

AD-A232 755

DTIC FILE COPY



COLLEGE PARK CAMPUS

Accuracy of the Reissner-Mindlin and the (1,1,2) Model for the Clamped-in Plate Problem

by

L. Li

and

I. Babuška

DTIC
ELECTE
MAR 06 1991
S B D

Technical Note BN-1117

December 1990



INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

DISTRIBUTION STATEMENT A

Approved for public release
Distribution Unlimited

01 2 22 019

15

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Note BN-1117	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Accuracy of the Reissner-Mindlin and the (1,1,2) Model for the Clamped-in Plate Problem		5. TYPE OF REPORT & PERIOD COVERED Final life of contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) L. Li ¹ and I. Babuska ²		8. CONTRACT OR GRANT NUMBER(s) ² ONR N00014-90-J-1030; NSF CCR-88-20279
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science & Technology University of Maryland College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE December 1990
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release: distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Plates; Reissner-Mindlin model		
20. ABSTRACT The paper shows that the Reissner-Mindlin plate model for clamped-in boundary condition does not capture the boundary layer behavior for the bending moments. This boundary layer is present in the 3 dimensional formulation. In contrast the (1,1,2) model shows this boundary layer. The strength of the boundary layer for the (1,1,2) model is analyzed.		

Accuracy of the Reissner-Mindlin and the (1,1,2) model
for the clamped-in plate problem

by

L. Li^{*}, I. BABUŠKA^{**}

TR-90-48-LB

November 1990

^{*}Department of Mathematics, University of Maryland, College Park, Maryland 20742 and Department of Mathematics Fudan University, Shanghai 20020, Peoples Republic of China.

^{**}Institute of Physical Science and Technology and Department of Mathematics, University of Maryland, College Park, Maryland 20742. The work of the author new partially supported by the office of Naval Research Grant N00014-90-J-1030 and by the National Science Foundation grant CCR-88-20279

Abstract

The paper shows that the Reissner-Mindlin plate model for clamped-in boundary condition does not capture the boundary layer behavior for the bending moments. This boundary layer is present in the 3 dimensional formulation. In contrast the (1,1,2) model shows this boundary layer. The strength of the boundary layer for the (1,1,2) model is analyzed.

Key words: plates, Reissner-Mindlin model.



Accession For	
NTIS GRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

1. Introduction

There are many plate models which try to approximate the solution of the three dimensional plate problem by a system of differential equations in two dimensions. Denoting the thickness of the plate by d , then the difference between the solution of these models and the three dimensional solution converges to zero (in the scaled energy norm) as $d \rightarrow 0$. The two best known models are the Kirchhoff and the Reissner-Mindlin model. The major difference between various models is the boundary layer behavior of the solution. The boundary layer leads to large differences in moments and shear forces in the boundary region.

If the boundary is smooth and the load is also smooth then the Kirchhoff solution is smooth up the boundary. In contrast, the Reissner-Mindlin solution has a boundary layer behavior whose strength depends on the type of boundary condition. In [1] [2] a rigorous analysis of the boundary layer behavior for the Reissner-Mindlin solution on a smooth domain is given. Among other things, it is shown that the boundary layer is strongest for the soft simple support and is very weak for the clamped-in boundary condition.

The Kirchhoff model, in general, gives accurate results in comparison with the 3 dimensional solution when the boundary layer is weak. Hence it could be expected that for the clamped boundary condition, the Kirchhoff as well as the Reissner-Mindlin model yield reliable and high quality results up to the boundary. However, this is not the case. The Reissner-Mindlin model completely "misses" in this case, the boundary layer which is stronger than any predicted by this model.

To show this, let us consider the problem of the uniformly loaded square plate $\omega = \{x_1, x_2 \mid |x_i| < \frac{1}{2}, i = 1, 2\}$ with thickness $d = 1/100$. We assume that the material of the plate is homogeneous, isotropic and its Poisson ratio

$\nu = 0.3$. In the Table 1 we show the moments $M_{1,1}(x_1,0)$, $M_{2,2}(x_1,0)$ for $0 < x_1 < 0,5$ computed from the 3 dimensional formulation, Reissner-Mindlin solution ($\kappa = 0.87$) and the solution of the model (1,1,2) which will be addressed in the Section 2.

Table 1. The moments $M_{1,1}(x_1,0)$ and $M_{2,2}(x_1,0)$

x_1	$M_{1,1}(x_1,0)$			$M_{2,2}(x_1,0)$				
	3D	RM	(1,1,2)	3D	RM	$\epsilon_{RM}\%$	(1,1,2)	$\epsilon_{112}\%$
0	-0.0229	-0.0229	-0.0229	-0.0229	-0.0229	0	-0.0229	0
.2000	-0.0157	-0.0157	-0.0157	-0.0163	-0.0163	0	-0.0163	0
.4000	+0.0164	+0.0163	+0.0164	+0.0026	+0.0027	0	+0.0026	0
.4900	0.0470	0.0470	0.0470	0.0141	0.0141	0	0.0141	0
.4930	0.0483	0.0483	0.0483	0.0145	0.0144	0	0.0144	0
.4990	0.0509	0.0509	0.0509	0.0168	0.0152	9.2	0.0170	1.4
.4993	0.0510	0.0510	0.0510	0.0176	0.0153	12.3	0.0179	2.1
.4999	0.0512	0.0512	0.0512	0.0207	0.0153	25.3	0.0212	2.8
.5000	0.0513	0.0513	0.0513	0.0220	0.0153	30.0	0.0220	0

We see that on the central line $x_2 = 0$ the moment $M_{1,1}(x_1,0)$ is very accurate, i.e. the error (with respect to the three dimensional solution) is less than 0.1% in all reported points for both the RM and (1,1,2) models. In addition the moment $M_{1,1}(x_1,0)$ is smooth up to the boundary. In contrast, the moment $M_{2,2}(x_1,0)$ is smooth up to the boundary only for the RM model, and the accuracy is very poor (30% error). The 3 dimensional and (1,1,2) models clearly show the boundary layer and the error of (1,1,2) model is practically acceptable.

To understand the boundary layer better let us define $\beta(x_1)$ so that

$$\exp\left[-\beta(x_1) \frac{0.5-x_1}{d}\right] = \frac{|M_{2,2}^{RM}(x_1,0) - M_{2,2}(x_1,0)|}{|M_{2,2}^{RM}(0.5,0) - M_{2,2}(0.5,0)|}$$

The function β characterizes the boundary layer and we can expect that for $|x_1-0.5|$ small, β is nearly constant. $M_{2,2}^{RM}(x_1,0)$ was used as the (smooth) base function. In Table 2 we show values of β for the 3 dimensional and the (1,1,2) model.

Table 2. The values of the function $\beta(x_1)$.

x_1	3D	(1,1,2)
0.4990	15.43	13.86
0.4996	18.17	13.41
0.4999	23.85	13.32
0.4993	24.13	13.31
0.4999	24.78	13.18

We see that the boundary layer of the three dimensional solution is still stronger than that of the model (1,1,2); nevertheless, as we have seen in Table 1, the values of the model (1,1,2) seems to be practically acceptable.

In section 3 we will show that for the (1,1,2) model we can expect

$$\beta \approx \sqrt{\frac{120}{1-\nu}} = 13.1.$$

The model (1,1,2) is very robust for all types of practically important boundary conditions. The RM model can yield good or bad results depending on boundary conditions. See also [3] for further details.

In Section 2 we will briefly mention the concept of hierarchic plate models. In Section 3 we will analyze the basic boundary layer property of the model (1,1,2).

Computations reported in this paper have been made by the program MSC|PROBE.

2. Hierarchic plate models.

Let $\Omega = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \omega, -\frac{d}{2} < x_3 < \frac{d}{2}\right\} \subset \mathbb{R}^3$ be the plate ω with thickness d . By Γ we denote the boundary of ω . Further we let

$$S = \left\{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \Gamma, -\frac{d}{2} < x_3 < \frac{d}{2}\right\}$$

$$R_{\pm} = \left\{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \Gamma, x_3 = \pm \frac{d}{2}\right\}.$$

We will consider the plate problem as the 3 dimensional solution of elasticity with homogeneous isotropic model with $\frac{1}{2}$ of normal load on upper and lower surface $R_{\pm}$.

As usual, denote the displacement vector by $u = (u_1, u_2, u_3)$, $\sigma = \{\sigma_{i,j}\}$, $\epsilon = \{\epsilon_{i,j}\}$ the stress and strain tensor respectively and by $\mathcal{E}^A(u)$ the strain energy expressed with the usual Hooke's matrix A ($\sigma = A\epsilon$).

The exact 3 dimensional solution of the clamped-in plate problem is the minimizer of

$$G^A(u) = \mathcal{E}^A(u) - Q(u)$$

over all $u \in \mathcal{K}(\Omega) = \{u \in (H^1(\Omega))^3, u = 0 \text{ on } S\}$,

and

$$Q(u) = \int_{\Omega} \frac{g}{2} (u_3(x_1, x_2, \frac{d}{2}) + u_3(x_1, x_2, -\frac{d}{2})) dx.$$

By the index A we emphasized the use of the Hooke's matrix A . The solution u under our assumption satisfies the symmetry conditions: $u_1(x_1, x_2, x_3) = -u_1(x_1, x_2, -x_3)$, $i = 1, 2$, $u_3(x_1, x_2, x_3) = u_3(x_1, x_2, -x_3)$.

Now let $n = (n_1, n_2, n_3)$, $n_i \geq 0$, $i = 1, 2, 3$. By the n -hierarchical solution u , we mean the minimizer of

$$G^B(u) = \mathcal{E}^B(u) - Q(u)$$

over the set $\mathcal{H}^{(n)}(\Omega) \subset \mathcal{H}(\Omega)$ of all functions of the form

$$(2.1) \quad u_i^{(n)}(x_1, x_2, x_3) = \sum_{j=0}^{n_i} u_{i,j}^{(n)}(x_1, x_2) x_3^j, \quad i = 1, 2, 3.$$

B denotes a Hooke's matrix, possibly with *modified* coefficients. Under our assumption the symmetry of the load yields $u_{1,j} = 0$ for j even and $i = 1, 2$ and $u_{3,j} = 0$ for j odd.

We have shown in [4] that the solution of the model (1,1,0) is identical with the Reissner-Mindlin solution provided a modified matrix B ($B \neq A$) is used. The solution of the model (1,1,2) mentioned in the Table 1 is computed for $B = A$.

We defined the solution for clamped-in plate only. It is obvious that for general homogeneous boundary conditions only the constraint of $\mathcal{H}(\Omega)$ on S has to be modified.

If the solution u is independent of x_2 and $u_2 = 0$, then the Reissner-Mindlin model becomes the Timoshenko beam and $\{u^{(n)}\}$ becomes a hierarchy of the beam solutions. From (2.1), we can compute the moments $M_{1,1}$ and $M_{2,2}$ in the usual way.

3) The hierarchy of the models.

In this section we will consider the hierarchy of the beam models for plain strain. This is the special case of the plate when the solution is independent of x_2 and $u_2 = 0$ (for example, an infinite strip plate).

The model (1,1,0) (with modified matrix B) leads to the well known Timoshenko beam equation whose solution has no boundary layer.

The model (1,1,2) uses the set of functions

$$u_1(x_1, x_3) = u_{1,1}(x_1)x_3$$

$$u_3(x_1, x_3) = u_{3,0}(x_1) + u_{3,2}(x_1)x_3^2$$

with $u_2 = 0$. The Euler equations for the functions $u_{1,1}$, $u_{3,0}$, $u_{3,2}$, of the minimizer of G^A are

$$(3.1) \quad -12 \mu d^{-2}(-u_{1,1} + u'_{3,0}) - (\lambda + 2\mu)u''_{1,1} + (2\lambda - \mu)u'_{3,2} = 0$$

$$-12 \mu d^{-2}(-u_{1,1} + u'_{3,0})' - \mu u''_{3,2} = 12q$$

$$4(\lambda + 2\mu)u_{3,2} - 2\lambda u'_{1,1} - \mu(-u_{1,1} + u'_{3,0})'$$

$$- \frac{3}{20} \mu d^2 u''_{3,2} = 3d^2 q$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

The boundary conditions for the clamped beam are $u_{1,1} = u_{3,0} = u_{3,2} = 0$. System 3.1 can be solved explicitly. Assuming for simplicity that $-\frac{1}{2} < x_1 < \frac{1}{2}$ and q is symmetric then the solution is

$$\begin{aligned} u_{1,1}(x_1) = & c_1 \left(e^{\frac{f}{d}x_1} - e^{-\frac{f}{d}x_1} \right) \frac{\nu}{\sqrt{30} \sqrt{1-\nu}} d + c_2 \left[\frac{1-\nu^2}{2E} \frac{1}{d^2} x_1 \right] \\ & - \frac{\nu(1+\nu)}{2E} \left[1 + \frac{\nu}{10} \right] d^2 \left[e^{\frac{f}{d}x_1} \int_0^{x_1} e^{-\frac{f}{d}y} q(y) dy + e^{-\frac{f}{d}x_1} \int_0^{x_1} e^{\frac{f}{d}y} q(y) dy \right] \\ & + \frac{\nu(1+\nu)}{E} \left[1 + \frac{\nu}{10} \right] d^2 \int_0^{x_1} q(y) dy + \frac{12(1+\nu^2)}{2E} \int_0^{x_1} \frac{(x_1-y)^2}{2} q(y) dy \end{aligned}$$

(3.2)

$$\begin{aligned}
u_{3,0}(x_1) = & C_1 \left[e^{\frac{f}{d}x_1} + e^{-\frac{f}{d}x_1} \right] \frac{\nu-5}{60} d^2 + C_2 \frac{1-\nu^2}{4E} \frac{1}{d^2} x_1^2 + C_3 \\
& + \frac{(1+\nu)(\nu+10)(5-\nu)\sqrt{1-\nu}}{40\sqrt{30} E} d^2 \left[e^{\frac{f}{d}x_1} \int_0^{x_1} e^{-\frac{f}{d}y} g(y) dy - e^{-\frac{f}{d}x_1} \int_0^{x_1} e^{\frac{f}{d}y} g(y) dy \right] \\
& + \frac{1+\nu}{10E} (\nu^2 + 5\nu - 20) d^2 \int_0^{x_1} (x_1 - y) g(y) dy + \frac{12(1-\nu^2)}{E} \int_0^{x_1} \frac{(x_1 - y)^3}{6} g(y) dy \\
u_{3,2}(x_1) = & C_1 \left[e^{\frac{f}{d}x_1} + e^{-\frac{f}{d}x_1} \right] + C_2 \frac{\nu(1+\nu)}{4E} \frac{1}{d^2} \\
& - \frac{3(1-\nu)}{E} \frac{\sqrt{1-\nu}}{2\sqrt{30}} (\nu+10) d \left[e^{\frac{f}{d}x_1} \int_0^{x_1} e^{-\frac{f}{d}y} g(y) dy - e^{-\frac{f}{d}x_1} \int_0^{x_1} e^{\frac{f}{d}y} g(y) dy \right] \\
& + 6 \frac{\nu(1+\nu)}{E} \int_0^{x_1} (x_1 - y) g(y) dy
\end{aligned}$$

where

$$f = \sqrt{\frac{120}{1-\nu}}$$

and C_1, C_2, C_3 are constants which are determined from the boundary condition at $x_1 = \frac{1}{2}$ (because of symmetry). Formulae (3.2) show that the boundary layer for the model (1,1,2) for the beam is of order $\sqrt{\frac{120}{1-\nu}}$ provided that the constant $C_1 \neq 0$. In the case of the clamped boundary for $\nu > 0$ in fact $C_1 \neq 0$. The moments $M_{1,1}(x_1)$ and $M_{2,2}(x_1)$ are computed from $u_{1,j}(x_1)$.

In the similar way also higher models could be investigated.

The behavior of the moment $M_{2,2}(x_1, 0)$ mentioned in the Section 1 is very close the the beam behavior. In fact we see in the Table 2 very good agreement with the strength of the boundary layer described here.

Finally we show another simple numerical example. Consider the plane

strain elasticity problem on the domain $\Omega = \{|x_1| < .5, |x_3| \leq 0.005\}$ with clamped-in boundary conditions for $|x_1| = 0.5$.

Consider the case when $E = 10^7$, $\nu = 0.3$ and a uniform load is imposed on the upper and lower side. Because of symmetry, only one half ($x_1 > 0$) of the domain will be considered. The solution of the model (1,1,2) is solved by the p-version of the finite element using only 2 elements $(0, 0.492)$, $(0.492, 0.5)$ of degree p . For the analysis of the p-version for the beam problem with relation to locking effects, we refer the reader to [5].

In Table 3, we show the strain energy $\mathcal{E}$ of the finite element solution as a function of p .

Table 3. Energy $\mathcal{E}$ of the finite element solution as function of p .

p	$\mathcal{E} \cdot 10^3$
2	0.313411
3	0.376942
4	0.378874
5	0.379102
6	0.379162
7	0.379176
8	0.379178

Table 4 shows the moment $M_{2,2}(x_1)$ (for the beam understood as special case of the plate) as function x_1 computed by elements of degree 8.

Table 4. The moment $M_{2,2}(x_1)$ for the (1,1,2) beam model

x_1	$M_{2,2}(x_1) \cdot 10^2$
0	-1.2496
0.2	-0.4966
0.4	+1.1503
0.45	1.7878
0.495	2.4225
0.4977	2.5216
0.4995	3.0536
0.49995	3.5062
0.5	3.5723

Approximating $10^2 M_{2,2}(x_1)$ by $A + B e^{-13.1 \left(\frac{0.5-x_1}{d} \right)} = \tilde{M}_{2,2}(x_1) 10^2$ with $A = 2.4673641$, $B = 1.1049693$ we get the values reported in Table 5. We also write the relative error in percent when values from Table 4 are used.

Table 5. The approximate moment $M_{2,2}(x_1)$ and its error ϵ

x_1	$\tilde{M}_{2,2} 10^2$	$\epsilon \%$
0.495	2.4689	1.9
0.4977	2.5216	0
0.4995	3.0413	0.4
0.4995	3.5022	0.1
0.5000	3.5723	0

We see that the values in the table have the predicted boundary layer behavior. We have very different behavior of $M_{1,1}(x_1)$ and $M_{2,2}(x_1)$. The equilibrium condition directly shows that the moment $M_{1,1}$ cannot have a boundary layer. On the other hand the boundary layer of displacement vectors has to lead to a boundary layer of $M_{2,2}$.

References

- [1] Arnold, D.N., Falk, R.S. [1989]. Edge effects in the Reissner-Mindlin

plate theory. In *Analytical and Computational Models of Shells*, ed. A.K. Noor, T. Belitsko, S. Simo ASME, 71-90.

- [2] Arnold, D.N. Falk, R.S. [1990]. The boundary layer for Reissner-Mindlin plate model: Soft simply supported, soft clamped and free plates, *J. Math. Anal.* 21, 281-312.
- [3] Babuška, I., Li, L. [1990]. The problem of plate modelings. Theoretical and computational results. Tech. Note BN-1116. Institute for Physical Science and Technology, University of Maryland, MD 20742.
- [4] Babuška, I., Li, L. [1990]. The h-p version of the finite element method in the plate modeling problem. Tech.Note BN-1115 Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742.
- [5] Li, L. [1990]. Discretization of the Timoshenko beam problem by the p and h-p versions of the finite element method. *Numer. Math.* 57, 413-420.

The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- **To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.**
- **To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.**
- **To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.**
- **To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.**
- **To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).**

Further information may be obtained from Professor I. Babuska, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.