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NASA Contractor Report 182085  
ICASE Report No. 90-57

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FOR PARTIAL DIFFERENTIAL EQUATIONS

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Contract No. NAS1-18605  
August 1990

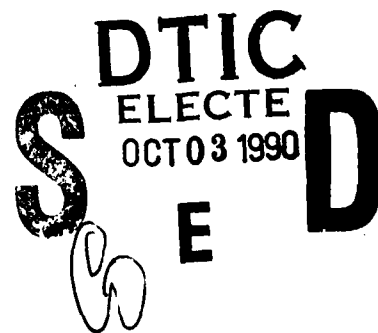
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# A Hybrid Perturbation-Galerkin Technique for Partial Differential Equations

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| Classification     |                                     |
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## ABSTRACT

A two-step hybrid perturbation-Galerkin technique for improving the usefulness of perturbation solutions to partial differential equations which contain a parameter is presented and discussed. In the first step of the method, the leading terms in the asymptotic expansion(s) of the solution about one or more values of the perturbation parameter are obtained using standard perturbation methods. In the second step, the perturbation functions obtained in the first step are used as trial functions in a Bubnov-Galerkin approximation. This semi-analytical, semi-numerical hybrid technique appears to overcome some of the drawbacks of the perturbation and Galerkin methods when they are applied by themselves, while combining some of the good features of each. The technique is illustrated first by a simple example. It is then applied to the problem of determining the flow of a slightly compressible fluid past a circular cylinder and to the problem of determining the shape of a free surface due to a sink above the surface. Solutions obtained by the hybrid method are compared with other approximate solutions, and its possible application to certain problems associated with domain decomposition is discussed.

\*Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

## 1. Introduction

A two-step hybrid analysis technique, which combines perturbation techniques with the Galerkin method, has been presented and discussed by the authors. It was applied to some singular perturbation problems in slender body theory [5], as well as to several classes of problems involving ordinary differential equations [2,6,7]. That technique also promises to be useful in the analysis of a very wide variety of partial differential equation type problems as well. In this paper we apply the method to some problems involving several independent variables. In particular, we demonstrate its usefulness by applying it to a linear boundary value problem, a nonlinear boundary value problem, and to a (nonlinear) free boundary value problem.

The method is based upon a hybrid technique which was apparently first studied by Ahmed K. Noor and collaborators in conjunction with the finite element analysis of geometrically nonlinear problems in structural mechanics (see Geer and Andersen [5] for several references). The Galerkin method has, of course, been known and used for a long time. However, a principle problem associated with its successful application lies in the choice of appropriate basis functions. In a series of papers Noor and his collaborators have shown for a variety of structural mechanics problems that the first few terms in a Taylor series expansion of the solution of a parameterized system of discretized equations can be particularly effective as Galerkin trial functions (or basis vectors). Subsequently, the present authors [2,5-7] have shown that the terms in either a regular or a singular perturbation expansion of the solution are also effective trial functions. In particular, we have demonstrated that the "reduced-basis" solutions can be useful for significantly larger values of the expansion parameter than the Taylor series or singular perturbation solutions on which they are based. A treatment of the reduced basis method from a mathematical point of view is given by Fink and Rheinbolt [3].

Some general observations about the technique are the following: 1) In many perturbation problems, much effort has to be expended to compute (analytically) each additional term in a perturbation expansion. Through the use of the proposed hybrid method, the known perturbation terms can be exploited more fully. 2) Another way of viewing the technique is to recognize that in many perturbation expansions the functional form of the higher-order terms can be well approximated by a linear combination of the lower-order terms. Thus, much of the effect of the higher order terms may be included by applying the reduced basis technique to a small number of lower order terms. 3) Since it is often possible to develop formal perturbation expansions for the solution about two or more values of the parameter, e.g. for small or large values of the parameter, the proposed method is a convenient way of combining the information contained in these different expansions. This is accomplished by allowing the

set of basis functions to include terms from the different perturbation expansions, which may be either regular or singular expansions (see [7] for several examples). 4) While the use of a Taylor series expansion is frequently limited by a finite radius of convergence, the proposed hybrid method can sometimes yield good results even well outside the radius of convergence (see [2] as well as our first example here), or even past a real singularity in the parameter [6]. In fact, when information from two or more expansions is employed, the method appears to provide meaningful (and often very accurate) approximations in "intermediate" regions of parameter values, e.g., in regions where the parameter is neither "small" nor "large" (see especially Geer and Andersen [7]). 5) The same general technique may also be applied in a fully numerical sense in that the perturbation functions themselves may be computed in discretized form. 6) However the perturbation functions are computed, the hybrid solutions share with the perturbation solutions the property that the accuracy is greatest near the expansion points and diminishes as the parameter moves away from these points. Usually a good indicator of accuracy is found by examining the difference in the hybrid solutions based on  $N - 1$  terms and  $N$  terms. Thus, it is possible to determine the range of parameter values for which the  $N$ -term expansion is valid.

In the following section, we describe the method in more detail and then apply it to a simple linear boundary value problem in Section 3. In Section 4, we apply the method to the problem of determining the flow of a slightly compressible fluid past a circular cylinder, while in Section 5 we apply it to the problem of determining the shape of a free surface induced by the presence of a sink above the surface. Each of the problems in sections 3-5 involves an elliptic boundary value problem, yet each serves as a model problem for a much wider class of problems. The problem of Section 3 is linear, and an arbitrary number of terms in its perturbation solution can be obtained in a straightforward manner. Thus, while an exact analytical solution is not known, it is possible to perform a rather detailed analysis of both the perturbation and hybrid results, including some explicit expressions for some of the hybrid solutions. In this problem the radius of convergence of the perturbation solution is finite and is limited by singularities which lie on the imaginary axis of the complex parameter plane. The problem of Section 4 involves a nonlinear partial differential equation, whose perturbation solution is tedious to obtain. Its radius of convergence is finite and appears to be limited by singularities which have real physical significance. The perturbation solution of the problem of Section 5 is the most difficult of the three problems to compute and is only an asymptotic (i.e., a nonconvergent) expansion [13]. For each of these problems we demonstrate that the hybrid method provides a much more useful solution than the perturbation solution alone. Some general comments about the method and a discussion of its possible application to certain problems associated with domain decomposition are discussed in Section 6.

## 2. Description of the Method

The method we wish to describe is a two-step hybrid analysis technique based on the successive use of perturbation expansion methods and the classical Bubnov-Galerkin approximation technique. To illustrate the general ideas of the method, suppose we are seeking (an approximation to) the solution  $u$  to the problem

$$(2.1) \quad \mathcal{L}(u, \varepsilon) = 0,$$

where  $\mathcal{L}$  is some partial differential operator and  $\varepsilon$  is a parameter. (Although we restrict our attention to two-dimensional elliptic boundary value problems in this paper, we believe that the method has a much wider range of applicability. Hence, we formulate the method in terms of a general operator  $\mathcal{L}$ .) Here (2.1) holds in some domain  $\mathcal{D}$ , and, in addition,  $u$  must satisfy certain conditions on the boundary of  $\mathcal{D}$ , which we denote by  $\partial\mathcal{D}$ . Without loss of generality, we can assume that these boundary conditions are homogeneous in  $u$ .

In the first step of the method, we generate the coordinate functions in a perturbation expansion of  $u$  about one or more specific values of the parameter  $\varepsilon$ , say about  $\varepsilon = \varepsilon_p$ ,  $p = 1, 2, \dots, P$ . In the second step, we construct new approximate solutions consisting of sums of some of these perturbation coordinate functions, each multiplied by an unknown amplitude, and then determine these amplitudes by using the Bubnov-Galerkin method.

To describe this idea in more detail, suppose that the solution to (2.1) can be expanded about  $\varepsilon = \varepsilon_p$  into a series of the form

$$(2.2) \quad u = \sum_{j=1}^{N_p} u^{(p,j)} \alpha_j^p(\varepsilon) + O(\alpha_{N_p+1}^p(\varepsilon)),$$

where  $\{\alpha_j^p(\varepsilon)\}$  is an appropriate asymptotic sequence of gauge functions and each  $u^{(p,j)}$  can be determined completely by standard perturbation methods (see, for example, Nayfeh [8]). By our assumption on the boundary conditions on  $u$ , each  $u^{(p,j)}$  satisfies homogeneous boundary conditions on  $\partial\mathcal{D}$ .

A subset of all of the perturbation functions  $\{u^{(p,j)}\}$  is now chosen as the set of coordinate functions for the Bubnov-Galerkin technique and approximations  $\tilde{u}$  for  $u$  are sought in the form

$$(2.3) \quad \tilde{u} = \sum_{j=1}^N u^{(j)} \delta_{j,N}(\varepsilon),$$

where the (unknown) parameters  $\{\delta_{j,N}(\varepsilon)\}$  represent the amplitudes of the coordinate functions  $u^{(j)}$ . Here each  $u^{(j)}$  is one of the perturbation coordinate functions  $u^{(p,j)}$ , and hence  $\tilde{u}$  satisfies the boundary conditions of the problem for any choice of amplitudes  $\{\delta_{j,N}\}$ . To

determine these amplitudes, we apply the Bubnov-Galerkin technique to the governing equation (2.1). Thus, we substitute (2.3) into (2.1) and require that the residual be orthogonal to the  $N$  coordinate functions over the domain  $\mathcal{D}$ , i.e.,

$$(2.4) \quad \int \int_{\mathcal{D}} \mathcal{L} \left( \sum_{j=1}^N u^{(j)} \delta_{j,N}(\varepsilon), \varepsilon \right) u^{(k)} dx = 0, \quad 1 \leq k \leq N.$$

Equations (2.4) represent a set of  $N$  equations for the  $N$  unknown amplitudes. While (2.4) must, in general, be solved numerically, solving it is much simpler than numerically solving (2.1). In particular, for a fixed value of  $\varepsilon$ , the solution to (2.4) is a point in  $N$ -dimensional space, where  $N$  is reasonably small, while the solution of (2.1) is a continuous function of the variables  $x$ .

We should note that this particular choice of coordinate functions overcomes the main drawback of the Bubnov-Galerkin method, which is the difficulty, from a practical point of view, of selecting a small number of good coordinate functions. By the way they are constructed, the perturbation coordinate functions are (under certain assumptions) elements of a set of functions which span the space of solutions in a neighborhood of their point of generation. Thus, they characterize the solution  $u$  in that neighborhood. We also observe that, in many applications, the functions  $u^{(p,i)}$  are determined by solving a set of linear equations, even though the original operator  $\mathcal{L}$  may be nonlinear.

For the three applications discussed in this paper we use  $P = 1$  only.

### 3. A Simple Example

To illustrate the basic features of the hybrid method, we consider the following simple two-dimensional example. We define  $u(x, y, \varepsilon)$  as the solution to the problem

$$(3.1) \quad \nabla^2 u + \varepsilon \sin(x) \cos(y) u_x + 2\varepsilon \sin(x) \sin(y) = 0, \quad (x, y) \in \mathcal{D},$$

$$(3.2) \quad \text{with } u = 0 \quad \text{for } (x, y) \in \partial\mathcal{D}.$$

Here  $\mathcal{D} = \{(x, y) : 0 \leq x, y \leq \pi\}$  and  $\partial\mathcal{D}$  denotes the boundary of  $\mathcal{D}$ . In (3.1), the symbol  $\nabla^2$  denotes the usual two-dimensional Laplacian operator, and the subscript  $x$  denotes differentiation with respect to  $x$ . The solution seems to be positive over the whole domain  $\mathcal{D}$  for positive values of  $\varepsilon$ . It is invariant under the  $180^\circ$  rotation  $x \rightarrow \pi - x$ ,  $y \rightarrow \pi - y$ . Equation (3.1) is also invariant under the transformation  $\varepsilon \rightarrow -\varepsilon$  and  $u(x, y) \rightarrow -u(x, \pi - y)$ . Thus, we may focus our attention on solutions for positive  $\varepsilon$ . Figure 1 shows surface and contour plots of the solution for three values of  $\varepsilon$ . For large values of  $\varepsilon$ , the solution exhibits a number of boundary layer properties (see Section 6).

Step One: For small values of  $\varepsilon$ , we construct a regular perturbation expansion of  $u$  in the form

$$(3.3) \quad u = \sum_{j=1}^N \varepsilon^j u^{(j)}(x, y) + O(\varepsilon^{N+1}),$$

where each of the perturbation coefficient functions  $u^{(j)}$  is independent of  $\varepsilon$ . To determine these functions, we substitute (3.3) into (3.1) and (3.2) and equate the coefficients of like powers of  $\varepsilon$ . In this way, we are led to the following system of equations, from which the  $u^{(j)}$  can be determined recursively.

$$(3.4) \quad \nabla^2 u^{(j)} = \begin{cases} -2 \sin(x) \sin(y) & \text{if } j = 1 \\ -\sin(x) \cos(y) u_x^{(j-1)} & \text{if } 2 \leq j \leq N, \end{cases}$$

$$(3.5) \quad u^{(j)} = 0 \quad \text{for } x = 0, \pi; \quad \text{and} \quad y = 0, \pi \quad \text{for } 1 \leq j \leq N.$$

In particular, using (3.4) and (3.5) we find

$$(3.6) \quad \begin{aligned} u^{(1)} &= \sin(x) \sin(y), \quad u^{(2)} = \left(\frac{1}{32}\right) \sin(2x) \sin(2y), \\ u^{(3)} &= \left(\frac{1}{128}\right) \left[ \left(\frac{1}{9}\right) \sin(3x) \sin(3y) + \left(\frac{1}{5}\right) \sin(3x) \sin(y) \right. \\ &\quad \left. - \left(\frac{1}{5}\right) \sin(x) \sin(3y) - \sin(x) \sin(y) \right], \\ u^{(4)} &= \left(\frac{1}{49152}\right) \sin(4x) \sin(4y) + \left(\frac{7}{76800}\right) \sin(4x) \sin(2y) \\ &\quad - \left(\frac{1}{19200}\right) \sin(2x) \sin(4y) - \left(\frac{1}{1920}\right) \sin(2x) \sin(2y). \end{aligned}$$

It is easy to determine the general form of each  $u^{(k)}$  and to construct a recurrence relation for the numerical coefficients involved. Thus, many more terms in the expansion can be computed in a straightforward manner.

Step Two: In the second step of the hybrid method, we use the perturbation coordinate functions  $\{u^{(j)}\}$  obtained in Step One to construct new approximate solutions  $\tilde{u}$  in the form

$$(3.7) \quad \tilde{u} = \sum_{n=1}^N \delta_{j,N}(\varepsilon) u^{(j)}(x, y),$$

where  $N$  is a relatively small integer and the new amplitudes  $\{\delta_{j,N}\}$  will be determined using the Galerkin technique. (We note that the  $\tilde{u}$  defined by (3.7) satisfy the boundary condition

(3.2) for any choice of the  $\{\delta_{j,N}\}$  because of the conditions (3.5) on the functions  $\{u^{(j)}\}$ . The procedure is as follows. We substitute (3.7) into (3.1) and require the residual to be orthogonal to each of the perturbation coordinate functions, i.e.,

$$\int \int_{\mathcal{D}} [\nabla^2 \tilde{u} + \varepsilon \sin(x) \cos(y) \tilde{u}_x + 2\varepsilon \sin(x) \sin(y)] u^{(k)} dx dy = 0,$$

which may be written in the form

$$(3.8) \quad \sum_{j=1}^N [c_{k,j} + \varepsilon d_{k,j}] \delta_{j,N}(\varepsilon) = \varepsilon b_j, \quad 1 \leq k \leq N,$$

where

$$(3.9) \quad c_{k,j} = \int \int_{\mathcal{D}} [\nabla^2 u^{(j)}] u^{(k)} dx dy, \quad d_{k,j} = \int \int_{\mathcal{D}} [\sin(x) \cos(y) u_x^{(j)}] u^{(k)} dx dy,$$

$$b_j = - \int \int_{\mathcal{D}} 2 \sin(x) \sin(y) u^{(k)} dx dy.$$

For this simple example, since the original problem is linear, the equations (3.8) to determine the  $\{\delta_{j,N}\}$  are linear. Further, since the integrals in (3.9) can be evaluated exactly, it is possible in this case to evaluate the amplitudes as rational functions of  $\varepsilon$ . However, we remark that for most applications (especially nonlinear ones) the evaluation of the amplitudes must be carried out numerically, typically by an iterative process for gradually increasing values of the parameter  $\varepsilon$ .

For  $N = 2$ , we use the expressions (3.6) to evaluate the coefficients  $\{c_{k,j}\}$ ,  $\{d_{k,j}\}$  and  $\{b_j\}$  explicitly and find

$$(3.10) \quad \delta_{1,2} = \frac{\varepsilon}{1 + \varepsilon^2/128}, \quad \delta_{2,2} = \frac{\varepsilon^2}{1 + \varepsilon^2/128}.$$

In a similar manner, using the symbolic manipulation system Mathematica [15], for  $N = 3$  we find

$$(3.11) \quad \delta_{1,3} = \varepsilon, \quad \delta_{2,3} = \frac{\varepsilon^2}{1 + \varepsilon^2/60}, \quad \delta_{3,3} = \varepsilon \delta_{2,3},$$

while for  $N = 4$

$$(3.12) \quad \delta_{1,4} = \frac{56524800 \varepsilon + 1216128 \varepsilon^3}{56524800 + 1216128 \varepsilon^2 + 2141 \varepsilon^4}, \quad \delta_{2,4} = \varepsilon \delta_{1,4},$$

$$\delta_{3,4} = \frac{56524800 \varepsilon^3}{56524800 + 1216128 \varepsilon^2 + 2141 \varepsilon^4}, \quad \delta_{4,4} = \varepsilon \delta_{3,4}.$$

Using Mathematica, we have obtained explicit expressions for the  $\delta_{j,N}$  for  $N \leq 8$ . In each case, we find that

$$(3.13) \quad \delta_{j,N} = \varepsilon^j + O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \rightarrow 0, \quad \text{for } 1 \leq j \leq N.$$

Hence we have verified that our  $N$ -term hybrid solution (3.7) agrees with the  $N$ -term perturbation solution (3.3) as  $\varepsilon \rightarrow 0$  through  $N = 8$ .

In Figure 2, we have plotted the relative  $L_2$ -error between numerical solutions to the problem (3.1)–(3.2) and either hybrid solutions (3.7) or perturbation solutions (3.3) for different values of  $N$ . Figure 2 demonstrates that: (1) there is a finite radius of convergence for the perturbation solutions; (2) the hybrid solutions are more accurate than the corresponding perturbation solutions, especially for larger values of  $\varepsilon$ ; and (3) for a fixed  $N$  the accuracy of the hybrid solutions gradually deteriorates as  $\varepsilon$  increases. We discuss this example further in Section 6. The numerical solutions were obtained using a simple second order finite difference method (coupled with SOR) on a 41 by 41 uniform grid. For large values of  $\varepsilon$ , the numerical solutions are more accurate than the perturbation and hybrid solutions, although for small  $\varepsilon$  the reverse is true.

#### 4. Circular Cylinder in Slightly Compressible Flow

We now apply our hybrid method to a problem involving a nonlinear partial differential equation. As a model of this type of problem, we consider the problem of determining the steady, two-dimensional flow of an inviscid, compressible, perfect gas past a circular cylinder of unit radius without circulation (see, e.g., Van Dyke [10]). If we introduce the usual polar coordinates  $(r, \theta)$  and express the fluid velocity  $\vec{q}$  in terms of a potential function  $\varphi$  as

$$\vec{q} = U \vec{\nabla} \left\{ (r + r^{-1}) \cos(\theta) + \varphi(r, \theta, M) \right\},$$

we find that, for  $r > 1$ ,  $\varphi$  satisfies

$$(4.1) \quad \begin{aligned} \mathcal{L}(\varphi, M) \equiv & \nabla^2 \varphi - \left(\frac{1}{2}\right) M^2 \{ [(1 - r^{-2}) \cos(\theta) + \varphi_r] Q_r \\ & + [-(r^{-1} + r^{-3}) \sin(\theta) + r^{-2} \varphi_\theta] Q_\theta + (\gamma - 1) Q \nabla^2 \varphi \} = 0, \end{aligned}$$

where

$$\begin{aligned} Q \equiv & r^{-4} - 2r^{-2} \cos(2\theta) + 2 \cos(\theta) (1 - r^{-2}) \varphi_r - 2 \sin(\theta) (r^{-1} + r^{-3}) \varphi_\theta \\ & + \varphi_r^2 + r^{-2} \varphi_\theta^2, \end{aligned}$$

with  $\varphi_r = 0$  on  $r = 1$ , and  $\varphi = O(r^{-1})$  as  $r \rightarrow \infty$ . Here  $M = U/c$  is the free stream Mach number,  $c$  is the speed of sound in the gas, and  $\gamma$  is the adiabatic ratio. For small values

of  $M$  the flow streamlines are shown in Figure 3. For subsonic flow, the maximum velocity occurs at the surface of the cylinder at  $\theta = \pi/2$  and  $\theta = 3\pi/2$ . We now apply the hybrid method as outlined in Section 2 to this (nonlinear) problem, where  $M$  plays the role of the parameter  $\varepsilon$ .

Step One: For small values of  $M$ , we use the regular perturbation method to obtain

$$(4.2) \quad \varphi(r, \theta, M) = \sum_{j=1}^N M^{2j} \varphi^{(j)}(r, \theta) + O(M^{2N+2}) \quad \text{as } M \rightarrow 0,$$

where each  $\varphi^{(j)}$  satisfies a Poisson equation in the region exterior to the cylinder, satisfies the condition  $\varphi_r^{(j)} = 0$  on  $r = 1$ , and is  $O(r^{-1})$  as  $r \rightarrow \infty$ . The general form of the perturbation coordinate functions  $\{\varphi^{(j)}\}$  is given by

$$(4.3) \quad \varphi^{(j)} = \sum_{k=0}^j f_{j,k}(r) \cos[(2k+1)\theta], \quad j \geq 1.$$

In particular, we find:

$$(4.4) \quad \begin{aligned} f_{1,0} &= \left(\frac{13}{12}\right) r^{-1} - \left(\frac{1}{2}\right) r^{-3} + \left(\frac{1}{12}\right) r^{-5}, \\ f_{1,1} &= -\left(\frac{1}{4}\right) r^{-1} + \left(\frac{1}{12}\right) r^{-3}. \end{aligned}$$

Van Dyke and Guttman [12] have computed 29 terms in the expansion (4.2) using a FORTRAN program and have presented a rather detailed analysis of the convergence of the series as  $N \rightarrow \infty$ . We have used the symbolic computation system Mathematica [15] to compute 5 of these terms in exact rational arithmetic, carrying  $\gamma$  as a parameter, and 11 terms for the special case when  $\gamma = 7/5$ . Although the general form of each  $\varphi^{(j)}$  is known and only certain numerical coefficients need to be determined, the amount of computational "labor" (using either purely numerical or symbolic computation) increases significantly as  $N$  increases. Consequently, it is desirable to obtain as much information as possible about the solution from the first few perturbation coefficient functions  $\{\varphi^{(j)}\}$ . This we do in Step Two below.

Step Two: We now seek new approximate solutions  $\tilde{\varphi}$  in the form

$$(4.5) \quad \tilde{\varphi}(r, \theta, M) = \sum_{j=1}^N \delta_{j,N}(M) \varphi^{(j)}(r, \theta),$$

where the new “amplitudes”  $\{\delta_{j,N}\}$  must be determined. To determine the  $\{\delta_{j,N}\}$ , we substitute (4.5) into the governing differential equation (4.1) and require that the residual be orthogonal to each of the perturbation coordinate functions  $\varphi^{(k)}$ , i.e.,

$$(4.6) \quad \int_0^{2\pi} \int_1^\infty \mathcal{L}(\tilde{\varphi}, M) \varphi^{(k)} r dr d\theta = 0, \quad 1 \leq k \leq N.$$

Equations (4.6) are a system of  $N$  (cubically nonlinear) equations to determine the  $N$  amplitudes  $\{\delta_{j,N}(M)\}$ . They must, in general, be solved numerically. However, this is straightforward to do using a standard method such as Newton’s method. In particular, by starting at “small” values of  $M$ , where we expect  $\delta_{j,N} \approx M^{2j}$ , and then proceeding to larger values of  $M$ , the solution of (4.6) can be obtained in an efficient manner.

For the special case when  $N = 1$ , we let  $\tilde{\varphi} = \delta_{1,1} \varphi^{(1)}$ , where  $\varphi^{(1)}$  is given by (4.3) and (4.4) with  $j = 1$ , and we find that equation (4.6) yields the following cubic equation for  $\delta = \delta_{1,1}$ :

$$(4.7) \quad \delta - M^2[1 + c_1 \delta + c_2 \delta^2 + c_3 \delta^3] = 0,$$

where

$$(4.8) \quad \begin{aligned} c_1 &= (4736 + 1111 \gamma)/7532, & c_2 &= (147139 + 158248 \gamma)/1084608, \\ c_3 &= (9182 + 815931 \gamma)/23861376. \end{aligned}$$

From (4.7) and (4.8), we see that

$$(4.9) \quad \delta = M^2 + c_1 M^4 + O(M^6) \quad \text{as } M \rightarrow 0,$$

and hence our hybrid solution reproduces the first term of the perturbation solution as  $M \rightarrow 0$ . Also, the cubic equation (4.7) has one negative real root and two positive real roots for  $0 < M < \tilde{M}_c \approx 0.5389247$ . For  $M > \tilde{M}_c$ , there are *no positive real roots* of this equation. In Section 6 we discuss a possible interpretation of  $\tilde{M}_c$  in relation to the convergence of the series (4.2).

In a similar manner, for  $N = 2, 3, 4$ , and  $5$ , we have used Mathematica to perform the integrations appearing in (4.6) and have expressed the resulting equations in exact rational arithmetic. We then expressed the solutions  $\{\delta_{j,N}\}$  as Taylor series in  $M^2$  and, in each case, we found that

$$(4.10) \quad \delta_{j,N} = M^{2j} + O(M^{2N+2}) \quad \text{as } M \rightarrow 0, \quad \text{for } 1 \leq j \leq N.$$

Thus, for these values of  $N$  we have verified that our hybrid solution (4.5) agrees with the first  $N$  terms in the perturbation expansion (4.2) as  $M \rightarrow 0$ . In addition, for each of these

values of  $N$ , we have also estimated the quantity  $\widetilde{M}_c$ , i.e., the value of  $M$  for which, if  $M > \widetilde{M}_c$ , the system (4.6) does not appear to have physically meaningful solutions. These values of  $\widetilde{M}_c$  are summarized in Table 1.

As an application of the results we have obtained, we estimate the critical Mach number  $M_*$  associated with this flow. Here  $M_*$  is defined as the value of the free-stream Mach number at which the flow locally becomes sonic for the first time and is determined as the solution of the equation

$$(4.11) \quad M_*^2 = 2/[(\gamma + 1)q_{\max}^2 - \gamma + 1].$$

Here  $q_{\max} = 2 - \partial\varphi/\partial\theta$  evaluated at  $r = 1$ , and  $\theta = \pi/2$  is the maximum surface speed. Van Dyke and Guttman [12] have used 29 terms in the expansion (4.2), along with a battery of numerical techniques (see also Andersen and Geer [1] and Van Dyke [11]), to obtain the estimate  $M_* = 0.39823780 \pm 0.0000001$ . In contrast, using our one-term hybrid approximation  $\tilde{\varphi}$ , we find that our approximation  $\tilde{q}_{\max}$  for  $q_{\max}$  is given by  $\tilde{q}_{\max} = 2 + (7/6)\delta_{1,1}$ . Using equations (4.7) and (4.8) to compute  $\delta_{1,1}$  for a particular value of  $M$ , we find from (4.11) that, for  $\gamma = 7/5$ , our one-term hybrid solution yields the approximation  $\widetilde{M}_* = 0.407257966$ . This value differs from Van Dyke and Guttman's estimate by only about 2.3%.

In a similar manner, we find approximations to both  $\widetilde{M}_c$  and  $\widetilde{M}_*$  using  $N = 2, 3, 4$ , and 5 terms in our hybrid approximation (4.5). The estimates we obtain, along with those obtained by using  $N$  terms in the perturbation expansion (4.2), are summarized in Table 1.

TABLE 1  
Approximations to  $M_*$

| N  | Perturbation | Hybrid   | $\widetilde{M}_c$ |
|----|--------------|----------|-------------------|
| 1  | 0.420943     | 0.407258 | 0.5389            |
| 2  | 0.409239     | 0.401704 | 0.4990            |
| 3  | 0.404577     | 0.399699 | 0.4740            |
| 4  | 0.402274     | 0.398948 | 0.4606            |
| 5  | 0.400979     | 0.398608 | 0.4513            |
| 6  | 0.400187     |          |                   |
| 7  | 0.399671     |          |                   |
| 8  | 0.399319     |          |                   |
| 9  | 0.399071     |          |                   |
| 10 | 0.398891     |          |                   |
| 11 | 0.398757     |          |                   |

In Section 6, we discuss certain implications of the values which appear in Table 1. For the present, we just note that the hybrid approximation to  $M_*$  based on  $N$  terms has about the same accuracy as the perturbation approximation based on  $2N$  terms. We have not tried to push our technique nearly far enough to begin to compete with Van Dyke's results. We only observe that with a small number of terms we have a better approximation.

## 5. Sink or Source Near a Free Surface

We now wish to apply our hybrid method to a problem for which the nonlinearity enters through the boundary conditions; in particular, a problem in which the boundary itself is unknown and must be determined as part of the solution to the problem. As a model of this type of problem, we consider the problem of determining the steady, two-dimensional shape of a free surface when a sink or source is placed above the surface. This problem has been studied using a variety of numerical techniques by several investigators, e.g., Tuck and Vanden-Broeck [9] and Vanden-Broeck and Keller [14], and also from a perturbation point of view by Vanden-Broeck, Schwartz and Tuck [13].

We consider a sink (or source) of strength  $Q$  located a distance  $h$  above the undisturbed height of a free surface. We introduce a rectangular coordinate system  $(hx, hy)$ , with gravity acting in the negative  $y$ -direction and with the origin a distance  $h$  below the sink (see Figure 4). We let the velocity potential be  $\Phi = Q \left\{ \left( \frac{1}{4\pi} \right) \log[x^2 + (y-1)^2] + \varphi(x, y) \right\}$  and denote the elevation of the free surface by  $y = \eta(x)$ . Then the problem to determine  $\varphi$  and  $\eta$  becomes

$$(5.1) \quad \nabla^2 \varphi = 0, \quad y > \eta(x),$$

with

$$(5.2) \quad B_1(\varphi, \eta) \equiv \varphi_y - \varphi_x \dot{\eta} - \left( \frac{1}{2\pi} \right) \frac{x \dot{\eta} + 1 - \eta}{x^2 + (\eta - 1)^2} = 0$$

and

$$(5.3) \quad B_2(\varphi, \eta) \equiv \eta - \varepsilon a \ddot{\eta} (1 + \dot{\eta}^2)^{-\frac{3}{2}} - \varepsilon (1 + \dot{\eta}^2) \left[ \varphi_x + \left( \frac{1}{2\pi} \right) \frac{x}{x^2 + (\eta - 1)^2} \right]^2 = 0$$

on  $y = \eta(x)$ . Here " $\dot{\cdot}$ "  $\equiv d/dx$ ,  $\varepsilon = Q^2/2gh^3$ , and  $a = 2\gamma h/\rho Q^2$ , where  $g$  is the acceleration due to gravity,  $\rho$  is the density of the fluid, and  $\gamma$  is the surface tension coefficient. Equation (5.2) is just the kinematic boundary condition on the free surface, while equation (5.3) follows from Bernoulli's equation and the condition on the jump in pressure at the free surface due to surface tension. We are interested in finding approximations to the solutions  $\varphi = \varphi(x, y, \varepsilon)$  and  $\eta = \eta(x, \varepsilon)$  to equations (5.1)–(5.3) for small values of  $\varepsilon$ .

Step One: For small values of  $\varepsilon$ , we look for regular perturbation solutions for  $\varphi$  and  $\eta$  in the form

$$(5.4) \quad \varphi = \sum_{j=0}^{N-1} \varepsilon^j \varphi^{(j)} + O(\varepsilon^N), \quad \eta = \sum_{j=1}^N \varepsilon^j \eta_j + O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \rightarrow 0.$$

Using the expressions (5.4) in equations (5.1)–(5.3), we find that the terms which are  $O(1)$  in  $\varepsilon$  in equations (5.1) and (5.2) yield the result

$$(5.5) \quad \varphi^{(0)} = \left( \frac{1}{4\pi} \right) \log(x^2 + (y+1)^2).$$

Then the terms which are  $O(\varepsilon)$  in equation (5.3) yield the expression

$$(5.6) \quad \eta_1 = \left( \frac{1}{\pi^2} \right) x^2 (1+x^2)^{-2}.$$

In a similar manner, the terms which are  $O(\varepsilon)$  in equations (5.1) and (5.2) yield the result

$$(5.7) \quad \varphi^{(1)} = \frac{1}{8\pi^3} \left[ \frac{3(x^2 - (y+1)^2)}{(x^2 + (y+1)^2)^2} + \frac{2(y+1)[(y+1)^2 - 3x^2]}{(x^2 + (y+1)^2)^3} \right],$$

and the terms which are  $O(\varepsilon^2)$  in (5.3) yield the expression

$$(5.8) \quad \eta_2 = -\frac{3}{2\pi^4} \left[ \frac{x^2(1 - 6x^2 + x^4)}{(1+x^2)^5} \right] + \frac{2a}{\pi^2} \left[ \frac{1 - 8x^2 + 3x^4}{(1+x^2)^4} \right].$$

Continuing this procedure with the aid of Mathematica, we find that the general form of the perturbation coefficient functions  $\varphi^{(j)}$  and  $\eta_j$ ,  $j \geq 1$ , is

$$(5.9) \quad \varphi^{(j)}(x, y) = (1/\pi^{2j+1}) \sum_{k=2}^{3j} c_{j,k} \psi^{(k)}(x, y),$$

$$\eta_j(x) = \sum_{k=0}^{j-1} a^k p_{j,k}(x^2) (1+x^2)^{-(3j-1-k)}.$$

Here each  $c_{j,k}$  is a constant, each  $p_{j,k}$  is a polynomial in  $x^2$ , and the functions  $\{\psi^{(k)}\}$  are defined by

$$\psi^{(0)} = \log[x^2 + (y+1)^2]^{\frac{1}{2}}, \quad \psi^{(k)} = (\partial/\partial y)\psi^{(k-1)} \quad \text{for } k \geq 1.$$

In particular, we find

$$\begin{aligned} \eta_3(x) = & \frac{x^2(45 - 1909x^2 + 6890x^4 - 2786x^6 + 9x^8 - 9x^{10})}{32\pi^6(1+x^2)^8} \\ & - \frac{a(12 - 853x^2 + 4008x^4 - 2658x^6 + 148x^8 - x^{10})}{4\pi^4(1+x^2)^7} \\ & - \frac{24a^2(2 - 33x^2 + 40x^4 - 5x^6)}{\pi^2(1+x^2)^6}, \end{aligned}$$

$$\begin{aligned}
c_{2,2} &= -c_{2,3} = (9 - 8\pi^2 a)/128, & c_{2,4} &= (27 - 80\pi^2 a)/384, \\
c_{2,5} &= (3 - 4\pi^2 a)/64, & c_{2,6} &= 3/640.
\end{aligned}$$

From the analysis of this perturbation series with zero surface tension by Vanden-Broeck *et al* [13] (see Section 6 for more discussion of their results), our formal series expansion of the solution for this example *does not converge* for any non-zero value of  $\varepsilon$ . However, despite this lack of convergence, we proceed to Step Two of our hybrid method.

Step Two: Using the expressions above for the perturbation coordinate functions  $\varphi^{(j)}(x, y)$  and  $\eta_j(x)$ , we look for new approximate solutions in the form:

$$(5.10) \quad \tilde{\eta}(x, \varepsilon) = \sum_{j=1}^N \delta_{j,N}(\varepsilon) \eta_j(x), \quad \tilde{\varphi}(x, y, \varepsilon) = \sum_{j=1}^N \delta_{N+j,N} \varphi^{(j-1)}(x, y),$$

where the new “amplitudes”  $\{\delta_{j,N}\}$  must be determined. (We note that  $\tilde{\varphi}$  satisfies  $\nabla^2 \tilde{\varphi} = 0$  for any choice of the amplitudes  $\{\delta_{j,N}\}$ .) To determine these amplitudes, we substitute the expressions (5.10) into the boundary conditions (5.2) and (5.3) and require that the residuals be orthogonal to appropriate sets of test functions  $\{\alpha_k\}$  and  $\{\beta_k\}$ , i.e.,

$$(5.11) \quad \int_{-\infty}^{\infty} B_1(\tilde{\varphi}, \tilde{\eta}) \alpha_k(x) dx = 0, \quad \int_{-\infty}^{\infty} B_2(\tilde{\varphi}, \tilde{\eta}) \beta_k(x) dx = 0, \quad 1 \leq k \leq N.$$

Once the test functions have been selected, equations (5.11) are a set of  $2N$  equations for the  $2N$  unknowns  $\{\delta_{j,N}, j = 1, 2, \dots, 2N\}$ . For simplicity we select as test functions  $\alpha_k = \beta_k = \eta_k(x)$ . The resulting hybrid solutions are discussed in the next section.

## 6. Discussion

The hybrid perturbation-Galerkin method, as we have described it here, is a semi-analytical, semi-numerical technique which appears to have the potential of being a useful tool both to complement and to supplement existing standard methods of analysis. It is semi-analytical in the sense that some of the analytical structure of the solution is determined by first constructing one or more perturbation approximations to the solution. It is semi-numerical in that new amplitudes of the perturbation coordinate functions typically are determined numerically by solving standard Galerkin equations associated with the problem. The method seems to have considerable potential to be an effective problem-solving tool because the perturbation coordinate functions appear to be very effective trial functions in the Galerkin approximation. Intuitively this is because, by the way they are constructed, they describe the basic nature of the solution, at least for parameter values near their point of generation.

In our first example (Section 3), we analyzed a simple, linear boundary-value problem on a bounded domain, for which an arbitrary number of terms in the regular perturbation series can be computed in a straightforward manner. By examining the singularities of the amplitudes  $\{\delta_{j,N}\}$  in our hybrid solution, it appears that the convergence of the perturbation series is limited by a pair of complex conjugate singularities located near or at  $\pm 6.70i$ . (This follows by analogy from the analysis of several similar problems, involving ordinary differential equations, for which the singularities of the  $\{\delta_{j,N}\}$  in the complex  $\varepsilon$ -plane converged to the singularities of the actual solution as the number of terms in the approximation increased. See especially [2]. In fact the location of the poles of the  $\{\delta_{j,N}\}$  which are closest to the origin form the sequence  $\pm 7.75i, \pm 7.15i, \pm 6.803i, \pm 6.721i, \pm 6.7048i, \pm 6.70283i, \dots$ . Another pair of singularities is indicated near  $\varepsilon \approx \pm 9i$ .) These singularities appear to have no immediate interpretation, but nonetheless limit the direct usefulness of the perturbation expansion.

In Figure 2, we have plotted the relative  $L_2$ -error between numerical solutions to problem (3.1)–(3.2) and perturbation or hybrid approximations. The figure clearly illustrates that the hybrid method allows us to obtain useful information about the solution well beyond the radius of convergence of the perturbation solution. Also, there is no indication that the range of convergence of the hybrid solution is limited.

In Figure 6, we have plotted the values of the solution at the center value  $x = y = \pi/2$  as calculated numerically and by the use of the hybrid technique, where the number of terms ( $N$ ) ranges from 1 to 8. It is interesting to note that our hybrid solutions appear to form an alternating sequence which brackets the true center point solution. Figure 1 shows that the solutions have their peak “elevations” at the center point.

The contour plots shown in Figure 7 demonstrate that higher-order perturbation functions can be very similar to one another. This suggests that the perturbation functions which are of higher order than these shown may be fairly well approximated by linear combinations of the functions shown (see observation (2) in the introduction). Figure 7 also demonstrates that the perturbation functions may have higher symmetries (group order = 4) than the solutions (group order = 2) which they are used to approximate. Such additional symmetries may be exploited in the steps of the hybrid method which are computed numerically. The symmetries of the perturbation functions follow a regular pattern which can be determined *a priori*.

From a singular perturbation point of view, as  $\varepsilon$  becomes large the solution to problem (3.1)–(3.2) takes on a somewhat complicated form. The leading term in the outer expansion of  $u$  is  $2(\pi - x)\tan(y)$  for  $0 < x \leq \pi, 0 \leq y < \pi/2$ , and is  $-2x\tan(y)$  for  $0 \leq x < \pi, \pi/2 < y \leq \pi$ . Boundary layers develop along  $x = 0$  for  $0 < y < \pi/2$  and along  $x = \pi$

for  $\pi/2 < y < \pi$ . In addition, an internal layer forms along  $y = \pi/2$ , as well as additional boundary layers at  $(0, \pi/2)$  and  $(\pi, \pi/2)$ . From the contour plots shown in Figure 1, it may be seen that the boundary and internal layers are beginning to be formed by the hybrid approximations as  $\varepsilon$  increases, even though these approximations are based on the small  $\varepsilon$  expansion of the solution, where no such layers are evident.

This last observation suggests that the hybrid method might be a useful tool for the general problem of determining an appropriate decomposition of the domain for a purely numerical solution to a specific problem. In particular, suppose that some "singular" behavior of the solution is anticipated for *large* values of a parameter appearing in the problem formulation. To help determine the location and, to some extent, the nature of this singular behavior, the hybrid method could be applied in the following way. First, a (regular) perturbation expansion of the solution is constructed for *small* values of the parameter. The hybrid method is then applied, using the small parameter perturbation coordinate functions, and then the behavior of the hybrid approximation is noted as the parameter value is increased. From the example of Section 3, as well as several other examples we have examined in some detail, it appears that the hybrid solution simulates at least some of the singular behavior of the exact solution as the parameter value is increased, and that this simulation becomes more accurate as the number of terms in the approximation is increased.

In our second example (Section 4), which involves an infinite domain, the regular perturbation expansion of  $\varphi$  is straightforward, although tedious, to compute. Frankl and Keldysh [4] have proved that  $\varphi$  is analytic in  $M^2$  about  $M^2 = 0$  and hence the perturbation expansion (4.2) is actually the Taylor series expansion of the exact solution. They did not, however, determine the radius of convergence  $M_c$  of the series. Through a careful analysis of the first 24 terms in the series, Van Dyke and Guttman [12] report that  $M_c = 0.402667605 \pm 0.000000005$ , which is about 1.11% greater than their estimate of  $M_*$ . If this were indeed the case, it would imply that shock free solutions exist for a continuous range of values of  $M$  above the critical Mach number. However, Van Dyke [private communication] now believes this result to be in error and that  $M_c = M_*$ .

Although Van Dyke and Guttman were unable to determine the nature of the singularity which limits the convergence of the series, we can safely assume that the singularity is related to the formation of a shock and hence represents a real singularity with a definite physical interpretation. The numbers  $\widetilde{M}_c$  presented in Table I are estimates for the value of  $M$  above which no physically meaningful hybrid solution to the problem exists. In this sense, they can be thought of as estimates of the radius of convergence of the series (4.2). If more of these estimates were computed and if it could be shown that they converge to the same limit as the estimates  $\widetilde{M}_*$ , this would lend support to the conjecture that  $M_c = M_*$ .

The solution to our third example (Section 5) involves two unknown functions (i.e., the potential function and the shape of the free surface), and the associated perturbation expansions are somewhat more difficult to compute than the expansions associated with the first two examples. However, the general form of the terms in each expansion can be determined and, in principle, only certain numerical constants need to be computed. Using a complex variable formulation of this problem (with zero surface tension) Vanden-Broeck *et al* [13] numerically computed these constants in the first 34 terms in each expansion. They analyzed the resulting series for the free surface and found it to be of exponential-integral character (i.e., the coefficient of  $\varepsilon^n$  behaves like  $n!$  for large  $n$ ). Consequently, the series diverges for all nonzero values of  $\varepsilon$ . They use a number of techniques to “sum” this divergent series and show that this analysis leads to a free surface shape with a jump discontinuity due to the location of a branch cut. One such free surface profile corresponding to our  $\varepsilon = \pi^2/5 = 1.974$  is indicated by a dashed line in Figure 5. Our hybrid solution with  $N = 2$  is shown by a solid line. Vanden-Broeck *et al* indicate that the jump discontinuity in their solution could be removed by applying a certain iterative procedure, but the corresponding profile was not displayed. Using a numerical, series truncation method on another complex variable formulation of the problem, Tuck and Vanden-Broeck [9] found a cusp like solution, corresponding in our notation to a value of  $\varepsilon = \varepsilon_c \approx 6.311$ . Vanden-Broeck and Keller [14], using a similar method, treated the same problem we are considering, except that a horizontal fixed surface is present at a finite distance above the sink. They found solutions corresponding to our problem for values of  $\varepsilon$  larger than  $\varepsilon_c$ , but did not find any steady solutions for  $0 < \varepsilon < \varepsilon_c$ . They indicate, however, that solutions with waves may exist for  $\varepsilon$  in the range  $0 < \varepsilon < \varepsilon_c$ .

The hybrid approximations presented in Section 5 appear to give “physically realistic” approximations to a steady solution to the problem, as indicated in Figures 4 and 5. In [2] we applied our hybrid method to an eigenvalue problem associated with an anharmonic oscillator. For this problem, the perturbation series also has a zero radius of convergence, but the hybrid approximations converge monotonically to the true solution as  $N$  increases. Based upon this experience, it is certainly possible that our hybrid approximations to the free surface for values of  $\varepsilon$  in the range  $0 < \varepsilon < \varepsilon_c$  are converging to a steady state solution. Obviously, this difficult problem needs much more investigation, e.g., either the development of a numerical method which will yield the steady solution for  $\varepsilon$  in the range  $0 < \varepsilon < \varepsilon_c$ , or a proof that no such solution exists in this range of parameter values.

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## FIGURE CAPTIONS

Figure 1. Surface and contour plots for 8-term hybrid and numerical solutions of the simple partial differential equation of Equations (3.1)–(3.2). The hybrid solutions shown for  $\varepsilon = 1$  and  $\varepsilon = 5$  are highly accurate. Some irregularities may be seen in the relatively flat portions of the hybrid solution for  $\varepsilon = 20$  which do not appear in the numerical solution for  $\varepsilon = 20$ . Nevertheless, the relative  $L_2$ -error is less than 1%.

Figure 2. Relative  $L_2$ -errors of perturbation and hybrid solutions of the simple PDE problem as a function of the parameter  $\varepsilon$ . The number of terms  $N$  ranges from 1 to 8. The log-log plots of the perturbation errors fall nearly on straight lines which intersect one another near the radius of convergence,  $R = 6.7026$ . The hybrid errors for given  $N$  and  $\varepsilon$  are significantly less than the perturbation errors for the same  $N$  and  $\varepsilon$  and show no signs of divergence.

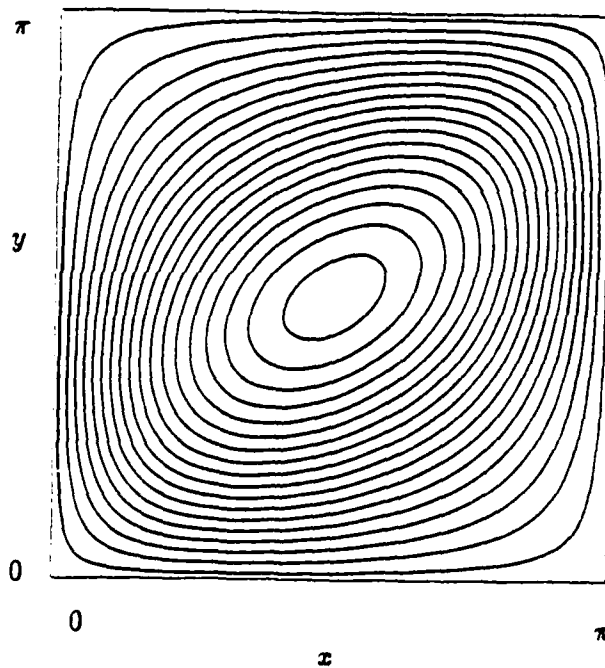
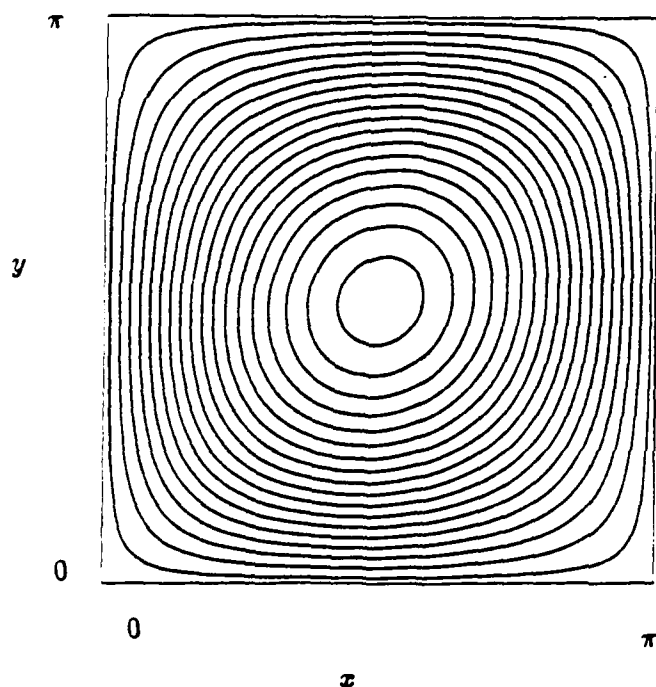
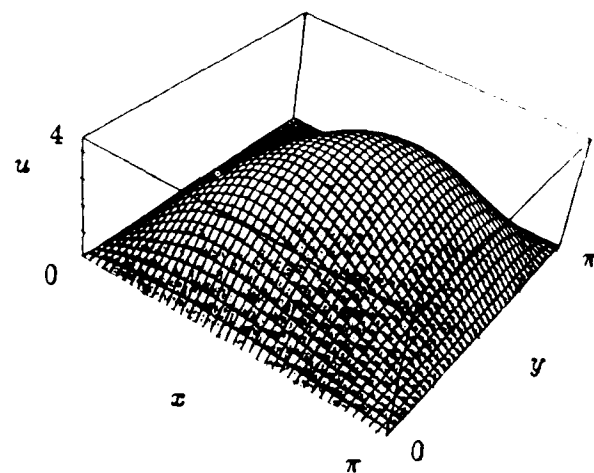
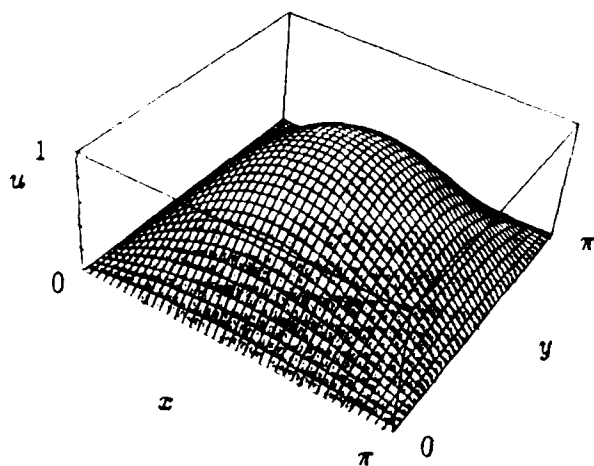
Figure 3. Streamlines for two-dimensional flow around a cylinder for Mach number  $M = 0$ , the low velocity limit of the solutions to Equation (4.1).

Figure 4. Free surface profile for the two-dimensional flow induced by a sink (black dot) above a liquid surface (see Equations (5.1)–(5.3)). The effects of surface tension are ignored. The horizontal and vertical scales differ.

Figure 5. Free surface profiles for  $\varepsilon = \pi^2/5$  with zero surface tension as computed by the hybrid method (solid line) and as computed by Vanden-Broeck et al (dashed line).

Figure 6. Maximum values of the solution ( $x = y = \pi/2$ ) of the simple PDE problem of Section 3 as computed numerically (dots) and by the hybrid method (solid lines) using from  $N = 1$  to  $N = 8$  terms. The results for  $N = 8$  are useful well beyond the radius of convergence,  $R$ , of the series expansion.

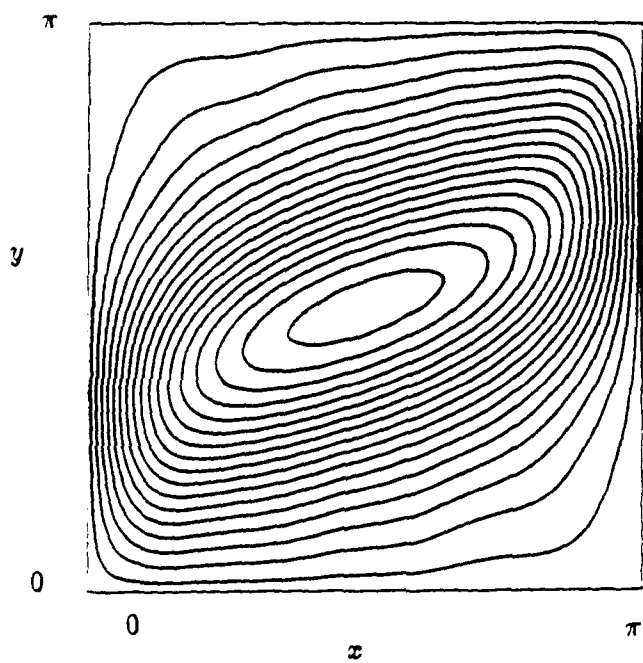
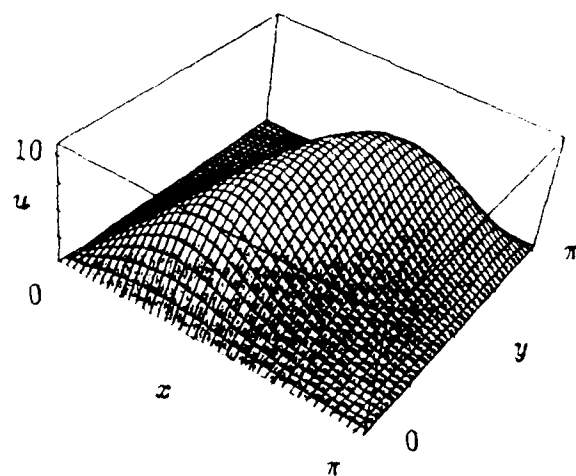
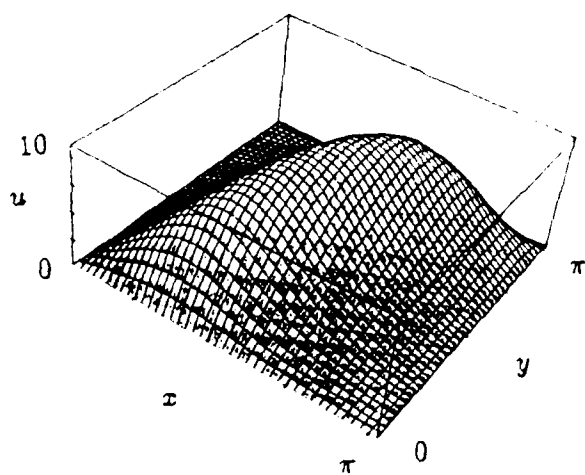
Figure 7. Contour plots of the first eight perturbation functions for the simple PDE problem of Section 3 demonstrate that the higher order perturbation functions can be very similar to one another and can have higher symmetries than the solutions they are used to approximate.



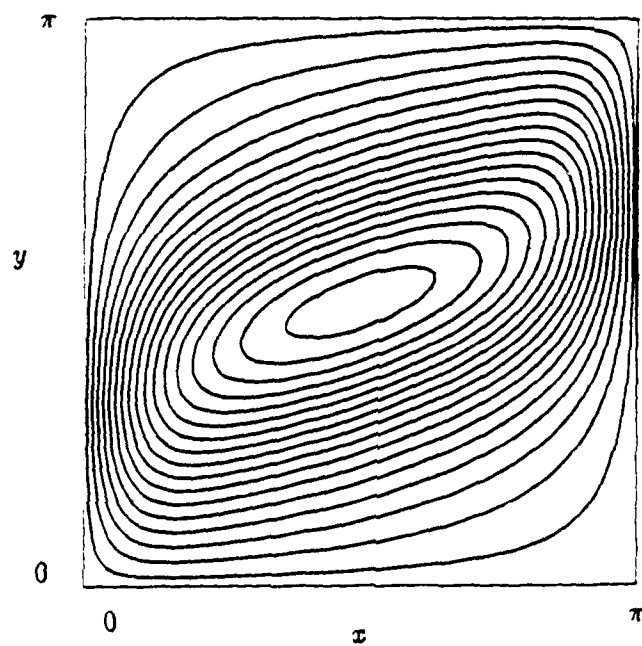
(a)  $\epsilon = 1$ , hybrid

(b)  $\epsilon = 5$ , hybrid

Figure 1



(c)  $\epsilon = 20$ , hybrid



(d)  $\epsilon = 20$ , numerical

Figure 1 (cont.)

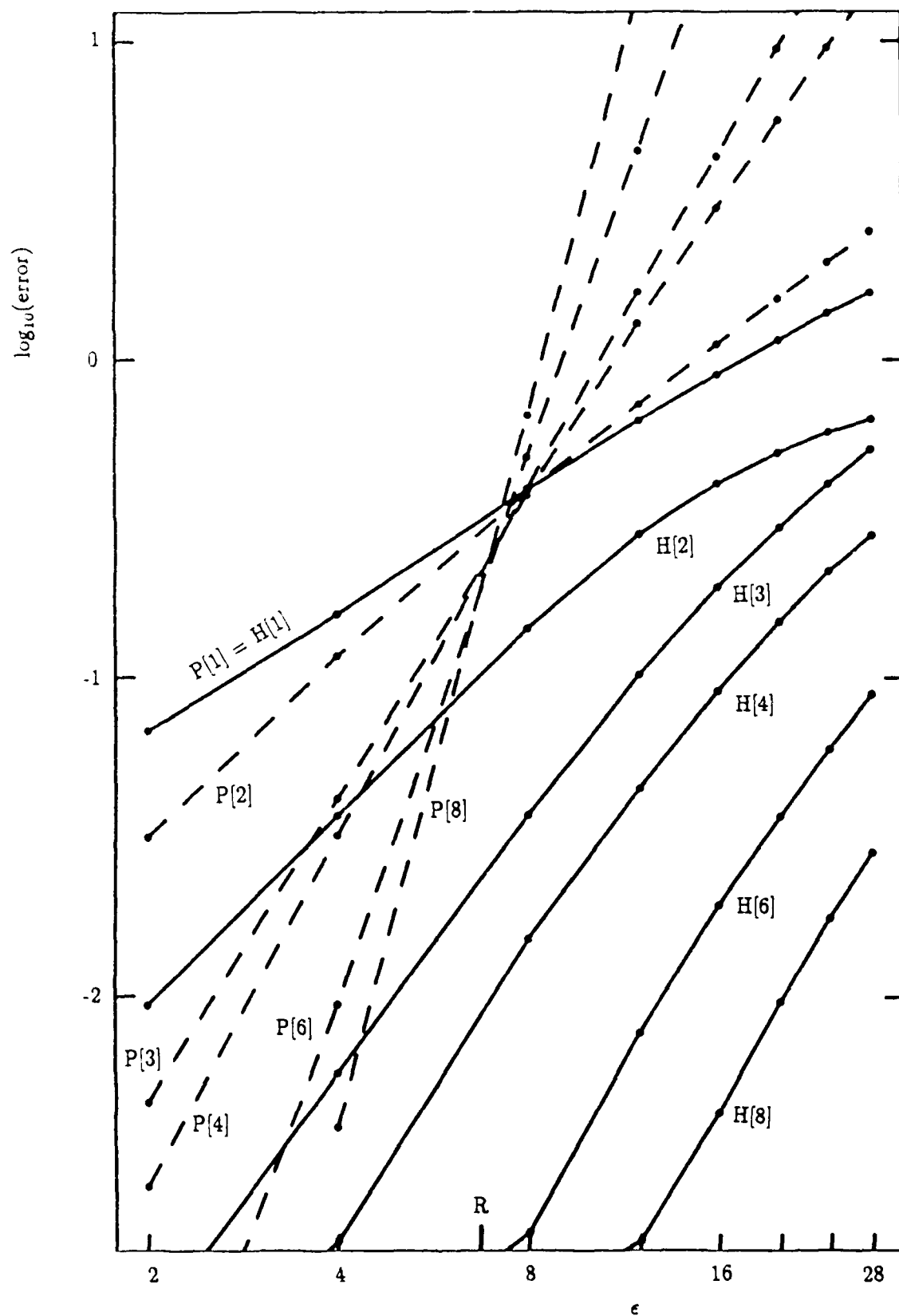
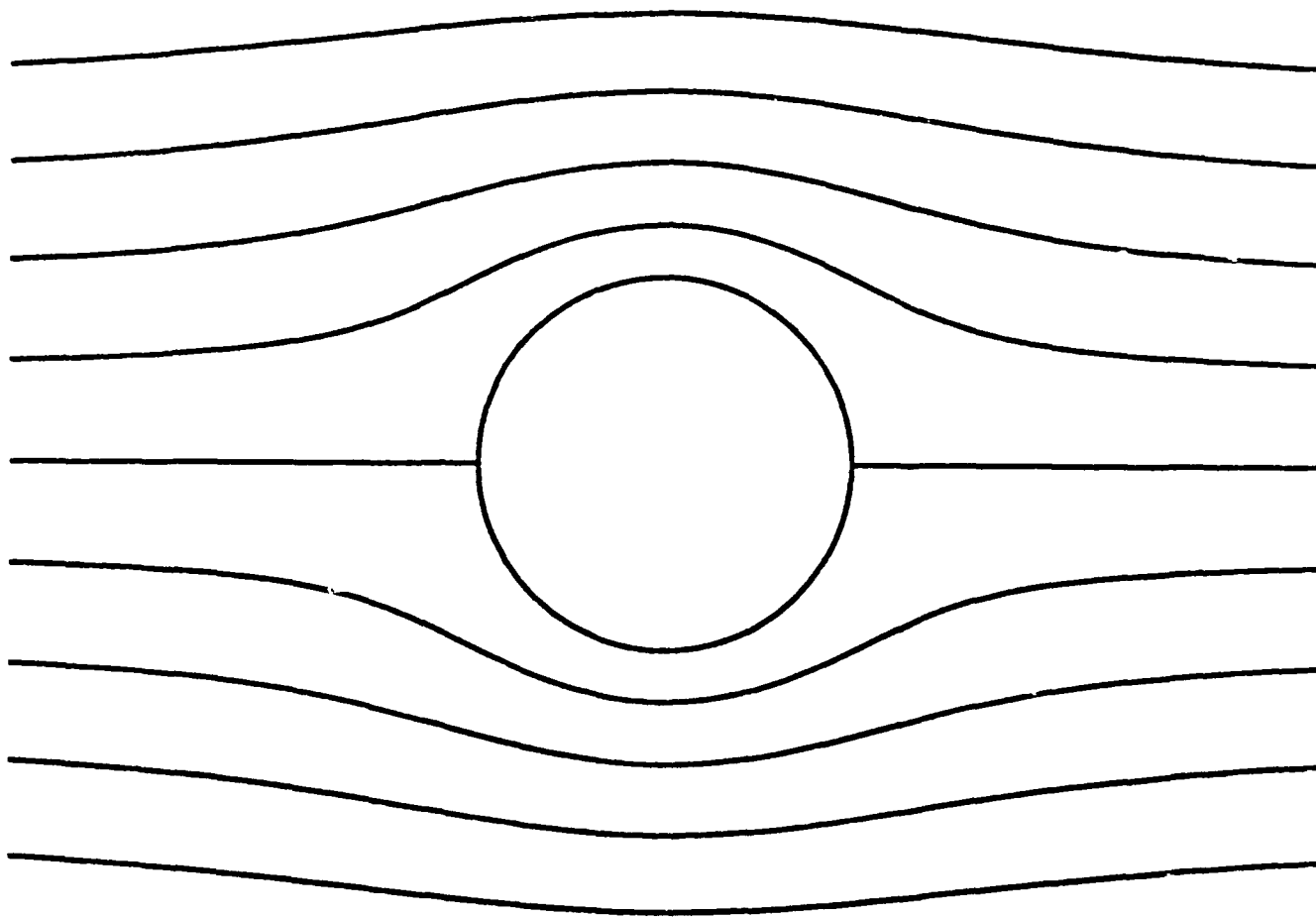


Figure 2



**Figure 3**

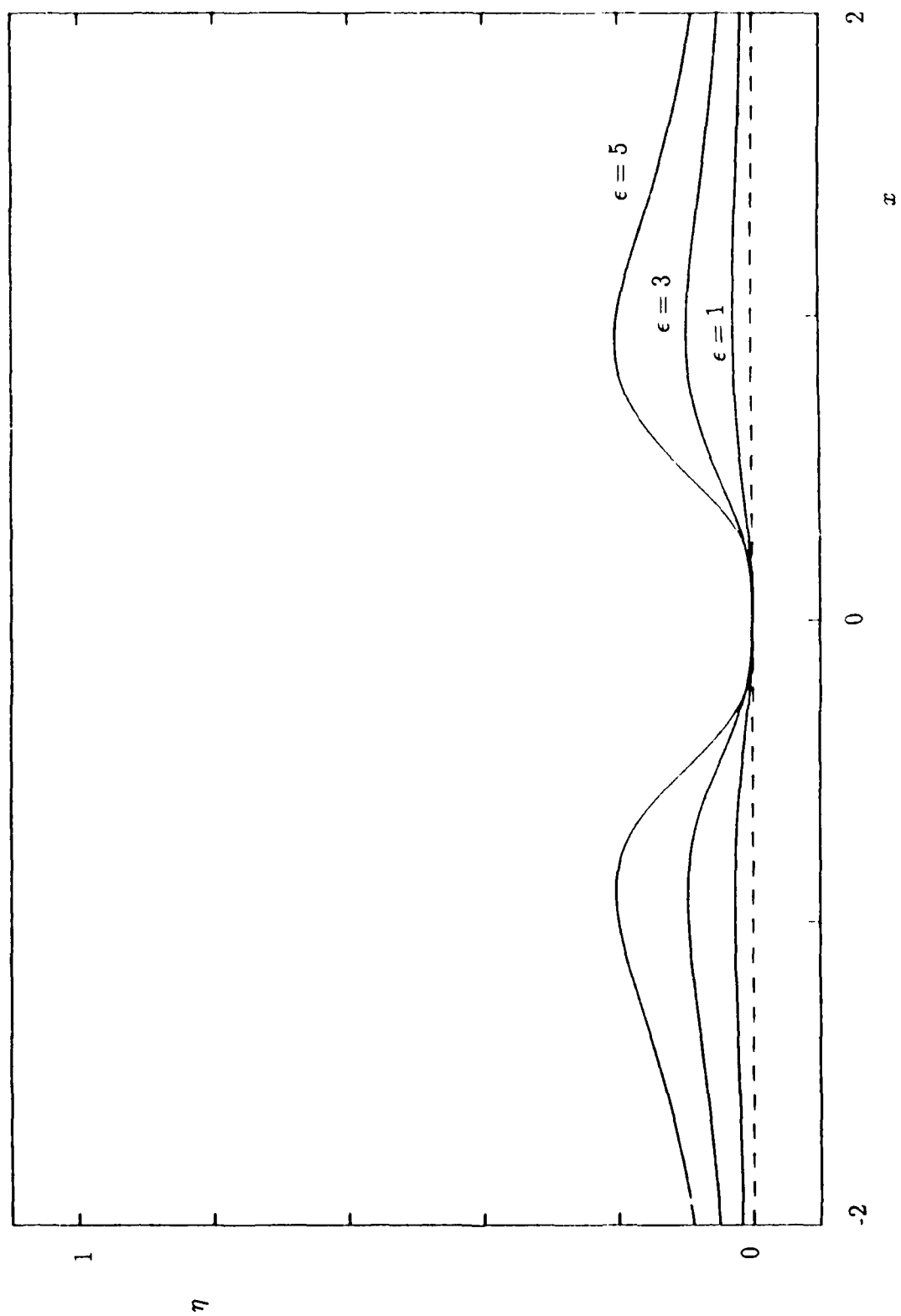


Figure 4

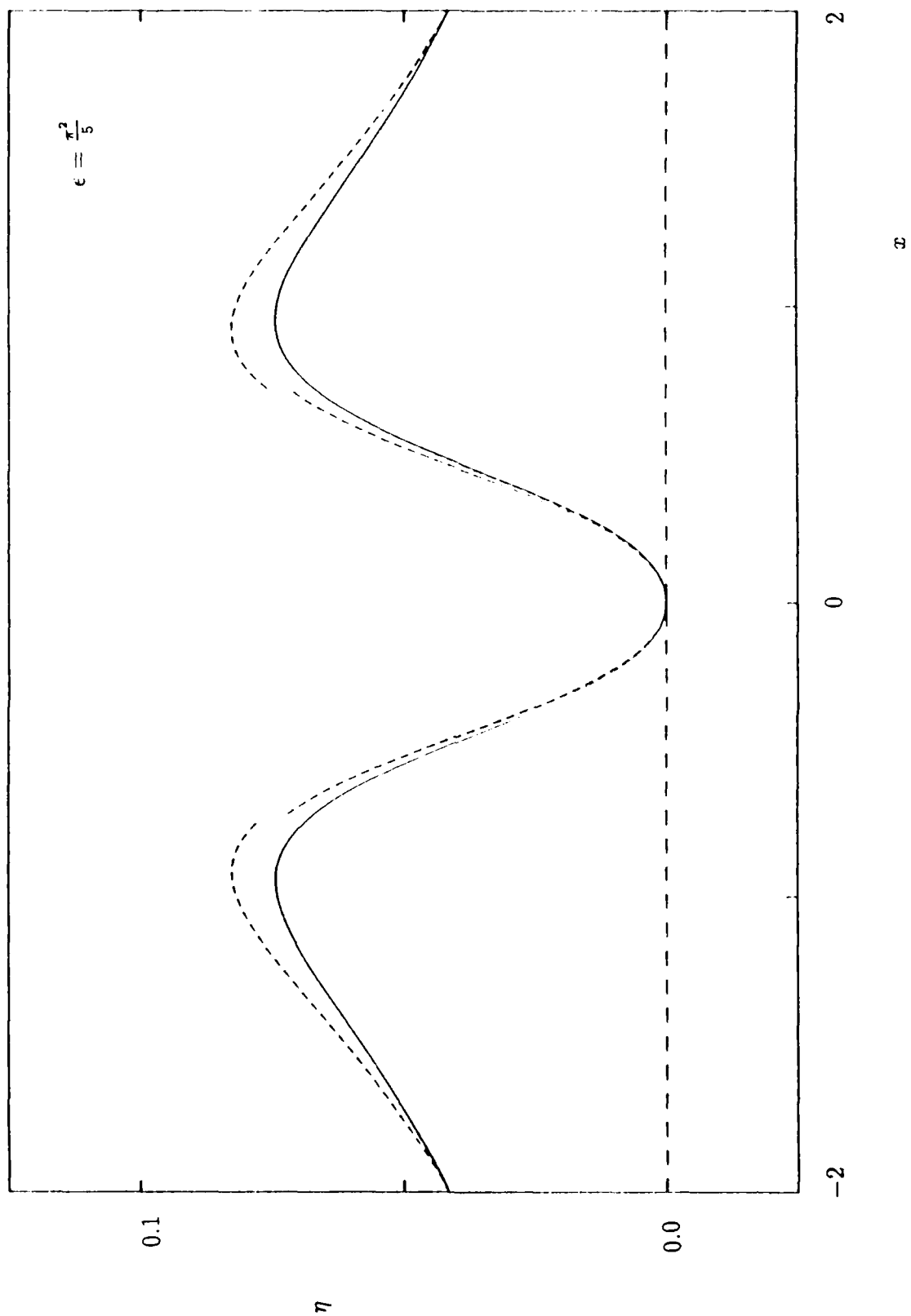


Figure 5

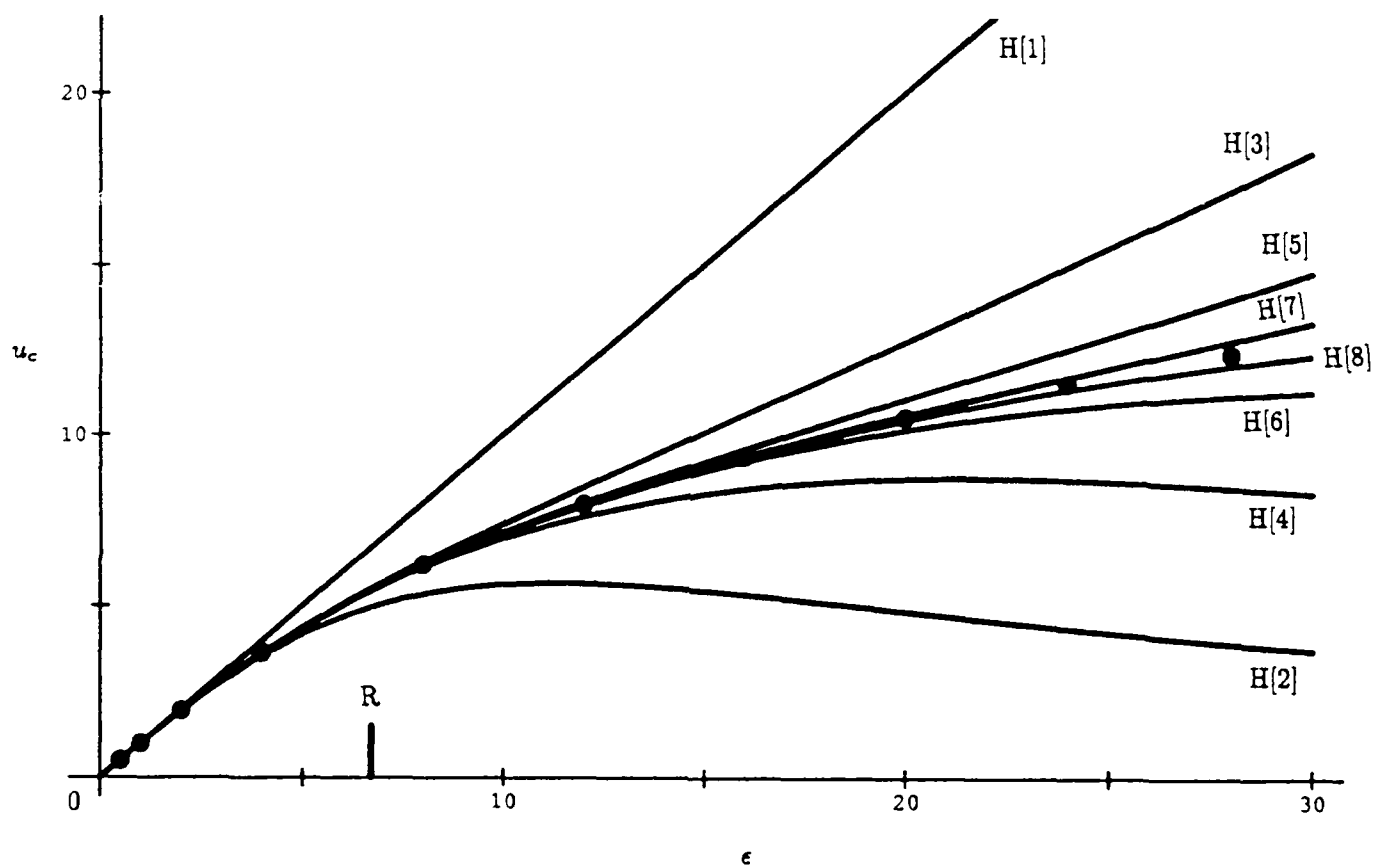
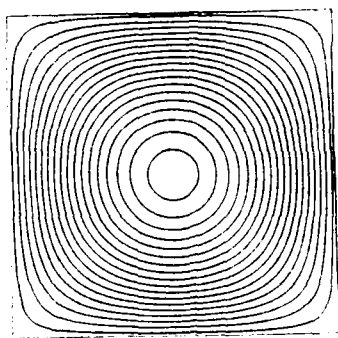
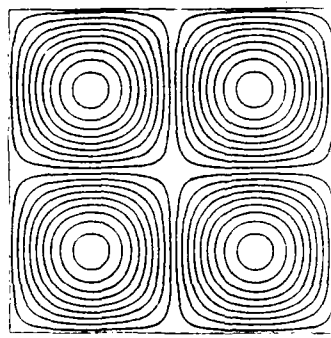


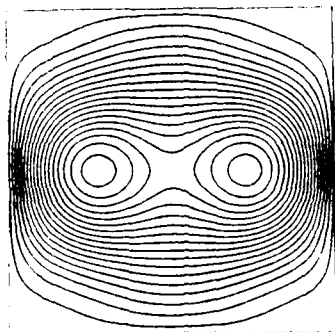
Figure 6



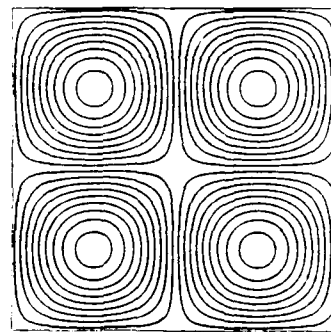
$i=1$



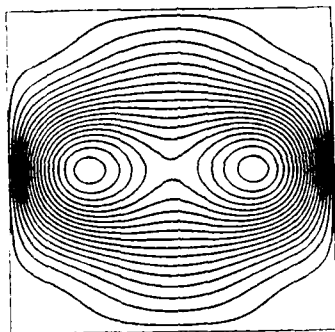
$i=2$



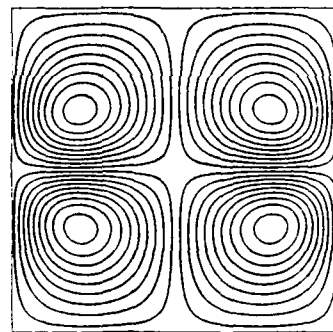
$i=3$



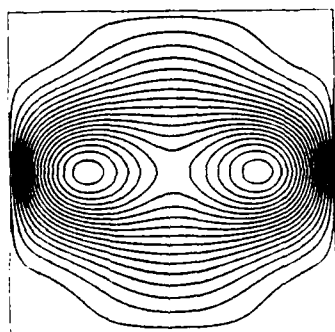
$i=4$



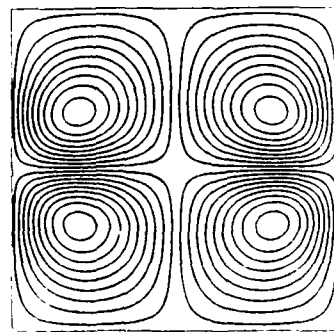
$i=5$



$i=6$



$i=7$



$i=8$

**Figure 7**



## Report Documentation Page

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                      |                             |                                                                                                          |                                                            |  |
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| 1. Report No.<br>NASA CR-182085<br>ICASE Report No. 90-57                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                      | 2. Government Accession No. |                                                                                                          | 3. Recipient's Catalog No.                                 |  |
| 4. Title and Subtitle<br><br>A HYBRID PERTURBATION-GALERKIN TECHNIQUE FOR PARTIAL DIFFERENTIAL EQUATIONS                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                      |                             |                                                                                                          | 5. Report Date<br>August 1990                              |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                      |                             |                                                                                                          | 6. Performing Organization Code                            |  |
| 7. Author(s)<br><br>James F. Geer<br>Carl M. Andersen                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |                                                      |                             |                                                                                                          | 8. Performing Organization Report No.<br>90-57             |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                      |                             |                                                                                                          | 10. Work Unit No.<br>505-90-21-01                          |  |
| 9. Performing Organization Name and Address<br>Institute for Computer Applications in Science and Engineering<br>Mail Stop 132C, NASA Langley Research Center<br>Hampton, VA 23665-5225                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                                                      |                             |                                                                                                          | 11. Contract or Grant No.<br>NAS1-18605                    |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                      |                             |                                                                                                          | 13. Type of Report and Period Covered<br>Contractor Report |  |
| 12. Sponsoring Agency Name and Address<br>National Aeronautics and Space Administration<br>Langley Research Center<br>Hampton, VA 23665-5225                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                      |                             |                                                                                                          | 14. Sponsoring Agency Code                                 |  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                      |                             |                                                                                                          |                                                            |  |
| 15. Supplementary Notes<br><br>Langley Technical Monitor:<br>Richard W. Barnwell<br><br>Final Report<br><br>To appear in the Proceedings of the<br>Workshop on Asymptotic Analysis and<br>Numerical Solution of PDEs                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                      |                             |                                                                                                          |                                                            |  |
| 16. Abstract<br><br>A two-step hybrid perturbation-Galerkin technique for improving the usefulness of perturbation solutions to partial differential equations which contain a parameter is presented and discussed. In the first step of the method, the leading terms in the asymptotic expansion(s) of the solution about one or more values of the perturbation parameter are obtained using standard perturbation methods. In the second step, the perturbation functions obtained in the first step are used as trial functions in the Bubnov-Galerkin approximation. This semi-analytical, semi-numerical hybrid technique appears to overcome some of the drawbacks of the perturbation and Galerkin methods when they are applied by themselves, while combining some of the good features of each. The technique is illustrated first by a simple example. It is then applied to the problem of determining the flow of a slightly compressible fluid past a circular cylinder and to the problem of determining the shape of a free surface due to a sink above the surface. Solutions obtained by the hybrid method are compared with other approximate solutions, and its possible application to certain problems associated with domain decomposition is discussed. |                                                      |                             |                                                                                                          |                                                            |  |
| 17. Key Words (Suggested by Author(s))<br><br>Perturbation method, Galerkin technique                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |                                                      |                             | 18. Distribution Statement<br><br>34 - Fluid Mechanics and Heat Transfer<br><br>Unclassified - Unlimited |                                                            |  |
| 19. Security Classif. (of this report)<br>Unclassified                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | 20. Security Classif. (of this page)<br>Unclassified |                             | 21. No. of pages<br>29                                                                                   | 22. Price<br>A03                                           |  |