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THE EFFECT OF WALL COMPLIANCE ON THE GÖRTLER VORTEX INSTABILITY

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THE EFFECT OF WALL COMPLIANCE ON
THE GÖRTLER VORTEX INSTABILITY [1]

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Abstract

The stability of the flow of a viscous incompressible fluid over a curved compliant wall to longitudinal Görtler vortices is investigated. The compliant wall is modeled by a particularly simple equation relating the induced wall displacement to the pressure in the overlying fluid. Attention is restricted to the large Görtler number regime; this regime being appropriate to the most unstable Görtler mode. The effect of wall compliance on this most unstable mode is investigated.

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I. INTRODUCTION

Our concern is with the effect of wall compliance upon the stability characteristics of the flow of an incompressible fluid over a wall of variable curvature. Recent theoretical work on the effects of wall compliance on the stability of fluid flows has been restricted to flows over flat compliant walls (here by flat we are referring to the unperturbed state of the wall). This work has incorporated various models of the resulting wall displacements eg. viscoelastic walls[1] and spring backed walls[2,3] (both isotropic and anisotropic). Consequently, this work has considered various aspects of the stability of Tollmien–Schlichting waves and the effect that the underlying wall motion has on their stability characteristics. A review of these results can be found in Carpenter[4].

The effect of wall compliance on flows over curved walls has recently been considered experimentally by Yurchenko, Babenko & Koslov[5]. They restricted their attention to centrifugal instabilities, namely Görtler vortices. Although no precise measurements of the growth rates of the initial disturbances have been presented by these authors their results for a compliant wall, in which they plot regions of the Görtler number/wavenumber space corresponding to maximum amplification rates of the initial disturbances, show a shift to the left as compared to the results obtainable for the a rigid wall. These results then suggest that the effect of the compliant wall has a stabilizing effect on the growth rate of the longitudinal vortex structure.

In the case of a rigid wall it has been shown by Denier, Hall & Seddougui[6] that there is a most unstable Görtler mode (ie a mode with maximum growth rate) in the parameter regime $G \gg 1$ (here G is the Görtler number) and has wavenumber given by

$k = O(G^{1/5})$ with growth rate $O(G^{3/5})$. Such a result was found by considering both the inviscid Görtler modes, found in the parameter regime $G \gg 1$ with wavenumber $k = O(1)$, and the right hand branch Görtler modes, found in the parameter regime $G \gg 1$ and wavenumber $k = O(G^{1/4})$. By their very nature the inviscid modes do not feel the effect of the wall and so we expect that these modes are virtually unaltered by the presence of a compliant wall. In the case of the right hand branch modes it is known from the work of Hall[7] that these modes are confined to a thin region internal to the flow and away from the wall. Hence, we expect that these modes are also unaffected by the presence of the compliant wall. However, in the case of the most unstable Görtler mode it is known[6] that these modes are confined to a thin wall layer of thickness $O(G^{-1/5})$. We anticipate that these modes will be strongly affected by the presence of the compliant wall. For these reasons we will confine our attention to the most unstable Görtler modes.

The outline of the paper is as follows. In §2 we formulate the problem under consideration and derive the governing equations for the fluid motion. The equation of motion for the wall displacement is assumed to have a particularly simple form (thus facilitating the analysis to be presented in §3); further refinements to the governing equation for the wall displacement can be made to take into account the effect of fluid substrates, elasticity of the compliant wall etc[4]. Such issues will not be pursued in this paper. Subsequently in §3 we will restrict our attention to the high Görtler number regime and consider the wavenumber regime $k = O(G^{1/5})$. The eigenvalue problem for the most unstable Görtler mode will be derived and in §4 we present some results obtained from a numerical solution

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of this problem. The implications of these results to the stability of flow over a curved compliant wall will be discussed.

II. FORMULATION OF THE PROBLEM

We consider the steady flow of a viscous incompressible fluid over a compliant wall of variable curvature. Assuming that L is a typical lengthscale over which the curvature changes, U_∞ is a typical flow velocity a great distance from the wall, and ν is the kinematic viscosity, we define a Reynolds number, Re, by

$$Re = \frac{U_\infty L}{\nu}, \tag{1}$$

and suppose that with respect to Cartesian axes x^*, y^*, z^* the unperturbed position of the wall is defined by

$$y^* = LRe^{-1/2}g(x^*/L). \tag{2}$$

Upon the imposition of some (suitably) small pressure perturbation the position of the wall is now defined by

$$y^* = LRe^{-1/2}\left\{g(x^*/L) + \delta\eta^*(x^*/L, Re^{1/2}z^*/L)\right\}, \tag{3}$$

where $\delta\eta^*$ is the wall displacement whose motion is assumed to be governed by[8]

$$LRe^{-1/2}\delta B_0\nabla^4\eta^* = -\delta p_1^*. \tag{4}$$

Here B_0 is the flexural rigidity of the compliant surface (we have assumed that the compliant surface is isotropic) and ∇^4 is the usual bi-harmonic operator

$$\nabla^4 = \partial_{x^*}^4 + 2\partial_{x^*}^2.\partial_{z^*}^2 + \partial_{z^*}^4.$$

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The term δp_1^* is the small pressure perturbation.

Defining nondimensional variables

$$(x, y, z) = (x^*, Re^{1/2}y^*, Re^{1/2}z^*)/L,$$

$$(u^+, v^+, w^+) = (u^*, Re^{1/2}v^*, Re^{1/2}w^*)/U_\infty,$$

$$p^+ = p^*/(\rho U_\infty^2), \quad p^* = p_0^* + \delta p_1^*,$$

we restrict attention to the limit $Re \to \infty$ and write

$$(u^+, v^+, w^+) = (\bar{u}, \bar{v}, 0) + \delta(\tilde{u}, \tilde{v}, \tilde{w}) + O(\delta^2),$$

$$p^+ = \bar{p}(x) + \delta Re^{-1/2}\tilde{p}(x, y, z) + O(\delta^2), \qquad p_1^+ = Re^{-1/2}\tilde{p}.$$

Here the basic velocity $(\bar{u}, \bar{v})$ depends only on x and y whereas the velocity perturbations $(\tilde{u}, \tilde{v}, \tilde{w})$ depend on all three dimensionless coordinates. Substitution of these expressions into the Navier–Stokes and continuity equations, taking the limit $\delta \to 0$ with Re held fixed, equating terms of order δ^0, δ^1 and taking the further limit $Re \to \infty$ in the resulting equations for $(\bar{u}, \bar{v}, 0, \bar{p})$ and $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p})$ gives, to leading order in powers of Re^{-1},

$$\bar{u}_x + \bar{v}_y = 0, \qquad \bar{u}\bar{u}_x + \bar{v}\bar{u}_y = -\bar{p}_x + \bar{u}_{yy}, \tag{5}$$

and

$$\tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0,$$

$$\bar{u}\tilde{u}_x + \bar{v}\tilde{u}_y + \tilde{u}\bar{u}_x + \tilde{v}\bar{u}_y = \tilde{u}_{yy} + \tilde{u}_{zz},$$

$$\bar{u}\tilde{v}_x + \tilde{v}\bar{v}_y + \tilde{u}\bar{v}_x + \bar{v}\tilde{v}_y = -\tilde{p}_y + \tilde{v}_{yy} + \tilde{v}_{zz},$$

$$\bar{u}\tilde{w}_x + \bar{v}\tilde{w}_y = -\tilde{p}_z + \tilde{w}_{yy} + \tilde{w}_{zz}, \tag{6}$$

together with the governing equation for the wall displacement

$$B\frac{\partial^4 \tilde{\eta}}{\partial z^4} = -\tilde{p}, \tag{7}$$

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where $B = Re^2 B_0/(\rho U_\infty^2 L^3)$ is a nondimensional constant which we initially take to be $O(1)$.

By writing down the Taylor series expansion for the no-slip conditions about $y = g(x)$ we obtain the boundary conditions

$$\bar{u} = \bar{v} = 0 \qquad y = g(x),$$

$$\tilde{u} = -\bar{u}_y \, \tilde{\eta}(x, z), \quad \tilde{v} = -\bar{v}_y \, \tilde{\eta}(x, z), \quad \tilde{w} = 0 \qquad y = g(x).$$

(8)

In the limit $y \to \infty$ we have the requirement

$$\bar{u} \to u_e(x), \quad (\tilde{u}, \tilde{v}, \tilde{w}) \to 0 \qquad y \to \infty.$$

(9)

where $u_e(x)$ is the dimensionless free stream velocity; in this paper we will confine our attention to the Blasius profile so that $u_e(x) = 1$. In order to highlight the effect of wall curvature we make use of the Prandtl transformation

$$y \to y + g, \quad \bar{v} \to \bar{v} + g' \bar{u}, \quad \tilde{v} \to \tilde{v} + g' \tilde{u},$$

whilst all other variables remain unchanged. The governing equations then become

$$\bar{u}_x + \bar{v}_y = 0, \qquad \bar{u} \bar{u}_x + \bar{v} \bar{u}_y = -\bar{p}_x + \bar{u}_{yy},$$

$$\bar{u} = \bar{v} = 0 \quad y = 0, \qquad \bar{u} \to 1 \quad y \to \infty,$$

(10)

and

$$\tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0,$$

$$\bar{u} \tilde{u}_x + \bar{v} \tilde{u}_y + \tilde{u} \bar{u}_x + \tilde{v} \bar{u}_y = \tilde{u}_{yy} + \tilde{u}_{zz},$$

$$\bar{u} \tilde{v}_x + \bar{v} \tilde{v}_y + \tilde{u} \bar{v}_x + \bar{v} \tilde{v}_y + G \chi(x) \bar{u} \bar{u} = -\bar{p}_y + \tilde{v}_{yy} + \tilde{v}_{zz},$$

$$\bar{u} \tilde{w}_x + \bar{v} \tilde{w}_y = -\tilde{p}_z + \tilde{w}_{yy} + \tilde{w}_{zz},$$

(11)

$$B \tilde{\eta}_{zzzz} = -\tilde{p},$$

$$\tilde{u} = -\bar{u}_y \tilde{\eta}, \quad \tilde{v} = \tilde{w} = 0 \qquad y = 0,$$

$$(\tilde{u}, \tilde{v}, \tilde{w}) \to 0 \qquad y \to \infty.$$

In (11) we have replaced $2g_{zz}$ by $G\chi(x)$ where G and χ will be refered to as the Görtler number and wall curvature respectively.

Finally taking the Fourier transform of the system (11) in z, eliminating w, p and the wall displacement η from the resulting equations gives

$$\bar{u}u_x + \bar{v}u_y + u\bar{u}_x + v\bar{u}_y = u_{yy} - k^2 u,$$

$$\left\{ \bar{u}_{xyy} + k^4 + k^2\bar{v}_y \right\} v + \bar{v}_x u_{yy} + \left\{ \bar{u}_{xxy} + k^2\bar{v}_x + k^2\chi G\bar{u} \right\} u$$

$$+ \left\{ \bar{u}_{yy} - \bar{u}\frac{\partial^2}{\partial y^2} + k^2\bar{u} \right\} v_x + 2\left\{ \bar{u}_{xy} + \bar{u}_x\frac{\partial}{\partial y} \right\} u_x \qquad (12a,b)$$

$$+ v_{yyyy} - \bar{v}v_{yyy} - \left\{ \bar{v}_y + 2k^2 \right\} v_{yy} + \left\{ \bar{u}_{xy} + k^2\bar{v} \right\} v_y = 0,$$

together with the boundary conditions

$$u = \frac{-\bar{u}_y}{Bk^6}\left\{ -(u_{xyy} + v_{yyy}) + k^2 u_x + \bar{u}(u_{xx} + v_{xy}) + \bar{v}(u_{xy} + v_{yy}) \right\} \qquad y = 0,$$

$$v = v' = 0 \qquad y = 0, \qquad\qquad (13)$$

$$u, v, v' \to 0 \qquad y \to \infty.$$

Here k is the transform variable, (u,v) are the transforms of $(\tilde{u}, \tilde{v})$. For $O(1)$ values of the wavenumber k and finite values of the Görtler number G the solution of the system (12,13) can only be found by numerical integration. However, in the asymptotic regime $G \gg 1$ progress can be made analytically and it this parameter regime which we consider in the following section. From the boundary condition (13) we see that the case of the rigid wall is recovered in the limit $B \to \infty$.

III. THE MOST UNSTABLE GÖRTLER MODE

In the case of a rigid wall it has been shown[6] that the system (12,13) has a mode with maximum growth rate in the parameter regime $G \gg 1$ and the wavenumber of this most unstable mode is $k = O(G^{1/5})$. Such a result was found by considering the structure of

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both the problem of the inviscid Görtler modes, which are found in the regime $k = O(1)$, $G \gg 1$, and that of the right hand branch modes, found in the parameter regime $G \gg 1$ and $k = O(G^{1/4})$.

By inspection of the structure of both the inviscid modes and the right hand branch modes[9] it is clearly seen that in the case now under consideration these modes are essentially unchanged from those relevant to the rigid wall problem. In fact, an analysis of the system (12,13) shows that in these two parameter regimes the effect of wall compliance is encountered as a lower order effect and does not contribute, at leading order, to the structure elucidated in Ref. 6. This result is easily seen if one considers the position of both the inviscid and right hand branch modes; the inviscid modes (by definition) do not feel the effect of the wall whereas the right hand branch modes are situated at a position internal to the flow regime and so will not feel the effect of the compliant wall.

For these reasons we will restrict our attention to effect of wall compliance on the most unstable Görtler mode. To consider this mode, we note[6] that this mode is confined to a wall layer of thickness $O(k^{-1})$ where the wavenumber scales as $k = O(G^{1/5})$. We then write

$$\varphi = ky, \qquad k = \bar{\lambda} G^{1/5}. \tag{14}$$

We expand the mean flow in this layer as

$$\bar{u} \sim \mu(x) k^{-1} \varphi + \cdots,$$

and expand the velocity field (u, v) and the surface displacement η as

$$u = \left\{ U_0 + G^{-1/5}U_1 + \cdots \right\} \exp \left(G^{3/5} \int^z \bar{\beta}(x)dx \right) ,$$

$$v = G^{2/5} \left\{ V_0 + G^{-1/5}V_1 + \cdots \right\} \exp \left(G^{3/5} \int^z \bar{\beta}(x)dx \right) , \qquad (15)$$

$$\eta = \left\{ \eta_0 + G^{-1/5}\eta_1 + \cdots \right\} \exp \left(G^{3/5} \int^z \bar{\beta}(x)dx \right) ,$$

where we have anticipated growth rates of $O(G^{3/5})$ (see Ref. 6). Substitution of (14) and

(15) into the governing equations (12a,b) we obtain, to leading order,

$$\left\{ \frac{d^2}{d\varphi^2} - \frac{\mu\bar{\beta}\varphi}{\bar{\lambda}^3} - 1 \right\} U_0 = \frac{\mu}{\bar{\lambda}^2}V_0,$$

$$\left\{ \frac{d^2}{d\varphi^2} - \frac{\mu\bar{\beta}\varphi}{\bar{\lambda}^3} - 1 \right\} \left\{ \frac{d^2}{d\varphi^2} - 1 \right\} V_0 = -\frac{\chi\mu\varphi U_0}{\bar{\lambda}^3}. \qquad (16)$$

An evaluation of the boundary conditions (13) shows that the effect of wall compliance

first becomes important when the compliance parameter $B = O(G^{-1/5})$. Hence, we define

$$B_0 = BG^{1/5}.$$

The boundary conditions appropriate to (16) are then found to be

$$U_0 = \frac{\mu}{B_0\bar{\lambda}^3} \frac{d^3V_0}{d\varphi^3}, \quad V_0 = 0, \quad V_0' = 0 \qquad \varphi = 0,$$

$$U_0 = 0, \quad V_0 = 0, \quad V_0' = 0 \qquad \varphi = \infty. \qquad (17)$$

As noted above, the case of the rigid wall is recovered in the limit $B_0 \to \infty$.

For computational purposes it is convenient to eliminate the parameters μ, χ from

(16) by writing

$$\bar{\lambda} = (\chi\mu^2)^{1/5}\lambda, \quad \bar{\beta} = (\chi^3\mu)^{1/5}\beta, \quad B_1 = B_0(\chi\mu^2)^{1/5}, \quad U_0 = \mu\tilde{U}_0, \quad V_0 = (\chi\mu^2)^{2/5}\tilde{V}_0,$$

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in which case (16,17) may be rewritten as

$$\left\{ \frac{d^2}{d\varphi^2} - \frac{\beta\varphi}{\lambda^3} - 1 \right\} \tilde{U}_0 = \frac{\tilde{V}_0}{\lambda^2},$$

$$\left\{ \frac{d^2}{d\varphi^2} - \frac{\beta\varphi}{\lambda^3} - 1 \right\} \left\{ \frac{d^2}{d\varphi^2} - 1 \right\} \tilde{V}_0 = -\frac{\varphi\tilde{U}_0}{\lambda^3},$$

$$\tilde{U}_0 = \frac{1}{\lambda^3 B_1} \frac{d^3\tilde{V}_0}{d\varphi^3}, \quad \tilde{V}_0 = \tilde{V}_0' = 0 \qquad \varphi = 0, \tag{18}$$

$$\tilde{U}_0 = \tilde{V}_0 = \tilde{V}_0' = 0 \qquad \varphi = \infty.$$

For a given value of the parameter B_1 the system (18) constitutes an eigenvalue problem for the growth rate $\beta = \beta(\lambda)$.

Before proceeding with a discussion of the results of our numerical integration of the system (18) we note that the limiting inviscid form of the governing equations is recovered by taking the limit $\lambda \to 0$ while the limiting form of the right hand branch governing equations are obtained by taking the limit $\lambda \to \infty$. In fact, from (18) it is easily seen that

$$\beta \sim \lambda^{1/2}, \qquad \lambda \to 0, \qquad \beta \sim \lambda^{-2}, \qquad \lambda \to \infty,$$

thus demonstrating that for some intermediate value of λ there exists a most amplified mode.

IV. RESULTS AND DISCUSSION

The system (18) was solved numerically using a finite difference scheme in the independent variable φ. For a given value of the parameter B_1 and a fixed value of the wavenumber λ an initial guess for the growth rate β was supplied and the program was run until convergence was achieved. The eigenvalue problem (18) was then solved by continuously iterating in the wavenumber λ to generate eigenvalues $\beta = \beta(\lambda; B_1)$. This procedure was repeated for various values of the parameter B_1.

For a given value of B_1 the system (18) has an infinite number of eigenvalues $\beta_n = \beta_n(\lambda; B_1)$. In Figure 1 we present a plot of the first eigenvalue β as a function of λ for various values of the parameter B_1; these eigenvalues correspond to the most unstable Görtler mode. In Table I the maximum eigenvalue β_{max} is given together with the corresponding critical wavenumber λ_{cr}. We see that the introduction of wall compliance has a stabilizing effect in that the maximum value of the growth rate is reduced. From Figure 1 we see that as the wall compliance is increased (ie B_1 is decreased) the maximum growth rate decreases until a threshold value is reached. Below this threshold value of B_1 the stabilization due to the wall compliance remains constant; from Table I we see that the stabilization is 5% compared to the case of the rigid wall.

From Table I we see that as the compliance parameter is decreased from the rigid wall value there is a slight increase in the critical wavenumber until a threshold value of the parameter B_1 is reached after which point the critical wavenumber decreases in magnitude.

In figures 2 and 3 we present the eigenfunctions for the maximum eigenvalues presented in Table I. The effect of wall compliance is to decrease the amplitude of the eigenfunctions. Otherwise there is little qualitative change from the case of the rigid wall.

In conclusion, the presence of a compliant wall has the effect of stabilizing the longitudinal vortex motion which is in qualitative agreement with the available experimental results[5]. However, this stabilization is relatively small and hence we may conclude that the stability of the flow over a curved compliant wall is essentially unchanged from that of the case of a rigid wall. It remains an open question as to whether either finite amplitude

effects or the presence of wall modes (ie modes induced by the motion of the compliant wall) will significantly influence the structure discussed in this paper.

V. ACKNOWLɔDGMENTS

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[9]see reference 6.

Table I. Maximum growth rate and critical wavenumbers.

B_1	$\beta_{\max}$	λ_{cr}
∞[a]	0.3135	0.475
100	0.3113	0.49
10	0.3039	0.513
1	0.2985	0.487
0.01	0.2977	0.476

[a] The case of the rigid wall.
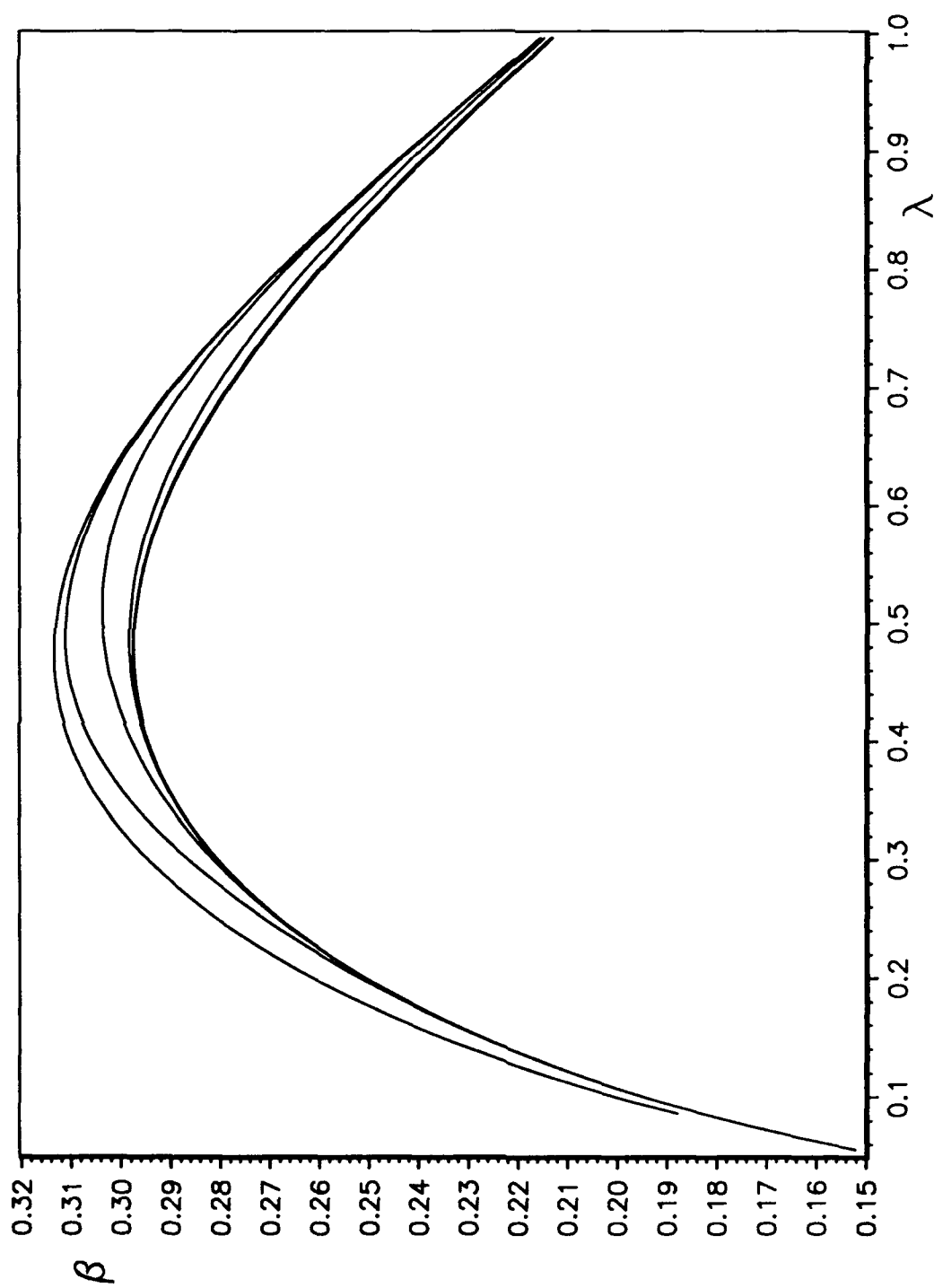
Figure 1. Plot of the first eigenvalue β of (18) versus λ for values of B_1 given in Table I.
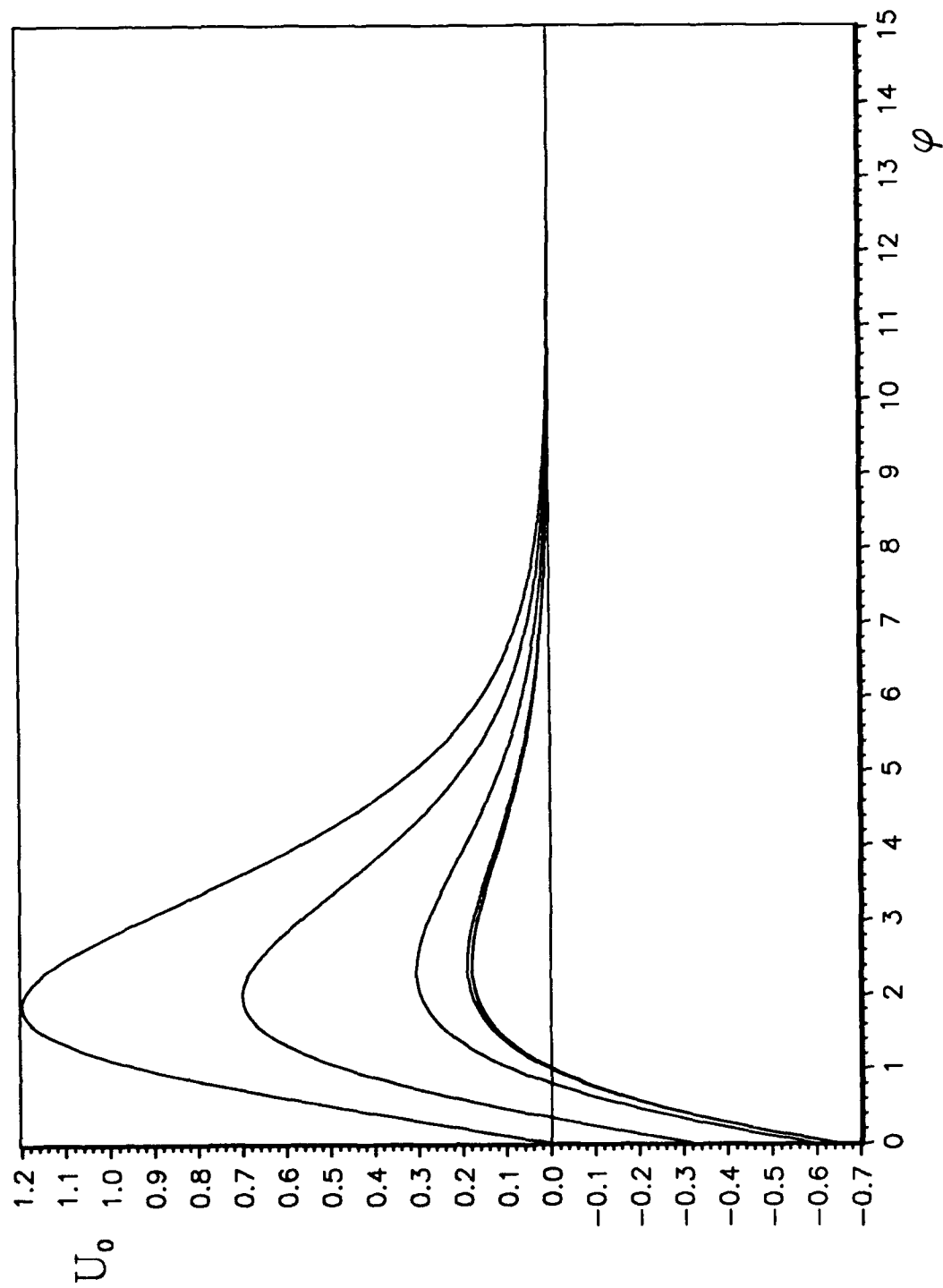
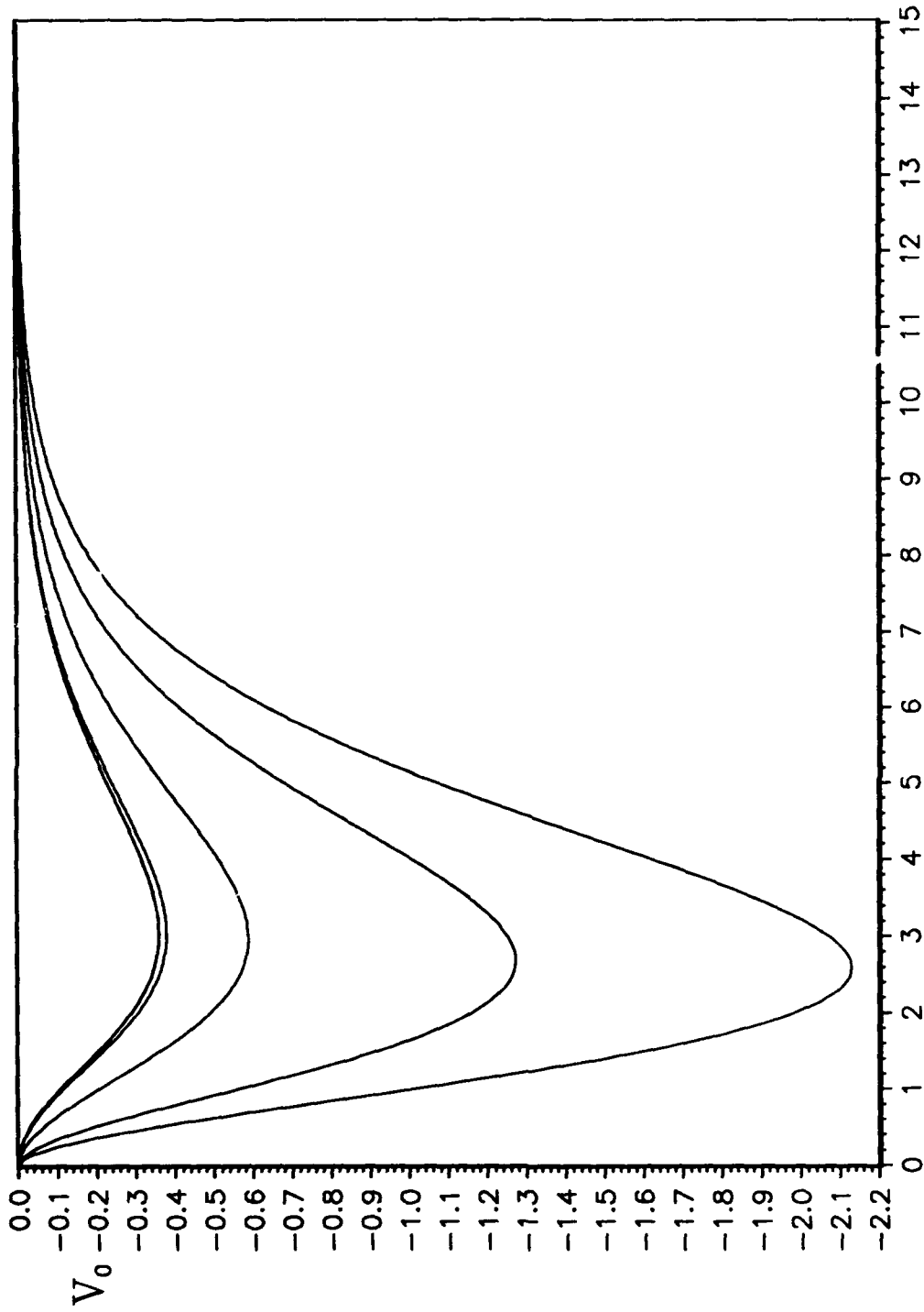**Figure 2.** The eigenfunction U_0 of (18) corresponding to the maximum eigenvalues of Table I.

Figure 3. The eigenfunction V_0 of (18) corresponding to the maximum eigenvalues of Table I.

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16. Abstract

The stability of the flow of a viscous incompressible fluid over a curved compliant wall to longitudinal Görtler vortices is investigated. The compliant wall is modeled by a particularly simple equation relating the induced wall displacement to the pressure in the overlying fluid. Attention is restricted to the large Görtler number regime; this regime being appropriate to the most unstable Görtler mode. The effect of wall compliance on this most unstable mode is investigated. (747)

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