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Andrew P. Bassom
Philip Hall

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Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, Virginia 23665–5225

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CONCERNING THE INTERACTION OF NON-STATIONARY CROSS-FLOW VORTICES IN A THREE-DIMENSIONAL BOUNDARY LAYER

Andrew P. Bassom[1] and Philip Hall[1]

Department of Mathematics
North Park Road
University of Exeter
Exeter, Devon
EX4 4QE
UNITED KINGDOM

ABSTRACT

Recently there has been much work devoted to considering some of the many and varied interaction mechanisms which may be operative in three-dimensional boundary layer flows. Here we are concerned with resonant triads of crossflow vortices. In contrast to much of the previous work we examine the effects of interactions upon resonant triads where each member of the triad has the property of being linearly neutrally stable; then the importance of the interplay between modes can be relatively easily assessed. We concentrate on investigating modes within the boundary layer flow above a rotating disc; this choice is motivated by the similarity between this disc flow and many important practical flows and, secondly, our selected flow is an exact solution of the Navier–Stokes equations which makes its theoretical analysis especially attractive. Firstly we demonstrate that the desired triads of linearly neutrally stable modes can exist within the chosen boundary layer flow and then subsequently obtain evolution equations to describe the development of the amplitudes of these modes once the interaction mechanism is accounted for. It is found that the coefficients of the interaction terms within the evolution equations are, in general, given by quite intricate expressions although some elementary numerical work shows that the evaluation of these coefficients is practicable. The basis of our work lends itself to generalisation to more complicated boundary layers and effects of detuning or non-parallelism could be provided for within the asymptotic framework.

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§1. Introduction.

The desire to understand the phenomenon of transition to turbulence in fluid flows has motivated a wealth of studies concerned with the instability of three-dimensional boundary layer flows. Here we address the problem of providing a self-consistent asymptotic description of the interactions within a resonant triad of travelling cross-flow vortices occurring in a three-dimensional boundary layer at a large Reynolds number. In particular, we relate our analysis to vortices in the boundary layer of the flow induced by a rotating disc. This flow is susceptible to instabilities similar to those which occur in flows over a swept wing; a situation which has practical relevance to the development of Laminar Flow Control wings. Further, there is an exact solution of the Navier Stokes equations which describes the rotating disc flow which makes this particular flow especially suitable for a theoretical analysis of the interaction of travelling cross-flow vortices.

The resonant interaction of a triad of waves is one of a number of possible mechanisms that play important roles in describing the nonlinear evolution of disturbances. Such an interacting triple is possible whenever three waves with wavenumbers $\mathbf{k}_j$ and frequencies Ω_j ($j = 1,2,3$) such that $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$ and $\Omega_1 = \Omega_2 + \Omega_3$ can coexist. This type of mechanism is the subject of our current investigation, but an excellent account of the variety of interaction phenomena which are possible within the scenario of fluid mechanics has been given by Craik (**1**).

The crossflow vortex instability structure considered here can only occur in three-dimensional boundary layers and was first examined both theoretically and experimentally by Gregory, Stuart & Walker (**2**) in the context of rotating disc flow. They showed with the china-clay technique that this instability takes the form of a regularly spaced pattern of equiangular spiral vortices which are stationary relative to the disc. In this paper GSW showed, using inviscid theory, that the stationary vortex mechanism is associated with an inflexional profile in which the inflexion point coincides with a point of zero velocity somewhere in the flow. Their calculation of the number of vortices expected to be seen in the flow above a rotating disc gave a number approximately four times greater than the observed value of about 30, but the angle of 13° between the axes of the vortices and the radius vector predicted was in good agreement with that seen in experiments. Brown (**3**), and later Cebeci & Stewartson (**4**) solved the Orr–Sommerfeld equation numerically and obtained values for the critical Reynolds number for transition much lower than that observed by Gregory *et. al.* and used by Stuart in his calculation for the number of vortices. Malik (**5**) found the neutral stability curve for these stationary disturbances and he showed the existence of a second stationary mode (this one however being viscous) which corresponds to zero wall shear stress of the crossflow velocity profile. The asymptotic structures (at high Reynolds numbers) of both this viscous mode and of the inviscid mode of GSW were obtained by Hall (**6**). The linear analysis presented in that paper has been extended by MacKerrell (**7**) and Gajjar (**8**) to include nonlinear effects.

The aforementioned investigations have concentrated on stationary modes. However, Faller & Kaylor (**9**) and Lilly (**10**), in their studies of periodic perturbations imposed on Ekman boundary layer flows, mentioned the existence of an instability with non-zero phase speed. Federov *et. al.* (**11**) showed that in rotating disc flow a similar instability occurs much earlier than the stationary mode observed in the experiments of Gregory *et. al.*. This instability also appeared as a pattern of spiral vortices, but which was moving relative to the disc. The number of these vortices was observed to be between 14 and 16, and had axes inclined at about 20° to the radius vector. Recently, Bassom & Gajjar (**12**) have conducted a theoretical analysis of some aspects of non-stationary vortices in rotating disc flow and, in particular, investigated the properties of neutrally stable nonlinear modes. Our present concern is with the time dependent version of the high Reynolds number inviscid crossflow vortex whose flow structure was elucidated by Hall (**6**). The modifications required to his analysis to incorporate the effects of unsteadiness are relatively straightforward and are considered in the following section.

There has been much work in relation to the important problem of the interaction of Tollmien-Schlichting disturbances with Görtler vortices (an instability associated with flows over curved surfaces). See, for example, (**13–17**) and the references therein. Also, some study has been made of the interaction between Tollmien-Schlichting waves and steady crossflow vortices, (**18–20**), and of the triad interaction between three unsteady crossflow vortices under consideration here. In particular, we refer to the work of Reed (**18, 19**) and El-Hady (**21**), who found that the growth of each of the instability modes could be substantially influenced by their mutual interaction mechanisms. Additionally, non-parallel effects were included and it was concluded that the evolution of the vortices was critically dependent upon the initial amplitudes of the modes.

However, these accounts ignored the crucial fact that for a completely rational description of the importance of the interaction process in the development of the disturbances each of the instability modes involved should be neutrally stable at leading order in their own rights. Otherwise, the vortex growth due to the interaction mechanism is no larger than would be experienced by the modes in the absence of the interaction, and the true importance of the mechanism is extremely difficult, if not impossible, to accurately assess. For this reason, our present aim is to first illustrate that a suitable triad of travelling crossflow vortices can exist in a three-dimensional boundary layer; crucially, these modes are chosen so as to ensure that each is individually neutrally stable at leading order.

Once the existence of the desired triad is established we move on to consider the problem when all three modes are simultaneously present. In the case of a single, infinitesimally small mode the asymptotic flow structure consists of a main layer (coincident with the majority of the boundary layer) augmented by a thin, viscous wall layer next to the surface of the disc and by at least one linear critical layer present somewhere in the flow. With all modes of the triad present, the corresponding flow structure becomes much more complicated, for now we have at least three critical layers (one for each mode) and sometimes more. It transpires that in order to obtain a rational description of the evolution of the triad each of the critical layers needs to be carefully analysed, for each contributes non-zero terms to each of the final, coupled amplitude equations for the three modes.

An important characteristic of our analysis is that each vortex is taken to be sufficiently small and to develop on appropriately slow length and timescales such that all the critical layers remain linear in character. We emphasize that although the equations obtained in this paper are specifically oriented towards the rotating disc flow, for the cases of other basic velocity profiles the derivation procedure would be essentially identical. The method adopted also allows nonparallel effects to be accounted for without formal difficulty. The linked triad of evolution equations, somewhat unfortunately, suffer from the handicap that they are very complicated and require much delicate non-trivial numerical work to establish the interaction coefficients for any particular flow. Here we perform some of the calculations for the rotating disc situation to demonstrate that the derivation of the evolution equations is practicable. The equations so derived have complex-valued coefficients and are not dissimilar in form to those derived by

Smith & Stewart (**22**) who were concerned with high frequency resonant triads within boundary layers. We consider that the principal result of the current work is that in three–dimensional boundary layers resonant triads of non–stationary crossflow vortices do exist. Further, given such a triad of modes which are neutrally stable at leading order, we show that we can develop a fully rational asymptotic account of the interaction processes.

The procedure for the remainder of the paper is as follows. In section 2 we briefly consider the asymptotic structure of a linear, travelling crossflow vortex in a three–dimensional boundary layer, making specific reference to the case of rotating disc flow as an example. Numerical work in section 3 establishes that the desired form of a resonant triad of modes exists for this special basic flow and, armed with this information, we make a careful study of the interaction mechanism in section 4 to enable us to derive a coupled triple of evolution equations for the modes. Finally, we perform some numerical work to establish the interaction coefficients, discuss a few solutions of the relevant evolution equations and we conclude with some discussion and suggestions for further work.

§2. The basic equations and flow structure for a linear crossflow vortex.

We have already stated that we intend to make an investigation of the interaction processes which take place between the three members of a resonant triad of crossflow vortices and a natural first step is to consider the disturbance structure for each individual mode in isolation. Here we concentrate on the rotating disc flow problem for definiteness, although the work can be trivially modified to account for other three–dimensional basic flows; for example the asymptotic suction profile or the attachment line flow, see Hall & MacKerrell (**23**). In the particular problem at hand, we consider the case in which the disc rotates about the z-axis with angular velocity Ω. Relative to cylindrical polar co–ordinates (r, θ, z) which rotate with the disc and in which r and z have been made dimensionless with some reference lengthscale L, the continuity and Navier–Stokes equations for an incompressible fluid in the region $z \geq 0$ are

$$\nabla . \mathbf{u} = 0, \tag{2.1a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} . \nabla)\, \mathbf{u} + 2\,(\boldsymbol{\Omega} \wedge \mathbf{u}) + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}. \tag{2.1b}$$

Here $\mathbf{u}$ is the velocity vector, $\mathbf{r}$ is the co-ordinate vector, ρ is the fluid density, p is the fluid pressure and ν the kinematic viscosity. The Reynolds number R for the flow is given by $R = \Omega L^2/\nu$ and is taken to be large throughout the following.

It is found convenient to define the small parameter $\epsilon = R^{-\frac{1}{6}}$. As the axes rotate with the disc, the basic flow is given by the Von-Kármán solution

$$\mathbf{u} = \mathbf{u_B} = L\Omega\left(r\bar{u}(\xi), r\bar{v}(\xi), \epsilon^3\bar{w}(\xi)\right), \qquad p = \rho L^2\Omega^2\epsilon^6\bar{p}(\xi), \tag{2.2}$$

where $z = \epsilon^3\xi$ and $\bar{u}, \bar{v}, \bar{w}$ satisfy the equations

$$\bar{u}^2 - (1+\bar{v})^2 + \bar{u}'\bar{w} = \bar{u}'', \qquad 2\bar{u}(1+\bar{v}) + \bar{v}'\bar{w} = \bar{v}'', \tag{2.3a, b}$$

$$\bar{w}' + 2\bar{u} = 0. \tag{2.3c}$$

Here primes denote differentiations with respect to ξ and the appropriate boundary conditions are

$$\bar{u} = \bar{v} = \bar{w} = 0 \quad \text{on} \quad \xi = 0; \quad \bar{u} \longrightarrow 0, \quad \bar{v} \longrightarrow -1 \quad \text{as} \quad \xi \longrightarrow \infty.$$

The structure of an infinitesimal unsteady crossflow vortex imposed on the above basic flow follows closely that given by Hall (**6**) for the steady case. The mode has wavelengths scaled on the boundary layer thickness and an O(1) wavespeed and so we consider perturbation quantities with r and θ dependence given by E, defined by

$$E \equiv \exp\left[\frac{i}{\epsilon^3}\left(\int^r \alpha(r,\epsilon)dr + \theta\beta(\epsilon) - t\Omega(\epsilon)\right)\right]. \tag{2.4a}$$

We shall restrict our attention to examining disturbances in a neighbourhood of some point $r = r_n$ and expand the wavenumbers and frequency in the forms

$$\alpha = \alpha_0 + \epsilon\alpha_1 + \dots, \quad \beta = \beta_0 + \epsilon\beta_1 + \dots, \quad \Omega = \Omega_0 + \epsilon\Omega_1 + \dots. \tag{2.4b}$$

In general these quantities will be complex but here we shall be concentrating on neutrally stable disturbances (at least to leading order) and so we take α_0, β_0 and Ω_0 to be real. The disturbance structure in the z-direction is sketched in figure (1). There is an inviscid layer (I) of thickness $O(\epsilon^3)$ (the same depth as the boundary layer), and to satisfy no-slip conditions at the wall a viscous layer (II) of thickness $O\left(\epsilon^{9/2}\right)$ must be present. If the basic flow is perturbed so that the total non-dimensional flow field is given by

$$(u, v, w, p) = (r\bar{u}, r\bar{v}, \epsilon^3\bar{w}, \epsilon^6\bar{p}) + \delta\left[(U, V, W, P)\,E + c.c.\right] + \ldots, \tag{2.5a}$$

where $\delta \ll 1$, then in zone (I) the disturbance quantities U, V, W, P (which are functions of ξ alone) expand in the forms

$$(U, V, W, P) = (U_0 + \epsilon U_1 + \ldots, V_0 + \epsilon V_1 + \ldots, W_0 + \epsilon W_1 + \ldots, P_0 + \epsilon P_1 + \ldots). \tag{2.5b}$$

Substituting expressions (2.5) in the continuity and the linearised Navier–Stokes equations yields at leading orders

$$i\left(\alpha_0 U_0 + \frac{\beta_0}{r_n} V_0\right) + W_0' = 0, \tag{2.6a}$$

$$i\left(\alpha_0 r_n \bar{u} + \beta_0 \bar{v} - \Omega_0\right) U_0 + r_n \bar{u}' W_0 = -i\alpha_0 P_0, \tag{2.6b}$$

$$i\left(\alpha_0 r_n \bar{u} + \beta_0 \bar{v} - \Omega_0\right) V_0 + r_n \bar{v}' W_0 = -\frac{i\beta_0}{r_n} P_0, \tag{2.6c}$$

and

$$i\left(\alpha_0 r_n \bar{u} + \beta_0 \bar{v} - \Omega_0\right) W_0 = -P_0'. \tag{2.6d}$$

If we define

$$\bar{U}_j = \alpha_j r_n \bar{u} + \beta_j \bar{v} - \Omega_j,$$

then we obtain the classical Rayleigh equation

$$\bar{U}_0\left(W_0'' - \gamma_0^2 W_0\right) = \bar{U}_0'' W_0, \tag{2.7a}$$

where $\gamma_0^2 \equiv \alpha_0^2 + \frac{\beta_0^2}{r_n^2}$. The required boundary conditions on (2.7a) are, firstly,

$$|W_0(\infty)| < \infty, \qquad W_0(0+) = 0, \tag{2.7b, c}$$

to avoid exponential growth at the edge of the boundary layer and to give the typical inviscid condition of tangential flow as the wall is approached. Technically, to reduce this slip velocity to zero at the surface of the disc, analysis of the viscous layer (II) is required but such analysis does not affect the leading order quantities α_0, β_0 and Ω_0 which are of the primary interest here.

The Rayleigh equation (2.7a) additionally has singularities at any positions where $\bar{U}_0 = 0$, and, if this occurs, the presence of an internal critical layer (at that position) is suggested. The details of the asymptotic analysis of such a (linear) critical layer follow well-known lines (see for example Drazin & Reid (**24**)) and so we merely sketch a few of the relevant results here. If in the vicinity of the critical layer, at $\xi = \bar{\xi}$ say, we have the Taylor expansion

$$\bar{U}_j = \sum_{r=1}^{\infty} \frac{B_{jr}}{r!} \left(\xi - \bar{\xi}\right)^r , \tag{2.8a}$$

where we have defined $B_{jk} \equiv \left(\alpha_j r_n \bar{u}^{(k)} + \beta_j \bar{v}^{(k)}\right)|_{\xi=\bar{\xi}}$, then the solution of (2.7a) as $\xi \longrightarrow \bar{\xi}+$ is given by

$$W_0 = \sum_{r=0}^{\infty} g_r \left(\xi - \bar{\xi}\right)^r + \sum_{r=1}^{\infty} G_r \left(\xi - \bar{\xi}\right)^r \ln\left(\xi - \bar{\xi}\right) , \tag{2.8b}$$

where g_0, g_1 are constants upon which the remaining coefficients $g_n \quad (n = 2, 3, \ldots)$, $G_n \quad (n = 1, 2, \ldots)$ depend. In particular, we have $G_1 = g_0 B_{02}/B_{01}$. The solution (2.8b) implies that U_0 and V_0 in (2.5) have logarithmic singularities as $\xi \longrightarrow \bar{\xi}+$ and to smooth these we need to invoke viscous effects in a thin layer (III in figure 1) surrounding $\xi = \bar{\xi}$. Balancing inertial and diffusive terms suggests that this zone is of thickness $O(\epsilon^4)$ and if we define $\xi = \bar{\xi} + \epsilon\psi$ here, where $\psi = O(1)$, the disturbance quantities develop as

$$U = \epsilon^{-1}\hat{U}_0 + \ln\epsilon\hat{U}_1 + \hat{U}_2 + \epsilon\ln\epsilon\hat{U}_3 + \epsilon\hat{U}_4 + \ldots, \tag{2.9a}$$

$$V = \epsilon^{-1}\hat{V}_0 + \ln\epsilon\hat{V}_1 + \hat{V}_2 + \epsilon\ln\epsilon\hat{V}_3 + \epsilon\hat{V}_4 + \ldots, \tag{2.9b}$$

$$W = \hat{W}_0 + \epsilon\ln\epsilon\hat{W}_1 + \epsilon\hat{W}_2 + \epsilon^2\ln\epsilon\hat{W}_3 + \epsilon^2\hat{W}_4 + \ldots, \tag{2.9c}$$

$$P = \hat{P}_0 + \epsilon\ln\epsilon\hat{P}_1 + \epsilon\hat{P}_2 + \epsilon^2\ln\epsilon\hat{P}_3 + \epsilon^2\hat{P}_4 + \ldots. \tag{2.9d}$$

Briefly, we find on substituting (2.9) into the linearised Navier Stokes equations that

$$\hat{U}_0 = -\frac{i}{|B_{01}|}\left(\alpha_0 - \frac{r_n \bar{u}'_{\bar{\xi}}\gamma_0^2}{B_{01}}\right)\bar{P}_0 \int_0^{\infty} \exp\left[-i\tau sgn(B_{01})\left(\psi + \frac{B_{10}}{B_{01}}\right)\right]\exp\left(-\frac{\tau^3}{3|B_{01}|}\right) d\tau, \tag{2.10a}$$

where $\bar{P}_0 = P_0(\bar{\xi})$. This solution, as $|\psi| \longrightarrow \infty$, matches with the solution for U_0 in (2.6) as $\xi \longrightarrow \bar{\xi}$. Across the critical layer we also find that

$$\hat{W}_0 = -\frac{i\gamma_0^2 \bar{P}_0}{B_{01}}, \tag{2.10b}$$

$$\hat{W}_1 = \frac{g_0 B_{02}}{B_{01}} \left(\psi + \frac{B_{10}}{B_{01}}\right), \tag{2.10c}$$

and

$$\frac{d^2\hat{W}_2}{d\psi^2} = \frac{iB_{02}g_0}{|B_{01}|} \int_0^\infty \exp\left[-i\tau sgn(B_{01})\left(\psi + \frac{B_{10}}{B_{01}}\right)\right] \exp\left(-\frac{\tau^3}{3|B_{01}|}\right) d\tau. \tag{2.10d}$$

If we integrate (2.10d) once and then apply the asymptotic results given by Haberman (**25**) concerning the behaviour of integrals of the general type

$$\int_0^\infty M(\tau)\left(\frac{e^{-i\tau\psi} - 1}{\tau}\right) d\tau,$$

for large argument ψ, we may show that the derivative $d\hat{W}_2/d\psi$ suffers a discontinuity across the critical layer and in particular that

$$\left[\frac{d\hat{W}_2}{d\psi}\right]_{\psi \longrightarrow -\infty}^{\psi \longrightarrow \infty} = \frac{i\pi B_{02}g_0}{|B_{01}|}. \tag{2.11}$$

The implication of the solutions (2.10), (2.11) is that the Frobenius solution (2.8b) for the eigenfunction W_0 as $\xi \longrightarrow \bar{\xi}+$ continues to hold in the limit $\xi \longrightarrow \bar{\xi}-$ so long as the replacement of $\ln(\xi - \bar{\xi})$ by $(\ln(\bar{\xi} - \xi) - i\pi sgn(B_{01}))$ in (2.8b) is made. Equivalently, we have that as $\xi \longrightarrow \bar{\xi}-$,

$$W_0 = \sum_{r=0}^{\infty} g_r\,(\xi - \bar{\xi})^r + \sum_{r=1}^{\infty} G_r\,(\xi - \bar{\xi})^r\,[\ln(\bar{\xi} - \xi) - i\pi sgn(B_{01})], \tag{2.12}$$

just below the critical layer. This is the well known phase shift of '$i\pi$' across a linear critical layer.

We now have a complete specification of the problem to determine the leading order neutral wavenumbers α_0, β_0 at some given frequency Ω_0. The Rayleigh equation (2.7a) needs to be solved subject to the boundary conditions (2.7b, c) and where the replacement given by (2.12) in the local series solution (2.8b) across the critical layers is applied. The analysis presented here is specific to the case of rotating disc flow, but may easily be modified to account for a variety of other three-dimensional basic velocity profiles.

§3. The isolation of resonant triads.

At the outset our aim was to verify that resonant triads (with the property that each member of the triad is neutrally stable at leading order) are possible within particular three-dimensional boundary layer flows. In the context of rotating disc flow, we therefore wish to seek ordered triples $\{(\alpha_{0j}, \beta_{0j}, \Omega_{0j})\,;\, j = 1,2,3\}$ such that

$$\alpha_{01} = \alpha_{02} + \alpha_{03}, \qquad \beta_{01} = \beta_{02} + \beta_{03}, \qquad \Omega_{01} = \Omega_{02} + \Omega_{03}, \tag{3.1}$$

and with the characteristic that each triple is an eigenvalue set of the system (2.7), (2.8) & (2.12). To this end a numerical approach was adopted.

A first concern is to find the location(s) of the critical layer(s). These occur wherever $\bar{U}_0 = 0$; i.e. wherever

$$\alpha_0 r_n \bar{u} + \beta_0 \bar{v} = \Omega_0, \tag{3.2a}$$

or, if we define the leading order wavespeed $c_0 = \Omega_0 / r_n \alpha_0$; wherever

$$U_* \equiv \bar{u} + \mu \bar{v} = c_0, \tag{3.2b}$$

with $\mu = \beta_0/(r_n \alpha_0)$. Hence, the 'effective' basic velocity profile U_* is a function of μ and this flow is sketched in figure (2). It is known that $\bar{u}'(0) = 0.510$, $\bar{v}'(0) = -0.616$ and then as the parameter μ varies the effective basic velocity acquires a number of distinct possible forms. The boundary conditions in (2.3) suggest that $U_* \longrightarrow -\mu$ as $\xi \longrightarrow \infty$ and for $\mu > \mu_c$ (≈ 0.8284) the profile is a monotonic decreasing function. For $0 < \mu < \mu_c$, U_* vanishes somewhere other than at the origin, and indeed when $\mu \approx 0.235$, the point of inflexion of the basic flow U_* coincides with the point at which it vanishes. In this case the boundary layer becomes susceptible to the inviscid stationary vortex found by GSW. For $\mu < 0$, U_* is always a positive function and for $\mu <\approx -0.6$ it is monotonic as well. We see from figure (2) that, depending upon the values of μ and c_0 in (3.2b), we may have either one or two critical layers.

The numerical scheme used in practice requires that a value of μ be preselected and then α_0, β_0 and Ω_0 are determined such that $\mu = \beta_0/(r_n \alpha_0)$. It is easily shown from (2.7a) that as $\xi \longrightarrow \infty$ the eigenfunction W_0 is proportional to $\exp(-\gamma_0 \xi)$. A suitably large value of ξ, say ξ_∞, was chosen and the asymptotic behaviour of the eigensolution was assumed to hold there. With these starting conditions, the

Rayleigh equation (2.7a) was integrated with a fourth order Runge–Kutta scheme to a point a small distance, say Δ, above the first critical layer encountered. There the series solution (2.8b) was assumed to hold and g_0 and g_1 were chosen so that the necessary matching conditions held. Then, using the replacement suggested by (2.12), the solution just below the critical layer was established which could then be used to integrate (2.7a) until either the next critical layer position was reached (in which case the process described above was repeated) or until $\xi = 0$. In general, the predicted value of $W_0(\xi = 0)$ is complex and non–zero and so a Newton iteration on the real values α_0 and c_0 was employed to ensure that the required condition (2.7c) was indeed met. Given the eigenvalues α_0 and c_0 for some chosen value μ it is then a simple task to retrieve the leading order azimuthal wavenumber β_0 and the frequency Ω_0. This numerical work was checked for independence from the chosen value ξ_∞, the step length used in the Runge–Kutta integration and of the distance Δ from the critical layer(s) at which the series solutions were applied.

The solutions of the eigenproblem for various values of the parameter μ (which is $\tan^{-1}\phi$, where ϕ is the angle which the considered disturbances make with the outward radial direction on the disc) are shown in figure (3), where we illustrate the dependence of the leading order radial wavenumber α_0 and the frequency Ω_0 on the wavenumber β_0. There are several noteworthy features of the solutions. Firstly, there are two distinct branches in solution space which are independent of each other and do not coalesce. On one branch (labelled I on figure (3)) we see that there is a non–trivial solution of the eigensystem with zero frequency and this corresponds to the inviscid stationary mode of (**2**). Here $\alpha_0 \approx 1.1$, $\beta_0 \approx 0.26r_n$ and $\mu \approx 0.235$, and then the basic profile U_* has a point of inflexion coincident with the point at which it vanishes. As the parameter μ increases we find that the frequency becomes negative and using the sketches of the basic flow in figure (2) we see that we are reduced to an eigenproblem with a single critical point. Appealing to the ideas of Stuart in (**2**), in the case of only one critical layer the neutral wavespeed c_0 is determined by the condition that the function U_* in (3.2b) must have a point of inflexion where $U_* = c_0$.

For a waveangle parameter μ less than the inviscid stationary mode value of 0.235 we now have two critical layers in the flow and the neutral frequency is positive. As $\mu \longrightarrow 0$ the wavenumbers and frequency become small. Finally for this branch, as $\mu \longrightarrow 0$ or $\mu \longrightarrow \infty$ then the wavenumbers in the radial and azimuthal directions become of different orders of magnitude and the symmetry of the governing equations (2.6) is lost. The structures which would govern these limiting cases are then not covered by the work of section 2 and it would be interesting to identify and analyse these structures.

On the second branch of solutions (labelled II on figure 3) it is immediately seen that only positive frequencies are found. In particular, we remark that as μ increases and approaches $\mu_c \approx 0.187$ (corresponding to a disturbance waveangle of 10.6^o) the wavenumbers and frequency become small. This suggests the existence of a longer wavelength disturbance with a small wavespeed and this structure corresponds to that of the 'upper-branch' mode investigated in (**12**). In this paper, small perturbations with wavenumbers $O(R^{\frac{5}{12}})$ and wavespeeds $O(R^{-\frac{1}{12}})$ were shown to be possible at waveangles between approximately 10.6^o and 39.6^o. As μ decreases from μ_c in our current problem and becomes negative the convergence of the numerical scheme employed becomes increasingly difficult to achieve and only the neutral solutions which can be reliably located by our method have been shown in figure (3). Indeed, for sufficiently negative μ the basic profile is monotonically increasing and has no point of inflexion. At these values of parameter, no neutral solutions of the sought type can exist and so it suggests the existence of some $\mu_* < 0$ at which our solutions cut out. It is seen from figure (2) that as μ decreases from zero the neutral disturbances have two critical layers up to the point where the basic velocity is no longer non-monotonic. Then the two critical layers must merge, cease to remain asymptotically distinct and a fresh analysis is required. Since the principal thrust of the present work is to study resonant triads we do not explore this aspect any further here. However, we remark that these considerations show that neutrally stable linearised disturbances are possible at propagation angles $> \tan^{-1}(\mu_*)$. There is therefore, a lower bound for this angle but no formal upper bound since from figure (3) we can see that neutral modes exist for $\mu_* < \mu < \infty$. This contrasts with the results of (**12**) where it was found that for linear disturbances the possible waveangles associated with upper branch neutral modes were bounded both above and below.

Given the eigenvalues of the problem specified by (2.7), (2.8) & (2.12) we sought resonant triad triples $(\alpha_{0j}, \beta_{0j}, \Omega_{0j})$ of waves which are each neutrally stable at leading order and which satisfy requirements (3.1). We found two such sets, in particular

$$\begin{aligned} &\alpha_{01} = 0.8514, \qquad &&\beta_{01} = 0.2648 r_n, \qquad &&\Omega_{01} = -3.137 \times 10^{-2}, \\ &\alpha_{02} = 0.7343, &&\beta_{02} = 0.0731 r_n, &&\Omega_{02} = 1.12 \times 10^{-2}, \\ &\alpha_{03} = 0.1171, &&\beta_{03} = 0.1917 r_n, &&\Omega_{03} = -4.25 \times 10^{-2}, \end{aligned} \tag{3.3a}$$

and

$$\begin{aligned} &\alpha_{01} = 0.5847, \qquad &&\beta_{01} = 0.2764 r_n, \qquad &&\Omega_{01} = -6.29 \times 10^{-2}, \\ &\alpha_{02} = 0.3605, &&\beta_{02} = 0.2699 r_n, &&\Omega_{02} = -7.16 \times 10^{-2}, \\ &\alpha_{03} = 0.2242, &&\beta_{03} = 6.46 \times 10^{-3} r_n, &&\Omega_{03} = 8.76 \times 10^{-3}. \end{aligned} \tag{3.3b}$$

We do not claim that these are the only possible triads but the important result here is that resonant triples with the sought neutral stability properties do indeed exist within a three-dimensional boundary layer. The type of analytical and numerical work presented here can be applied to other basic velocity profiles. Indeed very similar results as to the forms of the neutral stability curves in α_0/β_0 and Ω_0/β_0 spaces appear for the attachment line profile and, of course, given the existence of triads for a simple rotating disc flow we have no reason to doubt that a similar conclusion holds for other much more complicated and more physically realistic three-dimensional boundary layers.

This verification of the existence of resonant triads leads us on to an integral part of the current work– that of considering the disturbance structure of the three members of the triad when they are permitted to interact and evolve on a suitably slow timescale.

§4. An asymptotic description of the interaction mechanisms in a resonant triad of crossflow vortices.

We have illustrated that within the context of three-dimensional boundary layer flows resonant triads comprising of crossflow vortices can arise. We now investigate an asymptotic description of how the members of the triad may evolve in time and space once the interaction processes are accounted for. To ensure that the structures of the critical layers in the flow remain as simple as possible for this study we assume that the amplitude of each vortex is $O(\delta)$ where $\delta \ll 1$ is sufficiently small such that all the the critical layers are linear in character. Then, through the nonlinear terms in the Navier Stokes equations, it is found that each pair of modes combine to drive the third mode via terms of size $O(\delta^2)$ and this in turn suggests that the interaction mechanisms force the vortices to evolve on slow length and time scales of size $O(\delta)$.

If again we relate our working to the rotating disc problem as a concrete example, suppose that we wish to describe the interaction of three linear vortices with leading order wavenumbers α_{0j}, β_{0j} and of frequencies Ω_{0j} $(j = 1, 2, 3)$ such that the resonance conditions (3.1) hold. Then we define

$$E_j \equiv \exp\left[\frac{i}{\epsilon^3}\left(\int_{r_n}^{r} \alpha_j(r,\epsilon)dr + \theta\beta_j(\epsilon) - t\Omega_j(\epsilon)\right)\right], \tag{4.1a}$$

where

$$\alpha_j = \alpha_{0j} + \epsilon\alpha_{1j} + \ldots, \qquad \beta_j = \beta_{0j} + \epsilon\beta_{1j} + \ldots, \qquad \Omega_j = \Omega_{0j} + \epsilon\Omega_{1j} + \ldots, \tag{4.1b}$$

and $E_1 = E_2E_3$. We allow the disturbance vortices to develop on slow lengthscales $r_1 = \delta(r - r_n)$, $\theta_1 = \delta\theta$ and timescale $t_1 = \delta t$, so that in the governing Navier Stokes equations we formally use the methods of multiple scales to replace the r, θ and t derivatives according to the transformations

$$\begin{gathered}\frac{\partial}{\partial r} \longrightarrow \frac{\partial}{\partial r} + \delta\frac{\partial}{\partial r_1}, \qquad \frac{\partial}{\partial \theta} \longrightarrow \frac{\partial}{\partial \theta} + \delta\frac{\partial}{\partial \theta_1}, \\ \frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} + \delta\frac{\partial}{\partial t_1}.\end{gathered} \tag{4.2}$$

As previously, we choose to examine vortices in the neighbourhood of some point $r = r_n$ and then, in the main inviscid region (I) where $z = \epsilon^3\xi$, we are led to consider perturbation quantities of the forms

$$U = \delta\left[\sum_{j=1}^{3}(U_{00j} + \ldots)A_jE_j + c.c.\right] + \delta^2\left[\sum_{j=1}^{3}U_{10j}E_j + \ldots\right] + \ldots, \tag{4.3a}$$

$$V = \delta\left[\sum_{j=1}^{3}(V_{00j} + \ldots)A_jE_j + c.c.\right] + \delta^2\left[\sum_{j=1}^{3}V_{10j}E_j + \ldots\right] + \ldots, \tag{4.3b}$$

$$W = \delta\left[\sum_{j=1}^{3}(W_{00j} + \ldots)A_jE_j + c.c.\right] + \delta^2\left[\sum_{j=1}^{3}W_{10j}E_j + \ldots\right] + \ldots, \tag{4.3c}$$

$$P = \delta \left[\sum_{j=1}^{3} \left(P_{00j} + \ldots \right) A_j E_j + c.c. \right] + \delta^2 \left[\sum_{j=1}^{3} P_{10j} E_j + \ldots \right] + \ldots . \tag{4.3d}$$

In (4.3) the scaled amplitudes A_j are functions of the slow scales r_1, θ_1 and t_1 and $U_{00j}, V_{00j}, \ldots, U_{10j}, \ldots$ are functions of ξ alone. Substituting (4.2) and (4.3) into the Navier Stokes equations and comparing coefficients of E_j we find, as in section 2, that at leading orders W_{00j} satisfies the Rayleigh equation

$$\bar{U}_{0k} \left(W_{00k}'' - \gamma_{0k}^2 W_{00k} \right) - \bar{U}_{0k}'' W_{00k} = 0, \qquad k = 1, 2, 3, \tag{4.4}$$

where $\bar{U}_{jk} = \alpha_{jk} r_n \bar{u} + \beta_{jk} \bar{v} - \Omega_{jk}$ and $\gamma_{jk}^2 = \alpha_{jk}^2 + (\beta_{jk}^2 / r_n^2)$. The functions U_{00j} and V_{00j} may be found in terms of W_{00j} by suitable manipulations of the governing equations, which are precisely (2.6) with $\alpha_0, \beta_0, \Omega_0$ replaced by $\alpha_{0j}, \beta_{0j}, \Omega_{0j}$ and U_0, V_0, W_0, P_0 by $U_{00j}, V_{00j}, W_{00j}, P_{00j}$.

¿From here onward we concentrate on deriving the evolution equation for the vortex proportional to E_1. Given the obvious symmetry in the problem, together with the generalised way in which the amplitude equations will be derived, once the governing equation has been obtained for one mode it is formally straightforward to obtain the corresponding equations for the other two modes. At $O(\delta^2)$ in the Navier Stokes equations we find that the function W_{101} in (4.3c) satisfies the non-homogeneous Rayleigh equation

$$\begin{aligned} &\bar{U}_{01} \left(W_{101}'' - \gamma_{01}^2 W_{101} \right) - \bar{U}_{01}'' W_{101} = \\ &\quad - i W_{001} \left(\mathcal{L}'' - \frac{\bar{U}_{01}''}{\bar{U}_{01}} \mathcal{L} \right) A_1 - 2i \bar{U}_{01} W_{001} \left(\alpha_{01} \frac{\partial A_1}{\partial r_1} + \frac{\beta_{01}}{r_n} \frac{\partial A_1}{\partial \theta_1} \right) + I_* A_2 A_3, \end{aligned} \tag{4.5a}$$

where

$$\begin{aligned} I_* = i &\left[(\beta_{03}\alpha_{01} + \alpha_{02}\beta_{01}) \frac{U_{003} V_{002}}{r_n} + (\beta_{02}\alpha_{01} + \alpha_{03}\beta_{01}) \frac{U_{002} V_{003}}{r_n} + \alpha_{01}^2 U_{002} U_{003} + \right. \\ &\left. \frac{\beta_{01}^2}{r_n^2} V_{002} V_{003} \right]' + \left[W_{003} \left(\alpha_{01} U_{002} + \frac{\beta_{01}}{r_n} V_{002} \right)' + W_{002} \left(\alpha_{01} U_{003} + \frac{\beta_{01}}{r_n} V_{003} \right)' \right]' \\ &- i\gamma_{01}^2 \left[i \left(\alpha_{02} U_{003} W_{002} + \alpha_{03} W_{003} U_{002} + \frac{\beta_{02}}{r_n} V_{003} W_{002} + \frac{\beta_{03}}{r_n} V_{002} W_{003} \right) \right. \\ &\qquad \left. + W_{003} W_{002}' + W_{002} W_{003}' \right], \end{aligned} \tag{4.5b}$$

and

$$\mathcal{L} \equiv \left(\frac{\partial}{\partial t_1} + r_n \bar{u} \frac{\partial}{\partial r_1} + \bar{v} \frac{\partial}{\partial \theta_1} \right) . \tag{4.5c}$$

The appropriate boundary conditions on the function W_{01} are very similar to those for W_{001} in (4.4). In particular, to avoid unbounded growth as $\xi \longrightarrow \infty$ requires that $|W_{101}(\infty)| < \infty$ and for no disturbance at the surface of the disc, $W_{101}(0+) = 0$. Additionally, it is clear that appropriate jump conditions need to be imposed at the position of the critical layer(s). For ease of notation, suppose that the mode proportional to E_j has associated critical layers located at $\bar{\xi}_j$ and note that there could well be more than one critical point for each mode. Inspection of (4.5) suggests that singularities in the solution $W_{101}(\xi)$ may be expected at all the critical layers $\bar{\xi}_j$ for $j = 1, 2, 3$, not just at the positions $\bar{\xi}_1$.

The form of the solution of (4.5a) in the vicinity of the critical layers at $\bar{\xi}_j$ may be one of two distinct types. Close to the position $\bar{\xi}_1$ we can easily show that the right–hand side of (4.5a) is asymptotically proportional to $1/(\xi - \bar{\xi}_1)$ and from this we can deduce that for $0 < \xi - \bar{\xi}_1 \ll 1$,

$$W_{101} = \hat{X}_0 \log(\xi - \bar{\xi}_1) + O(1) + \ldots , \tag{4.6a}$$

where $\hat{X}_0$ is a constant which may be found analytically. Close to the other critical layers, $\bar{\xi}_j$ $(j = 2, 3)$, the behaviour of W_{101} is more complicated. Then we obtain, from (2.6) & (2.8), that

$$I_* \sim \frac{X_{0j}}{(\xi - \bar{\xi}_j)^3} + \frac{X_{1j}}{(\xi - \bar{\xi}_j)^2} + \frac{X_{2j}}{(\xi - \bar{\xi}_j)} + \ldots , \tag{4.6b}$$

and then

$$W_{101} \sim \frac{D_{0j}}{(\xi - \bar{\xi}_j)} + D_{1j} \ln(\xi - \bar{\xi}_j) + D_{2j} + D_{3j}(\xi - \bar{\xi}_j) \ln(\xi - \bar{\xi}_j) + \ldots , \tag{4.6c}$$

where the constants X_{kj}, D_{kj} are given in the Appendix. The solutions (4.6) are valid for $\xi > \bar{\xi}_j$ and clearly need to be modified for the region $\xi < \bar{\xi}_j$. This requires careful scrutiny of the critical layer properties in the vicinity of $\xi = \bar{\xi}_j$ in much the same way as was outlined in §2. The main difference now is that the algebraic manipulation needed within the critical layer region becomes much more complex due to the presence of all three modes. Rather than give details of the inevitably lengthy (but conceptually fairly straightforward) critical layer analysis, we elect to merely state that the result of this work is as is to be expected from the classical theory: that is that we obtain a '$i\pi$' phase shift in the logarithmic terms as characterised by (2.12). This then enables us to write down the behaviour of W_{101} for $0 < \bar{\xi}_j - \xi \ll 1$ and thence to continue the solution W_{101} into the region $\xi < \bar{\xi}_j$.

To derive the evolution equation for the amplitude A_1 we follow the usual process of multiplying (4.5a) by W_{001} and integrating over the range $[0, \infty)$, paying particular attention to the behaviour of the integrals close to the critical layers. Recalling the solutions (4.6) and the phase jumps across the critical layers, combining the integration with the boundary conditions imposed at $\xi = 0$ and at $\xi = \xi_\infty$ leads to the desired amplitude equation. After simplification, and substituting the definition of $\mathcal{L}$ given by (4.5c), we obtain

$$
\begin{aligned}
&\Bigg[\Bigg(\frac{2\pi B_{012}}{|B_{011}|B_{011}} r_n \bar{u}' W_{001}\left(W'_{001}\right)_{\text{finite part}} - \frac{\pi r_n \bar{u}''}{|B_{011}|} W_{001}^2 + \frac{\pi B_{012} r_n \bar{u}' W_{001}^2}{|B_{011}|B_{011}} \\
&\qquad - \frac{\pi r_n \bar{u}}{|B_{011}|}\left(\frac{B_{012}^2}{B_{011}^2} - \frac{B_{013}}{B_{011}}\right) W_{001}^2 - \frac{i\pi^2 r_n \bar{u} B_{012}^2 W_{001}^2}{B_{011}^2}\Bigg)_{\bar{\xi}_1} \\
&+ i \int_0^\infty \left(\frac{r_n \bar{u}''}{\bar{U}_{01}} W_{001}^2 - \frac{r_n \bar{u} \bar{U}''_{01}}{\bar{U}_{01}^2} W_{001}^2\right) d\xi + 2i \int_0^\infty \alpha_{01} r_n W_{001}^2 d\xi\Bigg] \frac{\partial A_1}{\partial r_1} \\
&+ \Bigg[\Bigg(\frac{2\pi B_{012}}{|B_{011}|B_{011}} \bar{v}' W_{001}\left(W'_{001}\right)_{\text{finite part}} - \frac{\pi \bar{v}''}{|B_{011}|} W_{001}^2 + \frac{\pi B_{012} \bar{v}' W_{001}^2}{|B_{011}|B_{011}} \\
&\qquad - \frac{\pi \bar{v}}{|B_{011}|}\left(\frac{B_{012}^2}{B_{011}^2} - \frac{B_{013}}{B_{011}}\right) W_{001}^2 - \frac{i\pi^2 \bar{v} B_{012}^2 W_{001}^2}{B_{011}^2}\Bigg)_{\bar{\xi}_1} \\
&+ i \int_0^\infty \left(\frac{\bar{v}''}{\bar{U}_{01}} W_{001}^2 - \frac{\bar{v} \bar{U}''_{01}}{\bar{U}_{01}^2} W_{001}^2\right) d\xi + 2i \int_0^\infty \beta_{01} W_{001}^2 d\xi\Bigg] \frac{\partial A_1}{\partial \theta_1} \\
&+ \Bigg[\Bigg(-\frac{\pi}{|B_{011}|}\left(\frac{B_{012}^2}{B_{011}^2} - \frac{B_{013}}{B_{011}}\right) W_{001}^2 - \frac{i\pi^2 B_{012}^2 W_{001}^2}{B_{011}^2}\Bigg)_{\bar{\xi}_1} \\
&\qquad\qquad - i \int_0^\infty \frac{\bar{U}''_{01}}{\bar{U}_{01}^2} W_{001}^2 d\xi\Bigg] \frac{\partial A_1}{\partial t_1} \\
&+ \left[\sum_{j=2}^{3}\left[\frac{i\pi B_{0j1}}{|B_{0j1}|}\left(D_{j1}\hat{g}_{j1} - D_{3j}\hat{g}_{j0}\right)\right]_{\bar{\xi}_j} - \int_0^\infty \frac{I_* W_{001}}{\bar{U}_{01}} d\xi\right] A_2 A_3 = 0.
\end{aligned}
\tag{4.7}
$$

In this amplitude equation we have defined

$$B_{0jk} = \left(r_n \alpha_{0j} \bar{u}^{(k)}(\xi) + \beta_{0j} \bar{v}^{(k)}(\xi)\right), \tag{4.8}$$

and have supposed that

$$W_{001} = \sum_{k=0}^{\infty} \hat{g}_{jk} \left(\xi - \bar{\xi}_j\right)^k,$$

close to the critical layers $\bar{\xi}_j$ $(j = 2, 3)$ at which this function does not have singularities. Otherwise, the notation is as previously defined.

In (4.7) the expression $⨍$ denotes the Hadamard (finite) part of the corresponding integral. Some of the integrals which are contained in the coefficients of $\partial A_1/\partial r_1$, $\partial A_1/\partial \theta_1$ and $\partial A_1/\partial t_1$ are singular at $\bar{\xi}_1$: the integral in the coefficient of $A_2 A_3$ is singular at the other critical points as well.

We notice that if the various integrals and summations were to be evaluated numerically and the formulae (4.7) rearranged appropriately, we could obtain an evolution equation for the E_1 mode which assumes the form;

$$\frac{\partial A_1}{\partial t_1} + \lambda_{11} \frac{\partial A_1}{\partial r_1} + \lambda_{12} \frac{\partial A_1}{\partial \theta_1} + \lambda_{13} A_2 A_3 = 0, \tag{4.9a}$$

where the coefficients $\lambda_{11}, \lambda_{12}, \lambda_{13}$ are some constants. By repeating the analysis outlined above we can, in an identical way, derive evolution equations for the other two modes of the types

$$\frac{\partial A_2}{\partial t_1} + \lambda_{21} \frac{\partial A_2}{\partial r_1} + \lambda_{22} \frac{\partial A_2}{\partial \theta_1} + \lambda_{23} A_1 A_3^* = 0, \tag{4.9b}$$

$$\frac{\partial A_3}{\partial t_1} + \lambda_{31} \frac{\partial A_3}{\partial r_1} + \lambda_{32} \frac{\partial A_3}{\partial \theta_1} + \lambda_{33} A_1 A_2^* = 0, \tag{4.9c}$$

where an asterisk on a quantity denotes the complex conjugate of that quantity.

Clearly, the numerical work required to actually evaluate the coefficients in (4.9) for any particular basic flow is nontrivial. However, to illustrate the type of evolution system which may be obtained we considered the computational procedure relating to the resonant triple given by (3.3a). For each member of the triad we computed the eigenfunction W_{00j} by solving the Rayleigh equation (2.7a) subject to boundary conditions (2.7b, c) and jump conditions (2.12). The corresponding radial and azimuthal velocity fields U_{00j}, V_{00j} may then be determined in terms of the functions $\bar{U}_{0j}$ and W_{00j} by eliminating the pressure from (2.6b, c). Since the problem (2.7) is linear we needed to fix some normalising constraint and this was chosen such that as $\xi \longrightarrow \infty$, W_{00j} was a real, positive function and that $\max\{0 \leq \xi < \infty : \quad |W_{00j}| = 1\}$.

At this stage the coefficients λ_{j1} and λ_{j2} in (4.9) were evaluated. The integrals involved in finding these unknowns are singular only at $\bar{\xi}_j$ and the analytic nature of these singularities may be deduced in a routine manner. To find the finite parts of the integrals across the positions $\bar{\xi}_j$, we subtracted the singular behaviour of the integrand in the neighbourhood of $\bar{\xi}_j$ and integrated the remaining (necessarily well behaved) function across $\bar{\xi}_j$. By utilising the formula given by Lighthill in **(26)** it was possible to deduce the Hadamard part of the subtracted singularity function and thence determine the value of the finite part of the original integral. The method worked well in practice and the corresponding results are given below.

We turn now to address the much more complicated task of determining the interaction coefficients λ_{j3} and again, as a concrete example, concentrate on the determination of λ_{13} for the method for calculating the other interaction coefficients follows in a very similar way. Then the function I_* is as defined in (4.5*b*) and was computed by using the information previously derived on U_{00j}, V_{00j} and W_{00j} and detailed above. It can be shown by straightforward analysis and using (4.6*b*) that the integrand $(I_* W_{001}/\bar{U}_{01})$ has singularities at each critical layer and to calculate this integral a technique was employed which is very similar to that already described. In the vicinity of each critical layer the singular part of the integrand was removed as before, except now we found that the remaining function was very erratic near the critical point. This is due to the fact that the integrand can only be determined numerically at the outset and the process involves the estimation of several derivatives of U_{00j}, V_{00j} which is a probable source of numerical error. On making the usual checks foı numerical stability, the determination of the finite part of the integral was found to be strongly influenced by the choice of the intervals around the critical points in which the singular behaviour was removed. This is obviously unreasonable and a more satisfactory procedure was found in which the integrand was written as the sum of the singular part and an interpolated polynomial in the vicinity of the critical points. Even so, in this case there was found to be some variation in the determined value of the coefficient λ_{13}, but it has to be remembered that this coefficient depends on a calculation in which four separate critical layers are traversed with a strong singularity needing to be accounted for at each critical layer. Consequently, the difficulties encountered in obtaining accurate values for the interaction coefficients in the evolution equations (4.9) are not surprising.

In this paper we have managed to obtain approximate values for the interaction coefficients. What we have shown is that determination of the interaction coefficients for the flow structure considered here is possible in practice, and, with the use of more sophisticated numerical methods, there is no reason to doubt that the accuracy of the coefficients could be markedly improved.

We now describe some properties of the solutions of the evolution equations and conclude with some discussion.

§5. Results and Discussion.

The numerical work described above leads to evolution equations which assume the forms

$$\frac{\partial A_1}{\partial t_1} = r_n(-0.114, 4.09 \times 10^{-3})\frac{\partial A_1}{\partial r_1} + (0.305, 0.111)\frac{\partial A_1}{\partial \theta_1} + (2.0, 0.6)A_2 A_3, \quad (5.1a)$$

$$\frac{\partial A_2}{\partial t_1} = r_n(7.52 \times 10^{-2}, -7.74 \times 10^{-2})\frac{\partial A_2}{\partial r_1} + (0.265, 0.117)\frac{\partial A_2}{\partial \theta_1} + (0.15, -0.15)A_1 A_3^*, \quad (5.1b)$$

$$\frac{\partial A_3}{\partial t_1} = r_n(-2.4 \times 10^{-2}, -0.14)\frac{\partial A_3}{\partial r_1} + (0.14, 0.13)\frac{\partial A_3}{\partial \theta_1} + (0.3, 0.4)A_1 A_2^*, \quad (5.1c)$$

for the resonant triple given by (3.3a) and (4.3). As already mentioned, the effect of having four critical layers within the flow makes the calculation of the interaction coefficients very sensitive and we believe that the values of these quantities quoted in (5.1) are correct to within about ten per cent.

Naturally, the evolution of the wave amplitudes A_1, A_2 and A_3 is very dependent upon the initial conditions and the behaviours of these quantities may, in general, only be determined by a numerical approach. We do not study this facet of the problem here but, instead, draw an analogy between the forms of (5.1) and those derived by Smith & Stewart (**22**) in their investigation of resonant triad nonlinear interaction in boundary layer transition. The evolution equations derived by these authors were a simpler form of (5.1), (effectively with $\partial/\partial r_1$ and $\partial/\partial\theta_1$ replaced by constants) with the interaction coefficients purely imaginary quantities. Smith & Stewart integrated their equations using a predictor–corrector formulation to march forward in time and the resulting evolution of the wave amplitudes took a very complicated form. These amplitudes generally increased with time although this trend was modified by the superposition of a very short period, large oscillation which had the overall effect of giving the function $|A_j(t_1)|$ a very spiky appearance. Similar types of evolution equations have been derived by Avis (**27**), who was concerned with resonant triads in axisymmetric flow on a cylindrical geometry, and by Bowles (**28**), who investigated triads within transonic flow. These authors obtained numerical solutions for the development of the mode amplitudes, and both found behaviour qualitative to that described in (**22**).

Very little is known about particular solutions of (5.1), although some work on related equations has been studied by Craik (**1, 29, 30**). No general analytic solutions for these equations have been found. However, an important property of the system

(5.1) is under what conditions these equations admit singularities in the solutions A_j, $(j = 1, 2, 3)$, either in a finite or an infinite time. Wang (**31**) has derived bounds for the solutions of three-wave interaction systems with arbitrary coupling coefficients. He showed that if

$$\frac{da_0}{dt} = i\omega_0 a_0 + \mu_0 a_1 a_2, \quad \frac{da_1}{dt} = i\omega_1 a_1 + \mu_1 a_0 a_2^*, \quad \frac{da_2}{dt} = i\omega_2 a_2 + \mu_2 a_0 a_1^*, \qquad (5.2)$$

with initial data $a_i(0) = a_{i0}$, $i = 1, 2, 3$: then explosive instabilities (instabilities that become unbounded eventually, possibly after infinite time) cannot exist if

$$\alpha < 0 \qquad \text{and} \qquad ||\mathbf{a}(0)|| < \sqrt{|\alpha|/\beta}, \qquad (5.3a)$$

where

$$\begin{aligned} \alpha \equiv \min\{ &\max\{\lambda_{00}, \lambda_{12}, \lambda_{21}\}, \max\{\lambda_{01}, \lambda_{10}, \lambda_{22}\}, \\ &\max\{\lambda_{00}, \lambda_{01}, \lambda_{22}\}, \max\{\lambda_{01}, \lambda_{12}, \lambda_{21}\}, \\ &\max\{\lambda_{02}, \lambda_{10}, \lambda_{21}\}, \max\{\lambda_{02}, \lambda_{11}, \lambda_{20}\}\}, \end{aligned} \qquad (5.3b)$$

$$\beta \equiv \frac{1}{2} \max_i \{|\mu_i|\}, \qquad \text{and} \qquad \lambda_{ij} \equiv |\mu_j| - 2\text{Im}\,(\omega_i)\,. \qquad (5.3c)$$

Here $||\mathbf{a}|| = ||\,(a_0, a_1, a_2)\,|| \equiv \left(\sum_{i=0}^{2} a_i a_i^*\right)^{1/2}$.

If we allow the disturbances proportional to (4.1) to be neutrally stable in space co-ordinates r_1 and θ_1 we can substitute

$$\frac{i\alpha_1}{r_n} \qquad \text{and} \qquad i\beta_1, \qquad (5.4)$$

for $\partial/\partial r_1$ and $\partial/\partial\theta_1$ in (5.1) with α_1 and β_1 real. Then using the results (5.3) we can find the region in (α_1, β_1) space for which explosive instabilities of the system (5.1) are impossible, at least for sufficiently small initial disturbance amplitudes. This information is illustrated in Figure (4) where we indicate the minimum size of the initial amplitude required in order to ensure that the possibility of explosive breakdown is maintained. Broadly speaking, we can conclude from figure (4) that once we are located in that part of (α_1, β_1) space at which a finite initial amplitude is required to enable explosive breakdown then by decreasing the radial wavenumber α_1 and/or by increasing the azimuthal wavenumber β_1 there needs to be a corresponding rise in the initial disturbance amplitude if the breakdown mechanism is to remain possible.

It should be noted that the point corresponding to the choice $\alpha_1 = \beta_1 = 0$ lies in the zone for which the possibility of explosive instability cannot be discounted irrespective of the initial conditions. Then the governing equations reduce to the forms

$$\frac{dA_1}{dt_1} = (2.0, 0.6)A_2A_3, \qquad \frac{dA_2}{dt_1} = (0.15, -0.15)A_1A_3^*, \qquad \frac{dA_3}{dt_1} = (0.3, 0.4)A_1A_2^*. \tag{5.5}$$

These are, of course, the evolution equations obtained by allowing the disturbances to evolve temporally whilst suppressing spatial dependence. It is a straightforward task to write $b_1 \equiv A_1^*$, $b_2 \equiv A_2$, $b_3 \equiv A_3$ and to normalise so that the system (5.5) may be rewritten as

$$\frac{db_i}{dt_1} = s_i b_j^* b_k^*, \qquad (i, j, k = 1, 2, 3 \quad \text{cyclically}), \tag{5.6}$$

where the interaction coefficients $s_i = \exp(i\phi_i)$. Craik **(29, 30)** has shown that 'phase-locked' solutions of systems analogous to (5.6) exist where $\Theta = \theta_1 + \theta_2 + \theta_3$ is constant and $\theta_i = phase(b_i)$. Then Θ must be a root of

$$\sum_{i=1}^{3} \tan(\phi_i - \Theta) = 0.$$

In this case the existence of roots of this equation is guaranteed and then, for appropriate initial conditions, the solutions b_i can exhibit a finite time singularity. It has to be remembered that in practice once the disturbance amplitudes become too large however, then the assumptions concerning the smallness of the amplitude within definitions (4.3) cease to remain valid and a revised asymptotic analysis is required.

Apart from the results relating to explosive instabilities described above, very little is known about solutions of system (5.2) for general parameter values. In the course of an investigation into interaction processes between crossflow vortices in practically important cases one would have to resort to extensive numerical calculations to provide a full description of the characteristics of the solution of the evolution equations. This provides scope for extending the elementary ideas presented here to the potentially more important interactions in experimentally generated boundary layers.

The derivation of the evolution equations relating to the second resonant triad as given by (3.3*b*) was also considered. However, for this triad, the mode proportional to E_3 has a critical layer very close to the wall. In practice, it was found that the numerical scheme described in §4 was too crude to accurately model the singular behaviour close to the wall to provide satisfactory convergence. This discovery points to the need to develop improved numerical procedures to compute the interaction coefficients.

The asymptotic arguments behind the derivation of the amplitude equations for modes with spatial and temporal dependences as in (4.1), (4.2) relied on the assumption that the amplitudes of the disturbances were $O(\delta)$ across the majority of the boundary layer, where $\delta \ll 1$. This condition was introduced primarily to ensure that the viscous critical layers within the flow assumed their simplest possible forms: i.e. linear critical layers encompassing the usual '$i\pi$' phase shift. We can formally relax this requirement by making the following observations. Recall that

$$E_j \equiv \exp\left[\frac{i}{\epsilon^3}\left(\int^r \alpha_j(r,\epsilon)dr + \theta\beta_j(\epsilon) - t\Omega_j(\epsilon)\right)\right], \tag{4.1a}$$

and suppose that, according to linear theory, the mode proportional to E_j is neutrally stable when

$$\begin{aligned} \alpha_j &= \alpha_{0j} + \epsilon\alpha_{1j} + \ldots\ldots + \epsilon^k\tilde{\alpha}_{kj} + \ldots, \\ \beta_j &= \beta_{0j} + \epsilon\beta_{1j} + \ldots\ldots + \epsilon^k\tilde{\beta}_{kj} + \ldots, \\ \Omega_j &= \Omega_{0j} + \epsilon\Omega_{1j} + \ldots\ldots + \epsilon^k\tilde{\Omega}_{kj} + \ldots. \end{aligned} \tag{5.7}$$

Suppose we allow the disturbance to evolve on slow space and time scales given by (4.2) when we replace

$$\delta\frac{\partial}{\partial r_1} \quad \text{by} \quad \epsilon^k\left(\frac{\partial}{\partial r_1} + i\tilde{\alpha}_{kj}\right), \tag{5.8}$$

and make similar replacements for $\delta(\partial/\partial\theta_1)$ and $\delta(\partial/\partial t_1)$. Then the analysis of the *interaction mechanism* will proceed in a very similar way as before, so long as the critical layers remain linear in form. Following the work of (**8, 20**), we know that the critical layers appearing in this type of disturbance structure remain linear in character until the disturbance amplitude reaches the size $O(\epsilon^2)$ at which point the critical layer configuration is modified due to the presence of (relatively) large mean flow corrections and the calculation of the phase shift across the layer becomes a numerical task. However, we can see that the derivation of the evolution triad of equations (4.7) may be trivially modified to account for the interaction between three modes of finite amplitudes, so long as these amplitudes remain less than $O(\epsilon^2)$.

To conclude, we believe that the major finding of our work here is that triads of linearly neutrally stable crossflow vortices can exist in three-dimensional boundary layers; further these triads are such that we have a resonant set of modes at leading orders in wavenumber/ wavespeed expansions based on the asymptotically large Reynolds number. Further, we have developed a fully rational description of the interaction processes which promises to reveal the degree to which these mechanisms can alter the stability characteristics of the flow. We have concentrated on a special three–dimensional flow as an illustrative example but the results obtained here suggest that the interaction process will be operative in more practically oriented boundary layer flows. Our work can be regarded as a first stage in a very complicated field: obvious generalisations could be tackled to explain situations in which, for example, we have critical layers of other than the simple linear kind; or in which the modes have slight detuning. In this context we note that Avis (**27**) has made a careful study of the effects of detuning on resonant triads in his axisymmetric flows. The analysis adopted here allows nonparallel effects to be accounted for without formal difficulties– these effects can play significant roles in practical applications. The largest drawback of our approach is that the computation of the interaction coefficients appears to be not straightforward, especially when the modes have critical points close together or when such critical points are near the bounding surface. We feel that to be able to use our approach to its best advantage further study needs to be devoted to developing more powerful and sophisticated techniques for resolution of the numerical problem as opposed to the fairly rudimentary ideas employed here. Overall however, we have here the first asymptotic description by which the importance of interaction mechanisms upon a resonant triad of linear neutrally stable crossflow vortices may be truly assessed.

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Appendix.

Here we briefly describe the derivation of the constants $\{X_{kj}, D_{kj}\}$ appearing in (4.6). We shall, for convenience, determine these quantities when $j = 3$ for then we may easily modify the results for $j = 2$.

With I_* as defined in (4.5b), we suppose that close to the critical layer $\bar{\xi}_3$ the eigenfunction W_{003} develops according to

$$W_{003} \sim g^{\dagger}_{03} + O\left((\xi - \bar{\xi}_3)\ln|\xi - \bar{\xi}_3|\right). \qquad (A1)$$

Further, in this region the mode proportional to E_2 has a regular Taylor expansion and so

$$(U_{002}, V_{002}, W_{002}) \sim (Q_{u02}, Q_{v02}, Q_{w02}) + (\xi - \bar{\xi}_3)\left(Q'_{u02}, Q'_{v02}, Q'_{w02}\right) + \ldots, \qquad (A2)$$

where the quantities $Q^{\{*\}}_{\{*02\}}$ are constants. Finally, we know that the behaviours of U_{003} and V_{003} can be obtained by suitable manipulations of the basic equations (2.6). Substituting this information into (4.5b) yie'' ..e behaviour

$$I_* \sim \frac{X_{03}}{(\xi - \bar{\xi}_3)^3} + \frac{X_{13}}{(\xi - \bar{\xi}_3)^2} + \frac{X_{23}}{(\xi - \bar{\xi}_3)} + \ldots, \qquad (A3)$$

where

$$X_{03} = -\frac{2iQ_{w02}|B_{031}|J}{\pi\beta_{03}B_{031}}(\alpha_{01}\beta_{03} - \alpha_{03}\beta_{01}), \qquad (A4)$$

$$X_{13} = \frac{|B_{031}|J}{\pi\beta_{03}B_{031}}(\alpha_{03}\beta_{01} - \alpha_{01}\beta_{03})\left[(\beta_{01} + \beta_{03})Q_{v02} + (\alpha_{01} + \alpha_{03})Q_{u02}\right]$$
$$- \frac{ig^{\dagger}_{03}B_{032}B_{011}Q_{w02}}{B^2_{031}} - \frac{iJ|B_{031}|}{\pi\beta_{03}B_{031}}(\alpha_{01}\beta_{03} - \alpha_{03}\beta_{01})Q'_{w02}, \qquad (A5)$$

and

$$X_{23} = \frac{ig^{\dagger}_{03}B_{032}B_{012}Q_{w02}}{B^2_{031}} - \frac{iQ_{w02}\gamma^2_{01}|B_{031}|}{\pi\beta_{03}B_{031}}(\alpha_{02}\beta_{03} - \alpha_{03}\beta_{02})J -$$
$$\frac{g^{\dagger}_{03}B_{032}}{B^2_{031}}\left[(\alpha_{01}Q_{u02} + \beta_{01}Q_{v02})B_{011} + (\alpha_{03}\beta_{01} - \alpha_{01}\beta_{03})\left(Q_{u02}\bar{v}'_{\bar{\xi}_3} - Q_{v02}r_n\bar{u}'_{\bar{\xi}_3}\right)\right]. \qquad (A6)$$

Here

$$J \equiv \frac{\pi\alpha_{03}B_{031}g^{\dagger}_{03}}{\gamma^2_{03}|B_{031}|} - \frac{\pi\bar{u}'_{\bar{\xi}_3}g^{\dagger}_{03}}{|B_{031}|}.$$

Then we can show that

$$W_{101} \sim \frac{D_{03}}{(\xi - \bar{\xi}_3)} + D_{13} \ln(\xi - \bar{\xi}_3) + D_{32} + D_{33} (\xi - \bar{\xi}_3) \ln(\xi - \bar{\xi}_3) + \ldots, \qquad (A7)$$

for $\xi > \bar{\xi}_3$ with

$$D_{03} = \frac{X_{03}}{2B_{010}},$$

$$D_{13} = \frac{2B_{011}}{B_{010}} D_{03} - \frac{X_{13}}{B_{010}},$$

and

$$D_{33} = \frac{B_{011}}{B_{010}} D_{13} + \gamma_{01}^2 D_{03} + \frac{X_{33}}{B_{010}}.$$

The quantities D_{13}, D_{33} appear in the evolution equation (4.7) for the mode proportional to E_1. Of course, we need to obtain quantities equivalent to those given above in order to traverse the critical layer(s) $\bar{\xi}_2$, but these can be deduced merely by interchanging the subscripts '2' and '3' in the relevant formulae above.

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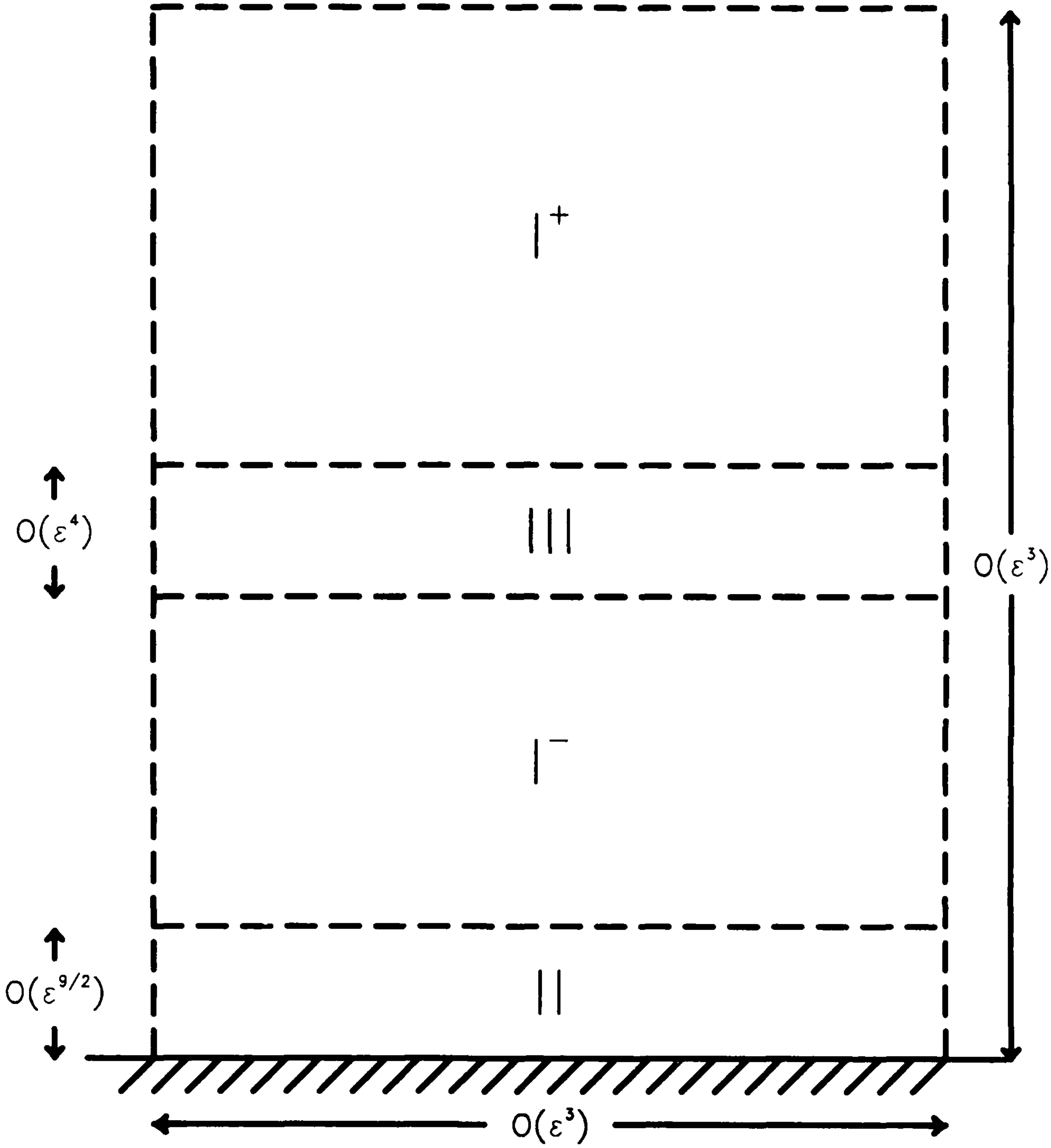

Figure (1).

Flow structure for the Rayleigh modes described in §2. The inviscid layer I is of thickness $O(\epsilon^3)$ (the same depth as the boundary layer and of the same size as the horizontal scale for the structure). In addition, we have a viscous wall layer (II) of depth $O(\epsilon^{9/2})$ and a critical layer (III) of thickness $O(\epsilon^4)$ at those positions where the 'effective' basic velocity profile $\bar{U}_0$ (see (2.7)) vanishes.

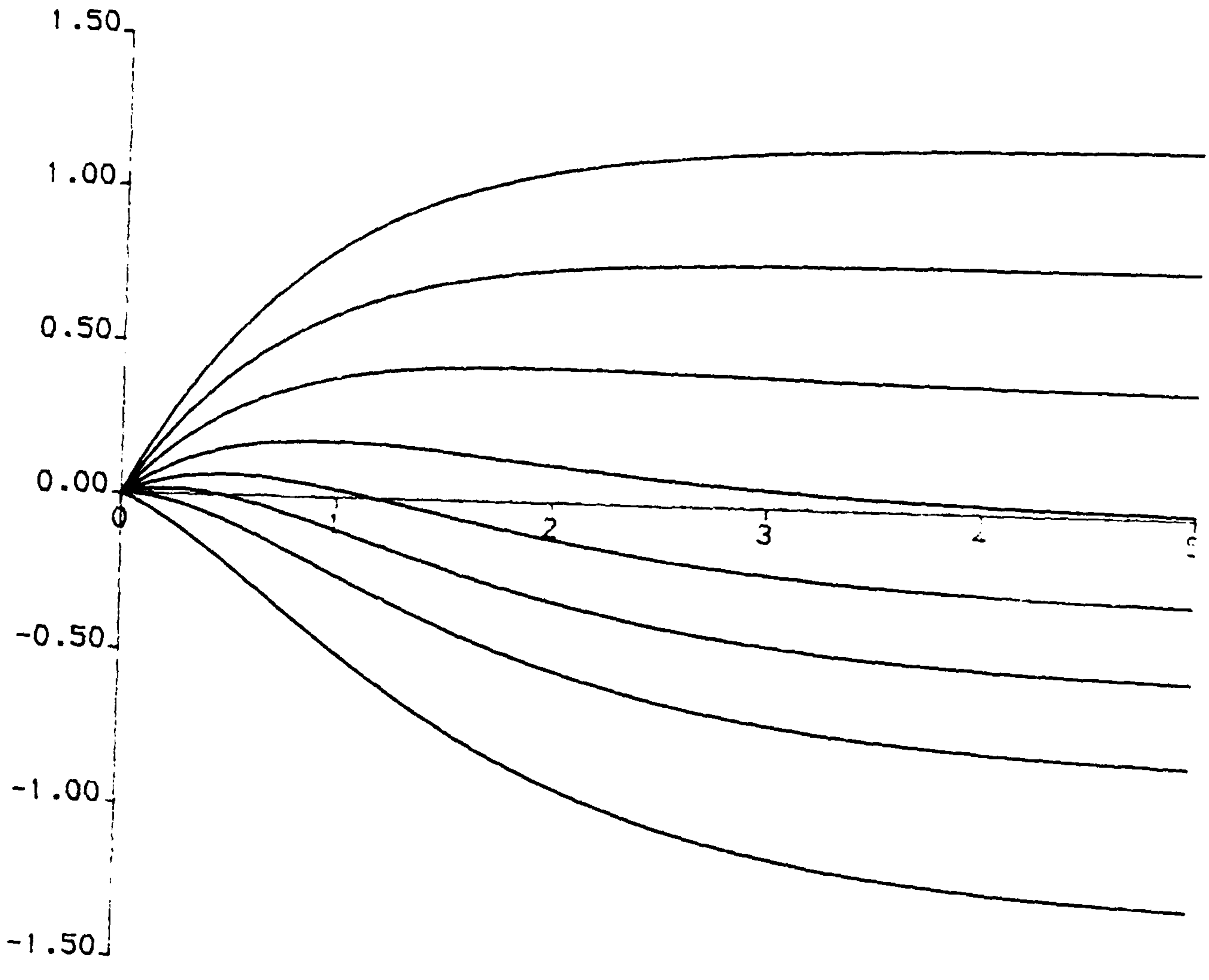

Figure (2).

a) Basic velocity profile $U_s \equiv \bar{u} + \mu\bar{v}$ as a function of the boundary layer coordinate ξ. (*i*) $\mu = -1.2$; (*ii*) $\mu = -0.8$; (*iii*) $\mu = -0.4$; (*iv*) $\mu = -0.0$; (*v*) $\mu = 0.3$; (*vi*) $\mu = 0.55$; (*vii*) $\mu = \mu_s = 0.8284$ and (*viii*) $\mu = 1.3$.

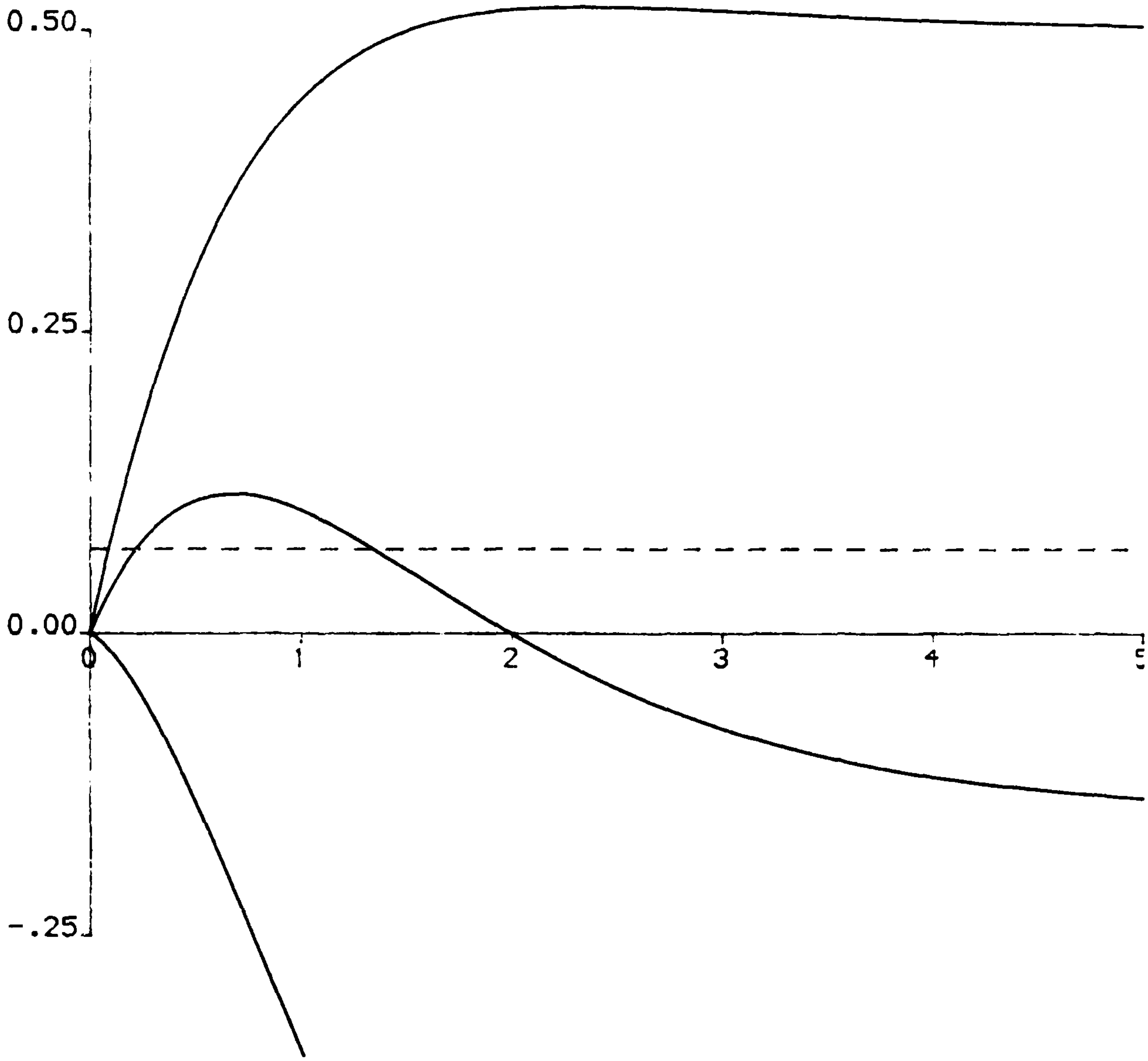

Figure (2).

b) Illustration that for $0 < \mu < \mu_c$ there can be either one or two points where $U_* = c_0$ for some given c_0. Hence there can be either one or two critical layers in the flow structure.

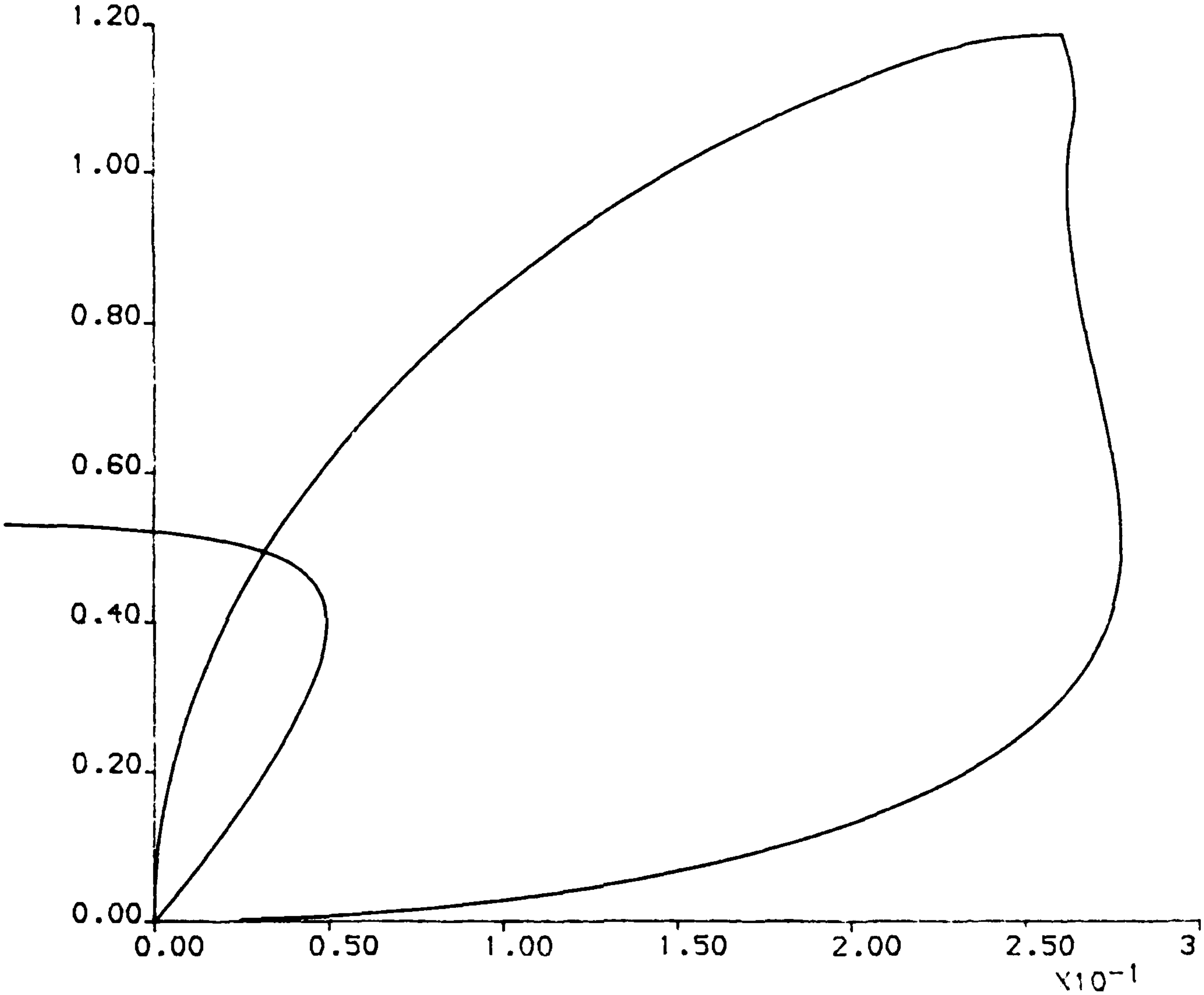

Figure (3).
Solutions of the eigenproblem (2.7) for various values of the waveangle parameter μ.

a) Radial wavenumber α_0 as a function of the azimuthal wavenumber β_0.

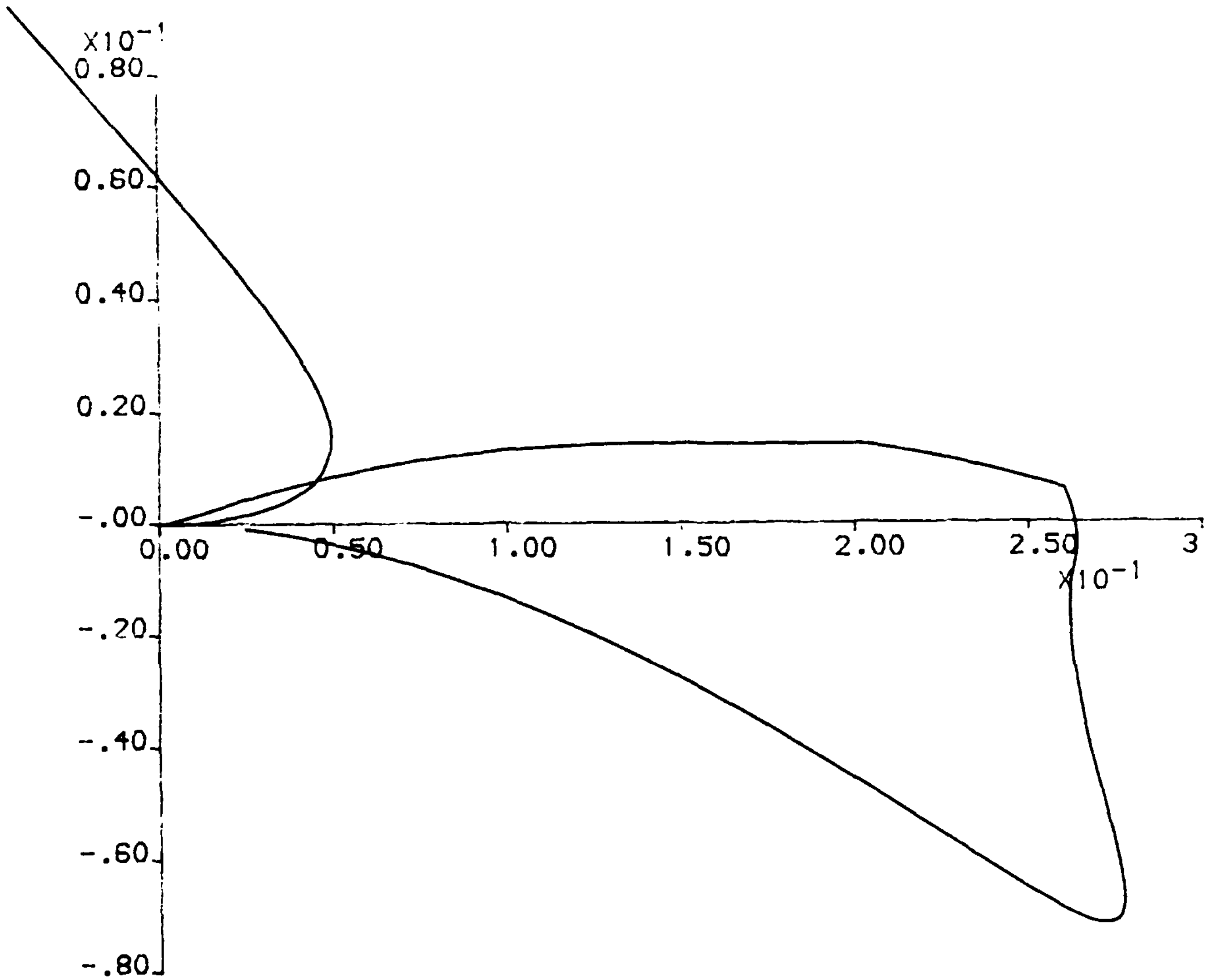

Figure (3).
Solutions of the eigenproblem (2.7) for various values of the waveangle parameter μ.

b) Frequency Ω_0 of mode as a function of β_0.

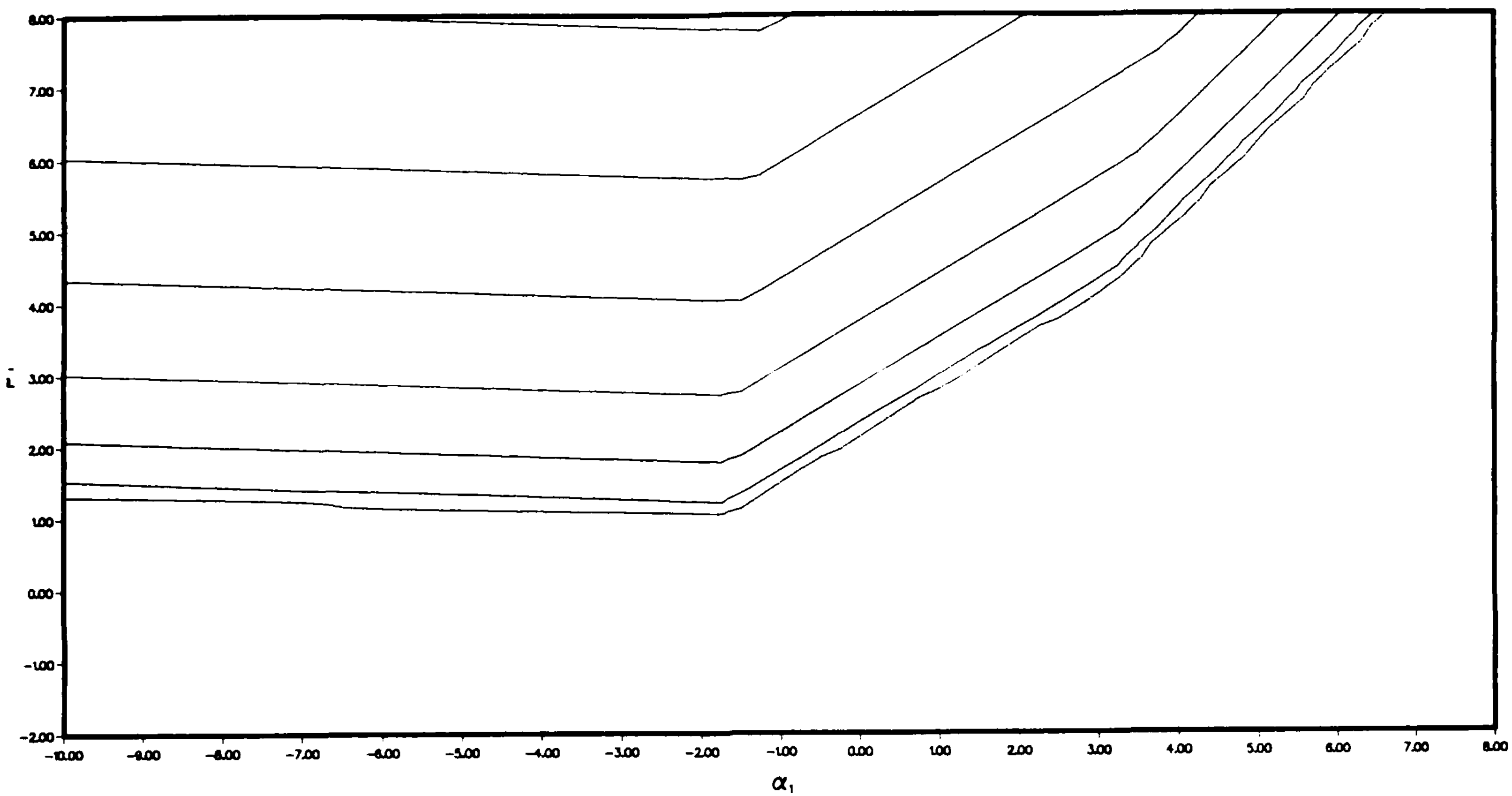

Figure (4).

Illustration of regions of (α_1, β_1) space in which (5.1), subject to the replacements (5.4), may be liable to explosive instability. Below and to the right of the zero contour such breakdown may occur irrespective of the initial disturbance amplitude. However, in the remainder of the phase space the breakdown is only possible providing the initial amplitude

$$A(0) \equiv \left(\sum_{i=1}^{3} A_i(0) A_i^*(0) \right)^{\frac{1}{2}}$$

is sufficiently large: the minimum values of $A(0)$ needed for this are indicated on the solid contours.