ARO 24834.4-MA

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AD-A232 334

LARGE AND MODERATE DEVIATION LIMIT THEOREMS FOR ARBITRARY SEQUENCES OF RANDOM VARIABLES, WITH APPLICATIONS

FINAL REPORT

JANUARY 3, 1991

U.S. ARMY RESEARCH OFFICE

DAAL03-88-K-0076

OLD DOMINION UNIVERSITY RESEARCH FOUNDATION

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (*Leave blank*)

2. REPORT DATE
1/3/91

3. REPORT TYPE AND DATES COVERED
FINAL (May 15, 1988 - November 15, 1990)

4. TITLE AND SUBTITLE
LARGE AND MODERATE DEVIATION LIMIT THEOREMS FOR ARBITRARY SEQUENCES OF RANDOM VARIABLES, WITH APPLICATIONS

5. FUNDING NUMBERS
DAAL03-88-K-0076

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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
ARO 25834.4-MA

11. SUPPLEMENTARY NOTES
The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

In this project we have obtained large deviation local limit theorems for ratio statistics $R_n = T_n/S_n$, where $\{T_n, n \geq 1\}$ is independent of $\{S_n > 0, n \geq 1\}$. We considered all the four occurances of lattice and nonlattice for T_n and S_n and obtained several interesting applications to our theorems. As an application to the multidimensional version of our result we obtained a generalization of an identity due to Ramanujan in infinite series. In the area of strong large deviation theorems we have obtained such theorems for ratio statistics $R_n = T_n/S_n$ under simple conditions on the moment generating functions of T_n and S_n. We also made significant progress in the area of moderate deviations. We obtained strong moderate deviation theorems for partial sums of m-dependent random variables. Some more research in this area is in progress.

14. SUBJECT TERMS
Large deviations, Moderate deviations

15. NUMBER OF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

Large and Moderate Deviation Limit Theorems For Arbitrary Sequences Of Random Variables, With Applications.

1. Statement of the problems studied.

In this research project we have obtained several limit theorems for arbitrary and dependent sequences of random variables. The limit theorems considered in this project fall into three categories namely, Large deviation local limit theorems, Strong large deviation theorems and Strong moderate deviation theorems. These three categories are dependent in the sense they all are subcategories of large deviation theory. The theory of large deviations and its many uses are well described in the book by Ellis(1985) and the monograph by Varadhan(1984). This work generalized several classical limit theorems that were obtained in the literature for independent and identically distributed random variables. In the next three sections we outline briefly the technical details of the work done under this contract.

2. Summary of the most important results.

2.1. Large Deviation Local Limit Theorems.

Let $\{R_n, n \geq 1\}$ be a sequence of random variables which converge in distribution to a random variable R. It is well known that convergence in distribution does not imply convergence of the corresponding density functions. Let g_n be the density function of R_n and g be the density function of R. An assertation of the form

$$g_n(r) \to g(r), \qquad \text{as } n \to \infty \tag{2.1.1}$$

is known as a local limit theorem. Now, suppose $R_n \to c$ in probability as $n \to \infty$. Let $\{r_n \, n \geq 1\}$ be a bounded sequence of real numbers. An asymptotic expression or the limit of $g_n(r_n)$ is known as a large deviation local limit theorem. In Chaganty and Sabnis(1990) we have obtained large deviation local limit theorems for ratio statistics $R_n = T_n/S_n$ where $\{T_n, n \geq 1\}$ is an arbitrary sequence of random variables independent of the positive sequence of random variables $\{S_n, n \geq 1\}$. In applications T_n represents an estimate ot the location parameter and S_n represents an estimate of the scale parameter based on a sample of size n and R_n is a test statistic used to test a hypothesis about the location parameter. The multivariate extensions of our results, that is, the case where T_n is a random vector and S_n is a positive random variable, yeilded very interesting identities. For example, let $T_n = (T_{1n}, \ldots, T_{kn})$, where $\{T_{1n}, n \geq 1\}, \ldots, \{T_{kn}, n \geq 1\}$ are independent sequences such that T_{jn} is distributed as Poisson(n), $1 \leq j \leq k$. Let S_n be also distributed as Poisson(n) and is independent of T_n. Let $(r_1, \ldots, r_k)$ be a vector of k positive integers, for $k \geq 1$. Considering the probability of the event $\{T_{jn}/S_n = r_j, 1 \leq j \leq k\}$ and applying the multivariate extension of our theorem, we obtain the following identity:

$$\sum_{s=0}^{\infty} \frac{n^s}{s!} \prod_{j=1}^{k} \frac{n^{r_j s}}{(r_j s)!} = \frac{\exp\left(n(1+r) \exp(-\sum r_j \tau_j)\right)}{(2\pi n)^{k/2} (\det(D))^{1/2}} \left[1 + O\left(\frac{1}{n}\right)\right] \tag{2.1.2}$$

as $n \to \infty$, where $r = \sum r_j$ and

$$\tau_j = \log r_j - \frac{\sum r_j \log r_j}{1+r}, \quad j = 1, \ldots, k$$

and D is a $k \times k$ matrix with $(p,q)\,th$ element given by

$$d_{pq} = \begin{cases} r_p\, r_q \ \exp(-\sum r_j \tau_j) & \text{if } p \neq q \\ r_p^2 \ \exp(-\sum r_j \tau_j) + \exp(\tau_p) & \text{if } p = q. \end{cases}$$

In the special case where $k = 4$ and $r_j = 1$, $1 \leq j \leq k$, the identity given by (2.1.2) agrees with the result of Ramanujan (see Collected papers of Ramanujan, page XXVI, VII(3), Edited by Hardy et. al (1962)).

2.2. Strong Large Deviation Theorems for Ratio Statistics.

Let $\{R_n, n \geq 1\}$ be a sequence of random variables such that $R_n/a_n \to 0$ for some sequence of real numbers $a_n \to \infty$. In most examples the probability of the event $\{R_n \geq x_n\}$ goes to zero exponentially fast whenever $x_n = O(a_n)$ and x_n is positive. The event $\{R_n \geq x_n\}$ for x_n positive is known in the literature as a large deviation event. Numerous authors including Cramér (1938), Chernoff (1952), Ellis (1984), Varadhan (1984) have obtained asymptotic expressions for log $P(R_n \geq x_n)$ under some conditions on the moment generating function of R_n. Such theorems are known as weak large deviation results as opposed to strong large deviation theorems, which obtain asymptotic expression for $P(R_n \geq x_n)$. In this research project we have obtained strong large deviation result for the ratio statistic $R_n = T_n/S_n$, where T_n and S_n are two independent sequences of random variables. This theorem is stated below as Theorem 2.2.1. A complete report is under preparation (see Chaganty(1990)). We shall continue to use the notation used in Section 2.1.

Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of nonlattice random variables with mgf $\phi_{1n}(z)$ and $\{S_n, n \geq 1\}$ be another sequence of positive random variables with mgf $\phi_{2n}(z)$. Assume that the two sequences are independent. Let ϕ_{1n} and ϕ_{2n} be nonvanishing and analytic in the region $\Omega = \{z \in C : |z| < c\}$, where $c > 0$ and C is the set of all complex numbers. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \to \infty$. Let $\psi_{in}(z) = (\log(\phi_{in}(z)))/a_n$ for $i = 1, 2$. Let $\{r_n\}$ be a positive bounded sequence of real numbers such that there exists τ_n satisfying $\psi'_{1n}(\tau_n) - r_n\psi'_{2n}(-r_n\tau_n) = 0$ and $0 < d < \tau_n < c_1 < c$.

THEOREM 2.2.1. *Assume that T_n and S_n satisfy the following conditions:*

(1) There exist $\beta_i < \infty$ such that $|\psi_i(z)| < \beta_i$ for $z \in \Omega$.

(2) There exists $\delta_1 > 0$ such that

$$\sup_{\delta \leq |t| \leq \lambda} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right| = o\left(\frac{1}{a_n}\right)$$

for all $0 < \delta < \delta_1$ and for each $\lambda > \delta_1$.

(3) There exists positive real number α_1 such that $\psi''_{1n}(\tau) > \alpha_1$.

Then

$$P\left(\frac{T_n}{S_n} \geq r_n\right) \sim \frac{\exp(a_n[\psi_{1n}(\tau_n) + \psi_{2n}(-r_n\tau_n)])}{\tau_n\left[2\pi a_n[\psi''_{1n}(\tau_n) + r_n^2\psi''_{2n}(-r_n\tau_n)]\right]^{1/2}}.$$

2.3. Strong Moderate Deviation Theorems.

Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d. random variables with mean 0 and variance 1. The theory of moderate deviations introduced by Rubin and Sethuraman(1965a) is concerned with evaluating the limit of

$$(2.4.1) \qquad \frac{1}{n} \log P\left(\frac{S_n}{n} > x_n\right)$$

where $S_n = X_1 + \cdots + X_n$ and $x_n = O(\sqrt{\log n/n})$ under moment conditions which are less restrictive than the assumption of finiteness of the moment generating function. In a subsequent paper Rubin and Sethuraman(1965b) showed that the limits of the expressions similar to the one given in (2.4.1) are useful to compare test statistics via Bayes risk efficiency. We shall call a result which gives the limit of the expression (2.4.1) a weak moderate deviation result. On the other hand a strong moderate deviation theorem gives the limit or an asymptotic expression to the probability of the event $\{S_n/n > x_n\}$, where $x_n = O(\sqrt{\log n/n})$. Most of the literature on the theory of moderate deviations, by far, is concentrated for statistics which are functions of i.i.d. random variables. In this research project we have obtained the following strong moderate deviation theorems for partial sums of stationary m-dependent sequence $\{X_n, n \geq 1\}$ of random variables. These results are contained in Chaganty(1989).

2.4. Gracefully Degrading Systems.

Several computer systems can be modelled by Gracefully Degrading Systems for reliability analysis. These systems use all the failure-free components to execute tasks. When a component failure is detected, these systems attempt to reconfigure to a system with one fewer components. These systems can be represented as falling somewhere between the extremes of ultra-reliable systems and high performance parallel systems in terms of the trade off between performance and reliability gained by the use of redundancy. In Chaganty and Mukkamala(1990) some important properties of the failure rate functions of gracefully degrading computer systems were stated and proved. In particular it was shown that the failure rate of a gracefully degrading system with i.i.d. DFR components is also DFR if the coverage probability is less than 1/2. This generalizes a well known result for series systems.

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3. Publications.

3.1. Published Abstracts.

1. Chaganty and Sabnis(1987). Saddlepoint Approximation for the density of the ratio of two statistics. Abstract no. 200-38, IMS Bulletin, p115, Vol. 16, No. 3.

2. Chaganty and Sabnis(1988). Multidimensional large deviation local limit theorems for ratio statistics. Abstract no. 207-54, IMS Bulletin, p259, Vol. 17, No. 3.

3. Chaganty(1989). Strong Moderate Deviation Theorems for m-Dependent Random Variables. Abstract no. 211-47, IMS Bulletin, p306, Vol. 18, No. 3.

4. Chaganty and Sethuraman(1990). Multidimensional Strong Large Deviation Theorems. Abstract no. CP-131, IMS Bulletin, p363, Vol. 19, No. 3.

3.2. Publications and Technical Reports.

1. Chaganty(1989). Strong Moderate Deviation Theorems for m-Dependent Random Variables. Technical Report.

2. Chaganty and Sabnis(1990). Large Deviation Local Limit Theorems for Ratio Statistics. To appear in Communications in Statistics.

3. Chaganty and Mukkamala(1990). A Note on the Failure Rate of a Gracefully Degrading System. Submitted to IEEE Transactions in Reliability.

4. Chaganty (1990). Strong large deviation theorems for ratio statistics. Under preparation.

5. Chaganty and Sethuraman(1990). Multidimensional Strong Large Deviation Theorems. Under preparation.

3.3. Presentations.

1. Strong Moderate Deviation Theorems for m-Dependent Random Variables (1989). International Symposium on Applied Probability, Sheffield, England.

2. Presented a series of twenty lectures on the "Theory of Large Deviations" at the Indian Statistical Institute, Bangalore Center.

3. Strong Moderate Deviation Theorems for m-Dependent Random Variables (1990). VAS meeting, Harrisonburg, Virginia.

4. Multidimensional Strong Large Deviation Theorems (1990). 2nd World Congress of the Bernoulli Society, Uppsala, Sweden.

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None.