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FINAL REPORT

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The Fast Adaptive Composite Grid Method and Algebraic Multigrid in Large Scale Computation

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I. Algebraic Multigrid Methods

Work in the last four years on algebraic multigrid (AMG) has progressed in a number of areas. The areas of application were mainly structures and fluid flow, although there were advances in basic ideas as well. Several general approaches for solving systems of equations were developed and applied to 2- and 3-d elasticity equations. [1]. Study of the equations on "thin" domains led to a method for solving some badly conditioned problems [2] by transforming the equations on coarser levels. A more general approach, called element interpolation, was developed that is more easily extended to other problems, and could be used in geometric multigrid methods as well.

AMG was also applied to problems in fluids, such as linear systems arising in flow equations on unstructured 2- and 3-d finite element meshes [3], and to Stokes flow [4]. A study of AMG applied to time-dependent Euler equations led to new ideas about the application of AMG to nonsymmetric problems, including a new coarsening strategy and a new view of the role of restriction in multigrid algorithms.

Work has also been done in somewhat more standard multigrid algorithms, guided by the understanding that AMG provides. A robust semicoarsening strategy was developed and applied to 3-d diffusion problems in a heterogeneous medium. Although used in the context of oil reservoir simulation [5], the basic ideas are useful in many areas of interest. Finally, Lagrangian and semi-Lagrangian multigrid ideas have been developed and tested for solving time-dependent 2-d incompressible Navier-Stokes equations using implicit time-stepping methods. Study of these methods in the context of oceanography and meteorology is currently under way.

II. Multilevel Adaptive Methods

The central focus of this part of the project has been on the fast adaptive composite grid method. The aim was its development as a basic methodology for fast and accurate solution of a wide variety of scientific problems, with focus on fluid flow (cf. [b]-[15]) and eigenvalue problems (cf. [16]-[17]). Special emphasis was placed on developing this methodology in the context of advanced computing systems, especially parallel processors (cf. [18]-[24]). Also, to provide the foundation for these methods, special finite volume discretization techniques were developed that are very effective for maintaining physical conservation. (cf. [25]-[26]).

To provide a sound footing and confidence in the capability and performance characteristics of these methods, the project team developed theoretical foundations for these discretization and solution methods cf. [27]-[32]).

Many of the results in this area are described in laymens terms in two popular articles (i.e., [33] and [34]) and in detailed technical terms in the recent SIAM book [35]. These articles and the book represent progress that the project experienced as a unification of the various components of multilevel adaptive methods, impelled by substantial progress in each area of research: improvements in efficiency in multigrid solvers, development of the highly effective finite volume element technique, advances in the fast adaptive composite grid methodology, and development of founding theory for each of these techniques. This includes:

Multigrid

Specific advances in basic multigrid (MG) solvers include the development of a new scheme for 3-D problems that eliminates certain inefficiencies of existing schemes. The basic approach here was to develop a scheme that works effectively for all types of anisotropies of the differential operator. A multigrid scheme was carefully developed and analyzed for the INTEL iPSC hypercube. This study was motivated by the common concern that conventional multigrid schemes must exhibit some parallel processing inefficiency since coarse grid processing must lead to processor ideal time. The results of this study proved this concern to be unfounded: practical use of parallel multigrid methods would exhibit negligible processor inefficiencies and, in any case, these schemes have optimal complexity in a parallel environment. A related development is a precise comparative analysis of the complexity of the Schwarz and multigrid methods in serial and parallel computation. Not surprisingly, the analysis established, in a limited setting, the superiority of multigrid for most cases of interest. In other words, simple domain decomposition schemes cannot compete with multigrid in practice.

FVE

The finite volume element method was developed by the project team in order to provide an efficient and accurate discretization method for the various (uniform and composite) grids used in multilevel processing. FVE has proved to exhibit several advantages, particularly for fluid flow applications: conservation is automatic (in fact, it exhibits a rigorous sense of local conservation, even on nonuniform and time evolving grids); it is valid at various discontinuities, including shocks; it is accurate even in the presence of grid interfaces; it is founded on the concept of approximation spaces, which is in contrast to usual methods based on a local truncation error (which can be tricky on nonuniform grids); it applies in natural ways for nonlinearities, general boundary conditions, nonuniformities, systems, and highly-convective flows; and it greatly simplifies the design of MG constructs (intergrid transfers, coarse grid operators, scaling, and norms). A major advance of the last project year was to establish a theoretical foundation for FVE. In fact, superconvergence results have been developed for general diffusion equations that show that FVE discretization error is accurate on irregular grids. The estimates obtained for FVE are better in some sense that the existing theory for any other basic discretization approach, including finite elements.

FAC

There are several areas of recent development in the fast adaptive composite grid (FAC) methodology.:

computational fluid dynamics FAC has been successfully applied to fluid flow problems of increasing complexity. To develop basic techniques, the initial work has focused on potential and planner cavity flows. However, work is now progressing into Euler and the full Navier-Stokes equations. One of the important aspects of this development is the study of FAC schemes for highly-convective (e.g., high Reynolds number) flow.

elliptic grid generation Many computational fluid flow simulations involve irregular boundaries and other phenomena that are treated by some form of grid generation. Unfortunately, grid mapping schemes tend to introduce several difficulties, including slow solvers (of the grid generation equations as well as the flow equations), complicated equations (local irregularities introduce global complications in transformed flow equations), and limited adaptive flexibility (grid point concentration is often controlled indirectly by certain "steering" functions). MG and FAC methods were applied to elliptic grid generation in the context of potential flow to overcome these difficulties: MG provides the fast solvers, FAC allows for efficiently adapted local grids, and a special version of FAC allow for local grid generation to avoid global complication of the flow equations.

high order Using a special hierarchical basis interpretation of FAC leads to a simple and very efficient mechanism for obtaining fourth and higher order discretization, without altering the basic discretization scheme. Numerical results show this to be a highly effective scheme, especially since it continues to allow local refinement.

domain decomposition Another interpretation of FAC, as a domain decomposition method, was developed and used as a basis for establishing a simple theory for approximate solvers.

self-adaptivity One of the difficult problems related to mesh refinement schemes is to decide where and how much additional resolution is needed. Consistent with the concept of multilevel schemes, the algorithm developed by the project team uses a Richardson extrapolation of the error on different levels of grids to provide an approximation of the error. This error approximation is then used to decide where and how much additional refinement is needed on the present grid. This information is then used to construct patches of refined grids which overlay the present grid. The work illustrated how the self-adaptive scheme can be sued in both nonlinear problems and time-dependent problems. The results for time-dependent problems include the refinement in the temporal direction as well as the spatial directions. Again a Richardson extrapolation is used to approximate the error and this approximation is used to determine both the time step on the coarsest grid and the local time steps on the patched grids. One of the focuses of this work is to demonstrate that FAC (and even more so, the self-adaptive FAC) is also a grid generation scheme. There is much work done in generating grids that are logically rectangular. The FAC grids (and of course the self-adaptive FAC grids) are not logically rectangular, but FAC includes a method for solving discretizations of partial differential equations on these irregular grids. This allows one to place grid patches where they are needed and not worry about how much the grids have to be contorted or stretched to keep them logically rectangular.

One of the problems related to mesh refinement schemes is the amount of work needed to solve the problem on the refinement meshes. For example, in all of the self-adaptive finite element schemes, when it is decided that more refinement is needed, the entire region is regridded placing additional elements where they are needed, and the problem is resolved. This process is very expensive. One of the advantages of FAC is that this is not necessary. But one must still be careful that the amount of work needed to generate the grids and solve on the resulting grids is sufficiently small. A complexity analysis was performed on both the FAC scheme and the self-adaptive algorithm showing that the problems can be gridded and solved with complexity proportional to the total number of grid points in the composite grid. This work assumes that multigrid is used to solve on each particular patch and shows that the self-adaptive process does not raise the order of work necessary to solve the problem.

<u>time-dependent problems</u> The need for mesh refinement schemes becomes more acute for time-dependent problems. As with the steady FAC algorithm, the scheme developed in the project allows for patches of finer grids to be inserted into a coarser grid and solves the problem on the resulting composite grid. Because often in regions where more resolution is needed more temporal resolution is also needed, the scheme allows for local time stepping on the patches. Though the software developed for the time-dependent FAC is general, all of the applications run are for parabolic problems. In addition, the project developed a proof of convergence for the time-dependent FAC scheme when applied to quasi-linear parabolic equations.

<u>parallel algorithms</u> FAC, like other local grid methods, has a fair degree of parallelism: separate refinement regions can be processed independently and each subgrid solver can be solved by a highly parallelized MG scheme. However, in a very large scale computing environment, full parallelism is hindered by the sequential way in which the refinement sequence is processed. Asychronous FAC (AFAC) was developed that allows for independent processing of all levels of refinement, without degradation in convergence rate. The full parallelism and efficiency of AFAC has been established in

simulated and actual parallel computation. These tests included incorporation of a simple by level load balance scheme that is allowed by the special AFAC approach.

theory

Recent theoretical developments for FAC and AFAC include fundamental variational theory for two-level and model multilevel schemes, a nonvariational FVE-type theory, and an analysis of complexity (including the case of a self-adaptive version of FAC).

III. Computational Fluid Dynamics

The CFD research has been aimed at gaining insights and obtaining a deeper understanding of the problems associated with numerical solution of fluid flow problems, and to developing techniques for dealing with these problems. The main numerical tool used was multigrid methods, although a number of the results and conclusions are not exclusively related to multigrid. The problems investigated were of incompressible flows, especially at high Reynolds numbers, and mostly steady flows, although time-dependent problems were also considered. For a general outline of our approach, see [49;Section 2].

The main results and conclusions of the research are as follows.

1. Even with the simplification of considering only incompressible flows there are a number of problems associated with the numerical solution of the fluid flow equations, each of which requires its own special handling. These include recirculation, boundary layers, singularities, poor approximation of certain fine-grid components by the coarse grids, poor advection of characteristic components by schemes with excessive artificial viscosity, treatment of time-dependence. In investigating these problems it is extremely important to separate between them in the model problems as well as in analysis. Otherwise, there is little hope in determining the cause of poor approximation and of poor (multigrid) convergence rates.

2. Problems with high-Reynolds recirculating flows require special considerations in the discretization, namely, that the local truncation errors be dominated by isotropic viscosity terms. This is true even in problems which include none of the other difficulties stated above, i.e., problems with very smooth solutions, for which the local truncations errors are vanishingly small compared with the O(1) solution itself throughout the domain of solution. Failure to comply with this rule, e.g., by attempting to employ upstream differences with cell-Reynolds numbers that are too high, may lead to spurious solutions even in very simple problems, such as flow between concentric rotating cylinders, for which an exact solution is known. See [46].

3. In the context of shear-driven (and) recirculating flows, where data propagates from the boundary into the interior of the domain through (elliptic) viscosity terms, smooth characteristic error components receive at most 0.5^q of the required correction from the coarse grid, where q is the order of approximation, even if the coarse-grid problem is solved exactly. We have developed methods for improving the coarse-grid correction and greatly accelerating convergence, while increasing the total work by only a small fraction. Analyses of the advection diffusion equation show that for first-order discretization the convergence factor can be reduced by these methods from 0.5 to 0.11 per two-level cycle, and the latter result can also be achieved by a special form of the multi-level W cycle for any number of levels. These predictions were realized experimentally for the advection diffusion equation, and convergence rates that are nearly as good were obtained with the incompressible Navier Stokes equations. See [55].

4. Flows that enter through some boundary, and whose general orientation can be followed, such as flows in ducts, introduce a different sort of problem - poor advection of data whose cross-stream frequencies are relatively high on the scale of the grid. The infinite-space analysis, which is useful in elliptic problems as well as in recirculating flow problems, is no longer sufficient. Instead, a half-space analysis, which brings into account the effect of the inflow boundary, has been used. This analysis and numerical experiments show that it is possible to employ effectively two different solution-mechanisms within the usual multigrid process, resulting in asymptotic convergence rates that are nearly as good as those obtained for the Poisson equation (even with second-order upstream discretization), and reduction of the algebraic error well below the level of truncation in a 1FMG algorithm. Also, the truncation error is shown to be reduced significantly by employing "narrow" upstream schemes, which also aid the solution process. See [51].

5. The techniques developed for entering flows are easily generalized to time-dependent problems, treating time as another spatial dimension in the discretization. Now the time dimension acts as the well-defined general orientation of the flow. This approach leads to an extremely efficient solver. Various multigrid techniques that have been developed for general time-dependent problems can be readily incorporated into this process.

6. Early calculations with the Stokes systems have shown that performance comparable with that of the Poisson solver is obtainable. Results with periodic boundary conditions were matched for the Dirichlet problem with negligible extra work (a single extra relaxation sweep along the boundaries after each full sweep).

IV. Ph.D. Students Produced

Fall, 1989	C. Liu, "Multilevel adaptive methods in computational fluid dynamics"
Spring, 1990	Z. Cai, "A theoretical foundation for the finite volume element method"
Fall, 1990	V. Henson, "Fourier methods of image reconstruction"

V. References

1. S. McCormick and J. Ruge, "Algebraic multigrid methods applied to problems in computational structural mechanics," State-of-the-Art Surveys on Computational Mechanics (A.K Noor and J.T. Oden, eds.), The American Society of Mechanical Engineers, New York, 1989.

2. J. Ruge, "A multigrid approach for elasticity problems on "thin" domains," Multigrid Methods: Theory, Applications and Supercomputing (S. McCormick, ed.), Marcel-Dekker, New York, 1988.

3. J. Ruge, "Algebraic multigrid methods (AMG) applied to fluid flow problems," Proceedings of the Sixth International Symposium on Finite Element Methods in Flow Problems, June, 1986, Antibes, France.

4. J. Ruge, "Algebraic multigird mehtods (AMG) applied to Stokes flow," Proceeding of the Seventh International Symposium on Finite Element Methods in Flow Problems, April, 1989, Huntsville, Alabama.

5. J.E. Dendy, J. Ruge, S. McCormick, T. Russell and S. Schaffer, "Multigrid methods for three-dimensional petroleum reservoir simulation," Proceeding of the SPE Symposium on Reservoir Simulation, February, 1989, Houston, Texas.

6. S. McCormick and J. Thomas, "The fast adaptive composite grid method (FAC) for elliptic boundary value problems," Math Comp. 46 (1986), 439-456.

7. S. McCormick, J. Thomas, and A. Thomas, "Application of the fast adaptive composite grid method to computational fluid dynamics," Proc. 5th Int. Conf. Num. Meths. in Laminar and Turbulent Flow, Montreal, Canada, 1987.

8. M. Heroux, S. McCormick, S. McKay, and J. Thomas, "Applications of the fast adaptive composite grid method," Lecture Notes in Pure and Applied Mathematics/110, Marcel-Dekker, May, 1988.

9. C. Liu and S. McCormick, "Multigrid, elliptic grid generation and the fast adaptive composite grid method for solving transonic potential flow equations," Lecture Notes in Pure and Applied Mathematics/110, Marcel-Dekker, May, 1988.

10. C. Liu and S. McCormick, "Multigrid, the rotated hybrid scheme, and the fast adaptive composite grid method for planar cavity flow," Proc. 12th JMACS Congress on Scientific Computation, Paris, July, 1988.

11. S. McCormick, M. McKay and J. Thomas, "Computational complexity of the fast adaptive composite grid (FAC) method," Applied Numerical Mathematics 6(1990), Special Issue on Domain Decomposition (W. Proskurowski, ed.)

12. S. McCormick, "The finite volume element technique and fast adaptive composite grid method for time-dependent differential equations," Proc. 7th International Conf. on Finite Element Methods in Flow Problems, Univ. of Alabama, Huntsville, April 3-7, 1989.

13. S. McCormick, "Multilevel adaptive schemes and domain decomposition methods," Proc. 5th Int. Symp. on Num. Meth. in Eng., Lausanne, Sept., 1989.

14. C. Liu and S. McCormick, "Multigrid, the fast adaptive composite (FAC) grid method for solving flow problems," Procs. Internat. Symp. on Comput. Fluid Dyns, Nagoya, Japan, Aug. 28-31, 1989.

15. S. McCormick and U. Ruede, "On local refinement higher order methods for elliptic partial differential equations," submitted.

16. J. Mandel and S. McCormick, "A multilevel variational method for Au = gBu on composite grids," J. Comp. Phys., 80 (1989), 442-450.

17. "Iterative solution of elliptic eigenproblems with refinement," submitted.

18. S. McCormick and D. Quinlan, "Multigrid load balancing for multiprocessors - an outline," preprint, CU-Denver, 1986.

19. W. Briggs, L. Hart, S. McCormick and D. Quinlan, "Multigrid methods on a hypercube," Lecture Notes in Pure and Applied Mathematics/110, Marcel-Dekker, May, 1988.

20. L. Hart and S. McCormick, "Asychronous multilevel adaptive methods for solving partial differential equations on multiprocessors: basic ideas," Parallel Computing 12(1989), 131-144.

21. Z. Cai and S. McCormick, "Computational complexity of the Schwarz alternating procedure," Internat. J. High Speed Computing 1 (1989), 1-28.

22. B. Dowell, M. Govett, S. McCormick and D. Quinlan, "Parallel asynchronous multilevel adaptive methods," Proc. 11th Int'l. Conf. on Num. Meths. in Fluid Dyns., Williamsburg, June 27 - July 1, 1988.

23. S. McCormick and D. Quinlan, "Dynamic grid refinement for partial differential equations in parallel computers," Proc. 7th International Conf. on Finite Element Methods in Flow Problems, Univ. of Alabama, Huntsville, April 3-7, 1989.

24. S. McCormick and D. Quinlan, "Asynchronous parallel adaptive methods for partial differential equations: performance results," Parallel Computing 12(1989), 145-156.

25. C. Liu and S. McCormick, "The finite volume-element method for planar cavity flow," Proc. 11th Int'l. Conf. on Num. Meths. in Fluid Dyns., Williamsburg, June 27 - July 1, 1988.

26. J. Mandel, S. McCormick and Z. Cai, "The finite volume element method for diffusion equations on general triangulations," SIAM J. Num. Anal., to appear.

27. J. Mandel S. McCormick and J. Ruge, "An algebraic theory for multigrid methods for variational problems," SIAM J. Num. Anal. 25 (1988), 91-110.

28. J. Mandel, S. McCormick and R. Bank, "Variational theory for multigrid methods," Chapter 5 in SIAM Frontiers in Applied Math., Vol. III on Multigrid Methods (S. McCormick, ed.), 1987.

29. J. Mandel and S. McCormick, "Iterative solution of elliptic equations with refinement: the two-level case," Proc. Second Conf. on Domain Decomposition, Los Angeles, Jan., 1988.

30. J. Mandel and S. McCormick, "Iterative solution of elliptic equations with refinement: the model multilevel case," Proc. Second Conf. on Domain Decomposition, Los Angeles, Jan., 1988.

31. Z. Cai and S. McCormick, "On the accuracy of the finite volume element method for diffusion equations on composite grids," SIAM J. Numer. Anal. 27(1990), 636-655.

32. S. McCormick and U. Ruede, "A finite volume convergence theory for the fast adaptive composite grid methods," submitted.

33. "Multigrid trends," SIAM News, Vol. 21, No. 1, Jan., 1988.

34. S. McCormick, "Computation: breaking the matrix speed limit," Nature Magazine (News and Views) 337 (1989), 205.

35. S. McCormick, <u>Multilevel Adaptive Methods for Partial Differential Equations</u>, SIAM, Philadelphia, 1989,162 pps.

36. D. Bai and A. Brandt, "Local mesh refinement multilevel techniques," SIAM J. Sci. Stat. Comp. 8 (1987) pp. 109-134.

37. A. Brandt and S. Ta'asan, "Multigrid method for nearly singluar and slightly indefinite problems," Multigrid Methods II (W. Hackbusch and U. Trottenberg, eds.), Springer-Verlag, 1986, pp. 100-122.

38. A. Brandt, D. Ron and D.J. Amit, "Multilevel approached to discrete-state and stochastic problems," Multigrid Methods II (W. Hackbusch and U. Trottenberg, eds.), Springer Verlag, 1986, pp. 66-99.

39. A. Brandt, "Multi-level approaches to large scale problems," Proc. ICM-86 (International Congress of Mathematicians, Berkeley, California, August 1986), pp. 1319-1334, 1987.

40. A. Brandt and A. Lanza, "Multigrid in general relativity: I. Schwarzschild spacetime," Class. Qunatum Grav. 5 1988, pp. 713-732.

41. A. Brandt, "Multilevel computations: review and recent developments," Multigrid Methods: Theory, Applications and Supercomputing (S.F. McCormick, ed.) Marcel-Dekker, 1988, pp. 35-62.

42. J. Ruge and A. Brandt, "A Multigrid approach for elasticity problems on "thin" domains," Multigrid Methods: Theory, Applications and Supercomputing (S.F. McCormick, ed.), Marcel-Dekker, 1988, pp. 541-555.

43. D. Kandel, E. Domany, D. Ron, A. Brandt and E. Loh, Jr., "Simulations without critical slowing down," Phys. Rev. Lett. 60, 1988, pp. 1591-1594.

44. A. Brandt and A.A. Lubrecht, "Multilevel matrix multiplication and fast solution of integral equations," J. Comp. Phys. 90, 1990, pp. 348-370.

45. A. Brandt, "Rigorous local mode analysis of multigrid," Math. Comp., submitted.

46. A. Brandt and I. Yavneh, "Inadequacy of first-order upwind difference scheme for some recuiculating flows," J. Comp. Phys., in press.

47. D. Kandel, E. Domany and A. Brandt, "Simulations without critical slowing down - Ising and 3-state Potts models," Phys. Rev. B., 40, 1989, pp. 330-344.

48. R. Ben-Av, A. Brandt, and S. Solomon, "The fermiomic matrix, instantons, zero modes and multigrid," Nuclear Physics B, in press.

49. A. Brandt, "The Weizmann Institute Research in Multilevel Computation: 1988 Report," Proc. 4th Copper Mountain Conf. on Multigrid Methods, (J. Mandel et al, eds.), SIAM, 1989, pp. 13-53.

50. A. Brandt, "The Scope of Multiresolution Iterative Computations," SIAM News 23, 1990, pp. 8-9.

51. A. Brandt and I. Yavneh, "On multigrid solution of high-Reynolds incompressible entering flows," J. Comp. Phys., submitted.

52. A. Brandt, M. Galun and D. Ron, "Optimal multigrid algorithms for calculating thermodynamic limits," Report, August 1990.

53. A. Brandt, "Multilevel computations of integral transforms and particle interactions with oscillatory kernels," Computer Phys. Communications, in press.

54. R. Ben-Av, A. Brandt, M. Harmatz, E. Katznelson, P.G. Lauwers, S. Solomon and K. Wolowesky, "Fermion simulation using parallel transported multigrid," Phys. Rev Lett., submitted.

55. A. Brandt and I. Yavneh, "Accelerated multigrid convergence for high-Reynolds recirculating flows," Report, August 1990.

56. D.S. Balsara and A. Brandt, "Multilevel methods for fast solution of N-body and hybrid systems," Multigrid Methods III (W. Hackbusch and U. Trottenberg, eds.) Springer Verlag, 1991.

57. V. Milulinsky and A. Brandt, "Multigrid methods for free boundary problems," Multigrid Methods III (W. Hackbusch and U. Trottenberg, eds.), Springer Verlag, 1991.

58. A. Brandt and J. Greenwald, "Parabolic multigrid revisited," Multigrid Methods III (W. Hackbusch and U. Trottenberg, eds.), Springer Verlag, 1991.

59. E. Gendler, "Multigrid methods for time-dependent parabolic equations," M.sc. Thesis, The Weizmann Institute of Science, Rehovot, Israel, 1986.

60. I. Ben-Zvi, "Optimal trajectory planning for robotic manipulators: a multigrid approach, M.Sc. Thesis, The Weizmann Institute of Science, Rehovot, Israel, 1988.

61. V. Mikulinsky, Multigrid methods for free boundary problems, M.Sc. Thesis, The Weizmann Institute of Science, Rehovot, Israel, 1987.

62. Z. Haras, "The 3-satisfiability problem: a multi-level annealing approach, M.Sc. Thesis, The Weizmann Institute of Science, Rehovot, Israel, 1988.

63. R. Kaminsky, "Multilevel solution of the long transportation problem, M.Sc. Thesis, The Weizmann Institute of Science, Rehovot, Israel, February 1989.

64. D. Ron, "Development of fast numerical solvers for problems of optimization and statistical mechanics, Ph.D. Thesis, The Weizmann Institute of Science, Rehovot, Israel, March 1899.

65. D. Sidilkover, "Numerical solution for steady state problems with discontinuities, Ph.D. Thesis, The Weizmann Institute of Science, Rehovot, Israel, 1989.

66. I. Yavney, "Interim .Report of Ph.D. work," The Weizmann Institute of Science, Rehovot, Israel, February 1989.