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Active Nonlinear Feedback Control for Aerospace Systems

Annual Report

David C. Hyland
Principal Investigator
Harris Corporation
MS 22/4842
Melbourne, FL
32901

For:

Air Force Office of Scientific Research (AFOSR)
Bolling Air Force Base
Washington, DC 20332

Attention:

Dr. Spencer Wu

December 1990

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HARRIS

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Although the theory of linear control systems is highly mature, *nonlinear* control-system design techniques remain relatively undeveloped. In real-world applications such as vibration suppression in flexible structures and large angle rigid-body spacecraft maneuvers, nonlinear plants generally require nonlinear controllers, while linear plants often benefit from the implementation of nonlinear controllers in the presence of structured plant uncertainty, actuator constraints, and nonquadratic performance criteria. This report discusses progress in several areas relating to the role of nonlinearities in feedback control. These area include Lyapunov function theory, chaotic controllers, statistical energy analysis, phase robustness, and optimal nonlinear control theory.

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Table of Contents

Abstract

Although the theory of linear control systems is highly mature, *nonlinear* control-system design techniques remain relatively undeveloped. In real-world applications such as vibration suppression for flexible structures and large angle rigid-body spacecraft maneuvers, nonlinear plants generally require nonlinear controllers, while linear plants often benefit from the implementation of nonlinear controllers in the presence of structured plant uncertainty, actuator constraints, and nonquadratic performance criteria. This report discusses progress in several areas relating to the role of nonlinearities in feedback control. These areas include Lyapunov function theory, chaotic controllers, Statistical Energy Analysis, phase robustness, and optimal nonlinear feedback control.

1.0 Introduction

1.0 Introduction

1.1 Review of Linear Multivariable Control Theory

The bulk of current control-system practice is based upon linear control theory. Classical single-loop design methods, whose basic development predates 1960, are widely utilized in practice. For high-performance multi-loop applications, modern multivariable techniques are finding their way into practice. A broad spectrum of linear multivariable control techniques has reached the graduate curriculum (see, for example, the in-depth textbook [1]). Moreover, the advanced development of such methods is reflected in the availability of several computer-aided design packages.

Within modern multivariable control theory there are several major thrusts of development that can be identified. From a state space perspective, the original work of Kalman and others has led to a rather complete theory of H_2-optimal linear-quadratic-Gaussian (LQG) control design [2–4]. Furthermore, an elegant state space theory within a geometric rather than optimization framework has been developed in [5]. Multivariable extensions of classical frequency-domain ideas have undergone significant development along a number of paths. For example, classical ideas have been generalized to the multivariable setting in [6,7], while an optimal design theory based upon a frequency-domain (H_∞) criterion was pioneered in [8] and further developed in numerous papers (see, e.g., [9,10]and references therein). We also note the development of further sophisticated approaches within an algebraic transfer function setting [11,12].

It is also worthwhile reviewing some recent trends in linear multivariable control, namely, robust control and controller simplification. Robust control refers to the need to effect desired closed-loop performance (e.g., tracking and disturbance rejection) in spite of plant modeling uncertainties. Within classical theory, the related concept of sensitivity plays a key role, while multivariable problems require more sophisticated approaches. Numerous robust control-design techniques have been developed under a variety of assumptions concerning the plant uncertainty. Unstructured uncertainty is addressable via H_∞ methods [9,10], while specialized techniques are required for more highly structured plant uncertainty; see, e.g., [13–16,I.20,I.30]. In addition, recent results concerning the state space solution of H_∞ problems yield greater unification of state space and frequency domain synthesis techniques [17–19,I.29].

The second trend in linear multivariable control theory we note here involves controller simplification issues. While modern design techniques such as LQG theory produce high-order controllers, it is desirable in practice to employ the simplest controller meeting design specifications. Here, "simplicity" may refer to dynamic dimension, number of digital operations, degree of decentralization, and other considerations affecting implementation, cost, reliability, etc. References [20–24,I.23,II.89,II.91] are representative of progress made in this area.

1.2 Nonlinear Control Theory

"Nonlinear control theory" refers to control theory in which either the controller or the plant (or both) is nonlinear. This theory is not as extensively developed as linear multivariable control theory. The principal approaches to nonlinear multivariable control design include local linearization, global linearization, the second method of Lyapunov, variable structure control, optimization-based methods, and differential-geometric methods. With these general classifications in mind, we can identify several advantages of nonlinear control over linear control. In this regard it is useful to consider three cases in which the theory is applied (see Figure 1.2–1): (*i*) nonlinear control for linear plants, (*ii*) linear control for nonlinear plants, and (*iii*) nonlinear control for nonlinear plants.

The role of nonlinearities in control theory can best be understood by reviewing the assumptions and limitations of standard linear-quadratic-Gaussian (LQG) theory. As its name implies, LQG theory is based upon three fundamental assumptions (Figure 1.2–2)

- the plant dynamics and measurement equations are linear in both the state and control variables
- the performance measure to be minimized is quadratic
- the plant disturbances and measurement noise are additive Gaussian white noise

In addition to these *explicit* assumptions the following *implicit* assumptions are crucial:

- the plant model is completely accurate
- mean-square control effort is limited

Under these assumptions, a major result of modern control theory [2] states that the optimal controller is given by the *linear* controller consisting of the Wiener-Kalman filter followed by the optimal linear-quadratic regulator. Hence in this case nonlinear controllers cannot improve performance.

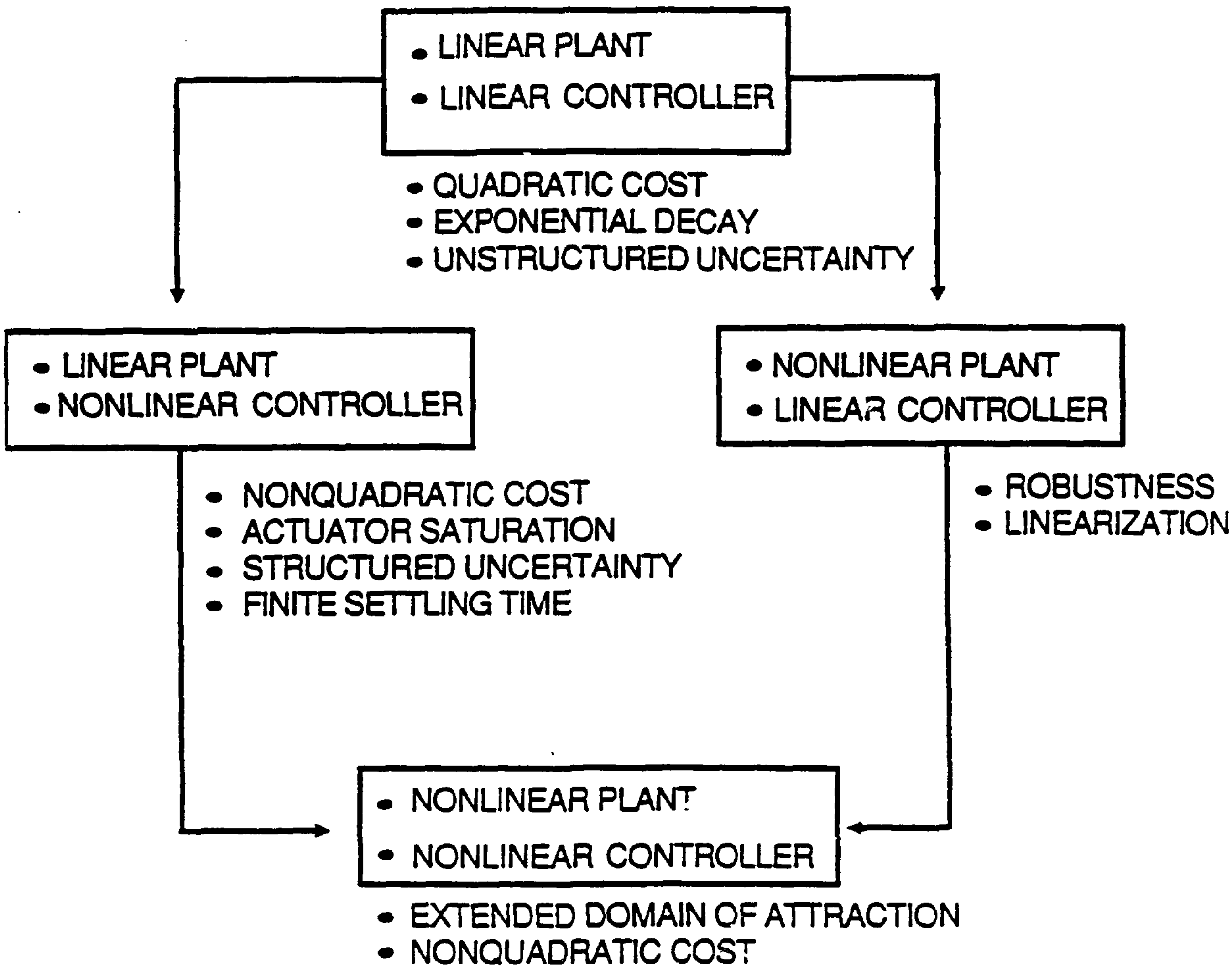


Figure 1.2–1. In nonlinear control theory, the plant and/or controller is nonlinear.

- QUADRATIC COST
- LINEAR DYNAMICS AND MEASUREMENTS
- ADDITIVE GAUSSIAN DISTURBANCES
- NO MODELING UNCERTAINTY
- MEAN-SQUARE ACTUATOR BOUNDS

↓

- LINEAR CONTROLLER IS OPTIMAL
  (LQG THEORY)

- NONLINEAR DYNAMICS AND/OR MEASUREMENTS
- NONGAUSSIAN, NONADDITIVE DISTURBANCES
- MODELING UNCERTAINTY
- NONQUADRATIC COST
- AMPLITUDE ACTUATOR BOUNDS

↓

- NONLINEAR CONTROLLER IS OPTIMAL

Figure 1.2–2. Linear controllers are generally optimal for only a narrow class of linear-quadratic-Gaussian problems.

Suppose, however, that not all of the assumptions of LQG theory are valid for a given problem, that is, one or more of the following conditions applies:

- the plant dynamics and/or measurement equation is nonlinear
- the disturbances are either nonadditive or non-Gaussian
- the relevant performance measure is nonquadratic
- the plant model is uncertain
- control effort is limited by amplitude (L_∞) or total fuel (L_1) constraints.

In real world applications, of course, *all* of these conditions apply, at least to some extent. The actual extent to which each one must be considered is problem-dependent. In each of these cases there is no reason to expect that a linear controller is optimal or even appropriate. Nevertheless, it is still desirable for a variety of reasons to seek linear controllers, and much of control theory has been directed toward this goal. Ultimately, however, we are faced with the following question: When is it necessary or advantageous to implement nonlinear controllers in place of linear controllers? Nonlinear controllers will generally entail more difficult performance validation and implementation complexity (Figure 1.2–3). Furthermore, we note that an additional level of controller complexity involves time-varying control (for either linear or nonlinear controllers) (Figure 1.2–4). We now examine the possible benefits of nonlinear controllers.

1.3 Linear Versus Nonlinear Controllers

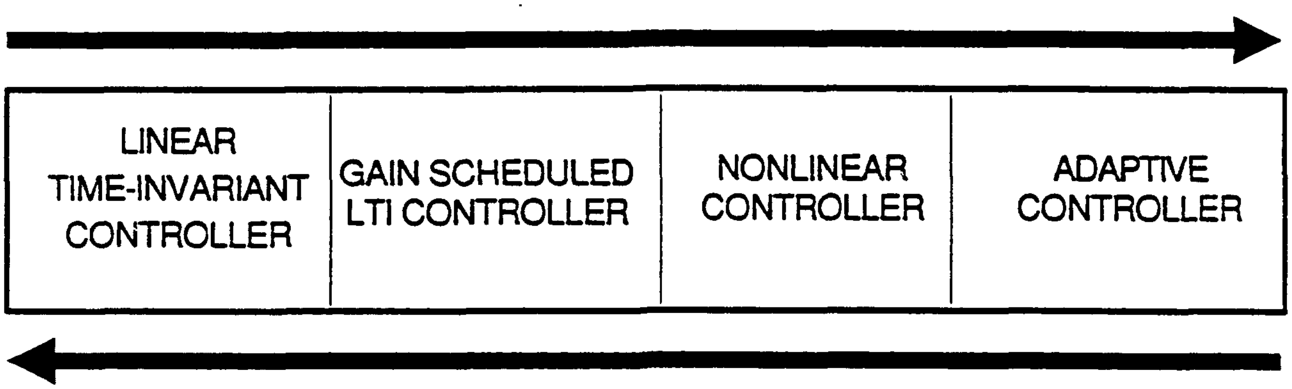
Let us first consider the problem of nonlinear plant dynamics. Such nonlinearities arise in a wide variety of engineering applications [25]. Nevertheless, linear control theory has been developed to deal with large classes of nonlinearities, for example, as bounded by a sector [26–28]. Lure's problem, the Aizermann conjecture, and the circle and Popov criteria are all traditional control theory topics dealing with nonlinearities.

In many applications, however, the nonlinearities are well modeled to the extent that their detailed structure can be exploited in control design. For example, in the case of a single rigid body we have Euler's equation

$$J\dot{\omega} + \omega \times J\omega = f(u),$$

where J denotes the moment of inertia, ω denotes angular velocity, and $f(u)$ denotes applied torque.

* GREATER GENERALITY
* IMPROVED PERFORMANCE FOR MODELING ACCURACY
* MORE COMPLEX TO IMPLEMENT
* HARDER TO VALIDATE



* MORE RESTRICTIVE CONTROLLER CLASS
* PERFORMANCE LIMITED BY MODELING ACCURACY
* SIMPLER TO IMPLEMENT
* EASIER TO VALIDATE

DESIGN GUIDELINES

* TRY TO MEET PERFORMANCE SPECIFICATIONS WITH SIMPLEST POSSIBLE CONTROL LAW
* IF SPECIFICATIONS CANNOT BE MET, THEN INCREASE CONTROL LAW COMPLEXITY AND ASSESS PERFORMANCE/IMPLEMENTATION/VALIDATION TRADEOFFS

Figure 1.2–3. Nonlinear controllers offer improved performance, but may entail greater implementation and validation complexity.

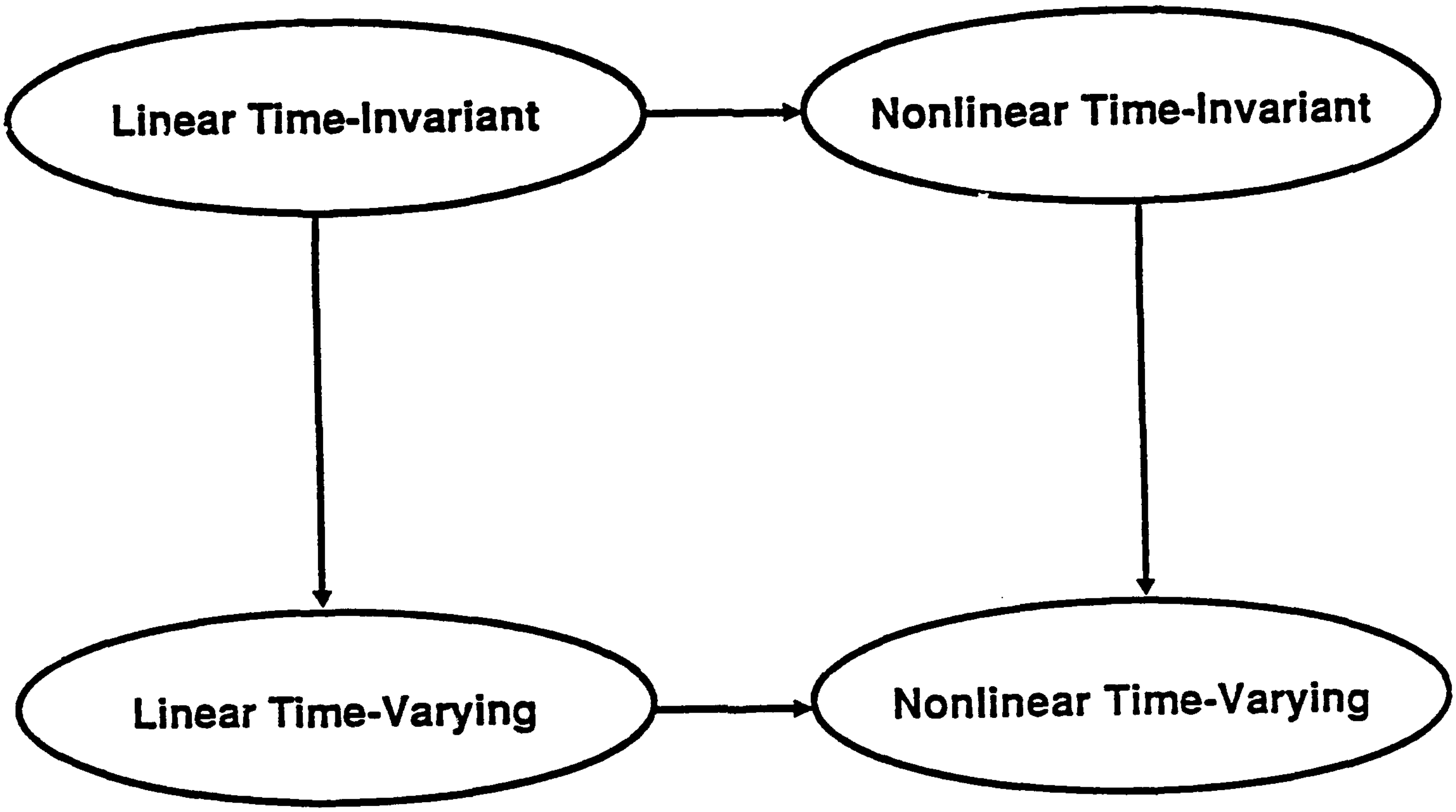


Figure 1.2–4. Linear and nonlinear controllers may be either time-invariant or time-varying.

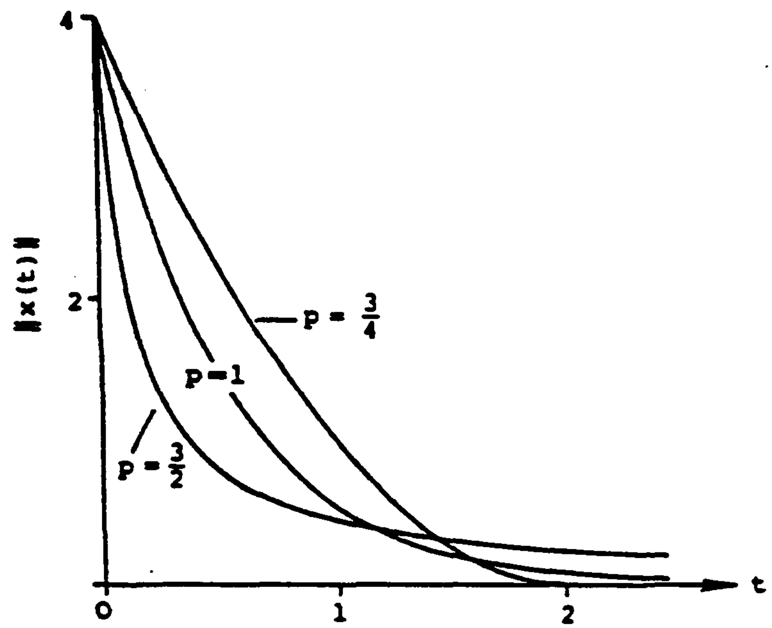
The quadratic gyroscopic term $\omega \times J\omega$ is significant in rapid maneuvers involving large structures. Since the structure of the nonlinearity in this case is crucial, we expect nonlinear controllers to play a role [29–57].

Next consider the problem of optimizing a nonquadratic performance measure. In this case it can generally be expected that linear controllers are not optimal. Time-optimal performance leads to bang-bang controllers, which are nonlinear, while higher-order (polynomial) performance measures lead to higher-order feedback laws [58–68]. For example, consider the effect of a nonquadratic performance measure as addressed in [68]. As shown in Figure 1.2–5, a super-linear state feedback (in this case a quadratic control) can efficiently regulate small amplitude signals, even driving the state to zero in finite time (if one neglects measurement and disturbance noise effects). A theory of sublinear control for finite-time control is developed in [69]. An additional performance aspect is transient behavior [70] which is difficult to capture by means of scalar performance measures.

There are, however, nonquadratic performance measures for which the optimal controller *is* linear. In particular, this is the case for H_∞ optimal control. For this problem the goal is to minimize the worst-case disturbance attenuation over all frequencies. The H_∞ problem differs mathematically from the LQG problem due to the modeling of disturbances and error signals as deterministic L_2 functions. Connections with the LQG setting can be established by means of an exponential-of-quadratic performance functional with white noise disturbances [71–84].

Problems involving uncertain plant models have motivated the subject of robust control theory. One approach to robust control involves modeling the uncertainty by means of the H_∞ norm and then applying H_∞ theory to guarantee robust stability and performance. In this case and for related problems in robust control, it has been shown that nonlinear controllers offer no advantage over linear controllers [85–90]. Though valuable, these results consider only restricted uncertainty characterizations (e.g., unstructured uncertainty) [86–88], very special performance measures (e.g., H_∞ performance) [85], or limited definitions of stability (e.g., quadratic stability) [90]. In fact, from the previous discussion on the optimality of nonlinear controllers for nonquadratic performance criteria, it is reasonable to conjecture that for a variety of system performance measures nonlinear controllers can yield better *robust* performance than linear controllers. In fact, it is even possible that the controller that solves the robust quadratic performance problem

$$\min_{u(t)}\left\{\max_{(\Delta A,\Delta B)\in\mathcal{U}}\int_0^\infty (x^{\mathrm{T}}R_1x+u^{\mathrm{T}}R_2u)dt\right\},$$
$$\dot{x}(t)=(A+\Delta A)x(t)+(B+\Delta B)u(t),$$



$$p = \frac{3}{2} \quad : \quad u = -2\|x\|x \Longrightarrow \text{ EFFICIENT REGULATION FOR LARGE AMPLITUDES}$$

$$p = 1 \quad : \quad u = -2x \Longrightarrow \text{ EXPONENTIAL DECAY (LINEAR)}$$

$$p = \frac{3}{4} \quad : \quad u = -2\|x\|^{-\frac{1}{2}}x \Longrightarrow \text{ EFFICIENT REGULATION FOR SMALL AMPLITUDES}$$

Figure 1.2–5. As shown in [68], nonlinear controls can be shaped to give efficient regulation for various, selected vibration amplitude regimes.

is a nonlinear controller. Results that indicate that nonlinear controllers can yield improved robustness properties are given in [91–94].

An additional advantage of nonlinear controllers is the ability to address actuator saturation limitations. In practice any electromechanical device used as a control actuator is subject to limitations on maximum force, torque output, power consumption, stroke, and angular speed limits. Thus, in reality, control-design optimization must account for constraints on the maximum value of actuator force or similar constraints on internal signals associated with the actuator dynamics. The simplest such realistic constraint takes the form of a pointwise bound on the actuator force output, i.e.,

$$|u(t)| \leq \bar{u}_{\max},$$

where $\bar{u}_{\max}$ is the largest physically possible magnitude of the actuator output. The above pointwise bound is an example of an L_∞ design constraint and differs crucially from L_2 constraints such as

$$\mathbb{E}[u^2] < \sigma^2$$

or

$$\int_0^\infty u^2(t)dt < \sigma^2,$$

which correspond to power and energy constraints, respectively.

Figure 1.2–6 illustrates the ramifications of pointwise bounded actuator constraints. Suppose that the plant is linear except for the physical constraint $|u(t)| \leq \bar{u}_{\max}$ and that the system is subject to an initial impulse disturbance. If one designs an optimal regulator using the integral square condition (for analytical convenience) as the constraint on the optimization problem, then the resulting controller is linear. Moreover, one can choose linear gains such that the peak actuator output is less than the physically imposed limit of $\bar{u}_{\max}$. Then (see the top half of Figure 1.2–6), following the initial disturbance, all signals, including the actuator output, decay exponentially. Note that although $u(t)$ just satisfies the physical constraint $|u(t)| < \bar{u}_{\max}$ for small t, for larger t, $|u(t)|$ is small. Thus, in this case actuator capability is wasted. On the other hand (see bottom half of Figure 1.2–6), it is possible to design a *sublinear* feedback control for which the actuator uses nearly its full capacity while the system state is driven to zero *faster* than exponentially. In fact, it is well known that minimal-time maneuvers actually require bang-bang control. Variable structure (nonlinear) controllers, which can be viewed as generalizations of bang-bang controllers, can also be used to control linear systems while efficiently utilizing actuator capabilities.

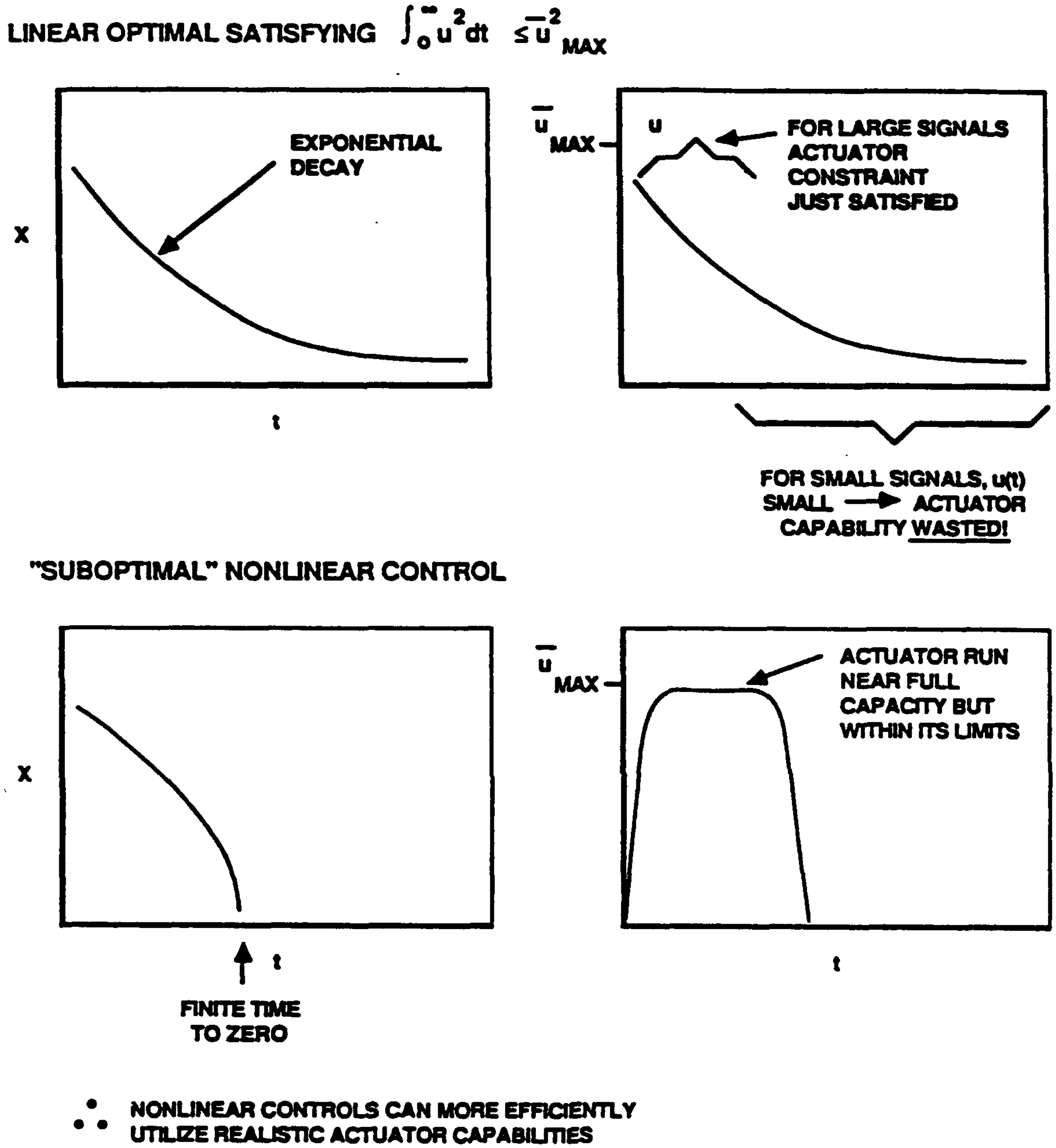


Figure 1.2–6. Nonlinear controllers can utilize actuators more efficiently than linear controllers in the presence of saturation bounds.

A specialized class of nonlinear controllers for linear plants is the class of adaptive controllers. In contrast to *fixed-gain* controllers, which maintain prespecified constants within the feedback law, *adaptive* controllers adjust feedback gains to improve closed-loop stability and performance when the plant is uncertain. Adaptive controllers generally utilize probing signals to excite the plant dynamics and thereby identify plant parameters. Feedback gains can then be adjusted to account for the identification data. The overall process of identification and adjustment clearly constitutes a nonlinear control law. Thus, the adaptive control literature can be viewed as a specialized subclass of nonlinear control, although for historical reasons this categorization is rarely utilized. For our purposes, viewing adaptive controllers as nonlinear controllers is particularly useful. For example, as discussed above, nonlinear controllers can be viewed as a specialized form of robust controllers for uncertain linear plants.

The distinction between nonlinear controllers and adaptive controllers has narrowed in recent years with the development of adaptive controllers not requiring explicit probing signals [95–102]. These results show that there exist nonlinear controllers that can stabilize generic classes of systems characterized by minimal a priori data. Although these controllers are usually thought of as adaptive since the feedback gains are continually adjusted, the feedback laws are clearly nonlinear controllers of special structure.

1.4 Overview of this Report

The central result of control system analysis and design is Lyapunov's method. The ability to construct a positive-definite functional that decays along system trajectories is sufficient to guarantee asymptotic stability. Design via Lyapunov functions need not be associated with the optimization of a performance measure although, as discussed in Section 5, the converse is often true, that is, optimal design may be predicated on a Lyapunov function. Hence, in our view, Lyapunov's method ultimately comprises the most fundamental technique in nonlinear (as well as linear) control theory.

This program is thus focussing on several problem areas relating to Lyapunov theory. The interrelationships among these areas is shown in Figure 1.4–1. In Section 2 we describe progress in analyzing energy flow in coupled mechanical systems. The results obtained thus far extend the foundations of Statistical Energy Analysis. In Section 3 we apply the results of Section 2 along with applications to the design of chaotic controllers for enhanced energy dissipation. Section 4 is devoted to progress in developing a theory of robustness due to phase properties. Finally, Section 5 discusses optimal nonlinear control theory.

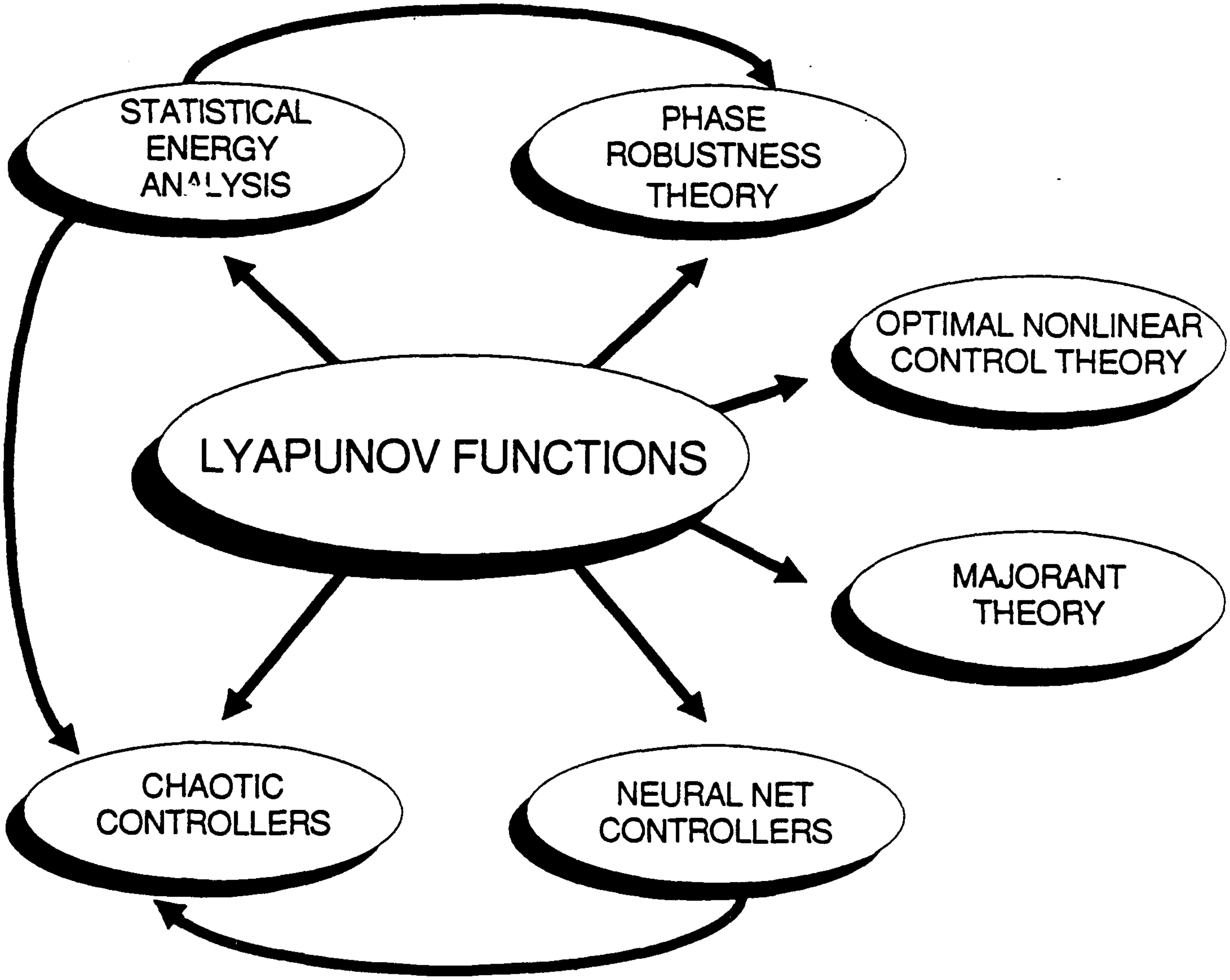


Figure 1.4–1. The program focusses on several related research problems relevant to nonlinear control.

2.0 Power Flow and Statistical Energy Analysis

2.0 Energy Flow and Statistical Energy Analysis

It is well known from thermodynamics that energy flows from hot objects to cold objects. It is less well known, however, that a similar phenomenon occurs in coupled mechanical systems with modal energy playing the role of temperature. Energy and power flow concepts, often called Statistical Energy Analysis (SEA), have proven to be useful tools for analyzing linear dynamic systems [237–249]. Hence this phase of the program is devoted to the further development of these ideas to support nonlinear analysis and design. In Section 3 these ideas are used to analyze and design chaotic feedback controllers.

2.1 Energy Flow in Coupled Dynamical Systems

The objective of SEA is to model energy flow among coupled dynamical subsystems. SEA was originally developed for acoustical analysis involving very large numbers of modes that may be poorly modeled. Many of the concepts of SEA as applied to high dimensional systems (such as equipartition of energy) have close connections with statistical mechanics of many particle systems. Although SEA theory has been widely applied, rigorous analytical results have been available only for identical couplings or for weak interactions. Under this program we have extended SEA theory to address an arbitrary number of subsystems with arbitrary coupling.

In this section we summarize results on SEA which are developed in the paper entitled "Power Flow, Energy Balance, and Statistical Energy Analysis for Large-Scale Interconnected Systems." This paper, which contains all details of the results reported here, appears in Appendix C.

To summarize these results consider the system

$$\dot{x} = Ax + Gx + w, \tag{1}$$

where the state $x \in \mathbb{C}^n$, the uncoupled dynamics matrix A is given by

$$\begin{aligned} A &= -\nu + j\Omega + H, \\ \nu &= \text{ diag } (\nu_1, \ldots, \nu_n) \in \mathbb{R}^{n \times n}, \quad \nu_i > 0, \\ H &= \text{ diag } (H_1, \ldots, H_n) \in \mathbb{C}^{n \times n}, \\ \Omega &= \text{ diag } (\Omega_1, \ldots, \Omega_n) \in \mathbb{R}^{n \times n}, \end{aligned}$$

and where G denotes the coupling among subsystems, that is,

$$G \in \mathbb{C}^{n \times n}, \quad G_{ii} = 0, \quad i = 1, \ldots, n.$$

The additive forcing $w(t)$ is taken to be white noise with intensity $V \geq 0$.

The first step of our approach is to note that for an output signal

$$z = Cx, \tag{2}$$

the steady-state mean-square response is given by

$$\begin{aligned} J &= \lim_{t\to\infty} \mathbb{E}[z^* z] \\ &= \operatorname{tr}[C^{\mathrm{T}} C Q], \end{aligned} \tag{3}$$

where

$$Q \triangleq \lim_{t\to\infty} \mathbb{E}[x x^*].$$

It is well known that the steady-state covariance Q is given by the algebraic Lyapunov equation

$$0 = AQ + QA^* + GQ + QG^* + V. \tag{4}$$

In many practical situations it can be argued (see Appendix C) that the principal contribution to J is due to the diagonal elements of Q. Hence our main result is based on a direct characterization of the diagonal elements of Q in terms of V, which is obtained by eliminating the off-diagonal elements of Q. To do this, we rewrite (4) as

$$0 = A\{Q\} + \{Q\}A^* + \{G\langle Q\rangle\} + \{\langle Q\rangle G^*\} + \{V\}, \tag{5}$$

$$0 = A\langle Q\rangle + \langle Q\rangle A^* + \langle G\langle Q\rangle\rangle + \langle\langle Q\rangle G^*\rangle + G\{Q\} + \{Q\}G^*, \tag{6}$$

where $\{\cdot\}$ and $\langle\cdot\rangle$ denote the diagonal part and off-diagonal part of a matrix, respectively. Here we have assumed for convenience that $\langle V\rangle = 0$.

Next, we apply Kronecker matrix algebra to solve (5), (6) for $\{Q\}$ in terms of $\{V\}$. To state the main result define the vector E of steady-state mean-square state energies

$$E = \begin{bmatrix} E_1 \\ \vdots \\ E_n \end{bmatrix},$$

where $E_i \triangleq Q_{ii}$, $i = 1, \ldots, n$, and the vector

$$\hat{V} = \begin{bmatrix} V_{11} \\ \vdots \\ V_{nn} \end{bmatrix},$$

which corresponds to $\{V\}$. Then we obtain the following consequence of (5), (6):

$$(\mu + \mathcal{P})E = \hat{V}, \tag{7}$$

where

$$\mu \triangleq \operatorname{diag}\{2\nu_1 - 2\mathrm{Re}(H_1), \ldots, 2\nu_n - 2\mathrm{Re}(H_n)\}, \tag{8}$$

and

$$\mathcal{P} = \hat{\mathcal{E}}^{\mathrm{T}}(\bar{G} \oplus G)\mathcal{E}_{\perp}[(\bar{A} \oplus A) \oplus (\bar{G} \oplus G)]^{-1}\mathcal{E}_{\perp}(\bar{G} \oplus G)\hat{\mathcal{E}}. \tag{9}$$

In (8) and (9), "Re" denotes real part, $\oplus$ denotes Kronecker sum, and $\mathcal{E}$, $\mathcal{E}_{\perp}$, and $\hat{\mathcal{E}}$ denote $n^2 \times n^2$ matrices of special structure whose element are ones and zeros. It can be shown that $\mathcal{P}$ is real.

To elucidate the meaning of (7) we can write its kth component as

$$\underbrace{(2\nu_k - 2\mathrm{Re}\ H_k)E_k}_{\substack{\text{power dissipated}\\ \text{by the } k\text{th mode}\\ \text{due to damping}}} + \underbrace{\Pi_k}_{\substack{\text{power flow}\\ \text{from the } k\text{th mode}\\ \text{to all other modes}\\ \text{due to coupling}}} = \underbrace{V_{kk},}_{\substack{\text{power input}\\ \text{due to external}\\ \text{disturbances}}} \tag{10}$$

where Π_k has the form

$$\Pi_k = \sum_{\ell=1}^{n} \mathcal{P}_{k\ell}E_{\ell}, \quad \mathcal{P}_{k\ell} \in \mathbb{R}. \tag{11}$$

The matrix $\mathcal{P}$ can be viewed as a power flow matrix, while relation (10) thus has the form of a power flow equation. To arrive at an energy balance relation we consider the case in which

$$\mathcal{P}_{k\ell} \le 0, \quad k \ne \ell, \quad k, \ell = 1, \ldots, n. \tag{12}$$

This occurs, for example, if the subsystem coupling is sufficiently weak. If, in addition, the couplings are energy conservative (for example, passive), then it can be shown that

$$\mathcal{P}_{kk} = \sum_{\substack{\ell=1\\ \ell\ne k}}^{n} |\mathcal{P}_{k\ell}|, \quad k = 1, \ldots, n. \tag{13}$$

Then, defining $\sigma_{k\ell} \triangleq |\mathcal{P}_{k\ell}|, \quad k \ne \ell$, so that $\sigma_{k\ell} \ge 0$, it follows from (11) that

$$\Pi_k = \sum_{\substack{\ell=1\\ \ell\ne k}}^{n} \sigma_{k\ell}(E_k - E_{\ell}). \tag{14}$$

In other words, power flow from the kth mode to all other modes is the sum of the individual power flows from mode k to mode ℓ, which are proportional to the energy differences $E_k - E_{\ell}$.

Note that power always flows from more energetic modes to less energetic modes (because of the nonnegativity of the coefficients $\sigma_{k\ell}$). Substituting (14) into (7) yields

$$\mu_k E_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \sigma_{k\ell}(E_k - E_\ell) = V_k, \tag{15}$$

which is an energy balance relation. Equations (10) and (15), which govern energy exchange among coupled oscillators, are completely analogous to the equations of thermal transfer with the modal energies playing the role of temperatures.

In physical situations involving nonconservative couplings, we have shown that although (13) no longer holds, it is still possible in the case of weak couplings to obtain a generalized power flow proportionality. In this case there exists a set of positive scale factors $D_k > 0, \quad k = 1, \ldots, n$, such that, with $\hat{E}_k \triangleq \frac{1}{D_k} E_k$, the energy difference power flow proportionality is given by

$$\Pi_k = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \hat{\sigma}_{k\ell}(\hat{E}_k - \hat{E}_\ell), \tag{16}$$

where $\hat{\sigma}_{k\ell} \triangleq D_\ell \sigma_{k\ell}$. Note that (16) is not merely a rewriting of (14) since in general $D_k \neq D_\ell$. With (16), the energy equation (7) assumes the form of a generalized energy balance relation given by

$$\hat{\mu}_k \hat{E}_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \hat{\sigma}_{k\ell}(\hat{E}_k - \hat{E}_\ell) = V_k, \tag{17}$$

where $k = 1, \ldots, n$. That is, there is a set of re-scaled energies such that (7) looks like the equations of thermal transfer.

Furthermore, while deriving energy difference power flow proportionality relations, we have also shown that the explicit expressions given for the power flow matrix $\mathcal{P}$ in the SEA literature are actually first-term approximations in a series expansion for $\mathcal{P}$. Indeed, it turns out that $\mathcal{P}$, which is given by a complicated expression involving ν, Ω, H, and G, agrees with the customary SEA expressions for "small" G. This in done by obtaining explicit expressions for the terms of a series expansion of $\mathcal{P}$ in ascending powers of the matrix elements of G.

Since the modal energies obey equations analogous to those of thermal transfer, it might be expected that if the coupling coefficients $G_{k\ell}$ are large compared to the modal dampings, then the energies should be approximately equal, that is,

$$E_1 \simeq E_2 \simeq \cdots \simeq E_n. \tag{18}$$

The paper in Appendix C provides a formulation and proof of this "energy equipartitioning" phenomenon.

3.0 Chaotic Controllers

3.0 Chaotic Controllers

In Section 2 we explored the notions of power flow and energy balance in interconnected systems. Our next goal is to apply these ideas to the analysis and design of nonlinear feedback controllers.

To do this we need only view the plant and controller as a pair of interacting subsystems. If disturbance rejection is an objective, then we seek to design a controller that maximizes power flow from the plant to the controller. Within an H_∞ context this idea has been explored in the recent paper

D. MacMartin and S. R. Hall, "An H_∞ Power Flow Approach to Control of Uncertain Structures," *Proc. Amer. Contr. Conf.*, pp. 3073–3080, San Diego, CA, May 1990.

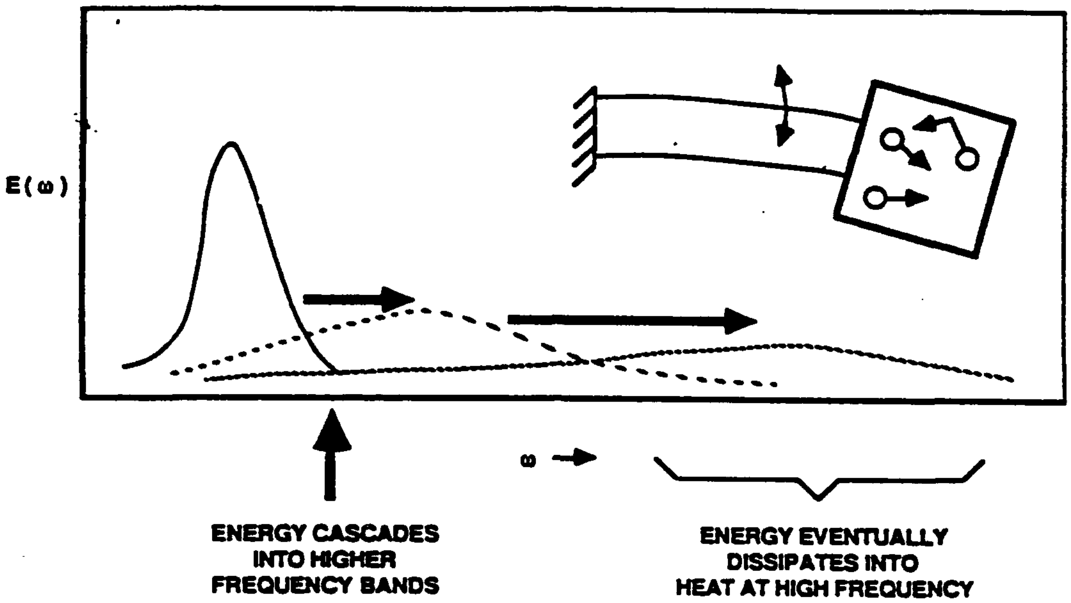
One of the main ideas discussed in this paper is that power flow out of the structure is maximized to the extent that the controller is able to match the impedance of the plant.

In this section we develop a nonlinear controller that exploits the phenomenon of chaos. Our principal goal is to demonstrate that this controller can enhance power flow from the plant to the controller by introducing nonlinearities that induce broadband spectral properties in the controller. A power flow analysis is then used to show that energy can be transferred more efficiently between arbitrary plant and compensator modes. Details of these results are given in "A Nonlinear Vibration Control Design with a Neural Network Realization" which appears in Appendix D.

3.1 Turbulence Model for Chaotic Controller Design

A unique feature of nonlinear systems is the energy cascade mechanism illustrated in Figure 3.1–1. Here, energy originally injected within some lower frequency band can be dispersed to higher frequency bands by virtue of coupling among the vibrational modes of the structure. Eventually the energy is transferred to very high frequencies where it is dissipated into heat by means of natural structural damping or by the action of an additional energy dissipative control law. Thus, a nonlinear controller such as illustrated in Figure 3.1–1 can be viewed as a catalyst for transmuting vibration more rapidly into heat.

The controller illustrated in Figure 3.1–1 can be realized by a purely mechanical device consisting of a chamber containing a number of particles of given mass that undergo free translational motion except for collisions with the chamber walls and with one another. This is essentially the



- ENERGY CASCADE MECHANISM OFFERS POTENTIAL FOR EXTREMELY ROBUST, RAPID ATTENUATION OF LOWER FREQUENCY VIBRATION.

- SUCH NONLINEAR CONTROLLERS ARE A *CATALYST* FOR TRANSMUTING VIBRATION MORE RAPIDLY INTO HEAT.

- COMPENSATOR (BY ITSELF) COULD BE CHAOTIC. BUT WHEN INTERCONNECTED WITH THE PLANT, ITS DAMPING PERFORMANCE IS EXTREMELY ROBUST.

Figure 3.1–1. Another unique aspect of nonlinear control is energy cascade via mechanical turbulence.

impact-damper control mechanism which received some attention in the 60's to mid 70's (for example, in connection with buffet alleviation in aircraft, see [164,165], but which was not subsequently pursued because of mechanical implementation difficulties. With present-day high-speed processors, however, such a nonlinear compensator can be implemented electromechanically using a colocated rate sensor/force actuator pair. However, it is not suggested that research be focused on the impact idea per se. Rather, such devices are mentioned here solely to illustrate the potential of *chaotic compensators* and to elucidate some fundamental aspects that might be suitably generalized within a rigorous design optimization theory. The term chaotic compensator is used because, considered by itself, the nonlinear controller displays chaotic motion. For example, suppose that one disconnects the chamber from the structure and measures the response to sinusoidal inputs. If there is no energy loss in collisions, then the system will display homoclinic tangles of great complexity. With some energy loss mechanisms, chaotic attractors will result. Thus, the compensator shown in Figure 3.1–1, when considered alone, is a chaotic system. However, the intriguing aspect here is that when this chaotic compensator is interconnected with the plant, its damping performance is quite effective and extremely robust. It is important for reliable implementation of effective compensation to understand and exploit the underlying mechanisms involved in this example.

3.2 Lyapunov Setting for the Chaotic Controller

Lyapunov theory provides the foundation for devising a controller that emulates the behavior of a chaotic compensator. Consider the plant with dynamics

$$\dot{x} = f_1(x) + f_2(x)u, \tag{1}$$

$$y = f_2^{\mathrm{T}}(x)x, \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, and dynamic feedback controller

$$\dot{x}_c = f_{c1}(x_c, y) + f_{c2}(x_c, y)y, \tag{3}$$

$$u = -f_{c2}^{\mathrm{T}}(x_c, y)x_c, \tag{4}$$

where $x_c \in \mathbb{R}^{n_c}$. Note that the controller uses only the available measurement y, although the plant is assumed to have a colocated-type symmetry as in a force-to-velocity model of a flexible structure. We assume that $f_1(\cdot)$ is dissipative, that is,

$$x^{\mathrm{T}} f_1(x) + f_1^{\mathrm{T}}(x)x < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{5}$$

and that $f_{1c}(\cdot, y)$ is also dissipative for all $y \in \mathbb{R}^m$. In this case the closed-loop system has the form

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}), \tag{6}$$

where

$$\tilde{x} \triangleq \begin{bmatrix} x \\ x_c \end{bmatrix}$$

and

$$\tilde{f}(\tilde{x}) \triangleq \begin{bmatrix} f_1(x) - f_2(x) f_{c2}^{\mathrm{T}}(x_c, y) x_c \\ f_{c1}(x_c, y) + f_{c2}(x_c, y) f_2^{\mathrm{T}}(x) x \end{bmatrix}$$

with y given by (2). Using the energy Lyapunov function $V(\tilde{x}) = \tilde{x}^{\mathrm{T}}\tilde{x}$ it is easy to show that $\frac{d}{dt}V(\tilde{x}) < 0$ along trajectories of (6).

Let us now specialize to the problem of vibration suppression. Hence consider the plant model

$$\dot{x} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -2\eta\Omega \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u, \qquad x \in \mathbb{R}^{2n},\ u \in \mathbb{R}^1, \tag{7}$$

where

$$\Omega \triangleq \text{ diag } \{\Omega_k\} = \text{ modal frequencies,}$$

$$\eta \triangleq \text{ diag } \{\eta_k\} = \text{ modal damping ratios,}$$

$$b = \text{ modal actuator influence coefficient, } b \in \mathbb{R}^n,$$

with scalar measurement

$$y = b^{\mathrm{T}} x_2. \tag{8}$$

Consider now the compensator

$$\dot{x}_c = \left(\begin{bmatrix} 0 & \bar{\Omega} \\ -\bar{\Omega} & -2\bar{\eta}\bar{\Omega} \end{bmatrix} + 2\alpha [0 \quad e^{\mathrm{T}}] x_c S \right) x_c + \kappa \begin{bmatrix} 0 \\ e \end{bmatrix} y^2, \tag{9}$$

$$u = -\kappa([0 \quad e^{\mathrm{T}}] x_c) y, \tag{10}$$

where $x_c \in \mathbb{R}^{2n_c}$, $\kappa > 0$, $\alpha > 0$, $e^{\mathrm{T}} = [1\ 1\ \ldots\ 1]$,

$$\bar{\Omega} = \text{ diag } \{\bar{\Omega}_i\},\ \bar{\Omega}_i > 0,\ i = 1, \ldots, n_c,$$

$$\bar{\eta} = \text{ diag } \{\bar{\eta}_i\},\ \bar{\eta}_i > 0,\ i = 1, \ldots, n_c,$$

and

$$S = \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ -1 & 0 & 1 & \ldots & 1 \\ -1 & -1 & 0 & \ldots & 1 \\ \vdots & & & & \vdots \\ -1 & -1 & -1 & \ldots & 0 \end{bmatrix} = -S^{\mathrm{T}}. \tag{11}$$

Note that the plant and compensator are of the form (1)–(4).

The choice of feedback controller can be understood by means of Figure 3.2–1. The underlying idea is to transfer vibrational energy from the structure to the controller as efficiently as possible and to exploit the natural dissipation of the controller. To do this, the controller dynamics equation (9) involves an input term proportional to y^2 to create higher-order harmonics of the natural structural frequencies. These harmonics are uniformly distributed to a portion of x_c by means of the vector $e^{\mathrm{T}} = [1\ 1\ \ldots\ 1]$. The compensator dynamics involve a dissipative linear term $\begin{bmatrix} 0 & \bar{\Omega} \\ -\bar{\Omega} & -2\bar{\eta}\bar{\Omega} \end{bmatrix}$ to set up its own modes of vibration. In addition, (9) involves a skew-symmetric term S that serves to uniformly distribute, or "mix," motion of all compensator states while performing modulation (that is, creation of higher harmonics) by means of $[0\ e^{\mathrm{T}}]x_c$. Finally, the control signal given by (10) again serves to modulate the measurement y by the compensator harmonics.

The intention of this compensator is to purposefully create chaos within the controller. There are two principal reasons for this intentional chaos. First, the structure itself has the ability to dissipate energy by means of the damping associated with its natural modes of vibration. Hence, by creating higher frequency harmonics, the compensator can efficiently distribute low-frequency energy, thereby exploiting the natural structural dissipation to the greatest possible extent.

The second motivation for this compensator structure, as already discussed, is to maximize the exchange of energy between the plant and compensator. Roughly speaking, energy will be transferred from the structure to the compensator if there is a significant level of impedance matching. The chaotic motion within the compensator serves to establish a broadband spectrum to enhance impedance matching and thus energy transfer.

To numerically demonstrate these concepts, we considered a 40th-order (20 modes between 1 and 20 rad/sec) lightly damped (.2% damping) plant model with a 40th-order compensator utilizing $\bar{\Omega} = \Omega$. To demonstrate the controller characteristics, we considered the closed-loop response from a nonzero initial condition. Specifically, the lowest frequency mode (1 rad/sec) was assigned an initial amplitude of unity and an initial velocity of zero, with all other modes at equilibrium.

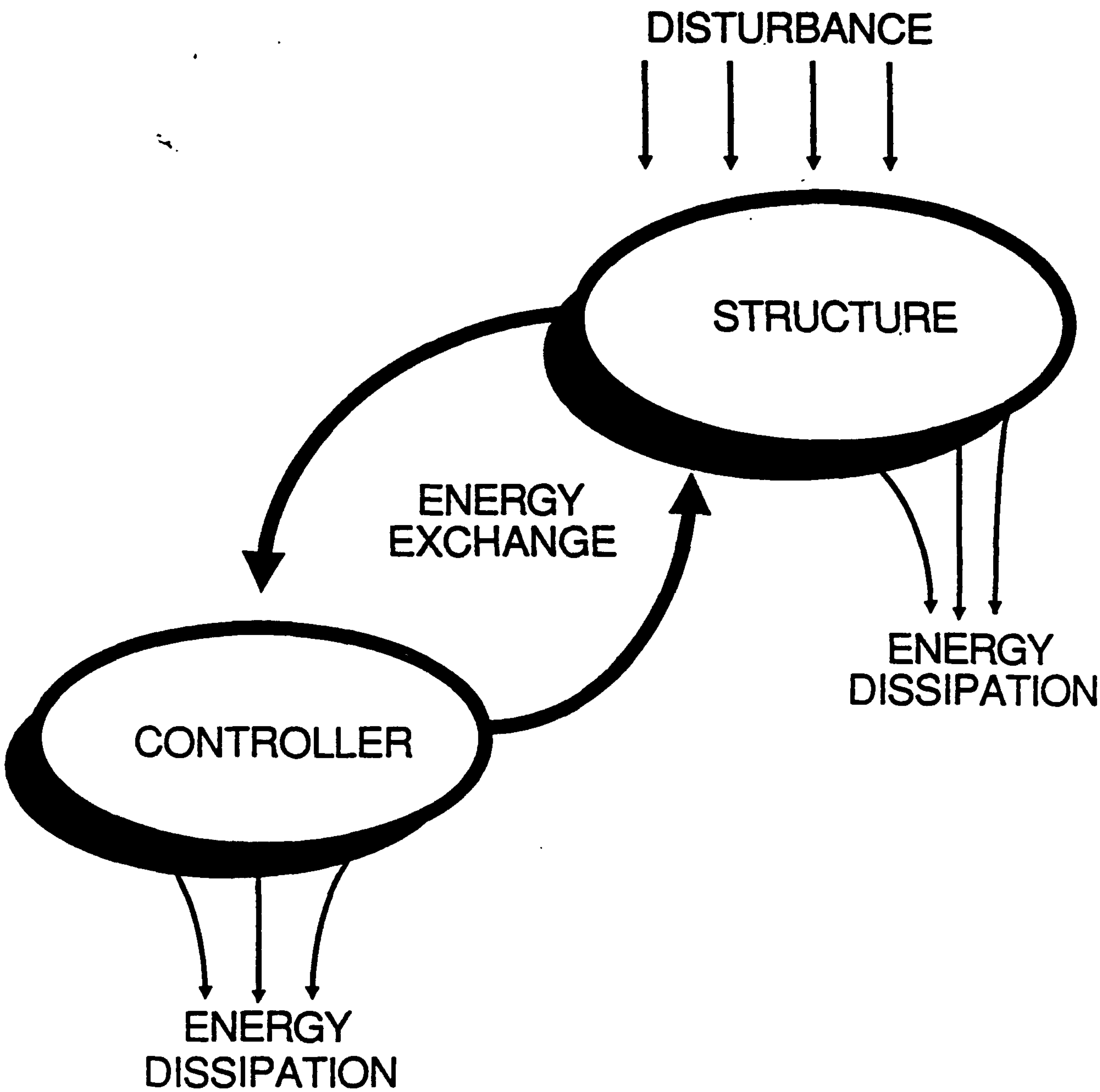


Figure 3.2–1. The controller serves as a mechanism for augmenting energy dissipation.

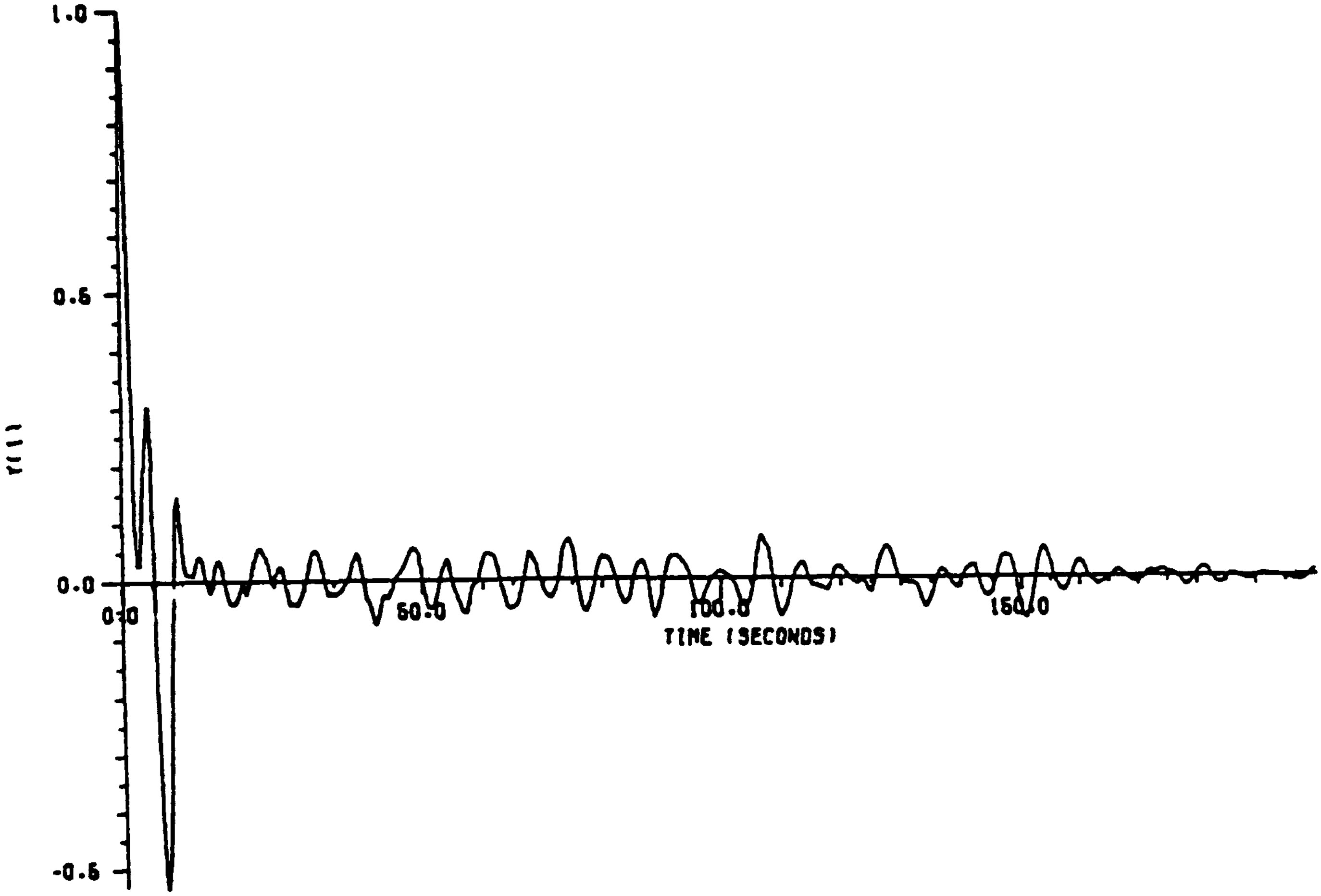


Figure 3.2–2. The time response of the lowest frequency mode exhibits rapid attenuation due to chaotic compensator dynamics.

Figure 3.2–2 shows how the amplitude of the first mode is quickly reduced to a low level with the remaining response composed of broadband motion. In addition, Figure 3.2–3 shows the spectrum of the measurement signal $y(t)$. This plot shows that the structure undergoes significant vibration outside of the modal bandwidth (approximately 4 Hz). This motion, which is due to the nonlinear coupling induced by the controller, shows that energy is transferred from low frequency to high frequency. Since the high frequency modes dissipate energy more efficiently than the low frequency modes (they go to zero like $e^{-\eta_k \omega_k t}$), the controller serves as an efficient mechanism for vibration suppression.

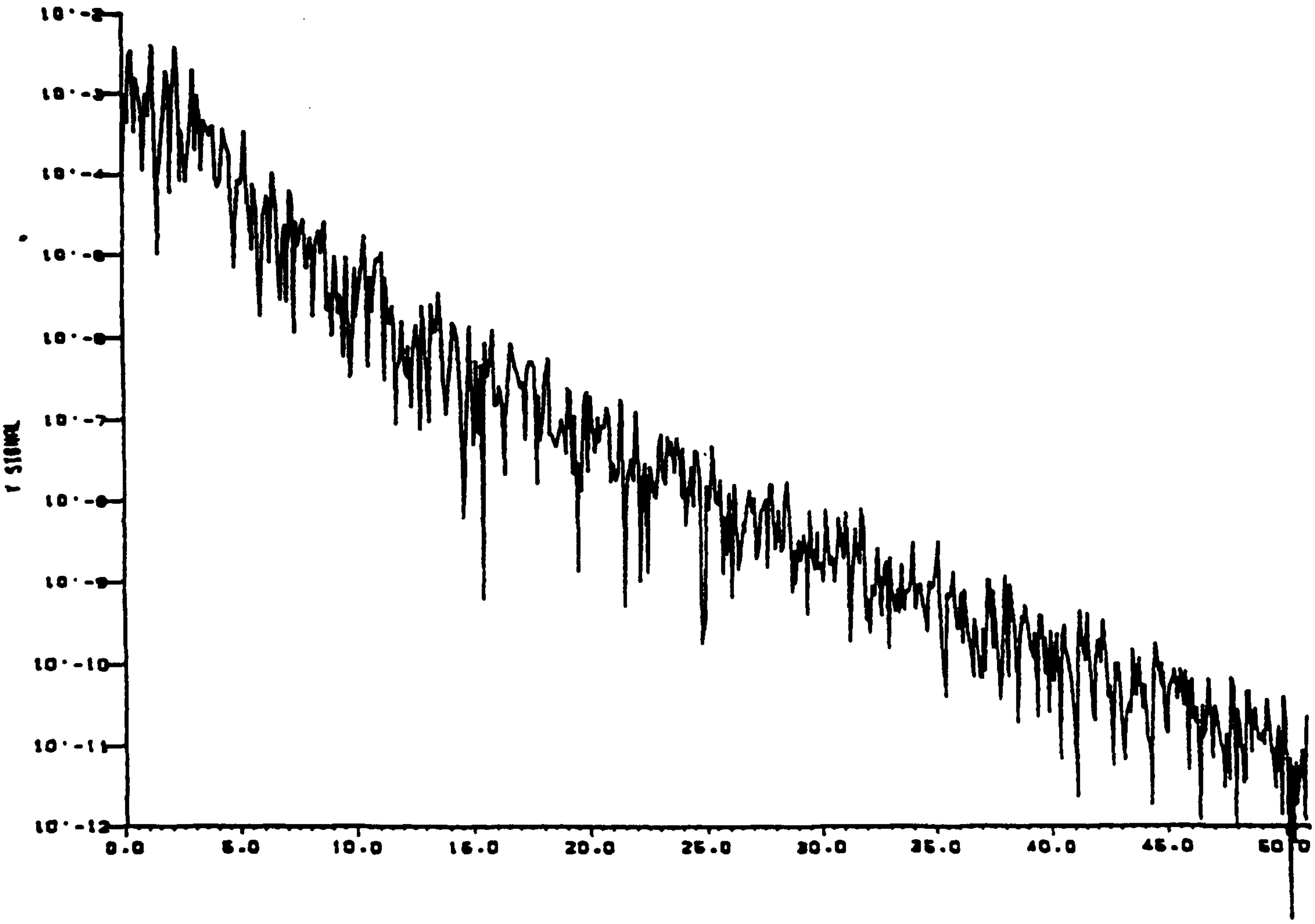


Figure 3.2–3. The spectrum of the measurement signal demonstrates significant out-of-band energy due to the nonlinear cascade mechanism.

4.0 Phase Robustness Theory

4.0 Phase Robustness Theory

H_∞ theory accounts for modeling uncertainty by bounding a weighted H_∞ norm characterization of plant uncertainty. The H_∞ norm does not account for phase, however, which can play an important role in robustness analysis. For example, the magnitude of plant uncertainty can be arbitrarily large as long as the phase of the uncertainty is such as to avoid instability.

Our interest in the role of phase information in robust control is based upon connections to power flow concepts. As will be seen, power flow and stability analysis involving passive systems can be extremely conservative if a small gain (H_∞) approach is used. What is lacking is the treatment of phase properties which become manifested in the structure of the quadratic Lyapunov function. The results described here can be used to guarantee robust stability and performance for both linear and nonlinear systems.

4.1 Positive Real Theory and Structured Lyapunov Functions

As a first step in developing a phase robustness theory, we shall demonstrate a link between phase properties and the structure of the Lyapunov function. Here we are considering Lyapunov functions of the form

$$V(x) = x^{\mathrm{T}} P x \tag{1}$$

where P is a positive definite matrix. We shall call $V(x)$ a structured Lyapunov function if P has internal structure. For example, P may be of the form

$$P = \begin{bmatrix} P_1 & & 0 & \\ & P_2 & & \\ & & \ddots & \\ & 0 & & P_r \end{bmatrix}, \tag{2}$$

where each diagonal block is also positive definite. We may, for example, also require that some of the diagonal blocks be repeated. Structured Lyapunov equations have been studied in

S. Boyd and Q. Yang, "Structured and Simultaneous Lyapunov Functions for Systems Stability Problems," *Int. J. Contr.* Vol. 49, pp. 2215–2240, 1990.

Now let us consider a simple case of robustness due to phase. Consider the plant

$$\dot{x} = Ax + Bu, \tag{3}$$

$$y = Cx, \tag{4}$$

with compensator

$$\dot{x}_c = A_c x_c + B_c y, \tag{5}$$

$$u = -C_c x_c. \tag{6}$$

Now assume that the plant is positive real and that the compensator is strictly positive real. By the Kalman-Yacubovitch (positive real) lemma there exist matrices L, L_c, P, and P_c such that

$$0 = A^{\mathrm{T}}P + PA + LL^{\mathrm{T}}, \tag{7}$$

$$PB = C^{\mathrm{T}}, \tag{8}$$

$$0 = A_c^{\mathrm{T}}P_c + P_cA_c + L_cL_c^{\mathrm{T}}, \tag{9}$$

$$P_cB_c = C_c^{\mathrm{T}}. \tag{10}$$

It is easy to see intuitively why the closed-loop system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x}, \quad \tilde{A} \triangleq \begin{bmatrix} A & -BC_c \\ B_cC & A_c \end{bmatrix} \tag{11}$$

is asymptotically stable, namely, because the phase shift of the loop transfer function (note the sign convention in (6)) is less than 180°. To see this from a Lyapunov function perspective, let $\tilde{P}$ satisfy

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}, \tag{12}$$

where

$$\tilde{R} = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^{\mathrm{T}} & R_2 \end{bmatrix} \tag{13}$$

is nonnegative definite. Expanding (12) with

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix} \tag{14}$$

yields

$$0 = A^{\mathrm{T}}P_1 + P_1A + (B_cC)^{\mathrm{T}}P_{12}^{\mathrm{T}} - P_{12}B_cC + R_1, \tag{15}$$

$$0 = A^{\mathrm{T}}P_{12} + P_{12}A_c + (B_cC)^{\mathrm{T}}P_2 - P_1BC_c + R_{12}, \tag{16}$$

$$0 = A_c^{\mathrm{T}}P_2 + P_2A_c - (BC_c)^{\mathrm{T}}P_{12} - P_{12}^{\mathrm{T}}BC_c + R_2. \tag{17}$$

If we set

$$R_1 = LL^{\mathrm{T}}, \; R_{12} = 0, \; R_2 = L_cL_c^{\mathrm{T}} \tag{18}$$

then (15)–(17) are satisfied by

$$P_1 = P,\ P_{12} = 0,\ P_2 = P_c. \tag{19}$$

To see that (16) is satisfied note that (8) and (10) imply

$$\begin{aligned}(B_cC)^{\mathrm{T}}P_2 - P_1BC_c &= (B_cC)^{\mathrm{T}}P_c - PBC_c \\ &= C^{\mathrm{T}}B_c^{\mathrm{T}}P_c - PBC_c \\ &= C^{\mathrm{T}}C_c^{\mathrm{T}} - C^{\mathrm{T}}C_c \\ &= 0.\end{aligned} \tag{20}$$

Hence the Kalman-Yacubovitch conditions yield

$$\tilde{P} = \begin{bmatrix} P & 0 \\ 0 & P_c \end{bmatrix}, \tag{21}$$

which shows that the Lyapunov function for the system "inherits" the Lyapunov function of the plant and compensator.

To contrast this situation with H_∞ theory, suppose $R_{12} = 0$ but that $(B_cC)^{\mathrm{T}}P_2 - P_1BC_c \neq 0$. Then P_{12} can be suppressed by letting

$$\|B_c\|,\ \|C_c\| << 1 \tag{22}$$

or

$$\max \mathrm{Re}\ \lambda(A_c) << 0. \tag{23}$$

However, (22) and (23) correspond to small gain for the feedback compensator. The phase result, however, does not require either (22) or (23). Thus we have shown that the Lyapunov function guaranteeing stability of this feedback interconnection has a particular internal structure. Since the stability is due to the phase properties of the plant and compensator, we can thus regard the Lyapunov structure as a manifestation of the phase aspects.

4.2 Ω-Bound Theory and Structured Covariances

Linear stochastic control theory is based on the second-moment statistic of the state variables. Letting Q denote the state covariance, in the steady state Q is given by the Lyapunov equation

$$0 = AQ + QA^{\mathrm{T}} + V. \tag{1}$$

Suppose now that A is uncertain, that is, A is replaced by $A + \Delta A$, where $\Delta A \in \mathcal{U}$, a given uncertainty set. Then (1) becomes

$$0 = (A + \Delta A)Q_{\Delta A} + Q_{\Delta A}(A + \Delta A)^{\mathrm{T}} + V. \tag{2}$$

To address (2) we introduce the notion of an Ω-bound which is a matrix function satisfying

$$\Delta A Q + Q \Delta A^{\mathrm{T}} \leq \Omega(Q), \text{ for all } Q \geq 0, \quad \Delta A \in \mathcal{U}. \tag{3}$$

That is, $\Omega(Q)$ bounds the uncertain terms in (2). We now consider the modified Lyapunov equation

$$0 = AQ + QA^{\mathrm{T}} + \Omega(Q) + V. \tag{4}$$

It is now easy to show that if (4) has a solution, then

$$Q_{\Delta A} \leq Q, \text{ for all } \Delta A \in \mathcal{U}. \tag{5}$$

The choice of Ω-bound will depend of course upon the uncertainty set $\mathcal{U}$. However, for a given set $\mathcal{U}$, there may be many Ω-bounds, and a "best" bound need not exist (they are only partially ordered). Two Ω-bounds that are convenient to work with are the *linear bound*

$$\Omega(Q) = \alpha Q + \alpha^{-1} \sum_{i=1}^{r} A_i Q A_i^{\mathrm{T}} \tag{6}$$

and *quadratic bound*

$$\Omega(Q) = D + QEQ. \tag{7}$$

By choosing a special quadratic bound, namely,

$$\Omega(Q) = \gamma^{-2} Q C^{\mathrm{T}} C Q \tag{8}$$

then (4) enforces an H_∞ norm bound (see [I.29]).

The problem with utilizing bounds such as (6) or (7) is that they may be extremely conservative. One reason these bounds are conservative is that (3) must be satisfied for all $Q \geq 0$ whether or not Q is the actual solution to (4). Moreover, these bounds may be conservative if the modeling uncertainty is large in magnitude but has bounded phase. Our approach to phase robustness theory was motivated by the stochastic theory developed in [II.1–II.12]. Using a multiplicative noise model with Stratonovich interpretation, Hyland proposed the Ω-operator

$$\hat{\Omega}(Q) = \sum_{i=1}^{r} [\tfrac{1}{2} A_i^2 Q + A_i Q A_i^{\mathrm{T}} + \tfrac{1}{2} Q A_i^{2\mathrm{T}}], \tag{9}$$

where, for a structural model in modal coordinates, each matrix A_i is a skew-symmetric matrix whose structure captures the effect of an uncertain modal frequency. A drawback of (9), however,

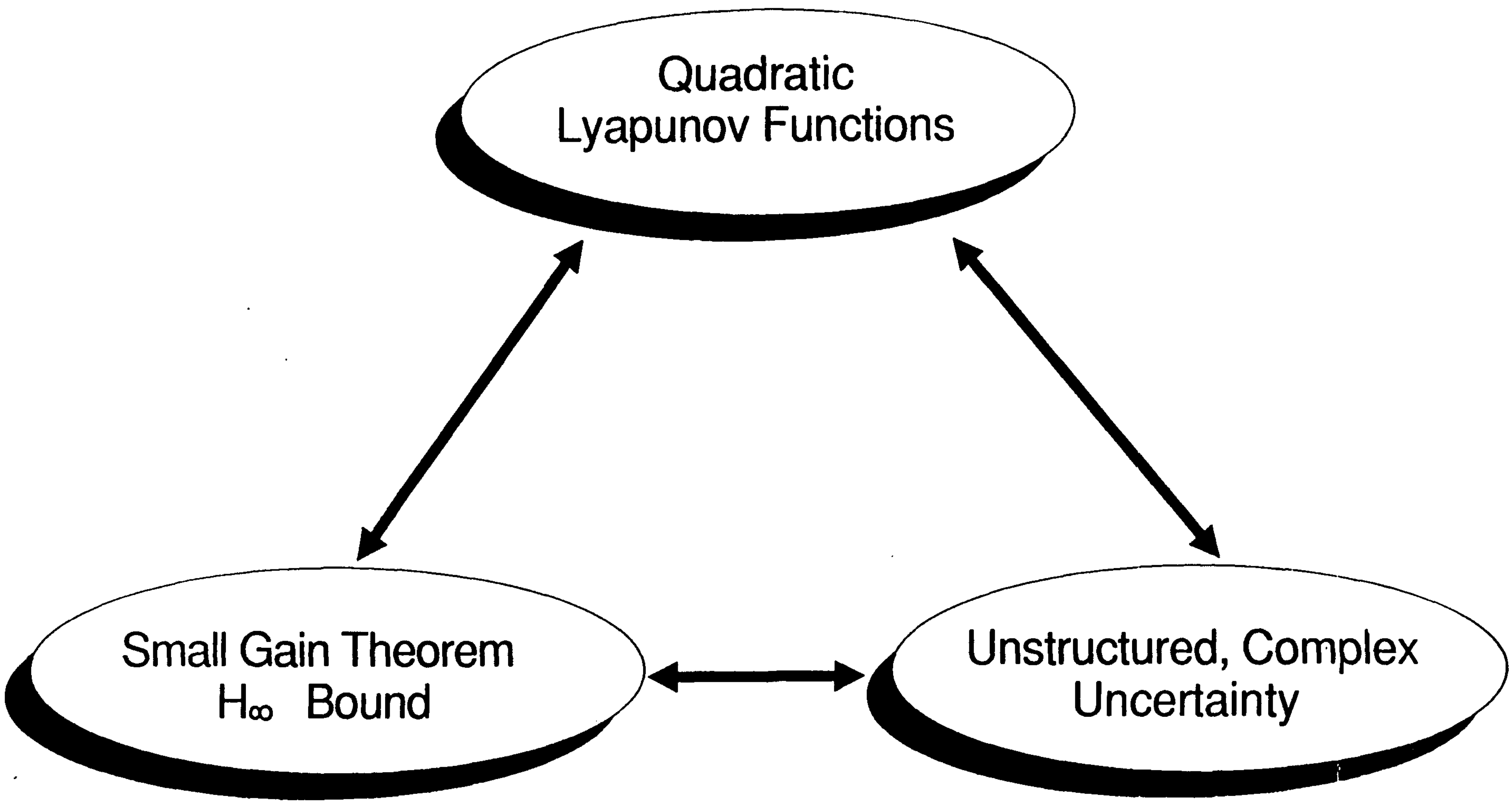


Figure 4.2–1. Small-gain results, or H_∞ theory, is related to unstructured Lyapunov functions and unstructured, complex uncertainty.

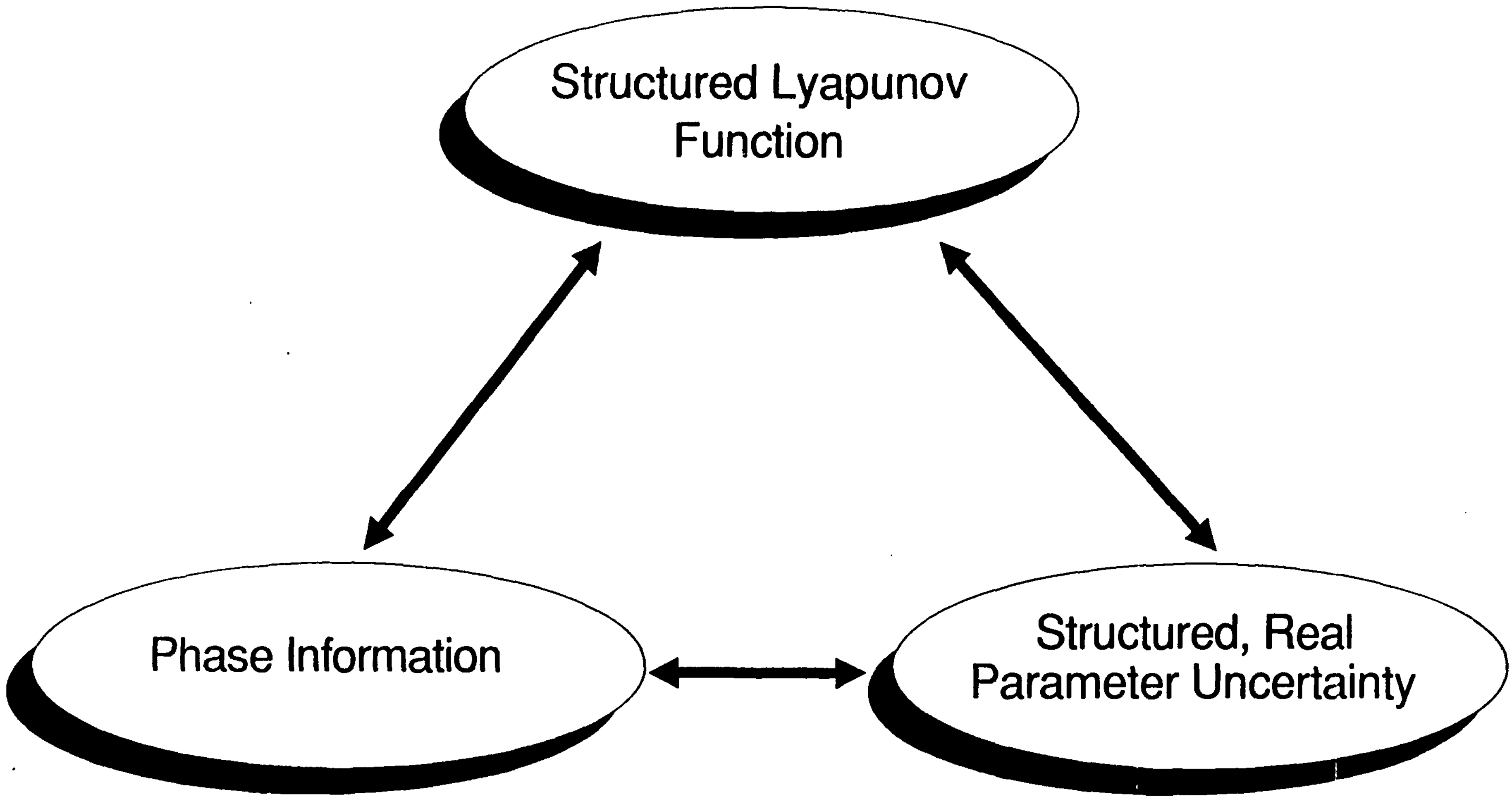


Figure 4.2–2. Phase information is manifested in structured Lyapunov functions and is directly related to structure, real-valued uncertainty.

is that $\hat{\Omega}(Q)$ is *indefinite*. Thus, in this case the modified Lyapunov equation (4) does *not* provide a bound for $Q_{\Delta A}$ and thus does not guarantee stability by means of standard techniques.

In summary, we note that there is an intricate interplay between phase information, real parameter uncertainty, and Lyapunov functions. The classical situation shown in Figure 4.2–1 is thus an extreme case of the more subtle situation addressed in Figure 4.2–2. Further discussion of these issues can be found in the paper "Real Parameter Uncertainty and Phase Information in the Robust Control of Flexible Structures," which appears in Appendix E.

4.3 Ω-Bounds for Positive Real Theory

To exploit the features of positive real transfer functions, we have developed a theory of robust controller synthesis with positive real uncertainty. The phase-bounded character of positive real transfer functions entails far less conservatism than small gain or H_∞ results when addressing real parameter uncertainty.

The results obtained thus far are detailed in the paper "Robust Stabilization with Positive Real Uncertainty: Beyond the Small Gain Theorem," which appears in Appendix F. This paper develops a state space theory of positive real transfer functions in terms of an algebraic Riccati equation. This characterization is more direct than the usual KYP characterization and provides the basis for state space controller synthesis techniques in the spirit of state space H_∞ theory.

More recently we have linked positive real theory with Ω-bound theory by showing that robust stability and robust H_∞ performance in the presence of positive real uncertainty are guaranteed by means of an Ω-bound. This connection has ramifications for nonlinear control. To see this, we recall that robust stability in the presence of sector-bounded nonlinearities is equivalent to a Nyquist circle criterion, which is equivalent to a positive real condition. Thus robustness to positive real uncertainty provides the means to guarantee stability with respect to a class of nonlinearities. Similar observations hold for the Popov criterion which also guarantees robustness for sector nonlinearities.

Our results provide the means for going beyond existing results in two respects. First, we can develop multivariable generalizations of the classical circle and Popov criteria using simplified Ω-bound theory. And, second, our techniques can be used for robust synthesis in addition to analysis as addressed by standard theory.

5.0 Optimal Nonlinear Feedback Control

5.0 Optimal Nonlinear Feedback Control

The methods and results discussed in Sections 2–5 are independent of optimality considerations. The purpose of this section is to discuss progress in developing an optimality-based control theory involving nonlinear controllers for linear and nonlinear plants.

As pointed out in Section 2, controller synthesis need not be based upon optimality criteria. For example, a controller can be constructed in accordance with a Lyapunov function to achieve stability, energy dissipation, etc. Nevertheless, there is strong motivation for developing an optimality-based theory.

Perhaps the prime motivations for developing an optimality-based theory is the success of such approaches in linear control theory. The well-known linear-quadratic-Gaussian control theory (LQG) is the major result in modern optimal multivariable feedback control theory. During the past decade, LQG theory has been extended to address numerous practical control-design issues such as disturbance attenuation, robustness, controller order, and pole placement (Figure 5–1). The resulting theory, known as Optimal Projection for Uncertain Systems (OPUS), has been extensively developed (see the reference list in Appendix B).

The second motivation for optimal nonlinear control theory is that it can drive the controller synthesis procedure within a class of candidate controllers. Specifically, as will be discussed later in this section, we can view a given Lyapunov function as providing the *framework* for controller synthesis by guaranteeing local or global asymptotic stability theory for a class of feedback controllers. The *actual* controller chosen for implementation can thus be the member of this candidate class that minimizes a specified performance function. The form of this functional is usually closely related to the structure of the Lyapunov function. In LQG theory, for example, the Lyapunov function is the familiar quadratic function $V(x) = x^{\mathrm{T}} Px$, while the gains are chosen to minimize a performance functional of the form $J = \operatorname{tr} PV$. In summary, then, Lyapunov function theory provides the *framework*, while optimization fixes the gains.

5.1 Optimal Nonlinear Feedback Control via Steady-State HJB Theory

The classical approach to optimal nonlinear control is to invoke the Maximum Principle. This result has been successful in characterizing solutions to problems such as minimum time control. Since the Maximum Principle does not explicitly guarantee stability via a Lyapunov function per se and does not directly lead to feedback controllers, we shall not adopt it as our principal

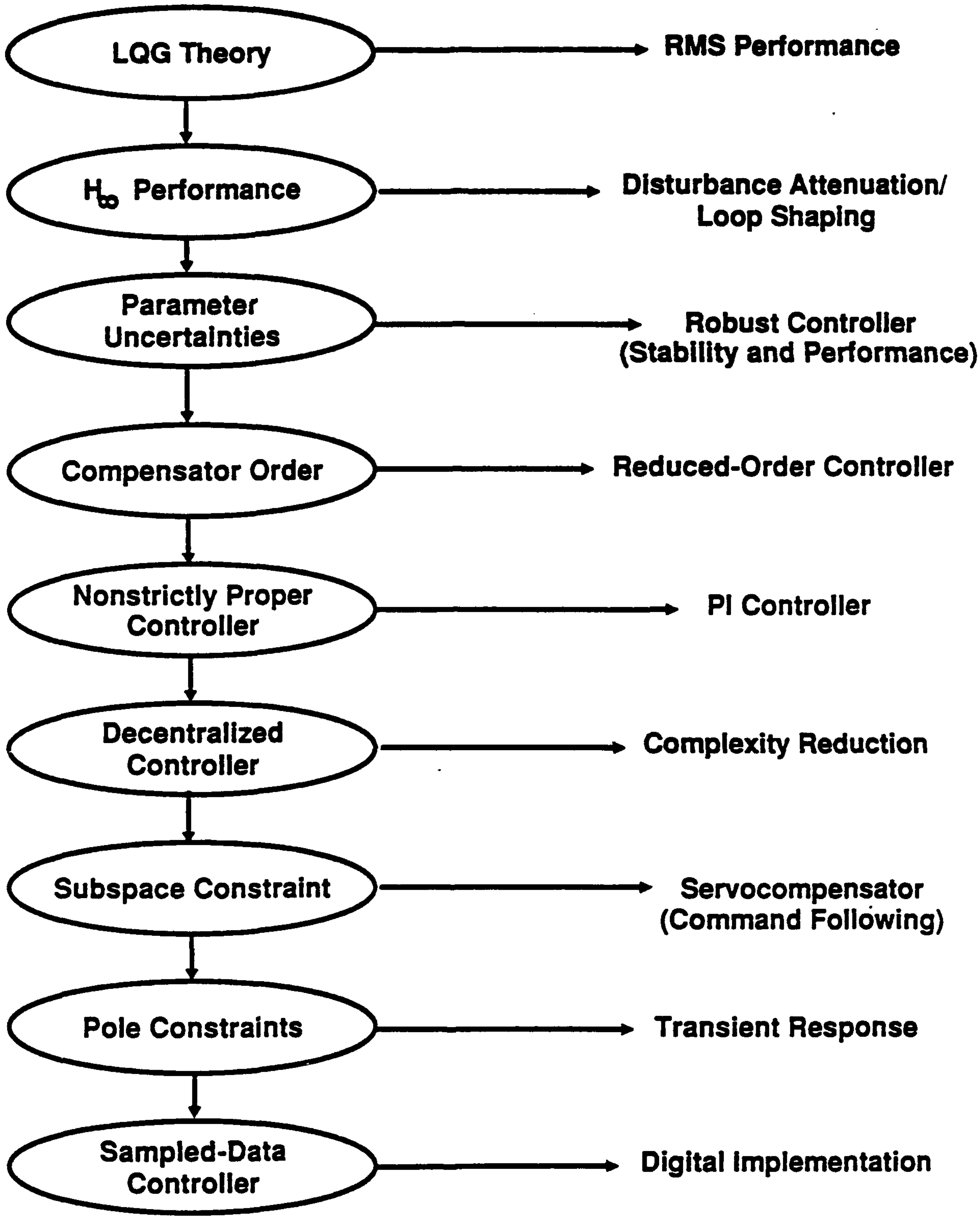


Figure 5–1. Optimal Projection for Uncertain Systems (OPUS) is an optimal linear control theory that systematically addresses a broad range of practical control design issues.

approach. Nevertheless, we bear in mind that the Maximum Principle does have links with the Hamilton-Jacobi-Bellman (HJB) approach which we *shall* consider and to which we now turn.

Hamilton-Jacobi-Bellman theory has its roots in the classical Hamilton-Jacobi partial differential equation as well as the dynamic programming technique of Bellman. In its most general form, the theory involves a partial differential equation whose solution yields an optimal controller. In recent years, this equation has attracted renewed interest with the discovery of generalized solutions [151,152].

If, in accordance with practical motivations, we restrict our attention to time-invariant systems on the infinite horizon with analytic data, the situation is considerably simplified. In this case the HJB partial differential equation reduces to a purely algebraic relationship.

To summarize the ideas involved we first consider the problem of evaluating a nonquadratic cost functional depending upon a nonlinear differential equation. It turns out that the cost functional can be evaluated in closed form so long as the cost functional is related in a specific way to an underlying Lyapunov function. The basis for the following development is the paper [60] by Bass and Weber. A more detailed treatment of these results is given in the paper "Nonquadratic Cost and Nonlinear Feedback Control" which appears in Appendix G.

For simplicity in the exposition here, we shall define all functions globally and assume that existence and uniqueness properties of the given differential equations are satisfied.

For the following result, let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $L\colon \mathbb{R}^n \to \mathbb{R}$. We assume $f(0) = 0$.

Lemma 1. Consider the system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{1}$$

with performance functional

$$J(x_0) = \int_0^\infty L(x(t))\mathrm{d}t. \tag{2}$$

Assume that

$$L(x) > 0,\ x \in \mathbb{R}^n,\ x \neq 0, \tag{3}$$

and assume there exists a C^1 function $V\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \tag{4}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{5}$$

$$L(x) = -V'(x)f(x), \quad x \in \mathbb{R}^n. \tag{6}$$

Then $x = 0$ is a globally asymptotically stable solution of (1) and, furthermore,

$$J(x_0) = V(x_0). \tag{7}$$

Proof. Let $x(t)$ satisfy (1). Then

$$\dot{V}(x(t)) \triangleq \frac{d}{dt}V(x(t)) = V^{-1}(x(t))f(x(t)). \tag{8}$$

Hence it follows from (6) that

$$\dot{V}(x(t)) = -L(x(t)).$$

Now (3) implies that

$$\dot{V}(x(t)) < 0, \quad x(t) \neq 0.$$

Since $V(x) > 0$, $x \neq 0$, it follows that $V(\cdot)$ is a Lyapunov function for (1) and that $x(t) \to 0$ as $t \to \infty$. Thus proves global asymptotic stability of the solution $x = 0$. Now (8) implies that

$$\begin{aligned} V(x(t)) - V(x_0) &= \int_0^t V'(x(s))f(x(s))\mathrm{d}s \\ &= -\int_0^t L(x(s))\mathrm{d}s. \end{aligned}$$

Letting $t \to \infty$ and noting $V(x(t)) \to 0$, it follows that

$$-V(x_0) = -\int_0^\infty L(x(t))\mathrm{d}t,$$

or, equivalently,

$$V(x_0) = J(x_0). \quad \square$$

The main feature of Lemma 1 is the role played by the Lyapunov function $V(x)$ in guaranteeing stability and for evaluating the functional $J(x_0)$. It can be recognized that $V(x)$ is the cost-to-go function in dynamic programming.

Let us illustrate Lemma 1 with a familiar example. Consider the linear system

$$\dot{x} = Ax, \quad x(0) = x_0, \tag{9}$$

with cost functional

$$J(x_0) = \int_0^\infty x^{\mathrm{T}} Rx \, \mathrm{d}t, \tag{10}$$

where $R \in \mathbb{R}^{n\times n}$ is positive-definite. If A is stable then there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$0 = A^{\mathrm{T}}P + PA + R. \tag{11}$$

Now define

$$V(x) = x^{\mathrm{T}}Px, \tag{12}$$

which satisfies (4) and (5). Furthermore, with $f(x) = Ax$ and $L(x) = x^{\mathrm{T}}Rx$ it follows that

$$\begin{aligned} V'(x)f(x) &= 2x^{\mathrm{T}}PAx \\ &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x \\ &= -x^{\mathrm{T}}Rx \\ &= -L(x) \end{aligned}$$

which verifies (6). Hence

$$J(x_0) = x_0^{\mathrm{T}}Px_0,$$

which is a familiar result from linear-quadratic theory.

To deal with more general situations, the following lemma, which appears in [60], will be useful.

Lemma 2. Let $A \in \mathbb{R}^{n\times n}$ be asymptotically stable and let h: $\mathbb{R}^n \to \mathbb{R}$ be a nonnegative-definite homogeneous p-form (p even). Then there exists a nonnegative-definite homogeneous p-form g: $\mathbb{R}^n \to \mathbb{R}$ such that

$$g'(x)Ax + h(x) = 0, \quad x \in \mathbb{R}^n. \tag{13}$$

In the quadratic case (13) yields the familiar result. To see this let $h(x) = x^{\mathrm{T}}Rx$ and $g(x) = x^{\mathrm{T}}Px$. Then (13) becomes

$$2x^{\mathrm{T}}PAx + x^{\mathrm{T}}Rx = 0,$$

or

$$x^{\mathrm{T}}(A^{\mathrm{T}}P + PA + R)x = 0,$$

which is satisfied by P given by (11). Now consider the nonquadratic cost functional

$$J(x_0) = \int_0^\infty [x^{\mathrm{T}}Rx + h(x)]\mathrm{d}t, \tag{14}$$

where

$$h(x) = \sum_{\nu=1}^{r} h_{2\nu}(x) \tag{15}$$

and, for $\nu = 1, \ldots, r$, $h_{2\nu}$: $\mathbb{R}^n \to \mathbb{R}$ is a nonnegative-definite homogeneous 2ν-form. We continue to assume that $x(t)$ satisfies (9), where A is stable. Now, let $g_{2\nu}$: $\mathbb{R}^n \to \mathbb{R}$ be the nonnegative-definite homogeneous 2ν-form satisfying

$$g'_{2\nu}(x)Ax + h_{2\nu}(x) = 0, \quad x \in \mathbb{R}^n, \quad \nu = 1, \ldots, r, \tag{16}$$

and define

$$g(x) = \sum_{\nu=1}^{r} g_{2\nu}(x). \tag{17}$$

Note that (15)–(17) imply

$$g'(x)Ax + h(x) = 0, \quad x \in \mathbb{R}^n. \tag{18}$$

Furthermore, define the positive-definite function

$$V(x) = x^{\mathrm{T}}Px + g(x), \tag{19}$$

where P satisfies (11). Now, to verify (6) we note that

$$\begin{aligned} V'(x)f(x) &= [2x^{\mathrm{T}}P + g'(x)]Ax \\ &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + \sum_{\nu=1}^{r} g'_{2\nu}(x)Ax \\ &= -x^{\mathrm{T}}Rx - \sum_{\nu=1}^{r} h_{2\nu}(x) \\ &= -L(x). \end{aligned}$$

Hence for $J(x_0)$ given by (14) we obtain

$$J(x_0) = V(x_0) = x_0^{\mathrm{T}}Px_0 + g(x_0). \tag{20}$$

Next consider in place of (9) the case in which the plant is nonlinear, for example,

$$\dot{x} = Ax + \sigma(x), \quad x(0) = x_0, \tag{21}$$

where $\sigma(0) = 0$ and we continue to assume that A is stable. Again, let g: $\mathbb{R}^n \to \mathbb{R}$ be given by (16) and (17) and define $V(x)$ by means of (19). It remains only to verify (6). Hence

$$\begin{aligned} V'(x)f(x) &= [2x^{\mathrm{T}}P + g'(x)][Ax + \sigma(x)] \\ &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + g'(x)Ax + [2x^{\mathrm{T}}P + g'(x)]\sigma(x) \\ &= -[x^{\mathrm{T}}Rx + h(x)] + [2x^{\mathrm{T}}P + g'(x)]\sigma(x) \\ &= -\{L(x) - [2x^{\mathrm{T}}P + g'(x)]\sigma(x)\}. \end{aligned}$$

Hence we see that (6) is *not* generally satisfied. However, we can salvage the situation be considering an auxiliary cost functional

$$\hat{J}(x_0) \triangleq \int_0^\infty \hat{L}(x(t))\mathrm{d}t, \tag{22}$$

where

$$\hat{L}(x) \triangleq L(x) - [2x^{\mathrm{T}}P + g'(x)]\sigma(x). \tag{23}$$

With this modification (3) must be satisfied with $L(x)$ replaced by $\hat{L}(x)$, that is,

$$L(x) > [2x^{\mathrm{T}}P + g'(x)]\sigma(x) \tag{24}$$

In the special case that

$$[2x^{\mathrm{T}}P + g'(x)]\sigma(x) \leq 0, \tag{25}$$

it follows that (24) is automatically satisfied (if (3) is satisfied) and, furthermore,

$$J(x_0) \leq \hat{J}(x_0). \tag{26}$$

That is, the auxiliary cost is an upper bound for the original cost.

By only a slight extension of Lemma 1, we can obtain sufficient conditions for characterizing optimal feedback controllers. Now let f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, where $f(0,0) = 0$, let L: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and define for $p \in \mathbb{R}^n$

$$H(x,p,u) \triangleq L(x,u) + p^{\mathrm{T}} f(x,u).$$

Theorem 1. Consider the controlled system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \tag{27}$$

with performance functional

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\mathrm{d}t. \tag{28}$$

Assume that there exist a C^1 function V: $\mathbb{R}^n \to \mathbb{R}$ and a function ϕ: $\mathbb{R}^n \to \mathbb{R}^m$ such that

$$V(0) = 0 \tag{29}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{30}$$

$$L(x, \phi(x)) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{31}$$

$$H(x, V'^{\mathrm{T}}(x), \phi(x)) = 0, \quad x \in \mathbb{R}^n, \tag{32}$$

$$H(x, V'^{\mathrm{T}}(x), u) \geq 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \tag{33}$$

Then with the feedback control $u(\cdot) = \phi(x(\cdot))$, the solution $x = 0$ of the closed-loop system is asymptotically stable and

$$J(x_0, \phi(x(\cdot)) = V(x_0). \tag{33}$$

Furthermore, the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$, that is,

$$J(x_0, \phi(x(\cdot)) = \min_{u(\cdot)} J(x_0, u(\cdot)). \tag{34}$$

Proof. Global asymptotic stability and result (33) follow directly from Lemma 1. We need only note that (31) can be written as

$$L(x, \phi(x)) = -V(x)f(x, \phi(x)), \quad x \in \mathbb{R}^n,$$

which corresponds to (6). It remains only to prove (34) using condition (32). For arbitrary $u(t)$ and for $x(t)$ satisfying (15) we have

$$\dot{V}(x(t)) = V'(x(t))f(x(t), u(t))$$

or

$$0 = -\dot{V}(x(t)) + V'(x(t))f(x(t), u(t)).$$

Hence

$$\begin{aligned} L(x(t), u(t)) &= -\dot{V}(x(t) + L(x(t), u(t) + V'(x(t)f(x(t), u(t)) \\ &= -\dot{V}(x(t) + H(x(t), V'^{\mathrm{T}}(x(t)), u(t)). \end{aligned}$$

Now using (32) and (33) we obtain

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty [-\dot{V}(x(t) + H(x(t), V'^{\mathrm{T}}(x(t)), u(t)]\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x(t)) + V(x_0) + \int_0^\infty H(x(t), V'^{\mathrm{T}}(x(t)), u(t))\mathrm{d}t \\ &= V(x_0) + \int_0^\infty H(x(t), V'^{\mathrm{T}}(x(t), u(t))\mathrm{d}t \\ &\geq V(x_0) \\ &= J(x_0, \phi(x(\cdot)), \end{aligned}$$

which yields (33). □

The principal feature of Theorem 1 is that the optimal control law $u = \phi(x)$ is a *feedback* controller. Furthermore, this control is optimal independently of the initial condition x_0.

Now let us illustrate Theorem 1 with some examples. We begin with the simplest case, namely, the linear quadratic regulator. Hence consider the controlled system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{36}$$

with performance functional

$$J(x_0, u(\cdot)) = \int_0^\infty [x^{\mathrm{T}} R_1 x + u^{\mathrm{T}} R_2 u] \mathrm{d}t, \tag{37}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and where R_1 and R_2 are positive definite. Define the feedback law

$$\phi(x) = -R_2^{-1} B^{\mathrm{T}} P x, \tag{38}$$

where P satisfies

$$0 = A^{\mathrm{T}} P + PA + R_1 - PSP, \tag{39}$$

where $S \triangleq BR_2^{-1}B^{\mathrm{T}}$. With $u = \phi(x)$, the closed-loop system (36) becomes

$$\dot{x} = \tilde{A}x, \quad x(0) = x_0, \tag{40}$$

where $\tilde{A} \triangleq A - SP$, while (37) and (39) can be written as

$$J(x_0, \phi(x(\cdot)) = \int_0^\infty x^{\mathrm{T}} \tilde{R} x \, \mathrm{d}t \tag{41}$$

and

$$0 = \tilde{A}^{\mathrm{T}} P + P\tilde{A} + \tilde{R}, \tag{42}$$

where $\tilde{R} \triangleq R_1 + PSP$. Thus the closed-loop system (40) with cost (41) has exactly the form of the example considered in (9)–(12). It remains only to show that $u = \phi(x)$ is the optimal control, which will be the case if (33) is satisfied. To show this, note that

$$\begin{aligned}
H(x, V'^{\mathrm{T}}(x), u) &= x^{\mathrm{T}} R_1 x + u^{\mathrm{T}} R_2 u + 2x^{\mathrm{T}} P(Ax + Bu) \\
&= x^{\mathrm{T}} R_1 x + u^{\mathrm{T}} R_2 u + x^{\mathrm{T}}(A^{\mathrm{T}} P + PA)x + 2x^{\mathrm{T}} PBu \\
&= x^{\mathrm{T}} PSPx + 2x^{\mathrm{T}} PBu + u^{\mathrm{T}} R_2 u \\
&= [R_2^{-1} B^{\mathrm{T}} Px + u]^{\mathrm{T}} R_2 [R_2^{-1} B^{\mathrm{T}} Px + u] \\
&\geq 0.
\end{aligned}$$

Note that it is now easy confirm (32) by setting $u = \phi(x) = -R_2^{-1} B^{\mathrm{T}} Px$.

We now apply Theorem 1 to an optimal control problem involving a nonquadratic cost. Hence consider the linear system (35) with cost

$$J(x_0, u(\cdot)) = \int_0^\infty [x^{\mathrm{T}} R_1 x + h(x) + u^{\mathrm{T}} R_2 u] \mathrm{d}t, \tag{43}$$

where $h(x)$ is given by (15). We shall consider a control law of the form

$$u = \phi(x) = \phi_L(x) + \phi_{NL}(x), \tag{44}$$

where $\phi_L(x)$ and $\phi_{NL}(x)$ are linear and nonlinear, respectively. Let $\phi_L(x)$ agree with the linear-quadratic solution, that is,

$$\phi_L(x) \triangleq -R_2^{-1} B^{\mathrm{T}} P x, \tag{45}$$

where P satisfies

$$0 = A^{\mathrm{T}} P + PA + R_1 - PSP. \tag{46}$$

Recall that (46) can be written as

$$0 = \tilde{A}^{\mathrm{T}} P + P\tilde{A} + \tilde{R}, \tag{47}$$

where $\tilde{A} \triangleq A - SP$ and $\tilde{R} \triangleq R_1 + PSP$.

For the nonlinear control $\phi_{NL}(x)$ let $g(x)$ be given by (17) where $g_{2\nu}(x)$ satisfies

$$g'_{2\nu}(x)\tilde{A}x + h_{2\nu}(x), \quad x \in \mathbb{R}^n, \quad \nu = 1, \ldots, r, \tag{48}$$

which is the same as (16) with A replaced by $\tilde{A}$. Now define

$$\phi_{NL}(x) = -\frac{1}{2} R_2^{-1} B^{\mathrm{T}} g'^{\mathrm{T}}(x) \tag{49}$$

and the Lyapunov function

$$V(x) = x^{\mathrm{T}} P x + g(x). \tag{50}$$

Next note that for $L(x, u)$ as in (43) we have

$$\begin{aligned} L(x, \phi(x)) &= x^{\mathrm{T}} R_1 x + h(x) + \phi^{\mathrm{T}}(x) R_2 \phi(x) \\ &= x^{\mathrm{T}} \tilde{R} x + h(x) + x^{\mathrm{T}} P S g'^{\mathrm{T}}(x) + \frac{1}{4} g'(x) S g'^{\mathrm{T}}(x). \end{aligned} \tag{51}$$

Furthermore, with $u = \phi(x)$ the system (36) becomes

$$\dot{x} = \tilde{A}x + B\phi_{NL}(x), \quad x(0) = x_0, \tag{52}$$

or

$$\dot{x} = \tilde{A}x - \frac{1}{2}Sg'^{\mathrm{T}}(x), \quad x(0) = x_0. \tag{53}$$

Returning to Theorem 1, it is clear that (29)–(31) are satisfied. However, note that

$$\begin{aligned} V'(x)f(x,\phi(x)) &= [2x^{\mathrm{T}}P + g'(x)][\tilde{A}x - \frac{1}{2}Sg'^{\mathrm{T}}(x)] \\ &= x^{\mathrm{T}}(\tilde{A}^{\mathrm{T}}P + P\tilde{A})x + g'(x)\tilde{A}x - x^{\mathrm{T}}PSg'^{\mathrm{T}}(x) - \frac{1}{2}g'(x)Sg'^{\mathrm{T}}(x) \\ &= -\big[x^{\mathrm{T}}\tilde{R}x + h(x) + x^{\mathrm{T}}PSg'^{\mathrm{T}}(x) + \frac{1}{2}g'(x)Sg'^{\mathrm{T}}(x)\big] \\ &= -\big[L(x,\phi(x)) + \frac{1}{4}g'(x)Sg'^{\mathrm{T}}(x)\big], \end{aligned}$$

so that

$$H(x, V^{-1}(x), \phi(x)) = -\frac{1}{4}g'(x)Sg'^{\mathrm{T}}(x), \tag{54}$$

which shows that (32) is *not* satisfied. However, if we define

$$\hat{L}(x,u) \triangleq L(x,u) + \frac{1}{4}g'(x)Sg'^{\mathrm{T}}(x), \tag{55}$$

then the auxiliary cost

$$\hat{J}(x_0, u(\cdot)) \triangleq \int_0^\infty \hat{L}(x(t), u(t))\mathrm{d}t \tag{56}$$

satisfies

$$J(x_0, u(\cdot)) \leq \hat{J}(x_0, u(\cdot)). \tag{57}$$

Finally, be defining

$$\hat{H}(x, V'^{\mathrm{T}}(x), u) \triangleq \hat{L}(x,u) + V'^{\mathrm{T}}(x)f(x,u), \tag{58}$$

it can be shown that

$$\hat{H}(x, V'^{\mathrm{T}}(x), u) = [u - \phi(x)]^{\mathrm{T}}R_2[u - \phi(x)]. \tag{59}$$

Hence (33) holds with $H(\cdot)$ replaced by $\hat{H}(\cdot)$. Consequently,

$$\begin{aligned} \hat{J}(x_0, \phi(x(\cdot)) &= V(x_0) \\ &= x_0^{\mathrm{T}}Px_0 + g(x_0) \\ &= \min_{u(\cdot)} \hat{J}(x_0, u(\cdot)). \end{aligned} \tag{60}$$

We next consider a special case of the above nonquadratic problem that leads to considerable simplification. This particular problem was considered in [63]. Suppose we require that $V(x)$ be of the form

$$V(x) = x^{\mathrm{T}}Px + \frac{1}{2}(x^{\mathrm{T}}Mx)^2 \tag{61}$$

so that $g(x) = \frac{1}{2}(x^{\mathrm{T}}Mx)^2$ where P satisfies (39) and M satisfies

$$0 = (A - SP)^{\mathrm{T}}M + M(A - SP) + R_1 + MSM. \tag{62}$$

Then $\phi(x)$ has the form

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}Px - R_2^{-1}B^{\mathrm{T}}(x^{\mathrm{T}}Mx)Mx. \tag{63}$$

Next we assume that $h(x)$ is given by

$$h(x) = (x^{\mathrm{T}}Mx)x^{\mathrm{T}}(R_1 + MSM)x. \tag{64}$$

With these definitions we note that

$$\begin{aligned} g'(x)\tilde{A}x &= (x^{\mathrm{T}}Mx)2x^{\mathrm{T}}M\tilde{A}x \\ &= (x^{\mathrm{T}}Mx)x^{\mathrm{T}}(\tilde{A}^{\mathrm{T}}M + M\tilde{A})x \\ &= (x^{\mathrm{T}}Mx)x^{\mathrm{T}}(R_1 + MSM)x \\ &= h(x), \end{aligned}$$

which verifies (48). Finally, define

$$L(x, u) = x^{\mathrm{T}}R_1x + h(x). \tag{65}$$

Following the previous development, we see that $\phi(x)$ given by (62) minimizes $\hat{J}(x_0, u(\cdot))$ defined by (55), where

$$\hat{L}(x, u) = x^{\mathrm{T}}R_1x + (x^{\mathrm{T}}Mx)x^{\mathrm{T}}(R_1 + MSM)x + (x^{\mathrm{T}}Mx)^2x^{\mathrm{T}}MSMx + u^{\mathrm{T}}R_2u. \tag{66}$$

Thus, be minimizing a sixth-order cost functional, the optimal control is a cubic feedback characterized by a pair of Riccati equations. The cost functional is somewhat artificial since it depends upon the solution of one of the Riccati equations.

We have thus shown that the problem considered by Speyer in [63] is a special case of the optimal nonquadratic cost problem addressed by Bass and Weber in [60]. Actually, however, the formulation of Speyer was a stochastic control problem based upon results of Wonham [153]. This formulation involves systems of the form

$$\dot{x} = Ax + Bu + D_1w, \tag{67}$$

where $D_1 w$ denotes additive white noise disturbances. (Speyer also considered multiplicative noise in [63] as well.) These disturbances lead to a modification of (46) of the form

$$0 = A^T P + PA + R_1 - PSP + (\text{tr } MV_1)M + 2MV_1 M, \tag{68}$$

Now note that (62) and (68) now constitute a pair of coupled Riccati equations.

Having reviewed the elements of a deterministic HJB theory as originated by Bass and Weber, out next goal is to develop a corresponding theory of stochastic control. Such a theory can be used for disturbance rejection for persistent disturbances. Our principal goal, however, is to generalize HJB theory to permit the design of fixed-structure controllers that operate on the available, possibly noisy, measurements. To our knowledge, no such theory currently exists, while progress in this direction is crucial for practical application of nonlinear control laws.

Appendix A
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Appendix B
List of Publications

I. Journal and Book Articles

I.1 D. C. Hyland and D. S. Bernstein, "The Optimal Projection Equations for Fixed–Order Dynamic Compensation," *IEEE Trans. Autom. Contr.*, Vol. AC–29, pp. 1034-1037, 1984.

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I.55 D. C. Hyland and D. S. Bernstein, "Power Flow, Energy Balance, and Statistical Energy Analysis for Large Scale Interconnected Systems," submitted to *J. Sound Vibr.*

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II. Conferences and Technical Reports

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Appendix C
"Power Flow, Energy Balance, and Statistical Energy Analysis for Large-Scale Interconnected Systems"

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Power Flow, Energy Balance, and Statistical Energy Analysis for Large-Scale Interconnected Systems

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Abstract

It is well known from thermodynamics that energy flows from hot objects to cold objects. It is less well known, however, that a similar phenomenon occurs in coupled mechanical systems with modal energy playing the role of temperature. Energy flow among coupled modes is the subject of Statistical Energy Analysis (SEA). Originally motivated by problems in acoustics involving numerous vibrational modes, SEA is based upon equations governing energy flow among individual modes or sets of modes. Such energy flow equations can be quite efficient in modeling the response of lightly damped structures. This paper has two goals. First, we derive a generalized formulation of power flow which allows *arbitrary* coupling of *arbitrary* strength. Previous theoretical results were limited to either identical couplings or weak interactions. These new results utilize Kronecker matrix algebra to derive an energy flow equation involving the diagonal elements of the solution to a Lyapunov equation. Analysis of the resulting equations, based upon M-matrix theory, yields generalized energy balance relations in the case of weak but arbitrary (possibly nonconservative) couplings.

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Notation

$\mathbb{E}$	expectation
$\mathbb{R}, \mathbb{C}$	real field, complex field
$\mathbb{R}^{r\times s}, \mathbb{C}^{r\times s}$	$r \times s$ real, complex matrices
$\mathbb{R}^{r}, \mathbb{C}^{r}$	$\mathbb{R}^{r\times 1}, \mathbb{C}^{r\times 1}$ (column vectors)
I_r or I	$r \times r$ identity matrix
j	$\sqrt{-1}$
$A_{k\ell}$	(k, ℓ)-element of $A \in \mathbb{C}^{r\times s}$
Re A, Im A	real, imaginary part of $A \in \mathbb{C}^{r\times s}$
$\bar{A}, A^{\mathrm{T}}, A^{*}$	complex conjugate, transpose, complex conjugate transpose of $A \in \mathbb{C}^{r\times s}$
$\{A\}, \langle A\rangle$	diagonal, off-diagonal part of $A \in \mathbb{C}^{r\times r}$ (see Section 2)
$\otimes, \oplus$, vec, vecd	See Appendix B
$A \geq\geq 0$	$A \in \mathbb{R}^{r\times s}$ is nonnegative (each entry of A is nonnegative)

1. Introduction

We are concerned with efficient methods for evaluating the steady-state statistical response of large-scale linear systems composed of many interconnected, high-dimensional subsystems. This problem arises from applications involving acoustical response, acoustical/structural interaction, high frequency vibration of mechanical systems, and dynamics and control of large space structures [1–25]. To illustrate the problem, suppose that each subsystem is well known and precisely characterized so that its eigenbasis is known. Then, assuming for convenience a semisimple eigenstructure, the kth subsystem considered in isolation is of the form

$$\dot{x}_k = \Lambda_k x_k + w_k, \quad k = 1, \ldots, r, \tag{1.1}$$

where, for $k = 1, \ldots, r$,

$$x_k \in \mathbb{C}^{n_k}, \quad \Lambda_k \triangleq \operatorname*{diag}_{i=1,\ldots,n_k} (\lambda_{ki}), \quad \lambda_{ki} \in \mathbb{C},$$

and w_k is a white noise process with Hermitian nonnegative-definite intensity $V_k \in \mathbb{C}^{n_k \times n_k}$. When the subsystems are interconnected, couplings are introduced among the subsystems in the form of perturbations to the individual subsystems. The subsystem dynamics in the interconnected case are given by

$$\dot{x}_k = \Lambda_k x_k + g_{kk} x_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{r} g_{k\ell} x_\ell + w_k, \quad k = 1, \ldots, r. \tag{1.2}$$

The matrix $g_{k\ell} \in \mathbb{C}^{n_k \times n_\ell}$, $k \neq \ell$, represents the effect of the ℓth subsystem on x_k, while the matrix $g_{kk} \in \mathbb{C}^{n_k \times n_k}$ represents an effective shift of Λ_k due to the interconnections. Our goal is to determine the steady-state second-moment response of the interconnected systems.

In the case of a large flexible structure consisting of several interconnected substructures, (1.1) represents the kth substructure, while the coupling terms in (1.2) arise from the mechanical interconnections among the substructures. Alternatively, in the case of acoustical/structural interaction, one wishes to predict the acoustical response of several acoustical spaces separated by elastic partitions (such as walls). Equation (1.1) then represents the modal dynamics of each acoustical chamber, while the coupling terms in (1.2) represent the dynamic couplings introduced by the elastic partitions. The power flow concept also has close connections with thermodynamics and circuit theory [26–34].

The problem posed by (1.2) is subsumed in the linear system model

$$\dot{x} = (-\nu + j\Omega)x + (H + G)x + w, \tag{1.3}$$

where

$$x \in \mathbb{C}^n,$$

$$\nu \triangleq \operatorname*{diag}_{k=1,\ldots,n} (\nu_k), \quad \nu_k \in \mathbb{R}, \quad \nu_k > 0, \quad k = 1, \ldots, n,$$

$$\Omega \triangleq \operatorname*{diag}_{k=1,\ldots,n} (\Omega_k), \quad \Omega_k \in \mathbb{R}, \quad k = 1, \ldots, n,$$

$$H \triangleq \operatorname*{diag}_{k=1,\ldots,n} (H_k), \quad H_k \in \mathbb{C}, \quad k = 1, \ldots, n,$$

$$G_{kk} = 0, \quad k = 1, \ldots, n, \quad G \in \mathbb{C}^{n \times n},$$

and w is white noise with Hermitian nonnegative-definite intensity $V \in \mathbb{C}^{n \times n}$. The diagonal matrix $-\nu + j\Omega \in \mathbb{C}^{n \times n}$ where $n = \sum_{\ell=1}^{r} n_\ell$ is a concatenation of all of the uncoupled subsystems in (1.1). The matrices H and G represent, respectively, the diagonal and off-diagonal portions of the perturbations due to subsystem interaction. We assume that the system (1.3) is asymptotically stable, that is, the spectrum of the matrix $-\nu + j\Omega + H + G$ is contained in the open left half plane. To study the steady-state, mean-square response of the system (1.3), suppose y defined by

$$y = Cx \tag{1.4}$$

is a response variable of interest, where $C \triangleq [C_1 \cdots C_n] \in \mathbb{C}^{1 \times n}$. Then it is well known [35] that the steady-state mean-square value of y is given by

$$\lim_{t \to \infty} \mathbb{E}[|y|^2] = \operatorname{tr}[C^* C Q], \tag{1.5}$$

where the steady-state covariance $Q \triangleq \lim_{t \to \infty} \mathbb{E}[xx^*] \in \mathbb{C}^{n \times n}$ is determined as the Hermitian nonnegative-definite solution to the Lyapunov equation

$$0 = (-\nu + j\Omega)Q + Q(-\nu - j\Omega) + (H + G)Q + Q(H + G)^* + V. \tag{1.6}$$

Note that due to symmetry, equation (1.6) represents $\frac{1}{2}n(n+1)$ scalar equations for the elements of Q.

Since (1.6) is a well-known equation with well-established solution techniques [36–38], the problem would appear to be solved. However, the difficulty in the application mentioned above is that the total system dimension n may be exceedingly large. For the example involving several acoustic spaces coupled by elastic partitions, each subsystem (a modest-sized room, say) can have millions of modes in the audio range. Thus the total dimension n can be of the order of $10^6 - 10^7$ while the coefficient matrix $(-\nu + j\Omega + H + G)$ is not necessarily either sparse or banded. Thus, the prediction of vibration response or sound pressure levels via the solution of (1.6) can be very cumbersome indeed. It is thus desirable to develop more efficient methods for estimating quantities such as $\mathbb{E}[|y|^2]$ which somehow circumvent the huge dimensionality of (1.6).

In this regard, many useful and important results and procedures have been developed. These are often referred to collectively as "Statistical Energy Analysis," or SEA [1–16]. SEA was developed for high-dimensional, lightly damped mechanical or acoustical systems for which there are passive mechanical energy-conservative interconnections among the subsystems. In the notation of (1.3), (1.4), this means that there is a basis in which $H + G$ is skew-Hermitian, that is,

$$H_k = j\hat{H}_k, \quad \hat{H}_k \in \mathbb{R}, \quad k = 1, \dots, n, \tag{1.7}$$

$$G^* = -G. \tag{1.8}$$

In the present paper, we develop results that deal with general coupling terms. These results are later specialized to couplings restricted by (1.7), (1.8).

The purpose of this paper is to elucidate some of the basic ideas of SEA in rigorous system-theoretic language and to provide generalizations of certain fundamental SEA results. Before summarizing these results, let us note that our problem formulation thus far in terms of a Lyapunov equation as in (1.6) already represents a point of departure from the techniques employed in [1–16]. Motivated by the literature on large scale systems theory [39], we utilize Kronecker matrix algebra [40,41] and M-matrix theory [39,42] as our principal mathematical tools. In an earlier paper [43] we used similar tools to analyze the stability and performance robustness of interconnected systems. The results of [43], which were themselves motivated by SEA, thus served as a precursor to the SEA extensions given in the present paper.

Perhaps the most fundamental tenet of SEA is that quantities such as $\mathbb{E}[|y|^2]$ can be estimated or approximately determined solely in terms of the "modal energies". In our notation the modal energies translate into the real, nonnegative diagonal elements

$$E_k \triangleq Q_{kk} = \lim_{t\to\infty} \mathbb{E}[|x_k|^2] \tag{1.9}$$

of the second-moment matrix Q. For example, if the system is a set of mechanical subsystems mechanically coupled, then E_k corresponds to the kinetic or potential energy of one of the vibrational modes of a subsystem. In Section 2 we discuss the various conditions under which it suffices to determine the E_k in order to evaluate the mean-square response of quantities of interest.

Having argued that mean-square response measures of interest can be deduced from knowledge of the modal energies, a second central tenet of SEA is that it is possible to formulate a set of

n linear equations that involve only the quantities E_k and that are sufficient to determine these quantities. Note that the diagonal elements of (1.6) give power flow relations of the form

$$\underbrace{(2\nu_k - 2\text{Re } H_k)E_k}_{\substack{\text{power dissipated}\\ \text{by the } k\text{th mode}\\ \text{due to damping}}} \quad + \quad \underbrace{\Pi_k}_{\substack{\text{power flow}\\ \text{from the } k\text{th mode}\\ \text{to all other modes}\\ \text{due to coupling}}} \quad = \quad \underbrace{V_{kk},}_{\substack{\text{power input}\\ \text{due to external}\\ \text{disturbances}}} \tag{1.10}$$

where Π_k is given by

$$\Pi_k \triangleq -\sum_{\substack{\ell=1\\ \ell\neq k}}^{n}(G_{k\ell}Q_{\ell k} + Q_{k\ell}\bar{G}_{k\ell}).$$

Statistical Energy Analysis asserts that the quantities Π_k can be evaluated as linear functions of the E_ℓ's alone so that a relationship holds of the form

$$\Pi_k = \sum_{\ell=1}^{n} \mathcal{P}_{k\ell}E_\ell, \quad \mathcal{P}_{k\ell} \in \mathbb{R}. \tag{1.11}$$

Thus, if (1.11) holds then by using (1.10) one need only solve n linear equations for the modal energies in place of solving the $\frac{1}{2}n(n+1)$ equations corresponding to the $n \times n$ Lyapunov equation (1.6). Relation (1.11) has been demonstrated in several special cases, namely, two coupled oscillators, n identical oscillators with identical coupling, and n nonidentical oscillators with weak inter-modal coupling [1–6]. In Section 3, however, and *without restrictions* (1.7), (1.8), we use Kronecker algebra to deduce directly from (1.6) that the modal energies E_k are determined by an energy equation of the form

$$(\mu + \mathcal{P})E = \hat{V}, \tag{1.12}$$

where

$$\mu \triangleq \operatorname*{diag}_{k=1,\ldots,n}(\mu_k), \quad \mu_k \triangleq 2\nu_k - 2\text{Re } H_k, \quad \mathcal{P} \in \mathbb{R}^{n\times n},$$

$$\hat{V} \triangleq \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \triangleq \begin{bmatrix} V_{11} \\ \vdots \\ V_{nn} \end{bmatrix}, \quad E \triangleq \begin{bmatrix} E_1 \\ \vdots \\ E_n \end{bmatrix} \triangleq \begin{bmatrix} Q_{11} \\ \vdots \\ Q_{nn} \end{bmatrix}.$$

By comparing (1.10) to (1.12) it can be seen that the expression (1.11) for Π_k is precisely the kth element of $\mathcal{P}E$. So long as the overall system is asymptotically stable, relations of the form (1.10) and (1.11) hold regardless of the number of modes or the magnitude of the couplings.

Further conditions on $\mu + \mathcal{P}$ that arise in the cases of two oscillators, n identical oscillators with identical coupling, or nonidentical oscillators with weak coupling lead to an energy difference power flow proportionality as in [11,12]. Specifically, suppose that

$$\mathcal{P}_{k\ell} \leq 0, \quad k \neq \ell, \quad k, \ell = 1, \ldots, n, \tag{1.13}$$

and

$$\mathcal{P}_{kk} = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} |\mathcal{P}_{k\ell}|, \quad k = 1, \ldots, n. \tag{1.14}$$

Then, defining $\sigma_{k\ell} \triangleq |\mathcal{P}_{k\ell}|, \quad k \neq \ell$, so that $\sigma_{k\ell} \geq 0$, it follows from (1.11) that

$$\Pi_k = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \sigma_{k\ell}(E_k - E_\ell). \tag{1.15}$$

In other words, power flow from the kth mode to all other modes is the sum of the individual power flows from mode k to mode ℓ, which are proportional to the energy differences $E_k - E_\ell$. Note that power always flows from more energetic modes to less energetic modes (because of the nonnegativity of the coefficients $\sigma_{k\ell}$). Substituting (1.15) into (1.12) yields

$$\mu_k E_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \sigma_{k\ell}(E_k - E_\ell) = V_k, \tag{1.16}$$

which is an energy balance relation. Equations (1.10) and (1.16), which govern energy exchange among coupled oscillators, are completely analogous to the equations of thermal transfer with the modal energies playing the role of temperatures.

In physical situations involving nonconservative couplings, we show that although (1.14) no longer holds, it is still possible in the case of weak couplings to obtain a generalized power flow proportionality. In this case there exists a set of positive scale factors $D_k > 0, \quad k = 1, \ldots, n$, such that, with $\hat{E}_k \triangleq \frac{1}{D_k} E_k$, the energy difference power flow proportionality is given by

$$\Pi_k = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \hat{\sigma}_{k\ell}(\hat{E}_k - \hat{E}_\ell), \tag{1.17}$$

where $\hat{\sigma}_{k\ell} \triangleq D_\ell \sigma_{k\ell}$. Note that (1.17) is not merely a rewriting of (1.15) since in general $D_k \neq D_\ell$. With (1.17), the energy equation (1.12) assumes the form of a generalized energy balance relation given by

$$\hat{\mu}_k \hat{E}_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \hat{\sigma}_{k\ell}(\hat{E}_k - \hat{E}_\ell) = V_k, \tag{1.18}$$

where $k = 1, \ldots, n$. That is, there is a set of re-scaled energies such that (1.12) looks like the equations of thermal transfer. This result, given in Section 4, generalizes (1.15), (1.16) to the case of weak but otherwise arbitrary (not necessarily conservative) modal couplings. These results are obtained by means of M-matrix theory [39,42].

While deriving energy difference power flow proportionality relations, we show that the explicit expressions given for $\dot{\mathcal{P}}$ in the SEA literature are actually first-term approximations in a series expansion for $\mathcal{P}$. Indeed, it turns out that $\mathcal{P}$, which is given by a complicated expression involving ν, Ω, H, and G, agrees with the customary SEA expressions for "small" G. This in done by obtaining explicit expressions for the terms of a series expansion of $\mathcal{P}$ in ascending powers of the matrix elements of G.

Since the modal energies satisfy equations analogous to those of thermal transfer, it might be expected that if the coupling coefficients $G_{k\ell}$ are large compared to the modal dampings, then the energies should be approximately equal, that is,

$$E_1 \simeq E_2 \simeq \cdots \simeq E_n. \tag{1.19}$$

Section 7 provides a formulation and proof of this "energy equipartitioning" phenomenon.

At this point, it is evident that this paper deals only with certain deterministic aspects of SEA. Rigorous exploration and extension of the "Statistical" aspect of Statistical Energy Analysis, which addresses the possibility of uncertainties in the system parameters and coupling coefficients, will form the subject of a future paper. Other extensions of the present paper are briefly mentioned in Section 8.

2. Characterization of System Response in Terms of the Modal Energies

Here we examine the conditions under which it suffices to compute only the modal energies (1.9) in order to estimate response quantities such as $\lim_{t\to\infty} \mathbb{E}[|y|^2]$. To carry out the necessary calculations, we shall utilize a somewhat unconventional notation for the diagonal and off-diagonal portions of a matrix. Specifically, for $M \in \mathbb{C}^{n\times n}$ define

$$\{M\} \triangleq \operatorname*{diag}_{k=1,\ldots,n} (M_{kk}), \quad \langle M\rangle \triangleq M - \{M\}.$$

For convenience, several identities involving these definitions are given in Appendix A.

Next we define the matrix

$$A \triangleq -\nu + j\Omega + H$$

and note that

$$A = \operatorname*{diag}_{k=1,\ldots,n} (A_k), \tag{2.1}$$

where

$$A_k \triangleq -\nu_k + j\Omega_k + H_k.$$

Then the Lyapunov equation (1.6) becomes

$$0 = AQ + QA^* + GQ + QG^* + V. \tag{2.2}$$

Using the identities of Appendix A to decompose the Lyapunov equation (1.6) into its diagonal and off-diagonal parts, we obtain (noting $A = \{A\}$ and $G = \langle G\rangle$)

$$0 = A\{Q\} + \{Q\}A^* + \{G\langle Q\rangle + \langle Q\rangle G^*\} + \{V\}, \tag{2.3}$$

$$0 = A\langle Q\rangle + \langle Q\rangle A^* + \langle G\langle Q\rangle + \langle Q\rangle G^*\rangle + G\{Q\} + \{Q\}G^* + \underline{\langle V\rangle}, \tag{2.4}$$

while (1.5) becomes

$$\lim_{t\to\infty} \mathbb{E}[|y|^2] = \operatorname{tr}[\{C^*C\}\{Q\}] + \underline{\operatorname{tr}[\langle C^*C\rangle\langle Q\rangle]}. \tag{2.5}$$

The underlined terms in (2.4) and (2.5) are zero when V and C^*C are diagonal. Furthermore, they can be neglected when the following conditions hold either separately or in combination:

i) The term $\langle V\rangle$ in (2.4) can be neglected when the modal excitation forces are uncorrelated, in which case $\langle V\rangle \simeq 0$. This occurs when excitations are spatially distributed with very short correlation length.

ii) The underlined terms in (2.4) and (2.5) are negligible when $\nu_k, \nu_\ell << | \Omega_k - \Omega_\ell |$, that is, the case of large modal frequency separation relative to modal damping.

iii) The underlined terms in (2.4) and (2.5) are negligible in the case of a distributed structural system with very high modal density wherein, for fixed k and for $\ell = 1, \ldots, n$, the real and imaginary parts of $G_{k\ell}, G_{\ell k}$, and $V_{k\ell}$ have many sign reversals for modes within any narrow frequency band. These sign reversals essentially cancel out the contributions of $\langle Q \rangle$ in (2.5) and the effect of $\langle V \rangle$ on $\{Q\}$ in (2.4) (see [13] for details).

When conditions *i*)–*iii*) are satisfied either separately or in combination, we have the approximate equations

$$0 = A\{Q\} + \{Q\}A^* + \{G\langle Q \rangle + \langle Q \rangle G^*\} + \{V\}, \tag{2.6}$$

$$0 = A\langle Q \rangle + \langle Q \rangle A^* + \langle G\langle Q \rangle + \langle Q \rangle G^* \rangle + G\{Q\} + \{Q\}G^*, \tag{2.7}$$

$$\lim_{t \to \infty} \mathbb{E}[| y |^2] = \mathrm{tr}[\{C^*C\}\{Q\}]. \tag{2.8}$$

These equations are exact when V and C^*C are diagonal; they are good approximations under conditions *i*)–*iii*). Note that these approximations have no impact on stability analysis.

The salient feature of (2.8) is that the response quantity involving y can be expressed in terms of the diagonal elements of Q. As mentioned in Section 1, the diagonal elements Q_{kk} have the physical significance of either kinetic or potential energies of the vibrational modes. Although we need only calculate the n diagonal elements of Q (the "system modal energies") to evaluate (2.8), it is still apparently necessary to solve an $n \times n$ Lyapunov equation to obtain all of Q. In fact, however, we now proceed to use Kronecker matrix algebra to eliminate the off-diagonal part $\langle Q \rangle$ from (2.6) and (2.7), thereby producing a system of only n equations determining $\{Q\}$, rather than the $\frac{1}{2}n(n+1)$ equations that characterize all of Q.

3. Determination of the Modal Energy Equations

Here we show that the decomposed Lyapunov equation (2.6), (2.7) can be reduced to a system of n equations involving only the modal energies E_k, that is, the diagonal elements of Q. To do this we employ the Kronecker matrix algebra, the basic definitions and identities of which are summarized in Appendix B. Note that the basic operators are the *vec* operator, which stacks the columns of a matrix into a vector, and the *vecd* operator, which stacks only the diagonal entries of a square matrix into a vector. Appendix B reviews the definition of the Kronecker product and sum along with identities (B.1) through (B.10), which are well known [40,41]. The remaining identities (B.11)–(B.18) are new and their proof is left to the reader. The matrices $\mathcal{E}$ and $\mathcal{E}_\perp$ are diagonal projection matrices that allow us to separate the entries of vec M corresponding to the diagonal and off-diagonal elements of a square matrix M (see (B.13) and (B.14)). We can now define

$$E \triangleq \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix} \triangleq \operatorname{vecd} Q = \begin{bmatrix} Q_{11} \\ Q_{22} \\ \vdots \\ Q_{nn} \end{bmatrix}, \tag{3.1}$$

$$\hat{V} \triangleq \operatorname{vecd} V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \triangleq \begin{bmatrix} V_{11} \\ V_{22} \\ \vdots \\ V_{nn} \end{bmatrix}, \tag{3.2}$$

which are real, nonnegative vectors since Q and V are Hermitian nonnegative-definite matrices. Furthermore, define

$$\mu \triangleq \underset{k=1,\ldots,n}{\operatorname{diag}} (\mu_k), \quad \mu_k \triangleq 2\nu_k - 2\text{Re } H_k. \tag{3.3}$$

and note that $\mu = -(\bar{A} + A) = -2\text{Re } A$.

Theorem 3.1. Assume that $A + G$ is asymptotically stable, let $Q \in \mathbb{C}^{n\times n}$ be the unique Hermitian nonnegative-definite solution to the Lyapunov equation (1.6), and define the nonnegative vectors $E, \hat{V} \in \mathbb{R}^n$ by (3.1), (3.2). If $\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp$ is nonsingular, then E and $\hat{V}$ satisfy

$$(\mu + \mathcal{P})E = \hat{V}, \tag{3.4}$$

where $\mu \in \mathbb{R}^{n\times n}$ is defined by (3.3) and $\mathcal{P} \in \mathbb{R}^{n\times n}$ is defined by

$$\mathcal{P} \triangleq \hat{\mathcal{E}}^{\mathrm{T}}(\bar{G} \oplus G)\mathcal{E}_\perp[\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}}. \tag{3.5}$$

Furthermore, $\mu + \mathcal{P}$ is nonsingular and its inverse $(\mu + \mathcal{P})^{-1}$ is a nonnegative matrix. Finally,

$$\lim_{t\to\infty} \mathbb{E}[|y|^2] = \sum_{k=1}^{n} |C_k|^2 E_k. \tag{3.6}$$

Proof. Applying the vec operator to (2.6) and using (B.7), (B.13), and (B.16) yields

$$\begin{aligned} 0 &= (\bar{A} \oplus A)\text{vec}\{Q\} + \text{vec}\{G\langle Q\rangle + \langle Q\rangle G^*\} + \text{vec}\{V\} \\ &= (\bar{A} \oplus A)\hat{\mathcal{E}}\,\text{vecd}\; Q + \mathcal{E}(\bar{G} \oplus G)\text{vec}\langle Q\rangle + \hat{\mathcal{E}}\,\text{vecd}\; V. \end{aligned} \tag{3.7}$$

Next applying the vec operator to (2.7) and using (B.7), (B.14), (B.15), (B.16), and $\mathcal{E}_\perp(\bar{A} \oplus A) = (\bar{A} \oplus A)\mathcal{E}_\perp$, yields

$$\begin{aligned} 0 &= (\bar{A} \oplus A)\text{vec}\langle Q\rangle + \text{vec}\langle G\langle Q\rangle + \langle Q\rangle G^*\rangle + (\bar{G} \oplus G)\text{vec}\{Q\} \\ &= (\bar{A} \oplus A)\text{vec}\langle Q\rangle + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp\text{vec}\langle Q\rangle + (\bar{G} \oplus G)\hat{\mathcal{E}}\,\text{vecd}\; Q \\ &= [\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]\text{vec}\langle Q\rangle + \mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}}\,\text{vecd}\; Q. \end{aligned} \tag{3.8}$$

Since $\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp$ is assumed to be nonsingular, (3.8) and (B.14) imply

$$\text{vec}\langle Q\rangle = -\mathcal{E}_\perp[\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}}\,\text{vecd}\; Q. \tag{3.9}$$

Substituting (3.9) into (3.7) yields

$$0 = [(\bar{A} \oplus A)\hat{\mathcal{E}} - \mathcal{E}(\bar{G} \oplus G)\mathcal{E}_\perp[\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}}]\text{vecd}\; Q + \hat{\mathcal{E}}\;\text{vecd}\; V. \tag{3.10}$$

Next note that

$$\hat{\mathcal{E}}^{\mathrm{T}}(\bar{A} \oplus A)\hat{\mathcal{E}} = \bar{A} + A = -\mu. \tag{3.11}$$

Multiplying (3.10) by $\hat{\mathcal{E}}^{\mathrm{T}}$ and using (3.11), (B.17), and (B.18) yields

$$0 = -[\mu + \hat{\mathcal{E}}^{\mathrm{T}}(\bar{G} \oplus G)\mathcal{E}_\perp[\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}}]\text{vecd}\; Q + \text{vecd}\; V,$$

or, using (3.5),

$$(\mu + \mathcal{P})\text{vecd}\; Q = \text{vecd}\; V,$$

which is (3.4).

To show that the $n \times n$ matrix $\mathcal{P}$ defined by (3.5) is real, take the complex conjugate of (3.5) and use (B.8) and (B.10)–(B.12) to obtain

$$\begin{aligned} \bar{\mathcal{P}} &= \hat{\mathcal{E}}^{\mathrm{T}}(G \oplus \bar{G})\mathcal{E}_\perp[A \oplus \bar{A} + \mathcal{E}_\perp(G \oplus \bar{G})\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(G \oplus \bar{G})\hat{\mathcal{E}} \\ &= \hat{\mathcal{E}}^{\mathrm{T}}U(\bar{G} \oplus G)U\mathcal{E}_\perp[U(\bar{A} \oplus A)U + \mathcal{E}_\perp(\bar{G} \oplus G)U\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp U(\bar{G} \oplus G)U\hat{\mathcal{E}} \\ &= \hat{\mathcal{E}}^{\mathrm{T}}(G \oplus \bar{G})\mathcal{E}_\perp[\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}} \\ &= \mathcal{P}. \end{aligned}$$

Next, to show that $\mu + \mathcal{P}$ is nonsingular, note that since $\hat{V} \in \mathbb{R}^n$ is an arbitrary nonnegative vector, the rank of $\mu + \mathcal{P}$ is n. Thus $\mu + \mathcal{P}$ is nonsingular. Furthermore, it can be seen that if $\hat{V}$ is the ith column of I_n, then the nonnegative solution E of (3.4) is the ith column of $(\mu + \mathcal{P})^{-1}$. Hence $(\mu + \mathcal{P})^{-1}$ is a nonnegative matrix. Finally, (3.6) follows from (2.8). $\square$

Remark 3.1. Suppose that G is symmetric, that is, $G = G^{\mathrm{T}}$, but not necessarily real. Then using (B.3) it is easy to show that $\mathcal{P}$ (which is real) is also symmetric. Hence in this case $\mu + \mathcal{P}$ and $(\mu + \mathcal{P})^{-1}$ are both real symmetric matrices.

As a separate result we state the following converse of Theorem 3.1.

Proposition 3.1. Assume that $\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp$ is nonsingular, let $\hat{V}, \mu$, and $\mathcal{P}$ be defined by (3.2), (3.3), and (3.5), and suppose there exists a nonnegative solution $E \triangleq \begin{bmatrix} E_1 \\ \vdots \\ E_n \end{bmatrix} \in \mathbb{R}^n$ to equation (3.4). Then the matrix $Q \in \mathbb{C}^{n \times n}$ defined by

$$Q \triangleq \operatorname*{diag}_{i=1,\ldots,n}(E_i) + \operatorname{vec}^{-1}([\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}} E), \tag{3.12}$$

is Hermitian and satisfies (1.6). If, in addition, $V_k > 0, \quad k = 1, \ldots, n$, and Q is positive definite, then $A + G$ is asymptotically stable.

Proof. The fact that Q given by (3.12) satisfies (1.6) follows by reversing the algebraic steps leading to (3.4). To show that Q is Hermitian, note that using (B.8)-(B.10) we have

$$\begin{aligned}
Q^* &= \operatorname*{diag}_{i=1,\ldots,n}(E_i) + \left[\operatorname{vec}^{-1}(\mathcal{E}_\perp[A \oplus \bar{A} + \mathcal{E}_\perp(G \oplus \bar{G})\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(G \oplus \bar{G})\hat{\mathcal{E}} E)\right]^{\mathrm{T}} \\
&= \operatorname*{diag}_{i=1,\ldots,n}(E_i) + \left[\operatorname{vec}^{-1}(\mathcal{E}_\perp[U(\bar{A} \oplus A)U + \mathcal{E}_\perp U(\bar{G} \oplus G)U\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp U(\bar{G} \oplus G)U\hat{\mathcal{E}} E)\right]^{\mathrm{T}} \\
&= \operatorname*{diag}_{i=1,\ldots,n}(E_i) + \left[\operatorname{vec}^{-1}(\mathcal{E}_\perp U[\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}} E)\right]^{\mathrm{T}} \\
&= \operatorname*{diag}_{i=1,\ldots,n}(E_i) + \operatorname{vec}^{-1}(\mathcal{E}_\perp[\bar{A} \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp]^{-1}\mathcal{E}_\perp(\bar{G} \oplus G)\hat{\mathcal{E}} E) \\
&= Q.
\end{aligned}$$

Finally, the stability of $A + G$ follows from standard Lyapunov theory [44, Lemma 12.2]. $\square$

Theorem 3.1 and Proposition 3.1 show that for the purpose of determining the diagonal entries of Q, that is, the modal energies $E_1, \ldots, E_n$, equation (3.4) is equivalent to equations (2.6) and (2.7). This verifies the tenet of Statistical Energy Analysis that there exists a system of n linear equations that determine the modal energies alone. Moreover, comparing the kth equation in (3.4) with the power balance relation (1.10), we now see that Π_k, the power flow from the kth mode to all other modes due to coupling, is given by

$$\Pi_k = \sum_{\ell=1}^{n} \mathcal{P}_{k\ell} E_\ell, \tag{3.13}$$

where $\dot{\mathcal{P}} \in \mathbb{R}$ is the (k,ℓ) element of $\mathcal{P}$. Thus the expression (1.11) is also verified. In the next section we further explore the structure of Π_k and $\mathcal{P}$ to derive a generalization of the energy difference power flow proportionality (1.15) for weak but arbitrary coupling matrices G.

4. Analysis of the Energy Equation: Energy Difference Power Flow Proportionality

In this section we analyze the energy equation (1.12) to determine conditions under which an energy difference power flow proportionality holds. Under the assumption that the off-diagonal elements of $\mathcal{P}$ are nonpositive, we obtain a generalized power flow proportionality involving scaled model energies. In Section 5 we then show that this result holds for weak, but otherwise arbitrary, couplings. Specializing further in Section 6 to the conservative case involving skew-Hermitian couplings, we obtain a power flow proportionality involving the actual (unscaled) modal energies.

The development requires several definitions and results from matrix theory [39,42]. A matrix $M \in \mathbb{R}^{n\times n}$ is called a Z-matrix if $M_{k\ell} \leq 0, \quad k \neq \ell \quad k,\ell = 1,\ldots,n$. Note that a Z-matrix $M \in \mathbb{R}^{n\times n}$ can always be placed in the form

$$M = \alpha I - N, \tag{4.1}$$

where $\alpha > 0$ and $N \geq\geq 0, \quad N \in \mathbb{R}^{n\times n}$. If (4.1) can be satisfied with $\alpha \geq \rho(N)$ (ρ denotes spectral radius), then M is called an *M-matrix*. If, furthermore, $\alpha > \rho(N)$, then, since $\det M \neq 0$, M is a nonsingular M-matrix. There are numerous (at least 50) equivalent conditions under which a Z-matrix is a nonsingular M-matrix [42]. We now summarize those conditions that will be used here. We shall call $B \in \mathbb{R}^{n\times n}$ ***diagonally dominant*** if

$$B_{kk} > \sum_{\substack{\ell=1\\ \ell\neq k}}^{n} |B_{k\ell}|, \quad k = 1,\ldots,n. \tag{4.2}$$

Lemma 4.1. Let $M \in \mathbb{R}^{n\times n}$ be a Z-matrix. Then the following are equivalent:

i) M is a nonsingular M-matrix,

ii) M is nonsingular and $M^{-1} \geq\geq 0$,

iii) the real part of each eigenvalue of M is positive,

iv) there exists positive diagonal $D \in \mathbb{R}^{n\times n}$ such that MD is diagonally dominant.

Proof. See conditions (N_{38}), (G_{20}), and (M_{35}) on pages 134–138 of [42]. □

Returning to the energy equation (1.12), we focus on the coefficient matrix $\mu + \mathcal{P}$. The crucial condition that $\mu + \mathcal{P}$ is a Z-matrix will be shown later for the case of weak, but otherwise arbitrary, couplings. First we recall from Theorem 3.1 that, under the assumptions of that Theorem, $\mu + \mathcal{P}$ is nonsingular and $(\mu + \mathcal{P})^{-1} \geq\geq 0$. Thus, condition *ii)* of Lemma 4.1 with $M = \mu + \mathcal{P}$ can be invoked to yield conditions *i)*, *iii)* and *iv)*.

Proposition 4.1. Suppose that the assumptions of Theorem 3.1 are satisfied and assume that $\mathcal{P}$ is a Z-matrix. Then

i) $\mu + \mathcal{P}$ is a nonsingular M-matrix,

ii) the real part of each eigenvalue of $\mu + \mathcal{P}$ is positive,

iv) there exists positive scalars $D_1, \ldots, D_n$ such that

$$D_k(\mu_k + \mathcal{P}_{kk}) > \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} D_\ell \, |\mathcal{P}_{k\ell}|, \quad k = 1, \ldots, n. \tag{4.3}$$

Proof. First note that since μ is a diagonal matrix, $\mu + \mathcal{P}$ is a Z-matrix if and only if $\mathcal{P}$ is a Z-matrix. Since, by Theorem 3.1, $(\mu + \mathcal{P})^{-1} \geq\geq 0$, condition *ii)* of Lemma 4.1 is satisfied with $M = \mu + \mathcal{P}$. Hence conditions *i)*, *iii)*, and *iv)* of Lemma 4.1 are also satisfied. Now it need only be noted that (4.3) is equivalent to (4.2) with $B = (\mu + \mathcal{P})D$ and $D = \operatorname{diag}_{k=1,\ldots,n}(D_k)$. $\square$

Remark 4.1. Suppose G is symmetric but not necessarily real. Then by Remark 3.1 $\mathcal{P}$ is symmetric. Since $\mu + \mathcal{P}$ is also symmetric, $\mu + \mathcal{P}$ has only real eigenvalues. It thus follows from condition *ii)* of Proposition 4.1 that $\mu + \mathcal{P}$ has only real positive eigenvalues. Hence in this case $\mu + \mathcal{P}$ is a symmetric positive-definite matrix.

Corollary 4.1. Suppose that the assumptions of Theorem 3.1 are satisfied and assume that $\mathcal{P}$ is a Z-matrix. Then there exist positive scalars $\hat{\mu}_k > 0, \quad k = 1, \ldots, n$, and nonnegative scalars $\hat{\sigma}_{k\ell} \geq 0, \quad k \neq \ell, \quad k, \ell = 1, \ldots, n$, such that

$$\hat{\mu}_k \hat{E}_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \hat{\sigma}_{k\ell}(\hat{E}_k - \hat{E}_\ell) = V_k, \quad k = 1, \ldots, n, \tag{4.4}$$

where $\hat{E}_k \triangleq \frac{1}{D_k} E_k, \quad k = 1, \ldots, n$.

Proof. Using (4.3) of Proposition 4.1, define $\hat{\mu}_k > 0$ by

$$\hat{\mu}_k \triangleq D_k(\mu_k + \mathcal{P}_{kk}) - \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} D_\ell \, |\mathcal{P}_{k\ell}|, \quad k = 1, \ldots, n. \tag{4.5}$$

Next note that with $\hat{E}_k \triangleq \frac{1}{D_k} E_k$ and, since $\mathcal{P}$ is assumed to be a Z-matrix, $\mathcal{P}_{k\ell} = -\,|\mathcal{P}_{k\ell}|, \quad k \neq \ell$, the kth equation of (1.12) yields

$$D_k(\mu_k + \mathcal{P}_{kk})\hat{E}_k - \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} D_\ell |\mathcal{P}_{k\ell}| \hat{E}_\ell = V_k. \tag{4.6}$$

Combining (4.5) and (4.6) yields

$$\hat{\mu}_k \hat{E}_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} D_\ell |\mathcal{P}_{k\ell}| (\hat{E}_k - \hat{E}_\ell) = V_k, \tag{4.7}$$

which implies (4.4) with $\hat{\sigma}_{k\ell} = D_\ell |\mathcal{P}_{k\ell}|$. □

Equation (4.4) can be viewed as a generalized energy balance relation since it involves scaled modal energies rather than the modal energies themselves. Furthermore, comparing (4.4) to (1.10) it follows that

$$\Pi_k = (\hat{\mu}_k - D_k \mu_k)\hat{E}_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \hat{\sigma}_{k\ell}(\hat{E}_k - \hat{E}_\ell). \tag{4.8}$$

That is, the power flow from the kth mode to all other modes is, aside from the offset term $(\hat{\mu}_k - D_k \mu_k)\hat{E}_k$, proportional to the difference between scaled modal energies. In Section 6 we show that under conservative couplings (4.8) becomes an actual (nonscaled) energy difference power flow proportionality. Next, however, we show that $\mathcal{P}$ is a Z-matrix under weak coupling.

5. Analysis of $\mathcal{P}$ in the Case of Weak Coupling

In the previous section the generalized power flow proportionality was based upon the assumption that $\mathcal{P}$ is a Z-matrix. In this section we show that this assumption is valid in the case of weak but otherwise arbitrary couplings. To do this we expand $\mathcal{P}$ in terms of powers of G and then show that the first term in the expansion is a Z-matrix.

To begin we define for convenience

$$\mathcal{A} \triangleq \bar{A} \oplus A, \quad \mathcal{G} \triangleq \bar{G} \oplus G \tag{5.1}$$

so that $\mathcal{P}$ defined by (3.5) can be written as

$$\mathcal{P} = \hat{\mathcal{E}}^{\mathrm{T}} \mathcal{G} \mathcal{E}_{\perp} (\mathcal{A} + \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp})^{-1} \mathcal{E}_{\perp} \mathcal{G} \hat{\mathcal{E}}. \tag{5.2}$$

For $r = 0, 1, 2, \ldots$, it is easy to confirm the identity

$$(\mathcal{A} + \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp})^{-1} = \sum_{i=0}^{r} (-\mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp})^{i} \mathcal{A}^{-1} + (-\mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp})^{r+1} (\mathcal{A} + \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp})^{-1}. \tag{5.3}$$

Combining (5.2) and (5.3) it follows that

$$\mathcal{P} = \sum_{i=0}^{r} \mathcal{P}_i + \mathcal{R}_r, \tag{5.4}$$

where

$$\mathcal{P}_i \triangleq \hat{\mathcal{E}}^{\mathrm{T}} \mathcal{G} \mathcal{E}_{\perp} (-\mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp})^{i} \mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \hat{\mathcal{E}} \tag{5.5}$$

and

$$\mathcal{R}_r \triangleq \hat{\mathcal{E}}^{\mathrm{T}} \mathcal{G} \mathcal{E}_{\perp} (-\mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp})^{r+1} (\mathcal{A} + \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp})^{-1} \mathcal{E}_{\perp} \mathcal{G} \hat{\mathcal{E}}. \tag{5.6}$$

Note that

$$\|\mathcal{R}_r\| = O(\|\mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp}\|^{r+3}) \tag{5.7}$$

for $\|\mathcal{G}\| \to 0$ (where $\|\cdot\|$ denotes arbitrary submultiplicative matrix norms). Clearly the error incurred in approximating $\mathcal{P}$ by $\sum_{i=0}^{r} \mathcal{P}_i$ depends on the size of $\mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp}$. This is consistent with the SEA literature since $\mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp}$ can be viewed as the ratio of modal coupling to modal damping. For $i = 0, 1$ we have

$$\mathcal{P}_0 = \hat{\mathcal{E}}^{\mathrm{T}} \mathcal{G} \mathcal{E}_{\perp} \mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \hat{\mathcal{E}}, \tag{5.8}$$

$$\mathcal{P}_1 = -\hat{\mathcal{E}}^{\mathrm{T}} \mathcal{G} \mathcal{E}_{\perp} \mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp} \mathcal{A}^{-1} \mathcal{E}_{\perp} \mathcal{G} \hat{\mathcal{E}}. \tag{5.9}$$

Lemma 5.1. Suppose that the assumptions of Theorem 3.1 are satisfied. Then

$$(\mathcal{P}_0)_{kk} = -2\mathrm{Re}[\sum_{\ell=1}^{n} G_{k\ell}G_{\ell k}\Gamma_{\ell k}], \quad k = 1, \ldots, n, \tag{5.10}$$

$$(\mathcal{P}_0)_{k\ell} = -2\,|G_{k\ell}|^2 \ \mathrm{Re}\ \Gamma_{\ell k}, \quad \ell \neq k, \quad \ell, k = 1, \ldots, n, \tag{5.11}$$

$$(\mathcal{P}_1)_{kk} = -2\mathrm{Re}[\sum_{\ell,m=1}^{n} G_{k\ell}\Gamma_{\ell k}G_{\ell m}\Gamma_{mk}G_{mk}], \quad k = 1, \ldots, n, \tag{5.12}$$

$$\begin{aligned}(\mathcal{P}_1)_{k\ell} = -2\mathrm{Re}\big[\sum_{m=1}^{n}(&G_{km}\Gamma_{mk}G_{m\ell}\Gamma_{\ell k}\bar{G}_{k\ell} + G_{km}\Gamma_{mk}\Gamma_{m\ell}G_{m\ell}\bar{G}_{k\ell}, \\ &+ G_{k\ell}\Gamma_{\ell k}\Gamma_{\ell m}\bar{G}_{m\ell}\bar{G}_{km})\big], \quad \ell \neq k, \quad \ell, k = 1, \ldots, n,\end{aligned} \tag{5.13}$$

where

$$\Gamma_{\ell k} \triangleq \frac{-1}{A_\ell + \bar{A}_k} = [\nu_\ell + \nu_k - (H_\ell + \bar{H}_k) + j(\Omega_k - \Omega_\ell)]^{-1}. \tag{5.14}$$

Proposition 5.1. Suppose that the assumptions of Theorem 3.1 are satisfied and assume, furthermore, that $\mu_k + \mu_\ell \geq 0, \quad k \neq \ell, \quad k, \ell = 1, \ldots, n$. Then $\mathcal{P}_0$ is a Z-matrix. If, furthermore, $\mu_k + \mu_\ell > 0, \quad k \neq \ell, \quad k, \ell = 1, \ldots, n, \quad G_{k\ell} \neq 0, \quad k \neq \ell, \quad k, \ell = 1, \ldots, n$, and $||G||$ is sufficiently small, then $\mathcal{P}$ is a Z-matrix.

Proof. From (5.11) we have

$$\begin{aligned}(\mathcal{P}_0)_{k\ell} &= -2\,|\,G_{k\ell}|^2 \ \mathrm{Re}\ \Gamma_{\ell k} \\ &= -\ |G_{k\ell}|^2\,(\mu_k + \mu_\ell)/\,|\,\Gamma_{\ell k}|^2 \\ &\leq 0.\end{aligned}$$

Hence, $\mathcal{P}_0$ is a Z-matrix. If, in addition, $\mu_k + \mu_\ell > 0$ and $G_{k\ell} \neq 0$, then $(\mathcal{P}_0)_{k\ell} < 0$. In this case $||G||$ sufficiently small implies $\mathcal{P}_{k\ell} < 0$ so that $\mathcal{P}$ is a Z-matrix. □

To understand the significance of Proposition 5.1, consider in place of (3.4) the approximate energy equation

$$(\mu + \mathcal{P}_0)E = \hat{V}. \tag{5.15}$$

If $||G||$ is small, that is, the coupling G is weak, then the norm of the residual $\mathcal{R}_0$ is of order $||\mathcal{A}^{-1}\mathcal{E}_\perp \mathcal{G}\mathcal{E}_\perp||^3$. Hence in this case (5.15) can serve as an approximation to (3.4).

Proposition 5.1 also shows that if $\mu_k + \mu_\ell > 0$ and $G_{k\ell} \neq 0$, $k \neq \ell, \quad k, \ell = 1, \ldots, n$, then $\mathcal{P}$ itself is a Z-matrix so that Corollary 4.1 can be applied. Note that if $G_{k\ell} = 0$ then $(\mathcal{P}_0)_{k\ell} = 0$ and thus the sign of $\mathcal{P}_{k\ell}$ depends on higher order terms in the expansion of $\mathcal{P}_{k\ell}$. It is interesting to note that if $G_{k\ell} = 0$ then $(\mathcal{P}_1)_{k\ell}$ is also zero so that in this case terms of even higher order play a role.

6. Specialization to the Case of Conservative Couplings

Many physical situations involve only passive or energy-conservative couplings among subsystems. This is the case considered in the SEA literature. To model this situation we assume that H and G are skew-Hermitian. If $V(x) = x^*x$ represents the total energy of the system, then it follows that energy dissipation along trajectories of the system (1.3) with $w = 0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x) = -2x^*\nu x < 0, \quad x \neq 0, \tag{6.1}$$

which is identical to the energy dissipation of the uncoupled system. Thus skew-Hermitian coupling has no effect on the total system energy. To analyze this case we begin with the following lemma which corresponds to equation (6) of [11].

Lemma 6.1. Suppose that the assumptions of Theorem 3.1 are satisfied and, furthermore, assume that G is skew-Hermitian. Then

$$\mathcal{P}e = 0, \tag{6.2}$$

where $e \triangleq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

Proof. It suffices to show that $(\bar{G} \oplus G)\hat{\mathcal{E}}e = 0$. Note

$$\begin{aligned}(\bar{G} \oplus G)\hat{\mathcal{E}}e &= (\bar{G} \oplus G)\hat{\mathcal{E}}\,\mathrm{vecd}\, I_n \\ &= (\bar{G} \oplus G)\mathrm{vec}\{I_n\} \\ &= (\bar{G} \oplus G)\mathrm{vec}\; I_n \\ &= \mathrm{vec}(G^* + G) \\ &= 0. \quad \square\end{aligned}$$

Since $e \neq 0$, $\mathcal{P}$ has a nontrivial nullspace and thus (6.2) implies that $\mathcal{P}$ is singular. Note that (6.2) can be written as

$$\sum_{\ell=1}^{n} \mathcal{P}_{k\ell} = 0, \quad k = 1, \ldots, n. \tag{6.3}$$

If, in addition, $\mathcal{P}$ is a Z-matrix, then (6.3) is equivalent to

$$\mathcal{P}_{kk} = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} |\mathcal{P}_{k\ell}|, \quad k = 1, \ldots, n. \tag{6.4}$$

Defining $\sigma_{k\ell} = |\mathcal{P}_{k\ell}| = -\mathcal{P}_{k\ell}, \quad k \neq \ell, \quad k,\ell = 1,\ldots,n$, it thus follows from (1.11)

$$\Pi_k = \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \sigma_{k\ell}(E_k - E_\ell). \tag{6.5}$$

Using (6.5) we can now obtain an energy difference power flow proportionality as a specialization of (4.4). This results is obtained directly and not by means of M-matrix theory which was used to derive (4.4).

Proposition 6.1. Suppose that the assumptions of Theorem 3.1 are satisfied, assume that $\mathcal{P}$ is a Z-matrix, and that G is skew-Hermitian. Then with $\sigma_{k\ell} \triangleq \ |\mathcal{P}_{k\ell}|, \quad k \neq \ell, \quad k,\ell = 1\ldots,n$, it follows that

$$\mu_k E_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} \sigma_{k\ell}(E_k - E_\ell) = V_k, \quad k = 1,\ldots,n. \tag{6.6}$$

Proof. Equation (6.6) is the kth equation of (3.4) using (6.4). $\square$

Remark 6.1. Proposition 6.1 does not state the $\mu_k > 0$, which is needed for (6.6) to have the physical interpretation of an energy balance relation. Note, however, that in (4.4) the coefficient $\hat{\mu}_k$ was shown to be positive by means of the diagonal dominance characterization of nonsingular M-matrices. Invoking this condition here would lead to a scaled energy balance relation in place of (6.7).

Remark 6.2. Suppose in addition to the assumption that G is Skew-Hermitian, we assume that Re $G = 0$. Then $G = j\hat{G}$, where $\hat{G}$ is a real symmetric matrix. Consequently, G is symmetric and thus Remark 3.1 implies that $\mathcal{P}$ is symmetric. Hence $\sigma_{k\ell} = \sigma_{\ell k}$ which shows that the power flow from the kth mode to the ℓth mode is equal to minus the power flow from the ℓth mode to the kth mode.

7. Equipartition of Energy

In the case of conservative couplings as considered in Section 6 we can show that energy equipartition occurs in the limit of strong coupling. Hence assume that G is a fixed skew-Hermitian coupling matrix and scale G by $\gamma > 0$ so that (1.12) is replaced by

$$(\mu + \mathcal{P}(\gamma))E = \hat{V}, \tag{7.1}$$

where

$$\mathcal{P}(\gamma) \triangleq \gamma^2 \hat{\mathcal{E}}^{\mathrm{T}} \mathcal{G} \mathcal{E}_{\perp} [\mathcal{A} + \gamma \mathcal{E}_{\perp} \mathcal{G} \mathcal{E}_{\perp}]^{-1} \mathcal{E}_{\perp} \mathcal{G} \hat{\mathcal{E}} \tag{7.2}$$

We are interested in evaluating

$$\bar{E} \triangleq \lim_{\gamma \to \infty} (\mu + \mathcal{P}(\gamma))^{-1} \hat{V}. \tag{7.3}$$

We sketch the main steps of the derivation.

It can be shown that

$$\bar{E} = \lim_{\gamma \to \infty} (\mu + \gamma \mathcal{P})^{-1} \hat{V}. \tag{7.4}$$

Now assume G is symmetric so that $\mathcal{P}$ is symmetric. Then by Corollary 7.6.3 of [45] it follows that

$$\lim_{\gamma \to \infty} (\mu + \gamma \mathcal{P})^{-1} = \mu^{-1} - \mu^{-1} \mathcal{P} \mu^{-\frac{1}{2}} (\mu^{-\frac{1}{2}} \mathcal{P} \mu^{-\frac{1}{2}})^{+} \mu^{-\frac{1}{2}}, \tag{7.5}$$

where $(\quad)^{+}$ denotes Moore-Penrose generalized inverse of $\mathcal{P}$ (or Drazin generalized inverse since $\mathcal{P}$ is symmetric). Now suppose that G is also skew-Hermitian. Then by Lemma 6.1, $\mathcal{P}e = 0$ so that

$$\mu^{-\frac{1}{2}} \mathcal{P} \mu^{-\frac{1}{2}} \mu^{\frac{1}{2}} e = 0 \tag{7.6}$$

Next suppose that $\mathcal{P}$ is a Z-matrix. Then it follows from Lemma 6.4.1 of [42] (by setting $\mu = \epsilon I$) that $\mathcal{P}$ is an M-matrix. Next assume $\mathcal{P}$ is irreducible, which is the case if all modes are mutually coupled. Then, since $\mathcal{P}$ is a singular irreducible M-matrix, it follows from Theorem 6.4.16 of [42] that rank $\mathcal{P} = n - 1$. Thus the null space of $\mathcal{P}$ is the one-dimensional subspace spanned by e. Now it can be seen that

$$I - \mu^{-\frac{1}{2}} \mathcal{P} \mu^{-\frac{1}{2}} (\mu^{-\frac{1}{2}} \mathcal{P} \mu^{-\frac{1}{2}})^{+} = \frac{\mu^{\frac{1}{2}} e e^{\mathrm{T}} \mu^{\frac{1}{2}}}{e^{\mathrm{T}} \mu e}. \tag{7.7}$$

Hence (7.4), (7.5), and (7.6) yield

$$\bar{E} = \frac{e e^{\mathrm{T}}}{e \mu e} \hat{V} = \frac{e(e^{\mathrm{T}} \hat{V})}{e \mu e} = (\frac{e^{\mathrm{T}} \hat{V}}{e^{\mathrm{T}} \mu e}) e \tag{7.8}$$

Hence

$$\bar{E}_1 = \bar{E}_2 = \cdots = \bar{E}_n = \frac{e^{\mathrm{T}} \hat{V}}{e^{\mathrm{T}} \mu e}, \tag{7.9}$$

which is an equipartition of energy.

8. Concluding Remarks

There are several issues and questions that remain to be explored:

1. How restrictive is the assumption that $A \oplus A + \mathcal{E}_\perp(\bar{G} \oplus G)\mathcal{E}_\perp$ is nonsingular? Can the inverse of this matrix be replaced by the inverse of a matrix of dimension $(n^2 - n) \times (n^2 - n)$ to account for the rank of $\mathcal{E}_\perp$?

2. It may be possible to redevelop the theory with real (as opposed to complex) models by allowing nonscalar blocks in the interconnection structure. The block Kronecker product [46] may be useful for such a formulation.

3. Further quantification of conditions *i*)–*iii*) of Section 2 may be useful. The theory may also be extendable to the case $\langle V \rangle \neq 0$.

4. It may be possible to develop transient (as opposed to steady-state) results for power flow.

5. It is well known that power flow can be modeled by time-averaging the unforced response of the system. Such a dual theory may provide further insights into the power flow phenomenon. Note that a time-averaging theory may require a dynamic model that is conservative rather than asymptotically stable.

6. Further analysis may reveal more general conditions under which $\mathcal{P}$ is a Z-matrix, particularly for the case of strong coupling.

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Appendix A. Identities Involving $\{\cdot\}$ and $\langle\cdot\rangle$.

For matrices $A, B \in \mathbb{C}^{n\times}$, the following identities are satisfied:

$$A = \{A\} + \langle A\rangle, \tag{A.1}$$

$$A = \{A\} \Leftrightarrow \langle A\rangle = 0, \tag{A.2}$$

$$A = \langle A\rangle \Leftrightarrow \{A\} = 0, \tag{A.3}$$

$$\{\langle A\rangle\} = 0, \quad \langle\{A\}\rangle = 0, \tag{A.4}$$

$$\big\{\langle A\rangle\{B\}\big\} = \big\{\{A\}\langle B\rangle\big\} = 0, \tag{A.5}$$

$$\{AB\} = \big\{\{A\}\{B\} + \langle A\rangle\langle B\rangle\big\}, \tag{A.6}$$

$$\langle\langle A\rangle\{B\}\rangle = \langle A\rangle\{B\}, \quad \langle\{A\}\langle B\rangle\rangle = \{A\}\langle B\rangle, \quad \langle\{A\}\{B\}\rangle = 0, \tag{A.7}$$

$$\langle AB\rangle = \{A\}\langle B\rangle + \langle A\rangle\{B\} + \langle\langle A\rangle\langle B\rangle\rangle. \tag{A.8}$$

Appendix B. Kronecker Matrix Algebra, Definitions and Identities

The following are basic definitions and identities:

vec and vec^{-1} Operators: For $A \in \mathbb{C}^{n\times m}$,

$$\mathrm{vec}\, A \triangleq \begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{n1} \\ A_{12} \\ \vdots \\ A_{22} \\ \vdots \\ A_{n2} \\ \vdots \\ A_{1m} \\ \vdots \\ A_{nm} \end{bmatrix}, \quad \mathrm{vec}^{-1}(\mathrm{vec}\, A) = A$$

vecd Operator: For $A \in \mathbb{C}^{n\times n}$,

$$\mathrm{vecd}\, A \triangleq \begin{bmatrix} A_{11} \\ A_{22} \\ \vdots \\ A_{nn} \end{bmatrix}.$$

Kronecker Product: For $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{p\times q}$,

$$A \otimes B \triangleq \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1m}B \\ A_{21}B & A_{22}B & \dots & A_{2m}B \\ \vdots & \vdots & & \vdots \\ A_{n1}B & A_{n2}B & \dots & A_{nm}B \end{bmatrix} \in \mathbb{R}^{np\times mq}$$

Kronecker Sum: For $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$,

$$A \oplus B \triangleq A \otimes I_m + I_n \otimes B \in \mathbb{C}^{nm\times nm}$$

Kronecker Algebra Identities: For compatible complex matrices A, B, C, D :

$$(A+B)\otimes C = A\otimes C + B\otimes C, \tag{B.1}$$

$$A\otimes(B+C) = A\otimes B + A\otimes C, \tag{B.2}$$

$$(A\otimes B)^{\mathrm{T}} = A^{\mathrm{T}}\otimes B^{\mathrm{T}}, \quad (A\oplus B)^{\mathrm{T}} = A^{\mathrm{T}}\oplus B^{\mathrm{T}}, \tag{B.3}$$

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD), \tag{B.4}$$

$$(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}, \tag{B.5}$$

$$\mathrm{vec}\ ABC = (C^{\mathrm{T}}\otimes A)\mathrm{vec}\ B, \tag{B.6}$$

$$\mathrm{vec}(AB+BC) = (C^{\mathrm{T}}\oplus A)\mathrm{vec}\ B. \tag{B.7}$$

Define the following special vectors and matrices whose dimensions will be inferred from the context in which they are used:

$e_r \triangleq$ column vector whose rth element is 1 and which is zero otherwise,

$E_{rs} \triangleq$ matrix whose (r,s)-element is 1 and which is zero otherwise ($E_{rs} = e_r e_s^{\mathrm{T}}$),

$U \triangleq \sum_{r,s} E_{rs}\otimes E_{rs}^{\mathrm{T}}$,

$E_r \triangleq E_{rr}$, where E_{rr} is square,

$\hat{\mathcal{E}} \triangleq \sum_r E_r\otimes e_r, \quad \mathcal{E} \triangleq \sum_r E_r\otimes E_r, \quad \mathcal{E}_\perp \triangleq I - \mathcal{E}.$

The following identities hold for compatible matrices A, B:

$$U^{-1} = U^{\mathrm{T}} = U, \tag{B.8}$$

$$\mathrm{vec}\ A^{\mathrm{T}} = U\mathrm{vec}\ A, \tag{B.9}$$

$$A \otimes B = U(B \otimes A)U, \quad A \oplus B = U(B \oplus A)U, \tag{B.10}$$

$$U\mathcal{E}_{\perp}U = \mathcal{E}_{\perp}, \quad U\mathcal{E}_{\perp} = \mathcal{E}_{\perp}U, \tag{B.11}$$

$$\hat{\mathcal{E}}^{\mathrm{T}}U = \hat{\mathcal{E}}^{\mathrm{T}}, \quad U\hat{\mathcal{E}} = \hat{\mathcal{E}}, \tag{B.12}$$

$$\mathrm{vec}\{A\} = \mathcal{E}\mathrm{vec}\ A = \mathcal{E}\mathrm{vec}\{A\}, \tag{B.13}$$

$$\mathrm{vec}\langle A\rangle = \mathcal{E}_{\perp}\mathrm{vec}\ A = \mathcal{E}_{\perp}\mathrm{vec}\langle A\rangle, \tag{B.14}$$

$$\mathcal{E} = \mathcal{E}^{\mathrm{T}} = \mathcal{E}^{2}, \quad \mathcal{E}_{\perp} = \mathcal{E}_{\perp}^{\mathrm{T}} = \mathcal{E}_{\perp}^{2}, \tag{B.15}$$

$$\mathrm{vec}\{A\} = \hat{\mathcal{E}}\mathrm{vecd}\ A, \tag{B.16}$$

$$\hat{\mathcal{E}}^{\mathrm{T}}\hat{\mathcal{E}} = I, \tag{B.17}$$

$$\hat{\mathcal{E}}^{\mathrm{T}}\mathcal{E} = \hat{\mathcal{E}}^{\mathrm{T}}. \tag{B.18}$$

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Appendix D
"A Nonlinear Vibration Control Design With a Neural Network Realization"

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A Nonlinear Vibration Control Design With a Neural Network Realization

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1. Background and Motivation

The nonlinear compensator design introduced in Section 2 and subsequently explored in the remainder of the paper was initially motivated by the problem of synthesizing control algorithms for vibration suppression in large flexible structures. Thus, to provide the basic background for the present development, consider a flexible structure instrumented, for vibration control purposes, with electromechanical actuators (to provide control forces) and electronic sensors (to provide measurements of structural motion used to construct appropriate drive signals for the actuators). In terms of modal coordinates, the structural plant model may be given as:

$$\dot{x} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -2\eta\Omega \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u; \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{2n}, \quad u \in \mathbb{R}^1 \quad (1)$$

where

$\Omega \triangleq \mathrm{diag}\,\{\Omega_k\}$ = modal frequencies

$\eta \triangleq \mathrm{diag}\,\{\eta_k\}$ = modal damping ratios

b = modal actuator influence coefficients, where for simplicity, we consider only one actuator so that $b \in \mathbb{R}'$

Again, for simplicity in the present exposition, we suppose that there is one rate sensor, collocated with the actuator. Then the sensor output, y, is given by:

$$y = b^T x_2 \quad (2)$$

The actuator input signal, u, is generally synthesized from the measurement signal, y. The generic form of a *linear* controller is:

$$u = -[K_D,\ K_V]\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \quad (3)$$

where $\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \in \mathbb{R}^{2n_c}$ is the state vector of a dynamic compensator. Assuming analog (continuous time) implementation of the controller, $\hat{x}$ evolves according to:

$$\dot{\hat{x}} = \begin{bmatrix} 0 & \hat{\Omega} \\ -\hat{\Omega} & -2\hat{\eta}\hat{\Omega} \end{bmatrix} \hat{x} + \begin{bmatrix} F_D \\ F_V \end{bmatrix} y \quad (4)$$

where K_D, K_V, F_D, F_V are constant gain matrices and

$$\hat{\Omega} = \mathrm{diag}\,\{\hat{\Omega}_i\}, \quad \hat{\Omega}_i > 0,\ i = 1\ldots,n$$
$$\hat{\eta} = \mathrm{diag}\,\{\hat{\eta}_i\}, \quad \hat{\eta}_i > 0,\ i = 1\ldots,n$$

It is seen that the generic linear compensator consists of a collection of oscillatory modes, as does the plant. As will be seen, to be effective, the compensator modal frequencies, $\hat{\Omega}_i$, must stand in a certain relationship to the plant frequencies.

To probe some of the limitations of linear compensation for structural vibration suppression, we first note that the action of the compensator can be understood in terms of its effect on the energy of vibration. As a measure of the amplitude of the k^{th} vibration mode, define:

$$E_k \triangleq \frac{1}{2} < x_{1k}^2 + x_{2k}^2 > \quad (5.a)$$

where < > denotes a time average over several periods of vibration. The physical significance of E_k is that it is the time average of the total mechanical energy (kinetic + potential) associated with the k^{th} plant mode.

Accordingly herein, E_k is termed the k^{th} plant mode energy. Similarly, we define the "k^{th}compensator mode energy":

$$\hat{E}_k \triangleq \frac{1}{2} < \hat{x}_{1k}^2 + \hat{x}_{2k}^2 > \quad (5.b)$$

which may also be interpreted physically as the electromagnetic energy stored in the inductive/capacitive elements of the analog controller electronics.

The system dynamics can be understood in terms of the energy sharing and power flow between structure and controller. It is known quite generally that equations of motion can be formed for the determination of the E_k's and $\hat{E}_k$'s alone. To illustrate this, consider an inherently stable form of the linear control law:

$$\begin{aligned} K_D &= 0, \quad K_V = \kappa\hat{b}^T; \\ F_D &= 0, \quad F_V = \kappa\hat{b} \end{aligned} \quad (6)$$

where κ is a real nonnegative constant and $\hat{b} \in IR^n$. This control is stable for all $b \in \mathbb{R}^{1n}$, $\hat{b} \in \mathbb{R}^n$ because $x^T x + \hat{x}^T \hat{x}$ is a Lyapunov function for the closed-loop system. Applying the principles of Statistical Energy Analysis [1], we obtain the following (approximate) equations-of-motion for the plant and compensator modal energies:

$$\frac{d}{dt}E_k = -2\eta_k\Omega_k E_k + \sum_\ell \sigma_{k\ell}(\hat{E}_\ell - E_k) \quad (7.a)$$
$$k = 1,\ldots,n$$

$$\frac{d}{dt}\hat{E}_k = -2\hat{\eta}_k\hat{\Omega}_k\hat{E}_k + \sum_\ell \sigma_{\ell k}(E_\ell - \hat{E}_k) \quad (7.b)$$
$$k = 1,\ldots,n_c$$

where:

$$\sigma_{k\ell} = \frac{1}{2}\kappa^2 b_k^2 \hat{B}_k^2 \left[\frac{\eta_k\Omega_k + \hat{\eta}_\ell\hat{\Omega}_\ell}{(\eta_k\Omega_k + \hat{\eta}_\ell\hat{\Omega}_\ell)^2 + (\Omega_k - \hat{\Omega}_\ell)^2)}\right] \quad (8)$$

(7) is a set of power balance relations displaying the way in which the feedback gains meditate the exchange of energy among the plant and compensator modes. (7.a), for example, states that the rate of change of the k^{th}plant mode energy equals the sum of the power loss due to dissipation $(-2\eta_k\Omega_k E_k)$ and the net power flow from the k^{th}plant mode into all the compensator modes $(\sum_\ell \sigma_{k\ell}(\hat{E}_\ell - E_k))$. The net power flows are seen to be proportional to the energy differences and, because of the nonnegativity of the coefficients $\sigma_{k\ell}$, power always flows from the higher energy mode to the lower energy mode. This energy exchange is more rapid, the larger is the power-flow coefficients $\sigma_{k\ell}$. An efficient linear controller design achieves its results by making the σ_{k_ℓ}'s as large as possible, to facilitate energy transfer from plant to compensator, and by choosing the $\hat{\eta}_k$'s (the compensator modal damping ratios) somewhat larger than the η_k's, thereby speeding up the dissipation of the energy transferred to the compensator.

Equation (8) shows that the power flow coefficients are inherently nonnegative and are sharply peaked functions of the frequency separation, $|\Omega_k - \hat{\Omega}_\ell|$, between plant and compensator modes. Thus, efficient linear control design (via Linear-Quadratic-Gaussian design, for example) maximizes the $\sigma_{k\ell}$'s by choosing the $\hat{\Omega}_k$ to nearly match the plant mode frequencies. This feature of quadratically optimal design, while it confers great efficiency, is also the source of major limitations. First, designed-for performance can be achieved only if the plant modal frequencies are accurately estimated in advance. In any case, a particular plant mode exchanges energy efficiently only with the compensator mode that matches its frequency.

The primary question addressed here is: Is it possible, by replacing the constant gains in (6) by functions of y and/or $\hat{z}$, to create a nonlinear compensator that achieves more efficient power flow between plant modes and compensator modes - i.e., energy exchange that is nearly independent of the modal frequency differences and that permits large power flow from any one plant mode simultaneously to *all* compensator modes? In the following Sections, we propose a nonlinear compensator design and investigate the design via both numerical simulations and analysis. Although the results are by no means complete, these exploratory investigations indicate an affirmative answer to the above question.

We find that, independently of plant modelling errors, the nonlinear compensator provides very effective vibration suppression. Moreover, the compensator can be viewed as the interconnection of very simple modular units and its effectiveness increases in proportion to the number of modules. This raises the question: Can the proposed nonlinear compensator be realised in a neural net? Accordingly, we demonstrate, in Section 5, that the nonlinear compensator can be implemented as a neural net with analog neurons.

2. A Nonlinear Compensator for Structural Vibration Suppression

With plant model (1) and (2), let us consider, in place of the linear controller (3), (4), (6), the nonlinear controller:

$$u = -\kappa y e^{\mathrm{T}} \hat{z}_2 \tag{9}$$

$$\dot{\hat{z}} = \begin{bmatrix} 0 & \hat{\Omega} \\ -\hat{\Omega} & -\hat{\eta}\hat{\Omega} \end{bmatrix} \hat{z} + \kappa y \begin{bmatrix} 0 \\ e \end{bmatrix} y \tag{10}$$

where:

$$e^{\mathrm{T}} \triangleq (1, 1, \ldots, 1)$$

In effect, we have replaced κ in (6) by κy where y is the sensor measurement signal (2), and κ is again a nonnegative constant whose magnitude indicates the controller "gain." We now study the dynamics of the closed-loop system defined by (1), (2), (9), and (10).

The intuitive reasoning behind the choices (9), (10) is as follows. First, although we retain the modal character of the linear compensator –i.e., the term $\begin{bmatrix} 0 & \hat{\Omega} \\ -\hat{\Omega} & -2\hat{\eta}\hat{\Omega} \end{bmatrix} \hat{z}$, the feedback gains are now chosen proportional to the measurement signal y in order to obtain a *quadratic* nonlinearity for the compensator as a whole. Motivated by analogies with fluid dynamic turbulence, a quadratic nonlinearity was desired in order to promote *chaotic** motion in the closed-loop system. This chaotic dynamics endows the signal y with a smooth, broad band power spectrum with no spikes or dominant harmonics. The resulting broad-band character of the "feedback gain", κy, is expected to give rise to very efficient power-flow from each plant mode to *all* compensator modes in a manner that is largely insensitive to the precise values of the structural modal frequencies.

To see if the above intuitive motions were correct, we first observed closed-loop performance via "brute force" numerical simulations for a particular model of the structural plant. Specific results and general observations are given in the next Section. Then using some of the general empirical observations from the simulations, a semi-empirical theory was developed in the form of a system of energy flow equations analogous to (7). These results are given in Section (4).

* In using the term "chaotic dynamics" herein, we refer to the operational definition given in [2] - namely, a system exhibits chaotic dynamics when, despite purely deterministic initial conditions and periodic inputs, its measured response exhibits smooth, continuous power spectra.

3. Numerical Simulations for an Example

For preliminary investigation of the compensator (9), (10), we performed numerical simulations for a particular example of the structural plant (1), (2). The example chosen is a string extended along $x \in [0, L]$, held fixed at both ends with uniform tension T. The potential differential equation for the lateral deflection, $w(x, t)$ is:

$$\rho \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial x^2} + f(x) \tag{11}$$

$$w(0) = w(L) = 0$$

where ρ is the constant lineal mass density and $f(x)$ is the force distribution due to a single control actuator. The modal decomposition of this system has the form:

$$w(x, t) = \sum_k \Psi_k(x) w_k(t) \tag{12}$$

$$\left(\int_0^L dx \Psi_k^2 = 1 \right)$$

$$\Psi_k(x) = \sqrt{\frac{2}{L}} \sin k\pi \frac{x}{L}$$

where, assuming uniform proportional damping, the modal coordinates w_k, satisfy:

$$\ddot{w}_k + 2\eta\Omega_k \dot{w}_k + \Omega_k^2 w_k = \frac{1}{\rho} \int_0^L dx \Psi_k(x) f(x) \tag{13}$$

Now, we nondimensionalize variables so that $\sqrt{\frac{T}{\rho}} \frac{t}{L} = 1$ and $\frac{1}{\rho} f = \hat{f} \sqrt{\frac{L}{2}}$ and suppose that $\hat{f}(x)$ arises from a point force actuator located at $x = \xi_a L$. Then:

$$\ddot{w}_k + 2\eta\Omega_k \dot{w}_k + \Omega_k^2 w_k = b_k u \tag{14}$$

$$\Omega_k = k$$

$$b_k = \sin k\pi\xi_a$$

Finally, assuming a collocated sensor and defining the plant state as $x^{\mathrm{T}} \triangleq (\Omega_1 w_1, \ldots, \Omega_n w_n, \dot{w}_1, \ldots, \dot{w}_n)$, the equations of motion are found to be identical in forms to (1) and (2) with:

$$\begin{aligned} \Omega_k &= k; \quad k = 1, \ldots, n \\ \eta_k &= \eta \ = \text{constant damping ratio} \\ b^{\mathrm{T}} &= [\sin \pi\xi_a \ \ \sin 2\pi\xi_a, \ldots, \sin n\pi\xi_a] \end{aligned} \tag{15}$$

With the above expressions and for a variety of choices of $\hat{\Omega}, \hat{\eta}$ and η, we conducted numerical simulations of the closed-loop systems consisting of (1), (2), (9) and (10) with various initial conditions. The qualitative results are not very sensitive to the choice of $\hat{\Omega}$ or the initial conditions. Some of these results are illustrated in Figs. 1 and 2, which pertain to the case $\hat{\eta}_k = \eta = 0.002$, $\hat{\Omega}_k = \Omega_k$, $\xi_a = 0.3$ and $n = 20$ (so that there are 80 states in the closed-loop simulation) and with initial conditions such that the first mode has unit displacement and velocity and all other states are zero –i.e.:

$$x^{\mathrm{T}}(0) = (1, 1, 0, \ldots, 0)$$

The simulations were obtained using a fourth-order Runge-Kutta integration routine. Special care has to be exercised in selecting a sufficiently small integration time-step, since, as will be seen, $y(t)$ exhibits very high frequency content for sufficiently large values of κ.

Fig. 1 shows time histories of the displacement response of the initially excited first mode and the corresponding time histories of the sensor measurement for three typical values of κ. For very small κ, the first mode response shows a lightly damped periodic motion dominated by the first mode frequency and $y(t)$ shows similar characteristics. Slightly larger values of κ result in weakly damped periodic motions with higher harmonics of the first mode frequency coming more into play.

On the other hand for κ above some critical threshold (which is roughly unity in this example), the response time histories exhibit a qualitative change. As illustrated by the middle plots in Fig. 1, the initially excited mode drops dramatically in amplitude after a relatively brief period and then is damped slowly thereafter. Neither $x_1(t)$ nor $y(t)$ exhibits any apparent periodicities and $y(t)$, in particular, shows evidence of higher frequency content. All of these tendencies are amplified for still larger values of κ (see the bottom of Fig. 1).

Further insight into the system dynamics is afforded by Fig. 2 which shows (under the same conditions as in Fig. 1) the time histories of the instantaneous modal energies (defined by equations (5) but without the time averaging) and the corresponding power spectra of $y(t)$. For small κ (top part of Fig. 2) energy sloshes back and forth among the first several plant modes and the power spectrum of y exhibits sharp isolated spikes. For κ above the threshold (middle part of Fig. 2), it is seen that the rapid initial drop off of the first mode energy is accompanied by a redistribution of energy into all the other modes, so that after a brief period, all the plant and compensator energies are roughly equal. During the period wherein modal energies are equalized, the total energy, E_T:

$$E_T \triangleq \sum_{k=1}^{n} E_k + \sum_{k=1}^{n_c} \hat{E}_k \tag{16}$$

does not appreciably decline. E_T is dissipated at a rather small rate consistent with the assumed plant and compensator damping ratios ($\eta = \hat{\eta} = 0.002$). Thus, the rapid decline in the initially excited mode observed in Fig. 1 is due not to direct energy dissipation but to the flow of the first mode energy into all other modes. Accompanying the modal energy equalization phenomenon, the peaks in the power spectrum of y have broadened and coalesced to form a continuous spectrum. Since the spectral peak broadening and coalescence is much larger than what can be attributed to damping and to the finite time period of the time sequence used to calculate the spectrum, it is apparent that the system undergoes chaotic motion for κ above the critical value.

The above tendencies are strengthened for still larger values of κ (bottom part of Fig. 2). Modal energy equalization occurs even sooner, the energy flow to all modes occurring at nearly the same rate; regardless of the relative values of the modal frequencies. The power spectrum of y is further smoothed and broadened. Indeed, the spectrum of y is nearly constant over the whole frequency band occupied by the plant and compensator modal frequencies.

The above findings tend to confirm the heuristic insights used in constructing the design (9), (10). In particular the quadratic nonlinearities in the compensator do trip the system into chaotic motion, resulting in a broad-band spectrum for $y(t)$ and very efficient energy flow among all the plant and compensator modes, even for modes having widely separated frequencies.

Surveying all the simulation results, we obtain additional observations regarding time-averaged correlation, autocorrelation functions and power spectra that prove useful in constructing a semi-empirical theory of the dynamics of the nonlinear compensator. With regard to correlations and autocorrelations, we have:

0.1 For t larger that $\sim$10 lowest mode periods, the separate modal coordinates x_{1k}, x_{2k}; $k = 1, \ldots, n$ are approximately uncorrelated (in the sense of time averaging)

0.2 Again for $t \gtrsim 10$ lowest mode periods, the autocorrelation coefficient of y:

$$\rho_y(t, \tau) \triangleq < y(t)y(t-\tau) > / < y^2(t) > \tag{17}$$

is approximately independent of t (i.e., it is weak-sense stationary)

In addition, we have observations concerning the changing character of the power spectra of the plant modal velocities, x_{2k} $k = 1, \ldots, n$, as κ increases. We denote the power spectrum of x_{2k} by $S_{x_{2k}}(\omega)$. Noting that the correlation coefficient, $\rho_{x_{2k}}(\tau)$, is the inverse Fourier transform of $S_{x_{2k}}(\omega)/\int_0^\infty d\omega S_{x_{2k}}$, the following observations also have direct import for correlation functions:

0.3 For very small κ, $S_{x_{2k}}$ exhibits isolated spikes at the frequencies of excited modes having half-power widths equal to $2\Omega_k\eta$

0.4 For larger κ, near the critical threshold value κ_c, $S_{x_{2k}}$ shows spikes at many additional modal frequencies. The width of the spikes grows to $\sim \kappa E_T^{\frac{1}{2}}$ (E_T given by (16)]. The value of κ_c appears to be roughly $\frac{1}{E_T^{\frac{1}{2}}}\delta\omega$, where $\delta\omega$ is the minimum separation between plant modal frequencies.

0.5 For $\kappa >> \kappa_c$, the spikes in $S_{x_{2k}}$ coalesce into a smooth, broad-band spectrum, which is approximately constant up to some frequency Ω_k and then drops off rapidly at higher frequencies. The value of Ω_k is roughly $\kappa E_T^{\frac{1}{2}}$.

4. A Semi-Empirical Theory for the Energy Dynamics of the Nonlinear Compensator

Here we use the simulation results and corresponding observations discussed in the last Section to construct a semi-empirical theory for the nonlinear compensator (9), (10). The theory takes the forma of approximate equations-of-motion for the time-averaged modal energies analogous to (7). These equations are then used to deduce various qualitative phenomena and provide a few useful design guidelines.

Space limitations preclude the full derivations, which will be given elsewhere. Here we attempt merely to sketch the development.

The first step is to form equations of motion for the "second moment matrix", $Q \triangleq \begin{pmatrix} x \\ \hat{x} \end{pmatrix} (x^T, \hat{x}^T)$, of the full closed-loop system (1), (2), (9) and (10). We then manipulate the equations to eliminate the cross-correlation terms in favor of the modal mean-squares x_k^2, $\hat{x}_m^2$; $k = 1 \ldots n$, $m = 1 \ldots n_c$, apply the time averaging operator and employ a perturbation expansion approach to obtain equations approximately valid for small κ. Neglecting terms of order κ^3 or smaller, we obtain the following equations for the time-averaged plant-mode energies, E_k, and compensator mode energies $\hat{E}_k$:

$$\begin{aligned} \dot{E}_k &= -2\eta_k\Omega_k E_k + \sum_\ell \vartheta_{k\ell}(\hat{E}_\ell - E_k), \quad k = 1 \ldots n \\ \dot{\hat{E}}_k &= -2\hat{\eta}_k\hat{\Omega}_k \hat{E}_k + \sum_\ell \vartheta_{\ell k}(E_\ell - \hat{E}_k), \quad k = 1 \ldots n_c \end{aligned} \tag{18}$$

where:

$$\begin{aligned} \vartheta_{k\ell} &= \kappa^2 b_k^2 \int_0^t d\tau < y(t)y(t-\tau) > \gamma\ k\ell(\tau) \\ \gamma_{k\ell}(t) &\triangleq \frac{1}{2} e^{-(\eta_k\Omega_k + \hat{\eta}_\ell\hat{\Omega}_\ell)t}[\cos(\omega_k - \hat{\omega}_\ell)t + \cos(\omega_k\hat{\omega}_\ell)t] \end{aligned} \tag{19}$$

and where $\omega_k, \hat{\omega}_k$ denote the damped natural frequencies:

$$\begin{aligned} \omega_k &\triangleq \Omega_k\sqrt{1-\eta_k^2} \\ \hat{\omega}_k &\triangleq \hat{\Omega}_k\sqrt{1-\hat{\eta}_k^2} \end{aligned} \tag{20}$$

Although developed for small κ, equations (18) appear to give correct results even for large κ. This may be due to the semi-er pirical manner in which explicit expressions for the coefficients $\vartheta_{k\ell}$ are derived, as discussed in the following.

It remains to express $\vartheta_{k\ell}$ explicitly in terms of the modal energies. To do this we use the empirical observations 0.1 through 0.5

given in the previous section. First, and immediate consequence of 0.1 and 0.2 is:

$$< y(t)y(t-\tau) > \simeq \sum_k b_k^2 < x_{2k}^2(t) > \rho_{x_k}(\tau) \tag{21}$$

where

$$\rho_{x_k}(\tau) \triangleq < x_{2k}(t)x_{2k}(t-\tau) > / < x_{2k}^2(t) > \tag{22}$$

However, neglecting higher-order terms in κ; $< x_{2k}^2 > \simeq E_k(t)$. Using this approximation in (21) and substituting the result into $\sigma_{k\ell}$ gives:

$$\sigma_{k\ell} \simeq \kappa^2 b_k^2 \sum_m b_m^2 E_m \Gamma_{mk\ell} \tag{23.a}$$

$$\Gamma_{mk\ell} \triangleq \int_0^t d\tau \rho_{x_m}(\tau)\gamma_{k\ell}(\tau) \tag{23.b}$$

Next we deduce the form of $\Gamma_{mk\ell}$ by considering two limiting cases: very small κ and very large κ. In the case of very small κ, 0.3 implies:

$$\rho_{x_m}(\tau) \simeq e^{-\eta \Omega_m \tau} \cos \omega_m \tau$$

This can be substituted into (23.b) and $\Gamma_{mk\ell}$ evaluated directly. For large κ, we use 0.4 and 0.5 together with dimensional analysis to deduce the asymptotic form of $\Gamma_{mk\ell}$. As the last step, an expression for $\Gamma_{mk\ell}$ is devised which correctly reduces to the expressions derived for the two limiting cases. The final result is:

$$\sigma_{k\ell} \simeq \kappa^2 b_k^2 \sum_m b_m^2 E_m \Gamma_{Mk\ell} \tag{24.a}$$

$$\begin{aligned} \Gamma_{mk\ell} = \frac{1}{4}(\Delta_m + \eta_k\Omega_k + \hat{\eta}_\ell\hat{\Omega}_\ell) \{ & \frac{1}{(\Delta_m + \eta_k\Omega_k + \hat{\eta}_\ell\hat{\Omega}_\ell)^2 + (\omega_m + \omega_k - \hat{\omega}_\ell)^2} \\ & \frac{1}{(\Delta_m + \eta_k\Omega_k + \hat{\eta}_\ell\hat{\Omega}_\ell)^2 + (\omega_m - \omega_k + \hat{\omega}_\ell)^2} \\ & \frac{1}{(\Delta_m + \eta_k\Omega_k + \hat{\eta}_\ell\hat{\Omega}_\ell)^2 + (\omega_m + \omega_k + \hat{\omega}_\ell)^2} \\ & \frac{1}{(\Delta_m + \eta_k\Omega_k + \hat{\eta}_\ell\hat{\Omega}_\ell)^2 + (\omega_m - \omega_k - \hat{\omega}_\ell)^2} \end{aligned} \tag{24.b}$$

$$\Delta_m \triangleq \eta\Omega_m + \kappa E_T^{\frac{1}{2}} \tag{24.c}$$

where $\omega_k, \hat{\omega}_k$ and E^T are defined in (20) and (16), respectively.

Equations (18) and (24) constitute a closed system of equations approximately describing the dynamics of the modal energies of the plant and compensator and may now be used to deduce various properties.

First, it should be noted that (18) are of the same structure as (7). The power flow from any one mode to all other modes is again proportional to the energy differences and because the coupling coefficients $\sigma_{k\ell}$ are all intrinsically positive, power always flows from the more energetic to the less energetic mode. These features ensure that the system will be driven toward equalisation of modal energies with the time scale for equalisation being dictated by the magnitude of $\kappa E_T^{\frac{1}{2}}$. This is consistent with the qualative observations of the last section. Furthermore, (24) shows that for sufficiently large κ; $\Gamma_{mk\ell}$ approaches the uniform limit $\frac{1}{\kappa E_T^{\frac{1}{2}}}$, so that:

$$\sigma_{k\ell} \simeq \frac{\kappa b_k^2}{E_T^{\frac{1}{2}}} < y^2 >$$

Thus, in contrast to (8), there is strong power flow from any one plant mode to all other compensator modes, regardless of the relative values of the plant compensator frequencies. This efficient energy sharing is a consequence of the nonlinearity introduced in the compensator design (9) and (10).

As a last topic, we use (18) and (24) to obtain simple quantitative estimates of the speed with which structural vibration energy can be drained away to the compensator via nonlinear design. For this purpose, consider the case wherein a set of n_d structural modes are directly excited and it is desired to reduce the vibration energy of these modes because of their determinous impact on system performance. We estimate the closed-loop response by taking account of the interactions between the compensator modes and only these directly excited modes.* For simplicity, suppose that all n_d modes are initially excited to the same energy:

$$E_k(0) = E_0 \quad \forall k = 1 \dots n_d$$

and that the compensator is initially quiescent; i.e., $\hat{E}_k^{(0)} = 0 \ \forall k$. Also let us estimate the magnitude of all the modal influence coefficients by some average value $\bar{b}$, i.e., $b_k^2 \simeq \bar{b}^2 \ \forall k$. Finally, since flexible structure vibration control is our motivating application, we assume small damping ratios for both plant and compensator; i.e., $\eta_k << 1.0$, $\hat{\eta}_k << 1.0$.

With the above conditions, suppose that $[0, \Delta\Omega]$ is the frequency band encompassed by all the initially excited modes. We need only choose the n_c compensator frequencies, $\hat{\Omega}_k$; $k = 1 \dots n_c$, *somewhere* in this band, because, as (24.b) shows, the choice of κ such that

$$\kappa E_T^{\frac{1}{2}} > 3\Delta\Omega \tag{25}$$

insures that all the $\Gamma_{mk\ell}$'s reduce to the same value, $\frac{1}{\kappa E_T^{\frac{1}{2}}}$, independently of the compensator frequencies. The design choice (25) implies, by use of (24.a), that:

$$\sigma_{k\ell} \simeq \frac{k\bar{b}^2}{E_T^{\frac{1}{2}}} \sum_m E_m \tag{26}$$

Using this result and the fact that during the initial period of energy equalisation, $\sum_m E_m$ can be estimated by E_T, equations (18) become:

$$\begin{aligned} \dot{E}_k &= -2\eta_k\Omega_k E_k + \kappa E_T^{\frac{1}{2}}\bar{b}^2 \sum_{\ell=1}^{n_c}(\hat{E}_\ell - E_k); \quad k = 1 \dots n_d \\ \dot{\hat{E}} &= -2\hat{\eta}_k\hat{\Omega}_k\hat{E}_k + \kappa E_T^{\frac{1}{2}}\bar{b}^2 \sum_{\ell=1}^{n_d}(E_\ell - \hat{E}_k); \quad k = 1 \dots n_c \\ E_k(0) &= E_0, \quad \hat{E}_k(0) = 0 \quad \forall k \end{aligned} \tag{27}$$

Note that the above equations imply:

$$\begin{aligned} \frac{d}{dt}E_T &= -2\left(\sum_k \eta_k\Omega_k E_k + \sum_k \hat{\eta}_k\hat{\Omega}_k\hat{E}_k\right) \\ E_T(0) &= n_d E_0 \end{aligned} \tag{28}$$

Evidently, the total energy is dissipated over a time scale of order $\frac{1}{\eta_k}$ or $\frac{1}{\hat{\eta}_k}$ (for some k). By virtue of our small damping ratio assumption and design choice (25), this time scale is much longer than the time scale over which the initial energy redistribution takes place. For investigation of the initial period of energy equalisation, therefore, we may treat E_T as a constant $\simeq n_d E_0$ and neglect the damping terms involving η_k and $\hat{\eta}_k$ in (27). With these approximations, one immediately obtains the following solutions for the relatively brief initial time period over which energy redistribution occurs:

$$\begin{aligned} E_k &= E_0 - \frac{n_c E_0}{n_d + n_c}(1 - e^{-t/T_k}); \quad k = 1 \dots n_d \\ \hat{E}_k &= \frac{n_d E_0}{n_d + n_c}(1 - e^{-t/T_k}); \quad k = 1 \dots n_c \end{aligned} \tag{29}$$

where:

$$T_k \triangleq \frac{1}{\kappa\bar{b}^2 E_T^{\frac{1}{2}} n_d(n_d + n_c)} \tag{30}$$

* This simplification actually results in an overestimation of the energies resident in the n_d excited modes.

Thus, after a time period of order T_κ, the energy initially residing in the plant structural vibration is drained away to the compensator modes and *all* modal energies are approximately equalised to:

$$t \geq T_\kappa : \quad E_k \simeq \frac{E_T}{n_d + n_c}, \quad \hat{E}_\kappa \simeq \frac{E_T}{n_d + n_c} \quad \forall k \qquad (31)$$

Now for $t >> T_\kappa$, we can characterise the evolution of the total energy, E_T, by using (31) in (28) to obtain:

$$t >> T_\kappa : \quad \frac{d}{dt} E_T \simeq -2\left(\sum_k \eta_k \Omega_k + \sum_k \hat{\eta}_k \hat{\Omega}_k\right) E_T \qquad (32)$$

Thus, over large time scales, the total energy is damped exponentially with an equivalent damping ratio that is a weighted average of *both* the plant and compensator damping ratios.

Another noteworthy feature of the above results is that both the time period, T_κ (equation (30)), needed for equalisation of energies and the value to which the modal energies are equalised (equations (31)) are universely proportional to the number of compensator modes. Thus, especially effective vibration suppression can be achieved by the nonlinear compensator if the number of its states can be made very large. This, suggests the question: Does the compensator (9), (10) have a simple repetitive structure can this structure be implemented as a neural net containing a large number of analog neurons?

5. A Neural Net Realisation of the Nonlinear Compensator

Equations (9) and (10) may be rewritten to reveal that the control signal u is the sum of n_c components:

$$u = \sum_{k=1}^{n_c} u_k \qquad (33)$$

where each u_k is the output of a *simple* nonlinear oscillator:

$$u_k = -\kappa y \hat{v}_k$$
$$\frac{d}{dt}\begin{pmatrix}\hat{\xi}_k \\ \hat{v}_k\end{pmatrix} = \begin{bmatrix} 0 & \hat{\Omega}_k \\ -\hat{\Omega}_k & -2\hat{\eta}_k\hat{\Omega}_k\end{bmatrix}\begin{pmatrix}\hat{\xi}_k \\ \hat{v}_k\end{pmatrix} + \kappa y^2 \begin{pmatrix}0 \\ 1\end{pmatrix} \qquad (34)$$

Thus, the nonlinear compensator is a combination of simple repetitive modules and this suggests that it can be efficiently implemented as a neural network.

To see that this is the case, define $g(\cdot)$ to be some antisymmetric sigmoidal function such that $g'(x)$ is maximum at $x = 0$, where $g'(0) = 1$. Then consider the replacement of (34) by:

$$u_k = -\kappa y g(\hat{v}_k)$$
$$\frac{d}{dt}\begin{pmatrix}\hat{\xi}_k \\ \hat{v}_k\end{pmatrix} = -\hat{\eta}_k\hat{\Omega}_k\begin{pmatrix}\hat{\xi}_k \\ \hat{v}_k\end{pmatrix} + \begin{bmatrix} 0 & \hat{\Omega}_k \\ -\hat{\Omega}_k & 0\end{bmatrix}\begin{pmatrix}g(\hat{\xi}_k) \\ g(\hat{v}_k)\end{pmatrix} + \kappa y^2 \begin{pmatrix}0 \\ 1\end{pmatrix} \qquad (35)$$

The above equations essentially reduce to (34) for small signal amplitudes such that $g(x) \simeq x$.

It is now easy to see that (33), (35) are equivalent to a system formed by interconnection of analog neurons, of the form given by Hopfield, illustrated here in Fig. 3. Using such neurons, we form neuron pairs in the manner shown in Fig. 4.a, such that each neuron pair implements one compensator mode. We finally interconnect n_c neuron pairs as shown in Fig. 4.b to obtain a system completely equivalent to (33), (35).

It is seen that this as an implementation involving very sparse neuronal interconnections. Note also, that just as $\frac{1}{2}\sum_k(x_{1k}^2 + x_{2k}^2) + \frac{1}{2}\sum_k(\hat{x}_{1k}^2 + \hat{x}_{2k}^2)$ is a Lyapunov function for the closed loop system (1), (2), (9), (10); so too, the quantity

$$J \triangleq \frac{1}{2}\sum_{k=1}^{n}(x_{1k}^2 + x_{2k}^2) + \sum_{k=1}^{n_c}\left(\int_0^{\hat{\xi}_k(t)} g(\chi)d\chi + \int_0^{\hat{v}_k(t)} g(\chi)d\chi\right) \qquad (36)$$

is a Lyapunov function for the systems (1), (2), (33), (35), since:

$$\frac{dJ}{dt} = -2\sum_{k=1}^{n}\eta_k\Omega_k x_{2k}^2 - \sum_{k=1}^{n_c}\hat{\eta}_k\hat{\Omega}_k(g(\hat{\xi}_k)\hat{\xi}_k + g(\hat{v}_k)\hat{v}_k) \qquad (37)$$

Thus the controller (33), (35) is inherently stable.

Summary and Conclusion

In this paper, we explored a novel type of nonlinear dynamic controller design that was originally motivated by certain issues in the problem of vibration suppression in flexible structures. Because of the particular form of quadratic nonlinearities, the controller provides extremely efficient energy exchange mechanisms capable of rapidly draining energy away from the structural plant. In addition the controller was shown to consist of the interconnection of simple repetitive modules and its effectively increases with the number of such modules and its effectively increases with the number of such modules. These features motivate the implementation of the nonlinear controller via the neural network architecture explored in the last Section. With the advent of appreciate analog neural net hardware, this suggested nonlinear compensation scheme could offer a very effective means of vibration suppression. For example suppose some 10 structural modes are significantly excited and must be suppressed and we employ the neural net controller involving a modest number of neurons, say 2000. Then $n_d = 10$, $n_c = 10^3$ and it is seen from (31) that the vibrational energy of each excited mode is quickly reduced to $\frac{n_d}{n_d + n_c}$ of its initial value – a reduction of more than a hundredfold.

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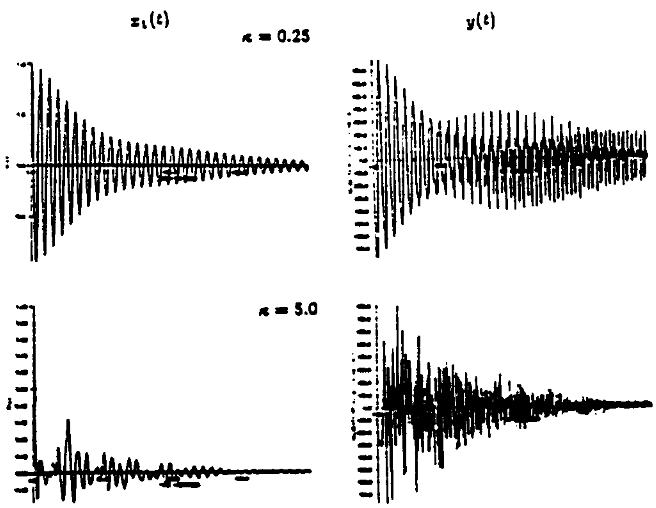

Fig. 1: Time histories of the initially excited modal displacement and the sensor measurement for $\hat{\Omega} = \hat{\Omega}_k$, $\hat{\eta}_k = 0.002$ and various values of κ.

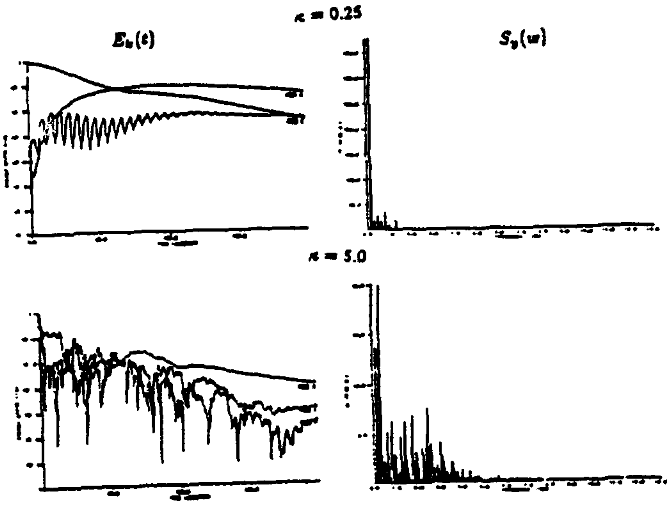

Fig. 2: Time histories (for the same conditions as Fig. 1) of the instantaneous model energies ($E_\kappa = \frac{1}{2}(x_{1\kappa}^2 + x_{2\kappa}^2)$, $\hat{E}_\kappa = \frac{1}{2}(\hat{x}_{1\kappa}^2 + \hat{x}_{2\kappa}^2)$) and the corresponding power spectra of the sensor measurement.

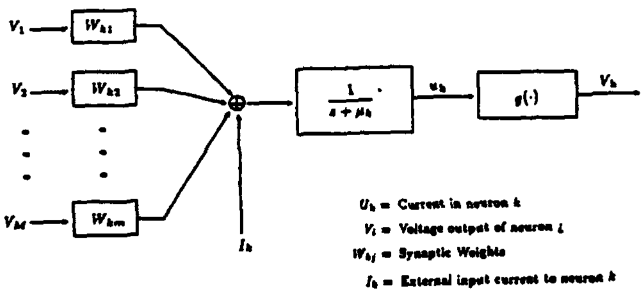

Fig. 3: Basic structure of an analog neuron following Hopfield (3).

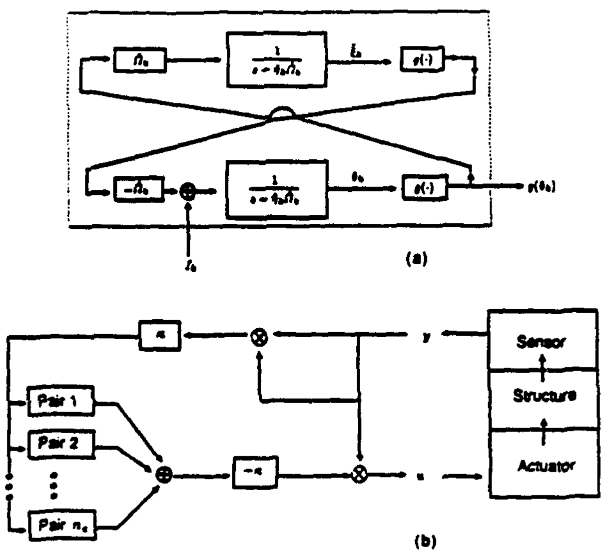

Fig. 4: Implementation of controller (33), (35) via a neural network:
(a) fundamental neuron pair, (b) overall architecture.

Appendix E

"Real Parameter Uncertainty and Phase Information in the Robust Control of Flexible Structures"

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Real Parameter Uncertainty and Phase Information in the Robust Control of Flexible Structures

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Abstract

Real parameter uncertainty and phase information play a key role in the analysis and synthesis of robust controllers for lightly damped flexible structures. In this paper we discuss the ramifications of this issue as it affects achievable performance in structural control. In this regard we review the state of knowledge in addressing real parameter and phase issues. The discussion is illustrated by examining robust controllers designed for the ACES structure at Marshall Space Flight Center. These controllers were designed by means of the Maximum Entropy generalised LQG methodology.

1. Introduction

Traditionally, spacecraft control-system designers have been primarily concerned with controlling rigid body attitude modes, while avoiding the excitation of flexible body dynamics. As performance requirements become more stringent and spacecraft become larger, control-system design must explicitly encompass flexible dynamic modes so as to actively suppress undesired structural vibration. Furthermore, for complex spacecraft, multi-input multi-output controllers with significant bandwidth may be required.

Since structural modeling and identification of large flexible structures in a 1-g environment possess inherent limitations, one of the key issues in structural control is robustness. Although robust control has undergone intensive development in the past two decades, there remain aspects of robust control that are relevant to structural control and that are largely unresolved. *These aspects are the role of real parameter uncertainty and phase information.* The purpose of this paper is to examine the impact of these issues on structural control, their interrelationship, and their manifestation within the analysis and synthesis of feedback systems.

2. Phase Stabilisation Versus Gain Stabilisation

From a classical control-design point of view, the issues of real parameter uncertainty and phase information are manifested in the fundamental concepts of gain and phase stabilisation. In terms of gain stabilisation, stability of a single-input single-output closed-loop system is insured by designing the controller so that the magnitude of the loop transfer function is less than unity in frequency regimes in which the phase is either known to be near 180° or is highly uncertain. In terms of phase stabilisation, stability is achieved by insuring that the phase of the loop transfer function is well behaved where the loop transfer function has gain greater than unity. Roughly speaking, phase stabilization can be used to allow high loop gain and thus achieve high performance in frequency regimes in which sufficient phase information is available, whereas gain stabilisation (e.g., rolloff) is needed to insure stability where the phase of a system is very poorly known. For further discussion of the distinction between phase and gain stabilization, see [1].

3. Structured Real Parameter Uncertainty Versus Unstructured Complex Parameter Uncertainty

A variety of approaches have been proposed for addressing uncertainty in the synthesis of robust controllers. These include H_∞ synthesis [2-5], quadratic Lyapunov functions [6-9], and the structured singular value [10,11]. All of these methods effectively treat the uncertain parameters as complex quantities, and are thus conservative with respect to real parameter uncertainty. If the uncertain parameters are known to be real, then special techniques are required to avoid conservatism [12-19].

To illustrate the conservatism of H_∞ theory in the presence of phase information, it need only be noted that $|e^{j\phi}| = 1$ regardless of the phase angle ϕ. Indeed, any robustness theory based upon norm bounds will suffer from the same shortcoming. Of course, every real parameter can be viewed as a complex parameter with phase $\phi = 0°$ or $\phi = 180°$. Since the existence of a *single* Lyapunov function for a norm-bounded uncertainty class is equivalent to a small-gain condition [9], much of Lyapunov theory exhibits a similar conservatism.

In structural modeling via finite element models, uncertainty in the mass, damping, and stiffness matrices is unavoidable. If the mass and stiffness matrix uncertainty is modeled as complex, unstructured perturbations, then the damping matrix is effectively perturbed as well. Indeed, damping is sometimes modeled as a complex stiffness [20, p. 194]. Difficulty arises when stiffness uncertainty is large relative to damping uncertainty, in which case complex stiffness uncertainty corresponds to a physically unrealisable unstable plant model.

4. Phase Information and Positive Real Transfer Functions

Phase information plays a fundamental role in structural control. For illustration, consider a flexible structure with a colocated rate sensor/force actuator pair and assume these devices are ideal. For such a system the transfer function from the actuator to sensor is known to be positive real, that is, to have phase lying between 90° and -90° [21,22]. In a negative feedback configuration, a controller for this plant that is strictly positive real cannot destabilise the system since the loop transfer function has phase less than -180° over all frequencies. Hence such a control system will be unconditionally robust to uncertainties in both natural frequencies and damping. Of course, these observations assume perfect sensors and actuators so as not to introduce additional phase lag. If the sensors and actuators do have significant dynamics, then the feedback law must be chosen so that the transfer function consisting of the cascaded sensor, compensator, and actuator dynamics is strictly positive real. If, in practice, positive realness can only be enforced over a limited frequency band, then loop gain rolloff is required when phase lags or phase uncertainties reach unacceptable levels.

By exploiting the stability guarantee due to the interconnection of positive real MIMO systems, robust positive real controllers have been studied for structural control [23-31]. A related approach involves using H_∞ design in conjunction with the bilinear transformation [32]. By using a Riccati equation to enforce a positive real constraint, robust controllers for positive real uncertainty were obtained in [33]. Related results appear in [34].

Alternative approaches to including phase information in analysis and synthesis include [35-38]. Ref. [39] extends the gain envelope approach of [40,41] to include a phase envelope as well. These envelopes are characterised by real parameters whose effect can then be addressed using real parameter robustness techniques.

An alternative approach to exploiting phase information is based on the concept of structured covariance matrix. Roughly speaking, robustness is not guaranteed by means of a Lyapunov function or covariance bound [8], but rather by means of a covariance matrix whose structure is insensitive to a given class of plant perturbations. This concept provides the basis for the generalised LQG synthesis technique known as Maximum Entropy design [42-48].

5. An Illustrative Example Using Maximum Entropy Synthesis

The ACES experimental testbed is located at NASA Marshall Space Flight Center. The basic test article, a spare Voyager Astromast, is a deployable, lightweight (about 5 pounds), lightly damped beam, approxi*mately 45 feet* in length. The Astromast is symmetric with a triangular cross section. Three longerons form the converse of the beam and extend continuously along its full length. The cross members, which give the beam its shape, divide the beam into 91 sections each having equal length and mass and similar elastic properties. When fully deployed, the Astromast exhibits a longitudinal twist of approximately 260 degrees.

The ACES configuration consists of an antenna and counterweight legs appended to the Astromast tip and the pointing gimbal arms at the Astromast base. The addition of structural appendages creates the "nested" modal frequencies characteristic of large space structures. Overall, the structure is very flexible and lightly damped. It contains many closely spaced, low frequency modes (more than 40 modes under 10 Hz). The ACES configuration is dynamically traceable to future space systems and is particularly responsive to the study of line-of-sight (LOS) issues.

The goal of the control design is to position the laser beam in the center of the detector. The disturbances were chosen to be position commands to the Base Excitation Table (BET). The BET motion is regulated by an analog controller which allows any type of BET movement within the frequency limitations of the hydraulic actuation system. In the discussion that follows we will consider only one single-input, single-output loop involving AGS-X, the x-torque of the Advanced Gimbal System, and BGYRO-X, the rotational rate of the base gyro.

For the AGS-X to BGYRO-X loop a model was developed by using the Eigensystem Realisation Algorithm. The ERA model was compared with the frequency response functions (FRF's) derived from the test data. The ERA model matched the FRF data fairly closely in magnitude although the modal frequencies do not exactly coincide. The ERA model differed even more from the FRF's in phase.

Control design for LOS performance, initially performed by using standard LQG techniques, required penalizing only the modes less than 3 Hz. Thus, high performance controllers were limited to having gain only at the modes less than 3 Hz. To avoid destabilizing the two higher frequency modes of the ERA model, the LQG controllers contained notches at the two corresponding frequencies.

The LQG controllers tended to be very sensitive to the phase uncertainty in the performance region, the frequency interval from DC to 3 Hz. They also were very sensitive to the frequency uncertainty in the two higher frequency modes. This control problem thus provides an excellent real-life example of phase uncertainty and real parameter (in this case frequency) uncertainty.

Robust control design was performed using the Maximum Entropy (ME) approach [49]. This approach allows the designer to directly account for real parameter uncertainty [42-48]. Figure 1 describes the influence of ME uncertainty design on the phase of a full-order compensator in the performance region. The phase of the LQG compensator varies widely (and wildly) over this frequency interval, implying that the Nyquist plot of the

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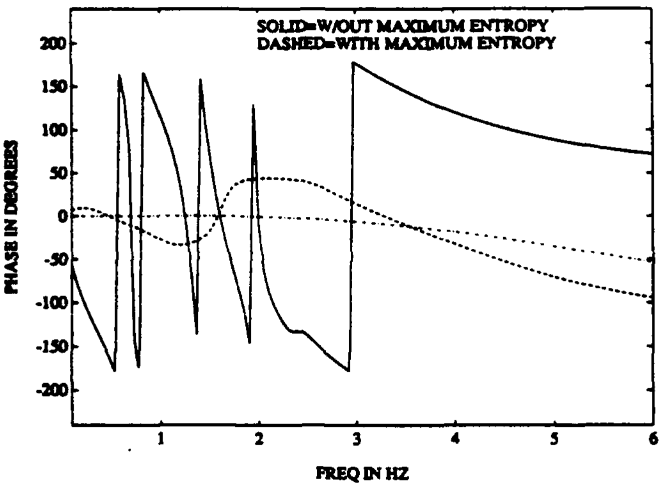
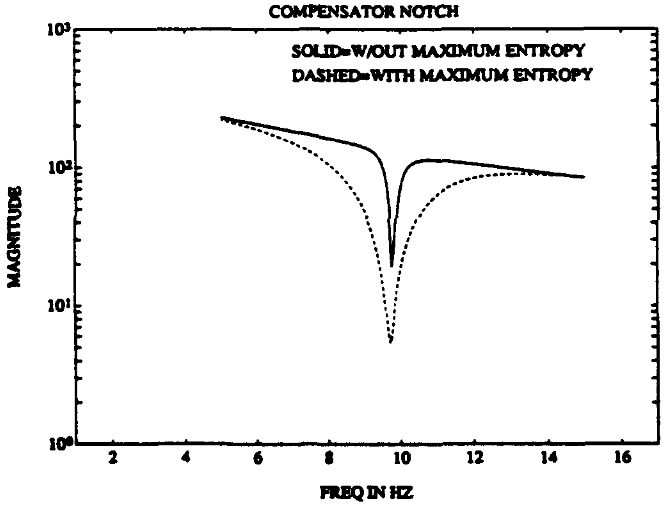
corresponding loop transfer function encircles the origin several times. As one would expect, these designs were nonrobust and, in fact, were unstable when implemented. However, the ME designs became positive real in the performance region tending toward rate feedback. Thus, the ME designs provided the needed stability robustness in the performance region. In addition, the ME designs robustified the LQG controller notches by increasing both the width and depth of the notches (Figure 2).

COMPENSATOR PHASE IN THE PERFORMANCE REGION



COMPENSATOR NOTCH



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Appendix F

"Robust Stabilization With Positive Real Uncertainty: Beyond the Small Gain Theorem"

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Robust Stabilization With Positive Real Uncertainty: Beyond the Small Gain Theorem

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Abstract

In many applications of feedback control, phase information is available concerning the plant uncertainty. For example, lightly damped flexible structures with colocated rate sensors and force actuators give rise to positive real transfer functions. Closed-loop stability is thus guaranteed by means of negative feedback with strictly positive real compensators. In this paper, the properties of positive real transfer functions are used to guarantee robust stability in the presence of positive real (but otherwise unknown) plant uncertainty. These results are then used for controller synthesis to address the problem of robust stabilization in the presence of positive real uncertainty. One of the principal motivations for these results is to utilize phase information in guaranteeing robust stability. In this sense these results go beyond the usual limitations of the small gain theorem and quadratic Lyapunov functions which may be conservative when phase information is available. The results of the paper are based upon a Riccati equation formulation of the positive real lemma and thus are in the spirit of recent Riccati-based approaches to bounded real (H_∞) control.

Key words: bounded real, positive real, robust stabilization, phase information

Running title: Positive Real Uncertainty

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1. Introduction

In many applications of feedback control, phase information is available concerning the plant uncertainty. For example, lightly damped flexible structures with colocated rate sensors and force actuators give rise to positive real transfer functions. Closed-loop stability is thus guaranteed by means of negative feedback with strictly positive real compensators. This principle has been widely used to design robust controllers for flexible structures [1–10].

The salient features of positive real transfer functions is that they are dissipative and phase bounded [11–25]. Hence the feedback interconnection of positive real transfer functions is guaranteed to be stable without requiring that a small gain condition be satisfied. Positive real design is thus potentially less conservative than bounded real (H_∞) design in the presence of phase information.

In this paper we utilize properties of positive real transfer functions to develop new conditions for robust stability and robust stabilizability. Although related results have been developed previously [26–30], this paper goes beyond earlier work by exploiting a Riccati equation formulation in the spirit of recent advances in H_∞ synthesis [31–37]. This is done in two different, but equivalent, ways. First we show that the Riccati equation used to enforce an H_∞ constraint can be transformed to yield a different Riccati equation that enforces a positive real constraint (Theorem 3.2). Alternatively, we show that the same Riccati equation can be obtained by manipulating the conditions of the positive real lemma (Proposition 3.3). Many of the techniques and transformations used in these steps are due to [17], which contains an extensive treatment of positive real and bounded real transfer functions.

Once the Riccati equation that enforces positive realness has been derived, robust stability can be guaranteed for a class of perturbations involving an arbitrary constant positive real matrix (see the set $\mathcal{U}$ defined by (4.6) and Theorem 4.1). The modeling of matrix uncertainty by means of a "fictitious" feedback loop (linear fractional transformation) is directly analogous to the small gain (H_∞) parameter uncertainty model of [37]. In our case, however, the class of uncertainties includes a phase constraint rather than a small gain condition (see Remark 4.1).

Having enforced robust stability for positive real uncertainty, we then proceed in Section 5 to give necessary and sufficient conditions for robust stabilizability in terms of a pair of coupled algebraic Riccati equations (Theorem 5.1). A robustly stabilizing feedback gain is then given in terms of the solutions to the Riccati equations. The stabilizability result is first stated for static output feedback and then specialized to the case of full-state feedback.

Finally, we close the paper by discussing connections between the positive real uncertainty modeling approach of this paper and the Maximum Entropy approach to robust control design of [10,38–43].

Notation:

$\mathbb{R}, \mathbb{R}^{r\times s}$	real numbers, $r \times s$ real matrices
$I_r, I; (\ )^{\mathrm{T}}, (\ )^*$	$r \times r$ identity matrix; transpose, complex conjugate transpose
$\mathrm{tr}, \rho(\ ), \sigma_{\max}(\ )$	trace, spectral radius, largest singular value
$\Vert H(s)\Vert_\infty$	$\sup_{\omega\in\mathbb{R}} \sigma_{\max}[H(j\omega)]$
n, m, m_0, ℓ	positive integers
A, B, C, K	$n\times n$, $n \times m$, $\ell \times n$, $m \times \ell$ matrices
B_0, C_0, D_0, F	$n \times m_0$, $m_0 \times n$, $m_0 \times m_0$, $m_0 \times m_0$ matrices

2. Preliminaries

In this section we establish key definitions and notational conventions that simplify the exposition in later sections. We begin with the definitions of positive real and bounded real transfer functions [11,17].

In this paper a *real-rational matrix function* is a matrix whose elements are rational functions with real coefficients. Furthermore, a *transfer function* is a real-rational matrix function each of whose elements is proper, i.e., finite at $s = \infty$. Finally, a *stable transfer function* is a transfer function each of whose poles is in the open left half plane. Note that the space of stable transfer functions is denoted in [44] by RH_∞, i.e., the real-rational subset of H_∞.

A square transfer function $G(s)$ is called *positive real* [17, p. 216] if 1) all elements of $G(s)$ are analytic for $\mathrm{Re}[s] > 0$ and 2) $G(s) + G^*(s)$ is nonnegative-definite for $\mathrm{Re}[s] > 0$. A square transfer function $G(s)$ is called *strictly positive real* [2,14] if 1) all elements of $G(s)$ are analytic for $\mathrm{Re}[s] \geq 0$ and 2) $G(j\omega) + G^*(j\omega)$ is positive definite for real ω. Finally, a square transfer function $G(s)$ is *strongly positive real* if it is strictly positive real and $D + D^{\mathrm{T}} > 0$, where $D \triangleq G(\infty)$. Note that strongly positive real implies strictly positive real, which further implies positive real. Furthermore, we note that if a transfer function is strictly positive real, then the system is stable and dissipative.

Next, we give the definition of bounded real. A transfer function $H(s)$ is ***bounded real*** [17] if and only if 1) all elements of $H(s)$ are analytic for $\text{Re}[s] \geq 0$ and 2) $I - H(j\omega)H^*(j\omega)$ is nonnegative definite for real ω. Equivalently, 2) can be replaced by [17, p. 307] 2′) $I - H(s)H^*(s)$ is nonnegative definite for $\text{Re}[s] > 0$. Alternatively, a transfer function $H(s)$ is bounded real if and only if $H(s)$ is stable and satisfies $\|H(s)\|_\infty \leq 1$.

Next we establish some notation involving state space realizations of transfer functions. Let [44]

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.1}$$

denote a state space realization of $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$. If $G(s)$ is square and $\det D \neq 0$, then

$$G^{-1}(s) \sim \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]. \tag{2.2}$$

Finally, if $G_1(s) = C_1(sI - A_1)^{-1}B_1 + D_1$ and $G_2(s) = C_2(sI - A_2)^{-1}B_2 + D_2$, then

$$G_1(s)G_2(s) \sim \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right]. \tag{2.3}$$

3. Riccati Equation Characterizations of Positive Real and Bounded Real Transfer Functions

In this section we provide explicit connections between positive real and bounded real transfer functions and their associated state-space realizations. Furthermore, we give Riccati equation characterizations of their resulting state-space realizations. Finally, we draw connections with the well-known positive real lemma [11,17,23].

We begin with a result [17] that relates bounded real transfer functions to positive real transfer functions via the Cayley (bilinear) transform. Throughout the paper γ denotes a positive number.

Lemma 3.1. If $\gamma^{-1}H(s)$ is an $m \times m$ bounded real transfer function with $\det[I_m - \gamma^{-1}H(s)] \neq 0$ for $\text{Re}[s] > 0$, then

$$G(s) \triangleq [I_m - \gamma^{-1}H(s)]^{-1}[I_m + \gamma^{-1}H(s)] \tag{3.1}$$

is positive real. Conversely, if $G(s)$ is an $m \times m$ positive real transfer function such that $G(s)$ is analytic for $\text{Re}[s] \geq 0$, then

$$\gamma^{-1}H(s) \triangleq [G(s) - I_m][G(s) + I_m]^{-1} \tag{3.2}$$

is bounded real.

Proof. Suppose $\gamma^{-1}H(s)$ is bounded real. Since $\det[I_m - \gamma^{-1}H(s)] \neq 0$ for $\mathrm{Re}[s] > 0$, it follows that $G(s)$ is analytic for $\mathrm{Re}[s] > 0$. Then with $G(s)$ defined by (3.1) it follows that $\gamma^{-1}H(s)$ satisfies (3.2). Thus, we obtain for $\mathrm{Re}[s] > 0$

$$\gamma^{-2}H(s)H^*(s) = [G(s) - I_m][G(s) + I_m][G^*(s) + I_m]^{-1}[G^*(s) - I_m] \leq I_m, \tag{3.3}$$

which implies

$$[G(s) + I_m]^{-1}[G^*(s) + I_m]^{-1} \leq [G(s) - I_m]^{-1}[G^*(s) - I_m]^{-1} \tag{3.4}$$

or, equivalently,

$$[G^*(s) + I_m][G(s) + I_m] \geq [G^*(s) - I_m][G(s) - I_m] \tag{3.5}$$

which further implies that $G(s)+G^*(s) \geq 0$ for $\mathrm{Re}[s] > 0$. Conversely, suppose $G(s)$ is positive real. Then, since $G(s)$ is assumed to be analytic for $\mathrm{Re}[s] \geq 0$, it is easy to show that $\det[G(s) + I_m] \neq 0$ for $\mathrm{Re}[s] \geq 0$. Therefore, $\gamma^{-1}H(s)$ defined by (3.2) is analytic for $\mathrm{Re}[s] \geq 0$. Then with $\gamma^{-1}H(s)$ defined by (3.2) it follows that $G(s)$ satisfies (3.1). Next, for $\mathrm{Re}[s] > 0$ we obtain

$$G(s)+G^*(s) = [I_m - \gamma^{-1}H(s)]^{-1}[I_m + \gamma^{-1}H(s)] + [I_m + \gamma^{-1}H^*(s)][I_m - \gamma^{-1}H^*(s)]^{-1} \geq 0. \tag{3.6}$$

Forming $[I_m - \gamma^{-1}H(s)](3.6)[I_m - \gamma^{-1}H^*(s)]$ yields

$$[I_m + \gamma^{-1}H(s)][I_m - \gamma^{-1}H^*(s)] + [I_m - \gamma^{-1}H^*(s)] + [I_m + \gamma^{-1}H^*(s)] \geq 0,$$

which implies $I_m - \gamma^{-2}H(s)H^*(s) \geq 0$ for $\mathrm{Re}[s] > 0$. $\square$

Next, we use the results of Lemma 3.1 to establish connections between the state space realizations of positive real and bounded real transfer functions.

Proposition 3.1. If $G(s)$ is a positive real transfer function with minimal realization

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{3.7}$$

then the bounded real transfer function $\gamma^{-1}H(s)$ defined by (3.2) has a minimal realization

$$\gamma^{-1}H(s) \sim \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}, \tag{3.8}$$

where

$$\hat{A} \triangleq A - B(I_m + D)^{-1}C, \tag{3.9}$$

$$\hat{B} \triangleq \sqrt{2}B(I_m + D)^{-1}, \tag{3.10}$$

$$\hat{C} \triangleq \sqrt{2}(I_m + D)^{-1}C, \tag{3.11}$$

$$\hat{D} \triangleq (D - I_m)(D + I_m)^{-1}. \tag{3.12}$$

Conversely, if $\gamma^{-1}H(s)$ is an $m \times m$ bounded real transfer function such that $\det[I_m - \gamma^{-1}H(s)] \neq 0$ for $\mathrm{Re}[s] > 0$ and with minimal realization

$$\gamma^{-1}H(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

then the positive real transfer function $G(s)$ defined (3.1) has a minimal realization

$$G(s) \sim \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right], \tag{3.13}$$

where

$$\bar{A} \triangleq A + B(I_m - D)^{-1}C, \tag{3.14}$$

$$\bar{B} \triangleq \sqrt{2}B(I_m - D)^{-1}, \tag{3.15}$$

$$\bar{C} \triangleq \sqrt{2}(I_m - D)^{-1}C, \tag{3.16}$$

$$\bar{D} \triangleq (I_m - D)^{-1}(I_m + D). \tag{3.17}$$

Proof. Given (3.7) it follows that the realizations of $G(s) - I_m$ and $G(s) + I_m$ are given by

$$[G(s) - I_m] \sim \left[\begin{array}{c|c} A & B \\ \hline C & D - I_m \end{array}\right], \quad [G(s) + I_m] \sim \left[\begin{array}{c|c} A & B \\ \hline C & D + I_m \end{array}\right].$$

Now, since $G(s)$ is positive real, it follows that $D + D^{\mathrm{T}} \geq 0$ which further implies that $I_m + D$ is invertible. Next, using (2.2) we have

$$[G(s) + I_m]^{-1} \sim \left[\begin{array}{c|c} A - B(I_m + D)^{-1}C & B(I_m + D)^{-1} \\ \hline -(I_m + D)^{-1}C & (D + I_m)^{-1} \end{array}\right].$$

Using (2.3), it now follows that $\gamma^{-1}H(s) = [G(s) - I_m][G(s) + I_m]^{-1}$ has a nonminimal realization

$$\gamma^{-1}H(s) \sim \left[\begin{array}{ccc} A - B(I_m + D)^{-1}C & 0 & B(I_m + D)^{-1} \\ -B(I_m + D)^{-1}C & A & B(I_m + D)^{-1} \\ (I_m - D)(I_m + D)^{-1}C & C & (D - I_m)(D + I_m)^{-1} \end{array}\right].$$

Next it follows from state-space manipulations that $\gamma^{-1}H(s)$ has a minimal state-space realization given by

$$\gamma^{-1}H(s) \sim \left[\begin{array}{c|c} A - B(I_m + D)^{-1}C & \sqrt{2}B(I_m + D)^{-1} \\ \hline \sqrt{2}(I_m + D)^{-1}C & (D - I_m)(D + I_m)^{-1} \end{array}\right].$$

Furthermore, Lemma 3.1 implies that $\gamma^{-1}H(s)$ is bounded real. Finally, the converse is shown in a similar fashion. $\square$

Having established connections between state-space realizations of positive real and bounded real transfer functions we proceed in the spirit of recent H_∞ results [32–37] to establish Riccati equation characterizations of positive real systems.

Theorem 3.1. Let $H(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $\sigma_{\max}(D) < \gamma$. If there exists an $n \times n$ nonnegative-definite matrix Q satisfying

$$0 = AQ + QA^{\mathrm{T}} + \gamma^{-2}(BD^{\mathrm{T}} + QC^{\mathrm{T}})(I_m - \gamma^{-2}DD^{\mathrm{T}})^{-1}(BD^{\mathrm{T}} + QC^{\mathrm{T}})^{\mathrm{T}} + BB^{\mathrm{T}}, \qquad (3.18)$$

then (A, B) is stabilizable if and only if

$$A \text{ is asymptotically stable.} \qquad (3.19)$$

Furthermore, in this case,

$$\|H(s)\|_\infty \leq \gamma. \qquad (3.20)$$

Conversely, if A is asymptotically stable and $\|H(s)\|_\infty < \gamma$, then there exists a unique nonnegative-definite matrix Q satisfying (3.18) and such that the eigenvalues of $A+\gamma^{-2}BD^{\mathrm{T}}(I_m-\gamma^{-2}DD^{\mathrm{T}})^{-1}C$ $+ \gamma^{-2}QC^{\mathrm{T}}(I_m - \gamma^{-2}DD^{\mathrm{T}})^{-1}C$ lie in the open left half plane. Furthermore, Q is the minimal solution to (3.18).

Proof. The asymptotic stability of A follows directly from Lyapunov theory while (3.20) follows from algebraic manipulation of (3.18); for details see [36]. The converse follows from the bounded real lemma [17, p. 308] or from spectral factor theory [31]. Finally, the proof of minimality is given in [45]. $\square$

Next, we utilize a transformation that converts a nonstrictly proper transfer function into a strictly proper transfer function both of which satisfy the same H_∞ bound. For convenience in stating this result define the notation

$$M \triangleq I_m - \gamma^{-2}DD^{\mathrm{T}}, \quad N \triangleq I_m - \gamma^{-2}D^{\mathrm{T}}D.$$

Note that M is positive definite if and only if N is positive definite.

Proposition 3.2. Let $H(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, and assume M and N are positive definite. Then A is asymptotically stable and

$$\|H(s)\|_\infty < \gamma \tag{3.21}$$

if and only if A' is asymptotically stable and

$$\|H'(s)\|_\infty < \gamma, \tag{3.22}$$

where $H'(s) \sim \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array}\right]$ and

$$A' \triangleq A + \gamma^{-2} B D^{\mathrm{T}} M^{-1} C, \tag{3.23}$$

$$B' \triangleq B N^{-1/2}, \tag{3.24}$$

$$C' \triangleq M^{-1/2} C. \tag{3.25}$$

Furthermore, (3.18) is equivalent to

$$0 = A'Q + QA'^{\mathrm{T}} + \gamma^{-2} Q C'^{\mathrm{T}} C' Q + B' B'^{\mathrm{T}}. \tag{3.26}$$

Proof. The results follow from Theorem 3.1 and algebraic manipulation. For details see [36]. □

Next, using Theorem 3.1 we give a Riccati equation characterization of positive real transfer functions. To do this we use (3.18) to imply that the transfer function corresponding to $(\hat{A}, \gamma\hat{B}, \hat{C}, \gamma\hat{D})$ has H_∞ norm less than γ. By Lemma 3.1 and Proposition 3.1 the resulting Riccati equation, i.e., (3.18) with (A, B, C, D) replaced by $(\hat{A}, \gamma\hat{B}, \hat{C}, \gamma\hat{D})$, implies that $G(s) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is positive real. To utilize Theorem 3.1 we require that $\sigma_{\max}(\gamma\hat{D}) < \gamma$ or, equivalently,

$$I_m - \hat{D}\hat{D}^{\mathrm{T}} > 0. \tag{3.27}$$

Now, using (3.12), this is equivalent to

$$I_m - (D - I_m)(D + I_m)^{-1}(D + I_m)^{-\mathrm{T}}(D - I_m)^{\mathrm{T}} > 0. \tag{3.28}$$

Since $(D + I_m)$ and $(D - I_m)$ commute, (3.28) implies

$$(I_m - D)(I_m - D^{\mathrm{T}}) < (I_m + D)(I + D^{\mathrm{T}}), \tag{3.29}$$

which implies that

$$D + D^{\mathrm{T}} > 0. \tag{3.30}$$

Thus, we restrict our attention to strongly positive real systems.

Theorem 3.2. Let $G(s) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, assume $\det[D + I_m] \neq 0$, define $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ by (3.9)–(3.12), and assume $\sigma_{\max}(\hat{D}) < 1$. If there exists an $n \times n$ nonnegative-definite matrix $\mathcal{Q}$ satisfying

$$0 = \hat{A}\mathcal{Q} + \mathcal{Q}\hat{A}^{\mathrm{T}} + (\hat{B}\hat{D}^{\mathrm{T}} + \mathcal{Q}\hat{C}^{\mathrm{T}})(I_m - \hat{D}\hat{D}^{\mathrm{T}})^{-1}(\hat{B}\hat{D}^{\mathrm{T}} + \mathcal{Q}\hat{C}^{\mathrm{T}})^{\mathrm{T}} + \hat{B}\hat{B}^{\mathrm{T}}, \tag{3.31}$$

$(\hat{A}, \hat{B})$ is stabilizable, and

$$\det[I_m - \hat{C}(sI_n - \hat{A})^{-1}\hat{B} - \hat{D}] \neq 0 \text{ for } \mathrm{Re}[s] > 0, \tag{3.32}$$

then

$$G(s) \text{ is strongly positive real.} \tag{3.33}$$

Conversely, if $\hat{A}$ is asymptotically stable and $G(s)$ is positive real, then there exists a unique nonnegative-definite matrix $\mathcal{Q}$ satisfying (3.31).

Proof. The result is a direct consequence of Theorem 3.1, Proposition 3.1 and Lemma 3.1. □

Remark 3.1. Using Proposition 3.2 we can represent (3.31) in the equivalent form

$$0 = \hat{A}'\mathcal{Q} + \mathcal{Q}\hat{A}' + \mathcal{Q}\hat{C}'^{\mathrm{T}}\hat{C}'\mathcal{Q} + \hat{B}'\hat{B}'^{\mathrm{T}}, \tag{3.34}$$

where

$$\hat{A}' \triangleq A - B(I_m + D)^{-1}C + \mathcal{Q}B(I_m + D)^{-1}\hat{D}^{\mathrm{T}}(I_m - \hat{D}\hat{D}^{\mathrm{T}})^{-1}(I_m + D)^{-1}C, \tag{3.35}$$

$$\hat{B}' \triangleq \sqrt{2}B(I_m + D)^{-1}(I_m - \hat{D}^{\mathrm{T}}\hat{D})^{-1/2}, \tag{3.36}$$

$$\hat{C}' \triangleq \sqrt{2}(I_m - \hat{D}\hat{D}^{\mathrm{T}})^{-1/2}(I_m + D)^{-1}C. \tag{3.37}$$

Remark 3.2. An interesting special case of Theorem 3.2 is the case $D = I_m$. Since $\hat{D} = 0$ (see (3.12)), (3.31) or, equivalently, (3.34) becomes

$$0 = (A - \frac{1}{2}BC)\mathcal{Q} + \mathcal{Q}(A - \frac{1}{2}BC)^{\mathrm{T}} + \frac{1}{2}\mathcal{Q}C^{\mathrm{T}}C\mathcal{Q} + \frac{1}{2}BB^{\mathrm{T}}. \tag{3.39}$$

Finally, we draw connections between Theorem 3.2 and the well-known positive real lemma used to characterize positive realness in the state-space setting [17].

Lemma 3.2. Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be an $m \times m$ transfer function with minimal realization (A, B, C, D). Then $G(s)$ is positive real if and only if there exist matrices $Q \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times p}$, and $W \in \mathbb{R}^{m\times p}$ with Q positive-definite and such that

$$0 = AQ + QA^{\mathrm{T}} + LL^{\mathrm{T}}, \tag{3.40}$$

$$0 = QC^{\mathrm{T}} - B + LW^{\mathrm{T}}, \tag{3.41}$$

$$0 = D + D^{\mathrm{T}} - WW^{\mathrm{T}}. \tag{3.42}$$

This form of the positive-real lemma is the dual of that given in [11,23], and the derivation is similarly dual. See [12] for further details on the dual positive real lemma.

The key question of interest here is the relationship between Q satisfying (3.40)–(3.42) and $\mathcal{Q}$ given by (3.31). To answer this question, we invoke the assumption that $D + D^{\mathrm{T}} > 0$ which, as noted earlier, is needed for the existence of $\mathcal{Q}$. Thus, once again, we restrict our attention to strongly positive real transfer functions. In this case, it follows from (3.42) that

$$WW^{\mathrm{T}} = D + D^{\mathrm{T}}. \tag{3.43}$$

Now, since $D + D^{\mathrm{T}} > 0$, W is nonsingular and thus (3.41) implies

$$L = (B - QC^{\mathrm{T}})W^{-\mathrm{T}}. \tag{3.44}$$

Using (3.44) it follows from (3.40) that

$$0 = AQ + QA^{\mathrm{T}} + (B - QC^{\mathrm{T}})W^{-\mathrm{T}}W^{-1}(B^{\mathrm{T}} - CQ) \tag{3.45}$$

or, since $(WW^{\mathrm{T}})^{-1} = W^{-\mathrm{T}}W^{-1}$,

$$0 = AQ + QA^{\mathrm{T}} + (B - QC^{\mathrm{T}})(D + D^{\mathrm{T}})^{-1}(B - QC)^{\mathrm{T}}. \tag{3.46}$$

Thus, we have shown that under the assumption that $D + D^{\mathrm{T}} > 0$, conditions (3.40)–(3.42) are equivalent to one Riccati equation given by (3.46). A similar result for the dual case appears in [17].

The next lemma connects the two Riccati equations (3.31) and (3.46).

Proposition 3.3. Assume $D + D^{\mathrm{T}} > 0$. Then the Riccati equation (3.46) is identical to the Riccati equation (3.31), or, equivalently, (3.34).

Proof. Using (3.46) it follows that

$$
\begin{aligned}
0 = {} & [A - B(D + D^{\mathrm{T}})^{-1}C]Q + Q[A - B(D + D^{\mathrm{T}})^{-1}C]^{\mathrm{T}} \\
& + QC^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}CQ + B(D + D^{\mathrm{T}})^{-1}B^{\mathrm{T}}.
\end{aligned} \tag{3.47}
$$

The result now follows from algebraic manipulation by noting that

$$(D + D^{\mathrm{T}})^{-1} = 2(I_m + D)^{-\mathrm{T}}[I_m - (D - I_m)(D + I_m)^{-1}(D + I_m)^{-\mathrm{T}}(D - I_m)^{\mathrm{T}}]^{-1}(I_m + D)^{-1}. \quad \square$$

Remark 3.3. Note that in the case $D = I_m$, Proposition 3.3 can readily be seen by comparing (3.39) and (3.46).

4. Robust Stability Problem with Positive Real Uncertainty

In this section we state the robust stability problem with positive real uncertainty. Consider the uncertain system

$$
\begin{aligned}
\dot{x}(t) &= [A - B_0F(I_m + D_0F)^{-1}C_0]x(t), \\
F + F^{\mathrm{T}} &\geq 0,
\end{aligned} \tag{Σ}
$$

when the inverse of $I_m + D_0F$ exists. It is useful to note that (Σ) can be viewed as a strongly positive real system (A, B_0, C_0, D_0) in a negative feedback configuration with the gain F (see Figure 4.1). That is,

$$\dot{x}(t) = Ax(t) + B_0u(t), \tag{4.1}$$

$$y(t) = C_0x(t) + D_0u(t), \tag{4.2}$$

with negative feedback

$$u(t) = -Fy(t), \tag{4.3}$$

where

$$F + F^{\mathrm{T}} \geq 0. \tag{4.4}$$

Thus, the question of interest is the stability of the uncertain system (Σ) with positive real uncertainty (4.4). However, before we proceed with this question we give a lemma on the existence of $(I_m + D_0F)^{-1}$ when the system (A, B_0, C_0, D_0) is strongly positive real.

Lemma 4.1. Let $D_0, F \in \mathbb{R}^{m_0 \times m_0}$, and assume that $D_0 + D_0^{\mathrm{T}}$ is positive definite and $F + F^{\mathrm{T}}$ is nonnegative definite. Then

$$\det[I_m + D_0F] \neq 0. \tag{4.5}$$

Proof. Since $D_0 + D_0^{\mathrm{T}}$ is positive definite, it follows from Lyapunov stability theory that $-D_0$ is asymptotically stable. Hence D_0 is nonsingular. Furthermore, it follows that $D_0^{-1} + D_0^{-\mathrm{T}} = D_0^{-1}(D_0 + D_0^{\mathrm{T}})D_0^{-\mathrm{T}}$ is positive definite. Since $F + F^{\mathrm{T}}$ is nonnegative definite, it follows that $D_0^{-1} + F + (D_0^{-1} + F)^{\mathrm{T}}$ is also positive definite. Hence $-(D_0^{-1} + F)$ is asymptotically stable and, consequently, $D_0^{-1} + F$ is nonsingular. Thus

$$\det[I_m + D_0 F] = (\det\ D_0)\det(D_0^{-1} + F) \neq 0. \quad \square$$

Next, we present the main result of this section which shows that the uncertain system (Σ) is robustly stable for all positive real uncertainty of the form (4.4). For the statement of the next result we define the uncertainty set

$$\mathcal{U} \triangleq \{\Delta A \in \mathbb{R}^{n\times n}:\quad \Delta A = -B_0 F(I_m + D_0 F)^{-1} C_0, \quad F + F^{\mathrm{T}} \geq 0\}, \tag{4.6}$$

where $B_0 \in \mathbb{R}^{n\times m_0}$, $C_0 \in \mathbb{R}^{m_0\times n}$, and $D_0 \in \mathbb{R}^{m_0\times m_0}$ are fixed matrices denoting the structure of the uncertainty and $F \in \mathbb{R}^{m_0\times m_0}$ is an uncertain matrix (see Figure 4.2).

Theorem 4.1. Let $G(s) \sim \left[\begin{array}{c|c} A & B_0 \\ \hline C_0 & D_0 \end{array}\right]$, where $A \in \mathbb{R}^{n\times n}$ is asymptotically stable. If $G(s)$ is strongly positive real, then $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Conversely, if $D_0 + D_0^{\mathrm{T}} > 0$ and $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then $G(s)$ is strongly positive real.

Proof. As shown in [2,14,18,21] a negative feedback configuration consisting of a positive real transfer function and a strictly positive real transfer function is stable. Under the assumption that $G(s)$ is strongly positive real, the dynamics matrix of the closed-loop system, which has the form $A - B_0 F(I_m + D_0 F)^{-1} C_0$, is asymptotically stable. Hence $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Conversely, if $D_0 + D_0^{\mathrm{T}} > 0$ and $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then it follows from the definition of hyperstability [21] that $G(s)$ is strongly positive real. $\square$

Remark 4.1. The key feature of the uncertainty set $\mathcal{U}$ is that the uncertain perturbation ΔA involves a phase constraint. To see this note that if $D_0 + D_0^{\mathrm{T}} > 0$ and $F + F^{\mathrm{T}} \geq 0$, then

$$F(I_m + D_0 F)^{-1} + [F(I_m + D_0 F)^{-1}]^{\mathrm{T}} = (I + D_0 F)^{-\mathrm{T}}[F + F^{\mathrm{T}} + F^{\mathrm{T}}(D_0 + D_0^{\mathrm{T}})F](I + D_0 F)^{-1} \geq 0.$$

However, the term $F(I_m + D_0 F)^{-1}$ is bounded in magnitude even though F is not. For example, if F is a scalar, then $|F(1 + D_0 F)^{-1}| \leq 1/D_0$. Thus the uncertainty set $\mathcal{U}$ incorporates both magnitude and phase constraints.

Remark 4.2. Theorem 4.1 implies that robust stability of the uncertain system (Σ) is equivalent to a positive real condition. This fact can be compared to the results of [37] where it is shown that the existence of a *fixed* Lyapunov function for the uncertain system

$$\begin{aligned} \dot{x}(t) &= (A + B_0 F C_0) x(t), \\ \sigma_{\max}(F) &\leq 1, \end{aligned} \tag{Σ'}$$

is equivalent to a small gain condition. However, since the present result involves a phase constraint not present a small gain condition (see Remark 4.1) one should expect to find a parameter dependent Lyapunov function for the uncertain systems (Σ) rather than a single Lyapunov function as in [37].

5. Robust Controller Synthesis for Positive Real Uncertainty

In this section we state the Robust Stabilizability Problem with Positive Real Uncertainty. The problem involves the set $\mathcal{U}$ given by (4.6) of uncertain perturbations ΔA of the nominal (A, B, C) system. The goal of the robust stability problem is to determine a static output feedback controller that stabilizes the plant for all variations in $\mathcal{U}$. See Figure 5.1.

Robust Stabilizability Problem with Positive Real Uncertainty. Determine $K \in \mathbb{R}^{m \times \ell}$ such that the closed-loop system consisting of the nth-order controlled plant

$$\dot{x}(t) = (A + \Delta A) x(t) + B u(t), \quad t \in [0, \infty), \tag{5.1}$$

measurements

$$y(t) = C x(t), \tag{5.2}$$

and output feedback controller

$$u(t) = K y(t), \tag{5.3}$$

is asymptotically stable for all $\Delta A \in \mathcal{U}$.

For each uncertain variation $\Delta A \in \mathcal{U}$, the closed-loop system can be written as

$$\dot{x}(t) = (A + BKC + \Delta A) x(t), \quad t \in [0, \infty). \tag{5.4}$$

The following result gives necessary and sufficient conditions for constructing a feedback gain K that solves the Robust Stabilizability Problem with Positive Real Uncertainty. For the statement of this result define

$$\nu \triangleq Q C^{\mathrm{T}} (C Q C^{\mathrm{T}})^{-1} C, \quad \nu_{\perp} \triangleq I_n - \nu.$$

$$R_0 \triangleq (D_0 + D_0^{\mathrm{T}})^{-1},$$

for arbitrary $Q \in \mathbb{R}^{n \times n}$ such that det $CQC^{\mathrm{T}} \neq 0$, and let R_1 and R_2 be arbitrary real $n \times n$ and $m \times m$ positive-definite matrices.

Theorem 5.1. There exists $K \in \mathbb{R}^{m \times \ell}$ that solves the Robust Stabilizability Problem with Positive Real Uncertainty if and only if there exist $n \times n$ positive-definite matrices Q, P satisfying

$$0 = (A - BR_2^{-1}B^{\mathrm{T}}P\nu - B_0R_0C_0)Q + Q(A - BR_2^{-1}B^{\mathrm{T}}P\nu - B_0R_0C_0)^{\mathrm{T}} + QC_0^{\mathrm{T}}R_0C_0Q + B_0R_0B_0^{\mathrm{T}}, \tag{5.5}$$

$$0 = (A - B_0R_0C_0 + QC_0^{\mathrm{T}}R_0C_0)^{\mathrm{T}}P + P(A - B_0R_0C_0 + QC_0^{\mathrm{T}}R_0C_0) + R_1 - PBR_2^{-1}B^{\mathrm{T}}P + \nu_\perp^{\mathrm{T}}PBR_2^{-1}B^{\mathrm{T}}P\nu_\perp. \tag{5.6}$$

Furthermore, one such gain K is given by

$$K = -R_2^{-1}B^{\mathrm{T}}PQC^{\mathrm{T}}(CQC^{\mathrm{T}})^{-1}. \tag{5.7}$$

Proof. (Sufficiency). Suppose there exist $n \times n$ positive-definite matrices Q, P satisfying (5.5) and (5.6). Then, with K given by (5.7), it follows that (5.5) is equivalent to

$$0 = (A + BKC - B_0R_0C_0)Q + Q(A + BKC - B_0R_0C_0)^{\mathrm{T}} + QC_0^{\mathrm{T}}R_0C_0Q + B_0R_0B_0^{\mathrm{T}}, \tag{5.8}$$

which further implies

$$0 = (A + BKC)Q + Q(A + BKC)^{\mathrm{T}} + (B_0 - QC_0^{\mathrm{T}})^{\mathrm{T}}(D_0 + D_0^{\mathrm{T}})^{-1}(B_0 - QC_0^{\mathrm{T}})^{\mathrm{T}}. \tag{5.9}$$

Furthermore, (5.6) is equivalent to

$$0 = (A + BKC - B_0R_0C_0 + QC_0^{\mathrm{T}}R_0C_0)^{\mathrm{T}}P + P(A + BKC - B_0R_0C_0 + QC_0^{\mathrm{T}}R_0C_0) + R_1 + K^{\mathrm{T}}R_2K. \tag{5.10}$$

Note that (5.10) is an auxiliary equation and is only needed for computing the gain K. Furthermore, note that (5.9) is equivalent to (3.46), or, equivalently (3.31). It now follows from Theorem 3.2 that $(A + BKC, B_0, C_0, D_0)$ is strongly positive real which, by Theorem 4.1, implies that $A + BKC + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$.

(Necessity). It follows from Theorem 3.2 and Proposition 3.3 that $(A + BKC, B_0, C_0, D_0)$ is strongly positive real if and only if there exists a nonnegative-definite solution Q to

$$0 = (A + BKC)Q + Q(A + BKC)^{\mathrm{T}} + (B_0 - QC_0^{\mathrm{T}})(D_0 + D_0^{\mathrm{T}})^{-1}(B_0 - QC_0^{\mathrm{T}})^{\mathrm{T}}. \tag{5.11}$$

Now it follows from compactness arguments that the functional $J(K) \triangleq \operatorname{tr} Q(R_1 + C^{\mathrm{T}}K^{\mathrm{T}}R_2KC)$ must have a global minimum on the set

$$\mathcal{S} \triangleq \{K \in \mathbb{R}^{m\times \ell}\colon\ A + BKC \text{ is asymptotically stable}\}$$

under the assumption that $B_0B_0^{\mathrm{T}}$ is positive definite. Note that $\mathcal{S}$ is not empty by the assumption that a robustly stabilizing K exists. In this case the necessary conditions for optimality of $J(K)$, which are equivalent to the existence of Q, P satisfying (5.5)–(5.6), must have a solution. □

Next, we specialize Theorem 5.1 to the full state feedback case. When the full state is available, i.e., $C = I_n$, the projection $\nu = I_n$ so that $\nu_\perp = 0$. In this case (5.7) becomes

$$K = -R_2^{-1}B^{\mathrm{T}}P \tag{5.12}$$

and (5.5), (5.6) specialize to

$$0 = (A - BR_2^{-1}B^{\mathrm{T}}P - B_0R_0C_0)Q + Q(A - BR_2^{-1}B^{\mathrm{T}}P - B_0R_0C_0)^{\mathrm{T}} + QC_0^{\mathrm{T}}R_0C_0Q + B_0R_0B_0^{\mathrm{T}}, \tag{5.13}$$

$$0 = (A - B_0R_0C_0 + QC_0^{\mathrm{T}}R_0C_0)^{\mathrm{T}}P + P(A - B_0R_0C_0 + QC_0^{\mathrm{T}}R_0C_0) + R_1 - PBR_2^{-1}B^{\mathrm{T}}P. \tag{5.14}$$

It is interesting to note that even in the full state feedback case the result involves two coupled Riccati equations.

A salient feature of (3.39) is the fact that the shift $-\frac{1}{2}BC$ to the matrix A can be nonpositive. That is, $-\frac{1}{2}BC$ can represent a *left* shift in contrast to the usual α-shift, which is a uniform open-loop right shift used to place the closed-loop poles to the left of $-\alpha$, where $\alpha > 0$[22]. The use of a left shift to the plant dynamics matrix has been used to model frequency uncertainty in lightly damped flexible structures [35–40]. Specifically, consider modal dynamics of the form

$$A = \text{block-diag}\left(\begin{bmatrix} -\eta_1 & \omega_1 \\ -\omega_1 & -\eta_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\eta_r & \omega_r \\ -\omega_r & -\eta_r \end{bmatrix}\right), \tag{5.15}$$

where $\eta_i > 0$ denotes the decay rate and ω_i denotes modal frequency. Also consider uncertainty of the form

$$\Delta A = \sum_{i=1}^{r} \sigma_i A_i, \tag{5.16}$$

where $\sigma_i \in [-\delta_i, \delta_i]$, $i = 1, \ldots, r$, are real, uncertain parameters with given bounds δ_i, and the matrices A_i are defined by

$$A_i = \text{block-diag}\left(0, \ldots, 0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right), \tag{5.17}$$

where the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ corresponds to the ith diagonal block of A. The skew symmetric structure of A_i accounts for uncertainty in the ith modal frequency ω_i. In [10, 38–43] the Maximum Entropy design approach is predicated upon a modified covariance (Lyapunov) equation of the form

$$0 = (A + S)Q + Q(A + S)^{\mathrm{T}} + \sum_{i=1}^{r} \delta_i^2 A_i Q A_i^{\mathrm{T}} + V, \tag{5.18}$$

where the shift S is defined by

$$S \triangleq \frac{1}{2} \sum_{i=1}^{r} \delta_i^2 A_i^2.$$

Note that S has the form

$$S = \text{block-diag}(-\frac{1}{2}\delta_1^2 I_2, \ldots, -\frac{1}{2}\delta_r^2 I_2)$$

so that S effectively shifts each mode to the left by introducing a (fictitious) augmentation to the open-loop damping. To relate (5.18) to (3.39), consider the case of a single uncertain modal frequency by setting $r = 1$. Furthermore, let

$$B_0 = C_0 = \delta_1 I_2, \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so that (with B, C replaced by B_0, C_0 in (3.39)) $-\frac{1}{2}B_0 C_0 = -\frac{1}{2}\delta_1^2 I_2 = \frac{1}{2}\delta_1^2 A_1^2 = S$. The remaining terms $\delta_1^2 A_1 Q A_1^{\mathrm{T}} + V$ in (5.18) can be shown to play a role similar to the terms $\frac{1}{2}\mathcal{Q}C^{\mathrm{T}}C\mathcal{Q} + \frac{1}{2}BB^{\mathrm{T}}$ in (3.39). See [46] for further details. Finally, the uncertain perturbations ΔA given by (4.6) have the form

$$\Delta A = -\delta_1^2 F(I_2 + D_0 F)^{-1}. \tag{5.19}$$

In the limiting case $D_0 \to 0$, setting $F = -\frac{\sigma_1}{\delta_1^2} A_1$ (so that $F + F^{\mathrm{T}} \geq 0$), (5.19) becomes

$$\Delta A = \sigma_1 A_1.$$

Hence $\mathcal{U}$ given by (5.16) can be used to capture frequency uncertainty of the form (5.16).

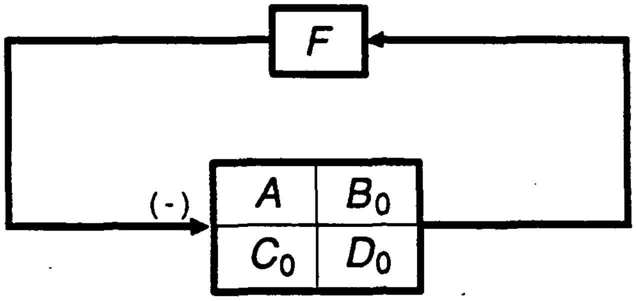

Figure 4.1.

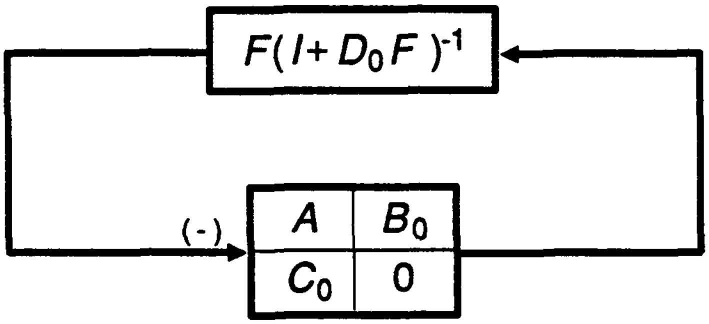

Figure 4.2.

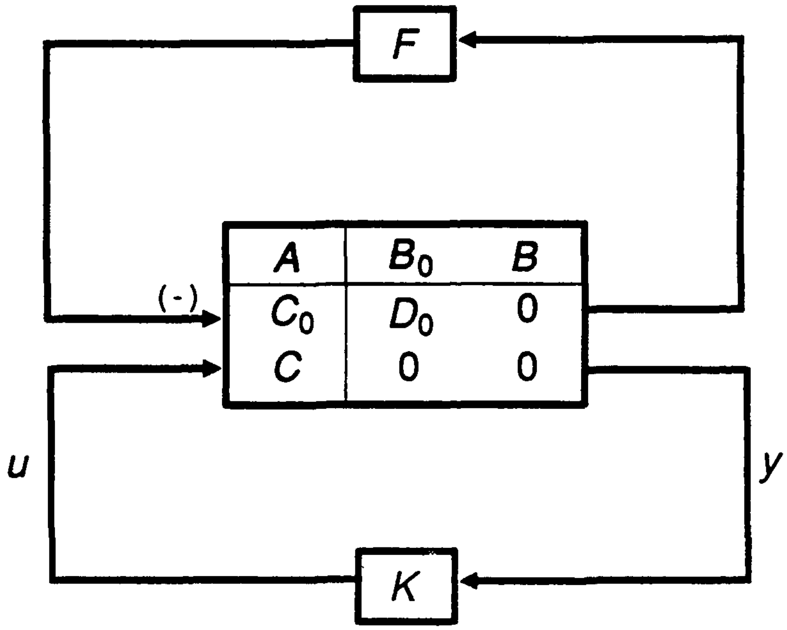

Figure 5.1.

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Appendix G

"Nonquadratic Cost and Nonlinear Feedback Control"

September 1990

Nonquadratic Cost and Nonlinear Feedback Control

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Abstract

Nonlinear controllers offer significant advantages over linear controllers in a variety of circumstances. Hence there has been significant interest in extending the linear-quadratic synthesis methodology to nonlinear-nonquadratic problems. The purpose of this paper is to review the current status of such efforts and to present, in a simplified manner, some of the basic ideas underlying these results. In particular, we focus on the classic paper by Bass and Webber (1966) and show its relationship to some results of Speyer (1976).

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1. Introduction

Linear-quadratic (LQ) control theory has been extensively developed over the past thirty years. In its most fundamental form, linear-quadratic control is based upon the following assumptions:

i) the plant dynamics and measurement equations are linear in both the state and control variables,

ii) the performance measure to be minimized is quadratic in the state and control variables,

iii) the plant disturbances and measurement noise are additive Gaussian white noise or L_2 signals.

In addition to these *explicit* assumptions the following *implicit* assumptions are crucial:

iv) the plant model is completely accurate,

v) the state and control variables are constrained in a mean-square or L_2 sense.

Under these assumptions it is well known that the optimal feedback controller is linear [1].

In many practical situations, however, one or more of these assumptions may be violated. For example, the plant and measurement equations may be nonlinear, the performance measure may be nonquadratic, the disturbances may be nongaussian or nonadditive, the plant model may be uncertain, or state variables and control effort may be limited by nonquadratic constraints. In such cases there is no reason to expect that the optimal controller is linear. Rather, it should be expected that nonlinear controllers will have better performance than the best linear controllers. For example, if the plant model is nonlinear then nonlinear controllers can be used to account for the global behavior of the plant [2–4]. Similarly, gain-scheduled controllers designed for multiple plant linearizations constitute a widely used class of nonlinear controllers [5].

In the case of optimal H_∞ performance or robust stabilizability in the presence of unstructured plant uncertainty, it has been shown [6–8] that nonlinear controllers offer no advantage over linear controllers. However, if the plant uncertainty is structured and if a quadratic Lyapunov function is assumed, then discontinuous nonlinear controllers have been shown to offer advantages over linear controllers [9–14]. Continuous approximations to the discontinuous controllers of [9,10] have been developed in [15,16]. Discontinuous controllers are also the focus of variable structure control which also addresses the problem of plant uncertainty [17–20]. It is also shown in [21] that nonlinear controllers can provide improved performance in a neighborhood of the worst case H_∞ disturbance attenuation.

Adaptive controllers can be viewed as nonlinear controllers that operate in the presence of significant plant uncertainty. Such controllers have been shown to stabilize uncertain systems that cannot be stabilized by means of linear controllers [22–29].

With regard to state and control constraints, one of the most common nonlinearities arising in applications is actuator saturation [30–32]. Linear-quadratic techniques, however, can, at best, only impose bounds on the L_2 norms of the state and control variables. Enforcing constraints of the form $\|x(t)\| \leq \alpha$ or $\|u(t)\| \leq \beta$ pointwise in time requires nonlinear controllers.

In view of the advantages of nonlinear controllers over linear controllers, it is not surprising that significant effort has been devoted to developing a theory of optimal nonlinear regulation [33–70]. The goal of the present paper is to provide a simplified framework for optimal nonlinear regulation in terms of nonquadratic cost functionals. In accordance with practical motivation, we restrict our attention to time-invariant systems on the infinite interval. In this case asymptotic stability is guaranteed by means of a Lyapunov function for the closed-loop system. This Lyapunov function is given as the solution to a steady-state form of the Hamilton-Jacobi-Bellman equation.

In future research, we intend to reverse the situation somewhat by fixing the structure of the Lyapunov function, cost functional, and feedback law prior to optimization. In this case the structure of the Lyapunov function can be viewed as providing the *framework* for controller synthesis by guaranteeing local or global asymptotic stability for a class of feedback controllers. The *actual* controller chosen for implementation can thus be the member of this candidate class that minimizes the given performance functional. In LQG theory, for example, the Lyapunov function is the familiar quadratic functional $V(x) = x^{\mathrm{T}}Px$, while the gains for the linear feedback control are chosen to minimize a quadratic performance functional. In summary, then, Lyapunov function theory provides the framework, while optimization fixes the gains.

2. Nonquadratic Cost Evaluation

In this section we investigate the role of Lyapunov functions in evaluating nonquadratic cost functionals. To expand upon the linear-quadratic case, we consider the problem of evaluating a nonquadratic cost functional depending upon a nonlinear differential equation. It turns out that the cost functional can be evaluated in closed form so long as the cost functional is related in a specific way to an underlying Lyapunov function. The basis for the following development is the paper [36] by Bass and Webber. Note that the results of this section make no explicit reference to control.

In accordance with practical motivations, we restrict our attention to time-invariant systems on the infinite horizon. Furthermore, for simplicity we shall define all functions globally and assume that existence and uniqueness properties of the given differential equations are satisfied.

For the following result, let f: $\mathbb{R}^n \to \mathbb{R}^n$ and L: $\mathbb{R}^n \to \mathbb{R}$. We assume $f(0) = 0$. Let $(')$ denote derivative.

Lemma 2.1. Consider the system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{2.1}$$

with performance functional

$$J(x_0) = \int_0^\infty L(x(t))\mathrm{d}t. \tag{2.2}$$

Furthermore, assume there exists a C^1 function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \tag{2.3}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{2.4}$$

$$V'(x)f(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{2.5}$$

$$L(x) + V'(x)f(x) = 0, \quad x \in \mathbb{R}^n. \tag{2.6}$$

Then $x(t) = 0$, $t \geq 0$, is a globally asymptotically stable solution to (2.1) with $x_0 = 0$. Furthermore,

$$J(x_0) = V(x_0), \quad x_0 \in \mathbb{R}^n. \tag{2.7}$$

Proof. Let $x(t)$, $t \geq 0$, satisfy (2.1). Then

$$\dot{V}(x(t)) \triangleq \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = V'(x(t))f(x(t)), \quad t \geq 0. \tag{2.8}$$

Hence it follows from (2.5) that

$$\dot{V}(x(t)) < 0, \quad t \geq 0, \quad x(t) \neq 0. \tag{2.9}$$

Thus, by (2.3), (2.4), and (2.9) it follows that $V(\cdot)$ is a Lyapunov function for (2.1) and thus $x(t) \to 0$ as $t \to \infty$ for all initial conditions x_0. This proves global asymptotic stability of the solution $x(t) = 0$, $t \geq 0$. Now (2.8) implies that

$$0 = -\dot{V}(x(t)) + V'(x(t))f(x(t))$$

and hence, by (2.6),

$$L(x(t)) = -\dot{V}(x(t)) + L(x(t)) + V'(x(t))f(x(t))$$
$$= -\dot{V}(x(t)).$$

Integrating over $[0,t)$ yields

$$\int_0^t L(x(s))\mathrm{d}s = -V(x(t)) + V(x_0).$$

Now letting $t \to \infty$ and noting that $V(x(t)) \to 0$, yields (2.7). $\square$

The main feature of Lemma 2.1 is the role played by the Lyapunov function $V(x)$ both in guaranteeing stability and in evaluating the functional $J(x_0)$. Let us illustrate this result with a familiar example. Consider the linear system

$$\dot{x} = Ax, \quad x(0) = x_0, \tag{2.10}$$

with cost functional

$$J(x_0) = \int_0^\infty x^{\mathrm{T}} Rx \,\mathrm{d}t, \tag{2.11}$$

where $R \in \mathbb{R}^{n\times n}$ is positive-definite. If A is asymptotically stable then there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$0 = A^{\mathrm{T}}P + PA + R. \tag{2.12}$$

Now define

$$V(x) = x^{\mathrm{T}}Px, \tag{2.13}$$

which satisfies (2.3) and (2.4). rthermore, with $f(x) = Ax$ and $L(x) = x^{\mathrm{T}}Rx$ it follows that

$$V'(x)f(x) = 2x^{\mathrm{T}}PAx = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x = -x^{\mathrm{T}}Rx = -L(x),$$

which verifies (2.5) and (2.6). Hence

$$J(x_0) = x_0^{\mathrm{T}}Px_0,$$

which is a familiar result from linear-quadratic theory.

Remark 2.1. Note that if (2.6) is valid, then (2.5) is equivalent to

$$L(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0. \tag{2.14}$$

More generally, assume A is asymptotically stable, let P be given by (2.12), and consider the case

$$L(x) = x^{\mathrm{T}}Rx + h(x), \tag{2.15}$$

$$f(x) = Ax + N(x), \tag{2.16}$$

$$V(x) = x^{\mathrm{T}}Px + g(x), \tag{2.17}$$

where $h(\cdot)$ and $g(\cdot)$ are nonquadratic and $N(\cdot)$ is nonlinear. To satisfy (2.6) we require that

$$x^{\mathrm{T}}Rx + h(x) + [2x^{\mathrm{T}}P + g'(x)][Ax + N(x)] = 0. \tag{2.18}$$

Our goal is to study (2.18) under a variety of choices for $h(\cdot), g(\cdot)$, and $N(\cdot)$. For convenience, rewrite (2.18) as

$$x^{\mathrm{T}}(A^{\mathrm{T}}P + PA + R)x + g'(x)Ax + h(x) + g'(x)N(x) = 0. \tag{2.19}$$

If A is asymptotically stable, then we can choose P to satisfy (2.12) as in the linear-quadratic case. Next, suppose $N(x) = 0$. Then (2.19) is satisfied if

$$g'(x)Ax + h(x) = 0. \tag{2.20}$$

The following lemma, which is quoted in [36], will be useful for satisfying (2.20).

Lemma 2.2. Let $A \in \mathbb{R}^{n\times n}$ be asymptotically stable and let h_k: $\mathbb{R}^n \to \mathbb{R}$ be a nonnegative-definite homogeneous k-form (k even). Then there exists a unique nonnegative-definite homogeneous k-form g_k: $\mathbb{R}^n \to \mathbb{R}$ such

$$g_k'(x)Ax + h_k(x) = 0, \quad x \in \mathbb{R}^n. \tag{2.21}$$

Proof. The result can be shown using the Kronecker product representation of multilinear functions. Details will appear in an expanded version of this paper. □

Suppose now that $h(x)$ is of the form

$$h(x) = \sum_{\nu=2}^{r} h_{2\nu}(x), \tag{2.22}$$

where, for $\nu = 2, \ldots, r$, $h_{2\nu}$: $\mathbb{R}^n \to \mathbb{R}$ is a nonnegative-definite homogeneous 2ν-form. Now, using Lemma 2.2, let $g_{2\nu}$: $\mathbb{R}^n \to \mathbb{R}$ be the nonnegative-definite homogeneous 2ν-form satisfying

$$g_{2\nu}'(x)Ax + h_{2\nu}(x) = 0, \quad x \in \mathbb{R}^n, \quad \nu = 2, \ldots, r, \tag{2.23}$$

and define

$$g(x) = \sum_{\nu=2}^{r} g_{2\nu}(x). \tag{2.24}$$

Since

$$g'(x) = \sum_{\nu=2}^{r} g'_{2\nu}(x),$$

summing (2.23) over ν yields (2.20). Since (2.6) is now satisfied, (2.9) implies that

$$J(x_0) = x_0^{\mathrm{T}} P x_0 + g(x_0). \tag{2.25}$$

As another illustration of condition (2.18), suppose that $V(x)$ is constrained to be of the form

$$V(x) = x^{\mathrm{T}} P x + \frac{1}{2}(x^{\mathrm{T}} M x)^2, \tag{2.26}$$

where P satisfies (2.12) and M is an $n \times n$ symmetric matrix. Then, with $N(x) = 0$ (2.18) yields

$$h(x) = -(x^{\mathrm{T}} M x) x^{\mathrm{T}} (A^{\mathrm{T}} M + MA) x. \tag{2.27}$$

If $\hat{R}$ is an $n \times n$ symmetric matrix and M is given by

$$0 = A^{\mathrm{T}} M + MA + \hat{R}, \tag{2.28}$$

then $h(x)$ satisfying (2.18) is of the form

$$h(x) = (x^{\mathrm{T}} M x)(x^{\mathrm{T}} \hat{R} x). \tag{2.29}$$

Thus, if $V(x)$ is of the form (2.26), then, by utilizing (2.28), condition (2.18) is satisfied if $L(x)$ has the form

$$L(x) = x^{\mathrm{T}} R x + (x^{\mathrm{T}} M x)(x^{\mathrm{T}} \hat{R} x). \tag{2.30}$$

In the next section we apply Lemma 2.1 to the problem of optimal nonlinear feedback control. The relation (2.18) shall play a key role with greater complexity arising from the fact that the nonlinear dynamics term $N(x)$ will be nonzero.

3. Optimal Control

In this section, we extend the development of Section 2 to obtain a characterization of optimal feedback controllers. These conditions are essentially a specialization of the Hamilton-Jacobi-Bellman (HJB) conditions for the time-invariant, infinite-horizon case. For this problem the HJB partial differential equation reduces to a purely algebraic relationship.

We begin with a notion of optimality involving only a Lyapunov function. Hence let $f\colon\ \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, assume that $f(0,0) = 0$, and consider the controlled system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0. \tag{3.1}$$

The control $u(\cdot)$ in (3.1) is restricted to the class of *admissible* controls consisting of measurable functions $u\colon\ [0,\infty) \to \mathbb{R}^m$ such that

$$u(t) \in \Omega, \quad t \geq 0, \tag{3.2}$$

where $\Omega \subset \mathbb{R}^m$ is given.

Definition 3.1. Let $V\colon\ \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. The function $\phi\colon\ \mathbb{R}^n \to \Omega$ is *optimal with respect to V* if

$$V'(x)f(x,\phi(x)) \leq V'(x)f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \Omega. \tag{3.3}$$

Note that if V satisfies (2.3), (2.4), and

$$V'(x)f(x,\phi(x)) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{3.4}$$

then V is a Lyapunov function for the closed-loop system

$$\dot{x}(t) = f(x(t), \phi(x(t))), \quad x(0) = x_0, \quad t \geq 0, \tag{3.5}$$

that is, system (3.1) with $u(t) = \phi(x(t)), \quad t \geq 0$. The inequality (3.3) thus characterizes feedback controllers that optimize the decay rate of the closed-loop system as measured by the Lyapunov derivative.

To illustrate Definition 3.1, consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.6}$$

where A is asymptotically stable and the control constraint set Ω is characterized by

$$|u_i(t)| \leq a_i, \quad t \geq 0, \quad i = 1, \ldots, m, \tag{3.7}$$

for given positive constants $a_1, \ldots, a_m$. Define the quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}}Px, \quad x \in \mathbb{R}^n, \tag{3.8}$$

where P is the unique positive-definite solution to

$$0 = A^{\mathrm{T}}P + PA + R \tag{3.9}$$

for an arbitrary $n \times n$ positive-definite matrix R. Then

$$V'(x)f(x,u) = -x^{\mathrm{T}}Rx + 2(B^{\mathrm{T}}Px)^{\mathrm{T}}u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \tag{3.10}$$

It is easy to see that a feedback control $\phi = (\phi_1, \ldots, \phi_m)^{\mathrm{T}}$: $\mathbb{R}^n \to \Omega$ that is optimal with respect to V is given by

$$\phi_i(x) = -a_i \operatorname{sgn}(B^{\mathrm{T}}Px)_i, \quad i = 1, \ldots, m. \tag{3.11}$$

Note that if $(B^{\mathrm{T}}Px)_i = 0$, then $V'(x)f(x,u)$ is independent of u_i. Hence the value of $\phi_i(x)$ has no effect on (3.3). If in place of (3.7) we constrain $u(t)$ by

$$\|u(t)\| \leq a, \quad t \geq 0, \tag{3.12}$$

where $\|\cdot\|$ denotes the Euclidean norm and $a > 0$, then a feedback function ϕ that is optimal with respect to V is given by

$$\phi(x) = \frac{-a}{\|B^{\mathrm{T}}Px\|}B^{\mathrm{T}}Px, \quad \text{if } B^{\mathrm{T}}Px \neq 0, \tag{3.13}$$

with $\phi(x)$ arbitrary if $B^{\mathrm{T}}Px = 0$.

We now turn to the problem of characterizing feedback controllers that minimize a performance functional. Let L: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and, for $p \in \mathbb{R}^m$, define

$$H(x,p,u) \triangleq L(x,u) + p^{\mathrm{T}}f(x,u).$$

Theorem 3.1. Consider the controlled system (3.1) with performance functional

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\mathrm{d}t. \tag{3.14}$$

Assume that there exist a C^1 function V: $\mathbb{R}^n \to \mathbb{R}$ and a function ϕ: $\mathbb{R}^n \to \Omega$ such that

$$V(0) = 0, \tag{3.15}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{3.16}$$

$$\phi(0) = 0, \tag{3.17}$$

$$V'(x)f(x,\phi(x)) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{3.18}$$

$$H(x, V'^{\mathrm{T}}(x), \phi(x)) = 0, \quad x \in \mathbb{R}^n, \tag{3.19}$$

$$H(x, V'^{\mathrm{T}}(x), u) \geq 0, \quad x \in \mathbb{R}^n, \quad u \in \Omega. \tag{3.20}$$

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the solution $x(t) = 0$, $t \geq 0$, of the closed-loop system is asymptotically stable, and

$$J(x_0, \phi(x(\cdot))) = V(x_0). \tag{3.21}$$

Furthermore, the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{U}(x_0)} J(x_0, u(\cdot)), \tag{3.22}$$

where

$$\mathcal{U}(x_0) \triangleq \{u(\cdot): \quad u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (3.1) satisfies } \lim_{t\to\infty} V(x(t)) = 0\}.$$

Proof. Global asymptotic stability and (3.21) are obtained by using (3.15)–(3.19) and applying Lemma 2.1 to the closed-loop system (3.5). To prove (3.22), let $u(\cdot) \in \mathcal{U}(x_0)$ and let $x(\cdot)$ be the solution to (3.1). Then it follows that

$$\dot{V}(x(t)) = V'(x(t))f(x(t), u(t))$$

or

$$0 = -\dot{V}(x(t)) + V'(x(t))f(x(t), u(t)).$$

Hence

$$\begin{aligned} L(x(t), u(t)) &= -\dot{V}(x(t)) + L(x(t), u(t)) + V'(x(t))f(x(t), u(t)) \\ &= -\dot{V}(x(t)) + H(x(t), V'^{\mathrm{T}}(x(t)), u(t)). \end{aligned}$$

Now using the fact that $u(\cdot) \in \mathcal{U}(x_0)$ along with (3.20) and (3.21), we obtain

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty [-\dot{V}(x(t)) + H(x(t), V'^{\mathrm{T}}(x(t)), u(t))]\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x(t)) + V(x_0) + \int_0^\infty H(x(t), V'^{\mathrm{T}}(x(t)), u(t))\mathrm{d}t \\ &= V(x_0) + \int_0^\infty H(x(t), V'^{\mathrm{T}}(x(t)), u(t))\mathrm{d}t \\ &\geq V(x_0) \\ &= J(x_0, \phi(x(\cdot)), \end{aligned}$$

which yields (3.22). $\square$

The principal feature of Theorem 3.1 is that the optimal control law $u = \phi(x)$ is a *feedback* controller. Furthermore, this control is an optimal stabilizing control *independent* of the initial condition x_0.

Remark 3.1. If (3.19) and (3.20) are satisfied, then it follows that

$$L(x,\phi(x)) + V'(x)f(x,\phi(x)) \leq L(x,u) + V'(x)f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \Omega. \tag{3.23}$$

If $L(x,u)$ is independent of u, then (3.23) is equivalent to

$$V'(x)f(x,\phi(x)) \leq V'(x)f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \Omega, \tag{3.24}$$

which is precisely condition (3.3). Thus, in this case conditions (3.19) and (3.20) imply that the feedback control $u(\cdot) = \phi(x(\cdot))$ is optimal with respect to V.

Now let us illustrate Theorem 3.1 with some examples. We begin with the simplest case, namely, the linear-quadratic regulator. Hence consider the controlled system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \geq 0, \tag{3.25}$$

with performance functional

$$J(x_0, u(\cdot)) = \int_0^\infty [x^{\mathrm{T}}R_1 x + u^{\mathrm{T}}R_2 u]\mathrm{d}t, \tag{3.26}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and where R_1 and R_2 are positive definite. Thus, $L(x,u)$ has the form

$$L(x,u) = x^{\mathrm{T}}R_1 x + u^{\mathrm{T}}R_2 u. \tag{3.27}$$

Furthermore, assume that $V(x)$ is quadratic, that is,

$$V(x) = x^{\mathrm{T}}Px, \tag{3.28}$$

where P is positive definite. Consequently, we have

$$H(x, V'^{\mathrm{T}}(x), u) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA + R_1)x + 2x^{\mathrm{T}}PBu + u^{\mathrm{T}}R_2 u.$$

If (3.19) and (3.20) are satisfied, then $u = \phi(x)$ must minimize $H(x, V'^{\mathrm{T}}(x), u)$. Hence, setting

$$\frac{\partial}{\partial u}H(x, V'^{\mathrm{T}}(x), u) = 0 \tag{3.29}$$

yields

$$u = \phi(x) = -R_2^{-1}B^{\mathrm{T}}Px. \tag{3.30}$$

To check (3.19) we note that

$$H(x, V'(x), \phi(x)) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA + R_1 - PSP)x,$$

where $S \triangleq BR_2^{-1}B^{\mathrm{T}}$. Thus, (3.19) holds if P is chosen to satisfy

$$0 = A^{\mathrm{T}}P + PA + R_1 - PSP. \tag{3.31}$$

The closed-loop system has the form

$$\dot{x} = (A - SP)x, \quad x(0) = x_0, \quad t \geq 0. \tag{3.32}$$

Writing (3.31) as

$$0 = (A - SP)^{\mathrm{T}}P + P(A - SP) + R_1, \tag{3.33}$$

it follows that $A - SP$ is asymptotically stable. Finally, it follows using (3.31) that

$$H(x, V'^{\mathrm{T}}(x), u) = (u + R_2^{-1}B^{\mathrm{T}}Px)^{\mathrm{T}}R_2(u + R_2^{-1}B^{\mathrm{T}}Px) \tag{3.34}$$

so that (3.20) is satisfied. In summary, the solution $u = \phi(x)$ to the linear-quadratic problem is given by (3.30) where P is the positive-definite solution to (3.31).

Next, we consider the case of a nonquadratic cost and nonlinear feedback control. Hence assume that $L(x,u)$, $f(x,u)$, and $V(x)$ are of the form

$$L(x,u) = x^{\mathrm{T}}R_1x + h(x) + u^{\mathrm{T}}R_2u, \tag{3.35}$$

$$f(x,u) = Ax + Bu, \tag{3.36}$$

$$V(x) = x^{\mathrm{T}}Px + g(x). \tag{3.37}$$

With this notation we have

$$H(x, V'^{\mathrm{T}}(x), u) = x^{\mathrm{T}}R_1x + h(x) + u^{\mathrm{T}}R_2u + [2x^{\mathrm{T}}P + g'(x)][Ax + Bu].$$

Again setting $\frac{\partial}{\partial u}H(x, V'^{\mathrm{T}}(x), u) = 0$ we obtain

$$u = \phi(x) = -R_2^{-1}B^{\mathrm{T}}Px - \frac{1}{2}R_2^{-1}B^{\mathrm{T}}g'(x). \tag{3.38}$$

This yields

$$H(x, V'^{\mathrm{T}}(x), \phi(x)) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA + R_1 - PSP)x + h(x) + g'(x)(A - SP)x - \frac{1}{4}g'(x)Sg'^{\mathrm{T}}(x). \tag{3.39}$$

To satisfy (3.19), let P satisfy (3.31) and require that

$$h(x) + g'(x)(A - SP)x - \frac{1}{4}g'(x)Sg'^{\mathrm{T}}(x) = 0, \quad x \in \mathbb{R}^n. \tag{3.40}$$

With (3.31) and (3.40) it follows that (3.19) is satisfied. Furthermore, it can be shown that

$$H(x, V'^{\mathrm{T}}(x), u) = [u - \phi(x)]^{\mathrm{T}} R_2 [u - \phi(x)] \tag{3.41}$$

so that (3.20) holds.

Returning to (3.40), let us consider the approach of [36]. Suppose that for $\nu = 2, \ldots, r$, $h_{2\nu}(x)$ is a given nonnegative-definite homogeneous 2ν-form. Since $A - SP$ is asymptotically stable, Lemma 2.2 implies that, for $\nu = 2, \ldots, r$, there exists a nonnegative-definite homogeneous 2ν-form $g_{2\nu}(x)$ satisfying

$$h_{2\nu}(x) + g'_{2\nu}(x)(A - SP)x = 0, \quad \nu = 2, \ldots, r. \tag{3.42}$$

Then (3.40) is satisfied with $h(x)$ and $g(x)$ defined by

$$h(x) = \sum_{\nu=2}^{r} h_{2\nu}(x) + \frac{1}{4} g'(x) S g'^{\mathrm{T}}(x), \tag{3.43}$$

$$g(x) = \sum_{\nu=2}^{r} g_{2\nu}(x). \tag{3.44}$$

As discussed in [36], the term $\frac{1}{4} g'(x) S g'^{\mathrm{T}}(x)$ appearing in $h(x)$ in (3.43) is somewhat artificial in the sense that it cannot be specified arbitrarily. It is interesting to note, however, that with $h(x)$ given by (3.43), the performance functional has the form

$$J(x_0, u(\cdot)) = \int_0^\infty [x^{\mathrm{T}} R_1 x + \sum_{\nu=2}^{r} h_{2\nu}(x) + u^{\mathrm{T}} R_2 u + \phi_{NL}^{\mathrm{T}}(x) R_2 \phi_{NL}(x)] \mathrm{d}t. \tag{3.45}$$

In (3.45), $\phi_{NL}(x)$ is the nonlinear part of the optimal feedback control, that is,

$$\phi(x) = \phi_L(x) + \phi_{NL}(x), \tag{3.46}$$

where

$$\phi_L(x) = -R_2^{-1} B^{\mathrm{T}} P x, \quad \phi_{NL}(x) = -\frac{1}{2} R_2^{-1} B^{\mathrm{T}} g'^{\mathrm{T}}(x). \tag{3.47}$$

As another example, suppose we require that $V(x)$ be of the form

$$V(x) = x^{\mathrm{T}} P x + \frac{1}{2}(x^{\mathrm{T}} M x)^2, \tag{3.48}$$

which corresponds to (3.37) with $g(x) = \frac{1}{2}(x^{\mathrm{T}} M x)^2$. Thus (3.38) specializes to

$$u = \phi(x) = -R_2^{-1} B^{\mathrm{T}} P x - R_2^{-1} B^{\mathrm{T}} (x^{\mathrm{T}} M x) M x \tag{3.49}$$

and (3.40) becomes

$$h(x) + (x^{\mathrm{T}}Mx)x^{\mathrm{T}}[(A - SP)^{\mathrm{T}}M + M(A - SP)]x - (x^{\mathrm{T}}Mx)^2 x^{\mathrm{T}}MSMx = 0. \tag{3.50}$$

To satisfy (3.50), let R be an arbitrary $n \times n$ symmetric matrix and, since $A - SP$ is asymptotically stable, let M be the unique symmetric solution to

$$(A - SP)^{\mathrm{T}}M + M(A - SP) + R = 0. \tag{3.51}$$

Now (3.50) is satisfied with

$$h(x) = (x^{\mathrm{T}}Mx)(x^{\mathrm{T}}Rx) + (x^{\mathrm{T}}Mx)^2 x^{\mathrm{T}}MSMx \tag{3.52}$$

A stochastic version of this problem was treated in [49]. In [49] the matrix R is chosen to be $R_1 + MSM$ so that (3.49) becomes a Riccati equation

$$(A - SP)^{\mathrm{T}}M + M(A - SP) + R_1 + MSM = 0. \tag{3.53}$$

See equation (23) of [49] setting $W_2 = 0$.

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