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Full-Information Item Bi-Factor Analysis

ONR Technical Report

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The purpose of this paper is to derive a bi-factor item-response model for binary response data, and to develop a corresponding method of parameter estimation. This restriction leads to a major simplification of the likelihood equations that (1) permits the statistical evaluation of problems of unlimited dimensionality, (2) permits conditional dependence among discrete and previously identified subsets of items, and (3) in some cases provides more parsimonious factor solutions than an unrestricted full-information item factor analysis might provide (*e.g.*, Bock and Aitkin, 1981). Keywords: Factor analysis, Psychometrics, Biometry. (RWJ)

ABSTRACT

A plausible s-factor solution for many types of psychological and educational tests is one in which there is one general factor and $s-1$ group or method related factors. The bi-factor solution results from the constraint that each item has a non-zero loading on the primary dimension α_{j1} and at most one of the $s-1$ group factors. This structure has been termed the "bi-factor" solution by Holzinger & Swineford, but it also appears in the work of Tucker and Joreskog. All attempts at estimating the parameters of this model have been restricted to continuously measured variables; it has not been previously considered in the context of item-response theory (IRT). It is conceivable, however, that the bi-factor structure might arise in IRT related problems.

The purpose of this paper is to derive a bi-factor item-response model for binary response data, and to develop a corresponding method of parameter estimation. This restriction leads to a major simplification of the likelihood equations that (1) permits the statistical evaluation of problems of unlimited dimensionality, (2) permits conditional dependence among discrete and previously identified subsets of items, and (3) in some cases provides more parsimonious factor solutions than an unrestricted full-information item factor analysis might provide (*e.g.*, Bock and Aitkin, 1981).

1 Introduction

Consider the case in which, for n variables, an s-factor solution exists in which there is one general factor and $s-1$ group or method related factors. The bi-factor solution constrains each item to have a non-zero loading on the primary dimension α_{j1} and on not more than one of the $s-1$ group factors (*i.e.*, $\alpha_{jh}, h = 2, \ldots, s$). For four items, the factor-pattern matrix might be

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ \alpha_{31} & 0 & \alpha_{33} \\ \alpha_{41} & 0 & \alpha_{43} \end{bmatrix}$$

This structure has been termed the "bi-factor" solution by Holzinger & Swineford (1937), inter-battery factor analysis by Tucker (1958), and is also one of the confirmatory factor analysis models considered by Joreskog (1969). In these applications, the model is restricted to test scores, assumed to be continuously distributed. It is easy, however to conceive of situations where the bi-factor pattern might arise at the item level. It is plausible for paragraph comprehension tests, for example, in which case the primary dimension describes the targeted aptitude and the additional factors describe knowledge of the content area within the paragraphs. In this context, items would be conditionally independent between paragraphs, but conditionally dependent within specific paragraphs.

The purpose of this paper is to derive an item-response model for binary response data that exhibit the bi-factor structure and to develop a corresponding method of parameter estimation. Of course, other types of tests that consist of items tapping different content areas would also be suitable for this type of analysis. As we will show, this restriction leads to a major simplification of the likelihood equations that (1) permits the statistical evaluation of problems of unlimited dimensionality, (2) permits conditional dependence among discrete and previously identified subsets of items, and (3) in some cases provides more parsimonious factor solutions than an unrestricted full-information item-factor analysis might provide (*e.g.*, Bock and Aitkin, 1981). In the following sections, we derive the likelihood and its first derivatives so that an EM solution to item bi-factor analysis may be obtained.

2 Likelihood Evaluation

Stuart (1958) showed that if n variables follow a standardized multivariate normal distribution where the correlation $\rho_{ij} = \sum_{h=1}^{s} \alpha_{ih}\alpha_{jh}$ and α_{ih} is nonzero for only one h, then the probability that the respective variables are simultaneously less than γ_j is,

$$P = \prod_{h=1}^{s} \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n_h} F((\gamma_j - \alpha_{jh} y)/(1-\alpha_{jh}^2)^{1/2}) \right] f(y) dy \tag{1}$$

where

$$f(t) = \exp(-\frac{1}{2}t^2)/(2\pi)^{1/2}$$

$$F(t) = \int_{-\infty}^{\gamma_j} f(t) dt$$

and n_h is the number of items loading on dimension h ($h = 1, \ldots, s$).

Equation (1) follows from the fact that if each variate is related to only a single dimension, then the s dimensions are independent, and the joint probability is simply the product of the s unidimensional probabilities. In the present context, this result only applies to the $s-1$ "nuisance" dimensions (*i.e.*, $h = 2, \ldots, s$); if a primary dimension exists, it will not be independent of the other $s-1$ dimensions. To compute this probability therefore requires a two-dimensional generalization of Stuart's (1958) original result.

To derive the two-dimensional result, we begin by noting that the probability of the primary dimension can be obtained using the formula of Dunnett and Sobel (1955),

$$P = \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n} F((\gamma_j - \alpha_{j1} y)/(1-\alpha_{j1}^2)^{1/2}) \right] f(y) dy, \tag{2}$$

which is valid as long as $\rho_{ij} = \alpha_i \alpha_j$. Of course, this directly implies a unidimensional problem. Combining the two results yields,

$$P = \int_{-\infty}^{\infty} \left\{ \prod_{h=2}^{s} \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n_h} F\left(\frac{\gamma_j - \alpha_{j1} z - \alpha_{jh} y}{\sqrt{1 - \alpha_{j1}^2 - \alpha_{jh}^2}} \right) \right] f(y) dy \right\} f(z) dz, \tag{3}$$

which can be approximated to any practical degree of accuracy using Gauss-Hermite quadrature (Stroud and Sechrest, 1966). What is important about this result is, if the assumptions are reasonable (as they clearly are for many IRT applications), then the probability of any response pattern can be obtained by a two-dimensional integration, regardless of the dimensionality s.

For example, if $y_j = \sum_{h=1}^{s} \alpha_{jh} \theta_h + \varepsilon_j$ and we assume that

$$
\begin{aligned}
y_j &\sim N(0,1), \\
\boldsymbol{\theta} &\sim N(\mathbf{0},\mathbf{I}), \text{ and} \\
\varepsilon_j &\sim N(0, 1 - \sum_{h=1}^{s} \alpha_{jh}),
\end{aligned}
$$

then the unconditional probability of observing score pattern $\mathbf{x} = \mathbf{x}_\ell$ is,

$$
P_\ell = \int_{-\infty}^{\infty} \left\{ \prod_{h=2}^{s} \left[\int_{-\infty}^{\infty} \prod_{j=1}^{n_h} [F(\theta_1, \theta_h)]^{x_{\ell j}} [1 - F(\theta_1, \theta_h)]^{1-x_{\ell j}} f(\theta_h) d\theta_h \right] \right\} f(\theta_1) d\theta_1, \tag{4}
$$

which can be approximated by,

$$
\hat{P}_\ell \cong \sum_{q_1}^{Q} \left\{ \prod_{h=2}^{s} \left[\sum_{q_h}^{Q} \prod_{j=1}^{n_h} [F(X_{q_1}, X_{q_h})]^{x_{\ell j}} [1 - F(X_{q_1}, X_{q_h})]^{1-x_{\ell j}} A(X_{q_h}) \right] \right\} A(X_{q_1}), \tag{5}
$$

where

$$
F(X_{q_1}, X_{q_h}) = F\left(-\frac{\gamma_j - \alpha_{j1} X_{q_1} - \alpha_{jh} X_{q_h}}{\sqrt{1 - \alpha_{j1}^2 - \alpha_{jh}^2}} \right),
$$

and $\mathbf{X}_q$ and $A(\mathbf{X}_q)$ are the nodes and corresponding weights of a Gauss-Hermite quadrature.

3 Marginal Maximum Likelihood Estimation

The parameters of the item bi-factor analysis model can be estimated by the method of marginal maximum likelihood using a variation of the approach described by Bock & Aitkin (1981). The parameters of this model include n "thresholds" or "intercepts", n primary factor loadings or "slopes" and a total of n factor loadings or slopes on the $h = 2, \ldots, s$ additional dimensions (*i.e.*, $\sum_{h=2}^{s} n_h = n$). The likelihood equations are derived as follows. Let and

$$\bar{N}_h(\mathbf{X}) = \sum_{\ell=1}^{S} r_\ell[E_{\ell h}(X_{q_1})]L_{\ell h}(X_{q_1}, X_{q_h})/P_\ell. \tag{13}$$

It should be noted that these equations are similar to those in the unrestricted case, except that in the bi-factor case, the conditional probability of response pattern $x_{\ell h}$ (*i.e.*, responses to items $j = 1, \ldots, n_h$ in subsection h for response pattern ℓ) is weighted by the factor, $E_{\ell h}(X_{q_1})$. Furthermore, since each item only appears in one subsection (h), the $\bar{N}$ now vary with h, in contrast to the unrestricted case. As such, the $\bar{N}_h$ denote the effective sample size for subset h at quadrature point (X_{q_1}, X_{q_h}). When weighted by $A(\mathbf{X})$ and summed over the quadrature nodes for each subsection, $\bar{N}_h$ yields the total number of respondents, whereas the corresponding weighting and summation for $\bar{r}_j$ yields the total number of respondents answering item j correctly.

From provisional parameter values, each E-Step yields $\bar{r}_j$ and $\bar{N}_h$, the expectations of the complete data statistics computed conditional on the incomplete data (see Bock, Gibbons, & Muraki, 1988). The subsequent M-step solves equation (10) using conventional maximum likelihood multiple probit analysis, substituting the provisional expectations of $\bar{r}_j$ and $\bar{N}_h$ (see Bock & Jones, 1968).

4 Illustration

To illustrate the application of the bi-factor IRT model, we have evaluated 20 items selected from an ACT natural science test, for a random sample of 1000 examinees (we are indebted to Terry Ackerman and Mark Reckase for these data). This test involves a series of questions regarding each of four paragraphs. For the purpose of this illustration, we selected the first 5 items from each of four paragraphs.

Table 1 displays the unrestricted promax-rotated 4-factor solution, which adequately fit these data (improvement in fit of a four-factor model over a three-factor model was $\chi^2_{17} = 31.59, p < .02$; the improvement in fit of five factors over four factors was not significant ($\chi^2_{16} = 18.44, p < .30$). Inspection of Table 1 reveals that each factor is dominated by items from a particular paragraph. In contrast, the estimated factor loadings for the bi-factor model (see Table 2) with $s = 5$ (*i.e.*, one primary dimension and four paragraph-specific dimensions) revealed a strong general ability dimension, as well as appreciable within paragraph associations. The fit of the restricted model was not significantly different from the fit of either the four-factor ($\chi^2_{45} = 23.83, p <$ .99) or the five-factor ($\chi^2_{60} = 43.22, p < .95$) unrestricted models. Inspection

$$
\begin{aligned}
P_\ell &= P(\mathbf{x} = \mathbf{x}_\ell) \\
&= \int_{\theta_1} \left\{ \prod_{h=2}^{s} \int_{\theta_h} \prod_{j=1}^{n_h} [F_j(\boldsymbol{\theta})]^{x_{\ell j}} [1 - F_j(\boldsymbol{\theta})]^{1-x_{\ell j}} f(\theta_h) d\theta_h \right\} f(\theta_1) d\theta_1 \\
&= \int_{\theta_1} \left\{ \prod_{h=2}^{s} \int_{\theta_h} L_{\ell h}(\theta) f(\theta_h) d\theta_h \right\} f(\theta_1) d\theta_1. \qquad (6)
\end{aligned}
$$

Then the log likelihood is,

$$
\log L = \sum_{\ell=1}^{S} r_\ell \, \log \, \hat{P}_\ell \qquad (7)
$$

where S denotes the number of unique response patterns. The derivative of the log marginal likelihood with respect to a general item parameter ν_j is as follows.
Let

$$
E_{\ell h}(\theta_1) = \frac{\left[\prod_{h=2}^{s} \int_{\theta_h} L_{\ell h}(\theta) f(\theta_h) d\theta_h\right]}{\int_{\theta_h} L_{\ell h}(\theta) f(\theta_h) d\theta_h}. \qquad (8)
$$

Then

$$
\frac{\partial \log L}{\partial \nu_j} = \sum_{\ell}^{S} \frac{r_\ell}{P_\ell} \left(\frac{\partial P_\ell}{\partial \nu_j} \right) \qquad (9)
$$

$$
= \sum_{\ell=1}^{S} \frac{r_\ell}{P_\ell} \int_{\theta_1} E_{\ell h}(\theta_1) \left\{ \int_{\theta_h} \left(\frac{x_{\ell j} - F_j(\boldsymbol{\theta})}{F_j(\boldsymbol{\theta})[1 - F_j(\boldsymbol{\theta})]} \right) L_{\ell h}(\boldsymbol{\theta}) \frac{\partial F_j(\boldsymbol{\theta})}{\partial \nu_j} f(\theta_h) d\theta_h \right\} f(\theta_1) d\theta_1. \qquad (10)
$$

Following Bock and Aitkin (1981), the marginal likelihood equations can be solved, using the EM algorithm of Dempster, Laird & Rubin (1977), by replacing the integrals with Gauss-Hermite quadratures and rearranging terms into the two-dimensional form:

$$
\sum_{q_1}^{Q} \sum_{q_h}^{Q} \frac{\bar{r}_j(\mathbf{X}) - \bar{N}_s(\mathbf{X}) F_j(\mathbf{X})}{F_j(\mathbf{X})[1 - F_j(\mathbf{X})]} \left(\frac{\partial F_j(\mathbf{X})}{\partial \nu_j} \right) A(X_{q_h}) A(X_{q_1}), \qquad (11)
$$

where

$$
\bar{r}_j(\mathbf{X}) = \sum_{\ell=1}^{S} r_\ell x_{\ell j} [E_{\ell h}(X_{q_1})] L_{\ell h}(X_{q_1}, X_{q_h}) / P_\ell \qquad (12)
$$

of the loadings within each paragraph reveals that the intra-paragraph item associations are quite variable.

As a computational note, we should point out that the numerical precision of the bi-factor solution represents a major improvement over the unrestricted solution. Given that the bi-factor solution only requires approximation of a two-dimensional integral, we were able to use 100 quadrature points (*i.e.*, 10 in each dimension) instead of the 243 quadrature points used in the unrestricted five factor solution, (*i.e.*, 3 in each dimension). Five factors probably represents the highest dimensional solution that is computational tractable at this time. Parameters of the unrestricted models were estimated using the TESTFACT program (Wilson, Wood & Gibbons, 1984).

5 A Simple Structure Model

Consider an orthogonal simple structure factor model in which each item loads on one and only one of s dimensions. This satisfies a complete simple structure model as defined by Thurstone (1947), which for measurement data could be evaluated using methods for confirmatory factor analysis (Joreskog, 1969). This is, of course, a simplification of the bi-factor model in which there is no primary dimension. In this case, the unconditional probability in (5) is reduced to the unidimensional form,

$$P_\ell \cong \prod_{h=1}^{s}\left[\sum_{q_h}^{Q}\left\{\prod_{j=1}^{n_h}[F(X_{q_h})]^{x_{\ell j}}[1-F(X_{q_h})]^{1-x_{\ell j}}\right\}A(X_{q_h})\right], \tag{14}$$

where

$$F(X_{q_h}) = F\left(-\frac{\gamma_j-\alpha_{jh}X_{q_h}}{\sqrt{1-\alpha_{jh}^2}}\right);$$

that is, (5) reduces to the product of the s independent unidimensional probabilities. The likelihood equations in (11) can then be approximated by,

$$\frac{\partial \log L}{\partial \nu_j} \cong \sum_{q_h}^{Q}\frac{\bar{r}_j(X_{q_h})-\bar{N}_h(X_{q_h})F_j(X_{q_h})}{F_j(X_{q_h})[1-F_j(X_{q_h})]}\left(\frac{\partial F_j(X_{q_h})}{\partial \nu_j}\right)A(X_{q_h}), \tag{15}$$

where

$$\bar{r}_j(X_{q_h}) = \sum_{\ell=1}^{S} r_\ell x_{\ell j}L_{\ell h}(X_{q_h})/e_h \tag{16}$$

and

$$\bar{N}_h(X_{q_h}) = \sum_{\ell=1}^{S} r_\ell L_{\ell h}(X_{q_h})/e_h. \tag{17}$$

In this case, e_h represents the constant

$$e_h = \sum_{q_h}^{Q} L_{\ell h}(X_{q_h})A(X_{q_h}),$$

and

$$P_\ell = \prod_{h=1}^{s} e_h$$

It is interesting to note that $\bar{r}_j$ and $\bar{N}_h$ now only contain information from the specific subset of items (h) for which item j is a member. This is, of course, due to the independence between the subsets that results from the simple structure.

Application of the simple structure model to the ACT natural science test example yields the item-parameters displayed in Table 3. Inspection of the parameter estimates in Table 3 reveals that removal of the primary factor increases the magnitude of the loadings on the individual paragraph dimensions. In terms of model fit, both the bi-factor model ($\chi^2_{20} = 336, p < .0001$) and the unrestricted four-factor model ($\chi^2_{65} = 361, p < .0001$) provide significant improvements in fit over the simple structure model, indicating that the test is in fact measuring a primary ability dimension and not merely four independent realms of knowledge.

6 Discussion

The bi-factor model presented here provides a natural alternative to the traditional conditionally-independent unidimensional IRT model. When potential sources of conditional dependence are known in advance, as in the case of paragraph comprehension tests or tests in which two or more methods of item presentation are involved, the item bi-factor solution provides an excellent alternative. An attractive by-product of this model is that it requires only the evaluation of a two-dimensional integral, regardless of the number of potential subtests, paragraphs, or content areas. These different content areas are, of course, assumed to be independent conditional on the primary ability dimension that the test was designed to measure. As such, the limitations on the dimensionality of the full-information item factor analysis model embodied in the TESTFACT program (Wilson, Wood & Gibbons, 1984), do not apply. Of course, the subsections (*e.g.*, paragraphs) must be known in advance.

In certain situations, for example psychiatric measurement (Gibbons, 1985), the existence of a primary dimension (*e.g.*, depression), is itself at question. In this case, comparison of the bi-factor and simple factor solutions presented here is of particular interest. Item bi-factor analysis could therefore help answer the question of whether depression is a unitary disorder or a mixture of a series of qualitatively distinct abnormalities; a question that has long plagued psychiatric researchers. Comparison of the fit of the bi-factor and simple structure models provides a tool for investigating such problems in psychiatric research and other areas as well.

Finally, those cases in which little is known about the structure of a particular test, but little confidence can be placed in the assumption of conditional independence, the more general solution presented by Gibbons *et. al.* (1989), using Clark's (1961) formulae for the moments of n jointly normal variables, could be used. This procedure uses a direct approximation to the multivariate normal distribution that underlies the item-response function, without restrictions on the form of the inter-item residual covariances. With it, the assumption of conditional independence is not required. Further work in this area is underway.

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Table 1

Full-Information Item Factor Analysis - Unrestricted Promax Solution
ACT Natural Science Test - 20 items and 1000 subjects

Item	γ_j	α_{j1}	α_{j2}	α_{j3}	α_{j4}
1	-.215	**.401**	-.005	-.036	.216
2	-.385	**.185**	-.019	-.007	.105
3	-.356	**.667**	-.070	-.081	-.081
4	-.098	**.619**	.013	.044	-.022
5	-.029	**.562**	-.092	-.059	.119
6	-.582	.129	.068	**.256**	.030
7	-.585	.184	-.211	**.419**	.102
8	-.137	-.037	-.061	**.025**	.172
9	-.246	.238	.063	**.362**	-.284
10	-.089	-.224	.128	**.620**	.060
11	-.049	.182	.135	-.034	**.311**
12	-.407	-.024	-.065	.124	**.320**
13	-.265	.247	.082	.020	**.173**
14	-.051	.137	.005	.007	**.585**
15	.040	.224	.129	-.045	**.295**
16	.345	.153	**.289**	-.122	-.109
17	.167	-.007	**.682**	.089	-.044
18	.172	-.096	**.520**	-.024	.120
19	.543	.008	**.500**	.067	.091
20	.672	-.073	**-.010**	.004	.163

Table 2

Full-Information Item Bi-Factor Analysis
ACT Natural Science Test - 20 items and 1000 subjects

Item	γ_j	α_{j1}	α_{j2}	α_{j3}	α_{j4}	α_{j5}
1	-.230	.524	.129			
2	-.392	.232	.115			
3	-.370	.411	.427			
4	-.118	.548	.278			
5	-.046	.489	.338			
6	-.593	.311		.277		
7	-.600	.376		.314		
8	-.138	.087		-.019		
9	-.259	.207		.390		
10	-.103	.226		.476		
11	-.062	.484			.141	
12	-.413	.261			.135	
13	-.277	.423			.199	
14	-.066	.573			.187	
15	.025	.492			.260	
16	.340	.112				.261
17	.150	.306				.662
18	.160	.240				.571
19	.528	.340				.493
20	.671	.061				.031

Table 3

Full-Information Simple Structure Item Factor Analysis
ACT Natural Science Test - 20 items and 1000 subjects

Item	γ_j	α_{j1}	α_{j2}	α_{j3}	α_{j4}
1	-.224	.482			
2	-.391	.251			
3	-.368	.571			
4	-.111	.612			
5	-.040	.585			
6	-.592		.408		
7	-.597		.467		
8	-.138		.032		
9	-.258		.429		
10	-.102		.509		
11	-.056			.489	
12	-.412			.297	
13	-.273			.449	
14	-.058			.591	
15	.031			.566	
16	.341				.282
17	.157				.732
18	.163				.616
19	.534				.597
20	.671				.057

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