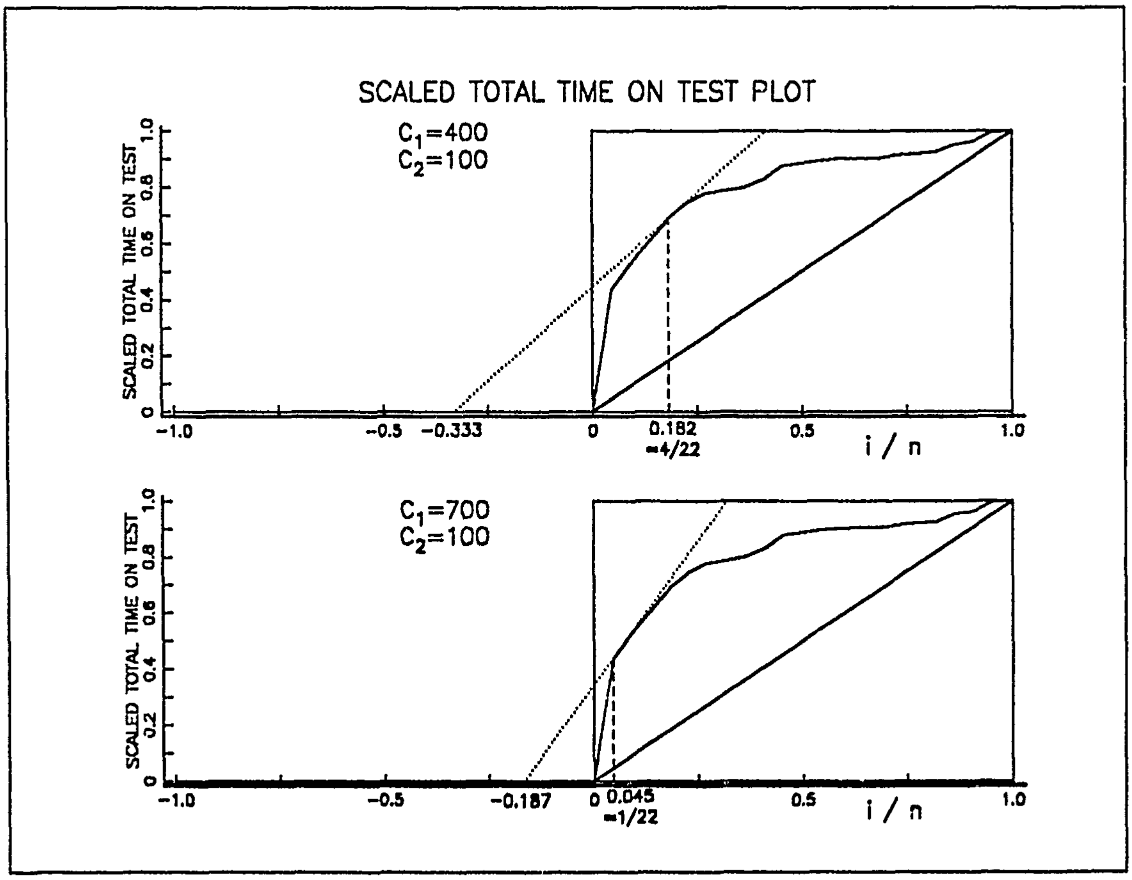

Figure 18. The total time on test plot for Caterpillar Tractor Engine data

We note that this procedure can be adapted to the sequential estimation procedure [Ref. 8] by replacing the empirical distribution as an estimator of F, by the product-limit estimator of F described in Chapter 2 and defining

$$\frac{T(x)}{T(Z_{(n)})} = \frac{\int_0^x \hat{S}(u)\, du}{\int_0^{Z_{(n)}} \hat{S}(u)\, du}$$

where

$$\hat{S}(u) = \prod_{\{i:\, Z_{(i)} \le u\}} \left(\frac{n-1}{n-i+1} \right)^{\delta_{(i)}}$$

is the product-limit estimator of $S \equiv 1 - F$. Then plot $\frac{T(Z_{(i)})}{T(Z_{(n)})}$ versus $\hat{F}(Z_{(i)}) = 1 - \hat{S}(Z_{(i)})$ to get the scaled "total time on test" plot. Following the same steps outlined for the *i.i.d.* case, plot $\frac{-C_2}{C_1 - C_2}$ on the x-axis and draw the tangent line to the scaled total time on test plot through the point $\frac{-C_2}{C_1 - C_2}$. The estimator of ϕ^* is $\hat{F}^{-1}(x_o)$ where x_o is the value on the x-axis, below the point where the tangent line intersects the scaled total time on test plot.

VI. CONCLUSIONS AND RECOMMENDATIONS

Under an age replacement policy, a system is replaced at failure or after being in service for t units of time, whichever comes first. The time t is called the scheduled censoring time or the scheduled replacement time. An is optimal replacement time $\phi^{\star}$ achieves the smallest long run expected cost. An important problem is the estimation of $\phi^{\star}$ when the form of the system's underlying life distribution is unknown. We show by example and through simulation that substantial cost savings can be effected using the sequential estimation procedure for $\phi^{\star}$described in Chapter 2.

An important part of our analysis is to provide practical guidelines for choosing $\xi_i = \phi^{\star}_{i-1} + a_i$, the scheduled replacement time to be used for the ith system, where $\phi^{\star}_{i-1}$ is the estimator of $\phi^{\star}$ based on $(i-1)$ replacements, and $\{a_n\}$ is a fixed sequence of constants. From our analysis, in the early stages of sampling we would rather choose large values for a_i so that ξ_i tends to be greater than $\phi^{\star}$. If the scheduled replacement times ξ_i are too small in the early stages, then most of the system life times are censored. Thus, very little information is collected about the underlying distribution. This leads to poor estimates for $\phi^{\star}$. After a large number of replacements, the estimates of the underlying distribution and therefore $\phi^{\star}$ are relatively good. A good age replacement policy will keep the scheduled censoring time closely approximate the optimal replacement time $\phi^{\star}$. Therefore the best performing $\{a_n\}$ sequences have smaller a_i for large i.

The proper choice of $\{a_n\}$ sequence is a very difficult. The best choice of an $\{a_n\}$ sequence for sequential estimation of age replacement policy depends on the system's underlying life distribution which is generally unknown, the costs C_1, C_2, and the sample size N at which sampling stops. From our simulation, we see that when the underlying distribution is close to exponential the $\{a_n\}$ sequences should have larger values and should not decrease as fast as those whose failure rate increases more rapidly. For large sample sizes $\{a_n\}$ sequences should decrease faster in early stages than small sample sizes. In addition as the cost ratio $\frac{C_1}{C_2}$ increases, the best performing $\{a_n\}$ sequence have smaller values.

Future research might be concerned with the following subjects:

- Sequential estimation when the underlying life distribution F comes from a parametric family of distributions.
- How to use the data to estimate the $\{a_n\}$ sequence.
- If an imperfect repair model is permitted, then the next failure depends on system history, failure distributions are not independently and identically distributed (*i.i.d.*), how to change our Sequential Estimation Procedure to fit new situation.
- Minimizing long run expected costs per unit time gives the same optimal ϕ^* for $C_1 = 2$, $C_2 = 1$ and costs for $C_1 = 2{,}000{,}000$, $C_2 = 1{,}000{,}000$. Clearly minimizing long run expected costs is not appropriate under all circumstances. Other cost functions need to be considered.

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