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**LINEAR-TIME SEPARATION ALGORITHMS
FOR THE THREE-INDEX
ASSIGNMENT POLYTOPE**

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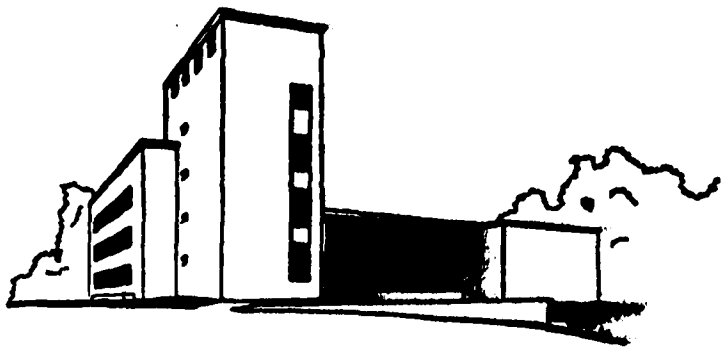
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Abstract

Balas and Saltzman [1] identified several classes of facet inducing inequalities for the three-index assignment polytope, and gave $O(n^4)$ separation algorithms for two of them. We give $O(n^3)$ separation algorithms for these two classes of facets, and also for a third class. Since the three-index assignment problem has n^3 variables, these algorithms are linear-time and their complexity is best possible.

Keywords:

- Three-index assignment
- Facets
- Cliques
- Odd holes



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1. Introduction

Consider three disjoint n -sets and the collection of all weighted triplets with one element in each n -set. The three-index assignment (or three-dimensional matching) problem asks for a minimum-weight set of triplets that partitions the union of the three n -sets. It can be stated as the following 0-1 programming problem:

$$\begin{aligned}
 \min \quad & \sum \{c_{ijk}x_{ijk} : i \in I, j \in J, k \in K\}, \\
 \text{s.t.} \quad & \sum \{x_{ijk} : j \in J, k \in K\} = 1, \quad \forall i \in I, \\
 & \sum \{x_{ijk} : i \in I, k \in K\} = 1, \quad \forall j \in J, \\
 & \sum \{x_{ijk} : i \in I, j \in J\} = 1, \quad \forall k \in K, \\
 & x_{ijk} \in \{0, 1\}, \quad \forall i, j, k,
 \end{aligned} \tag{1.1}$$

where I, J and K are disjoint sets with $|I| = |J| = |K| = n$. Let A be the coefficient matrix of the constraint set of (1.1). Then $R = I \cup J \cup K$ is the row index set of A . Let S be the column index set of A . Let G_A be the intersection graph of A . Then S is the node set of G_A . Let

$$P = \{x \in \mathbb{R}^{n^3} : Ax = e, x \geq 0\},$$

where $e = (1, \dots, 1)^T \in \mathbb{R}^{3n}$. Then

$$P_I = \text{conv}\{x \in \{0, 1\}^{n^3} : x \in P\}$$

is the three-index assignment polytope of order n .

Balas and Saltzman [1] described the cliques of G_A as belonging to three distinct classes and showed that cliques in two of the three classes induce inequalities that define facets of P_I . Furthermore, they gave an $O(n^4)$ procedure for finding a facet-defining clique-inequality violated by a given noninteger $x \in P$, or showing that no such inequality exists. They also described in [1] the odd holes of G_A and identified two classes of facets associated with odd holes that are easy to generate. One class has coefficients of 0 and 1, the other class coefficients of 0, 1 and 2.

In Section 2 of this paper, we give two $O(n^3)$ separation algorithms for the above mentioned two classes of facet defining clique-inequalities. In Section 3, we give an $O(n^3)$ separation algorithm for facet inducing inequalities attributed with lifted odd holes of length 5, having coefficients equal to 0 or 1. Since the number of variables of the three-index assignment problem is n^3 , the complexity of these procedures is of the lowest possible order. Our procedures can be used to speed up the performance of the branch and bound algorithm

for the three-index assignment problem developed in [2], which uses facet inducing inequalities as ingredients of a Lagrangian relaxation to obtain strong lower bounds. Our procedures can be extended to more general cases. In Section 4, we present an $O(n^{h+1})$ procedure for detecting a violated facet-defining inequality of P_I , which is associated with a $2h + 1$ -hole of G_A and belongs to a subclass of the first odd-hole-associated class of facets of [1].

We use the same notation as [1]. Especially, a^s means a column of A associated with $s \in S$. We may specify $s = \{i, j, k\}$ for $i \in I, j \in J, k \in K$. For a set $Q \subseteq R$, we use Q_I, Q_J and Q_K to denote the parts of Q in I, J and K respectively.

For literature on the three-index assignment problem, also see [3], ..., [8].

2. Facets Induced by Cliques

For every $s \in S$, let

$$C(s) = \{t \in S : a^s \cdot a^t \geq 2\}.$$

Then the node set $C(s)$ induces a clique of size $3n - 2$ in G_A [1, Proposition 2.2]. Such a clique is called a clique of class 2. Furthermore, for $n \geq 3$, the inequality

$$x(C(s)) \equiv \sum \{x_t : t \in C(s)\} \leq 1 \quad (2.1)$$

defines a facet of P_I for every $s \in S$ [1, Theorem 3.3]. Given a noninteger $x \in P$, an $O(n^4)$ procedure was presented in [1, Section 4] to detect whether any inequality induced by a clique of class 2 is violated. We now present an $O(n^3)$ procedure to do this.

Algorithm 2.1. Suppose that x is a noninteger point in P . Let ν be an integer greater than or equal to 4.

Step 1. Let $d_s = 0$ for all $s \in S$.

Step 2. Check x_t for all $t \in S$. If

$$x_t \geq \frac{1}{\nu n}, \quad (2.2)$$

then set $d_s := d_s + x_t$ for all $s \in C(t)$. If $d_s > 1$, then stop: (2.1) is violated by x with this s . Otherwise, continue.

Step 3. For $s \in S$, if

$$d_s > \frac{\nu - 3}{\nu}, \quad (2.3)$$

then check whether the inequality (2.1) associated with s is violated and if so, then stop. Else continue. ■

Theorem 2.2. *Algorithm 2.1 determines in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 2. The value of ν which minimizes the complexity of the algorithm is 6.*

To prove this theorem we need two lemmas. The first lemma is essentially Lemma 4.1 of [1]. We repeat it here for convenience:

Lemma 2.3. *For any $x \in P$ and any positive number λ , the number of components with value $\geq \lambda$ is $\leq n/\lambda$. ■*

Lemma 4.1 of [1] states this for $1/\lambda$ integer, but the result holds for reals as well. The proof in [1] is somewhat indirect. In fact, the sum of components of $x \in P$ is always n , and all the components of x are nonnegative. The conclusion thus follows.

Lemma 2.4. *For any $x \in P$ and any positive number λ , the number of $s \in S$ such that $x(C(s)) \geq \lambda$ is $\leq n(3n - 2)/\lambda$.*

Proof: Consider

$$\sum \{x(C(s)) : s \in S\}.$$

For each $t \in S$, the number of distinct s such that $t \in C(s)$ is $3n - 2$. Thus,

$$\sum \{x(C(s)) : s \in S\} = (3n - 2) \sum \{x_t : t \in S\} = (3n - 2)n.$$

Since $x(C(s)) \geq 0$ for each s , the conclusion follows. ■

Proof of Theorem 2.2: We first show that Algorithm 2.1 works. By Proposition 2.2 of [1], $|C(s)| = 3n - 2$ for every $s \in S$. Suppose that (2.1) is violated for an $s \in S$. Then

$$\begin{aligned} d_s &:= \sum \{x_t : t \in C(s), x_t \geq \frac{1}{\nu n}\} \\ &> 1 - \sum \{x_t : t \in C(s), x_t < \frac{1}{\nu n}\} \\ &\geq 1 - \frac{3n - 2}{\nu n} \geq \frac{\nu - 3}{\nu}, \end{aligned}$$

i.e., (2.3) holds and the violation is discovered in Step 3. Therefore, Algorithm 2.1 determines whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 2.

We now consider the complexity of the algorithm. The complexity of Step 1 is n^3 . According to Lemma 2.3, there are at most νn^2 components of x satisfying (2.2). For each t , the number of s in $C(t)$ is $3n - 2$. Hence, the complexity of Step 2 is $\nu n^2(3n - 2)$. Finally, if (2.3) holds, then

$$x(C(s)) \geq d_s \equiv \sum \{x_t : t \in C(s), x_t \geq \frac{1}{\nu n}\} > \frac{\nu - 3}{\nu}.$$

By Lemma 2.4, the number of s such that (2.3) holds is not greater than $\frac{\nu n(3n-2)}{\nu-3}$. The number of basic operations to check (2.1) is $3n - 2$. We also need to use n^3 basic steps to check (2.3). Therefore, the complexity of Step 3 is $n^3 + \frac{\nu n(3n-2)^2}{\nu-3}$. Hence, the overall complexity of Algorithm 2.1 is $O(n^3)$. The value of ν that minimizes the maximum of the two polynomials expressing the complexity of Steps 2 and 3 is easily seen to be 6. ■

We now consider facets induced by cliques of class 3 [1]. Suppose that $s, t \in S$ such that

$$a^s \cdot a^t = 0. \tag{2.4}$$

Let $s = (i_s, j_s, k_s)$, $t = (i_t, j_t, k_t)$, $t_1 = (i_s, j_t, k_t)$, $t_2 = (i_t, j_s, k_t)$, $t_3 = (i_t, j_t, k_s)$, and

$$C(s, t) = \{s, t_1, t_2, t_3\}.$$

Then the node set $C(s, t)$ induces a (4-)clique in G_A [1, Proposition 2.3]. Such a clique is called a clique of class 3. The number of cliques of class 3 is $\frac{1}{4}n^3(n - 1)^3$ [1, Corollary 2.5]. Furthermore, for $n \geq 4$, the inequality

$$x(C(s, t)) \equiv \sum \{x_t : t \in C(s, t)\} \leq 1 \tag{2.5}$$

defines a facet of P_I for every $s, t \in S$ satisfying (2.4) [1, Theorem 3.7]. Given a noninteger $x \in P$, an $O(n^4)$ procedure was also presented in [1, Section 4] to detect whether any inequality induced by a clique of class 3 is violated. We now present an $O(n^3)$ procedure to do this.

Algorithm 2.5. Suppose that x is a noninteger point in P . Check x_s for $s \in S$. If

$$\frac{1}{4} < x_s < 1, \tag{2.6}$$

then check x_p for $p \in S$ satisfying

$$a^s \cdot a^p = 1. \tag{2.7}$$

If

$$x_p > \frac{1 - x_s}{3}, \tag{2.8}$$

then for $t \in S$ such that $a^t \cdot a^s = 0$ and

$$a^t \cdot a^p = 2, \quad (2.9)$$

check whether (2.5) is violated or not. ■

Theorem 2.6. *Algorithm 2.5 determines in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 3.*

Proof: We first show that Algorithm 2.5 works. Let $C(s, t)$ be the node set of a clique of class 3. Since $|C(s, t)| = 4$, if $x \in P$ violates (2.5), then x has at least one component $> \frac{1}{4}$. By Proposition 2.4 of [1], there is no loss of generality in assuming that this happens for the components indexed by s , i.e., that $x_s > \frac{1}{4}$. Since $x \in P$, we have

$$\sum \{x_{i,j,k} : j \in J, k \in K\} = 1, \quad (2.10)$$

$$\sum \{x_{ij,k} : i \in I, k \in K\} = 1, \quad (2.11)$$

$$\sum \{x_{ijk} : i \in I, j \in J\} = 1. \quad (2.12)$$

If $x_s = 1$, then $x_p = 0$ for any p such that $a^p \cdot a^s = 1$, by (2.10), (2.11) and (2.12). In this case, for any t such that $a^t \cdot a^s = 0$, the left hand side of (2.5) is 1. Thus, if $x \in P$ violates (2.5), $x_s < 1$. This justifies (2.6). Since $|C(s, t)| = 4$, if (2.5) is violated, then (2.8) must hold for at least one $p \in C(s, t)$, $p \neq s$. Therefore, this algorithm determines whether a given $x \in P$ violates a facet defining inequality induced by a clique of class 3.

We now consider the complexity issue. The complexity to check all s is $O(n^3)$. According to Lemma 2.3, there are at most $4n$ components of x satisfying (2.6). For each such component x_s , there are $(n-1)^2$ components x_p satisfying (2.7). So the complexity to find all pairs $s, p \in S$ satisfying (2.6) and (2.7) is $O(n^3)$. By (2.10), (2.11) and (2.12), for given x_s satisfying (2.6), there are at most 6 p such that (2.7) and (2.8) hold. When s and p are fixed and (2.7) is satisfied, there are $n-1$ t satisfying (2.4) and (2.9). One needs 3 additions and one comparison to check (2.5). Therefore, the complexity of the remaining part of the algorithm is $O(n^2)$. Hence, the overall complexity of Algorithm 2.5 is $O(n^3)$. ■

In the proof, we see that the main work is to check all $s \in S$. This work cannot be reduced. So this order cannot be further improved. Similarly, the order in Theorem 2.2 is also the minimum.

By [1], only these two classes of cliques induce facets of P_I . Combining Theorems 2.2 and 2.6, we have

Theorem 2.7. *One can determine in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality induced by a clique of G_A . ■*

3. Facets Associated with Five-Holes

Assume that $n \geq 3$. Let $Q \subseteq R$, $|Q| = 2h + 1$ for some positive integer h with $h \leq n - 1$, $1 \leq |Q_L| \leq h$, $L = I, J, K$, and let

$$S(Q) := \{s \in S : \sum \{a_q^s : q \in Q\} \geq 2\}. \quad (3.1)$$

Then the inequality

$$\sum \{x_s : s \in S(Q)\} \leq h \quad (3.2)$$

defines a facet of P_I [1, Theorem 6.1]. These facets can be regarded as lifted from odd hole inequalities. The number of distinct inequalities (3.2) is $O(2^{3n})$ [1, Proposition 6.5]. In particular, facet defining inequalities (2.1) are special cases of (3.2) with $h = 1$.

Denote the set of Q satisfying the above requirements with a fixed h by S^h . Let $\Phi_h = \{S(Q) : Q \in S^h\}$, where $S(Q)$ is defined by (3.1). Thus, Algorithm 2.1 is an $O(n^3)$ algorithm to detect violated facets induced by sets in Φ_1 . In this section, we present an $O(n^3)$ procedure to detect whether there exists a facet induced by a set in Φ_2 , which is violated by a given noninteger $x \in P$. Notice that the number of facets defined by (3.2) with $h = 2$ is $O(n^5)$. Without loss of generality, we may assume that such an x does not violate any inequality (2.1), i.e., assume that Algorithm 2.1 was applied to x first.

Suppose that $x \in P$. For $i \in I, j \in J, k \in K$, we will denote

$$x(i, j, K) := \sum \{x_{ijk'} : k' \in K\}, \quad (3.3)$$

$$x(i, J, k) := \sum \{x_{ij'k} : j' \in J\}, \quad (3.4)$$

$$x(I, j, k) := \sum \{x_{i'jk} : i' \in I\} \quad (3.5)$$

In the following, we establish some formulas to calculate $x(S(Q))$ for $Q \in S^2$. We consider two cases.

First, suppose that $s, t \in S$ satisfy

$$a^s \cdot a^t = 1.$$

Then

$$Q \equiv s \cup t \in S^2.$$

Assume that $s = (i_s, j_s, k_s), t = (i_t, j_t, k_t)$. It is not difficult to see that if $k_s = k_t$, then

$$x(S(Q)) = x(C(s)) + x(C(t)) + x(i_s, j_t, K) + x(i_t, j_s, K) - x_{i_s, j_t, k_s} - x_{i_t, j_s, k_s} \quad (3.6)$$

if $j_s = j_t$, then

$$x(S(Q)) = x(C(s)) + x(C(t)) + x(i_s, J, k_t) + x(i_t, J, k_s) - x_{i_s, j_t, k_t} - x_{i_t, j_s, k_s} \quad (3.7)$$

and if $i_s = i_t$, then

$$x(S(Q)) = x(C(s)) + x(C(t)) + x(I, j_s, k_t) + x(I, j_t, k_s) - x_{i_s, j_s, k_t} - x_{i_t, j_t, k_s}. \quad (3.8)$$

Next, suppose that $s, t \in S$ satisfy

$$a^s \cdot a^t = 2. \quad (3.9)$$

Again assume that $s = (i_s, j_s, k_s), t = (i_t, j_t, k_t)$. Then define for the ordered pair (s, t) ,

$$x(s, t) := x(i_t, J, k_t) + x(i_t, j_t, K) - 2x_{i_t, j_t, k_t}, \quad (3.10)$$

if $i_s \neq i_t$;

$$x(s, t) := x(i_t, j_t, K) + x(I, j_t, k_t) - 2x_{i_t, j_t, k_t}, \quad (3.11)$$

if $j_s \neq j_t$;

$$x(s, t) := x(I, j_t, k_t) + x(i_t, J, k_t) - 2x_{i_t, j_t, k_t}, \quad (3.12)$$

if $k_s \neq k_t$. Notice that in (3.10) $j_t = j_s$ and $k_t = k_s$, in (3.11) $i_t = i_s$ and $k_t = k_s$, and in (3.12) $i_t = i_s$ and $j_t = j_s$. Thus, (3.10) can be written with j_s and k_s substituted for j_t and k_t , and there are similar expressions for (3.11) and (3.12).

In the case that (3.9) holds, we may expand $s \cup t$ to a set $Q \in S^2$ by adding an adequate element. For example, if $i_s \neq i_t$, we may add $j \neq j_s$ and let

$$Q_I = \{i_s, i_t\}, Q_J = \{j_s, j\}, Q_K = \{k_s\}, \quad (3.13)$$

or we may add $k \neq k_s$ and let

$$Q_I = \{i_s, i_t\}, Q_J = \{j_s\}, Q_K = \{k_s, k\}. \quad (3.14)$$

Notice that there are $2(n-1)$ ways to expand a given pair of s and t satisfying (3.9) to a set $Q \in S^2$. We may also calculate $x(S(Q))$ by expressions similar to (3.6)-(3.8). For example, in case (3.13), one can check that

$$x(S(Q)) = x(C(s)) + x(s, t) + x(i_s, j, K) + x(i_t, j, K) + x(I, j, k_s) - 2x_{i_s, j, k_s} - 2x_{i_t, j, k_s}, \quad (3.15)$$

where $x(s, t)$ is given by (3.10). Note that if $i_s \neq i_t$, there are two possibilities, namely, (3.13) and (3.14). Therefore, there are six cases totally, i.e., two cases for $i_s \neq i_t$, two cases for $j_s \neq j_t$, and two cases for $k_s \neq k_t$. In each of these six cases, there is an expression of type (3.15) to calculate $x(S(Q))$. Later, we will refer to these as "expressions of type (3.15)".

We are now ready to describe our algorithm:

Algorithm 3.1. Suppose that x is a noninteger point in P . Suppose (2.1) holds for all $s \in S$.

Step 1. Let $d_s = 0$ for all $s \in S$.

Step 2. Check x_t for all $t \in S$. If

$$x_t \geq \frac{1}{12n}, \quad (3.16)$$

then let $d_s := d_s + x_t$ for all $s \in S$ satisfying

$$a_s \cdot a_t \geq 2.$$

Step 3. For all $s \in S$, if

$$d_s \geq \frac{1}{12}, \quad (3.17)$$

then set $d_s := x(C(s))$.

Step 4. For all $i \in I, j \in J, k \in K$, calculate $x(i, j, K)$, $x(i, J, k)$ and $x(I, j, k)$ by (3.3), (3.4) and (3.5).

Step 5. For all $i \in I, j \in J, k \in K$, form the sets

$$L(i, J, K) := \{s \in S : d_s > \frac{1}{3}, i_s = i\},$$

$$L(I, j, K) := \{s \in S : d_s > \frac{1}{3}, j_s = j\}$$

and

$$L(I, J, k) := \{s \in S : d_s > \frac{1}{3}, k_s = k\}$$

Store them.

Step 6. For each $s = (i_s, j_s, k_s) \in S$ such that $d_s > \frac{1}{2}$, check d_t for each $t \in L(s)$, where

$$L(s) := (L(i_s, J, K) \cup L(I, j_s, K) \cup L(I, J, k_s)) \setminus \{s\}. \quad (3.18)$$

If $a^t \cdot a^s = 1$ and $d_t > (2 - d_s)/3$, calculate $x(S(Q))$ by the appropriate expression (3.6), (3.7) or (3.8), with d_p substituted for $x(C(p))$, $p = s, t$.

If $a^t \cdot a^s = 2$ and $x(s, t) > (2 - d_s)/3$, calculate $x(S(Q))$ by the appropriate expression of type (3.15), with d_p substituted for $x(C(p))$, $p = s, t$, for every $Q \in S^2$ such that $s \cup t \subseteq Q$.

If $x(S(Q)) > 2$ for some Q , stop: the corresponding inequality (3.2) is violated by x . Otherwise, continue. ■

Theorem 3.2. Algorithm 3.1 determines in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality (3.2) with $h = 2$.

We prove several auxiliary results.

Lemma 3.3. After Step 3 of Algorithm 3.1, for all $s \in S$,

$$d_s \leq x(C(s)). \quad (3.19)$$

If $x(C(s)) > \frac{1}{3}$, then $d_s = x(C(s))$.

Proof: It is obvious that (3.19) holds. Suppose that $x(C(s)) > \frac{1}{3}$. Then

$$\begin{aligned} d_s &= \sum \{x_t : t \in C(s), x_t \geq \frac{1}{12n}\} > \frac{1}{3} - \sum \{x_t : t \in C(s), x_t < \frac{1}{12n}\} \\ &\geq \frac{1}{3} - \frac{3n-2}{12n} \geq \frac{1}{12}, \end{aligned}$$

i.e., $d_s \geq \frac{1}{12}$ holds and in this case d_s is set equal to $x(C(t))$ in Step 3. ■

Lemma 3.4. At Step 6,

$$|L(s)| \leq 27n, \quad (3.20)$$

$$|\{t \in L(s) : a^s \cdot a^t = 2, x(s, t) > \frac{1}{3}\}| \leq 18, \quad (3.21)$$

where $L(s)$ is defined by (3.18) and $x(s, t)$ is defined by (3.10), (3.11) and (3.12).

Proof: Suppose that $s = (i_s, j_s, k_s)$.

On the one hand, by Lemma 3.3 and the definition of $L(i_s, J, K)$ at Step 5,

$$\sum \{x(C(t)) : t \in L(i_s, J, K)\} \geq \frac{1}{3} |L(i_s, J, K)|.$$

On the other hand,

$$\begin{aligned} &\sum \{x(C(t)) : t \in L(i_s, J, K)\} \\ &= \sum_{j_t, k_t} \{x(C(t)) : t = (i_s, j_t, k_t)\} \\ &\leq \sum_{j_t, k_t} [\sum_i x_{ij_t k_t} + \sum_j x_{i_s j_t k_t} + \sum_k x_{i_s j_t k}] \\ &= \sum_i \sum_{j_t, k_t} x_{ij_t k_t} + \sum_{j_t} \sum_{j_t, k_t} x_{i_s j_t k_t} + \sum_{k_t} \sum_{j_t, k} x_{i_s j_t k} \end{aligned}$$

$$= \sum_i 1 + \sum_{j_i} 1 + \sum_{k_i} 1 = 3n.$$

Thus, $|L(i_s, J, K)| \leq 9n$. Similarly, we can show that $|L(I, j_s, K)| \leq 9n$ and $|L(I, j_s, K)| \leq 9n$. From the definition of $L(s)$, we then have (3.20).

It is also seen that

$$\begin{aligned} & \sum \{x(s, t) : t \in S, a^s \cdot a^t = 2\} \\ & \leq 2 \sum_{j, k} x_{i, j, k} + 2 \sum_{i, k} x_{i, j, k} + 2 \sum_{i, j} x_{i, j, k} = 6. \end{aligned}$$

Hence, (3.21) holds. ■

Lemma 3.5. *Suppose that $Q \in S^2$. Let*

$$H(Q) := \{s \in S : \sum \{a_q^s : q \in Q\} = 3\}.$$

If the inequality (3.2) with $h = 2$ is violated by x , then there are $s, t \in H(Q)$, $s \neq t$ such that

- (i). $x(C(s)) > \frac{1}{2}$;
- (ii). *either $a^s \cdot a^t = 1$ and $x(C(t)) > (2 - x(C(s)))/3$, or $a^s \cdot a^t = 2$ and $x(s, t) > (2 - x(C(s)))/3$.*

Proof: Without loss of generality, assume that $Q_I = \{i_1, i_2\}$, $Q_J = \{j_1, j_2\}$, $Q_K = \{k\}$. Then $H(Q) = \{(i_1, j_1, k), (i_1, j_2, k), (i_2, j_1, k), (i_2, j_2, k)\}$, and

$$2 < x(S(Q)) \leq x(C((i_1, j_1, k))) + x(C((i_1, j_2, k))) + x(C((i_2, j_1, k))) + x(C((i_2, j_2, k))).$$

Then at least one term on the right hand side of the above inequality is greater than $\frac{1}{2}$. Without loss of generality, assume it is the first term. Let $s = (i_1, j_1, k)$. Then (i) holds.

Denote $t_1 = (i_2, j_2, k)$, $t_2 = (i_1, j_2, k)$, $t_3 = (i_2, j_1, k)$. Then

$$2 < x(S(Q))$$

$$\leq x(C(s) + x(C(t_1)) + x(i_1, j_2, K) + x(i_2, j_1, K) - x_{i_1 j_2 k} - x_{i_2 j_1 k}, \quad (\text{by (3.6)})$$

$$\leq x(C(s)) + x(C(t_1)) + x(s, t_2) + x(s, t_3). \quad (\text{by (3.11) and (3.10)})$$

Notice that $a^s \cdot a^{t_1}$ and $a^s \cdot a^{t_2} = a^s \cdot a^{t_3} = 2$. Thus,

$$2 - x(C(s)) < x(C(t_1)) + x(s, t_2) + x(s, t_3).$$

At least one term at the right hand side of above inequality is greater than one third of the left hand side of this inequality. Hence, (ii) holds. ■

Proof of Theorem 3.2: By Lemmas 3.5 and 3.3, if the inequality (3.2) with $h = 2$ is violated by x for some Q , there are $s, t \in S$ such that $d_s > \frac{1}{2}$, and either $a^s \cdot a^t = 1$ and $d_t > (2 - d_s)/3$, or $a^s \cdot a^t = 2$ and $x(s, t) > (2 - d_s)/3$.

Thus if $t \in L(s)$, i.e., if $d_t > \frac{1}{3}$, then Step 6 will detect the violation. To see that this is the case, notice that if $a^s \cdot a^t = 1$, then

$$d_t > (2 - d_s)/3 = (2 - x(C(s)))/3 \geq \frac{1}{3},$$

(since $x(C(s)) \leq 1$ for all $s \in S$), and if $a^s \cdot a^t = 2$, then

$$\begin{aligned} d_t = x(C(t)) &\geq x(s, t) && \text{(from the definitions)} \\ &> (2 - d_s)/3 = (2 - x(C(s)))/3 && \geq \frac{1}{3}, \end{aligned}$$

Thus in both cases $t \in L(s)$.

Consider now the complexity of Algorithm 3.1. Steps 1 and 2 are obviously $O(n^3)$. In Step 3, the number of $s \in S$ such that $d_s \geq \frac{1}{12}$ is $\leq 12n(3n - 2)$ (From Lemma 2.4 and the fact that $d_s \leq x(C(s))$); and for each number s , calculating $x(C(s))$ requires $3n - 3$ additions; thus the complexity of Step 3 is also $O(n^3)$. Further, each of Steps 4 and 5 again requires $O(n^3)$ operations.

Finally, consider the complexity of Step 6. By Lemmas 3.3 and 2.4, the number of s such that $d_s > \frac{1}{2}$ is $O(n^2)$. From Lemma 3.4, the cardinality of $L(s)$ is $O(n)$. Forming $L(s)$ itself is $O(1)$, since its constituent sets have been calculated and stored in Step 5. Thus, checking d_t for each $t \in L(s)$ for all s such that $d_s > \frac{1}{2}$ is $O(n^3)$. If $a^s \cdot a^t = 1$ and $d_t > (2 - d_s)/3$, calculating $x(S(Q))$ is $O(1)$, since the expressions $x(C(s))$, $x(C(t))$ and $x(i, j, K)$, $x(i, J, k)$, $x(I, j, k)$ have been calculated in Step 3 and Step 4 respectively. On the other hand, from Lemma 3.4 the number of those $t \in L(s)$ (for given s) such that $a^s \cdot a^t = 2$ and $x(s, t) > \frac{1}{3}$ is $O(1)$. For each such t , there are $2(n - 1)$ ways to expand the pair (s, t) to a set $Q \in S^2$, and for each such set Q , calculating $x(S(Q))$ is again $O(1)$, since the constituent terms of the expressions of type (3.15) needed for this have already been calculated. Thus, the complexity of dealing with all pairs (s, t) satisfying $a^s \cdot a^t = 2$ is again $O(n^3)$, which leaves the total complexity of Step 6 at $O(n^3)$. ■

4. Facets Associated with Other Odd Holes

It is somewhat more complicated to detect violated facets defined by (3.2) with $h > 2$. We first discuss some subclasses of S^h and Φ_h . Assume that $n \geq 3$. Let \tilde{S}^h be the subcollection of S^h such that for any

$Q \in \bar{S}^h$ there exists an $L = I, J$, or K such that $|Q_L| = 1$. Note that in this case $|Q_{L'}| = h$ for $L' \in \{I, J, K\}$, $L' \neq L$. Let $\bar{\Phi}_h = \{S(Q) : Q \in \bar{S}^h\}$. Clearly, $\bar{S}^1 = S^1$, $\bar{S}^2 = S^2$, but \bar{S}^h is a proper subset of S^h for $h > 2$. Assume that $h \geq 3$. We will show that

Theorem 4.1. *For a given $x \in P$, it is possible in $O(n^{h+1})$ steps to detect whether there exists a facet of P_I , which is defined by (3.2) with $Q \in \bar{S}^h$ and violated by x .*

Again, we prove several auxiliary results.

Suppose that $x \in P$ does not violate any inequality

$$x(S(Q)) \leq r. \quad (4.1)$$

with $Q \in \bar{S}^r$, $r = 1, 2, \dots, h-1$, but violates (4.1) with $h = r$ and a set $Q \in \bar{S}^h$. Suppose that h is fixed, not dependent on n .

Lemma 4.2. *Let $H(Q)$ be defined as in Lemma 3.5. Then there exists an $s \in H(Q)$ such that*

$$x(C(s)) > \frac{1}{h}. \quad (4.2)$$

Proof: Clearly,

$$h < x(S(Q)) \leq \sum \{x(C(s)) : s \in H(Q)\}.$$

However, for $Q \in \bar{S}^h$, $|H(Q)| \leq h^2$. The conclusion thus follows. ■

Since $Q \in \bar{S}^h$, without loss of generality, assume that $Q_K = \{k\}$. Then $|Q_I| = |Q_J| = h$. Suppose now that $s_1 = (i_1, j_1, k) \in H(Q)$ and (4.2) holds for $s = s_1$.

Lemma 4.3. *One of the following claims holds.*

(i). $Q_I = \{i_1, i_2, \dots, i_h\}$, where i_1, i_2, \dots, i_h are distinct, and $Q_J \supseteq \{j_1, j_2, \dots, j_\tau\}$, where $\tau \leq h$, j_1, j_2, \dots, j_τ are distinct, such that

$$x(C(s_\nu)) > \frac{1}{2h-1}, \quad (4.3)$$

for $\nu = 2, \dots, h$, where $s_\nu = (i_\nu, j_\nu, k)$, and $j_\nu \in \{j_1, \dots, j_\tau\}$ for $\tau < j \leq h$;

(ii). $Q_I \supseteq \{i_1, i_2, \dots, i_\tau\}$, where $\tau \leq h$, i_1, i_2, \dots, i_τ are distinct, and $Q_J = \{j_1, j_2, \dots, j_h\}$, where j_1, j_2, \dots, j_h are distinct, such that (4.3) holds, for $\nu = 2, \dots, h$, where $s_\nu = (i_\nu, j_\nu, k)$, and $i_\nu \in \{i_1, \dots, i_\tau\}$ for $\tau < i \leq h$.

Proof: Suppose that (4.3) holds for s_ν , $\nu = 2, \dots, \tau$ such that there are no repetitions in $\{i_1, \dots, i_\tau\}$ and $\{j_1, \dots, j_\tau\}$. Suppose that such a collection is maximal, i.e., for any $i \in Q_I \setminus \{i_1, \dots, i_\tau\}$ and $j \in Q_J \setminus \{j_1, \dots, j_\tau\}$,

$$x(C((i, j, k))) \leq \frac{1}{2h-1}. \quad (4.4)$$

If the conclusion of this lemma does not hold, then there exists $i_0 \in Q_I \setminus \{i_1, \dots, i_\tau\}$ and $j_0 \in Q_J \setminus \{j_1, \dots, j_\tau\}$ such that

$$x(C((i_0, j_\nu, k))) \leq \frac{1}{2h-1} \quad (4.5)$$

$$x(C((i_\nu, j_0, k))) \leq \frac{1}{2h-1}, \quad (4.6)$$

for $\nu = 1, \dots, \tau$. However,

$$x(S(Q)) \leq x(S(Q')) + \sum \{x(C(i_0, j, k)) : j \in Q_J, j \neq j_0\} + \sum \{x(C(i, j_0, k)) : j \in Q_J\}, \quad (4.7)$$

where Q' is defined by $Q'_I = Q_I \setminus \{i_0\}$, $Q'_J = Q_J \setminus \{j_0\}$, $Q'_K = Q_K = \{k\}$. By (4.4), (4.5) and (4.6), each term in the right hand side of (4.7), except the first term, is not greater than $\frac{1}{2h-1}$. The number of such terms is $2h-1$. Clearly, $Q' \in \bar{S}^{h-1}$. Thus, $x(S(Q')) \leq h-1$ by our assumption. Thus, the right hand side of (4.7) is not great than h . This forms a contradiction with our assumptions on x and Q . Therefore, this lemma holds. ■

The following is a brief proof of Theorem 4.1. Cumbersome details are omitted.

Proof of Theorem 4.1: By changing 12 to $4(2h-1)$ in (3.16), and (3.17), we may calculate all $x(C(s)) > \frac{1}{2h-1}$ in $O(n^3)$ steps. Check $s \in S$ with $x(C(s)) \geq \frac{1}{h}$. The number of such s is $O(n^2)$. Let $s_1 = s = (i_1, j_1, k_1)$. We first check $Q \in \bar{S}^h$ such that $s \subseteq Q$ and Q has a structure as described in Lemma 4.3 (i).

Let $k = k_1$. For $\tau = 2, \dots, h$, find all possible $s_\nu = (i_\nu, j_\nu, k)$ for $\nu = 2, \dots, \tau$ such that (4.3) holds and there are no repetitions in $\{i_1, \dots, i_\tau\}$ and $\{j_1, \dots, j_\tau\}$. (We may find s_2, \dots, s_τ sequentially.) Similarly to Lemma 3.4, one may see that the number of $t = (i', j', k)$ such that

$$x(C(t)) > \frac{1}{2h-1}$$

is $O(n)$. Thus, the number of such (s_2, \dots, s_τ) is $O(n^{\tau-1})$. Then we find all possible $s_\nu = (i_\nu, j_\nu, k)$ for $\nu = \tau+1, \dots, h$ such either (i) or (ii) of Lemma 4.3 holds. With an argument similar to Lemma 3.4, one may see that the number of such s_ν is $O(1)$. In order to expand $s_1 \cup \dots \cup s_h$ to a set $Q \in \bar{S}^h$, we need to add it with

$$\{j_{\tau+1}, \dots, j_h\} \subseteq J \setminus \{j_1, \dots, j_\tau\}.$$

Therefore, the number of expansion ways is $O(n^{h-\tau})$. For each Q produced in this way, we may use formulas similar to (3.6)-(3.8), (3.15) to calculate $x(S(Q))$ and check whether

$$x(S(Q)) \leq h. \quad (4.8)$$

The complexity of such formulas are $O(1)$. Summing up the above discussion, one see that the complexity of check (4.8) for all $Q \in \bar{S}^h$ such that $s \subseteq Q$ and Q has a structure as described in Lemma 4.3 (i), for a fixed s , is $O(n^{h-1})$.

The case of Lemma 4.3 (ii) and the cases, where i_1 or j_1 in s is fixed, are similar. Since the number of such s is $O(n^2)$, the overall complexity of such a procedure is $O(n^{h+1})$. This completes the proof of this theorem. ■

In general, the order of procedures to detect violated facets induced by sets in Φ_h for $h \geq 3$ may be higher than $O(n^{h+1})$. For example, let $h = 4$. We may find $O(n^6)$ triples of (s, t, p) such that $s, t, p \in S$, $x(C(s)), x(C(t)), x(C(p)) > \frac{1}{4}$, and

$$a^s \cdot a^t = 0, a^t \cdot a^p = 0, a^s \cdot a^p = 0.$$

Then $Q := s \cup t \cup p \in S^4$. We need to check whether

$$x(S(Q)) \leq 4$$

or not. Thus, the complexity of such a procedure is at least $O(n^6)$. But $6 > h+1 \equiv 5$. Hence, the complexity for detecting violated facets induced by sets in Φ_h may be higher than the complexity corresponding to $\bar{\Phi}_h$ for $h \geq 3$.

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