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APPLICATION OF THE THEORY OF OPTIMAL CONTROL TO THE DEVELOPMENT OF TERMINAL SEARCH PATTERNS

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RESEARCH AND TECHNOLOGY DEPARTMENT

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The theory of optimal search, as usually presented, treats the development of a search effort allocation over a space X which is optimal in some sense, given an underlying target position distribution. In other words, when the location of a target is not precisely known and when the detection characteristics of the search mechanism are well-defined, one can prescribe a search effort function on both space and time to achieve a practical MOE (measure of effectiveness). The best measure of effectiveness is usually considered to be the probability of detection, or target acquisition, under a time budget constraint.

Optimal search theory, surprisingly, usually does not embrace in a rigorous, methodical fashion the generation of vehicle path processes which reproduce, to some

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FOREWORD

The theory of optimal search, as usually presented, treats the development of a search effort allocation over a space $\bar{X}$ which is optimal in some sense, given an underlying target position distribution. In other words, when the location of a target is not precisely known and when the detection characteristics of the search mechanism are well-defined, one can prescribe a search effort function of both space and time to achieve a practical MOE (measure of effectiveness). The best measure of effectiveness is usually considered to be the probability of detection, or target acquisition, under a time budget constraint.

Optimal search theory, surprisingly, usually does not embrace in a rigorous, methodical fashion the generation of vehicle path processes which reproduce, to some level of approximation, the optimal allocation of effort. Rather, more often than not, heuristic search mechanisms, such as parallel sweeping or radial tracking, are designed to replicate the effort distribution. The purpose of the report is primarily to examine, from first principles, the underlying variational problems and to obtain, if possible, the best solutions satisfying reasonable cost criteria. We show that it is possible to generate sensible paths when one knows that the target is stationary. Also, those conditions under which the best solution can be approximated by a logarithmic spiral are presented. Finally, in the case of a moving target, we present a *stochastic vehicle search pattern*, whose expected path, for example, could be followed by a search vehicle.

The present document treats the terminal search phase of an overall engagement strategy synthesis model for acquistion of an enemy target. The work was sponsored by the Naval Surface Warfare Center (NSWC) under an independent research grant, the principal investigator for the overall effort being Dr. B. C. Meyers (N35).

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CHAPTER 1

INTRODUCTION

In a standard work on search and detection,[1] the following comments are found: "It is natural to expect that a subject called 'Search Theory' will be mainly preoccupied with assessing the consequences of following a particular search path, and perhaps even with discovering paths that are in some sense optimal for the searcher. This is not the case–such problems are rare in Search Theory...There are two good reasons for this: One is that most searchers would not benefit from knowing the optimal path unless it happened to be easy to follow, and the other is that a 'path' is not a convenient mathematical object in most cases." This kind of philosophy may have some merit, especially with regard to the statement that most searchers would not benefit from knowledge of an optimal, complicated path, which would, in all likelihood, not be conveniently traversable by a searcher. On the other hand, it can be argued that an understanding of the optimal track is at least useful insofar as it provides a benchmark against which the performance of a "simple track process" can be evaluated. Furthermore, one might be able to approximate the complicated path by a reasonably simple one, say, by a chordal or polygonal line approximation to the optimal track. As to the comment that an optimal path is usually not a convenient mathematical object, hopefully we shall be able in this document to provide convincing evidence to the contrary.

Indeed, in this report, we shall utilize optimal control theory to develop optimal path solutions for the problem of acquistion of a stationary target. It will be seen that such theory can be invoked to yield a "convenient mathematical object," albeit someone might argue that he would not consider the path to be practical. The development of this part of the work is due to the second author of this report, J. W. Wingate, whose expertise is in the field of optimal control. The first author, E. A. Cohen, Jr., analyzed the logarithmic spiral search and compared that path with the optimal one insofar as asymptotic behavior is concerned. The dynamic model was also developed by the first author.

It is our hope that the methodology presented here will serve two purposes: (1) fill what we consider to be an obvious hiatus in the search theory literature and (2) provide a theoretical foundation for better practical, computational studies. Heretofore, authors [2,3,4] seem to have emphasized various practical search mechanisms; they apparently have not addressed the issue as to behavior of their search mechanisms in comparison with some standard.

CHAPTER 2

OPTIMAL CONTROL SOLUTIONS FOR A STATIONARY GAUSSIAN TARGET

GENERAL THEORY

Before addressing the particular problems of concern to us, it is useful to review some of the theoretical optimal control theory concepts pertinent to search. To that end let us consider the following special case of the general control problem of Lagrange: [5]

$$\text{Minimize} \int_0^T L(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \text{ subject to the constraints on}$$

$\mathbf{x}(t), \mathbf{u}(t)$ that

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), 0 \leq t \leq T, \tag{2-1}$$

$$\mathbf{u}(t) \in U, \tag{2-2}$$

$$\mathbf{x}(0) = \mathbf{a}; g^i(\mathbf{x}(T)) = 0, 1 \leq i \leq m, \tag{2-3}$$

where U is a set of vector functions $\mathbf{u}(t)$ (called controls) and the $g^i, 1 \leq i \leq m$, are scalar functional forms to be equated to zero at the terminal time T (terminal constraints). In this special case of the control problem of Lagrange, the initial vector $\mathbf{a}$ is given and the interval $[0, T]$ is fixed. Furthermore, the vector $\mathbf{x}$ belongs to an n-dimensional Euclidean space R^n, and $\mathbf{u}$ belongs to a k-dimensional Euclidean space R^k. The functions L and $\mathbf{f}$ are to be continuous in $t, \mathbf{x}$, and $\mathbf{u}$ and to have continuous first partial derivatives $L_t, L_{\mathbf{x}}, \mathbf{f}_t$, and $\mathbf{f}_x$. That is, L and $\mathbf{f}$ are required to be smooth functions. In addition, the functions g^i are to be smooth. In contrast, the control $\mathbf{u}(t)$ will generally be assumed to be piecewise continuous.

For this special case, the Maximum Principle of Optimal Control Theory takes the following form:

Define the function H of $(t, \mathbf{x}, \mathbf{u}, \mathbf{p})$ by

$$H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) = \mathbf{p} \cdot \mathbf{f}(t, \mathbf{x}, \mathbf{u}) - \lambda_0 L(t, \mathbf{x}, \mathbf{u}),$$

where λ_0 is a scalar constant. The function H is known as a Hamiltonian. If

$$\mathbf{x}_0 : \mathbf{x}_0(t), \mathbf{u}_0(t), 0 \leq t \leq T,$$

is a solution to our minimization problem stated at the outset, there exist multipliers $\lambda_j, 0 \leq j \leq m$, not all zero, and an absolutely continuous function $\mathbf{p}(t)$ (understood componentwise) such that

$$\lambda_0 \geq 0; \tag{2-4}$$

the Euler-Lagrange equations

$$\dot{\mathbf{x}}_0(t) = H_{\mathbf{p}}(t, \mathbf{x}_0(t), \mathbf{u}_0(t), \mathbf{p}(t))$$

$$\dot{\mathbf{p}}(t) = -H_{\mathbf{x}}(t, \mathbf{x}_0(t), \mathbf{u}_0(t), \mathbf{p}(t)), \tag{2-5}$$

$0 \leq t \leq T$, hold (the dot denoting differentiation); the Weierstrass condition:

$$H(t, \mathbf{x}_0(t), \mathbf{u}_0(t), \mathbf{p}(t)) = \max_{\mathbf{u} \in U} H(t, \mathbf{x}_0(t), \mathbf{u}, \mathbf{p}(t)), 0 \leq t \leq T \tag{2-6}$$

obtains; the equation

$$\frac{d}{dt}(H(t, \mathbf{x}_0(t), \mathbf{u}_0(t), \mathbf{p}(t)) = H_t(t, \mathbf{x}_0(t), \mathbf{u}_0(t), \mathbf{p}(t)), 0 \leq t \leq T \tag{2-7}$$

holds (d/dt denoting a *total derivative* and H_t representing a partial derivative with respect to $t - \mathbf{x}, \mathbf{u}$, and $\mathbf{p}$ being held fixed); $\mathbf{x}_0$ satisfies the imposed boundary conditions given by Equation (2-3); and $\mathbf{p}$ satisfies the natural boundary (or transversality conditions:

$$\sum_{i=1}^{m} \lambda_i g_{\mathbf{x}}^i(\mathbf{x}(T)) + \mathbf{p}(T) = 0. \tag{2-8}$$

If $\lambda_0 > 0$), it can be normalized to a convenient, positive constant (say 1). When $\lambda_0 = 0$, the solution is said to be *abnormal.* Such abnormality usually arises when too many constraints are imposed.

In practice, Differential Equations (2-5) will often be complex and would need to be solved numerically if a direct approach were necessary. If the solution to this system can be inferred by other means, say through qualitative properties of the system (sign properties of right-hand side, for example) and through characteristics of the Weierstrass condition, namely, Equation (2-6), then we are able to ascertain the answer indirectly. We shall see the advantages of such artifices in the sequel.

THE OPTIMAL SEARCHER MOTION PROBLEM

Let us now investigate the details of the optimal searcher motion problem from the perspective of optimal control theory. We shall develop optimal solutions for two practical situations: (A) The target is stationary in two-space (R^2), and its location is governed by a circular Gaussian Law:

$$p(\mathbf{x}) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x_1^2 + x_2^2}{2\sigma^2}\right) \tag{2-9}$$

and (B) the target is stationary in two-space, and its position follows an elliptic Gaussian Law:

$$p(\mathbf{x}) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right)\right). \tag{2-10}$$

In Case (B), we can always orient the axis of our coordinate system (in principal directions) so as to eliminate correlation effects. We shall develop our solution for Case (A) first, utilizing the theory of Lagrange just described. For (B), a different approach, subsuming (A), will be presented.

For Case (A), let $\mathbf{r}(t) = (x_1(t), x_2(t))$ represent the searcher's position as a function of time, and let us suppose that the searcher is to follow a constant speed trajectory which moves outward from the origin (the mode of the distribution). Then the searcher motion is described by

$$\dot{\mathbf{r}}(t) = \mathbf{v}(t), (\mathbf{r} \in R^2),$$

where $||\mathbf{v}(T)|| = v$, and $\mathbf{r} \cdot \mathbf{v} \geq 0$. The latter condition follows from the requirement that $||\mathbf{r}||$ is nondecreasing, since $d(||\mathbf{r}||^2)/dt = d(\mathbf{r} \cdot \mathbf{r})/dt = 2\mathbf{r} \cdot \mathbf{v} \geq 0$.

Two issues pertinent to Problem (A) need to be addressed: (1) an optimal allocation of effort corresponding to the target distribution and (2) a cost effectiveness criterion to measure the efficiency of any given search path in *matching the optimal allocation.* Point (1) has been resolved in the literature.[6] We shall first review that argument.

The target distribution function for Problem (A) has been given by Equation (2-9). One also needs a detection law, which gives the probability of acquisition of a target known to be located at some position $\mathbf{x}$ as a function of $\mathbf{x}$ and an effort density z. The *simplest law* of exponential type is afforded by

$$b(\mathbf{x}, z) = 1 - e^{-az}, \mathbf{x} \in X, a > 0, z \geq 0, \tag{2-11}$$

where a is some known quantity. Note that $b(\mathbf{x}, z)$, as given by Equation (2-11), does not depend explicitly upon $\mathbf{x}$. However, there may well be an *implicit dependence* in the sense that z could be a function of $\mathbf{x}$. In fact, one would generally expect that the latter functional coupling would be present. Equation (2-11) is certainly reminiscent of the Koopman law of random search,[2] but it differs conceptually in the sense that Koopman's Law gives an *unconditional probability of detection*, whereas Equation (2-11) is definitely a conditional one. It is important to realize that the optimal allocation which we derive now is predicated on the kind of detection law assumed. If Equation (2-11) turned out to be an inadequate description, one would have to substitute for it another detection law and consequently derive another allocation. Conceptually, the methodology which will be presented should not generally be affected; only the optimal effort distribution need be modified.

The quantity z of Equation (2-11) is to be an effort density, i.e., in our case, z is to be given by $z = f(\mathbf{x})$ such that

$$\int_X f(\mathbf{x})\, d\mathbf{x} = A_s, \tag{2-12}$$

where A_s is the total area to be searched, using a sensor of width W. Explicitly, $A_s = WL$, where L is the length of the search path. Thus z is dimensionless and is indeed an areal density of search in the truest sense of the term. Now, if one uses an effort density $f(\mathbf{x})$ and any given *conditional detection law* $b(\mathbf{x}, f(\mathbf{x}))$ for that effort density, one can express the unconditional chance of detection by

$$P(f) = \int_X p(\mathbf{x}) b(\mathbf{x}, f(\mathbf{x}))\, d\mathbf{x}, \tag{2-13}$$

so that $P(f)$ is obviously a functional (a mapping of functions to the real numbers). We want to obtain a function $f^*(\mathbf{x})$ which maximizes $P(f)$ under the constraint of Equation (2-12). That is, we want an allocation of resources which maximizes the acquisition probability. Note that a "cookie-cutter effect", in which $b(\mathbf{x}, z) \simeq 1$ for a negligible effort expenditure z, is achieved by choosing a to be a large positive number.

The rest of the argument for Issue (1) above is furnished, following Stone,[6] as follows: for a fixed $\lambda > 0$, one maximizes the pointwise Lagrangian with respect to z, given $\mathbf{x}$:

$$L(\mathbf{x}, z) = p(\mathbf{x})(1 - e^{-az}) - \lambda z.$$

One finds that the optimal solution $z^* = f_\lambda^*$ is

$$z^* = f_\lambda^*(\mathbf{x}) = \frac{1}{a}\{\ln[ap(\mathbf{x})/\lambda]\}^+,$$

where

$$\{s\}^+ \equiv \begin{cases} s, & s \geq 0 \\ 0, & s < 0. \end{cases}$$

Theory presented by Stone then indicates that f_λ^* is optimal over all functions f satisfying

$$\int_X f(\mathbf{x})\, d\mathbf{x} \leq \int_X f_\lambda^*(\mathbf{x})\, d\mathbf{x}. \tag{2-14}$$

Therefore, if one finds a $\lambda > 0$ such that Equation (2-12) holds, he will have the optimal allocation. The remainder of the argument follows that of Stone,[6] there worked out for the case $a = 1$. We have, first of all, the following optimal allocation, given in polar coordinates (r, θ) :

$$f_{\lambda_1}^*(r, \theta) = \begin{cases} \frac{1}{a}\left[-\ln(2\pi\sigma^2\lambda_1) - \frac{r^2}{2\sigma^2}\right] & , r^2 \leq -2\sigma^2 \ln(2\pi\sigma^2\lambda_1), \\ 0 & , r^2 > -2\sigma^2 \ln(2\pi\sigma^2\lambda_1), \end{cases} \tag{2-15}$$

where $\lambda_1 \equiv \lambda/a$. Integrating the cost function over the space X, as Stone does, we find that

$$C(f_{\lambda_1}^*) = \pi\sigma^2[\ln(2\pi\sigma^2\lambda_1)]^2/a. \tag{2-16}$$

Equating A_s to the right hand side of Equation (2-16) and solving for λ_1, we have

$$\lambda_1 = \frac{1}{2\pi\sigma^2} \exp\left[-\left(\frac{aA_s}{\pi\sigma^2}\right)^{\frac{1}{2}}\right]. \tag{2-17}$$

Substituting Equation (2-17) for λ_1 into Equation (2-15), we find that

$$f_\lambda^*(r, \theta) = \begin{cases} \frac{1}{a}\left[\left(\frac{aA_s}{\pi\sigma^2}\right)^{\frac{1}{2}} - \frac{r^2}{2\sigma^2}\right] & , r^2 \leq 2\sigma^2\left(\frac{aA_s}{\pi\sigma^2}\right)^{\frac{1}{2}} \\ 0 & , r^2 > 2\sigma^2\left(\frac{aA_s}{\pi\sigma^2}\right)^{\frac{1}{2}}. \end{cases} \tag{2-18}$$

Note that the argument presented so far does not require a constant speed search. If we do assume that the searcher travels at constant speed v, then $A_s = WvT$, where T is the total time devoted to the search. Observe that, at any point (r,θ), $f^*_\lambda(r,\theta) = O(1/a^{\frac{1}{2}})$ as a tends to $+\infty$, i.e., $f^*_\lambda(r,\theta)$ tends to zero as a tends to $+\infty$; it does so like $a^{-\frac{1}{2}}$. Thus, as the detection law approaches that of a cookie cutter (for any effort z, the detection probability is 1, provided the target is at $\mathbf{x}$), the effort needed at any point tends to zero, but, at the same time, the effort is supported over a disk whose radius tends to infinity like $a^{\frac{1}{2}}$. Clearly, the searcher, under these circumstances, should "move out from the origin more and more rapidly," i.e., he should spend very little time in any given area trying to locate a target. The mathematics seems to reflect this intuitive line of reasoning.

Suppose that, instead of assignation of effort over the plane, one decides a priori to restrict his attention to some disk D centered on the origin. Now $P(D)$, the chance that the target lies in D, is given by

$$\begin{aligned} P(D) &= \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_0^R \exp\left(\frac{-r^2}{2\sigma^2}\right) r\,dr\,d\theta \\ &= \frac{1}{\sigma^2}\int_0^R \exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr \\ &= 1 - \exp\left(-\frac{R^2}{2\sigma^2}\right). \end{aligned} \tag{2-19}$$

Thus, from Equation (2-19), if $P = P(D)$ is given in advance, we can easily find R. In fact, $R = \sigma[2\ln(1/(1-P))]^{\frac{1}{2}}$. Now, assuming that the target belongs to D, the target distribution becomes a conditional one, namely,

$$P_D(\mathbf{x}) = \begin{cases} \dfrac{1}{2\pi\sigma^2 P(D)}\exp\left(-\dfrac{x_1^2+x_2^2}{2\sigma^2}\right) & ,\mathbf{x}\in D \\ 0 & ,\mathbf{x}\notin D. \end{cases} \tag{2-20}$$

With this new target distribution, one now attempts to find the optimal allocation of effort. We have

$$z^*_D = f^*_{\lambda,D}(\mathbf{x}) = \frac{1}{a}\{\ln[ap_D(\mathbf{x})/\lambda]\}^+, \mathbf{x}\in D, \tag{2-21}$$

so that, if $R^2(P,\lambda_1,\sigma) = \min(R^2, -2\sigma^2\ln(2\pi\sigma^2\lambda_1 P))$,

$$f^*_{\lambda,D}(r,\theta) = \begin{cases} \dfrac{1}{a}\left[-\ln(2\pi\sigma^2\lambda_1 P(D) - \dfrac{r^2}{2\sigma^2}\right] & ,r^2 \le R^2(P,\lambda_1,\sigma) \\ 0 & ,r^2 > R^2(P,\lambda_1,\sigma). \end{cases} \tag{2-22}$$

Thus, if we define $\lambda_2 = \lambda_1 P(D)$, we find that the cost over the disk is

$$C(f^*_{\lambda,D}(r,\theta)) = \begin{cases} \dfrac{\pi R^2}{a}\left[-\ln(2\pi\sigma^2\lambda_2) - \dfrac{R^2}{4\sigma^2}\right] & ,0 < \lambda_2 \le \dfrac{1-P}{2\pi\sigma^2}, \\ \dfrac{\pi\sigma^2}{a}[\ln(2\pi\sigma^2\lambda_2)]^2 & ,\dfrac{1-P}{2\pi\sigma^2} \le \lambda_2 \le \dfrac{1}{2\pi\sigma^2}. \end{cases} \tag{2-23}$$

The first line of Equation (2-23) corresponds to the case where $R^2(P,\lambda_1,\sigma) = R^2$. In that event $f^*_{\lambda,D}$ is nonzero over the entire disk $\rho \le R$. In fact, $0 < \lambda_2 \le (1-P)/2\pi\sigma^2$ if and only if

$R^2 \leq -2\sigma^2 \ln(2\pi\sigma^2\lambda_2)$. Equating the right side of line 1 of Equation (2-23) to $A_s = WvT$, one finds that $\lambda_2 = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{aWvT}{\pi R^2} - \frac{R^2}{4\sigma^2}\right)$. Inserting this relation for λ_2 into the inequality $0 < \lambda_2 \leq (1-P)/2\pi\sigma^2$, one finds that $A_s \geq \pi\sigma^2 \ln^2(1-P)/a$. Also, the allocation density can be found in terms of WvT by using in line 1 of Equation (2-23) the same equality for λ_2. It follows that

$$f^*_{\lambda,D}(r,\theta) = \frac{1}{a}\left(\frac{aWvT}{\pi R^2} + \frac{R^2 - 2r^2}{4\sigma^2}\right), r \leq R, WvT \geq \pi\sigma^2(\ln^2(1-P))/a. \tag{2-24}$$

In addition, we would require that $R \leq vT$, so that a searcher could reach the circle $\rho = R$ at time T.

On the other hand, consider the second line of Equation (2-23). Then we find, of course, that $0 < A_s \leq \pi\sigma^2 \ln^2(1-P)/a$, corresponding to $\frac{1-P}{2\pi\sigma^2} \leq \lambda_2 \leq \frac{1}{2\pi\sigma^2}$. Now $\lambda_2 = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{\sigma}\left(\frac{WvTa}{\pi}\right)^{\frac{1}{2}}\right)$. Inserting this equality into the allocation density, we have, when $WvT \leq \pi\sigma^2(\ln^2(1-P))/a$ and $R_1^2 \equiv -2\sigma^2 \ln(2\pi\sigma^2\lambda_2) = 2\sigma\left(\frac{WvTa}{\pi}\right)^{\frac{1}{2}}$,

$$f^*_{\lambda,D}(r,\theta) = \begin{cases} \frac{1}{a}\left(\frac{1}{\sigma}\left(\frac{WvTa}{\pi}\right)^{\frac{1}{2}} - \frac{r^2}{2\sigma^2}\right) & , r^2 \leq R_1^2 \leq R^2 \\ 0 & , R_1^2 \leq r^2 \leq R^2. \end{cases} \tag{2-25}$$

Equation (2-25) has the same form as Equation (2-18), the only difference being that, instead of having a location distribution over the entire plane, we have one restricted to the disk $r \leq R$. When the effort to be expended does not exceed the threshold $\pi\sigma^2(\ln^2(1-P))/a$, we have a situation similar to that encountered when the size of the disk is unrestricted. Any theory developed for that case would then be applicable for Equation (2-25) also. The situation for Equation (2-24) is quite interesting; however, further details concerning it will be deferred for the moment.

Let us now return to Problem (A). The polar form presented in Equation (2-18) will be quite useful to us in the development of a cost effectiveness model, which is the goal of Step (2). Of course, the radially symmetric nature of Problem (A) introduces an element of simplicity upon which we can certainly capitalize. Note, for example, that Equation (2-18) includes no θ dependence. Even so, generation of a cost criterion which is ideally suited to the task of path effectiveness assessment is a complex issue. In some sense, one would like to match the allocation density prescibed by Equation (2-18) with the allocation density *which evolves from any particular choice of path*. However, as Figure 2-1 indicates, the latter density is a very complicated entity because of the complex fashion in which any particular spatial point is generally covered by search area, even for a path with increasing radius vector in time. Note, in Figure 2-1, that calculation of the actual search density would involve contributions from several strips of width W and that, the closer the spiral arms are to one another, the harder such a calculation would become.

Clearly we need a cost criterion based on a simpler, but still meaningful, concept. Therefore, instead of concerning ourselves with the allocation density, let us attempt to match a *measure of cumulative allocation*. Since Problem (A) exhibits radial symmetry, one is inclined to match cumulative allocations within disks centered on the origin. That is to say, suppose that a curve

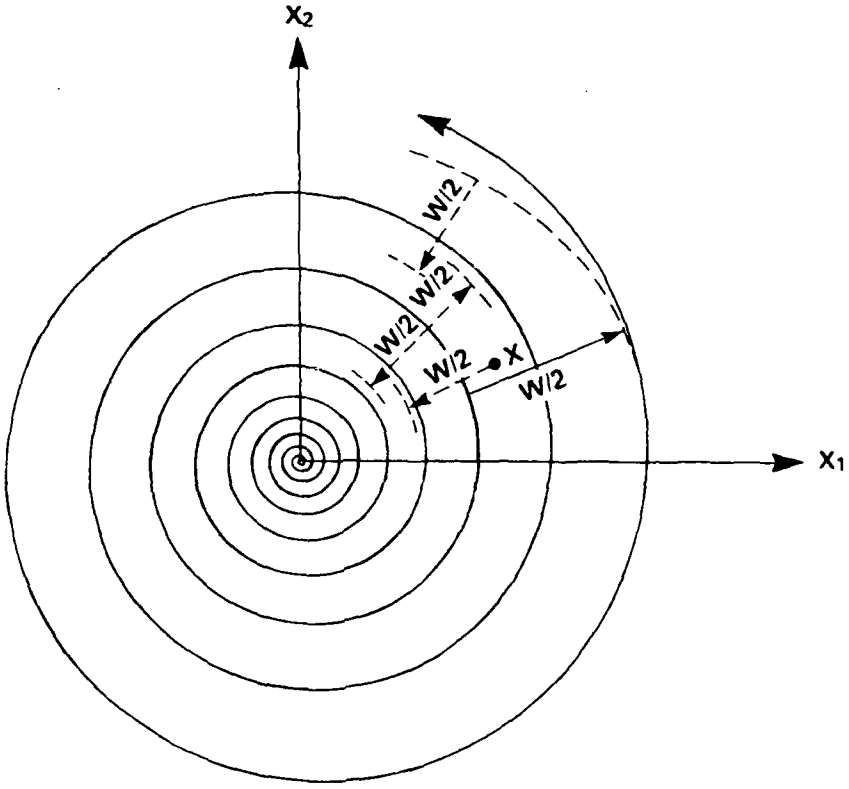

FIGURE 2-1. SEARCH PATH DENSITY ASSOCIATED WITH TYPICAL POINT x

such as that displayed in Figure 2-1 is expressed in polar form as a function of time, given by $(\rho(t), \theta(t))$, and that we agree to compute the total area searched using such a path up to time $t = t_0$. For a constant speed of search v, that area would just be Wvt_0. The cumulative effort function to be matched within the disk of radius $\rho(t_0)$ is obtained by integration of the right hand side of Equation (2-18) over that disk. We denote the latter quantity by $E(\rho(t_0))$, where $\rho(t_0) = \|r(t_0)\|$. To simplify matters, we agree to ignore search area which intersects the disk $\{\mathbf{r} : \|\mathbf{r}\| \leq \rho(t_0)\}$ for time $t > t_0$. Hopefully, that contribution is relatively small. Thus, we are led to the error (cost) criterion of mean-square type:

$$C(\mathbf{r}(t)) = \int_0^T (c(t, \rho(t)))^2 \, dt, \qquad c(t, \rho) \equiv Wvt - E(\rho), \tag{2-26}$$

which is at least a tractable quantity. Criterion (2-26) does suffer from one difficulty, viz., that peculiar oscillatory paths $\mathbf{r}(t)$ could arise which optimize it despite the fact that such tracks are unrealistic candidates. To alleviate that situation, one might require that both $\rho(t)$ and $\theta(t)$ be strictly increasing functions or that $\rho(t)$ be strictly increasing and $\theta(t)$ be strictly decreasing.

Integration of Equation (2-18) over a disk of radius $\rho(t)$ leads to the quantity

$$E(\rho(t)) = \frac{\pi \rho^2(t)}{2a\sigma^2} \left(R^2(T) - \frac{\rho^2(t)}{2} \right), \tag{2-27}$$

where

$$\frac{R^2(T)}{2\sigma^2} = \left(\frac{aWvT}{\pi\sigma^2} \right)^{\frac{1}{2}}.$$

We demand that our path start from the origin and intersect the circle $\|\mathbf{r}\| = R(T)$ at time $t = T$. That is, we have the boundary conditions $\mathbf{r}(0) = \mathbf{0}; \|\mathbf{r}(T)\| = R(T)$. The quantities W, v, T, σ, and R are, for the present purpose, to be treated as given positive numbers.

The circular symmetry of problem (A) allows us to reduce a 2-D problem to a 1-D one in terms of the radial variable $\rho = \|\mathbf{r}\|$. Anticipating future developments, after production of the radial solution part, the azimuthal motion can be determined by choosing the azimuthal speed such that the total speed is v. Note that $v^2 = \dot{x}^2(t) + \dot{y}^2(t) = \dot{\rho}^2(t) + \rho^2(t)\dot{\theta}^2(t)$ (where a dot denotes differentiation), so that $|\dot{\rho}(t)| \leq v$, a natural restriction on the radial rate. Therefore, the radial motion satisfies

$$\dot{\rho}(t) = u,$$

where $0 \leq u \leq v$. Also, $\rho(0) = 0$ and $\rho(T) = R(T)$. The cost functional to be considered is that of Equation (2-26). Let us set $c(t, \rho(t)) = 0$ for every t. Appealing to Equation (2-27), one then finds that

$$\overline{\rho}(t) = R(T)\left(1 - (1 - t/T)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{2-28}$$

Shown in Figure 2-2 is the curve $\rho = \overline{\rho}(t)$, together with those regions for which $c < 0$ and $c > 0$. Since $\dot{\overline{\rho}}(t) = +\infty$ at both $t = 0$ and $t = T$, it is clear that our speed constraint on the radial motion will not allow us to secure a path for which $c(t, \rho(t))$ is always zero, i.e., $\overline{\rho}(t)$ cannot be followed near the endpoints.

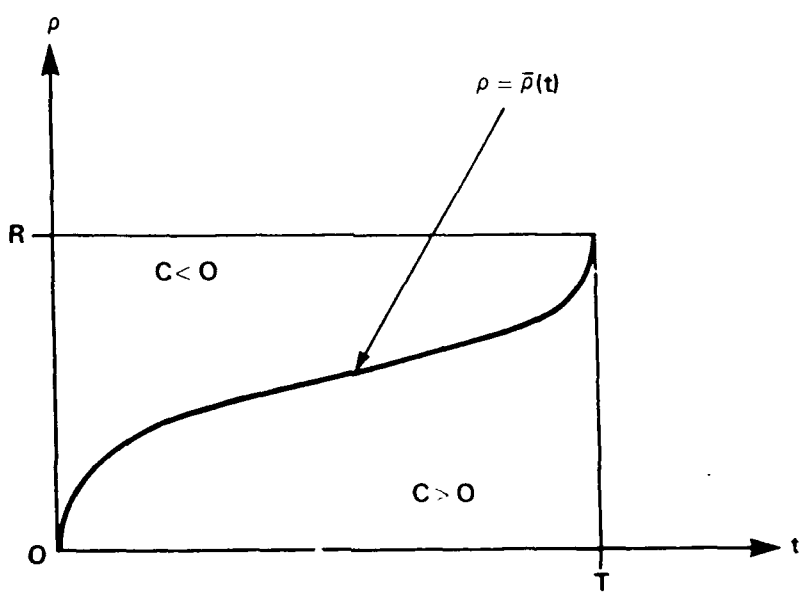

FIGURE 2-2. CURVE $\rho = \overline{\rho}(t)$ AND REGIONS $c < 0$ AND $c > 0$

We now develop the necessary conditions for optimality, using the optimal control theory results of Inequality (2-4) and Equations (2-5) to (2-8). The Hamiltonian in our case is

$$H = p(t)u - \lambda_0[c(t, \rho)]^2.$$

We are looking for an optimal pair $\rho_0(t), u_0(t)$. Hamiltonian theory shows that one can find an absolutely continous function $p(t)(0 \leq t \leq T)$ and a nonnegative multiplier λ_0 such that

$$(\lambda_0, p(t)) \neq (0,0) \qquad (0 \leq t \leq < T)$$

and such that the Euler-Lagrange equations hold:

$$\dot{\rho}_0(t) = u_0(t) \tag{2-29a}$$

$$\dot{p}(t) = -2\lambda_0 c(t, \rho_0(t)) E'(\rho_0(t)), \qquad 0 \leq t \leq T. \tag{2-29b}$$

(Both the dot and the prime denote differentiation operations. Throughout the text, we shall utilize both notations interchangeably.) In addition, we have the boundary conditions

$$\rho_0(0) = 0; \rho_0(T) = R \tag{2-30}$$

mentioned previously and the Weierstrass condition of Equation (2-6):

$$p(t)u_0(t) - \lambda_0[c(t, \rho_0(t))]^2 = \max_{0 \leq u \leq v} (p(t)u - \lambda_0[c(t, \rho_0(t))]^2).$$

Thus

$$u_0(t) = \begin{cases} v, & p(t) > 0 \\ 0, & p(t) < 0, 0 \leq t \leq T, \end{cases} \tag{2-31}$$

no information being provided about $u_0(t)$ when $p(t) = 0$. The condition where $p(t) = 0$ is known as a singular control situation.

The multiplier λ_0 must be nonnegative. Suppose that $\lambda_0 = 0$. Then, of course, $p(t)$ can never vanish, and its continuity implies that it has the same sign for all t in $[0, T]$. If $p(t)$ were negative for all t, Equation (2-31) would indicate that $u_0(t) = 0$ everywhere. Then Equation (2-29a) leads to the conclusion that $\rho_0(t)$ is constant for all t. Since $\rho_0(0) = 0$, indeed $\rho_0(t) = 0$ for all t. The trajectory would be reduced to a single point, and obviously a one-point trajectory is not a solution to our problem. Thus the sign of $p(t)$ is positive for all t. Now Equation (2-31) implies that $u_0(t) = v$ for all $t \in (0, T)$. Appealing again to Equation (2-29a) and Equation (2-29b), together with the fact that $\rho_0(0) = 0$, one has purely radial motion $\rho = vt, \theta =$constant. In order that we meet the boundary condition $\rho(T) = R(T)$, we must have $R(T) = vT$. Now this is a tenable motion (and indeed the only one possible) when, in fact, $R(T) = vT$. One must admit that this instance is trivial; so, without loss of generality, let us assume that $R < vT$.

Assume then that $\lambda_0 > 0$. As mentioned previously, we may normalize it to be unity. Let us suppose that $p(T) < 0$. Then, since $p(t)$ is continuous, $p(t) < 0$ within some ϵ neighborhood of $t = T$, say on $(T - \epsilon, T)$. Condition (2-31) shows that $u_0(t) = 0$ on the ϵ-interval, and Equation (2-29a) says that $\rho_0(t)$ is constant on the same interval. The boundary condition $\rho_0(T) = R$ implies that $\rho_0(t) = R, t \in (T - \epsilon, T)$. We assert that $\rho_0(t) = R, 0 \leq t \leq T$. Suppose, to the contrary, that there exists

a t, say $t = t_1$, where $t_1 < T - \epsilon$, and that $\rho_0(t_1) < R$. By continuity of $\rho_0(t)$, we may assume, without loss of generality, that $(t, \rho_0(t))$ lies in the region $c < 0$ of Figure 2-2 for $t \geq t_1$. Appealing to Equation (2-29b) and the fact that $E'(\rho_0(t)) \geq 0, t \geq t_1$, one finds that $\dot{p}(t) \geq 0, t \geq t_1$. Therefore, $p(t) < 0$ in $[t_1, T]$. Condition (2-31) shows that $u_0(t) = 0$ in $[t_1, T]$. Thus, $\rho_0(t)$ is constant in $[t_1, T]$. Since $\rho_0(t) = R$ when $t \in (T - \epsilon, T), \rho_0(t) = R$ in $[t_1, T]$. We have a contradiction. It follows that $\rho_0(t) = R, 0 \leq t \leq T$. From Equation (2-29a) and Equation (2-29b), since $E'(\rho_0(t)) = E'(R) = 0, \dot{p}(t) = 0$ for all t, and so $p(t) = p(T)$. We have $\rho_0(t) = R, u_0(t) = 0, p(t) = p(T)$ for $0 < t < T$. Such a state of affairs is in conflict with the fact that $\rho_0(0) = 0$. Suppose that $p(T) = 0$. We claim that again there exists an $\epsilon > 0$ such that $u_0(t) = 0$ when t belongs to $(T - \epsilon, T)$. Suppose, to the contrary, that, for every $\epsilon > 0$, there exists t_ϵ such that $u_0(t_\epsilon) > 0$. Now we know that there exists a $\delta > 0$ such that $c(t, \rho_0(t)) < 0$ in $(T - \delta, T)$. We see that this is true by checking Figure 2-2 and noting that, unless the curve $\rho = \rho_0(t)$ lies in the region $c < 0$ for t sufficiently close to T, the radial rate requirement would be violated. In fact, the only way $\rho = \rho_0(t)$ could lie completely in $c > 0$ for t close to T is that $\dot{\rho}_0(T) = +\infty$. Such a condition is a clear violation of physical requirements. Also, if $c = 0$ at a point t_0 sufficiently near $t = T$, then since $\dot{\overline{\rho}}(T) = +\infty$, the slope of the line joining $(t_0, \overline{\rho}(t_0))$ to $(T, \overline{\rho}(T))$ must exceed v. The law of the mean for derivatives then implies the existence of a point $t_1 > t_0$ such that $\dot{\rho}_0(t_1) > v$ (a clear violation). Suppose then that we have $t = t_\epsilon$ such that $T - \delta < t_\epsilon < T$ and $u_0(t_\epsilon) > 0$. Clearly, $\rho_0(t_\epsilon) < R$, for otherwise there would exist a $t > t_\epsilon$ for which $\rho_0(t) > R$ (by the definition of a derivative). Thus, $E'(\rho_0(t_\epsilon)) > 0, c(t_\epsilon, \rho_0(t_\epsilon)) < 0$, and $\lambda_0 > 0$, so that $\dot{p}(t_\epsilon) > 0$. Now, appealing to Equation (2-29a) and Equation (2-29b) again, $\dot{p}(t) \geq 0$ in $[t_\epsilon, T]$. Therefore, $p(t)$ is monotone increasing in that interval, and, since $p(T) = 0, p(t) \leq 0$ in $[t_\epsilon, T]$. Suppose that $p(t_\epsilon) = 0$. Since $\dot{p}(t_\epsilon) > 0$, there would exist a $t > t_\epsilon$ such that $p(t) > 0$, which is impossible. Hence, $p(t_\epsilon) < 0$. From Equation (2-31), $u_0(t_\epsilon) = 0$, and we have a contradiction to our original statement. Therefore, there exists some $\epsilon > 0$ such that, for $t \in [T - \epsilon, T], u_0(t) = 0$. The rest of the argument proceeds as for the case where $p(T) < 0$. We are again led to the fact that our initial condition $\rho(0) = 0$ cannot be satisfied. Finally, then, $p(T)$ must be positive, and the continuity of $p(t)$, together with Equation (2-31), means that $u_0(t) = v$ on the final part of the motion. As we have shown previously, $c < 0$ on the final part of the motion, so that Equation (2-29a) and Equation (2-29b) can be invoked once more to show that $\dot{p}(t) > 0$ there.

Remember that, as long as $p(t)$ remains positive, $u_0(t) = v$, so that motion is radial, and thus, in Figure 2-2, $\rho_0(t)$ is a straight line running backwards from the point $(T, R(T))$. Thus, we are within the region $c < 0$ until the line $\rho_0 = v(t - T) + R(T)$ intersects $\rho = \overline{\rho}(t)$. We said that, on this part of the motion, $\dot{p}(t) > 0$. It follows that $p(t)$ decreases as we work backward along the line $\rho_0 = vt + k(k = R(T) - vT)$ from the point $(T, R(T))$. Radial motion will cease only if $p(\bar{t}) = 0$ for some $\bar{t}$. Let us examine the first $\bar{t}$ for which this phenomenon occurs. (If $p(t) > 0$ for all t, radial motion $\rho_0(t) = vt, 0 \leq t \leq T$, is the only motion, and then our situation is again the trivial scenario in which $R = vT$.) We have three possibilities: (a) $p(\bar{t}) = 0$ and $c(t, \rho_0(t)) < 0 (\bar{t} \leq t < T)$; (b) $p(t) > 0$ when

the path backwards from time T crosses $c = 0$; (c) $p(t)$ vanishes just when the backwards trajectory (following a straight line with slope v) crosses the curve $c = 0$.

First we check Case (a). In this instance, from the Euler-Lagrange equations, $\dot{p}(\bar{t}) > 0$, so that p changes sign at $\bar{t}$. Therefore, $p(t) < 0$ in some interval $(\bar{t} - \delta, \bar{t})$. From the Weierstrass condition, $u_0(t) = 0$ in that same interval, and it follows that $\rho_0(t) = \rho_0(\bar{t})$ in $(\bar{t} - \delta, \bar{t})$. It is now possible to argue, in a similar vein as before, that $u_0(t) = 0$ on, $[0, \bar{t}]$, so that $\rho_0(t) = \rho_0(\bar{t})$ for $t < \bar{t}$. The initial condition $\rho_0(0) = 0$ cannot be satisfied, so that (a) is untenable. Figure 2-3 is an illustration of Case (a).

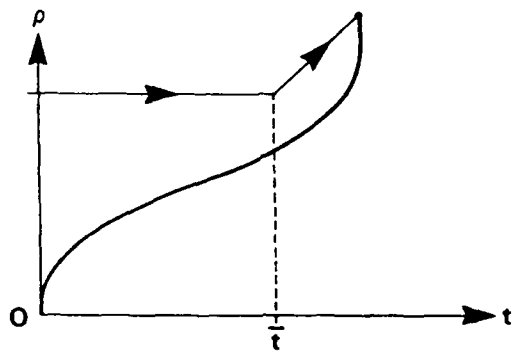

FIGURE 2-3. $p(\bar{t}) = 0$ AND $c(t, \rho_0(t)) < 0, \bar{t} \leq t < T$

For Case (b), we are really in an adverse situation, for, from the Euler-Lagrange equations, $\dot{p} < 0$ when $c > 0$. Thus $p(t)$ increases as we work backward toward $t = 0$ from the crossing point and hence is always positive. The Weierstrass condition forces us to continue on a radial course; and, when $R < vT$, we cannot satisfy our initial condition $\rho_0(0) = 0$. This situation is illustrated in Figure 2-4 and Figure 2-5. In Case (b), $p(t)$ cannot vanish.

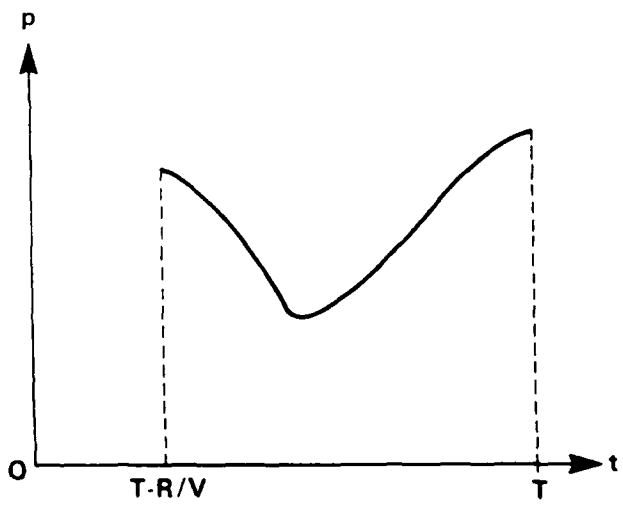

FIGURE 2-4. BEHAVIOR OF $p(t)$ IN CASE (b)

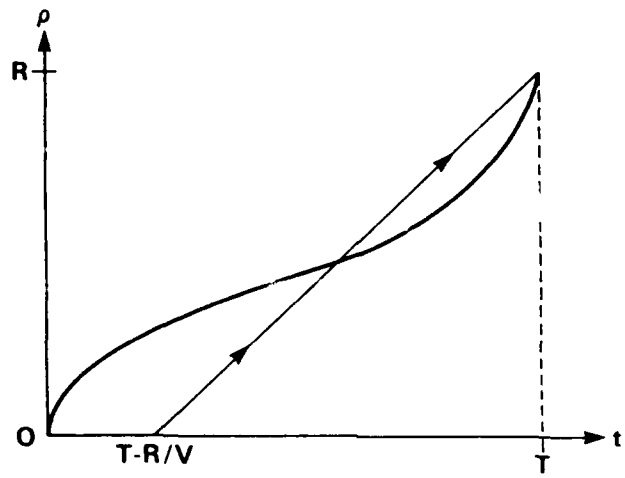

FIGURE 2-5. BEHAVIOR OF $\rho(t)$ IN CASE (b)

The remaining possibility is that p vanishes just when the backwards trajectory crosses the curve $c = 0$. The trajectory cannot cross $c = 0$ at that point, since we would be forced back into a situation like that which occurs in Case (b), namely, that one where $\dot{p} < 0$ for $c > 0$. A radial solution would be the result, and that possibility cannot occur, since again ρ would vanish at $t = T - R/v$. We must therefore follow the curve $c = 0$ (so that $p(t) = 0$ along it) until we get to the initial part of the trajectory, which has the same properties as that for the backward portion from $(T, R(T))$. The situation is illustrated in Figure 2-6 and Figure 2-7. The solution has three component parts. From 0 up to some time $t_1, u_0(t) = v$ and $c > 0$; from t_1 to t_2, $u_0(t) = \dot{\overline{\rho}}(t), c = 0$, and $\rho_0(t) = \overline{\rho}(t)$; from t_2 to $T, u_0(t) = v$ and $c < 0$. In $[0, t_1]$ p is positive; on $[t_1, t_2]$ it vanishes (singular control problem); on $[t_2, T]$ it is again positive. On the singular portion, $u_0(t) < v$, so that azimuthal motion is required to bring the speed up to v. On the nonsingular portions, motion is radial. The switching times t_1 and t_2 are easily found, since we know the equations for $\overline{\rho}$ and for the straight-line portions of ρ_0. We have

$$vt_1 = R\left(1 - (1 - t_1/T)^{\frac{1}{2}}\right)^{\frac{1}{2}} \tag{2-32}$$

$$R - v(T - t_2) = R\left(1 - (1 - t_2/T)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{2-33}$$

To solve Equation (2-32), we first define $K = R/vT, 0 < K < 1$. Since $vKT = R, t_1 \leq KT$. In fact, from Figure 2-8, one sees that $t_1 < KT$. Let $s_1 = t_1/KT = vt_1/R$. Then, from Equation (2-32),

$$s_1 = \left(1 - (1 - Ks_1)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \qquad 0 < s_1 < 1,$$

so that s_1 satisfies

$$1 - s_1^2 = (1 - Ks_1)^{\frac{1}{2}}, \qquad 0 < s_1 < 1. \tag{2-34}$$

From Equation (2-34), upon squaring both sides, we find that s_1 must be a solution of the following equation:

$$s(s^3 - 2s + K) = 0, 0 < s < 1.$$

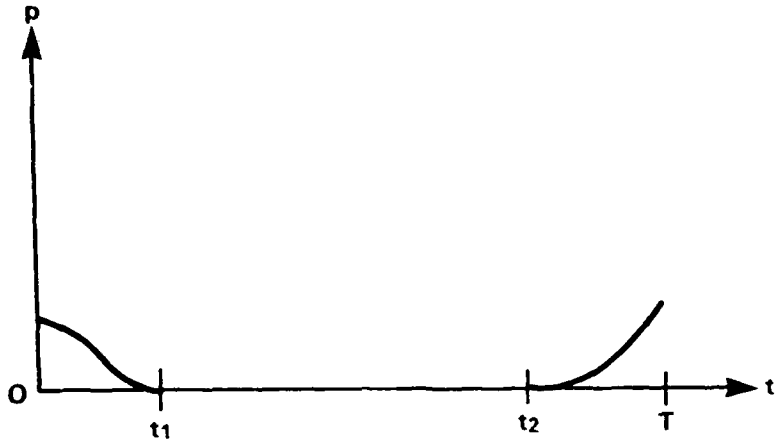

FIGURE 2-6. BEHAVIOR OF $p(t)$ IN CASE (c)

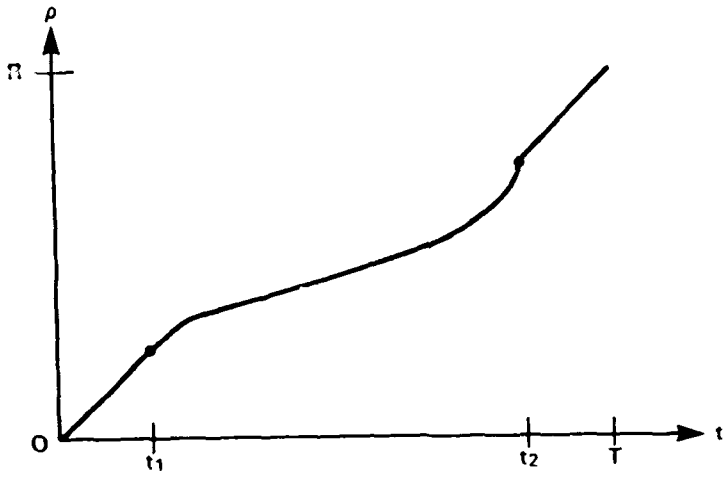

FIGURE 2-7. BEHAVIOR OF $\rho(t)$ IN CASE (c)

Clearly $s_1 = 0$ is not the solution desired, for we demand that $0 < s_1 < 1$. Evaluating the cubic factor of the left member of Equation (2-35) at $s = -\infty$, we obtain $-\infty$, at $s = 0$ we obtain $K > 0$, at $s = 1$ we have $K - 1 < 0$, and, finally, at $s = +\infty$, we get $+\infty$. Therefore, the cubic changes sign in $(-\infty, 0), (0, 1)$, and $(1, \infty)$. It follows that all roots of the cubic are real and simple, one zero occurring in each of the three aforenamed intervals. We desire the root in $(0, 1)$. Applying standard formulas for solving cubics, [7] one has

$$s_1 = (2\sqrt{6}/3)\cos\left(\frac{1}{3}(\mathrm{Cos}^{-1}(-3K\sqrt{6}/8) + 4\pi)\right), \tag{2-36}$$

which one can easily show (appealing to simple inequalities) is the smaller positive root. As is customary, Cos^{-1} represents the principal arccos function, whose range is $[0, \pi]$. For the solution to Equation (2-33), refer to Figure 2-9. It is clear from the definition of K that a line with slope

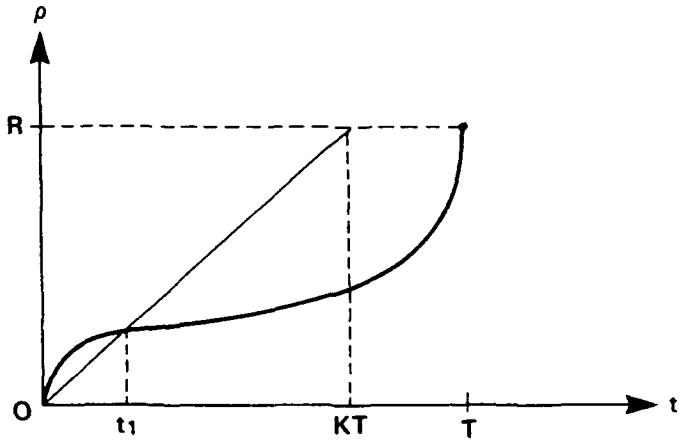

FIGURE 2-8. FIRST SWITCHING TIME FOR OPTIMAL SEARCHER MOTION

v emanating from (T, R) must have intersected the t axis at time $(1-K)T$. We have previously defined the time t_2 to be that time for which the same line intersects the curve $\rho = \overline{\rho}(t)$. Let $s_2 = (1 - t_2/T)/K = v(T - t_2)/R$. Then, dividing Equation (2-33) by R, one finds that

$$1 - s_2 = \left(1 - (Ks_2)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{2-37}$$

whence

$$(Ks_2)^{\frac{1}{2}} = 2s_2 - s_2^2 \tag{2-38}$$

Squaring both sides of Equation (2-38) and noting that $s_2 \neq 0$ (equivalently, $t_2 \neq T$), we see that s_2 satisfies the equation

$$s^3 - 4s^2 + 4s - K = 0 \tag{2-39}$$

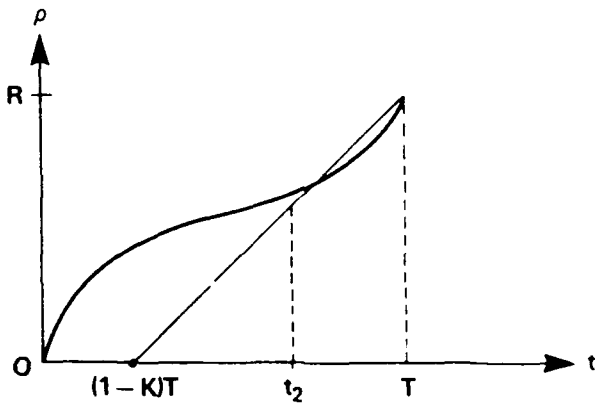

FIGURE 2-9. SECOND SWITCHING TIME FOR OPTIMAL SEARCHER MOTION

The cubic polynomial in Equation (2-39) is negative when $s = 0$, positive when $s = 1$, negative when $s = 2$, and certainly positive in a neighborhood of $s = +\infty$. It follows that all its roots are real and positive and that there is precisely one root in $(0, 1)$. Using a standard procedure for converting a cubic to another cubic with a missing quadratic term,[7] let $s = u + 4/3$ in Equation (2-39). Then one has the polynomial

$$p(u) = \left(u + \frac{4}{3}\right)^3 - 4\left(u + \frac{4}{3}\right)^2 + 4\left(u + \frac{4}{3}\right) - K, \tag{2-40}$$

to be re-expressed in standard form as a polynomial in u. To do this, think of the right hand side of Equation (2-40) as a Taylor expansion of the polynomial about $u = -\frac{4}{3}$. Then express $p(u)$ as $p(u) = p(0) + p'(0)u + \frac{p''(0)}{2!}u^2 + \frac{p'''(0)}{3!}u^3$, and one has the canonical polynomial form. One finds that

$$u^3 - 4u/3 + (16 - 27K)/27 = 0. \tag{2-41}$$

We want the *most negative root* of the left member of Equation (2-41), since that root will yield the *least positive root* of the left side of Equation (2-39). Using simple inequality relationships, together with standard formulas,[7] one finds that

$$s_2 = \frac{4}{3}\left(1 + \cos\left(\frac{1}{3}\left(\mathrm{Cos}^{-1}\left(\frac{27}{16}K - 1\right) + 2\pi\right)\right)\right), \tag{2-42}$$

and we know that $t_2 = T - Rs_2/v$, with s_2 given by Equation (2-42). In terms of K, one has then

$$t_1 = T\left(\frac{2}{3}K\sqrt{6}\cos\left(\frac{1}{3}\mathrm{Cos}^{-1}\left(-\frac{3K\sqrt{6}}{8}\right) + \frac{4\pi}{3}\right)\right) \tag{2-43}$$

$$t_2 = T\left(1 - \frac{4K}{3}\left(1 + \cos\left(\frac{1}{3}\left(\mathrm{Cos}^{-1}\left(\frac{27K}{16} - 1\right)\right) + \frac{2\pi}{3}\right)\right)\right). \tag{2-44}$$

We give now a procedure for obtaining an optimal spiral motion, provided that $v > R(T)/T$. Let $(\rho_0(t), \theta_0(t))$ represent the optimal spiral motion. Then we want $\rho_0(t) = vt, \theta_0(t) = c_1, 0 \le t \le t_1$, and $\rho_0(t) = v(t - T) + R(T), \theta_0(t) = c_2, t_2 \le t \le T$, where c_1, c_2 are constants. When $t \in [t_1, t_2]$, we want $\rho_0(t) = \overline{\rho}(t)$. From the relation $\dot{\rho}^2(t) + \rho^2(t)\dot{\theta}^2(t) = v^2$, we can secure $\dot{\theta}(t)$. In fact, let $f(t) = d\overline{\rho}^2(t)/dt$, and observe that $\dot{\rho}_0(t) = f(t)/2\rho_0(t)$. Then we have $\overline{\rho}^2(t)\dot{\theta}_0^2(t) = (v^2 - f^2(t)/4\overline{\rho}^2(t))$, so that $\dot{\theta}_0(t) = \pm(4v^2\overline{\rho}^2(t) - f^2(t))^{\frac{1}{2}}/\overline{\rho}^2(t)$. If we choose the positive sign for $\dot{\theta}_0(t)$, we are assured of obtaining a spiral motion. One would, of course, need to integrate $\dot{\theta}_0(t)$ numerically, using the initial condition $\theta_0(t_1) = c_1$, in order to secure $\theta_0(t)$ on $[t_1, t_2]$. This type of procedure may obviously be used for any functional form $\overline{\rho}(t)$ whose derivative does not exceed v in the required interval of t values. In particular, when we develop subsequently the optimal allocation for preassigned circular disks, one could straightforwardly obtain the optimal path in the same way. Therefore, we shall not need to develop these paths again for such situations.

The cost functional has a nonzero integrand on $(0, t_1)$ and on (t_2, T). The integral over $(0, t_1)$, using Equation (2-26) and Equation (2-27), is

$$\begin{aligned} J_1 &= \int_0^{t_1} \left(Wvt - \frac{\pi v^2 t^2}{2a\sigma^2} \left(R^2 - \frac{v^2 t^2}{2} \right) \right)^2 dt \\ &= \int_0^{t_1} \left(\frac{\pi^2 v^8 t^8}{16a^2\sigma^4} - \frac{\pi^2 R^2 v^6 t^6}{4a^2\sigma^4} + \frac{\pi W v^5 t^5}{2a\sigma^2} + \frac{\pi^2 v^4 R^4 t^4}{4a^2\sigma^4} - \frac{\pi W v^3 R^2 t^3}{a\sigma^2} + W^2 v^2 t^2 \right) dt \\ &= \frac{\pi^2 v^8 t_1^9}{144a^2\sigma^4} - \frac{\pi^2 R^2 v^6 t_1^7}{28a^2\sigma^4} + \frac{\pi W v^5 t_1^6}{12a\sigma^2} + \frac{\pi^2 v^4 R^4 t_1^5}{20a^2\sigma^4} - \frac{\pi W v^3 R^2 t_1^4}{4a\sigma^2} + \frac{W^2 v^2 t_1^3}{3}, \end{aligned} \tag{2-45}$$

and the integral over (t_2, T) is

$$J_2 = \int_{t_2}^{T} \left(Wvt - \frac{\pi[v(t-T)+R]^2}{2a\sigma^2} \left(R^2 - \frac{(v(t-T)+R)^2}{2} \right) \right)^2 dt. \tag{2-46}$$

It is convenient to make the change of variable $u = T - t$ in Equation (2-46). We have

$$J_2 = \int_0^{T-t_2} \left(Wv(T-u) - \frac{\pi(R-vu)^2}{2a\sigma^2} \left(R^2 - \frac{(R-vu)^2}{2} \right) \right)^2 du. \tag{2-47}$$

The integrand of the right hand side of Equation (2-47) is

$$\begin{aligned} j_2(u) &= W^2 v^2 (T-u)^2 - \frac{\pi W v (T-u)(R-vu)^2}{a\sigma^2} \left(R^2 - \frac{(R-vu)^2}{2} \right) \\ &\quad + \frac{\pi^2 (R-vu)^4}{4a^2\sigma^4} \left(R^2 - \frac{(R-vu)^2}{2} \right)^2 \\ &= W^2 v^2 (T-u)^2 - \pi R^2 W v (T-u)(R^2 - 2Rvu + v^2u^2)/a\sigma^2 \\ &\quad + \pi W v (T-u)(R^4 - 4R^3vu + 6R^2v^2u^2 - 4Rv^3u^3 + v^4u^4)/2a\sigma^2 \\ &\quad + \pi^2 R^4 (R^4 - 4R^3vu + 6R^2v^2u^2 - 4Rv^3u^3 + v^4u^4)/4a^2\sigma^4 \\ &\quad - \pi^2 R^2 (R^6 - 6R^5vu + 15R^4v^2u^2 - 20R^3v^3u^3 + 15R^2v^4u^4 - 6Rv^5u^5 \\ &\qquad + v^6u^6)/4a^2\sigma^4 \\ &\quad + \pi^2 (R^8 - 8R^7vu + 28R^6v^2u^2 - 56R^5v^3u^3 + 70R^4v^4u^4 - 56R^3v^5u^5 \\ &\qquad + 28R^2v^6u^6 - 8Rv^7u^7 + v^8u^8)/16a^2\sigma^4. \end{aligned} \tag{2-48}$$

Expanding the right hand side of Equation (2-48), one finds that

$$\begin{aligned} j_2(u) =& W^2 v^2 (T-u)^2 + \frac{\pi R^4}{16a^2\sigma^4}(\pi R^4 - 8aWvT\sigma^2) + \frac{\pi W R^4 vu}{2a\sigma^2} \\ &+ \frac{\pi R^2 v^2 u^2}{2a^2\sigma^4}(4aWvT\sigma^2 - \pi R^4) + \frac{\pi R v^3 u^3}{2a^2\sigma^4}(\pi R^4 - 4aW\sigma^2(R + vT)) \\ &+ \frac{\pi v^4 u^4}{8a^2\sigma^4}(4\sigma^2 aW(vT + 4R) + 7\pi R^4) - \frac{\pi v^5 u^5}{2a^2\sigma^4}(W\sigma^3 a + 4\pi R^3) \\ &+ \frac{3\pi^2 R^2 v^6 u^6}{2a^2\sigma^4} - \frac{\pi^2 R v^7 u^7}{2a^2\sigma^4} + \frac{\pi^2 v^8 u^8}{16a^2\sigma^4}. \end{aligned} \tag{2-49}$$

Integration of the right side of Equation (2-49) leads to

$$\begin{aligned} J_2 = & \frac{W^2v^2(T^3-t_2^3)}{3} + \frac{\pi R^4}{16a^2\sigma^4}(\pi R^4 - 8aWvT\sigma^2)(T-t_2) + \frac{\pi W R^4 v(T-t_2)^2}{4a\sigma^2} \\ & - \frac{\pi R^2 W v^3 (T-t_2)^4}{2a\sigma^2} + \frac{\pi v^4 (T-t_2)^5}{40a^2\sigma^4}(4\sigma^2 aW(vT+4R) + 7\pi R^4) \\ & - \frac{\pi v^5 (T-t_2)^6}{12a^2\sigma^4}(Wa\sigma^2 + 4\pi R^3) + \frac{3\pi^2 R^2 v^6 (T-t_2)^7}{14a^2\sigma^4} - \frac{\pi^2 R v^7 (T-t_2)^8}{16a^2\sigma^4} \\ & + \frac{\pi^2 v^8 (T-t_2)^9}{144a^2\sigma^4}. \end{aligned} \tag{2-50}$$

We shall investigate the nature of both J_1 and J_2 in Chapter 3 when we compare the efficiency of the optimal design with that of a logarithmic spiral search.

Let us consider again the situation where effort is to be assigned to some preassigned disk D, whose probability $P(D)$ is given a priori. Remember that Equation (2-24) and Equation (2-25) give the optimal effort density assigned to the disk, depending upon whether or not the search effort WvT exceeds or does not exceed the threshold $\pi\sigma^2(\ln^2(1-P))/a$. We would like to know what the optimal search path is in these two situations. The scenario for Equation (2-25) obviously parallels that for the unconstrained case, so that the character of the solution just obtained is preserved. Figure 2-7 then applies, with $R_1(T)$ replacing $R(T)$. For Equation (2-24), the cumulative allocation function over a disk with center at the origin and with radius $r \le R$ is

$$\frac{\pi r^2}{a}\left(\frac{aWvT}{\pi R^2} + \frac{R^2 - r^2}{4\sigma^2}\right). \tag{2-51}$$

Referring to our cost criterion (2-26) and setting $c(t,\rho(t)) = 0$, we find, using Expression (2-51), that

$$\overline{\rho}^2(t) = \frac{4WvT\sigma^2 a + \pi R^4 - [(4WvT\sigma^2 a + \pi R^4)^2 - 16\pi R^4\sigma^2 aWvt]^{\frac{1}{2}}}{2\pi R^2}. \tag{2-52}$$

From Equation (2-52), it follows that

$$\overline{\rho}(t) = \frac{1}{R}\left(\frac{4WvT\sigma^2 a + \pi R^4}{2\pi}\right)^{\frac{1}{2}}\left[1 - (1 - 16\pi R^4\sigma^2 aWvt/(4WvT\sigma^2 a + \pi R^4)^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{2-53}$$

Let $b(T) = 16\pi R^4\sigma^2 aWv/(4WvT\sigma^2 a + \pi R^4)^2$ and $L(t) = b(T)t$. One sees that, when $WvT > \pi\sigma^2(\ln^2(1-P))/a$ (the condition under which Equation (2-24) applies), $4WvT\sigma^2 a > \pi R^4$, so that $L(T) < 1$. Obviously, $L(0) = 0$. Set

$$r_1(t) = [1 - (1 - L(t))^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{2-54}$$

so that $\overline{\rho}(t) = \frac{1}{R}\left(\frac{4WvT\sigma^2 a + \pi R^4}{2\pi}\right)^{\frac{1}{2}} r_1(t)$. We want to know the behavior of $r_1(t)$. One finds that

$$r_1'(t) = \frac{b(T)}{4}[1 - (1 - L(t))^{\frac{1}{2}}]^{-\frac{1}{2}}(1 - L(t))^{-\frac{1}{2}}. \tag{2-55}$$

From Equation (2-55) $r_1'(0) = +\infty$, and $0 < r_1'(T) < \infty$ when $4WvT\sigma^2 a > \pi R^4$. Of course, when $4WvT\sigma^2 a = \pi R^4, r_1'(T) = +\infty$, and $\overline{\rho}(t)$ looks like the curve displayed in Figure 2-2, where the slope at

both $t = 0$ and $t = T$ is $+\infty$. As T increases such that $4WvT\sigma^2 a > \pi R^4$, the derivative of $\overline{\rho}(t)$ at $t = T$ begins to decrease and, for a while, should be larger than v. Thus, the character of our optimal solution should be precisely the same until such time T_0 when $\overline{\rho}'(T_0) = v$. As T continues to increase, we shall show that $\overline{\rho}'(T)$ decreases, so that, for $T \geq T_0$, we can assert that the radial part of the solution contains only two parts instead of three, namely, a part $\rho = vt$ from the origin to a point $(t_1, \overline{\rho}(t_1))$, followed by $\overline{\rho}(t)$ itself all the way from $t = t_1$ to $t = T$. Another fact will be of interest: As T increases, the point of inflection present in Figure 2-2 moves progressively to the right and finally disappears leaving $\overline{\rho}(t)$ as a concave, strictly increasing function for T sufficiently large. We now establish these points mathematically.

To establish the inflectional behavior of $\overline{\rho}(t)$, let us determine $r_1''(t)$. Using Equation (2-55), we have

$$4r_1''(t) = b^2(T)\{-\frac{1}{4}[1-(1-L(t))^{\frac{1}{2}}]^{-\frac{3}{2}}(1-L(t))^{-1} + \frac{1}{2}[1-(1-L(t))^{\frac{1}{2}}]^{-\frac{1}{2}}(1-L(t))^{-\frac{3}{2}}\}, \tag{2-56}$$

so that

$$16r_1''(t) = b^2(T)[1-(1-L(t))^{\frac{1}{2}}]^{-\frac{3}{2}}(1-L(t))^{-\frac{3}{2}}[2-3(1-L(t))^{\frac{1}{2}}]. \tag{2-57}$$

From Equation (2-57), $r_1''(t) = 0$ (and hence $\overline{\rho}''(t) = 0$) if and only if $2-3(1-L(t))^{\frac{1}{2}} = 0$. It follows that $L(t_{\inf}) = b(T)t_{\inf} = \frac{5}{9}$, where $t_{\inf}$ means that time at which the curve $\overline{\rho}(t)$ has an inflection point. Note now that $b(T)$ is a strictly decreasing function of T and tends to zero like $\frac{1}{T^2}$ as T tends to infinity. Thus indeed $t_{\inf} = 5/(9b(T))$ increases as T increases and eventually must exceed T, so that the inflectional character of $\overline{\rho}(t)$ within $[0, T]$ is no longer present. One of our points is thus established. We want next to show that $\overline{\rho}'(T)$ is a decreasing function of T, and we also want to find that value of T, say T_0, such that $\overline{\rho}'(T_0) = v$. The point $T = T_0$ then will represent that value of T beyond which $\overline{\rho}(t)$ has only two parts, one a nonsingular part and the other a singular part (corresponding to $\lambda_0 = 0$). Differentiating Equation (2-52) with respect to t, we have

$$2\overline{\rho}(t)\overline{\rho}'(t) = \frac{4Wv\sigma^2 aR^2}{[(4WvT\sigma^2 a + \pi R^4)^2 - 16\pi WvT\sigma^2 aR^4]^{\frac{1}{2}}}. \tag{2-58}$$

Putting $t = T$ into Equation (2-58), we have

$$\overline{\rho}(T)\overline{\rho}'(T) = \frac{2Wv\sigma^2 aR^2}{4WvT\sigma^2 a - \pi R^4}. \tag{2-59}$$

Since $\overline{\rho}(T) = R$, we have

$$\overline{\rho}'(T) = \frac{2Wv\sigma^2 aR}{4WvT\sigma^2 a - \pi R^4}. \tag{2-60}$$

From Equation (2-60) $\overline{\rho}'(T)$ is a strictly decreasing function of T and indeed tends to zero like $1/T$ as T tends to infinity. Setting $\overline{\rho}'(T) = v$ in Equation (2-60), we find that

$$T_0 = R(\pi R^3 + 2W\sigma^2 a)/4Wv\sigma^2 a. \tag{2-61}$$

One last point to be made is that, when $T = T_0, t_{inf} < T_0$. Otherwise, since $R < vT$, we would have a logical contradiction. To see this, refer to Figure 2-10. Since $\overline{p}(t)$ in this figure is concave and increasing, $\overline{p}'(t) > v$ for all t in $(0,T)$. Now, since $R < vT_0$, the slope of the line shown joining the origin to (T_0, R) is indeed smaller than v. By the law of the mean for derivatives, there exists some t, say $t = t_1$, such that $\overline{p}'(t_1) = \frac{R}{T_0} = kv$, where $k < 1$. This possibility is, however, precluded by the fact that $\overline{p}'(t) > v$ *for all* t *in* $(0,T_0)$. So Figure 2-10 presents an impossible situation. Thus, the inflectional behavior of the optimal solution fails to exist for some $T_{inf} > T_0$, where T_0, as before, is that value of t for which $\overline{p}'(T) = v$.

We next discuss Problem (B): The target is stationary in two-space, and its position follows the elliptic Gaussian Law given by Equation (2-10). First we present an argument which will, simultaneously with the evolution of the optimal path, establish the existence of optimal solutions. We note that Hamiltonian theory only affords a necessity argument: "If an optimal solution to the problem exists, it is necessary that certain conditions obtain." It is nice to know that there indeed exists at least one solution to the optimal control problem.

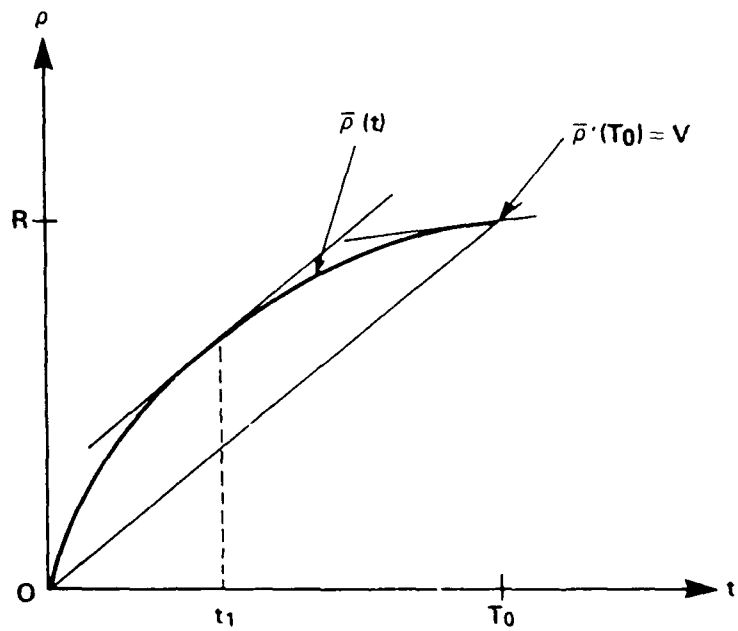

FIGURE 2-10. ASSUMPTION THAT OPTIMAL SOLUTION IS CONCAVE WHEN $\overline{p}'(T) = v$

For Problem (B), instead of attempting to match cumulative effort in circular disks, we now attempt to match it within "elliptic disks." The bounding ellipses become the level surfaces of the density given by Equation (2-10). Our (local) Lagrangian is

$$L(\mathbf{x}, \lambda, z) = p(\mathbf{x})(1 - e^{-az}) - \lambda z, \tag{2-62}$$

so that $L_z = ap(\mathbf{x})e^{-az} - \lambda$. L_z equals zero when $z^* = \frac{1}{a}\ln\left(\frac{ap(\mathbf{x})}{\lambda}\right) = \frac{1}{a}\ln\left(\frac{p(\mathbf{x})}{\lambda_1}\right)$, where $\lambda_1 = \lambda/a$, as before. The optimal solution, analogous to that for Case (A), is just $z^* = \frac{1}{a}\left[\ln\left(\frac{p(\mathbf{x})}{\lambda_1}\right)\right]^+$. Now $p(\mathbf{x}) \geq \lambda_1$ if

and only if $\|\mathbf{r}\|_Q^2 \equiv \left(\frac{x_1}{\sigma_1}\right)^2 + \left(\frac{x_2}{\sigma_2}\right)^2 \leq -2\ln(2\pi\sigma_1\sigma_2\lambda_1)$. Thus,

$$f^*_{\lambda_1}(\mathbf{x}) = \begin{cases} \frac{1}{a}\left[-\ln(2\pi\sigma_1\sigma_2\lambda_1) - \frac{\|\mathbf{r}\|_Q^2}{2}\right] & , \|\mathbf{r}\|_Q^2 \leq -2\ln(2\pi\sigma_1\sigma_2\lambda_1) \\ 0 & , \text{ otherwise.} \end{cases} \tag{2-63}$$

Here $\|\mathbf{r}\|_Q^2 = \mathbf{r}^T Q\mathbf{r}$, where $\mathbf{r}^T = (x_1, x_2)$ and Q is the diagonal matrix $Q = \text{diag}(1/\sigma_1^2, 1/\sigma_2^2)$. The total search cost is again WvT, where v is the speed in the (x_1, x_2)-plane. Integrating $f^*_{\lambda_1}(\mathbf{x})$ over $\mathbf{x}$ space, one finds that

$$WvT = \frac{\pi\sigma_1\sigma_2}{a}\ln^2(2\pi\sigma_1\sigma_2\lambda_1), \tag{2-64}$$

so that Equation (2-63) becomes

$$f^*_{\lambda_1}(\mathbf{x}) = \begin{cases} \frac{1}{a}\left[\left(\frac{aWvT}{\pi\sigma_1\sigma_2}\right)^{\frac{1}{2}} - \frac{\|\mathbf{r}\|_Q^2}{2}\right] & , \|\mathbf{r}\|_Q^2 \leq 2\left(\frac{aWvT}{\pi\sigma_1\sigma_2}\right)^{\frac{1}{2}} \\ 0 & , \text{otherwise.} \end{cases} \tag{2-65}$$

Integrating the right hand side of Equation (2-65) over the elliptic disk D: $\{\mathbf{x} : \left(\frac{x_1}{\sigma_1}\right)^2 + \left(\frac{x_2}{\sigma_2}\right)^2 \leq r^2 \leq R^2(T)\}$, where $R^2(T) \equiv 2\left(\frac{aWvT}{\pi\sigma_1\sigma_2}\right)^{\frac{1}{2}}$, one finds that the cumulative effort $E(\|\mathbf{r}\|_Q)$ is

$$E(\|\mathbf{r}\|_Q) = \frac{\pi\sigma_1\sigma_2\|\mathbf{r}\|_Q^2}{2a}\left[R^2(T) - \frac{\|\mathbf{r}\|_Q^2}{2}\right]. \tag{2-66}$$

Our cost functional is now

$$\begin{aligned} J = C(\mathbf{r}(t)) &= \int_0^T (c(t, \mathbf{r}(t)))^2\, dt \\ &= \int_0^T (Wvt - E(\|\mathbf{r}(t)\|_Q))^2\, dt, \end{aligned} \tag{2-67}$$

with $E(\|\mathbf{r}(T)\|_Q)$ afforded by Equation (2-66). The reader may check the fact that $E(\|\mathbf{r}(T)\|_Q) = E(R(T)) = WvT$, the total search effort, provided we require $\|\mathbf{r}(T)\|_Q = R(T)$. The searcher motion is, of course, given by $\dot{\mathbf{r}}(t) = \mathbf{v}$, where the usual (Euclidean) norm of $\mathbf{v}$, namely, $\|\mathbf{v}\|$, is just the speed v. Furthermore, we want $\mathbf{r}(0) = \mathbf{0}$, and we desire that $\|\mathbf{r}(t)\|_Q$ be monotone increasing; thus

$$\mathbf{r} \cdot Q\mathbf{v} \geq 0. \tag{2-68}$$

The problem which we want to address is that of determining whether or not there exists a solution to the problem

$$\min C(\mathbf{r}(t)), \tag{2-69}$$

where $\mathbf{r}(0) = \mathbf{0}, \mathbf{r}(t)$ is piecewise continuous on $[0, T], \|\dot{\mathbf{r}}(t)\| = v$ whenever $\dot{\mathbf{r}}(t)$ is defined, and $\|\mathbf{r}(T)\|_Q = R(T)$, where Q is any positive definite symmetric matrix. Note, of course, that the standard Euclidean norm is used on the derivative of $\mathbf{r}(t)$. We cannot state categorically that such a problem

will have a solution. On the other hand, if, instead of requiring that $\|\dot{\mathbf{r}}(t)\| = v$ except for a finite number of values of t, we demand only that $\|\dot{\mathbf{r}}(t)\| \leq v$, standard theory[8] indicates that a solution can indeed be found.

Let X_0 be the set of functions $\mathbf{r}(t)$ in the domain of our original problem and X_1 be the set of $\mathbf{r}(t)$ for the new problem. Clearly $X_0 \subseteq X_1$. We claim that X_0 is dense in X_1 in the sup norm topology. That is, given any $\mathbf{f}(t)$ in X_1 and $\epsilon > 0$ arbitrary, but positive, there exists $\mathbf{f}_\epsilon(t)$ in X_0 such that

$$\sup_{0 \leq t \leq T} \|\mathbf{f}(t) - \mathbf{f}_\epsilon(t)\| \leq \epsilon.$$

The situation is illustrated in Figure 2-11 ,which shows a "zigzag function $\mathbf{f}_\epsilon(t)$" approximating $\mathbf{f}(t)$. Here $\mathbf{f}_\epsilon(t) = \mathbf{f}(t)$ for $0 \leq t \leq t_1$, and $\mathbf{f}_\epsilon(t) \neq \mathbf{f}(t)$ on $t_1 < t < T$. Clearly, the amplitude and frequency of the zigzag can be so controlled that the searcher speed is precisely v at all points of the path $\mathbf{r} = \mathbf{f}_\epsilon(t)$ and that $\|\mathbf{f}_\epsilon(t) - \mathbf{f}(t)\| \leq \epsilon$ for all t in $[0, T]$. Furthermore, the following assertion is now obvious:

$$\inf_{\mathbf{r}(t) \in X_0} J = \min_{\mathbf{r}(t) \in X_1} J.$$

We don't know a priori that J attains its infimum on X_0, but we can examine the minimizing solution in X_1 and (if it is not in X_0) approximate it by a point of X_0 (i.e., a function of t) as closely as we like.

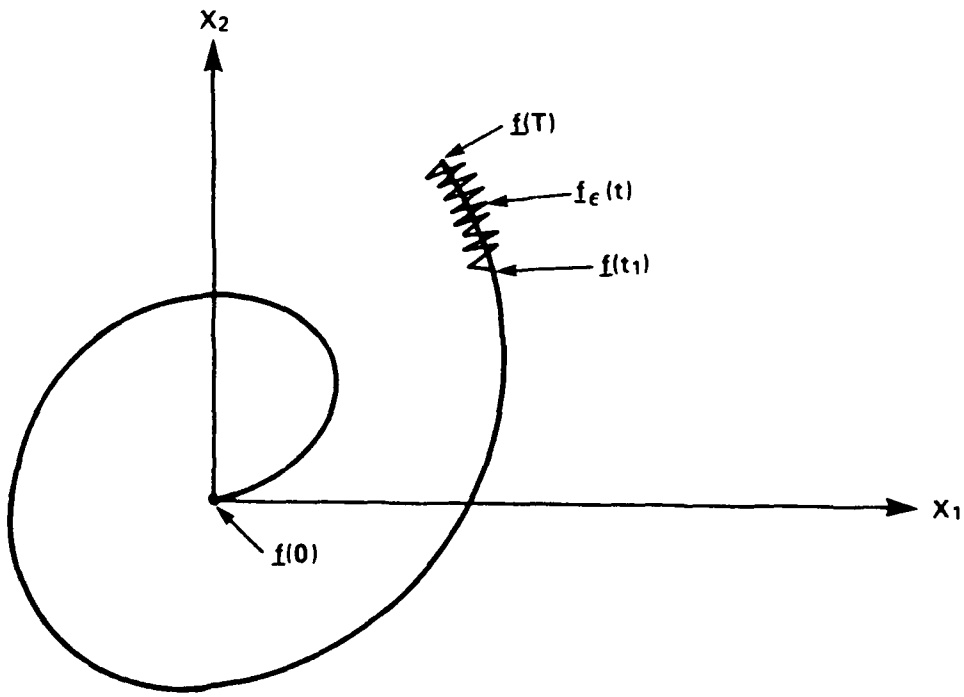

FIGURE 2-11. ZIGZAG FUNCTION APPROXIMATING A GIVEN FUNCTION

We asserted previously that $\dot{\mathbf{r}}(t)$ is piecewise continuous on $[0,T]$, i.e., its components $\dot{x}_i(t), i = 1,2$, are piecewise continuous on that interval. It can then be shown that $\rho(t) = ||\mathbf{r}(t)||_Q$ is a Lipschitz function and that, therefore, it is certainly absolutely continuous. [9] We have, for any t_1, t_2 belonging to $[0,T]$,

$$\begin{aligned} |\rho(t_1) - \rho(t_2)| &= |\, ||\mathbf{r}(t_1)||_Q - ||\mathbf{r}(t_2)||_Q \,| \\ &\leq ||\mathbf{r}(t_1) - \mathbf{r}(t_2)||_Q \\ &\leq |\int_{t_1}^{t_2} ||\dot{\mathbf{r}}(t)||_Q \, dt| \\ &\leq |\int_{t_1}^{t_2} \max_{||\dot{\mathbf{r}}(t)|| \leq v} ||\dot{\mathbf{r}}(t)||_Q \, dt|. \end{aligned} \tag{2-70}$$

It remains to find

$$\max_{||\dot{\mathbf{r}}(t)|| \leq v} ||\dot{\mathbf{r}}(t)||_Q, \tag{2-71}$$

i.e., the maximum of the Q-norm of the velocity vector when its ordinary Euclidean norm does not exceed v. In other words, we want to find a vector $\mathbf{x}_0$ whose usual norm does not exceed v and which maximizes the expression $\mathbf{x}'Q\mathbf{x}$. Now, since Q is symmetric, there exists an orthogonal matrix P[10] such that $Q = P'DP$, where D is a diagonal matrix and the entries on the diagonal, say $1/\sigma_1^2$ and $1/\sigma_2^2$, are the eigenvalues of matrix Q. Assume that $\sigma_2 \leq \sigma_1$. The form $\mathbf{x}'Q\mathbf{x}$ is $\mathbf{y}'D\mathbf{y}$ in $\mathbf{y}$ coordinates, where $\mathbf{y} = P\mathbf{x}$. Since P is orthogonal, it is well-known [10] that $||\mathbf{y}||_2 = ||\mathbf{x}||_2$ in the usual sense, so that $||\mathbf{y}||_2 \leq v$. Now

$$\begin{aligned} \mathbf{y}'D\mathbf{y} &= y_1^2/\sigma_1^2 + y_2^2/\sigma_2^2 \\ &\leq (y_1^2 + y_2^2)/\sigma_2^2 \\ &\leq v^2/\sigma_2^2. \end{aligned}$$

Clearly, the vector $\mathbf{y}_0 = (0,v)^T$ in the y-coordinate frame is a candidate $\mathbf{y}$ which maximizes our quadratic form. This vector is an eigenvector of length v corresponding to $1/\sigma_2^2$ for the diagonal matrix D. Therefore, $\mathbf{x}_0 = P'\mathbf{y}_0$ is likewise an eigenvector of Q for the eigenvalue $1/\sigma_2^2$. It follows that, whenever $\dot{\mathbf{r}}(t)$ exists (it is indeed piecewise continuous), its Q-norm is largest when $\dot{\mathbf{r}}(t)$ is an eigenvector of Q corresponding to $1/\sigma_2^2$. This would occur, in particular, when $\mathbf{r}(t)$ is on the semiminor axis of the ellipse $E = \{\mathbf{y} : y_1^2/\sigma_1^2 + y_2^2/\sigma_2^2 = 1\}, \dot{\mathbf{r}}(t)$ points along the semiminor axis away from the origin, and $||\dot{\mathbf{r}}(t)||_2 = v$. Thus, from Inequality (2-70), we have

$$|\rho(t_1) - \rho(t_2)| \leq v|t_2 - t_1|/\sigma_2, \tag{2-72}$$

so that $\rho(t)$ is Lipschitz continuous on $[0,T]$ with Lipschitz constant v/σ_2. By Lipschitz continuity, $\dot{\rho}(t)$ exists almost everywhere on $[0,T]$ and is Lebesgue-summable, the integral of $\dot{\rho}(t)$ being $\rho(t)$ itself. From Inequality (2-72) we discover that, *wherever* $\dot{\rho}(t)$ *exists*, it is bounded in absolute value by v/ρ_2, i.e., $|\dot{\rho}(t)| \leq v/\rho_2$ almost everywhere on $[0,T]$.

Suppose that the $c = 0$ path qualitatively appears as shown in Figure 2-12. We shall show later that, for our effort function given by Equation (2-66), this assertion is valid. Then, in analogy with Case (A), consider the path, as shown, which moves out from the origin along a semiminor axis at speed $v(\dot{\rho} = v/\sigma_2)$ until $c = 0$, whereupon $t = t_1$, then follows the $c = 0$ curve at a speed less than v *while still moving along the semiminor axis* until a certain time t_2, whence it switches back to speed v, continuing to move along the semiminor axis up to the terminal time T, at which $\|\mathbf{r}(T)\|_Q = R(T)$. The switching time t_2 is determined, as shown in Figure 2-12, by the point of intersection of the line $\rho = \frac{v}{\sigma_2}(t - T) + R(T)$ and the curve $c = 0(\rho = \overline{\rho}(t))$. Note that, *at the moment*, we are only considering *a path which coincides with the semiminor axis of the ellipse* $\|\mathbf{r}\|_Q = R(T)$. Let $\mathbf{r}_*$ be the above path, with corresponding Q-norm ρ_*, and compare it with any other admissible path $\tilde{\mathbf{r}}$ (with Q-norm $\tilde{\rho}$).

From our above analysis, the maximum value of $\dot{\rho}(t)$, when it exists, is v/σ_2, so that, almost everywhere on both $[0, t_1]$ and $[t_2, T], \dot{\tilde{\rho}}(t) \leq v/\sigma_2 = \dot{\rho}_*(t)$. Consider the interval $[0, t_1]$. Let $f(t) = \tilde{\rho}(t) - \rho_*(t)$ there, and we see that $\dot{f}(t) \leq 0$ almost everywhere. Thus, for any $t \in [0, t_1]$,

$$\int_0^t \dot{f}(r)\, dr \leq 0,$$

so that

$$f(t) - f(0) \leq 0.$$

But $f(0) = 0$, since both paths must emanate from the origin. It follows that $f(t) \leq 0$ on $[0, t_1]$, i.e., that $\tilde{\rho}(t) \leq \rho_*(t)$ there. On the other hand, we also know that, for t in $[t_2, T], \dot{f}(t) \leq 0$ almost everywhere, so that

$$\int_t^T \dot{f}(r)\, dr \leq 0.$$

Thus,

$$f(T) - f(t) \leq 0.$$

Now $f(T) = 0$, since both paths must satisfy the condition $\|\mathbf{r}(T)\|_Q = R(T)$. So $f(t) \geq 0$ on $[t_2, T]$, implying that $\tilde{\rho}(t) \leq \rho_*(t)$ there ence, appealing to Figure 2-12 again and to the nature of $c(t, \mathbf{r}(t))$, we see, for example, that

$$c(t, \tilde{\mathbf{r}}(t)) \geq c(t, \mathbf{r}_*(t)) \geq 0, t \in [0, t_1]. \tag{2-73}$$

The first of inequalities (2-73) follows because $\tilde{\rho}(t) \leq \rho_*(t)$ in $[0, t_1]$, so that $E(\tilde{\rho}(t)) \leq E(\rho_*(t))$ there. Thus $c(t, \tilde{\mathbf{r}}(t)) = Wvt - E(\tilde{\rho}(t)) \geq Wvt - E(\rho_*(t)) = c(t, \mathbf{r}_*(t))$. The second inequality, namely, $c(t, \mathbf{r}_*(t)) \geq 0$, is valid because $\mathbf{r}_*(t))$ lies in the region $c > 0$ of Figure 2-12. Also, by definition of $\mathbf{r}_*(t)$,

$$|c(t, \mathbf{r}(t))| \geq |c(t, \mathbf{r}_*(t))| = 0, t \in [t_1, t_2]. \tag{2-74}$$

Finally, by reasoning paralleling that for Inequalities (2-73),

$$c(t, \tilde{\mathbf{r}}(t)) \leq c(t, \mathbf{r}_*(t)) \leq 0, t \in [t_2, T]. \tag{2-75}$$

It follows that $C(t,\bar{\mathbf{r}}(t)) \geq C(t,\mathbf{r}_*(t))$, i.e., that $\mathbf{r}_*$ is optimal in the class X_1.

As illustrated in Figure 2-11, we can approximate $\mathbf{r}_*$ by a path in X_0 (for which the speed is always v) as closely as we like. (As we previously stated, a zigzag function will do.) Now if, instead of a zigzag path, we can arrange *a spiral path in* X_0 which connects $(t_1,\mathbf{r}_*(t_1))$ to $(t_2,\mathbf{r}_*(t_2))$ *while also maintaining* $c = 0$, we will have an optimal path in X_0. In other words, we can certainly obtain an ϵ-approximate solution in X_0 to an optimal solution in X_1, but there is some doubt that a minimizer exists in X_0 for any choice of the pair (σ_1,σ_2). However, it is easily shown that, when $\sigma_1 = \sigma_2$, a spiral solution which meets the requirements can be generated if only $R < vT$. One might conjecture then that, when $\sigma_1 R < vT$, an elliptic spiral meeting the conditions could likewise be obtained (the speed being larger that that required to proceed from the origin along the semimajor axis of the bounding ellipse, arriving at the boundary at time T). In any event, for every pair $(\sigma_1,\sigma_2), 0 < \sigma_2 < \sigma_1$, there *hopefully exists* $v_0(\sigma_1,\sigma_2,T)$ such that, when $v > v_0(\sigma_1,\sigma_2,T)$, spiral motion is possible.

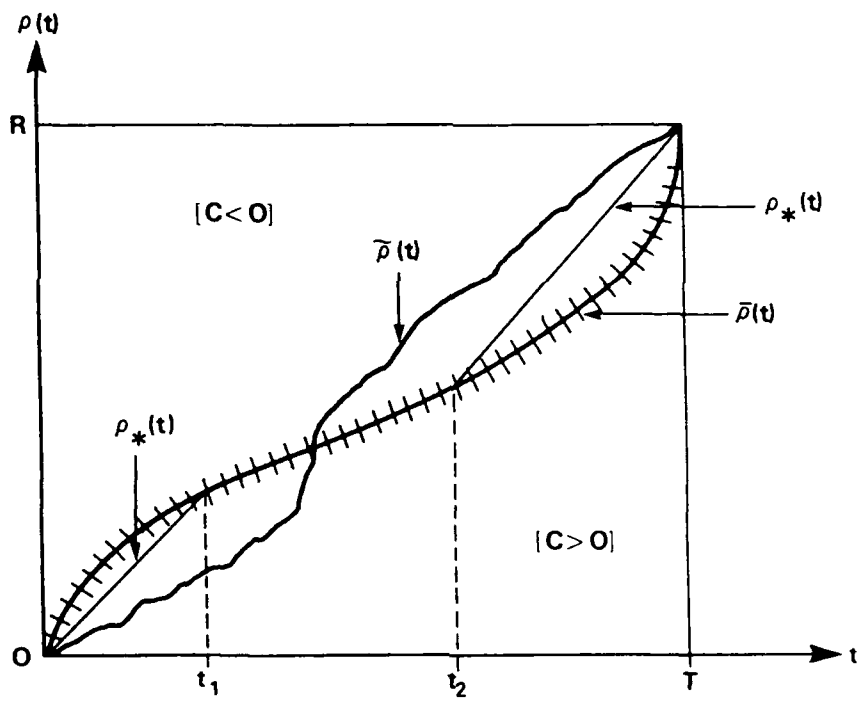

FIGURE 2-12. OPTIMAL SEARCHER VERSUS NONOPTIMAL SEARCHER MOTION IN ELLIPTIC CASE

We now show that the $c \equiv 0$ curve in Figure 2-12 has the qualitative character depicted there. In fact, it is possible to prove that $\bar{\rho}(t)$ is exactly the same function as that obtained for Case (A), i.e., that Equation (2-28) holds, with $\bar{\rho}(t) = \|\bar{\mathbf{r}}(t)\|_Q$. Indeed, from Equation (2-66) and from the fact that $\bar{\rho}(0) = 0$

$$c(t,\bar{\mathbf{r}}(t)) = Wvt - \frac{\pi\sigma_1\sigma_2\bar{\rho}(t)^2}{2a}[R^2(T) - \frac{\bar{\rho}^2(t)}{2}] = 0$$

if and only if

$$\overline{\rho}^2(t) = R^2(T)[1 - (1 - 4aWvt/\pi\sigma_1\sigma_2 R^4(T))^{\frac{1}{2}}]. \tag{2-76}$$

Now $R^2(T) = 2(aWvT/\pi\sigma_1\sigma_2)^{\frac{1}{2}}$, so that, from Equation (2-76), $\overline{\rho}^2(t) = R^2(T)[1 - (1 - t/T)^{\frac{1}{2}}]$, and we again have Equation (2-27), but this time with an interpretation for $\overline{\rho}(t)$ consistent with an elliptic norm.

Just as in the circular case, it is possible to design a procedure for generating an optimal spiral search pattern, provided $v_0(\sigma_1, \sigma_2, T)$ exists for any (σ_1, σ_2). The procedure (algorithm) which we now present should allow one in practice to devise the path if it is possible to do so. Let

$$\frac{x_1^2(t)}{\sigma_1^2} + \frac{x_2^2(t)}{\sigma_2^2} = \overline{\rho}^2(t) \equiv f(t), \tag{2-77}$$

and let us require that

$$\dot{x}_1^2(t) + \dot{x}_2^2(t) = v^2. \tag{2-78}$$

Making the transformation $y_1(t) = x_1(t)/\sigma_1, y_2(t) = x_2(t)/\sigma_2$ in Equation (2-77) and Equation (2-78), we find an equivalent pair of relations, namely,

$$y_1^2(t) + y_2^2(t) = f(t) \tag{2-79}$$

and

$$\sigma_1^2\dot{y}_1^2(t) + \sigma_2^2\dot{y}_2^2(t) = v^2. \tag{2-80}$$

Let $r_y^2(t) = y_1^2(t) + y_2^2(t)$ and $y_1(t) = r_y(t)\cos\theta_y(t), y_2(t) = r_2(t)\sin\theta_y(t)$. Then one sees that

$$\begin{aligned}
&\sigma_1^2\big(\dot{r}_y^2(t)\cos^2\theta_y(t) - r_y(t)\dot{r}_y(t)(\sin 2\theta_y(t))\dot{\theta}_y(t) \\
&+r_y^2(t)(\sin^2\theta_y(t))\dot{\theta}_y^2(t)\big) + \sigma_2^2\big(\dot{r}_y^2(t)\sin^2\theta_y(t) \\
&+r_y(t)\dot{r}_y(t)(\sin 2\theta_y(t))\dot{\theta}_y(t) + r_y^2(t)(\cos^2\theta_y(t))\dot{\theta}_y^2(t)\big) = v^2.
\end{aligned} \tag{2-81}$$

Now we use the fact that $\dot{r}_y(t) = \dot{f}(t)/2r_y(t)$, substitute this relation into Equation (2-81), and solve the quadratic equation in $\dot{\theta}_y(t)$ to obtain

$$\dot{\theta}_y(t) = \frac{\dot{f}(t)(\sigma_1^2 - \sigma_2^2)\sin(2\theta_y(t)) \pm 2f_1(t))}{4f(t)f_1(t)}, \tag{2-82}$$

where

$$f_1(t) \equiv \left(4v^2 f(t)f_2(t) - \sigma_1^2\sigma_2^2\dot{f}^2(t)\right)^{\frac{1}{2}}$$

and

$$f_2(t) \equiv \sigma_1^2\sin^2\theta_y(t) + \sigma_2^2\cos^2\theta_y(t).$$

To insure that $\dot{\theta}_y(t)$ is positive, let us choose the positive sign in Equation (2-82). Using the initial condition $\theta_y(t_1) = \pi/2$, we could then solve Equation (2-82) numerically to obtain $\theta_y(t), t_1 \le t \le t_2$. Now

$x_1(t) = \sigma_1 y_1(t) = \sigma_1 r_y(t)\cos\theta_y(t), x_2(t) = \sigma_2 y_2(t) = \sigma_2 r_y(t)\sin\theta_y(t)$, and $r_x^2(t) = r_y^2(t)(\sigma_1^2\cos^2\theta_y(t)+\sigma_2^2\sin^2\theta_y(t))$. Also, $\theta_x(t) = \tan^{-1}(x_2(t)/x_1(t)) = \tan^{-1}(\sigma_2\tan\theta_y(t)/\sigma_1)$. Since the inverse tangent and tangent functions are both strictly increasing functions of their arguments, $\theta_x(t)$ is an increasing function of t whenever $\theta_y(t)$ is. The pair $(r_x(t), \theta_x(t))$ thus obtained would represent an optimal solution on $[t_1, t_2]$, given $r_x(t_1) = vt_1, \theta_x(t_1) = \pi/2$ as initial information. The solution found thus far would, however, not be satisfactory, since we require that, at time $t_2, (x_1(t_2), x_2(t_2))$ is to be on the semiminor axis of the bounding ellipse. To rectify this situation, we start from the point $r_x(t_2) = \sigma_2 R(T) - v(T - t_2), \theta_x(t_2) = \pi/2$ on the semiminor axis and allow θ_x to increase as time decreases away from t_2. To implement such a procedure, again choose the positive sign in Equation (2-82). Now let $(r_x^f(t), \theta_x^f(t))$ be the forward solution obtained by marching from $t = t_1$ forward to $t = t_2$, and let $(r_x^b(t), \theta_x^b(t))$ be the backward solution obtained by receding from $t = t_2$ to $t = t_1$. Then, if the forward and backward solutions intersect at some point $t = t_{INT}$, an optimal solution on $[t_1, t_2]$ would be $(r_x^o(t), \theta_x^0(t))$, where $(r_x^o(t), \theta_x^o(t)) = (r_x^f(t), \theta_x^f(t)), t_1 \le t \le t_{INT}$, and $r_x^o(t), \theta_x^o(t)) = (r_x^b(t), \theta_x^b(t)), t_{INT} \le t \le t_2$. It is possible to show that they will intersect. To see this, examine first the circular situation. Refer to Figure 2-13. If we consider the total angular increment from time t_1 to time t_2, divide the increment in half, and then reflect the portion of the curve from point $(r(t_h), \theta(t_h))$ to $(r(t_2), \theta(t_2))$ about the line shown, we have an equivalent path with respect to allocation considerations. In so doing, the point $(r(t_2), \theta(t_2))$ gets mapped to $(r(t_2), \pi/2)$. We can do this because, *in the circular case, the allocation density is radially symmetric.* Let ϵ_1 and ϵ_2 be small and positive. Imagine that one were to change the circle of radius $R(T)$ into an ellipse whose semiminor axis is aligned with the x_2 direction and has length $(1 - \epsilon_1)(R(T)$ and whose semimajor axis is aligned with the x_1 direction and has length $(1 + \epsilon_2)R(T)$. Then, *by continuity*, the modified optimal path P_c (c for circular) would be converted into an optimal path P_e. That path P_e must start on the semiminor axis, namely, the x_2 -axis, and end on the semiminor axis, namely, the x_2 -axis again. The backward trajectory portion of P_e will simply be a modification of the dashed part of P_c leading from $(r_1(t_2), \pi/2)$ back to $(r(t_h), \theta(t_h))$, and the forward part of P_e will correspond to the forward portion of P_c. Therefore, such an intersection point is expected to exist, in general; since, as the ellipticity is progressively changed, maintaining the x_2 -axis as the semiminor axis, the optimal path must change in such fashion that one starts on the x_2 -axis and terminates on the same axis.

The argument here is *not a proof of existence of an optimal solution* in X_0. It should be taken rather as a *plausibility argument.* We are using what mathematicians call a homotopy in order to derive optimal solutions for new situations from those obtained under old conditions which are *close to the ne v ones.* One may use such a concept to obtain numerical solutions to Equation (2-82), employing the so-called Picard method.[11] As input to the method, one might start with $\theta_y(t)$ obtained for the circular case.

Parallel to the circular case, one may assume that all effort is to be devoted to a preassigned

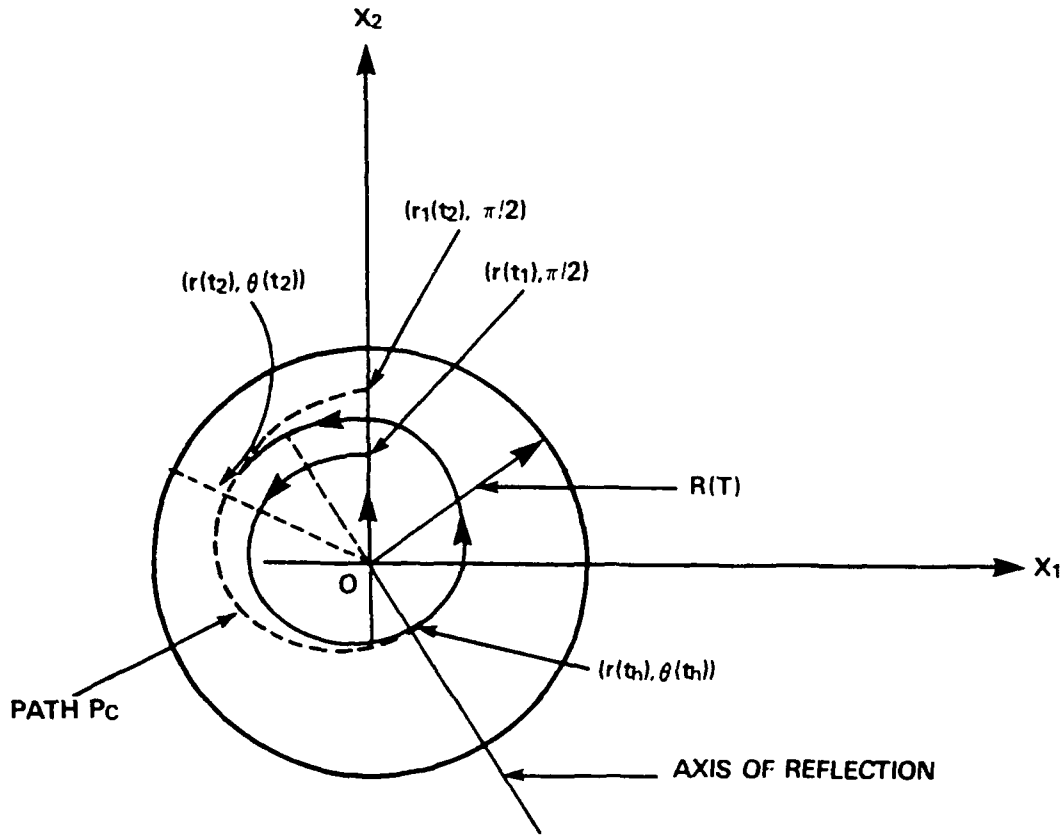

FIGURE 2-13. ALTERATION OF PATH IN CIRCULAR CASE TO MEET PATH CONSTRAINT

elliptic disk D_e. Suppose that we represent this disk by $D_e = \{(x_1, x_2) : (x_1/\sigma_1)^2 + (x_2/\sigma_2)^2 \leq R^2)$. Then we have

$$P(D_e) = \iint_{\mathbf{x} \in D_e} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right)\right) dx_1\, dx_2. \tag{2-83}$$

Letting $y_1 = x_1/\sigma_1, y_2 = x_2/\sigma_2$, Equation (2-83) becomes

$$P(D_e) = \frac{1}{2\pi} \iint_{\mathbf{y} \in D} \exp(-\frac{1}{2}(y_1^2 + y_2^2)) dy_1 dy_2, \tag{2-84}$$

where D is a circular disk of radius R. If we now change to polar coordinates, i.e., let $y_1 = r\cos\theta, y_2 = r\sin\theta$, Equation (2-84) becomes

$$P(D_e) = (1 - \exp(-R^2/2)). \tag{2-85}$$

From Equation (2-85) we have

$$R = [-2\ln(1 - P(D_e))]^{\frac{1}{2}},$$

so that, given $P(D_e)$, one knows R and hence the disk D_e. Parallel to Equation (2-20), one has the conditional Gaussian distribution

$$P_{D_e}(\mathbf{x}) = \begin{cases} \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2}+\frac{x_2^2}{\sigma_2^2}\right)\right)/P(D_e) & ,\mathbf{x}\in D_e \\ 0 & ,\mathbf{x}\notin D_e. \end{cases} \tag{2-86}$$

We use Equation (2-21) with $p_D(\mathbf{x})$ replaced by $p_{D_e}(\mathbf{x})$ in Equation (2-86). As before, let $\lambda_1 = \lambda/a$. Then $p_{D_e}(\mathbf{x}) \geq \lambda_1$ if and only if $\|\mathbf{x}\|_Q^2 \leq -2\ln(2\pi\sigma_1\sigma_2\lambda_1 P)$. So

$$f^*_{\lambda,D_e}(\mathbf{x}) = \begin{cases} \frac{1}{a}\left[-\ln(2\pi\sigma_1\sigma_2\lambda_1 P(D_e)) - \frac{\|\mathbf{x}\|_Q^2}{2}\right] & ,\|\mathbf{x}\|_Q^2 \leq R_m^2 \\ 0 & ,\|\mathbf{x}\|_Q^2 > R_m^2, \end{cases}$$

where $R_m^2 = R^2(P,\lambda_1,\sigma_1,\sigma_2) \equiv \min(R^2, -2\ln(2\pi\sigma_1\sigma_2\lambda_1 P))$. Now define $\lambda_2 \equiv \lambda_1 P(D_e)$, and compute the cost over the elliptic disk. We have

$$C(f^*_{\lambda,D_e}(\mathbf{x})) = \begin{cases} \frac{\pi R^2\sigma_1\sigma_2}{a}\left[-\ln(2\pi\sigma_1\sigma_2\lambda_2) - \frac{R^2}{4}\right] & ,0 < \lambda_2 \leq \frac{1-P}{2\pi\sigma_1\sigma_2} \\ \frac{\pi\sigma_1\sigma_2}{a}\left[\ln(2\pi\sigma_1\sigma_2\lambda_2)\right]^2 & ,\frac{1-P}{2\pi\sigma_1\sigma_2} \leq \lambda_2 \leq \frac{1}{2\pi\sigma_1\sigma_2}. \end{cases} \tag{2-87}$$

The first line of Equation (2-87) corresponds to the case where $R^2(P,\lambda_1,\sigma_1,\sigma_2) = R^2$. Then f^*_{λ,D_e} is nonzero over the entire elliptic disk $\|\mathbf{x}\|_Q \leq R$. Equating the right hand side of line 1 of Equation (2-87) to $A_s = WvT$, one finds that $\lambda_2 = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(-\frac{R^2}{4} - \frac{aWvT}{\pi R^2\sigma_1\sigma_2}\right)$. Inserting this relation for λ_2 into $0 < \lambda_2 \leq (1-P)/2\pi\sigma_1\sigma_2$, we find that $A_s \geq \pi(\ln(1-P))^2\sigma_1\sigma_2/a$. Also, substituting the λ_2 value obtained above into the allocation density, we have

$$f^*_{\lambda,D_e}(\mathbf{x}) = \frac{1}{a}\left(\frac{aWvT}{\pi R^2\sigma_1\sigma_2} + \frac{R^2 - 2\|\mathbf{x}\|_Q^2}{4}\right), \|\mathbf{x}\|_Q \leq R, WvT \geq \pi\sigma_1\sigma_2(\ln^2(1-P))/a. \tag{2-88}$$

In addition, note that a searcher can reach the boundary of an elliptic disk after T units of time if and only if he can do so along its semiminor axis. Thus we should require that $R\sigma_2 \leq vT$, i.e., that $R \leq vT/\sigma_2$.

Now consider the second line of Equation (2-87). Then $0 < A_s \leq \pi(\ln^2(1-P))\sigma_1\sigma_2/a$, corresponding to $\frac{1-P}{2\pi\sigma_1\sigma_2} \leq \lambda_2 \leq \frac{1}{2\pi\sigma_1\sigma_2}$. Also, $\lambda_2 = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\left(\frac{aWvT}{\pi\sigma_1\sigma_2}\right)^{\frac{1}{2}}\right]$. Inserting this equality into the allocation density, we have, when $WvT \leq \pi(\ln^2(1-P))\sigma_1\sigma_2/a$,

$$f^*_{\lambda,D_e}(\mathbf{x}) = \begin{cases} \frac{1}{a}\left[\left(\frac{aWvT}{\pi\sigma_1\sigma_2}\right)^{\frac{1}{2}} - \frac{\|\mathbf{x}\|_Q^2}{2}\right] & ,\|\mathbf{x}\|_Q^2 \leq R_1^2 \leq R^2 \\ 0 & ,R_1^2 \leq \|\mathbf{x}\|_Q^2 \leq R^2, \end{cases} \tag{2-89}$$

where $R_1^2 \equiv -2\ln(2\pi\sigma_1\sigma_2\lambda_2) = 2(aWvT/\pi\sigma_1\sigma_2)^{\frac{1}{2}}$. Equation (2-89) has the same form as Equation (2-65), the difference between the two being that Equation (2-89) presumes a distribution over a preassigned elliptic disk and that Equation (2-65) presumes one over the entire $\mathbf{x}$ plane. When

the effort to be expended does not exceed $\pi(\ln^2(1-P))\sigma_1\sigma_2/a$, the situation is parallel to that of *unrestricted disk size.* The theory just formulated for Case (B) then dictates the optimal path. The optimal path process for Equation (2-88) remains to be generated. We shall next develop this process.

The cumulative allocation function (cumulative effort) over a subdisk $\|\mathbf{x}\|_Q \leq r$ of the elliptic disk $\|\mathbf{x}\|_Q \leq R$ when $WvT \geq \pi\sigma_1\sigma_2(\ln^2(1-P))/a$ is found by integration of Equation (2-88) over that elliptic subdisk. We find that

$$E(\|\mathbf{r}\|_Q) = E(\rho) = \frac{\pi\sigma_1\sigma_2\rho^2}{a}\left(\frac{aWvT}{\pi R^2\sigma_1\sigma_2} + \frac{R^2-\rho^2}{4}\right), \rho \leq R. \tag{2-90}$$

Referring to our cost criterion (2-67) and setting $c(t,\mathbf{r}(t)) = 0$, we find, using Equation (2-90) and the fact that $\overline{\rho}(0) = 0$, that

$$\overline{\rho}^2(t) = \frac{4aWvT + \pi\sigma_1\sigma_2R^4 - \left[(4aWvT + \pi\sigma_1\sigma_2R^4)^2 - 16\pi a\sigma_1\sigma_2R^4Wvt\right]^{\frac{1}{2}}}{2\pi\sigma_1\sigma_2R^2}. \tag{2-91}$$

From Equation (2-91) it follows that

$$\overline{\rho}(t) = \frac{1}{R}\left(\frac{4aWvT + \pi\sigma_1\sigma_2R^4}{2\pi\sigma_1\sigma_2}\right)^{\frac{1}{2}}\left[1 - \left(1 - \frac{16\pi a\sigma_1\sigma_2R^4Wvt}{(4aWvT + \pi\sigma_1\sigma_2R^4)^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{2-92}$$

Let $b(T) = 16\pi a\sigma_1\sigma_2R^4Wv/(4aWvT+\pi\sigma_1\sigma_2R^4)^2$ and $L(t) = b(T)t$. One sees that $WvT > \pi\sigma_1\sigma_2(\ln^2(1-P))/a$ if and only if $4aWvT > \pi\sigma_1\sigma_2R^4$, so that $L(T) < 1$. Obviously, $L(0) = 0$. Set

$$r_1(t) = \left[1 - (1 - L(t))^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{2-93}$$

the same form as Equation (2-54). Then $\overline{\rho}(t) = \frac{1}{R}\left(\frac{4aWvT+\pi\sigma_1\sigma_2R^4}{2\pi\sigma_1\sigma_2}\right)^{\frac{1}{2}} r_1(t)$. Just as in the circular case, we can assert that $r_1'(0) = +\infty$ and that $0 < r_1'(T) < \infty$ when our new condition $WvT > \pi\sigma_1\sigma_2(\ln^2(1-P))/a$ applies. Of course, when $4aWvT = \pi\sigma_1\sigma_2R^4, r_1'(T) = +\infty$, and $\overline{\rho}(t)$ looks like the curve displayed with hatch marks in Figure 2-12. As T increases such that $4aWvT > \pi\sigma_1\sigma_2R^4$, the derivative of $\overline{\rho}(t)$ at $t = T$ begins to decrease and, for a while, is larger than v. From the right side of Equation (2-55), which is also symbolically the derivative of the right side of Equation (2-93), we see that $r_1'(T)$ tends to zero as T tends to infinity. We shall show that $\overline{\rho}'(T)$ is a strictly decreasing function, so that, at precisely one value of T, say $T = T_0, \overline{\rho}'(T_0) = v/\sigma_2$. For $T > T_0$, the line segment portion of $\rho_*(t)$ in Figure 2-12 disappears, and it becomes possible to follow the $c = 0$ curve all the way from $t = t_1$ to $t = T$. The inflectional nature of $r_1(t)$ is precisely the same as in the circular case, and the analysis presented there can be repeated verbatim here. Indeed $t_{\text{inf}} = 5/(9b(T))$ again, with $b(T)$ a strictly decreasing function of T, behaving like $1/T^2$ as T tends to infinity. Thus, as in the circular case, the inflectional character of $\overline{\rho}(t)$ is no longer present when T is sufficiently large.

We would now like to show, as in the circular case, that $\overline{\rho}'(T)$ is a strictly decreasing function of T, and we want to ascertain T_0, where $\overline{\rho}'(T_0) = v/\sigma_2$. From Equation (2-91), we have

$$\overline{\rho}(T)\overline{\rho}'(T) = \frac{2aWvR^2}{4aWvT - \pi\sigma_1\sigma_2 R^4}. \tag{2-94}$$

Since $\overline{\rho}(T) = R(T)$, one finds that

$$\overline{\rho}'(T) = \frac{2aWvR}{4aWvT - \pi\sigma_1\sigma_2\rho^4}. \tag{2-95}$$

Thus $\overline{\rho}'(T)$, as in the circular case, is a decreasing function of T, tending to zero like $1/T$. Setting $\overline{\rho}'(T) = v/\sigma_2$ in Equation (2-95), we have

$$T_0 = \frac{R\sigma_2(\pi R^3\sigma_1 + 2aW)}{4aWv}. \tag{2-96}$$

Also, similar to the argument used for Figure 2-10, we can claim that $T_{\inf} > T_0$, so that, whenever $T \geq T_{\inf}, \overline{\rho}(t)$ would appear as shown in Figure 2-10, where, of course, for the elliptic case, $\overline{\rho}(t) = \|\mathbf{r}(t)\|_Q$.

Parallel to Case (A), we would next like to consider the order of the error when one substitutes $\rho_*(t)$ into Equation (2-67). (We shall only investigate the error for the unconditional elliptic Gaussian distribution, i.e., the one corresponding to a moving boundary ellipse.) On $0 \leq t \leq t_1, \rho_*(t) = vt/\sigma_2$; using Equation (2-66) and Equation (2-67), we have the following parallel to J_1 for the circular case (given in Equation (2-45)):

$$\begin{aligned} J_1 &= \int_0^{t_1} \left(Wvt - \frac{\pi\sigma_1 v^2 t^2}{2a\sigma_2}\left[R^2(T) - \frac{v^2t^2}{2\sigma_2^2}\right]\right)^2 dt \\ &= \frac{\pi^2\sigma_1^2 v^8 t_1^9}{144a^2\sigma_2^6} - \frac{\pi^2 R^2 v^6 t_1^7 \sigma_1^2}{28a^2\sigma_2^4} + \frac{\pi\sigma_1 W v^5 t_1^6}{12a\sigma_2^3} \\ &\quad + \frac{\pi^2\sigma_1^2 v^4 R^4 t_1^5}{20a^2\sigma_2^2} - \frac{\pi\sigma_1 W v^3 R^2 t_1^4}{4a\sigma_2} + \frac{W^2 v^2 t_1^3}{3}. \end{aligned} \tag{2-97}$$

For J_2, one has

$$\begin{aligned} J_2 &= \int_{t_2}^{T} \left(Wvt - \frac{\pi\sigma_1\sigma_2(R + v(t-T)/\sigma_2)^2}{2a}\left[R^2 - \frac{(R + v(t-T)/\sigma_2)^2}{2}\right]\right)^2 dt \\ &= \int_0^{T-t_2} \left(Wv(T-u) - \frac{\pi\sigma_1\sigma_2(R - vu/\sigma_2)^2}{2a}\left[R^2 - \frac{(R - vu/\sigma_2)^2}{2}\right]\right)^2 du, \end{aligned} \tag{2-98}$$

upon making the change of variable u=T-t. Expanding the integrand of the right member of

Equation (2-98) and integrating the result term by term, we find that

$$
\begin{aligned}
J_2 = & \frac{W^2 v^2 \left(T^3 - t_2^3\right)}{3} + \frac{\pi\sigma_1\sigma_2}{a}\{\frac{R^4\left(\pi\sigma_1\sigma_2 R^4 - 8aWvT\right)(T-t_2)}{16a} \\
& + \frac{WvR^4(T-t_2)^2}{4} - \frac{R^2Wv^3(T-t_2)^4}{2\sigma_2^2} \\
& + \frac{v^4\left(7\pi\sigma_1\sigma_2R^4 + 4aW\left(4R\sigma_2 + vT\right)\right)(T-t_2)^5}{40a\sigma_2^4} \\
& - \frac{v^5\left(4\pi\sigma_1R^3 + Wa\right)(T-t_2)^6}{12a\sigma_2^4} \\
& + \frac{3\pi\sigma_1R^2v^6(T-t_2)^7}{14a\sigma_2^5} - \frac{\pi\sigma_1Rv^7(T-t_2)^8}{16a\sigma_2^6} \\
& + \frac{\pi\sigma_1v^8(T-t_2)^9}{144a\sigma_2^7}\}.
\end{aligned}
\tag{2-99}
$$

The values for t_1 and t_2 are again obtainable through Equation (2-43) and Equation (2-44), but with a different definition for $R(T)$. That is, Equation (2-43) and Equation (2-44) yield t_1 and t_2, respectively, when $K = \frac{R}{vT}$ and $R^2(T)$ is given by $2\left(\frac{aWvT}{\pi\sigma_1\sigma_2}\right)^{\frac{1}{2}}$ instead of $2\sigma^2\left(\frac{aWvT}{\pi\sigma^2}\right)^{\frac{1}{2}} = 2\sigma\left(\frac{aWvT}{\pi}\right)^{\frac{1}{2}}$. We shall investigate the nature of J_1 and J_2 as T tends to infinity in Chapter 3, which we turn to next.

CHAPTER 3

COMPARISON OF THE EFFECTIVENESS OF OPTIMAL CONTROL STRATEGIES WITH LOGARITHMIC SPIRAL DESIGNS

Up to this point, we have developed the optimal allocations of effort for stationary targets whose distributions are known to be Gaussian, and we have devised procedures for generating optimal search strategies when these are expected to exist. Optimal search plans, even if they turn out not to be practically useful, serve as a basis of comparison; after all, they represent the best that one can do under the stated conditions of the problem. The purpose of this chapter is (1) to develop logarithmic spiral solutions to our problem for both the circular and elliptic Gaussian cases and (2) to compare the asymptotic properties of the optimal solutions developed in Chapter 2 with those for the spirals. We shall discover that the optimal solutions yield a mean-square error of the order of $T^{-\frac{3}{2}}$ as T tends to infinity, v being held constant, and that the logarithmic spiral solutions behave like T^3 in this regard. On the other hand, we shall find that, *if we properly choose v as a function of T*, we can drive both the cost for the spiral and the optimal solution toward the same asymptotic behavior, thus making the logarithmic spiral computationally attractive.

Let us begin by studying the properties of a conventional logarithmic spiral, which we might like to adapt to the situation where the target is stationary and its location follows a circular Gaussian law. The general form of the logarithmic spiral is $\rho = c_1 e^{c_2\theta}$, where c_1 and c_2 are given positive quantities, which may depend upon the budget time T, the speed v of the searcher, the searcher's detection width W, and, finally, the variance σ^2 of the target's location. It is important to note that c_1 and c_2 are to be constants once such parameters are adjusted. The problem of interest thus becomes: Find ρ and θ as functions of time t, where $0 \leq t \leq T$, such that $\rho = c_1 e^{c_2\theta}$, the speed along the spiral is v, and the searcher meets the boundary of a known circular disk $\rho \leq R(T)$ at time $t = T$. Then compute the cost function, given by Equation (2-26), where $\rho = \rho(t)$ is the radial component of the logarithmic spiral.

Our target distribution for the circular Gaussian case is given by Equation (2-9), where $r^2 = x_1^2 + x_2^2$, of course. Since the target is most likely to be at or close to the origin of the (x_1, x_2) plane, we agree to start the spiral at the origin (or perhaps at ϵ distance away from it) and spiral out to the boundary of the disk $\rho \leq R(T)$, on which the optimal effort happens to be supported. The cumulative effort to be approximated is given, in fact, by Equation (2-27). Note that, when $\theta = -\infty$, $\rho = 0$, so that one has the initial information: At time $t = 0$, $\rho = 0$ and $\theta = -\infty$. We shall see that, when speed is the sole requirement to be matched, one can design a spiral which is traversable

at that speed from the origin to the boundary of disk $\rho \leq R(T)$, reaching the boundary at time T, provided, of course, that $R(T) < vT$.

Now the length of a curve $\rho = f(\theta)$ between two points (ρ_1, θ_1) and (ρ_2, θ_2), where ρ and θ are polar coordinates, is given by[12]

$$L = \int_{\theta_0}^{\theta_1} \left(f^2(\theta) + \dot{f}^2(\theta)\right)^{\frac{1}{2}} d\theta, \tag{3-1}$$

so that, when $\rho = c_1 e^{c_2\theta}, c_1 > 0, c_2 > 0, \theta_0 = -\infty$, and $\theta_1 = \theta$,

$$\begin{aligned} L &= c_1 \left(1 + c_2^2\right)^{\frac{1}{2}} \int_{-\infty}^{\theta} e^{c_2 u}\, du \\ &= \frac{c_1 \left(1 + c_2^2\right)^{\frac{1}{2}}}{c_2} e^{c_2\theta}. \end{aligned} \tag{3-2}$$

If the searcher moves at constant speed v, then the right side of Equation (3-2) must equal vt, where t is the time required to go from the origin to the point (ρ, θ). That is, we must have

$$\frac{c_1 \left(1 + c_2^2\right)^{\frac{1}{2}}}{c_2} e^{c_2\theta} = vt. \tag{3-3}$$

Now, of course, $\rho(\theta) = c_1 e^{c_2\theta}$; so that, considering the search width W, the area of coverage up to the time that a searcher meets the boundary of a circular disk $r \leq r_0$ (including recoverage if it exists) is just

$$LW = \left(1 + c_2^2\right)^{\frac{1}{2}} r_0 W / c_2. \tag{3-4}$$

In particular, when $t = T$, one requires that $r_0 = R(T)$, so that the second member of Equation (3-4) must be equated to WvT. Then we have

$$vT = \left(1 + c_2^2\right)^{\frac{1}{2}} R(T)/c_2, \tag{3-5}$$

so that, necessarily, using the definition of $R(T)$,

$$\begin{aligned} v^2T^2 &= \left(1 + c_2^2\right) R^2(T)/c_2^2 \\ &= 2\left(1 + c_2^2\right) \sigma (aWvT/\pi)^{\frac{1}{2}}/c_2^2. \end{aligned} \tag{3-6}$$

Defining $c_3 = 1/c_2^2$, one can readily solve Equation (3-6) for c_3 (and hence for c_2) as a function of v, T, W, and σ. Note, from Equation (3-5), that, when vT and $R(T)$ are close to each other, c_2 is very large, implying that the *spiral moves out from the origin very rapidly* in order to meet the requirement that the circle $\rho = R(T)$ be intercepted at time T. On the other hand, when vT is large relative to $R(T)$, Equation (3-5) implies that c_2 will be close to zero, so that one builds up effort in the neighborhood of 0 before he transits to the boundary of the disk. Observe also that $R(T) < vT$ if and only if $v^3 > 4aW\sigma^2/\pi T^3$. In particular, one value of v could be used for all budget times $T \geq T_0$, say.

It appears now that the parameter c_1 serves solely as a scaling parameter, not essentially affecting our analysis. Therefore, let $c_1 = 1$ and $c_2 = c$. We would like to examine the nature of the effort function $E(r), 0 \leq r \leq R(T)$, as given by Equation (2-27). Note first that $E(0) = E'(0) = 0$. Also, $E'(r) = \pi r\left(R^2 - r^2\right)/a\sigma^2$, which is positive for $r < R$ and zero for $r = R$. Furthermore, one has $E''(r) = \pi\left(R^2 - 3r^2\right)/a\sigma^2$. Clearly, then, on $0 \leq r \leq R(T)$, $E''(r)$ has precisely one root;in addition, $E''(r)$ goes from positive to negative values through the root. That is to say, $E(r)$ has a point of inflection. Define $E_1(r;k) = kWr, k > 0$, and let us find that value or those values of k which minimize the following cost functional:

$$C_1(k) = \int_0^{R(T)} \left[E(r) - E_1(r;k)\right]^2 dr. \tag{3-7}$$

From Equation (3-7), we have

$$C_1'(k) = -2W \int_0^{R(T)} r[E(r) - kWr]\, dr, \tag{3-8}$$

which vanishes when

$$\begin{aligned} k^* &= \frac{3}{WR^3(T)} \int_0^{R(T)} rE(r)\, dr \\ &= \frac{3}{WR^3(T)} \int_0^{R(T)} \left(\frac{\pi r^3}{2a\sigma^2} R(T) - \frac{\pi r^5}{4a\sigma^2}\right) dr \\ &= \frac{\pi R^3(T)}{4aW\sigma^2}. \end{aligned} \tag{3-9}$$

It is clear from Equation (3-8) (derivative passes from a negative value through zero to a positive value) that k^* is a (unique) minimizing point for the right side of Equation (3-7). Observe from Equation (3-9) that $k^* > 1$ whenever $\pi R^3(T) > 4aW\sigma^2$. However, by definition of $R(T)$, we have that

$$\frac{vT}{R(T)} = \frac{\pi R^3(T)}{4aW\sigma^2}.$$

so that, indeed, $k^* > 1$ when $R(T)$ exceeds vT, a condition which one naturally requires in order for the searcher to meet the boundary of the disk $\rho(t) \leq R(T)$ at time T. Such being the case, $k^* = (1 + c^*)^{\frac{1}{2}}/c^*$, where c^* is the solution to Equation (3-5). In other words, the best linear least squares fit to the optimal cumulative effort function $E(r)$, $0 \leq r \leq R(T)$, must be the logarithmic spiral $\rho(t) = exp(c^*\theta(t))$ which matches the problem conditions. The situation is depicted in Figure 3-1.

The form of $\rho(t)$ and $\theta(t)$ can now be readily determined through the condition $\dot{\rho}^2(t) + \rho^2(t)\dot{\theta}^2(t) = v^2$. For any c and $\rho(t) = e^{c\theta(t)}$, one has

$$c^2 e^{2c\theta}\dot{\theta}^2(t) + e^{2c\theta}\dot{\theta}^2(t) = v^2,$$

so that

$$\dot{\theta}^2(t) = \frac{v^2 e^{-2c\theta(t)}}{1 + c^2}. \tag{3-10}$$

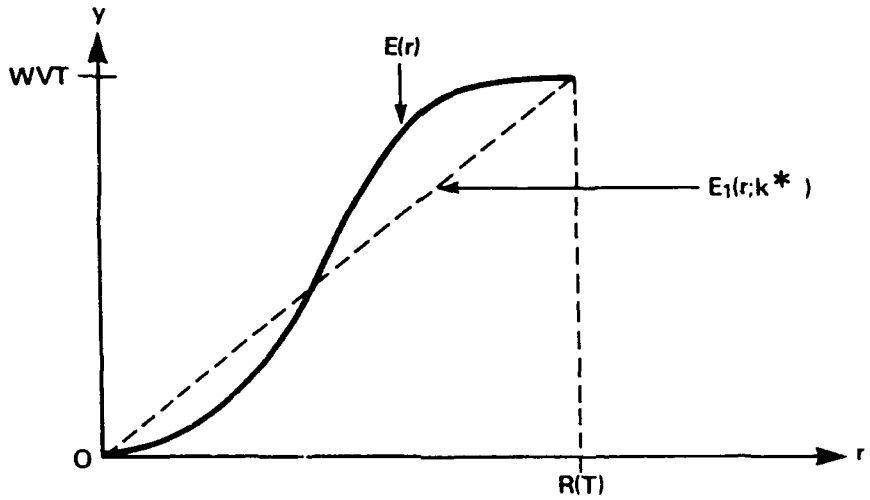

FIGURE 3-1. CUMULATIVE EFFORT VERSUS OPTIMAL LINEAR EFFORT

Since θ is to be a strictly increasing function of time, Equation (3-10) implies that

$$\dot{\theta}(t) = \frac{ve^{-c\theta(t)}}{(1+c^2)^{\frac{1}{2}}}. \tag{3-11}$$

Using the initial condition $\theta(0) = -\infty$ and separating variables in Equation (3-11), one finds that

$$\theta(t) = \frac{1}{c}\ln\left(\frac{cvt}{(1+c^2)^{\frac{1}{2}}}\right). \tag{3-12}$$

When $c = c^*$, so that $c^*/(1+c^{*2})^{\frac{1}{2}} = R(T)/vT$, one has, from Equation (3-12),

$$\theta(t) = \frac{1}{c^*}\ln\left(\frac{R(T)t}{T}\right). \tag{3-13}$$

Now $\dot{\rho}(t) = c^* e^{c^*\theta(t)}\dot{\theta}(t)$, so that, appealing to Equation (3-13) and the condition $\rho(T) = R(T)$, we have

$$\rho(t) = \frac{R(T)t}{T}. \tag{3-14}$$

The error functional $C_1(k)$, as exhibited in Equation (3-7), is analogous to that of Equation (2-26), the difference being that integration in the former is with respect to r, and integration in the latter is with respect to t. We can, of course, make a change of variable from r to t for a given choice $r = r(t)$. In particular, suppose that we use Equation (3-14). Then we have

$$\begin{aligned} C_1(k^*) &= \frac{R(T)}{T}\int_0^T [E(\rho(t)) - Wvt]^2\,dt \\ &= \frac{R(T)}{T}C(\mathbf{r}(t)), \end{aligned} \tag{3-15}$$

so that $C(\mathbf{r}(t))$ for the logarithmic spiral chosen is just

$$C(\mathbf{r}(t)) = \frac{TC_1(k^*)}{R(T)}. \tag{3-16}$$

We next examine the behavior of $C_1(k^*)$ as T tends to infinity. It will be shown that $C_1(k^*) = O(T^{\frac{9}{4}})$, so that, from Equation (3-16), $C(\mathbf{r}(t)) = O(T^3)$, $\mathbf{r}(t)$ being given by the logarithmic spiral. One has

$$C_1(k^*) = \int_0^{R(T)} \left[E(r) - \frac{\pi R^3(T) r}{4a\sigma^2}\right]^2 dr$$
$$= \int_0^{R(T)} E^2(r)\,dr - \frac{\pi R^3(T)}{2a\sigma^2}\int_0^{R(T)} rE(r)\,dr + \frac{\pi^2 R^6(T)}{16a^2\sigma^4}\cdot\frac{R^3(T)}{3}. \tag{3-17}$$

Let us consider the right member of Equation (3-17). It is

$$\frac{\pi^2}{4a^2\sigma^4}\int_0^{R(T)} r^4\left(R^4(T) - R^2(T)r^2 + \frac{r^4}{4}\right)dr - \frac{\pi^2 R^3(T)}{4a^2\sigma^4}\int_0^{R(T)} r^3\left(R^2(T) - \frac{r^2}{2}\right)dr + \frac{\pi^2 R^9(T)}{48a^2\sigma^4}, \tag{3-18}$$

which reduces to one term, namely,

$$C_1(k^*) = \frac{\pi^2 R^9(T)}{2520a^2\sigma^4}. \tag{3-19}$$

Now $R(T) = (2\sigma)^{\frac{1}{2}}(aWvT/\pi)^{\frac{1}{4}}$, so that

$$C_1(k^*) = \frac{2(2\sigma)^{\frac{1}{2}}a^{\frac{1}{4}}(WvT)^{\frac{9}{4}}}{315\pi^{\frac{1}{4}}}. \tag{3-20}$$

From Equation (3-16), one sees that

$$c(\mathbf{r}(t)) = \frac{2(WvT)^2 T}{315}. \tag{3-21}$$

Let us now compare the right side of Equation (3-21) with the cost obtained for the optimal solution. The latter is the sum of two quantities obtained in Chapter 2, namely, the second member of Equation (2-45) for J_1 and the second member of Equation (2-50) for J_2. First, let us consider J_1. J_1 depends critically on the nature of t_1, so we want to study the asymptotic behavior of t_1 as T tends to infinity. Now t_1 is given by the right side of Equation (2-43), and we observe that it depends on the quantity $K = R(T)/vT = (2\sigma)^{\frac{1}{2}}\left(aW/\pi v^3T^3\right)^{\frac{1}{4}}$. Since K tends to zero as T tends to infinity, suppose that we let $x = 3K\sqrt{6}/8$ and study $Cos^{-1}(-x)$. Let α be such that

$$Cos^{-1}(-x) = \frac{\pi}{2} + \alpha. \tag{3-22}$$

From Equation (3-22), it follows that

$$-x = \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha, \tag{3-23}$$

so that $x = \sin\alpha$. Since x is close to zero when T is large, we see from Equation (3-22) that α will also be small. Thus, in Equation (3-23), we may replace $\sin\alpha$ by α itself and claim that, for small x,

$$Cos^{-1}(-x) \sim \frac{\pi}{2} + x, \tag{3-24}$$

the right side being the linear part of the Maclaurin series for $Cos^{-1}(-x)$. It follows that

$$\begin{aligned} t_1 &\sim \frac{2KT\sqrt{6}}{3}\cos\left[\frac{1}{3}\left(\frac{\pi}{2}+\frac{3K\sqrt{6}}{8}\right)+\frac{4\pi}{3}\right] \\ &= \frac{2KT\sqrt{6}}{3}\cos\left(\frac{3\pi}{2}+\frac{K\sqrt{6}}{8}\right) \\ &= \frac{2KT\sqrt{6}}{3}\sin\left(\frac{K\sqrt{6}}{8}\right). \end{aligned} \tag{3-25}$$

Now $\sin\left(\frac{K\sqrt{6}}{8}\right) \sim \frac{K\sqrt{6}}{8}$, so that, from Equation (3-25),

$$\begin{aligned} t_1 &\sim \frac{k^2T}{2} \\ &= \frac{R^2(T)}{2v^2T} \\ &= \sigma\left(\frac{aW}{\pi v^3T}\right)^{\frac{1}{2}}. \end{aligned} \tag{3-26}$$

Therefore, t_1 behaves like $T^{-\frac{1}{2}}$ as T becomes large. From Equation (2-45), we see that each term of J_1 tends to zero as t_1 tends to infinity, so that J_1 itself tends to zero. Indeed, its behavior is of the order of the last three terms, so that J_1 goes to zero like $T^{-\frac{3}{2}}$ as T goes to infinity.

Next consider J_2, given by the right side of Equation (2-50). We examine t_2, as provided by the second member of Equation (2-44). Again, t_2 depends upon K. Note first that

$$Cos^{-1}(-x) = \pi - Cos^{-1}(x), \tag{3-27}$$

valid for all x if we agree that Cos^{-1} is to be the principal arccos operator. Now let $x = 1 - y^2$, and consider $Cos^{-1}(1-y^2)$. We find that, for small positive y (x close to 1),

$$\begin{aligned} Cos^{-1}(1-y^2) &= Sin^{-1}\left(y\sqrt{2-y^2}\right) \\ &\sim y\sqrt{2}, \end{aligned} \tag{3-28}$$

where Sin^{-1} is understood as the principal arcsin function. Thus we find that

$$\begin{aligned} t_2 &\sim T\left(1-\frac{4K}{3}\left(1+\cos\left(\frac{1}{3}\left(\pi-\frac{3\sqrt{6K}}{4}\right)+\frac{2\pi}{3}\right)\right)\right) \\ &= T\left(1-\frac{4K}{3}\left(1+\cos\left(\pi-\frac{\sqrt{6K}}{4}\right)\right)\right) \\ &= T\left(1-\frac{4K}{3}\left(1-\cos\left(\frac{\sqrt{6K}}{4}\right)\right)\right) \\ &= T\left(1-\frac{8K}{3}\sin^2\frac{\sqrt{6K}}{8}\right) \\ &\sim T\left(1-\frac{K^2}{4}\right), \end{aligned} \tag{3-29}$$

so that t_2 tends to infinity like T itself as T tends to infinity. Thus, t_2^3 tends to infinity like T^3 as T goes to infinity. Now let us study carefully the expansion of J_2, as given by Equation (2-50). One finds, after a simple calculation and upon making use of Equation (3-29) and the definition of $R(T)$, that the first three terms may be combined to give

$$\frac{(T-t_2)\left(WvTK^2\right)^2}{48}, \tag{3-30}$$

which is $O\left(T^{-\frac{3}{2}}\right)$. The remaining terms in the expansion will likewise be seen to be $O\left(T^{-\frac{3}{2}}\right)$. The result is that J_2, along with J_1, is $O\left(T^{-\frac{3}{2}}\right)$. Therefore, the cost for the optimal solution is of order $T^{-\frac{3}{2}}$, compared to T^3 for the logarithmic spiral. It appears that the optimal solution is clearly superior to the spiral.

Next let us consider the elliptical Gaussian case. We want to define a curve called an *elliptical logarithmic spiral*, which, in the circular case, would become our conventional spiral. As indicated in Figure 3-1, the radial part of the circular spiral was linear; indeed, we found that $\rho(t) = R(T)t/T$. Therefore, for the elliptic case, again set $\rho(t) = R(T)t/T$, but with $\rho^2(t) = (x_1^2(t)/\sigma_1^2 + x_2^2(t)/\sigma_2^2)$. That is, $\rho(t)$ is now the elliptic norm of $\mathbf{r}(t) = (x_1(t), x_2(t))$. At this point, we invoke the procedure that we employed in Chapter 2 to obtain the optimal search path, but we employ that method in order to generate an elliptical spiral. Thus, let $y_1(t) = x_1(t)/\sigma_1$ and $y_2(t) = x_2(t)/\sigma_2$. In Equation (2-77) replace $f(t)$ by $\rho^2(t)$, where $\rho(t) = R(T)t/T$. In the nonphysical plane, one again has Equation (2-80) and Equation (2-81), and we substitute $\dot{r}_y(t) = \dot{f}(t)/2r_y(t)$ into Equation (2-81), *using our new f(t).* One then solves the nonlinear differential equation (2-82), subject to a convenient initial condition on θ_y. By analogy with the circular case, one could make $\theta_y(0)$ *large and negative*, for example. Also, one should choose the plus sign in Equation (2-82) to insure that the curve spirals outward from the origin.

Having defined our elliptic logarithmic spiral, let us, as in the circular Gaussian case, compare the cost function for the spiral with the cost function for the optimal path. Appealing to Equation (2-66) and Equation (2-67), we find that the cost for the logarithmic spiral is the same as that for the circular one, i.e., we again have Equation (3-21). Furthermore, the forms for J_1 and J_2, together with those for both t_1 and t_2, are still valid. Therefore, we find the same essential behavior that we obtained in the circular Gaussian case.

There is one situation in which the asymptotic behavior of the cost function for the spiral mimics that for the optimal solution. Suppose that one decides, for some reason, to let v vary with the budget time T such that $v(T) = cT^{-\alpha}$, where c is some positive constant. For the circular case, one knows that $R(T) < vT$ when $v^3 > 4aW\sigma^2/\pi T^3$, i.e., when $v > \left(\frac{4aW\sigma^2}{\pi}\right)^{\frac{1}{3}}/T$. This inequality can be obtained in two reasonable ways, namely, (1) by choosing any constant $c > (4aW\sigma^2/\pi)^{\frac{1}{3}}$ and letting $v = c/T$ or (2) by setting $v = (4aW\sigma^2/\pi)^{\frac{1}{3}}/T^{\alpha}$, $0 < \alpha < 1$. The former procedure will force the distance covered in time T to be a constant for all T and will therefore clearly not lead to a useful search

strategy. In contrast, Procedure (2) allows the search to expand as the budget time T increases, and we might like to adjust α so as to minimize the cost. Therefore, let us assume that

$$v = \left(\frac{4aW\sigma^2}{\pi}\right)^{\frac{1}{3}} / T^{\alpha}, \qquad 0 < \alpha < 1. \tag{3-31}$$

Then, appealing to Equation (3-21), we find that

$$C(\mathbf{r}(t)) = \frac{2W^2T^{3-2\alpha}}{315}, \qquad 0 < \alpha < 1. \tag{3-32}$$

Thus, if we let $\alpha = 1 - \epsilon$, where ϵ is small and positive, we can almost make the cost C linear in the time T and, at the same time, assure ourselves that the search expands in time. On the other hand, examine the optimal solution for the circular case. Consider J_1, as given by Equation (2-45). Noting that t_1 behaves like $T^{(3\alpha-1)/2}$ as T tends to infinity, it follows that $v^8 t_1^9 = O\left(T^{(11\alpha-9)/2}\right)$, which, for $\alpha = 1 - \epsilon$, is *almost* $O(T)$. The other terms in J_1 are all found to manifest the same behavior pattern, so that,*loosely speaking*, $J_1 = O(T)$. Similarly, one finds that the right side of Equation (2-50) is essentially $O(T)$, implying that the cost for the optimal solution is *essentially* $O(T)$, *provided that* $v(T) = O(1/T^{\alpha}), \alpha = 1 - \epsilon$, and ϵ is small and positive. It is clear that, for the elliptic case, the same statements must hold, since, apart from a few constants, the forms of J_1, J_2, t_1, t_2, and the cost for the spiral are the same. The choice of $v(T) = O(1/T^{\alpha})$, with α close to, but less than, 1 has the effect of *equilibrating the terms in* J_1 *and* J_2, *so that they all behave like* T. The cost per unit time, namely, $C(\mathbf{r}(t))/T$, is *almost* $O(1)$.

CHAPTER 4

AN OPTIMAL STOCHASTIC PROCESS FOR THE CASE OF A MOVING TARGET

The main effort of this report has been devoted to ascertaining the nature of optimal paths for acquiring targets known to be stationary in some two-dimensional Euclidean reference frame. If the target is moving, it is possible to adapt the theory we have developed so far in order to maximize one's chance of finding it (or, equivalently, to minimize the mean time to acquire it). In fact, we shall be led to a stochastic process, as opposed to a deterministic one, which one or more search vehicles must follow within the time T allowed.

The idea is conceptually very simple. Suppose, as shown in Figure 4-1, that we have located the nominal target position at some point in time, which we shall denote by t=0. That is to say, we know the nominal target range R_T and bearing B_T at $t = 0$. If we knew the target's velocity vector $\mathbf{v}_T$ and were willing to travel at a closing speed v_0 to intercept the most likely (nominal) position of the target after $t_I = R_T/v_0$ units of time, then we would have a deterministic trajectory for acquisition. In fact, we see that

$$\begin{aligned} \mathbf{x}_m(t) &= (\mathbf{v}_T + \mathbf{v}_0)t_I + \mathbf{v}_T(t - t_I) + \mathbf{x}_s(t - t_I) \\ &= \mathbf{v}_0 t_I + \mathbf{v}_T t + \mathbf{x}_s(t - t_I), \qquad 0 < t - t_I \leq T, \end{aligned} \tag{4-1}$$

where $\mathbf{x}_m(t)$ is our path for acquiring the moving target, t_I is nominal point intercept time, $\mathbf{v}_0$ is own ship (or possibly own missile) velocity when $\mathbf{v}_T = \mathbf{0}$, $\mathbf{v}_T$ is target velocity, and $\mathbf{x}_s(t-t_I)$ is our trajectory plan for detecting a stationary target. Since our search *in the moving frame of reference* (the frame stationary relative to the target) must start at $t = t_I$, the total time budget T is calculated relative to such intercept time. Shown in Figure 4-1 are some of the level curves of the target's Gaussian location distribution *in the moving reference frame.* The searcher's total velocity is $\mathbf{v}_0 + \mathbf{v}_T$.

So far there is only one distribution function for the target which needs to be considered, namely, that of the target within its own frame of reference. Since we know the target's velocity, we need only *travel with the target* while we move toward the intercept point (mode of the Gaussian distribution) and *continue to follow the target motion thenceforward.*

Now let us suppose that $\mathbf{v}_T$ is not known with certainty, but that it is some constant vector, whose distribution function is known a priori. One example of such a distribution function would be the following: The target speed v_T lies between two known limits v_L and v_U, and the target heading

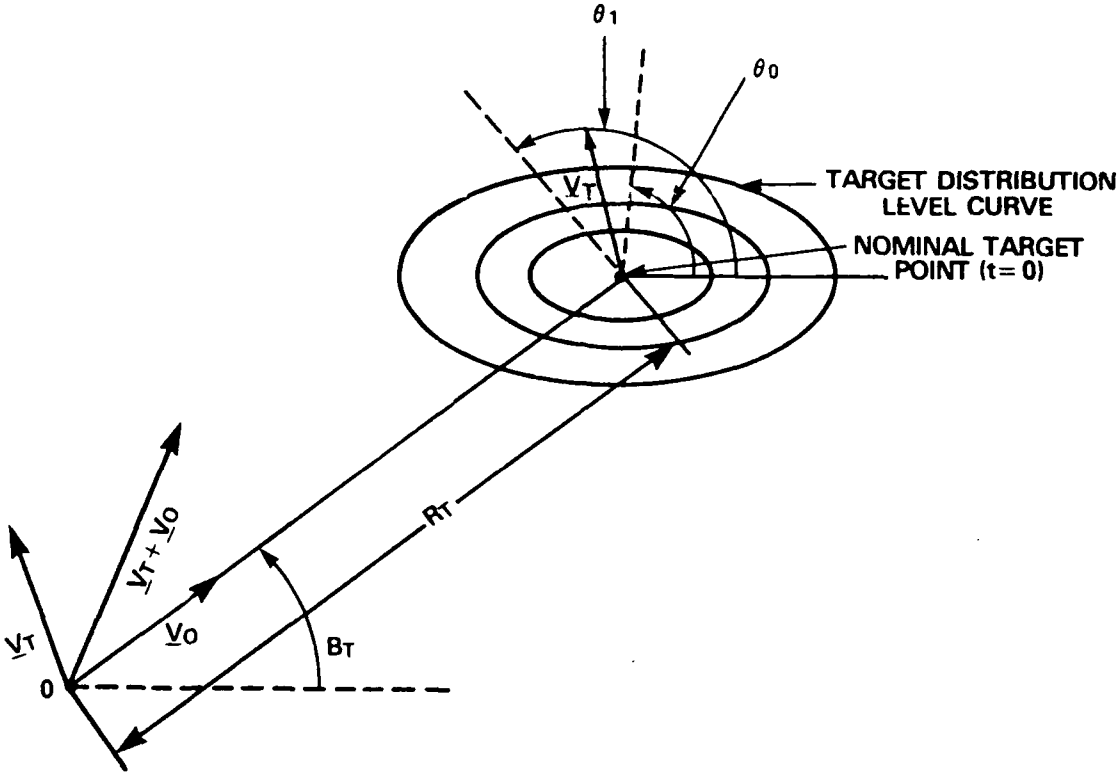

FIGURE 4-1. SEARCHER-TARGET GEOMETRY FOR MOVING TARGET DETECTION PROBLEM

θ_T is between θ_L and θ_U. Furthermore,

$$p(v_T, \theta_T) = \frac{1}{(v_U - v_L)(\theta_U - \theta_L)}, \qquad v_L \le v_T \le v_U, \theta_L \le \theta_T \le \theta_U. \tag{4-2}$$

That is, the target velocity is uniform over a known range of angles and speeds. Then we may conceive of Equation (4-1) as a *stochastic path process* for acquiring the target, and we could, in fact, design strategies for target acquisition based on our knowledge of the process. If, for example, one is limited to a single search vehicle, then it might be reasonable to follow the mean of the process and hope for the best. Thus,

$$\overline{\mathbf{x}}_m(t) = E(\mathbf{x}_m(t)) = \mathbf{v}_0 t_I + tE(\mathbf{v}_T) + \mathbf{x}_s(t - t_I), \qquad t_I < t \le t_I + T, \tag{4-3}$$

so that $\overline{\mathbf{x}}_m(t)$ is again a deterministic search mechanism. On the other hand, if several vehicles could be released from point O in Figure 4-1 at roughly the same time, we could try to follow the stochastic process itself at some level. That is, let

$$\mathbf{x}_m^{(k)}(t) = \mathbf{v}_0 t_I + \mathbf{v}_T^{(k)} t + \mathbf{x}_s(t - t_I), \qquad 1 \le k \le NV, \tag{4-4}$$

where $\mathbf{x}_m^{(k)}(t)$ is realization k of stochastic motion $\mathbf{x}_m(t)$ for search vehicle k, NV being the number of vehicles available.

One might imagine several variants of the stochastic motion we have presented here, depending upon known characteristics of target motion and/or searcher motion. For example, suppose that the target is known to be decelerating over time in some given direction, having started from either a known or unknown speed at time 0. Then the generic equation (4-1) is still operative; only the distribution given by Equation (4-2) need be altered. One candidate might be

$$p(v_T^{(I)}, \theta_T, d_T) = \frac{1}{(v_U^I - v_L^I)(\theta_U - \theta_L)(d_U - d_L)}, \tag{4-5}$$

where $v_T^{(I)}$ is the initial target speed and d_T is the constant target deceleration (a negative deceleration is interpretable as an acceleration).

Of course, one may learn over time (if there is enough time to do so), thus allowing one to distribute a number of search vehicles on a tentative basis. These vehicles might be cheap, dispensable items. Once the target's motion is known with greater certainty, a more expensive search mechanism could be employed, perhaps along the expected course (presumably then having small variance characteristics).

CHAPTER 5

SUMMARY AND FUTURE WORK

We have shown in this report how one can use optimal control theory to design tracking procedures which maximize the acquisition probability for either stationary or moving targets. It is assumed that one is given a time budget T in which to operate, that one may choose his search speed v within reasonable operating limits, and that one will initiate search at the most likely target point. We have compared the efficiency of optimal search plans with logarithmic spiral plans, which are obviously suboptimal, and we found the following: If speed is not chosen carefully as a function of budget time, the optimal search is clearly superior to the spiral search. However, if one is careful to adjust the search speed and, in particular, if one moves relatively slowly, both optimal search and logarithmic spiral search have similar characteristics for large budget times.

If one does not(or cannot) initiate search from the most likely target point, he could be forced to pursue a different track. However, in certain circumstances it would be expected that a simple reversal of an optimal path mechanism or a logarithmic spiral search would be applicable. For example, in a radially symmetric situation, as would be encountered when the target distribution is circular normal, if one had to start search on the boundary of some disk *at a prescribed point*, working into the disk, he should probably reverse the path which would have normally evolved from the center (where the target was most likely to be). On the other hand, an elliptical distribution would present a different situation if one had to start at a point not on the minor axis. That case has not been studied in this report.

Future work would address the important issue of tracking targets in the presence of either false targets or aggressor targets whose influence regions overlap those of the target of interest. Also, numerical comparisons between optimal and suboptimal paths developed here might be of interest, as well as comparisons between them and paths actually utilized in practical situations.

REFERENCES

1. Washburn, A. R., *Search and Detection*, Military Applications Section, Operations Research Society of America, May 1981.

2. Koopman, B. O., *Search and Screening*, Pergamon Press, New York, NY, 1980.

3. Reber, R. K., *A Theoretical Evaluation of Various Search Salvage Procedures for Use with Narrow-Path Locators, Part I, Area and Channel Searching*, Tech. Rep. 117, Bureau of Ships, Dept. of the Navy, 1 Jan 1965.

4. *National Search and Rescue Manual*, United States Coast Guard, Washington, DC, 1959.

5. Hestenes, M. R., *Calculus of Variations and Optimal Control Theory*, Wiley, New York, NY, 1966.

6. Stone, L. D., *Theory of Optimal Search*, Academic Press, New York, NY, 1975.

7. Hodgman, C. D., *Mathematical Tables from Handbook of Chemistry and Physics*, Tenth Edition, Chemical Rubber Publishing Co., Cleveland, OH, 1954.

8. Lee, E. B. and Markus, L., *Foundations of Optimal Control Theory*, Wiley, New York, NY, 1967, pp. 259-262.

9. Natanson, I. P., *Theory of Functions of a Real Variable, Vol. 1*, Ungar Publishing Co., New York, NY, 1955.

10. Bellman, R., *Introduction to Matrix Analysis*, McGraw-Hill, New York, NY, 1960.

11. Birkhoff, G. and Rota, G., *Ordinary Differential Equations*, Blaisdell, Waltham, MA, 1962.

12. Randolph, J. F. and Kac, M., *Analytic Geometry and Calculus*, Macmillan, New York, NY, 1954.

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