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HEAVY-TRAFFIC ANALYSIS OF MULTI-TYPE QUEUEING UNDER PROBABILISTICALLY LOAD-PREFERENTIAL SERVICE ORDER

Donald P. Gaver
J. A. Morrison

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This report was prepared by:

Donald P. Gaver

DONALD P. GAVER
Professor of Operations Research

Reviewed by:

P. Purdue

PETER PURDUE
Professor and Chairman
Department of Operations Research

Released by:

Dean of Faculty and Graduate Studies

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D. P. Gaver

Naval Postgraduate School
Monterey, California 93943

J. A. Morrison

AT&T Bell Laboratories
Murray Hill, New Jersey 07974

ABSTRACT

A model of queueing for a single server by several types of customers (messages, or jobs), with a simple dynamic priority rule, is considered. The rule is equivalent to *selecting the next server occupant type with a probability proportional to the number of* that type enqueued. The situation studied here occurs in fields such as computer and communication system performance analysis, in operational analysis of logistics systems, and in the repair of elements of a manufacturing system. It is assumed that the population sizes of the items of different types are large, and that the mean service rates are correspondingly large, in comparison with the service demand rates. Moreover, it is assumed that the system is in heavy traffic. Under these assumptions, asymptotic approximations are derived for the steady-state means and covariances of the number of items of different types either waiting or being served. Numerical comparisons with simulated results show excellent agreement.

Key words. asymptotics, dynamic priority rules, logistics, repair problems

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D. P. Gaver

Naval Postgraduate School
Monterey, California 93943

J. A. Morrison

AT&T Bell Laboratories
Murray Hill, New Jersey 07974

1. INTRODUCTION

This paper is concerned with the analysis of queueing for a single server by several types of customers (messages, or jobs) under heavy traffic. A simple dynamic priority rule is presumed to schedule next service among waiting customers when the server becomes available. The rule is equivalent to selecting the next server occupant type with a probability proportional to the number of that type enqueued. For generalizations, with special attention to logistics and repair problems, see Gaver, Jacobs and Pilnick [1]. Problems similar to that considered here have been studied by Towsley [2], Yao and Buzacott [3] and by others.

The situation studied here occurs in fields such as computer and communication system performance analysis, and also in operational analysis of logistics systems. It is also possible to find it in manufacturing, where each of several types of machines suffers occasional breakdown and is eligible for repair. The total productivity of the system depends on having a sufficient number of each machine type in operation; the present scheduling discipline assists in this objective, although others (to be discussed in a subsequent paper) are more effective. Note that the current discipline closely resembles a *processor-sharing* scheduling rule which is frequently used for controlling multiprocessors in computer systems.

In a specific model formulation, there are r types of demand-producing items, and K_i is the population size of items of type i. If $N_i(t)$ denotes the number of items of type i waiting for service, or being served, at time t, then the probability that a new type i demand for service is initiated in $(t, t+dt)$ is $\lambda_i[K_i - N_i(t)]dt + o(dt)$, so that λ_i is the service demand rate for items of type i. The service time of items of type i is exponentially distributed with rate ν_i, so that the mean service time is $1/\nu_i$; service times are all independent. Thus all queue lengths are finite $(N_i(t) \le K_i)$ and a long-run or steady-state distribution will always exist, as is true of the simple repairman problem [4].

A particular example of the probabilistically load-preferential scheduling rules, e.g. considered by Gaver, Jacobs and Pilnick [1], is the following. Let $\mathbf{N}(\tau+) = (N_1(\tau+), N_2(\tau+), \ldots, N_r(\tau+))$ denote the state of the system at time $\tau+$ immediately after the service of an item is completed. Then, for $\mathbf{N}(\tau+) \ne \mathbf{0}$, an item of type i is selected for service with probability

$$q_i(\mathbf{N}(\tau+)) = \frac{c_i N_i(\tau+)}{\sum_{j=1}^{r} c_j N_j(\tau+)}, \tag{1.1}$$

where $c_j > 0$ for $j \in (1, 2, \ldots, r)$. Such a service schedule gives preference, with appropriate weights, to those types of items of which there are more waiting for service. If one of the queues gets long compared to the others, then it is likely that an item from that queue will be selected for service next. Hence this scheduling rule may be regarded as a variant of serving the longest queue first.

For $\mathbf{N}(t) \neq \mathbf{0}$, let $I(t)$ denote the type of item which is being served at time t. It is clear from the assumptions made that $(\mathbf{N}(t), I(t), t \geq 0)$ is a finite-state-space Markov process in continuous time with integer-valued vector state space. In principle, the joint probability distributions

$$P\{\mathbf{N}(t) = \mathbf{n}, I(t) = i \mid \mathbf{N}(0) = \mathbf{n}(0)\} = P_i(\mathbf{n}, t; \mathbf{n}(0)), \quad i \in (1, 2, \ldots, r), \tag{1.2}$$

where $0 \leq n_j \leq K_j$, $j \in (1, 2, \ldots, r)$, can be found, given the initial condition $\mathbf{n}(0)$, by solving a system of $r \prod_{j=1}^{r} (K_j + 1)$ linear differential equations, the Kolmogorov forward equations [4]. All probabilistic quantities of interest can be found from such equations, or similar backward equations. In practice such solutions involve extensive computing, so it is of interest to proceed otherwise.

To simplify the analysis, we make two basic assumptions. The first is that the system is *large*, i.e. that the population sizes $K_i = a\,\alpha_i$, where $a >> 1$ and $\alpha_i = O(1)$. Since a fast server is needed to accommodate a large system, it is also assumed that $\nu_i = a\,\mu_i$, where $\mu_i = O(1)$. The second assumption is that the system is in *heavy traffic* circumstances, meaning that $\sum_{i=1}^{r} \lambda_i \alpha_i / \mu_i > 1$. This condition ensures that the server is extremely unlikely to be idle. Under these circumstances, Gaver, Jacobs and Pilnick [1] derived a diffusion approximation, and obtained numerical results for the time-dependent problem.

In this paper we consider the steady-state problem under the same circumstances, and derive asymptotic approximations for $a >> 1$ to the means and covariances of the number of items of different types in the system, i.e. either waiting or being served. The lowest order approximation to the means agrees with that obtained from the diffusion approximation [1]. In this paper we also derive the first order correction term to the means. Our approximation to the covariances differs from that obtained by the diffusion approximation, when the service rates ν_i are unequal. However, numerical results indicate that the difference is not very large.

In §2 we formulate the problem, and introduce generating functions. We then introduce the scalings corresponding to a large system in heavy traffic, and look for an asymptotic expansion in inverse powers of a. The leading term in the expansion is obtained in §2, and the first order correction term is derived in §3. Asymptotic approximations to the means and covariances of the number of items of different types in the system are obtained in §4. Numerical comparisons, which show that the asymptotic and simulated results are in excellent agreement, are presented in §5. In the appendix we give an alternate derivation of the lowest-order asymptotic approximation to the means and covariances, and indicate how we obtain the first order correction to the means by this method. We also derive the lowest-order asymptotic approximation to the joint probability density function.

2. AN ASYMPTOTIC ANALYSIS

Without special boundary conditions the setup described in the previous section is an irreducible finite Markov chain and hence possesses a steady-state or long-run solution

$$\lim_{t \to \infty} P_i(\mathbf{n}, t; \mathbf{n}(0)) = p_i(\mathbf{n}), \qquad i \in (1, 2, \ldots, r), \tag{2.1}$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_r)$. We note that $p_i(\mathbf{n}) = 0$ if $n_i = 0$, and $p_i(\mathbf{n}) = 0$ unless $\mathbf{0} \le \mathbf{n} \le \mathbf{K}$, $\mathbf{n} \ne \mathbf{0}$, where $\mathbf{K} = (K_1, K_2, \ldots, K_r)$. The steady-state probability that the system is empty is denoted by $p(\mathbf{0})$. For $\mathbf{0} \le \mathbf{n} \le \mathbf{K}$, $\mathbf{n} \ne \mathbf{0}$ and $\mathbf{n} \ne \mathbf{K}$, $q_i(\mathbf{n})$ is the probability that, when an item departs and leaves the system in state $\mathbf{n}$, then an item of type i goes into service. We assume, for the time being, only that $q_i(\mathbf{n}) = 0$ if $n_i = 0$, $q_i(\mathbf{n}) = 1$ if $n_j = 0$ for $j \ne i$ and $1 \le n_i \le K_i$, and $\sum_{i=1}^{r} q_i(\mathbf{n}) = 1$ for $\mathbf{0} \le \mathbf{n} \le \mathbf{K}$, $\mathbf{n} \ne \mathbf{0}$ and $\mathbf{n} \ne \mathbf{K}$.

We denote by $\mathbf{e}_i$ the vector with components $e_{ij} = \delta_{ij}$. Then, in a standard manner, we obtain

$$\sum_{j=1}^{r} \lambda_j K_j p(0) = \sum_{j=1}^{r} \nu_j p_j(\mathbf{e}_j), \tag{2.2}$$

$$\left[\sum_{j=1}^{r} \lambda_j (K_j - \delta_{ij}) + \nu_i\right] p_i(\mathbf{e}_i) = \lambda_i K_i p(0) + \sum_{j=1}^{r} \nu_j p_j(\mathbf{e}_i + \mathbf{e}_j), \tag{2.3}$$

and, for $n_i \neq 0$, $\mathbf{n} \neq \mathbf{e}_i$ and $0 \leq \mathbf{n} \leq \mathbf{K}$,

$$\left[\sum_{j=1}^{r} \lambda_j (K_j - n_j) + \nu_i\right] p_i(\mathbf{n})$$

$$= \sum_{j=1}^{r} \lambda_j (K_j - n_j + 1) p_i(\mathbf{n} - \mathbf{e}_j) + q_i(\mathbf{n}) \sum_{j=1}^{r} \nu_j p_j(\mathbf{n} + \mathbf{e}_j). \tag{2.4}$$

We introduce the generating functions

$$u_i(\mathbf{x}) = \sum_{0 \leq \mathbf{n} \leq \mathbf{K}} p_i(\mathbf{n}) x_1^{n_1} \cdots x_r^{n_r}, \tag{2.5}$$

and note that $u_i(0) = 0$. We now multiply equations (2.2), (2.3) and (2.4) by 1, x_i and $x_1^{n_1} \cdots x_r^{n_r}$, respectively, and sum on i and $\mathbf{n}$. It is found that

$$\sum_{j=1}^{r} \lambda_j (1 - x_j) \left\{ K_j \left[p(0) + \sum_{i=1}^{r} u_i(\mathbf{x}) \right] - x_j \sum_{i=1}^{r} \frac{\partial u_i}{\partial x_j} \right\} = \sum_{j=1}^{r} \frac{\nu_j}{x_j} (1 - x_j) u_j(\mathbf{x}). \tag{2.6}$$

This equation, which holds for general $q_i(\mathbf{n})$ with $\sum_{i=1}^{r} q_i(\mathbf{n}) = 1$, will be useful later. The equation is vacuous if $\mathbf{x} = \mathbf{1}$, so that one of the original equations is redundant. The normalization condition is

$$p(0) + \sum_{i=1}^{r} u_i(\mathbf{1}) = 1. \tag{2.7}$$

We now consider the particular case

$$q_i(\mathbf{n}) = \frac{c_i n_i}{\sum_{l=1}^{r} c_l n_l}, \tag{2.8}$$

where $c_l > 0$ for $l \in (1, 2, \ldots, r)$, and multiply equations (2.3) and (2.4) by $c_i x_i$ and $\sum_{l=1}^{r} c_l n_l x_1^{n_1} \cdots x_r^{n_r}$, respectively, and sum on $\mathbf{n}$. After some manipulations it is found that

$$\sum_{l=1}^{r} c_l x_l \frac{\partial}{\partial x_l} \left[\sum_{j=1}^{r} \lambda_j K_j (1 - x_j) u_i \right]$$

$$= p(\mathbf{0}) c_i \lambda_i K_i x_i + c_i x_i \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{r} \frac{\nu_j u_j}{x_j} \right) - \sum_{l=1}^{r} c_l x_l \frac{\partial}{\partial x_l} (\nu_i u_i)$$

$$+ \sum_{l=1}^{r} c_l x_l \frac{\partial}{\partial x_l} \left[\sum_{j=1}^{r} \lambda_j x_j (1 - x_j) \frac{\partial u_i}{\partial x_j} \right]. \tag{2.9}$$

We are interested in the means and covariances of the number of items of different types in the system, namely $E(n_j)$ and $E(n_j n_k) - E(n_j)E(n_k)$. We note, from (2.5), that

$$E(n_j) = \sum_{i=1}^{r} \frac{\partial u_i}{\partial x_j}(\mathbf{1}), \quad E(n_j n_k) = \sum_{i=1}^{r} \left[\frac{\partial^2 u_i}{\partial x_j \partial x_k}(\mathbf{1}) + \delta_{jk} \frac{\partial u_i}{\partial x_j}(\mathbf{1}) \right]. \tag{2.10}$$

We now introduce the scalings corresponding to a large system in heavy traffic, and let

$$K_j = a\,\alpha_j, \quad \nu_j = a\,\mu_j, \quad j \in (1, 2, \ldots, r); \quad a >> 1, \tag{2.11}$$

where

$$\sum_{j=1}^{r} \frac{\lambda_j \alpha_j}{\mu_j} > 1. \tag{2.12}$$

Since, from (2.10), we are interested in the behavior of $u_i(\mathbf{x})$ in the neighborhood of $\mathbf{x} = \mathbf{1}$, we also let

$$x_j = 1 - \xi_j / a, \quad u_j(\mathbf{x}) = \psi_j(\boldsymbol{\xi}), \quad j \in (1, 2, \ldots, r). \tag{2.13}$$

Then, from (2.6), (2.7) and (2.9), we have

$$p(0) + \sum_{i=1}^{r} \psi_i(0) = 1\,, \tag{2.14}$$

$$\sum_{j=1}^{r} \lambda_j \xi_j \left\{ \alpha_j \left[p(0) + \sum_{i=1}^{r} \psi_i(\xi) \right] + \left(1 - \frac{\xi_j}{a}\right) \sum_{i=1}^{r} \frac{\partial \psi_i}{\partial \xi_j} \right\} = \sum_{j=1}^{r} \frac{\mu_j \xi_j}{(1-\xi_j/a)} \psi_j(\xi)\,. \tag{2.15}$$

and

$$\begin{aligned}
&\mu_i \sum_{l=1}^{r} c_l \left(1 - \frac{\xi_l}{a}\right) \frac{\partial \psi_i}{\partial \xi_l} - c_i \sum_{j=1}^{r} \mu_j \frac{(1-\xi_i/a)}{(1-\xi_j/a)} \frac{\partial \psi_j}{\partial \xi_i} - \frac{c_i \mu_i \psi_i}{a(1-\xi_i/a)} \\
&+ \frac{p(0)}{a} c_i \lambda_i \alpha_i \left(1 - \frac{\xi_i}{a}\right) + \frac{1}{a} \sum_{j=1}^{r} \lambda_j \alpha_j \xi_j \sum_{l=1}^{r} c_l \left(1 - \frac{\xi_l}{a}\right) \frac{\partial \psi_i}{\partial \xi_l} \\
&+ \frac{1}{a} \sum_{j=1}^{r} c_j \lambda_j \alpha_j \left(1 - \frac{\xi_j}{a}\right) \psi_i + \frac{1}{a} \sum_{j=1}^{r} c_j \lambda_j \left(1 - \frac{\xi_j}{a}\right) \left(1 - \frac{2\xi_j}{a}\right) \frac{\partial \psi_i}{\partial \xi_j} \\
&+ \frac{1}{a} \sum_{j=1}^{r} \lambda_j \xi_j \left(1 - \frac{\xi_j}{a}\right) \sum_{l=1}^{r} c_l \left(1 - \frac{\xi_l}{a}\right) \frac{\partial^2 \psi_i}{\partial \xi_j \partial \xi_l} = 0\,.
\end{aligned} \tag{2.16}$$

We assume an asymptotic expansion of the form

$$\psi_i(\xi) \sim \psi_i^{(0)}(\xi) + \frac{1}{a} \psi_i^{(1)}(\xi) + \cdots\,, \qquad p(0) \sim 0\,, \tag{2.17}$$

since we expect $p(0)$, the probability that the system is empty, to be exponentially small in a. Then, to lowest order, from (2.16) we obtain

$$\mu_i \sum_{l=1}^{r} c_l \frac{\partial \psi_i^{(0)}}{\partial \xi_l} - c_i \sum_{j=1}^{r} \mu_j \frac{\partial \psi_j^{(0)}}{\partial \xi_i} = 0\,. \tag{2.18}$$

These equations are satisfied by

$$\psi_i^{(0)} = \frac{c_i}{\mu_i} \frac{\partial \theta}{\partial \xi_i}\,, \tag{2.19}$$

where the function $\theta(\xi)$ is to be determined. But, from (2.15) and (2.17), we have

$$\sum_{j=1}^{r} \lambda_j \xi_j \sum_{i=1}^{r} \left(\alpha_j \psi_i^{(0)} + \frac{\partial \psi_i^{(0)}}{\partial \xi_j} \right) = \sum_{j=1}^{r} \mu_j \xi_j \psi_j^{(0)}\,. \tag{2.20}$$

We look for a solution of the form

$$\theta(\xi) = \theta(0)\exp\left(-\sum_{l=1}^{r}\beta_l\xi_l\right), \tag{2.21}$$

so that, from (2.19),

$$\psi_i^{(0)}(\xi) = -\frac{c_i\beta_i}{\mu_i}\theta(\xi), \qquad \frac{\partial\psi_i^{(0)}}{\partial\xi_j} = \frac{ci\beta_i}{\mu_i}\beta_j\theta(\xi). \tag{2.22}$$

We let

$$A = \sum_{i=1}^{r}\frac{c_i\beta_i}{\mu_i}. \tag{2.23}$$

Then (2.20) is satisfied if

$$A\lambda_j(\alpha_j-\beta_j) = c_j\beta_j, \quad \text{i.e.,} \quad \beta_j = \frac{A\lambda_j\alpha_j}{(A\lambda_j+c_j)}. \tag{2.24}$$

Since we want $\beta_j > 0$, $j \in (1, 2, \ldots, r)$, we want $A > 0$. But, (2.23) and (2.24) imply that

$$F(A) \equiv \sum_{j=1}^{r}\frac{c_j\lambda_j\alpha_j}{\mu_j(A\lambda_j+c_j)} - 1 = 0. \tag{2.25}$$

But $F(0) > 0$, because of assumption (2.12), and $F(+\infty) = -1$. Since $F(A)$ is a decreasing function of A for $A > 0$, it follows that there is a unique solution $A > 0$ of $F(A) = 0$. From the normalization condition (2.14), and (2.17), we have

$$\sum_{i=1}^{r}\psi_i^{(0)}(0) = 1. \tag{2.26}$$

It follows from (2.22) and (2.23) that

$$\theta(0) = -\frac{1}{A}. \tag{2.27}$$

This completes the determination of $\theta(\xi)$, and hence $\psi_i^{(0)}(\xi)$.

In the next section we determine $\psi_i^{(1)}(\xi)$. The reader who is not interested in the details may proceed to §4, where the asymptotic approximations to the means and covariances are evaluated.

3. THE CORRECTION TERM

We now consider the first order correction term $\psi_i^{(1)}(\xi)$ in the asymptotic expansion (2.17). It follows from (2.16) that

$$\mu_i \sum_{l=1}^{r} c_l \frac{\partial \psi_i^{(1)}}{\partial \xi_l} - c_i \sum_{j=1}^{r} \mu_j \frac{\partial \psi_j^{(1)}}{\partial \xi_i} = \mu_i \sum_{l=1}^{r} c_l \xi_l \frac{\partial \psi_i^{(0)}}{\partial \xi_l} + c_i \sum_{j=1}^{r} \mu_j(\xi_j - \xi_i) \frac{\partial \psi_j^{(0)}}{\partial \xi_i} + c_i \mu_i \psi_i^{(0)}$$
$$- \sum_{j=1}^{r} \lambda_j \alpha_j \xi_j \sum_{l=1}^{r} c_l \frac{\partial \psi_i^{(0)}}{\partial \xi_l} - \sum_{j=1}^{r} c_j \lambda_j \alpha_j \psi_i^{(0)}$$
$$- \sum_{j=1}^{r} c_j \lambda_j \frac{\partial \psi_i^{(0)}}{\partial \xi_j} - \sum_{j=1}^{r} \lambda_j \xi_j \sum_{l=1}^{r} c_l \frac{\partial^2 \psi_i^{(0)}}{\partial \xi_j \partial \xi_l}. \qquad (3.1)$$

It is found, from (2.21), (2.22) and (2.24), after some reduction, that

$$\mu_i \sum_{l=1}^{r} c_l \frac{\partial \psi_i^{(1)}}{\partial \xi_l} - c_i \sum_{j=1}^{r} \mu_j \frac{\partial \psi_j^{(1)}}{\partial \xi_i}$$
$$= \frac{c_i \beta_i}{A \mu_i} \theta(\xi) \left[\sum_{j=1}^{r} c_j^2 \beta_j - A c_i \mu_i + A \mu_i \sum_{j=1}^{r} c_j \beta_j (2\xi_j - \xi_i) - \sum_{l=1}^{r} c_l \beta_l \sum_{j=1}^{r} c_j \beta_j \xi_j \right]. \qquad (3.2)$$

We note that if we sum (3.2) on i, and use (2.23), then ɔoth sides of the equation are identically ze. ɔ.

In view of (2.21), we let

$$\psi_i^{(1)}(\xi) = \chi_i^{(1)}(\xi) \exp\Big(- \sum_{k=1}^{r} \beta_k \xi_k\Big), \qquad (3.3)$$

and we define

$$B = \sum_{j=1}^{r} c_j \beta_j, \qquad D = \sum_{j=1}^{r} c_j^2 \beta_j. \qquad (3.4)$$

Then, since $\theta(0) = -1/A$,

$$\mu_i \sum_{l=1}^{r} c_l \left(\frac{\partial \chi_i^{(1)}}{\partial \xi_l} - \beta_l \chi_i^{(1)} \right) - c_i \sum_{j=1}^{r} \mu_j \left(\frac{\partial \chi_j^{(1)}}{\partial \xi_i} - \beta_i \chi_j^{(1)} \right)$$

$$= -\frac{c_i \beta_i}{A^2 \mu_i} \left[D - Ac_i\mu_i - AB\mu_i\xi_i + (2A\mu_i - B) \sum_{j=1}^{r} c_j\beta_j\xi_j \right]. \tag{3.5}$$

We look for a solution of the form

$$\chi_i^{(1)}(\xi) = \frac{c_i}{\mu_i} \left[\frac{\partial \phi^{(1)}}{\partial \xi_i} - \beta_i \phi^{(1)}(\xi) \right] + a_i + \sum_{k=1}^{r} b_{ik}\, \xi_k\,, \tag{3.6}$$

and note that the terms involving $\phi^{(1)}$ are annihilated by the operator on the left side of (3.5), i.e. they provide a solution of the homogeneous equation.

It is found that (3.5) is satisfied by (3.6) if

$$c_i\beta_i \sum_{j=1}^{r} \mu_j b_{jk} - B\mu_i b_{ik} = \frac{c_i\beta_i}{A^2\mu_i} \left[(B - 2A\mu_i)c_k\beta_k + AB\mu_i\delta_{ik} \right], \tag{3.7}$$

and

$$c_i\beta_i \sum_{j=1}^{r} \mu_j a_j - B\mu_i a_i = c_i \sum_{j=1}^{r} \mu_j b_{ji} - \mu_i \sum_{l=1}^{r} c_l b_{il} + \frac{c_i\beta_i}{A^2\mu_i} (Ac_i\mu_i - D). \tag{3.8}$$

If we sum (3.7) and (3.8) on i, and use (2.23) and (3.4), then both sides of each equation are identically zero. Since we are looking for a particular solution of the inhomogeneous equation, we may take

$$\sum_{j=1}^{r} \mu_j b_{jk} = 0, \qquad \sum_{j=1}^{r} \mu_j a_j = 0. \tag{3.9}$$

Then (3.7) and (3.8) imply that

$$b_{ik} = \frac{c_i \beta_i}{AB\mu_i} \left[\left(2 - \frac{B}{A\mu_i} \right) c_k \beta_k - B\delta_{ik} \right], \tag{3.10}$$

and

$$a_i = \frac{2c_i\beta_i}{AB^2\mu_i} (D - Bc_i). \tag{3.11}$$

It remains to determine the single function $\phi^{(1)}(\xi)$ in (3.6). But, from (2.15), (2.17) and (2.20), we have

$$\sum_{j=1}^{r} \lambda_j \xi_j \sum_{i=1}^{r} \left(\alpha_j \psi_i^{(1)} + \frac{\partial \psi_i^{(1)}}{\partial \xi_j} \right) - \sum_{j=1}^{r} \mu_j \xi_j \psi_j^{(1)}$$

$$= \sum_{j=1}^{r} \lambda_j \xi_j^2 \sum_{i=1}^{r} \frac{\partial \psi_i^{(0)}}{\partial \xi_j} + \sum_{j=1}^{r} \mu_j \xi_j^2 \psi_j^{(0)}. \tag{3.12}$$

It follows, from (2.21)-(2.24), (2.27) and (3.3), that

$$\frac{1}{A} \sum_{j=1}^{r} c_j \beta_j \xi_j \sum_{i=1}^{r} \chi_i^{(1)} + \sum_{j=1}^{r} \lambda_j \xi_j \sum_{i=1}^{r} \frac{\partial \chi_i^{(1)}}{\partial \xi_j} - \sum_{j=1}^{r} \mu_j \xi_j \chi_j^{(1)}$$

$$= \frac{1}{A} \sum_{j=1}^{r} (c_j - A\lambda_j) \beta_j \xi_j^2. \tag{3.13}$$

Then, from (3.6), we obtain

$$\sum_{j=1}^{r} \lambda_j \xi_j \sum_{i=1}^{r} \frac{c_i}{\mu_i} \frac{\partial^2 \phi^{(1)}}{\partial \xi_i \partial \xi_j} + \frac{1}{A} \sum_{j=1}^{r} c_j \beta_j \xi_j \sum_{i=1}^{r} \frac{c_i}{\mu_i} \frac{\partial \phi^{(1)}}{\partial \xi_i} - \sum_{j=1}^{r} (A\lambda_j + c_j) \xi_j \frac{\partial \phi^{(1)}}{\partial \xi_j}$$

$$= \frac{1}{A} \sum_{j=1}^{r} (c_j - A\lambda_j) \beta_j \xi_j^2 - \frac{1}{A} \sum_{j=1}^{r} c_j \beta_j \xi_j \sum_{i=1}^{r} \left(a_i + \sum_{k=1}^{r} b_{ik} \xi_k \right)$$

$$- \sum_{j=1}^{r} \lambda_j \xi_j \sum_{i=1}^{r} b_{ij} + \sum_{j=1}^{r} \mu_j \xi_j \left(a_j + \sum_{k=1}^{r} b_{jk} \xi_k \right). \tag{3.14}$$

We look for a solution of the form

$$\phi^{(1)}(\xi) = \kappa + \sum_{k=1}^{r} \gamma_k \xi_k + \frac{1}{2} \sum_{k=1}^{r} \sum_{l=1}^{r} \omega_{kl} \xi_k \xi_l, \tag{3.15}$$

where we assume, without loss of generality, that

$$\omega_{kl} = \omega_{lk} . \tag{3.16}$$

We define

$$g_{jk} = \frac{c_j\beta_j}{A} \sum_{i=1}^{r} b_{ik} - \mu_j b_{jk} + \frac{1}{A}(A\lambda_j - c_j)\beta_j\delta_{jk} , \tag{3.17}$$

$$\Gamma_j = \sum_{i=1}^{r} \frac{c_i}{\mu_i}\omega_{ij} , \tag{3.18}$$

$$h_j = \lambda_j\Gamma_j + \frac{c_j\beta_j}{A}\sum_{i=1}^{r} a_i + \lambda_j \sum_{i=1}^{r} b_{ij} - \mu_j a_j , \tag{3.19}$$

and

$$\epsilon = \sum_{i=1}^{r} \frac{c_i\gamma_i}{\mu_i} . \tag{3.20}$$

Then, it follows from (3.14)-(3.20) that

$$\sum_{j=1}^{r}\sum_{k=1}^{r} \xi_j\xi_k \left[\frac{c_j\beta_j}{A}\Gamma_k - (A\lambda_j + c_j)\omega_{jk} + g_{jk}\right] + \sum_{j=1}^{r}\xi_j\left[\frac{\epsilon}{A}c_i\beta_j - (A\lambda_j + c_j)\gamma_j + h_j\right] = 0 . \tag{3.21}$$

In view of the symmetry in (3.16), we deduce from (3.21) that

$$[(A\lambda_j + c_j) + (A\lambda_k + c_k)]\omega_{jk} = \frac{1}{A}(c_j\beta_j\Gamma_k + c_k\beta_k\Gamma_j) + g_{jk} + g_{kj} . \tag{3.22}$$

Hence,

$$\omega_{jk} = \frac{(c_j\beta_j\Gamma_k + c_k\beta_k\Gamma_j)}{A[(A\lambda_j + c_j) + (A\lambda_k + c_k)]} + f_{jk} , \tag{3.23}$$

where

$$f_{jk} = \frac{(g_{jk} + g_{kj})}{[(A\lambda_j + c_j) + (A\lambda_k + c_k)]} = f_{kj} . \tag{3.24}$$

We define

$$C = \sum_{i=1}^{r} \frac{c_i \beta_i}{\mu_i^2}. \tag{3.25}$$

Then, from (2.23), (3.4), (3.10), (3.17) and (3.24), we obtain

$$g_{jk} = \frac{1}{A^2} c_j \beta_j c_k \beta_k \left(\frac{1}{\mu_j} - \frac{1}{\mu_k} - \frac{C}{A} \right) + \lambda_j \beta_j \delta_{jk}, \tag{3.26}$$

and

$$f_{jk} = \frac{2(\lambda_j \beta_j \delta_{jk} - \frac{C}{A^3} c_j \beta_j c_k \beta_k)}{[(A\lambda_j + c_j) + (A\lambda_k + c_k)]}. \tag{3.27}$$

From (3.18) and (3.23) we have

$$\Gamma_k = \frac{1}{A} \sum_{j=1}^{r} \frac{c_j (c_j \beta_j \Gamma_k + c_k \beta_k \Gamma_j)}{\mu_j [(A\lambda_j + c_j) + (A\lambda_k + c_k)]} + \sum_{j=1}^{r} \frac{c_j}{\mu_j} f_{jk}, \tag{3.28}$$

which is a linear system of equations for Γ_k, $k \in (1, 2, \ldots, r)$, with known coefficients, which has to be solved numerically, in general. Once the Γ_k have been calculated, ω_{jk} is given by (3.23). Also, h_j is given by (3.19). We define

$$G = \sum_{i=1}^{r} \frac{c_i^2 \beta_i}{\mu_i}. \tag{3.29}$$

Then, from (2.23), (3.4), (3.10) and (3.11), we obtain

$$h_j = \lambda_j \Gamma_j + c_j \beta_j \left[\lambda_j \left(\frac{2}{B} - \frac{C}{A^2} - \frac{1}{A\mu_j} \right) + \frac{2}{A^2 B} (A c_j - G) \right]. \tag{3.30}$$

Next, from (3.21), we deduce that

$$(A\lambda_j + c_j)\gamma_j = \frac{\epsilon}{A} c_j \beta_j + h_j. \tag{3.31}$$

Hence, from (3.20), we find that

$$\epsilon \left[1 - \frac{1}{A} \sum_{j=1}^{r} \frac{c_j^2 \beta_j}{\mu_j (A\lambda_j + c_j)} \right] = \sum_{j=1}^{r} \frac{c_j h_j}{\mu_j (A\lambda_j + c_j)}. \tag{3.32}$$

But, from (2.23),

$$1 - \frac{1}{A} \sum_{j=1}^{r} \frac{c_j^2 \beta_j}{\mu_j (A\lambda_j + c_j)} = \sum_{j=1}^{r} \frac{c_j \beta_j \lambda_j}{\mu_j (A\lambda_j + c_j)} > 0. \tag{3.33}$$

Hence ϵ is determined, and then γ_j is given by (3.31).

It remains to determine the constant κ in (3.15). But from the normalization condition (2.14), and (2.17) and (3.3),

$$0 = \sum_{i=1}^{r} \psi_i^{(1)}(0) = \sum_{i=1}^{r} \chi_i^{(1)}(0). \tag{3.34}$$

It follows from (3.6) and (3.15) that

$$\sum_{i=1}^{r} \left[\frac{c_i}{\mu_i} (\gamma_i - \beta_i \kappa) + a_i \right] = 0. \tag{3.35}$$

Hence, from (2.23), (3.11), (3.20) and (3.29), we obtain

$$\kappa = \frac{\epsilon}{A} + \frac{2}{A^2 B^2} (AD - BG). \tag{3.36}$$

This completes the determination of $\phi^{(1)}(\xi)$, and hence $\chi_i^{(1)}(\xi)$ and $\psi_i^{(1)}(\xi)$.

4. THE MEANS AND COVARIANCES

We now evaluate the asymptotic approximations to the means and covariances of the number of items of different types in the system. From (2.10) and (2.13) we have

$$E(n_j) = -a \sum_{i=1}^{r} \frac{\partial \psi_i}{\partial \xi_j}(0), \quad E(n_j n_k) = a^2 \sum_{i=1}^{r} \frac{\partial^2 \psi_i}{\partial \xi_j \partial \xi_k}(0) + \delta_{jk} E(n_j). \tag{4.1}$$

Hence, from (2.17), we obtain the asymptotic approximations

$$E(n_j) \sim - \sum_{i=1}^{r} \left[a \frac{\partial \psi_i^{(0)}}{\partial \xi_j}(0) + \frac{\partial \psi_i^{(1)}}{\partial \xi_j}(0) + \cdots \right], \tag{4.2}$$

and

$$E(n_j n_k) - \delta_{jk} E(n_j) \sim \sum_{i=1}^{r} \left[a^2 \frac{\partial^2 \psi_i^{(0)}}{\partial \xi_j \partial \xi_k}(0) + a \frac{\partial^2 \psi_i^{(1)}}{\partial \xi_j \partial \xi_k}(0) + \cdots \right]. \tag{4.3}$$

From (2.22), (2.23) and (2.27) it follows that

$$-\sum_{i=1}^{r} \frac{\partial \psi_i^{(0)}}{\partial \xi_j}(0) = \beta_j . \tag{4.4}$$

Also, from (3.3), (3.6) and (3.15), we have

$$\frac{\partial \psi_i^{(1)}}{\partial \xi_j}(0) = \frac{c_i}{\mu_i}(\omega_{ij} - \beta_i \gamma_j) + b_{ij} - \beta_j \chi_i^{(1)}(0). \tag{4.5}$$

Hence, from (2.23), (3.10), (3.18), (3.25) and (3.34), we obtain

$$\sum_{i=1}^{r} \frac{\partial \psi_i^{(1)}}{\partial \xi_j}(0) = \Gamma_j - A\gamma_j + c_j \beta_j \left(\frac{2}{B} - \frac{C}{A^2} - \frac{1}{A\mu_j} \right). \tag{4.6}$$

From (4.2), (4.4) and (4.6) we find the asymptotic approximations to the means,

$$E(n_j) \sim a\beta_j + \left[A\gamma_j - \Gamma_j + c_j \beta_j \left(\frac{1}{A\mu_j} + \frac{C}{A^2} - \frac{2}{B} \right) \right] + \cdots . \tag{4.7}$$

The quantities β_j are given by (2.24), where A is the positive root of (2.25), and B and C are given by (3.4) and (3.25). Also Γ_j, $j \in (1, 2, ..., r)$, satisfy the linear system of equations (3.28), subject to (3.27), and γ_j is determined from (3.29)-(3.32).

Next, from (2.19), (2.21), (2.23) and (2.27), it follows that

$$\sum_{i=1}^{r} \frac{\partial^2 \psi_i^{(0)}}{\partial \xi_j \partial \xi_k}(0) = \beta_j \beta_k . \tag{4.8}$$

Hence, from (4.2)-(4.4) and (4.8), we obtain

$$E(n_j n_k) - E(n_j) E(n_k)$$

$$\sim a \left\{ \beta_j \delta_{jk} + \sum_{i=1}^{r} \left[\frac{\partial^2 \psi_i^{(1)}}{\partial \xi_j \partial \xi_k}(0) + \beta_j \frac{\partial \psi_i^{(1)}}{\partial \xi_k}(0) + \beta_k \frac{\partial \psi_i^{(1)}}{\partial \xi_j}(0) \right] \right\} + \cdots . \tag{4.9}$$

But, from (3.3) and (3.34),

$$\sum_{i=1}^{r}\left[\frac{\partial^2\psi_i^{(1)}}{\partial\xi_j\,\partial\xi_k}(0)+\beta_j\frac{\partial\psi_i^{(1)}}{\partial\xi_k}(0)+\beta_k\frac{\partial\psi_i^{(1)}}{\partial\xi_j}(0)\right]=\sum_{i=1}^{r}\frac{\partial^2\chi_i^{(1)}}{\partial\xi_j\,\partial\xi_k}(0)\,. \qquad (4.10)$$

Also, from (2.23), (3.6), (3.15) and (3.16), we have

$$\sum_{i=1}^{r}\frac{\partial^2\chi_i^{(1)}}{\partial\xi_j\,\partial\xi_k}(0) = -\,A\,\omega_{jk}\,. \qquad (4.11)$$

From (4.9)-(4.11) we find the asymptotic approximations to the covariances,

$$E(n_j n_k) - E(n_j)E(n_k) \sim a(\beta_j\delta_{jk} - A\omega_{jk}) + \cdots\,, \qquad (4.12)$$

where ω_{jk} is given by (3.23) and (3.27). We note that the covariances, as well as the means, are of order a.

We now consider those systems for which

$$\lambda_j = \lambda,\quad c_j = c,\quad j \in (1,2,\ldots,r)\,, \qquad (4.13)$$

since it is then possible to explicitly solve (2.25) for A and (3.28) for Γ_k. We note that assumption (2.12) is now

$$\lambda\sum_{j=1}^{r}\frac{\alpha_j}{\mu_j} > 1\,. \qquad (4.14)$$

The solution of (2.25) is found to be

$$A = c\left(\sum_{j=1}^{r}\frac{\alpha_j}{\mu_j} - \frac{1}{\lambda}\right). \qquad (4.15)$$

Also, from (2.24),

$$\beta_j = \frac{A\lambda\alpha_j}{(A\lambda+c)}\,, \qquad (4.16)$$

and, from (3.27),

$$f_{jk} = \frac{\beta_j}{(A\lambda+c)} \left(\lambda\delta_{jk} - \frac{c^2C}{A^3}\beta_k\right), \tag{4.17}$$

where, from (3.25),

$$C = c\sum_{i=1}^{r} \frac{\beta_i}{\mu_i^2}. \tag{4.18}$$

We define

$$\Omega = \sum_{j=1}^{r} \frac{\Gamma_j}{\mu_j}. \tag{4.19}$$

Then, from (3.28) and (4.15)-(4.17),

$$A(2A\lambda+c)\Gamma_k = c^2\Omega\beta_k + 2cA\beta_k\left(\frac{\lambda}{\mu_k} - \frac{cC}{A^2}\right). \tag{4.20}$$

If we divide (4.20) by μ_k and sum, we find that

$$\Omega = \frac{C}{A^2\lambda}(A\lambda-c). \tag{4.21}$$

Hence, from (4.20),

$$\Gamma_k = \frac{c\beta_k}{(2A\lambda+c)}\left[\frac{2\lambda}{\mu_k} - \frac{cC}{A^3\lambda}(A\lambda+c)\right]. \tag{4.22}$$

It follows from (3.23), (4.17) and (4.22) that

$$\beta_j\delta_{jk} - A\omega_{jk}$$

$$= \frac{c\beta_j\delta_{jk}}{(A\lambda+c)} + \frac{c^2\beta_j\beta_k}{(A\lambda+c)(2A\lambda+c)}\left[\frac{C}{A^3\lambda}(2A^2\lambda^2+2A\lambda c+c^2) - \lambda\left(\frac{1}{\mu_j}+\frac{1}{\mu_k}\right)\right], \tag{4.23}$$

which gives an explicit expression for the asymptotic approximations (4.12) to the covariances.

It remains to calculate the first order correction term to the means in (4.7). The quantities h_j, ϵ and γ_j are obtained in a straightforward manner from (3.29)-(3.32) and

(4.22). It is then found from (4.7) that

$$E(n_j) \sim a\beta_j + \frac{c^3\beta_j}{A(A\lambda+c)(2A\lambda+c)}\left(\frac{1}{\mu_j}-\frac{C}{A}\right) + \cdots . \tag{4.24}$$

Since our approximation (4.12) to the covariances differs from that obtained by the diffusion approximation [1], we give an alternate derivation of our results in the appendix. Corresponding to the scalings in (2.11) we let $n_j = a\beta_j + \sqrt{a}\, v_j$ in (2.4), with $q_i(\mathbf{n})$ given by (2.8). We then develop an asymptotic expansion in inverse powers of $\sqrt{a}$ for $\phi_i(\mathbf{v}) = a^{r/2} p_i(\mathbf{n})$. It is found that, to lowest order, $\mu_i \phi_i(\mathbf{v}) \sim c_i\beta_i \Phi^{(0)}(\mathbf{v})$, where $A\,\Phi^{(0)}(\mathbf{v})$ is a multivariate Gaussian probability density function. From this we are able to obtain the lowest order approximations to the means and covariances, and we again obtain (4.12) and $E(n_j) = a\beta_j + O(1)$. Although we omit the rather lengthy details, we have in fact carried out the analysis to the next order in the asymptotic expansion, and have verified that it leads to the approximation (4.7) to $E(n_j)$. It is of interest that the asymptotic expansions of the densities are in inverse powers of $\sqrt{a}$, whereas the expansions of the moments are in inverse powers of a.

The difference in the equation for the covariances between our approximation and the diffusion approximation [1] is pointed out in the appendix.

5. NUMERICAL EXAMPLES

In order to check the accuracy of the approximations proposed, several systems were both simulated and approximated. In all cases studied the agreement between the asymptotic expansion approximation and simulation was excellent for both mean and standard deviation of the steady-state distribution. The mean of the diffusion approximation agrees with the lowest-order asymptotic approximation; the standard deviation of the current diffusion approximation agrees precisely with the lowest-order asymptotic approximation only when the service rates are identical, but correspondence is quite close numerically for all situations investigated to date.

Here are the systems, and their properties.

System 1
($r = 5$)

i:	1	2	3	4	5
K_i:	100	110	120	130	140
λ_i:	1.3	1.3	1.3	1.3	1.3
ν_i:	50	100	300	400	450
c_i:	1	1	1	1	1

Means					
Asymptotic:	81.40	89.54	97.68	105.82	113.96
Simulation:	81.14	89.39	97.47	105.60	113.74
(95% Conf.)	(81.04, 81.24)	(89.23, 89.54)	(97.37, 97.57)	(105.42, 105.79)	(113.53, 113.95)

Std. Deviations					
Asymptotic:	3.73	4.38	4.93	5.23	5.50
Diffusion:	3.75	4.38	4.91	5.20	5.47
Simulation:	3.75	4.38	4.89	5.24	5.56
(95% Conf.)	(3.67, 3.82)	(4.31, 4.45)	(4.80, 4.97)	(5.17, 5.31)	(5.48, 5.65)

System 2
$(r = 5)$

i:	1	2	3	4	5
K_i:	50	100	150	200	250
λ_i:	1.3	1.3	1.3	1.3	1.3
ν_i:	50	100	300	400	450
c_i:	1	1	1	1	1

Means					
Asymptotic:	39.18	78.37	117.55	156.73	195.91
Simulation:	39.10	78.17	117.35	156.26	195.57
(95% Conf.):	(39.03, 39.18)	(78.05, 78.28)	(117.12, 117.57)	(155.95, 156.57)	(195.30, 195.84)

Std. Deviations					
Asymptotic:	2.78	4.25	5.85	7.13	8.34
Diffusion:	2.85	4.33	5.90	7.24	8.56
Simulation:	2.78	4.21	5.90	7.19	8.33
(95% Conf.):	(2.75, 2.80)	(4.16, 4.27)	(5.81, 6.00)	(7.13, 7.25)	(8.26, 8.40)

The simulations were carried out on an IBM 3033 computer at the Naval Postgraduate School, using the LLRANDOMII random number operating package; Lewis and Uribe [5]. Time-dependent queue lengths were simulated: an event clock was advanced at either job arrivals or service completions, and queue lengths were suitably incremented or decremented. The current queue lengths were recorded at fixed discrete time steps; 500 independent replications were recorded. A batch mean process was utilized to obtain the confidence limits. For further details see Pilnick [6].

For the two systems displayed, and for many others explored, the leading term asymptotic agreement with simulation was well within the 95% confidence limits displayed for the latter. Diffusion approximation was also good, but slightly less accurate than the asymptotics described here. A full description of the diffusion approximation approach taken is given in Gaver, Jacobs and Pilnick [1]. The small size of the standard deviations as compared to their means is noticeable: certainly there is little resemblance of the current system behavior to that of a system with multi-type Poisson arrivals for service. In the latter situation the queue length standard deviation will be close to the corresponding mean queue lengths in heavy traffic. Of course in the case of multi-type Poisson arrivals there will be a steady-state solution only if a suitable traffic intensity for the system is less than unity; no such condition need be satisfied for the finite systems considered in this paper.

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APPENDIX

We here give an alternate derivation of the asymptotic approximations to the means and covariances of the number of items of different types in the system. Corresponding to the scalings in (2.11), we let

$$n_j = a\,\beta_j + \sqrt{a}\ v_j, \qquad p_i(\mathbf{n}) = \phi_i(\mathbf{v})/a^{r/2}, \tag{A1}$$

and it will be shown that β_j is given by (2.24). From (2.4) and (2.8), we have

$$\mu_i\phi_i(\mathbf{v}) + \sum_{j=1}^{r} \lambda_j\left(\alpha_j-\beta_j-\frac{v_j}{\sqrt{a}}\right)[\phi_i(\mathbf{v}) - \phi_i(\mathbf{v}-\mathbf{e}_j/\sqrt{a})]$$
$$= \frac{1}{a}\sum_{j=1}^{r} \lambda_j\phi_i(\mathbf{v}-\mathbf{e}_j/\sqrt{a}) + \frac{c_i(\beta_i+v_i/\sqrt{a})}{\sum_{k=1}^{r} c_k(\beta_k+v_k/\sqrt{a})}\sum_{j=1}^{r}\mu_j\phi_j(\mathbf{v}+\mathbf{e}_j/\sqrt{a}). \tag{A2}$$

If we expand in inverse powers of $\sqrt{a}$, we obtain

$$\mu_i\phi_i(\mathbf{v}) + \sum_{j=1}^{r} \lambda_j\left(\alpha_j-\beta_j-\frac{v_j}{\sqrt{a}}\right)\left[\frac{1}{\sqrt{a}}\frac{\partial\phi_i}{\partial v_j}(\mathbf{v}) + \cdots\right] - \frac{1}{a}\sum_{j=1}^{r}\lambda_j[\phi_i(\mathbf{v}) + \cdots]$$
$$= \frac{c_i}{B}\left[\beta_i + \frac{1}{\sqrt{a}}\left(v_i - \frac{\beta_i}{B}\sum_{k=1}^{r} c_k v_k\right) + \cdots\right]\sum_{j=1}^{r}\mu_j\left[\phi_j(\mathbf{v}) + \frac{1}{\sqrt{a}}\frac{\partial\phi_j}{\partial v_j}(\mathbf{v}) + \cdots\right], \tag{A3}$$

where B is given by (3.4).

If we sum (A2) with respect to i, we find that

$$\sum_{i=1}^{r}\mu_i\phi_i(\mathbf{v}) + \sum_{j=1}^{r}\lambda_j\left(\alpha_j-\beta_j-\frac{v_j}{\sqrt{a}}\right)\sum_{i=1}^{r}[\phi_i(\mathbf{v})-\phi_i(\mathbf{v}-\mathbf{e}_j/\sqrt{a})]$$
$$= \frac{1}{a}\sum_{j=1}^{r}\lambda_j\sum_{i=1}^{r}\phi_i(\mathbf{v}-\mathbf{e}_j/\sqrt{a}) + \sum_{j=1}^{r}\mu_j\phi_j(\mathbf{v}+\mathbf{e}_j/\sqrt{a}). \tag{A4}$$

It follows that

$$\sum_{j=1}^{r} \lambda_j \left(\alpha_j - \beta_j - \frac{v_j}{\sqrt{a}}\right) \sum_{i=1}^{r} \left[\frac{\partial \phi_i}{\partial v_j}(\mathbf{v}) - \frac{1}{2\sqrt{a}} \frac{\partial^2 \phi_i}{\partial v_j^2}(\mathbf{v}) + \cdots \right]$$

$$= \frac{1}{\sqrt{a}} \sum_{j=1}^{r} \lambda_j \sum_{i=1}^{r} [\phi_i(\mathbf{v}) + \cdots] + \sum_{j=1}^{r} \mu_j \left[\frac{\partial \phi_j}{\partial v_j}(\mathbf{v}) + \frac{1}{2\sqrt{a}} \frac{\partial^2 \phi_j}{\partial v_j^2}(\mathbf{v}) + \cdots \right]. \quad (A5)$$

We assume an asymptotic expansion of the form

$$\phi_i(\mathbf{v}) \sim \phi_i^{(0)}(\mathbf{v}) + \frac{1}{\sqrt{a}} \phi_i^{(1)}(\mathbf{v}) + \frac{1}{a} \phi_i^{(2)}(\mathbf{v}) + \cdots . \quad (A6)$$

Then, from (A3), we obtain

$$\mu_i \phi_i^{(0)}(\mathbf{v}) = \frac{c_i \beta_i}{B} \cdot \sum_{j=1}^{r} \mu_j \phi_j^{(0)}(\mathbf{v}) = c_i \beta_i \Phi^{(0)}(\mathbf{v}), \quad (A7)$$

where $\Phi^{(0)}(\mathbf{v})$ is to be determined. But, from (A5),

$$\sum_{j=1}^{r} \lambda_j (\alpha_j - \beta_j) \sum_{i=1}^{r} \frac{\partial \phi_i^{(0)}}{\partial v_j} = \sum_{j=1}^{r} \mu_j \frac{\partial \phi_j^{(0)}}{\partial v_j}. \quad (A8)$$

It follows from (2.23) and (A7) that

$$\sum_{j=1}^{r} [A \lambda_j (\alpha_j - \beta_j) - c_j \beta_j] \frac{\partial \Phi^{(0)}}{\partial v_j} = 0. \quad (A9)$$

This equation is satisfied if (2.24) holds, which we assume to be the case.

Next, from (A3) and (A6), we have

$$\mu_i \phi_i^{(1)}(\mathbf{v}) - \frac{c_i \beta_i}{B} \sum_{j=1}^{r} \mu_j \phi_j^{(1)}(\mathbf{v}) = \frac{c_i}{B} \left(v_i - \frac{\beta_i}{B} \sum_{k=1}^{r} c_k v_k \right) \sum_{j=1}^{r} \mu_j \phi_j^{(0)}(\mathbf{v})$$

$$+ \frac{c_i \beta_i}{B} \sum_{j=1}^{r} \mu_j \frac{\partial \phi_j^{(0)}}{\partial v_j} - \sum_{j=1}^{r} \lambda_j (\alpha_j - \beta_j) \frac{\partial \phi_i^{(0)}}{\partial v_j}. \quad (A10)$$

Hence, from (2.24) and (A7), we obtain

$$\mu_i \phi_i^{(1)}(\mathbf{v}) = c_i \beta_i \Phi^{(1)}(\mathbf{v}) + c_i \Big(v_i - \frac{\beta_i}{B} \sum_{k=1}^{r} c_k v_k \Big) \Phi^{(0)}(\mathbf{v})$$

$$+ c_i \beta_i \Big(\frac{1}{B} - \frac{1}{A\mu_i} \Big) \sum_{j=1}^{r} c_j \beta_j \frac{\partial \Phi^{(0)}}{\partial v_j}, \qquad \text{(A11)}$$

where $\Phi^{(1)}$ is to be determined. But, from (A5) and (A6),

$$\sum_{j=1}^{r} \lambda_j (\alpha_j - \beta_j) \sum_{i=1}^{r} \frac{\partial \phi_i^{(1)}}{\partial v_j} - \sum_{j=1}^{r} \mu_j \frac{\partial \phi_j^{(1)}}{\partial v_j}$$

$$= \sum_{j=1}^{r} \lambda_j v_j \sum_{i=1}^{r} \frac{\partial \phi_i^{(0)}}{\partial v_j} + \frac{1}{2} \sum_{j=1}^{r} \lambda_j (\alpha_j - \beta_j) \sum_{i=1}^{r} \frac{\partial^2 \phi_i^{(0)}}{\partial v_j^2}$$

$$+ \sum_{j=1}^{r} \lambda_j \sum_{i=1}^{r} \phi_i^{(0)}(\mathbf{v}) + \frac{1}{2} \sum_{j=1}^{r} \mu_j \frac{\partial^2 \phi_j^{(0)}}{\partial v_j^2}. \qquad \text{(A12)}$$

From (A7), (A11) and (A12), with the help of (2.23), (2.24) and (3.25), it is found, after considerable reduction, that

$$\frac{1}{A} \sum_{j=1}^{r} c_j \beta_j \Big(\frac{1}{\mu_j} - \frac{C}{A} \Big) \sum_{k=1}^{r} c_k \beta_k \frac{\partial^2 \Phi^{(0)}}{\partial v_j \, \partial v_k} - \sum_{j=1}^{r} c_j \beta_j \frac{\partial^2 \Phi^{(0)}}{\partial v_j^2}$$

$$+ \frac{1}{A} \sum_{k=1}^{r} \frac{c_k}{\mu_k} \sum_{j=1}^{r} c_j \beta_j \frac{\partial}{\partial v_j} (v_k \Phi^{(0)}) - \sum_{j=1}^{r} (A\lambda_j + c_j) \frac{\partial}{\partial v_j} (v_j \Phi^{(0)}) = 0, \qquad \text{(A13)}$$

an equation involving $\Phi^{(0)}$ only. From (A1) the normalization condition implies asymptotically, from the Euler-Maclaurin summation formula [7], that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{r} \phi_i(\mathbf{v}) dv_1 \cdots dv_r = \int_{-\infty}^{\infty} \sum_{i=1}^{r} \phi_i(\mathbf{v}) d\mathbf{v} \sim 1. \qquad \text{(A14)}$$

Hence, from (A6),

$$\int_{-\infty}^{\infty} \sum_{i=1}^{r} \phi_i^{(0)}(\mathbf{v}) d\mathbf{v} = 1, \qquad \int_{-\infty}^{\infty} \sum_{i=1}^{r} \phi_i^{(1)}(\mathbf{v}) d\mathbf{v} = 0. \qquad \text{(A15)}$$

It follows from (2.23) and (A7) that

$$\int_{-\infty}^{\infty} \Phi^{(0)}(\mathbf{v})d\mathbf{v} = \frac{1}{A}. \tag{A16}$$

We let

$$M_i = A \int_{-\infty}^{\infty} v_i \Phi^{(0)}(\mathbf{v})d\mathbf{v}. \tag{A17}$$

Then from (A13), after multiplication by v_i and some integrations by parts, we obtain

$$(A\lambda_i + c_i)M_i = c_i\beta_i \frac{\Delta}{A}, \qquad \Delta = \sum_{k=1}^{r} \frac{c_k}{\mu_k} M_k. \tag{A18}$$

Hence,

$$\left[1 - \frac{1}{A} \sum_{i=1}^{r} \frac{c_i^2\beta_i}{\mu_i(A\lambda_i + c_i)}\right] \Delta = 0. \tag{A19}$$

It follows from (3.33) that $\Delta = 0$, and then from (A18) that $M_i = 0$. Consequently, from (A1), (A6), (A7), (A15) and (A17), $E(n_j) = a\beta_j + O(1)$. We have in fact carried out the analysis to the next order in the expansion (A6), and have verified that it leads to the asymptotic approximation (4.7) to $E(n_j)$. However, we omit the rather lengthy details.

Next we consider the covariances, and let

$$M_{il} = A \int_{-\infty}^{\infty} v_i v_l \Phi^{(0)}(\mathbf{v})d\mathbf{v}. \tag{A20}$$

If we integrate by parts and use (A16), we obtain

$$A \int_{-\infty}^{\infty} v_i v_l \frac{\partial^2 \Phi^{(0)}}{\partial v_j \partial v_k} d\mathbf{v} = \delta_{ij}\delta_{lk} + \delta_{lj}\delta_{ik}, \tag{A21}$$

and

$$A \int_{-\infty}^{\infty} v_i v_l \frac{\partial}{\partial v_j}(v_k \Phi^{(0)})d\mathbf{v} = -(\delta_{ij}M_{lk} + \delta_{lj}M_{ik}). \tag{A22}$$

Hence, from (A13), we find that

$$[(A\lambda_i+c_i)+(A\lambda_l+c_l)]M_{il}-\frac{1}{A}\sum_{k=1}^{r}\frac{c_k}{\mu_k}(c_i\beta_iM_{lk}+c_l\beta_lM_{ik})$$

$$=2c_i\beta_i\delta_{il}+\frac{1}{A}c_i\beta_ic_l\beta_l\left(\frac{2C}{A}-\frac{1}{\mu_i}-\frac{1}{\mu_l}\right). \tag{A23}$$

If we let

$$M_{il}=\beta_l\delta_{il}-A\,\omega_{il}, \tag{A24}$$

then we obtain equation (3.23) for ω_{jk}, where Γ_j and f_{jk} are given by (3.18) and (3.27). The asymptotic approximations (4.12) to the covariances follow from (A1), (A6), (A7), (A20) and (A24).

We remark that in the diffusion approximation [1] to the covariances, the off-diagonal terms on the right-hand side of equation (A23) are absent, although the diagonal terms agree exactly.

We will now show that $A\,\Phi^{(0)}(\mathbf{v})$ is a zero-mean multivariate Gaussian probability density function. We introduce the characteristic function

$$\chi^{(0)}(\mathbf{y})=A\int_{-\infty}^{\infty}e^{i\mathbf{v}\cdot\mathbf{y}}\,\Phi^{(0)}(\mathbf{v})\,d\mathbf{v}, \quad \mathbf{v}\cdot\mathbf{y}=\sum_{l=1}^{r}v_ly_l, \tag{A25}$$

where $i=\sqrt{-1}$. If we integrate by parts, we obtain

$$A\int_{-\infty}^{\infty}e^{i\mathbf{v}\cdot\mathbf{y}}\,\frac{\partial^2\Phi^{(0)}}{\partial v_j\,\partial v_k}\,d\mathbf{v}=-\,y_jy_k\,\chi^{(0)}(\mathbf{y}), \tag{A26}$$

and

$$A\int_{-\infty}^{\infty}e^{i\mathbf{v}\cdot\mathbf{y}}\,\frac{\partial}{\partial v_j}\,(v_k\,\Phi^{(0)})\,d\mathbf{v}=-\,y_j\,\frac{\partial\chi^{(0)}}{\partial y_k}. \tag{A27}$$

It follows from (A13) that

$$\left[\sum_{j=1}^{r} c_j \beta_j y_j^2 - \frac{1}{A} \sum_{j=1}^{r} c_j \beta_j \left(\frac{1}{\mu_j} - \frac{C}{A}\right) \sum_{k=1}^{r} c_k \beta_k y_j y_k\right] \chi^{(0)}(\mathbf{y})$$

$$- \frac{1}{A} \sum_{k=1}^{r} \frac{c_k}{\mu_k} \sum_{j=1}^{r} c_j \beta_j y_j \frac{\partial \chi^{(0)}}{\partial y_k} + \sum_{j=1}^{r} (A\lambda_j + c_j) y_j \frac{\partial \chi^{(0)}}{\partial y_j} = 0. \tag{A28}$$

But, from (A16), (A17), (A20) and (A25), since $M_j = 0$, we have

$$\chi^{(0)}(0) = 1, \quad \frac{\partial \chi^{(0)}}{\partial y_j}(0) = 0, \quad - \frac{\partial^2 \chi^{(0)}}{\partial y_j \, \partial y_k}(0) = M_{jk}. \tag{A29}$$

It is straightforward to verify, with the help of (A23), that (A28) and (A29) are satisfied by

$$\chi^{(0)}(\mathbf{y}) = \exp\left(- \frac{1}{2} \sum_{l=1}^{r} \sum_{m=1}^{r} M_{lm} y_l y_m\right), \tag{A30}$$

which establishes the desired result [8].

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Defense Communications Agency 1
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78150 Le Chesnay Cedex
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Department of Operations Research
University of Canterbury
Christchurch, NEW ZEALAND

68. Koh Peng Kong 1
OA Branch, DSO
Ministry of Defense
Blk 29 Middlesex Road
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Nuffield College
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Department of Mathematical Statistics
University of Birmingham
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Statistics Laboratory
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Cambridge
ENGLAND

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Statistics Laboratory
16 Mill Lane
Cambridge
ENGLAND

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