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Maximum Likelihood Estimation applied to Target Motion Analysis using Second Order Derivatives

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ABSTRACT (UNCLASSIFIED)

The tracking problem of unknown marine platforms using passive sonar measurements is generally referred to as *Target Motion Analysis* (TMA). This report describes the application of a *Maximum Likelihood Estimation* (MLE) method to obtain position and velocity estimates of a platform when bearing angle and frequency measurements are available from a passive sonar system. The frequency measurements are related to one or more cardinal frequency peaks from the radiated frequency spectrum by the platform.

The MLE method offers the opportunity to use a multi-leg model, i.e. the platform to be localized is assumed to move according to a piecewise linear track, where each part is referred to as a leg. On each leg constant course and speed is assumed. Additionally, bearing and frequency measurements related to bottom reflections of the acoustical signals can be used.

In ref. [Gmelig Meyling, 1989-1] a Newton-type optimization method using first and second order derivative information of the residual functions is proposed. By using the analytic expressions of the derivatives as decribed in this report a major reduction of computation time is accomplished.

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SAMENVATTING (ONGERUBRICEERD)

In het algemeen wordt het volgprobleem m.b.t. onbekende platforms op zee waarbij passieve sonar metingen worden gebruikt *Target Motion Analysis* (TMA) genoemd. Dit rapport beschrijft de toepassing van een *Maximum Likelihood schattingsmethode* (MLE) om de bewegingsparameters van een platform te schatten uit peilings- en frequentiemetingen die m.b.v. een passief sonarsysteem worden bepaald. De frequentiemetingen zijn gerelateerd aan een of meer pieken uit het door het platform uitgezonden frequentiespectrum.

De MLE methode geeft de mogelijkheid een *multi-leg* model te gebruiken, d.w.z. het platform wordt verondersteld te varen volgens een stuksgewijs lineaire baan, waarbij elk stuk een *poot* genoemd wordt. Op elke poot wordt een constante koers en vaart verondersteld. Tevens is het mogelijk metingen te gebruiken die afkomstig zijn van bodemgereflecteerde akoestische signalen.

In ref. [Gmelig Meyling, 1989-1] is een Newton-type optimalizatiemethode voorgesteld die gebaseerd is op de eerste en tweede afgeleiden van de residufuncties. Een belangrijke reduktie in de benodigde rekentijd wordt bereikt door gebruik te maken van de in dit rapport beschreven analytische uitdrukkingen voor de afgeleiden.

1 INTRODUCTION

The tracking problem of unknown marine platforms using sonar measurements is generally referred to as ***Target Motion Analysis*** (TMA). This report describes the application of a ***Maximum Likelihood Estimation*** (MLE) method to obtain position and velocity estimates of a platform when bearing angle and frequency measurements are available from a passive sonar system, such as a towed array. The frequency measurements are related to one or more cardinal frequency peaks from the radiated frequency spectrum by the platform.
The estimation problem is formulated as a Maximum Likelihood Estimation problem (MLE) which is both non-linear and poorly conditioned. In literature [Lindgren 1978, Aidala 1979, 1982, 1983] much attention is paid to the application of ***Kalman Filters*** (KF) to solve bearings-only TMA problems. In [Nardone, 1984] the fundamental properties if the bearins-only TMA problem are discussed. An extended Kalman filter for the bearing- and frequency measurement TMA problem is proposed in [Ockeloen and Willemsen, 1982]. However, in many situations Kalman filters suffer from unacceptable bias and slow convergence caused by inappropriate linearization of the non-linear TMA equations.
By using a proper numeric optimization method to obtain an ML estimate the disadvantages of Kalman filters have been overcome at the cost of more computational effort.
The optimization method in [Gmelig Meyling, 1989] is based on an ***Corrected Gauss-Newton*** method combined with a ***Newton*** method and an ***Active Set*** method to account for additional constraints on the parameters to be estimated. The MLE approach offers the opportunity to use a multi-leg model, i.e. the platform to be localized is assumed to move according to a piecewise linear track, where each part is referred to as a leg. On each leg uniform linear motion is assumed. The multi-leg model is described as a linear state model in Cartesian coordinates with non-linear measurement equations. The MLE problem, however, is not formulated in cartesian coordinates since it aims to estimate parameters

which are used during real-time operation on board. The method also accounts for bearing and frequency measurement equations related to bottom reflections of the acoustic signals.
The Gauss-Newton method requires only the first order derivative information of the residuals, whereas the Newton method also makes use of the second order derivatives. In both methodes either finite difference approximations or explicit analytic expressions of the derivatives can be used. However, applying the explicit analytic expressions leads to a large reduction in computation time, which is much more attractive for real-time operation of the method.
This report has two objectives. First it recapitulizes the theoretical background of the MLE method in view of the TMA problem. Second, analytic expressions for the first and second order derivatives of the residual with respect to the target parameters are presented. The expressions are required by the Modified Newton Method as proposed in [Gmelig Meyling, 1989-1].

The report is organized as follows. In Chapter 2 the TMA problem is formulated. Section 2.2 introduces the multi-leg motion model, Section 2.3 discusses the parameters to be estimated and their relation to the multi-leg model. In Section 2.4 the measurement equations are formulated for both direct-path and bottom bounce propagation.
In Chapter 3 the MLE method is introduced and some properties of the MLE are discussed such as bias and the Cramer-Rao lowerbound. In Chapter 4 analytic expressions for the residual derivatives are determined.
Readers who are interested in the results only are recommended to skip reading Chapter 4 and just to focus on the tables of Chapter 4, which are recapitulized in Appendix B.

2 TMA MODEL

2.1 Preliminaries

Throughout this report the following notation is used:

- Vectors and matrices are denoted in bold lower and upper case respectively,
- Superscripts in parenthesis denote *leg-index* numbers.
- Variables related to own ship or target ship information have the superscript OS or TS respectively,
- Similar superscripts are used for *direct-path* (DP) and *bottom-bounce* (BB) variables.

Two cartesian coordinate systems (X,Y) will be used where the x- and y-axis are related to the geographical east and north. *Absolute Cartesian coordinates* are defined such that the origin represents a fixed geographical point, usually the own-ship position at the time instant of the first measurement. The second, a *relative Cartesian coordinate system*, moves along the own-ship track such that the own-ship stays at the origin.
The advantage of Cartesian coordinates is that the absolute or relative motions of both platforms can be described straightforward. So, the relative uniform linear motion at time instant t_k is represented by $[x_k, y_k, \dot{x}_k, \dot{y}_k]$ where x_k, y_k are the relative position and $\dot{x}_k$, $\dot{y}_k$ are the relative velocity components. The variables with superscript OS and TS are absolute position and velocity components:

$$\begin{aligned} x_k &= x_k^{TS} - x_k^{OS} \\ y_k &= y_k^{TS} - y_k^{OS} \\ \dot{x}_k &= \dot{x}_k^{TS} - \dot{x}_k^{OS} \\ \dot{y}_k &= \dot{y}_k^{TS} - \dot{y}_k^{OS} \end{aligned} \tag{2.1}$$

The position and velocity of the target will also be described in polar coordinates $[b_k, R_k, C_k, v_k]$, where b_k is the absolute bearing angle taken from the geographical north (the y-axis of the cartesian system), R_k is the range between target ship and own ship, C_k is the target ship course angle taken from north, and v_k is the target ship speed at time t_k. Note that the polar position coordinates are *relative* with respect to the own ship while the polar velocity components describe the *absolute* velocity of the target platform. The following relations between Cartesian and polar variables hold:

$$
\begin{aligned}
b_k &= \mathrm{ATAN2}(x_k, y_k) \\
R_k &= (x_k^2 + y_k^2)^{\frac{1}{2}} \\
C_k &= \mathrm{ATAN2}(\dot{x}_k^{TS}, \dot{y}_k^{TS}) \\
v_k &= (\dot{x}_k^{TS^2} + \dot{y}_k^{TS^2})^{\frac{1}{2}}
\end{aligned}
\qquad (2.2)
$$

where the ATAN2 function is defined by

$$
\mathrm{ATAN2}(x,y) = \begin{cases} \mathrm{ARCTAN}(x/y) & y > 0 \\ \pi + \mathrm{ARCTAN}(x/y) & y < 0,\ x > 0 \\ -\pi + \mathrm{ARCTAN}(x/y) & y < 0,\ x < 0 \\ \pi/2 & y = 0,\ x > 0 \\ -\pi/2 & y = 0,\ x < 0 \end{cases} \qquad (2.3)
$$

Figure 2.1 illustrates the definition of the position and velocity parameters just described.

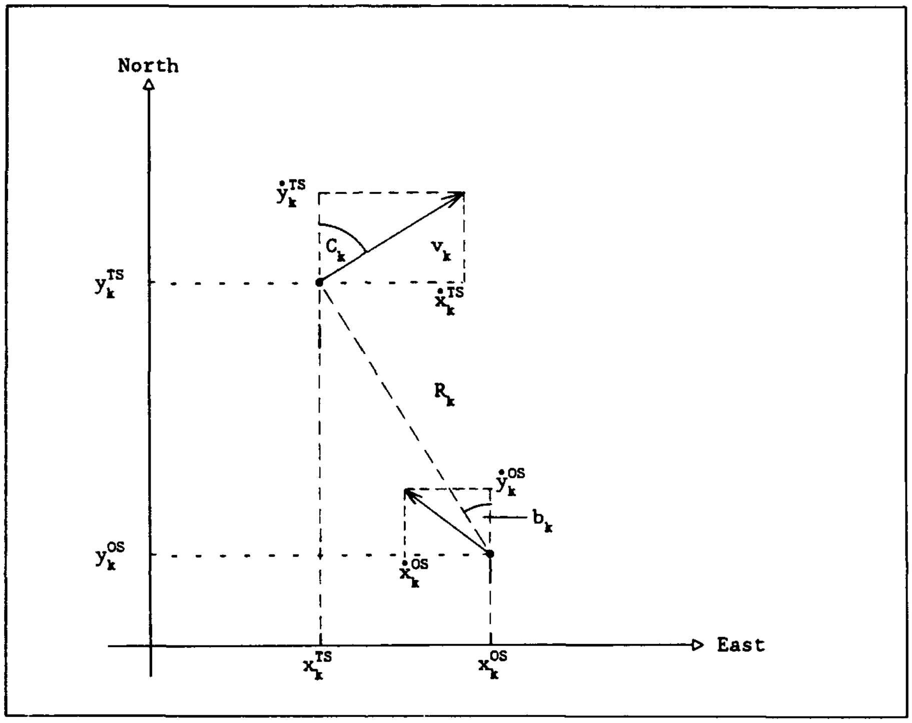

Fig. 2.1: Position and velocity variables at time t_k

2.2 Multi-Leg Model

The purpose of TMA is to estimate the position, velocity and frequency parameters of an unknown platform from a set measurements (z_k) at measurement times (t_k). The measurements consist of bearing angles and frequency measurements from the sonar system, which can be obtained from direct path or bottom bounce trajectories to the targetship. Moreover, measurements about range, speed or course of the platform can be added to the set. In the following we assume piecewise uniform linear motion of the ***target ship*** (TS) and known position and velocity of the ***own ship*** (OS).

The TS motion can be described by the following difference equations in cartesian coordinates:

$$x_N^{TS} = x_k^{TS} + \sum_{i=1}^{m} v_x^{(i)} \cdot T_{N,k}^{(i)}$$

$$y_N^{TS} = y_k^{TS} + \sum_{i=1}^{m} v_y^{(i)} \cdot T_{N,k}^{(i)} \qquad (2.4)$$

where $v_x^{(i)}$, $v_y^{(i)}$ are the absolute velocity components of the target ship which are assumed to be constant during leg i. The total number of legs is equal to m:

$$\dot{x}_k^{TS} = v_x^{(i)} \qquad \forall k: \quad \tau^{(i-1)} \le t_k < \tau^{(i)} \quad , \; i = 1,\ldots,m \qquad (2.5)$$
$$\dot{y}_k^{TS} = v_y^{(i)}$$

where $\tau^{(i-1)}$ and $\tau^{(i)}$ indicate the beginning and end time of leg i, for i = 1, ...,m. In this report the manoeuvre $\tau^{(i)}$ of the target ship are assumed to be known, i.e. these values may be obtained by a manoeuvre detection procedure. The time periods $T_{N,k}^{(i)}$ are defined as

$$T_{N,k}^{(i)} = \max(\tau^{(i)} - t_k,\ 0) + \max(\tau^{(i-1)} - t_N,\ 0) - \max(\tau^{(i)} - t_N,\ 0) - \max(\tau^{(i-1)} - t_k,\ 0) \qquad (2.6)$$

The muli-leg track is illustrated by Figure 2.2.

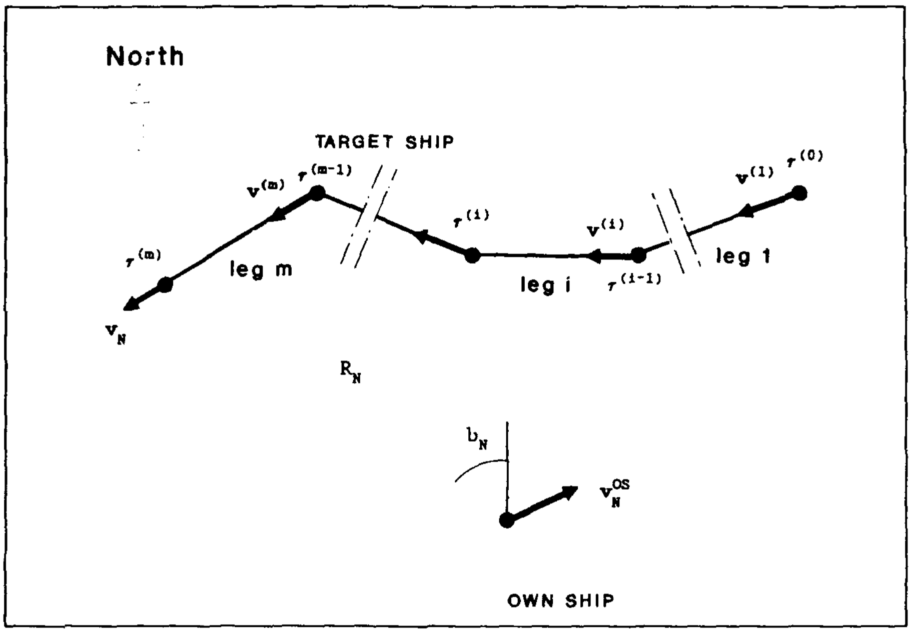

Fig. 2.2: The multi-leg model.

The multi-leg model is determined by the following discrete-time state equation:

$$\mathbf{x}_N = \Phi_{N,k}\mathbf{x}_k \tag{2.7}$$

where

$$\mathbf{x}_k = [\ x_k^{TS} \quad y_k^{TS} \quad v_x^{(1)} \quad v_y^{(1)}, \ \ldots\ldots, \ v_x^{(m)} \quad v_y^{(m)}\ ]^T \tag{2.8}$$

$$
\Phi_{N,k} = \begin{bmatrix}
1 & 0 & T^{(1)}_{N,k} & 0 & T^{(2)}_{N,k} & 0 & \dots & T^{(m)}_{N,k} & 0 \\
0 & 1 & 0 & T^{(1)}_{N,k} & 0 & T^{(2)}_{N,k} & \dots & 0 & T^{(m)}_{N,k} \\
0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1
\end{bmatrix} \tag{2.9}
$$

If the model contains m legs the state x_k contains $2(m+1)$ variables which have to be estimated. Additional to these variables we also like to estimate q unknown source frequency variables f_{01}, f_{02}, ..., f_{0q}. These variables are assumed to result from cardinal peaks in the frequency spectrum of the target platform. The corresponding frequency measurements contain a doppler shift caused by the radial motion components of OS and TS. Section 2.3 discusses the corresponding measurement equations in detail. A frequency measurement related to time t_k and source frequency j will be denoted by f_{kj}.

The state equation (2.7) is now expanded by q additional equations:

$$
f_{0j}(t_N) = f_{0j}(t_k) \qquad \forall k \tag{2.10}
$$

The result is a $2(m+1)+q$ dimensional state vector

$$
x_k = [\ x_k \quad y_k \quad v_x^{(1)} \quad v_y^{(1)}, \ \dots, \ v_x^{(m)} \quad v_y^{(m)} \quad f_{01}, \ \dots, \ f_{0q}\ ]^T \tag{2.11}
$$

and a transition matrix

$$\Phi_{N,k} = \left[\begin{array}{c|ccc|c} I_2 & T^{(1)}_{N,k}\cdot I_2 & \ldots & T^{(m)}_{N,k}\cdot I_2 & O_{2,q} \\ \hline O_{2m,2} & & I_{2m} & & O_{2m,q} \\ \hline O_{q,2} & & O_{q,2m} & & I_q \end{array}\right] \tag{2.12}$$

where I_n and $O_{m,n}$ indicate a $n \times n$ identity matrix and a $m \times n$ zero matrix respectively. In the sequel of this report we will mainly use $\Phi_{k,N}$ which is the inverse of $\Phi_{N,k}$, i.e. $T^{(i)}_{N,k} = - T^{(i)}_{k,N}$.

2.3 Polar coordinates

For operational usage it is more convenient to use relative polar position coordinates and absolute polar velocity coordinates per leg. We introduce the polar state vector as

$$y_k = [\ b_k \quad R_k \quad C^{(1)} \quad v^{(1)}, \ldots, \quad C^{(m)} \quad v^{(m)} \quad f_{01}, \ldots, f_{0q}\ ] \tag{2.13}$$

with b_k the absolute bearing angle, R_k the range, $C^{(i)}$, $v^{(i)}$ the Course and Speed of the platform at leg i and f_{0j} the j^{th} source frequency. The transition from y_k to y_N can be determined by using the vector transformation function F: $\mathbf{x} \to \mathbf{y}$ and its inverse G: $\mathbf{y} \to \mathbf{x}$:

$$\begin{aligned} y_k &= F(\mathbf{x}_k) \\ \mathbf{x}_k &= G(\mathbf{y}_k) \end{aligned} \tag{2.14}$$

The transformations F and G are determined by

$$F: \quad \begin{aligned} b_k &= \text{ATAN2}(x_k, y_k) \\ R_k &= (x_k^2 + y_k^2)^{\frac{1}{2}} \\ C^{(i)} &= \text{ATAN2}(v_x^{(i)}, v_y^{(i)}) \\ v^{(i)} &= (v_x^{(i)2} + v_y^{(i)2})^{\frac{1}{2}} \\ f_{0j} &= f_{0j} \end{aligned} \qquad \begin{aligned} \\ \\ i &= 1, \ldots, m \\ \\ j &= 1, \ldots, q \end{aligned} \tag{2.15}$$

$$G: \quad \begin{aligned} x_k &= R_k \cdot \text{SIN}\, b_k \\ y_k &= R_k \cdot \cos b_k \\ v_x^{(i)} &= v^{(i)} \cdot \text{SIN}\, C^{(i)} \\ v_y^{(i)} &= v^{(i)} \cdot \cos C^{(i)} \\ f_{0j} &= f_{0j} \end{aligned} \qquad \begin{aligned} \\ \\ i &= 1, \ldots, m \\ \\ j &= 1, \ldots, q \end{aligned}$$

Note that the own-ship position x_k^{OS}, y_k^{OS} is used implicitely by the functions $F(\mathbf{x}_k)$ and $G(\mathbf{y}_k)$.
The transition from $\mathbf{y}_N$ to $\mathbf{y}_k$ and vice versa is now determined by

$$\begin{aligned} \mathbf{y}_N &= F(\Phi_{N,k} \cdot G(\mathbf{y}_k)) \\ \mathbf{y}_k &= F(\Phi_{k,N} \cdot G(\mathbf{y}_N)) \end{aligned} \tag{2.16}$$

Note that the own ship position are assumed to be known in order to be able to carry out the transformation.

2.4 Measurement Equations

The set of measured data generally consists of bearing angles and frequency data which result either from a direct acoustic path (DP) or from bottom bounce reflections (BB). The acoustic path is assumed to be known, e.g., by using an acoustic propagation prediction model.

Moreover, note that the multi-leg model does not take account of variations in depth. Other type of measurements such as range, course and speed of the target may be exploited (see Section 2.4.2). These measurements may be available from other platforms or from information about the underwater sound propagation conditions. Section 2.4.1. deals with typical passive sonar measurements.

2.4.1 Direct path and bottom-bounce measurements

In case of direct path acoustics the equations describing the measured bearings and frequencies are relative simple

$$z_k^b = b_k + \nu_k^b$$
$$z_j^f = f_{kj} + \nu_{kj}^f \tag{2.17}$$

where b_k is defined by (2.2) and f_{kj} is specified by the following relation:

$$f_{kj} = f_{0j} - f_{0j}\frac{\dot{r}_k}{c} \tag{2.18}$$

The scalar c represents the sound velocity in water (1500m/s) and $\dot{r}_k$ is the range rate (the relative radial velocity component) at time t_k. The measurement noise is represented by Gaussian noise processes ν_k^b and ν_{kj}^f with normal distribution functions $N(0,\sigma_k^b)$ and $N(0,\sigma_{kj}^f)$.
The second term of the right-hand side in equation (2.18) is an approximation of the doppler shift. Here we have assumed that the measurement noise ν_{kj}^f has a larger magnitude than the approximation error of the doppler term. Moreover, equation (2.2) and (2.18) only hold in cases where the OS and TS do not differ in depth. In case the target and the own ship in reality move in different horizontal planes or the bearing angles are related to bottom bounce reflections, the line of sight is projected onto a horizontal plane. Figure 2.3 illustrates the underlying model for the direct-path and bottom-bounce measurements. The elevation angle ψ depends on the range and the difference in depth D between TS and OS. The constant D is either equal to the real difference

D_{DP} or to the apparent difference in depth D_{BB} in case of direct path or bottom bounce signals, respectively:

$$
\begin{aligned}
D_{DP} &= D_{TS} - D_{OS} \\
D_{BB} &= 2D_{DB} - D_{TS} - D_{OS}
\end{aligned}
\tag{2.19}
$$

where D_{DP}, D_{BB} and D_B are the target ship depth, the own ship depth and the sea depth. In most long-range cases the difference D_{TS} -D_{OS} is negligible. The measured absolute bearing angles b^{DP} and b^{BB} are obtained from:

$$
\begin{aligned}
b_k^{DP} &= C_k^{OS} + \beta_k^{DP} \\
b_k^{BB} &= C_k^{OS} + \beta_k^{BB}
\end{aligned}
\tag{2.20}
$$

where β_k^{DP} or β_k^{BB} is the angle between the projected line of sight and the OS course as measured by the towed array sonar. β_k^{DP} and β_k^{BB} depend on the elevation angle ψ. If α^{DP} and α^{BB} are defined as the cosine of the elevation angles ψ_{DP} and ψ_{BB} respectively,

$$
\begin{aligned}
\alpha^{DP} &= \cos\psi_{DP} = \frac{R}{(R^2+D_{DP}^2)^{\frac{1}{2}}} \\
\alpha^{BB} &= \cos\psi_{BB} = \frac{R}{(R^2+D_{BB}^2)^{\frac{1}{2}}}
\end{aligned}
\tag{2.21}
$$

then the bearing angles β_k^{DP} and β_k^{BB} are equal to

$$
\begin{aligned}
\beta_k^{DP} &= \begin{cases} \mathrm{ARCCOS}(\alpha_k^{DP} \cos\beta_k)\ , & 0 \le \beta_k \le \pi \\ -\mathrm{ARCCOS}(\alpha_k^{DP} \cos\beta_k)\ , & -\pi \le \beta_k < 0 \end{cases} \\
\beta_k^{BB} &= \begin{cases} \mathrm{ARCCOS}(\alpha_k^{BB} \cos\beta_k)\ , & 0 \le \beta_k \le \pi \\ -\mathrm{ARCCOS}(\alpha_k^{BB} \cos\beta_k)\ , & -\pi \le \beta_k < 0 \end{cases}
\end{aligned}
\tag{2.22}
$$

where β_k represents the actual azimuth of the TS with respect to the OS course:

$$b_k = C_k^{OS} + \beta_k \tag{2.23}$$

In case of large range values the difference in depth D_{TS}-D_{OS} can be neglected. The result is that b_k^{DP} becomes equal to b_k of (2.15).

When receiving bottom-bounce signals the frequency measurements will have a different doppler component compared to the direct-path case. The relative radial velocity between TS and OS is represented by $\dot{r}$. The doppler shift is proportional to the projection of $\dot{r}$ on the bearing line of the BB bearing angle:

$$\dot{r}^{DP} = \dot{r} \cos \psi_{DP} \quad = \alpha^{DP} \cdot \dot{r} \tag{2.24}$$

$$\dot{r}^{BB} = \dot{r} \cos \psi_{BB} \quad = \alpha^{BB} \cdot \dot{r} \tag{2.25}$$

The result is

$$f_{kj}^{DP} = f_{0j}(1 - \alpha_k^{DP}\frac{\dot{r}_k}{c})$$

$$f_{kj}^{BB} = f_{0j}(1 - \alpha_k^{BB}\frac{\dot{r}_k}{c}) \tag{2.26}$$

where f_{0j} is the source frequency and c is the sound velocity.
In case of bottom-bounce the measurements are denoted by

$$\begin{aligned} z_k^b &= b_k^{BB} + \nu_k^b \\ z_{kj}^f &= f_{kj}^{BB} + \nu_{kj}^f \end{aligned} \tag{2.27}$$

where ν_k^b and ν_{kj}^f are Gaussian noise processes as defined above.

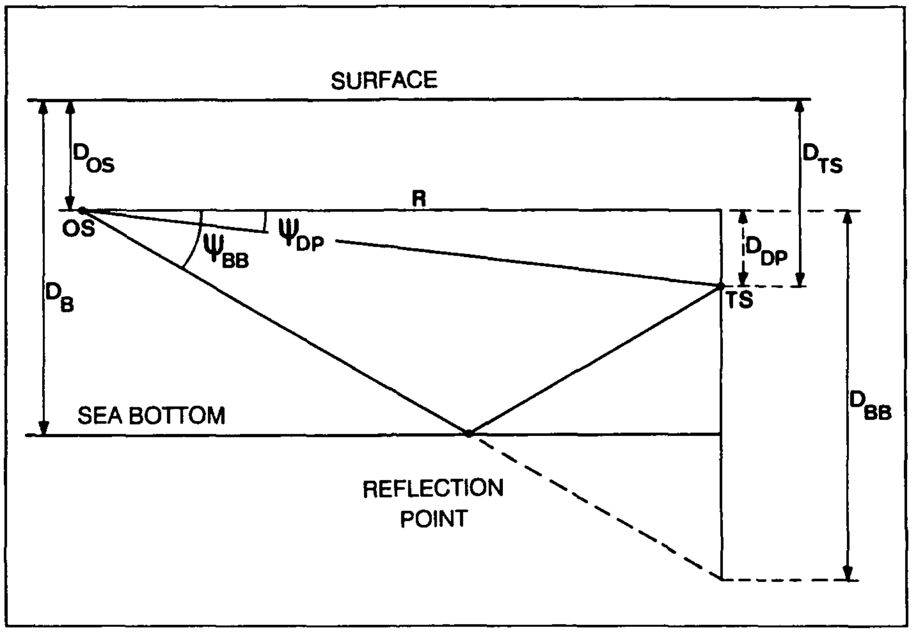

Fig. 2.3: Bottom-bounce bearing and frequency measurements.

Note that b_k^{DP} and f_{kj}^{DP} in (2.20, 2.22) and (2.27) reduce to simple range independent forms if the difference in depth D^{DP} between OS and TS is assumed to be zero. Moreover, if the angle between the line of sight and the towed array is 90 degrees one cannot distinguish DP from BB situations.

2.4.2 Additional measurements.

The additional measurements about range, course and speed are denoted by

$$\begin{aligned} z_k^R &= R_k + \nu_k^R \\ z_k^{C(i)} &= C^{(i)} + \nu_k^C \\ z_k^{v(i)} &= v^{(i)} + \nu_k^v \end{aligned} \tag{2.28}$$

where $\nu_k^R \sim N(0,\sigma_k^R)$, $\nu_k^C \sim N(0,\sigma_k^C)$ and $\nu_k^v \sim N(0,\sigma_k^v)$ are Gaussian noise processes.

3 MAXIMUM LIKELIHOOD ESTIMATION

3.1 TMA formulated as an MLE problem

This Section deals with the formulation of the general TMA estimation problem assuming Gaussian distributed measurement noise. Suppose the vector y_N must be estimated from the measurement set Z_N which contains all types of measurements z_k discussed in Section 2.4. Let's denote z_k^* as the variable to be measured at t_k:

$$\begin{aligned} z_k^* &= h_k(y_k) \\ z_k &= z_k^* + \nu_k^z \qquad\qquad (3.1) \\ Z_N &= \{z_k,\ k = 1,\dots,N\} \end{aligned}$$

where ν_k^z denotes the Gaussian noise process N $(0,\sigma_k^z)$ corresponding to z_k as defined in Section 2.4. The type of the function $h_k(y_k)$ depends on the measurement type at time t_k, i.e. h_k is defined by (2.20) through (2.23) in case of a bearing measurement or (2.26) in case of a frequency measurement. The a posteriori probability density function is denoted by $p(y_N|Z_N)$. The ***maximum a posteriori*** (MAP) estimate is found for $y_N = y_{N|N}$ such that $p(y_N|Z_N)$ attains its maximum value. Using the Bayes rule and taking the logarithm leads to

$$\ln p(y_N|Z_N) = \ln p(Z_N|y_N) + \ln p(y_N) - \ln p(Z_N) \qquad (3.2)$$

Since the last term does not depend on y_N the MAP estimate can be found by minimizing the function

$$l(y_N) = -\ln p(Z_N|y_N) - \ln p(y_N) \qquad (3.3)$$

where the last term represents the a priori knowledge on y_N.
If there is no a priori knowledge the MAP estimate reduces to the maximum likelihood estimate, i.e. the last term of (3.3) is constant when $p(y_N)$ is a uniform probability density function. Note that when a priori knowledge is regarded as an additional measurement which also has Gaussian statistics the MLE problem is in fact a special case of MAP estimation.

The MLE problem is formulated as

$$\min_{y_N} -\mathrm{LN}\ p(Z_N|y_N) \tag{3.4}$$

Since the measurements in Z_N are assumed to be uncorrelated and distributed according to $N(0,\sigma_k^2)$, i.e.

$$p(z_k|z_k^*) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_k^2}\ \mathrm{EXP}\ \left(-\tfrac{1}{2}\left(\frac{z_k-z_k^*}{\sigma_k^2}\right)^2 \right) \tag{3.5}$$

The likelihood function $p(Z_N|y_N)$ can be expressed as

$$\begin{aligned} p(Z_N|y_N) &= p(z_1,\dots,z_N|y_N) = \prod_{k=1}^{N} p(z_k|y_N) \\ &= \prod_{k=1}^{N} p(z_k|z_k^*) \\ &= \left((2\pi)^{-N/2}\prod_{k=1}^{N}\frac{1}{\sigma_k^2}\right)\mathrm{EXP}\left(-\tfrac{1}{2}\sum_{k=1}^{N}\left(\frac{z_k-z_k^*}{\sigma_k^2}\right)^2\right) \end{aligned} \tag{3.6}$$

The negative log-likelihood function becomes equal to

$$-\mathrm{LN}\ p(Z_N|y_N) = \tfrac{1}{2}\ell(Z_N,y_N) + \mathrm{LN}\ (2\pi)^{N/2} + \sum_{k=1}^{N}\mathrm{LN}\ \sigma_k^2 \tag{3.7}$$

where $\ell(y_N)$ is equal to

$$\ell(Z_N, y_N) = \sum_{k=1}^{N} \left(\frac{z_k - z_k^*}{\sigma_k^z}\right)^2 \tag{3.8}$$

In the following the ***weighted residual*** $r_k(y)$ is defined as

$$r_k(y) = \frac{z_k - h_k(y)}{\sigma_k^z} \tag{3.9}$$

Further we denote the estimate of y_k and z_k^* based on Z_N by $y_{k|N}$ and $z_{k|N}$ respectively. Now the MLE problem is reformulated as a non-linear optimization problem:

$$\min_{y_{N|N}} \quad \ell(Z_N, y_{N|N}) \tag{3.10}$$

A necessary condition for obtaining the ML estimate is

$$\frac{\partial \ell(Z_N, y_{N|N})}{\partial y_{N|N}} = 0$$

which leads to

$$g(y_{N|N}) = \sum_{k=1}^{N} r_k \cdot \frac{\partial r_k(y_{N|N})}{\partial y_{N|N}} = 0 \tag{3.11}$$

However, since the squared terms are non-linear in $y_{N|N}$ we have to use numerical methods to solve the MLE problem.

3.2 Numerical methods

In this section, only the outline of the Gauss-Newton and Newton method is given. Extensive details can be found in [Gmelig Meyling, 1989-1]. The Newton method uses the gradient $g(y_{N|N})$ related to some estimate $y_{N|N}$ to obtain a better estimate in the sense that the objective function $\ell(Z_N, y_{N|N})$ has a lower value than $\ell(Z_N, y_{N|N})$.
The Newton method assumes a local quadratic behaviour of the objective function, i.e. the function is written as a Taylor series expansion of two terms:

$$\ell(y + s) \approx \ell(y) + g(y)^T s + \tfrac{1}{2} s^T G(y)\, s \tag{3.12}$$

where

$$G(y) = \frac{\partial^2 \ell(y)}{\partial y \partial y^T} \tag{3.13}$$

(See Appendix A for details on vector/matrix differentials).
The minimum of the right-hand side is obtained if s satisfies

$$G(y) \cdot s = -g(y) \tag{3.14}$$

s is referred to as the *Newton direction*.
By defining the N dimensional vector $f(y)$ containing the residuals $r_k(y_{k|N})$, $k = 1, \ldots, N$ and the $N \times n$ Jacobian matrix J by

$$\begin{aligned} f(y) &= [r_1(y),\ r_2(y),\ \ldots\ldots,\ r_N(y)]^T \\ J(y) &= \frac{\partial f(y)}{\partial y} \end{aligned} \tag{3.15}$$

g(y) and G(y) are rewritten as

$$g(y) = J(y)^T f(y)$$
$$G(y) = J(y)^T J(y) + Q(y) \tag{3.16}$$

where Q(y) is equal to

$$Q(y) = \sum_{k=1}^{N} r_k(y) \frac{\partial^2 r_k(y)}{\partial y \partial y^T} \tag{3.17}$$

The Newton direction is found by solving

$$(J(y)^T J(y)+Q(y))\cdot s = -J(y)^T f(y) \tag{3.18}$$

In order to find a descent direction the matrix G must be positive definite. However, when the residuals are large G is not guaranteed to be positive definite because of Q. A better alternative is to neglect Q. Since J^TJ is positive definite, except in special cases where dependency between elements of y_N occurs [Gmelig Meyling and de Vlieger, 1989], s is guaranteed to be a descent direction. This method is referred to as the *Gauss-Newton* method. A numerical robust way to solve (3.18) is to use a *singular value decomposition* (SVD) of J(y):

$$J(y) = U \Sigma V^T \tag{3.19}$$

where U is a $N \times n$ orthogonal matrix, Σ a $n \times n$ diagonal matrix and V a $n \times n$ orthonormal matrix. The diagonal of Σ contains the *singular values* of matrix J in descending order:

$$\Sigma = \mathrm{DIAG}[\ \sigma_1 \ \ldots\ldots\ \sigma_n\ ], \qquad \sigma_i \geq \sigma_{i+1} \tag{3.20}$$

Moreover the following relations hold:

$$
\begin{aligned}
&V^T V = VV^T = U^T U = I_n \\
&J^T J = V\,\Sigma^2 V^T \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.21)\\
&(J^T J)^{-1} = V\,\Sigma^{-2} V^T \qquad\qquad \sigma_i > 0, \quad i=1,\ldots,n
\end{aligned}
$$

The matrix V contains the eigenvectors of J^TJ and the squared singular values σ_1^2 through σ_n^2 are the eigenvalues of J^TJ. Note that if J does not have full rank one or more singular values will be zero and a pseudo inverse of J^TJ must be used in determining a search direction, i.e. the search is performed in a lower dimensional space. In fact the SVD decomposition accounts for the bad conditioning of the TMA problem. When some singular values are much lower than others one can cancel those directions corresponding to the small singular values [Gmelig Meyling, 1989]. Bad conditioning is related to measurement noise and geometry of the own ship and the target ship tracks.

The outline of a Newton-type optimization method is as follows:

1. Start with an arbitrary estimate y for i=1
2. Determine a search direction **s** (Newton or Gauss-Newton) by solving equation (3.18).
3. Perform a line search along the direction **s**
4. Check convergence criteria. If not satisfied increase iteration counter i and goto step 2.
5. Calculate the inverse of **G** which is an estimate of the error covariance of the maximum-likelihood estimate $y^{MLE}_{N|N} = y$

Details on Newton-type methods can be found in [Gill, Murray and Wright]. Specific details on Newton-type methods for TMA problems are described in [Gmelig Meyling].

3.3 Unbiased Estimates and the Cramer-Rao lower bound

The estimation error $e(Z)$ is defined by

$$e(Z) = y_{N|N} - y_N \tag{3.22}$$

Any unbiased estimation $y_{N|N}$ of the parameter vector y_N is characterized by

$$E(e(Z)|y_N) = \int_Z e(Z)\ p(Z|y_N)dZ = 0 \tag{3.23}$$

where $E(e|y)$ denotes the mathematical expectation of e knowing y. If we consider $y_{N|N}$ as an biased estimate of y_N with bias $\lambda(y_N)$ the expected value can be written as

$$E(e(Z)|y_N) = \lambda(y_N) \tag{3.24}$$

Differentiation with respect to y_N leads to

$$\int_Z e(Z)\ \frac{\partial\ \text{LN}(p(Z|y_N))}{\partial y_N} p(Z|y_N)dZ = I_n + \frac{\partial\lambda(y_N)}{\partial y_N} \tag{3.25}$$

Now the error covariance matrix Λ_e is defined as follows

$$\begin{aligned}\Lambda_e &= \int_Z [e(Z) - \lambda(y_N)][e(Z) - \lambda(y_N)]^T p(Z|y_N)dZ \\ &= E([e(Z) - \lambda(y_N)][e(Z) - \lambda(y_N)]^T|y_N)\end{aligned} \tag{3.26}$$

The vector $\partial \text{LN}(p(Z|y_N))/\partial y_N$ is a stochastic vector with expectation equal to zero and a covariance matrix equal to the matrix M:

$$M = E\left(\left(\frac{\partial \mathrm{LN}\ p(Z|y_N)}{\partial y_N}\right)\left(\frac{\partial \mathrm{LN}\ p(Z|y_N)}{\partial y_N}\right)^T \mid y_N \right)$$
$$= -E\left(\frac{\partial^2 \mathrm{LN}\ p(Z|y_N)}{\partial y_N \partial y_N^T} \mid y_N \right) \tag{3.27}$$

The $n \times n$ matrix M is called the ***Fisher information matrix***. The Cramer-Rao theorem [v.Trees, Eykhoff] states that for any unbiased estimate of $y_{N|N}$ the following inequality holds (see appendix C):

$$\Lambda_e \geq M^{-1} \tag{3.28}$$

The meaning of this theorem is that for each unbiased estimate the error covariance matrix has a lower bound, which is the inverse of the Fisher information matrix. The lower bound can be determined exactly if the real target state y_N is known. When Monte Carlo experiments are used to show the performance of the MLE method it is thus possible to determine the lower bound. When the estimation method is ***efficient*** the error covariance matrix is equal to the lower bound. The error covariance matrix can be either estimated from the Monte Carlo simulation results or by using the matrix $(J(y_{N|N})^T J(y_{N|N}) + Q(y_{N|N}))^{-1}$. The estimated error covariance matrix can be compared to the lower bound in order to get an efficiency measure of the MLE.

Using the definitions of g and G the matrix M can be rewritten as

$$M = E(g(y_N)g(y_N)^T | y_N) = J(y_N)^T J(y_N) \tag{3.29}$$

Unfortunately it cannot be proven that $y_{N|N}$ is unbiased. Moreover one is not able to determine an analitic expression of the bias $\lambda(y_N)$. Experimental results in [Gmelig Meyling and de Vlieger, 1989-2] show however that in many cases the MLE will be unbiased and one can use the Cramer-Rao lowerbound to verify the efficiency of the MLE method.

3.4 Quality of Fit and Confidence regions.

Given an estimate y the likelihood was defined by (3.6) and (3.8):

$$p(Z|y) = \{(2\pi)^{-N/2} \prod_{k=1}^{N} (\sigma_k^2)^{-1}\} \exp(-\tfrac{1}{2}\ell(Z,y))$$

$$\ell(Z_N,y_N) = \sum_{k=1}^{N} \left(\frac{Z_k - Z_k^*}{\sigma_k^2}\right)^2 \qquad (3.30)$$

The quantity $\ell(Z,y)$ at its minimum is distributed according to a chi-square distribution function for N-n degrees of freedom, where n is equal to the dimension of y. Suppose we have a realization Z_R (a set of measurements) and an estimate y^{MLE}. Other possible realizations Z which lead to other MLE estimates y with a likehood $\ell(Z,y)$ such that

$$\ell(Z,y) > \ell(Z_R,y^{MLE}) \qquad (3.31)$$

result in MLE solutions which do not fit the data Z as well as y^{MLE} fits Z_R. Hence the probability $Pr\{\ell(Z,y)>\ell(Z_R,y^{MLE})\}$ can be used as a quantitive measure for the quality of fit of y^{MLE}:

$$Pr\{\ell(Z,y)>\ell(Z_R,y^{MLE})\} = Q(\frac{N-n}{2}, \tfrac{1}{2}\ell(Z_R,y^{MLE})) \qquad (3.32)$$

Q(v,a) is refered to as the incomplete gamma function:

$$Q(v,a) = \frac{1}{\Gamma(v)} \int_{a}^{\infty} x^{v-1} \exp(-x)dx \qquad (3.33)$$

If the measurement noise realizations resulted into realization Z_R with a low probability of occurrence the likelihood of the ML estimate will be low and the probability 1-Pr will indicate that there is a high chance that someother realization would fit better. However, the quality of fit may not be confused with the confidence in the estimate. One may obtain a bad fit, for instance because of a bad measurement, but still obtain an excellent estimate and visa versa.

Let's now consider all possible estimates y for which the likelihood is larger or equal to a lowerbound $\alpha\ p(Z=Z_R|y^{MLE})$, for $0 \le \alpha \le 1$:

$$\alpha\ p(Z=Z_R|y^{MLE}) \le p(Z=Z_R|y) \le p(Z=Z_R|y^{MLE}) \tag{3.34}$$

Hence we select all estimates around y^{MLE} for which

$$\ell(Z_R,y^{MLE}) \le \ell(Z_R,y) \le \ell(Z_R,y^{MLE}) - 2\ \mathrm{LN}\ \alpha \tag{3.35}$$

The set {y} satisfying condition (3.35) defines a likelihood region R in the state space. The probability that y lies in the region R given the measurement set Z_R is:

$$\Pr(y \in R|Z=Z_R) = \int_R p(y|Z=Z_R)dy \tag{3.36}$$

By using the Bayes rule we obtain

$$p(y|Z=Z_R) = \frac{p(Z=Z_R|y)p(y)}{\int_y p(Z=Z_R|y)p(y)dy} \tag{3.37}$$

Note that the nominator of (3.37) is a normalization constant. The desired probability can be calculated if p(y) is available. Remember that p(y) represents the a priory knowledge about the target track and that the MLE method does not account for this knowledge. The worst case situation is at best represented by a uniform distribution:

$$p(y) = \begin{cases} 1/\Delta^n & -\Delta/2 \le y_i \le \Delta/2, \quad i=1,..,n \\ 0 & y_i \le -\Delta/2 \text{ or } y_i \le \Delta/2 \end{cases} \tag{3.38}$$

Hence, for $\Delta \to \infty$ the a priori knowledge vanishes. Substituting (3.38) into (3.37) results in a probability

$$\Pr(y \in R \mid Z = Z_R) = \frac{\int_R p(Z = Z_R \mid y)\,dy}{\int_y p(Z = Z_R \mid y)\,dy} \tag{3.39}$$

Substitution of (3.30) leads to

$$\Pr(y \in R \mid Z = Z_R) = \frac{\int_R \mathrm{EXP}(-\tfrac{1}{2}\ell(Z_R, y))\,dy}{\int_y \mathrm{EXP}(-\tfrac{1}{2}\ell(Z_R, y))\,dy} \tag{3.40}$$

which is the probability that the real target parameters lie in a confidence region R defined by (3.35) based on the knowledge of the measurement set Z_R. In general the boundary of R will not be an ellipsoid because of the nonlinear behaviour of the likelihood function. The shape of R can be found by using a Monte Carlo Integration approach, i.e. by producing a sufficiently large number of random vectors y, which are drawn from a uniform distribution function that fully covers the region R, one can display those vectors y graphically that satisfie the condition (3.35). Simultaneously one can approximate the integrals of (3.40) numerically by using both the accepted and the rejected vectors. The displayed points are scattered in the region R and hence show the shape of R.

4 RESIDUAL DERIVATIVES

The application of Newton-type algorithms implies the usage of first and second order derivatives of the estimate $z_{k|N}$ with respect to the state estimate $y_{N|N}$. This Chapter is concerned with the determination of these derivatives. First we focus on bearing and frequency residuals with respect to direct-path and bottom-bounce measurements. Then range, course and speed residuals are considered in Section 4.2. In Section 4.3 second derivatives of the residuals are determined.

4.1 First derivatives of bearing and frequency residuals

The general vector form of the first derivative of a residual $r_k(y_{k|N})$ is equal to

$$\frac{\partial r_k(y_{k|N})}{\partial y_{N|N}} = -\frac{1}{\sigma_k^2}\;\frac{\partial h_k(F(\Phi_{k,N}G(y_{N|N})))}{\partial y_{N|N}} \tag{4.1}$$

Throughout this Chapter we just concentrate on the derivatives of h_k instead of $r_k(y_{k|N})$. According to the rules of Appendix A the following expression is obtained:

$$\frac{\partial h_k(y_{k|N})}{\partial y_{N|N}} = \frac{\partial G(y_{N|N})}{\partial y_{N|N}}\cdot\Phi^T_{k,N}\cdot\frac{\partial F(x_{k|N})}{\partial x_{k|N}}\cdot\frac{\partial h_k(y_{k|N})}{\partial y_{k|N}} \tag{4.2}$$

which can also be denoted as

$$\frac{\partial z^*_{k|N}}{\partial y_{N|N}} = \frac{\partial x_{N|N}}{\partial y_{N|N}}\cdot\Phi^T_{k,N}\cdot\frac{\partial z^*_{k|N}}{\partial x_{k|N}} \tag{4.3}$$

where $z^*_{k|N} = h_k(y_{k|N})$.

In most cases computational efficiency is improved by calculating the derivatives from explicit analytic expressions instead of using (4.2) or (4.3). However (4.2) and (4.3) will serve as a guide to derive the individual expressions of each element of each type of residual derivative. First, consider the direct-path bearing estimate $b_{k|N}$. The result of equation (4.3) is

$$\frac{\partial b_{k|N}}{\partial y_{N|N}} = \frac{\partial x_{N|N}}{\partial y_{N|N}} \cdot \Phi^T_{k,N} \cdot \frac{\partial \mathrm{ATAN2}(x_{k|N}, y_{k|N})}{\partial x_{k|N}} \tag{4.4}$$

Further we have

$$\frac{\partial x_{N|N}}{\partial y_{N|N}} = \begin{bmatrix} R_{N|N}\cos b_{N|N} & -R_{N|N}\sin b_{N|N} & \cdots & 0_{2,2} & \cdots & 0_{2,2} \\ \sin b_{N|N} & \cos b_{N|N} & \ddots & & & \vdots \\ \vdots & & \ddots & v(i)\cos C(i) & -v(i)\sin C(i) & \vdots \\ 0_{2,2} & & \cdots & \sin C(i) & \cos C(i) & \cdots\ 0_{2,2} \\ \vdots & & & & \ddots & \vdots \\ 0_{q,2} & & \cdots & 0_{q,2} & \cdots & I_q \end{bmatrix} \tag{4.5}$$

Note that (4.5) is a block diagonal matrix with m+1 submatrices of dimension 2 × 2 and one q × q identity matrix on its diagonal. Therefore it is much more convenient to rewrite (4.4) as a number of expressions with respect to bearing, range, speed and course of each leg, and each source frequency:

$$\frac{\partial b_{k|N}}{\partial b_{N|N}} = R_{N|N}\cos(b_{N|N})\cdot\frac{y_{k|N}}{R^2_{k|N}} - R_{N|N}\sin(b_{N|N})\cdot\frac{-x_{k|N}}{R^2_{k|N}}$$

$$\frac{\partial b_{k|N}}{\partial R_{N|N}} = \sin(b_{N|N})\cdot\frac{y_{k|N}}{R^2_{k|N}} + \cos(b_{N|N})\cdot\frac{-x_{k|N}}{R^2_{k|N}}$$

$$\frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} = v^{(i)}_{N|N}\cos(C^{(i)}_{N|N})\cdot\frac{y_{k|N}T^{(i)}_{k,N}}{R^2_{k|N}} - v^{(i)}_{N|N}\sin(C^{(i)}_{N|N})\cdot\frac{-x_{k|N}T^{(i)}_{k,N}}{R^2_{k|N}} \tag{4.6}$$

$$\frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} = \sin(C^{(i)}_{N|N})\cdot\frac{y_{k|N}T^{(i)}_{k,N}}{R^2_{k|N}} + \cos(C^{(i)}_{N|N})\cdot\frac{-x_{k|N}T^{(i)}_{k,N}}{R^2_{k|N}}$$

$$\frac{\partial b_{k|N}}{\partial f_{0j|N}} = 0$$

The result of equations (4.6) is shown in table 4.1.

$$\frac{\partial b_{k|N}}{\partial b_{N|N}} = \frac{R_{N|N}}{R_{k|N}}\cos(b_{N|N} - b_{k|N})$$

$$\frac{\partial b_{k|N}}{\partial R_{N|N}} = \frac{1}{R_{k|N}}\sin(b_{N|N} - b_{k|N})$$

$$\frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} = \frac{T^{(i)}_{k,N}}{R_{k|N}} v^{(i)}_{N|N}\cos(b_{k|N} - C^{(i)}_{N|N}) \qquad i = 1, \ldots, m$$

$$\frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} = -\frac{T^{(i)}_{k,N}}{R_{k|N}}\sin(b_{k|N} - C^{(i)}_{N|N}) \qquad i = 1, \ldots, m$$

$$\frac{\partial b_{k|N}}{\partial f_{0j|N}} = 0$$

Table 4.1 ***First derivatives of Direct-path bearing estimate*** $b_{k|N}$.

The estimated direct-path frequency equation is denoted by

$$f_{kj|N} = f_{0j|N}\left(1 - \frac{\dot{r}_{k|N}}{c}\right) \tag{4.7}$$

where $f_{0j|N}$ is the estimate of source frequency f_{0j} as a result of the set Z_N. The derivative with respect to $y_{N|N}$ can be derived according to (4.3). It is however easier to obtain the derivative straight-forward from

$$\frac{\partial f_{kj|N}}{\partial y_{N|N}} = -\frac{f_{0j}}{c}\frac{\partial \dot{r}_{k|N}}{\partial y_{N|N}} + \frac{\partial f_{0j|N}}{\partial y_{N|N}}\left(1 - \frac{\dot{r}_{k|N}}{c}\right) \tag{4.8}$$

Note that $\partial f_{0j|N}/\partial y_{N|N}$ is equal to $e_{2(m+1)+j}$ where e_i is the identity vector with a unity element at the i^{th} position. Substitution in (4.8) leads to

$$\frac{\partial f_{kj|N}}{\partial y_{N|N}} = -\frac{f_{0j}}{c}\frac{\partial \dot{r}_{k|N}}{\partial y_{N|N}} + \left(1 - \frac{\dot{r}_{k|N}}{c}\right) e_{2(m+1)+j} \tag{4.9}$$

The first term of (4.9) depends on the range-rate derivative $\dot{r}_{k|N}$:

$$\dot{r}_{k|N} = (v_x^{(i)} - \dot{x}_k^{OS})\sin b_{k|N} + (v_y^{(i)} - \dot{y}_k^{OS})\cos b_{k|N} \qquad t^{(i-1)} \le t_k < t^{(i)} \text{ or } t_k = t^{(m)} \tag{4.10}$$

For the sake of convenient notation we introduce $v_{k|N}^{TAN}$ as

$$v_{k|N}^{TAN} = (v_x^{(i)} - \dot{x}_k^{OS})\cos b_{k|N} - (v_y^{(i)} - \dot{y}_k^{OS})\sin b_{k|N} \qquad t^{(i-1)} \le t_k < t^{(i)} \text{ or } t_k = t^{(m)} \tag{4.11}$$

Equivalent expressions are :

$$\dot{r}_{k|N} = (v^{(i)}_{k|N} \sin C^{(i)}_{k|N} - \dot{x}^{OS}_{k}) \sin b_{k|N} + (v^{(i)}_{k|N} \cos C^{(i)}_{k|N} - \dot{y}^{OS}_{k}) \cos b_{k|N}$$

$$v^{TAN}_{k|N} = (v^{(i)}_{k|N} \sin C^{(i)}_{k|N} - \dot{x}^{OS}_{k}) \cos b_{k|N} - (v^{(i)}_{k|N} \cos C^{(i)}_{k|N} - \dot{y}^{OS}_{k}) \sin b_{k|N}$$

Differentiation of (4.10) with respect to $y_{N|N}$ leads to

$$\frac{\partial \dot{r}_{k|N}}{\partial y_{N|N}} = v^{TAN}_{k|N} \frac{\partial b_{k|N}}{\partial y_{N|N}} + v^{(i)}_{k|N} \sin(b_{k|N} - C^{(i)}_{N|N}) \frac{\partial C^{(i)}_{N|N}}{\partial y_{N|N}} + \cos(b_{k|N} - C^{(i)}_{N|N}) \frac{\partial v^{(i)}_{N|N}}{\partial y_{N|N}}$$

$$\frac{\partial v^{TAN}}{\partial y_{N|N}} = - \dot{r}_{k|N} \frac{\partial b_{k|N}}{\partial y_{N|N}} + v^{(i)}_{k|N} \cos(b_{k|N} - C^{(i)}_{N|N}) \frac{\partial C^{(i)}_{N|N}}{\partial y_{N|N}} - \sin(b_{k|N} - C^{(i)}_{N|N}) \frac{\partial v^{(i)}_{N|N}}{\partial y_{N|N}} \tag{4.12}$$

Note that $C^{(i)}_{k|N}) = C^{(i)}_{N|N})$ for all k since the TS course at leg i remains constant. The result is a function of the bearing derivatives:

$$\frac{\partial \dot{r}_{k|N}}{\partial y_{N|N}} = v^{TAN}_{k|N} \frac{\partial b_{k|N}}{\partial y_{N|N}} + v^{(i)}_{k|N} \sin(b_{k|N} - C^{(i)}_{N|N})\, e_{2i+1} + \cos(b_{k|N} - C^{(i)}_{N|N})\, e_{2i+2}$$

$$\frac{\partial v^{TAN}}{\partial y_{N|N}} = - \dot{r}_{k|N} \frac{\partial b_{k|N}}{\partial y_{N|N}} + v^{(i)}_{k|N} \cos(b_{k|N} - C^{(i)}_{N|N})\, e_{2i+1} - \sin(b_{k|N} - C^{(i)}_{N|N})\, e_{2i+2} \tag{4.13}$$

Substituting the bearing derivatives of Table 4.1 in (4.13) leads to the result summarized in Table 4.2.

$$\frac{\partial f_{kj|N}}{\partial b_{N|N}} = \frac{-f_{0j|N}}{c} v^{TAN}_{k|N} \frac{R_{N|N}}{R_{k|N}} \cos(b_{N|N} - b_{k|N})$$

$$\frac{\partial f_{kj|N}}{\partial R_{N|N}} = \frac{-f_{0j|N}}{c} v^{TAN}_{k|N} \frac{1}{R_{k|N}} \sin(b_{N|N} - b_{k|N})$$

$$\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} = \frac{-f_{0j|N}}{c} v^{(i)}_{N|N} \sin(b_{k|N} - C^{(i)}_{N|N})$$

$$t^{(i-1)} \le t_k < t^{(i)} \text{ or } t_k = t^{(m)}$$

$$- \frac{f_{0j|N}}{c} v^{(i)}_{N|N} \frac{v^{TAN}_{k|N}\, T^{(i)}_{k,N}}{R_{k|N}} \cos(b_{k|N} - C^{(i)}_{N|N})$$

$$\frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}} = \frac{-f_{0j|N}}{c} \cos(b_{k|N} - C^{(i)}_{N|N})$$

$$t^{(i-1)} \le t_k < t^{(i)} \text{ or } t_k = t^{(m)}$$

$$+ \frac{f_{0j|N}}{c} \frac{v^{TAN}_{k|N}\, T^{(i)}_{k,N}}{R_{k|N}} \sin(b_{k|N} - C^{(i)}_{N|N})$$

$$\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} = - \frac{f_{0j|N}}{c} v^{(i)}_{N|N} \frac{v^{TAN}_{k|N}\, T^{(i)}_{k,N}}{R_{k|N}} \cos(b_{k|N} - C^{(i)}_{N|N})$$

$$t_k \le t^{(i-1)} \text{ or } t_k > t^{(i)}$$

$$\frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}} = \frac{f_{0j|N}}{c} \frac{v^{TAN}_{k|N}\, T^{(i)}_{k,N}}{R_{k|N}} \sin(b_{k|N} - C^{(i)}_{N|N})$$

$$t_k \le t^{(i-1)} \text{ or } t_k > t^{(i)}$$

$$\frac{\partial f_{kj|N}}{\partial f_{0j|N}} = 1 - \frac{\dot{r}_{k|N}}{c}$$

$$\frac{\partial f_{kj|N}}{\partial f_{0i|N}} = 0 \qquad i \ne j$$

Table 4.2 *First derivatives of direct-path frequency estimate* $f_{kj|N}$.

4.2 First derivatives of range, course and speed.

The first derivatives of the course and speed estimates are obtained straight forward:

$$\frac{\partial C^{(i)}_{k|N}}{\partial y_{N|N}} = e_{2i+1}$$
$$\frac{\partial v^{(i)}_{k|N}}{\partial y_{N|N}} = e_{2i+2} \qquad (4.14)$$

where e_j is the identity vector with a unity element at the j^{th} position.
The Range derivatives are obtained by applying the general equation (4.3):

$$\frac{\partial R_{k|N}}{\partial y_{N|N}} = \frac{\partial x_{N|N}}{\partial y_{N|N}} \cdot \Phi^T_{k,N} \cdot \frac{\partial R_{k|N}}{\partial x_{k|N}} \qquad (4.15)$$

where

$$\frac{\partial R_{k|N}}{\partial x_{k|N}} = [\ \sin b_{k|N} \ \ \cos b_{k|N} \ \ 0 \ \ldots\ldots \ 0\ ]^T \qquad (4.16)$$

Carrying out the matrix multiplications leads to the results in Table 4.3.

$$\frac{\partial R_{k|N}}{\partial b_{N|N}} = -R_{N|N}\ \mathrm{SIN}(b_{N|N} - b_{k|N})$$

$$\frac{\partial R_{k|N}}{\partial R_{N|N}} = \cos(b_{N|N} - b_{k|N})$$

$$\frac{\partial R_{k|N}}{\partial C^{(i)}_{N|N}} = T^{(i)}_{k,N}.v^{(i)}_{N|N}.\mathrm{SIN}(b_{k|N} - C^{(i)}_{N|N}) \qquad i = 1, \ldots, m$$

$$\frac{\partial R_{k|N}}{\partial v^{(i)}_{N|N}} = T^{(i)}_{k,N} \cos(b_{k|N} - C^{(i)}_{N|N}) \qquad i = 1, \ldots, m$$

$$\frac{\partial R_{k|N}}{\partial f_{0j|N}} = 0$$

Table 4.3 *First derivatives of Range estimate* $R_{k|N}$.

4.3 First Derivatives of Bottom-bounce bearing and frequency

The bottom-bounce bearing and frequency derivatives are obtained from the BB-expressions of Section 2.4:

$$b^{BB}_k = \begin{cases} C^{OS}_k + \mathrm{ARCCOS}(\alpha^{BB}_k \cos\beta_k)\ , & 0 \le \beta_k \le \pi \\ C^{OS}_k - \mathrm{ARCCOS}(\alpha^{BB}_k \cos\beta_k)\ , & -\pi \le \beta_k < 0 \end{cases} \qquad (4.17)$$

The variable $\alpha_{k|N}$ is defined as

$$\alpha_{k|N} = \frac{R_{k|N}}{(R^2_{k|N} + D^2_{BB})^{1/2}} \qquad (4.18)$$

in correspondence to the definitions in (2.21) and (2.19)[1].
Similar, the bottom-bounce frequency estimate is equal to

$$f^{BB}_{kj|N} = f_{0j|N}\,(1 - \alpha_{k|N}\,\frac{\dot{r}_{k|N}}{c}) \tag{4.19}$$

The derivative of b^{BB} can be expressed as

$$\frac{\partial b^{BB}_{k|N}}{\partial y_{N|N}} = \frac{\partial \mathrm{ARCCOS}(u)}{\partial u}\left[- \alpha_{k|N}\mathrm{SIN}\,\beta_{k|N}\,\frac{\partial b_{k|N}}{\partial y_{N|N}} + \cos\beta_{k|N}\,\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}\,\frac{\partial R_{k|N}}{\partial y_{N|N}}\right] \tag{4.20}$$

where $u = \alpha_{k|N}\cos\beta_{k|N}$ and

$$\frac{\partial\ \mathrm{ARCCOS}(u)}{\partial u} = \frac{-1}{(1-\alpha^2_{k|N}.\cos^2(b_{k|N}))^{1/2}} \tag{4.21}$$

Further

$$\frac{\partial \alpha_{k|N}}{\partial R_{k|N}} = \frac{D^2_{BB}}{(R^2_{k|N} + D^2_{BB})^{3/2}} \tag{4.22}$$

The range derivative in (4.20) can be found from table 4.3. Hence the result can be found from

$$\frac{\partial b^{BB}_{k|N}}{\partial y_{N|N}} = \frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}}\,\frac{\partial b_{k|N}}{\partial y_{k|N}} + \frac{\partial b^{BB}_{k|N}}{\partial \alpha_{k|N}}\,\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}\,\frac{\partial R_{k|N}}{\partial y_{k|N}} \tag{4.23}$$

1) Remember that we still assume D_{TS}-D_{OS} to be zero. Note that when D_{TS}-D_{OS} is substantial large, we also have to reformulate the direct-path bearing and frequency equations by using the bottom-bounce equations and replace D_{BB} by D_{DP}.

where

$$\frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}} = \frac{\alpha_{k|N} \text{SIN}(\beta_{k|N}).\text{SIGN}(\beta_{k|N})}{(1-\alpha^2_{k|N}\cos^2(b_{k|N}))^{1/2}}$$

$$\frac{\partial b^{BB}_{k|N}}{\partial \alpha_{k|N}} = -\frac{\cos(\beta_{k|N})}{(1-\alpha^2_{k|N}.\cos^2(b_{k|N}))^{1/2}} \qquad (4.24)$$

$$\frac{\partial \alpha_{k|N}}{\partial R_{k|N}} = \frac{D^2_{BB}}{(R^2_{k|N} + D^2_{BB})^{3/2}}$$

The direct-path bearing and the range derivatives can be found from the tables 4.1 and 4.3 respectively. There is one trivial case:

$$\frac{\partial b^{BB}_{k|N}}{\partial f_{0j|N}} = 0 \qquad j = 1, \ldots, q \qquad (4.25)$$

The results have been summarized in Table 4.4

$$\beta_{k|N} = b_{k|N} - C_k^{OS}$$

$$\frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}} = \frac{\alpha_{k|N} \mathrm{SIN}(\beta_{k|N}) \,.\, \mathrm{SIGN}(\beta_{k|N})}{(1-\alpha^2_{k|N}\cos^2(\beta_{k|N}))^{1/2}}$$

$$\frac{\partial b^{BB}_{k|N}}{\partial \alpha_{k|N}} = -\frac{\cos(\beta_{k|N})}{(1-\alpha^2_{k|N} \,.\, \cos^2(\beta_{k|N}))^{1/2}}$$

$$\frac{\partial \alpha_{k|N}}{\partial R_{k|N}} = \frac{4d^2}{(R^2_{k|N} + 4d^2)^{3/2}}$$

$$\frac{\partial b^{BB}_{k|N}}{\partial b_{N|N}} = \frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}} \overset{\text{Table 4.1}}{\overset{\downarrow}{\frac{\partial b_{k|N}}{\partial b_{N|N}}}} + \frac{\partial b^{BB}_{k|N}}{\partial \alpha_{k|N}} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \overset{\text{Table 4.3}}{\overset{\downarrow}{\frac{\partial R_{k|N}}{\partial b_{N|N}}}}$$

$$\frac{\partial b^{BB}_{k|N}}{\partial R_{N|N}} = \frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}} \frac{\partial b_{k|N}}{\partial R_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial \alpha_{k|N}} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial R_{N|N}}$$

$$\frac{\partial b^{BB}_{k|N}}{\partial C^{(i)}_{N|N}} = \frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial \alpha_{k|N}} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial C^{(i)}_{N|N}}$$

$$i = 1, \ldots, m$$

$$\frac{\partial b^{BB}_{k|N}}{\partial v^{(i)}_{N|N}} = \frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}} \frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial \alpha_{k|N}} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial v^{(i)}_{N|N}}$$

$$\frac{\partial b^{BB}_{k|N}}{\partial f_{0j|N}} = 0$$

Table 4.4 ***First Derivatives of Bottom-bounce bearing.***

The BB frequency $f^{BB}_{kj|N}$ can be written as a function of the source frequency $f_{0j|N}$, the estimated frequency $f_{kj|N}$ and the factor $\alpha_{k|N}$:

$$f^{BB}_{kj|N} = f_{0j|N} + \alpha_{k|N}(f_{kj|N} - f_{0j|N}) \tag{4.26}$$

Remember that $f_{kj|N}$ is equal to $f^{DP}_{kj|N}$ if D_{DP} is zero!
The first derivative can be calculated straightforward by using Table 4.2 and 4.3:

$$\begin{aligned} \frac{\partial f^{BB}_{kj|N}}{\partial y_{N|N}} = \; & \alpha_{k|N} \frac{\partial f_{kj|N}}{\partial y_{N|N}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial y_{N|N}} \\ & + (1-\alpha_{k|N}) \frac{\partial f_{0j|N}}{\partial y_{N|N}} \end{aligned} \tag{4.27}$$

The result is shown in Table 4.5.

Table 4.2 ↓   Table 4.4, Table 4.3 ↓ ↓

$$\frac{\partial f^{BB}_{kj|N}}{\partial b_{N|N}} = \alpha_{k|N} \frac{\partial f_{kj|N}}{\partial b_{N|N}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial b_{N|N}}$$

$$\frac{\partial f^{BB}_{kj|N}}{\partial R_{N|N}} = \alpha_{k|N} \frac{\partial f_{kj|N}}{\partial R_{N|N}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial R_{N|N}}$$

$$\frac{\partial f^{BB}_{kj|N}}{\partial C^{(i)}_{N|N}} = \alpha_{k|N} \frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial C^{(i)}_{N|N}}$$

$$i = 1, \ldots, m$$

$$\frac{\partial f^{BB}_{kj|N}}{\partial v^{(i)}_{N|N}} = \alpha_{k|N} \frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial v^{(i)}_{N|N}}$$

$$\frac{\partial f^{BB}_{kj|N}}{\partial f_{0j|N}} = \left(1 - \alpha_{k|N} \frac{\dot{r}_{k|N}}{c}\right)$$

Table 4.5 *First Derivatives of Bottom-bounce frequency.*

4.4 Second derivatives of range, bearing and frequency residuals

The second derivatives of direct path and bottom-bounce bearing and frequency estimates are obtained from the Tables 4.1 through 4.5 of the previous sections. First we focus on direct path derivatives. From the first order derivative $\partial b_{k|N}/\partial b_{N|N}$ in Table 4.1 we obtain

$$\frac{\partial^2 b_{k|N}}{\partial\bullet\,\partial b_{N|N}} = -\frac{R_{N|N}}{R_{k|N}^2}\cos(b_{N|N}-b_{k|N})\frac{\partial R_{k|N}}{\partial\bullet} + \frac{1}{R_{k|N}}\cos(b_{N|N}-b_{k|N})\frac{\partial R_{N|N}}{\partial\bullet} + \frac{R_{N|N}}{R_{k|N}}\sin(b_{N|N}-b_{k|N})\left(\frac{\partial b_{k|N}}{\partial\bullet}-\frac{\partial b_{N|N}}{\partial\bullet}\right) \tag{4.29}$$

where $\bullet$ indicates one of the estimated variables in $y_{N|N}$. Note that the second term is only non-zero if $\bullet$ indicates $R_{N|N}$. Similar, when $\bullet$ is equal to $b_{N|N}$ the term with $\partial b_{N|N}/\partial\bullet$ will become non-zero. Many of the terms can be expressed easily by using first order derivatives. In this way the second derivatives can be expressed by:

$$\frac{\partial^2 b_{k|N}}{\partial\bullet\,\partial b_{N|N}} = -\frac{1}{R_{k|N}}\frac{\partial b_{k|N}}{\partial b_{N|N}}\frac{\partial R_{k|N}}{\partial\bullet} + \frac{1}{R_{N|N}}\frac{\partial b_{k|N}}{\partial b_{N|N}}\frac{\partial R_{N|N}}{\partial\bullet} + R_{N|N}\frac{\partial b_{k|N}}{\partial R_{N|N}}\left(\frac{\partial b_{k|N}}{\partial\bullet}-\frac{\partial b_{N|N}}{\partial\bullet}\right) \tag{4.30}$$

This procedure is repeated to obtain second derivatives from all the first derivatives of Table 4.1:

$$\frac{\partial^2 b_{k|N}}{\partial\bullet\ \partial R_{N|N}} = -\frac{1}{R^2_{k|N}}\ \mathrm{SIN}(b_{N|N} - b_{k|N})\frac{\partial R_{k|N}}{\partial\bullet}$$

$$-\frac{1}{R_{k|N}}\cos(b_{N|N} - b_{k|N})\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial b_{N|N}}{\partial\bullet}\right)$$

$$\frac{\partial^2 b_{k|N}}{\partial\bullet\partial C^{(i)}_{N|N}} = -\frac{T^{(i)}_{k,N}}{R^2_{k|N}}\ v^{(i)}_{N|N}\cos(b_{k|N} - C^{(i)}_{N|N})\ \frac{\partial R_{k|N}}{\partial\bullet} \tag{4.31}$$

$$-\frac{T^{(i)}_{k,N}}{R_{k|N}}\ v^{(i)}_{N|N}\ \mathrm{SIN}(b_{k|N} - C^{(i)}_{N|N})\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial\bullet}\right)$$

$$+\frac{T^{(i)}_{k,N}}{R_{k|N}}\cos(b_{k|N} - C^{(i)}_{N|N})\ \frac{\partial v^{(i)}_{N|N}}{\partial\bullet}$$

$$\frac{\partial^2 b_{k|N}}{\partial\bullet\partial v^{(i)}_{N|N}} = \frac{T^{(i)}_{k,N}}{R^2_{k|N}}\ \mathrm{SIN}(b_{k|N} - C^{(i)}_{N|N})\ \frac{\partial R_{k|N}}{\partial\bullet}$$

$$-\frac{T^{(i)}_{k,N}}{R_{k|N}}\cos(b_{k|N} - C^{(i)}_{N|N})\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial\bullet}\right)$$

$$\frac{\partial^2 b_{k|N}}{\partial\bullet\partial f_{0j}} = 0$$

By using the first order derivatives expressions as much as possible the result in Table 4.6 is obtained.

$$\frac{\partial^2 b_{k|N}}{\partial \cdot \, \partial b_{N|N}} = -\frac{1}{R_{k|N}} \frac{\partial b_{k|N}}{\partial b_{N|N}} \frac{\partial R_{k|N}}{\partial \cdot} + \frac{1}{R_{N|N}} \frac{\partial b_{k|N}}{\partial b_{N|N}} \frac{\partial R_{N|N}}{\partial \cdot}$$

$$+ R_{N|N} \frac{\partial b_{k|N}}{\partial R_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \cdot} - \frac{\partial b_{N|N}}{\partial \cdot} \right)$$

$$\frac{\partial^2 b_{k|N}}{\partial \cdot \, \partial R_{N|N}} = -\frac{1}{R_{k|N}} \frac{\partial b_{k|N}}{\partial R_{N|N}} \frac{\partial R_{k|N}}{\partial \cdot} - \frac{1}{R_{N|N}} \frac{\partial b_{k|N}}{\partial b_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \cdot} - \frac{\partial b_{N|N}}{\partial \cdot} \right)$$

$$\frac{\partial^2 b_{k|N}}{\partial \cdot \, \partial C^{(i)}_{N|N}} = -\frac{1}{R_{k|N}} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} \frac{\partial R_{k|N}}{\partial \cdot} + \frac{1}{v^{(i)}_{N|N}} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} \frac{\partial v^{(i)}_{N|N}}{\partial \cdot}$$

$$+ v^{(i)}_{N|N} \frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \cdot} - \frac{\partial C^{(i)}_{N|N}}{\partial \cdot} \right)$$

$$i = 1, \ldots, m$$

$$\frac{\partial^2 b_{k|N}}{\partial \cdot \, \partial v^{(i)}_{N|N}} = -\frac{1}{R_{k|N}} \frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} \frac{\partial R_{k|N}}{\partial \cdot}$$

$$- \frac{1}{v^{(i)}_{N|N}} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \cdot} - \frac{\partial C^{(i)}_{N|N}}{\partial \cdot} \right)$$

$$i = 1, \ldots, m$$

$$\frac{\partial^2 b_{k|N}}{\partial \cdot \, \partial f_{0j}} = 0$$

Note: $\bullet$ is an element of $\mathbf{y}_{N|N}$

Table 4.6 *Second Derivatives of Direct-Path bearing.*

In the following the second order derivatives of the direct-path frequency estimate f_{kj} are obtained in a similar way:

$$\frac{\partial^2 f_{kj|N}}{\partial \bullet\ \partial b_{N|N}} = + \frac{R_{N|N}}{R^2_{k|N}} \frac{f_{0j|N}}{c} v^{TAN}_{k|N} \cos(b_{N|N} - b_{k|N}) \frac{\partial R_{k|N}}{\partial \bullet}$$
$$- \frac{1}{R_{k|N}} \frac{f_{0j|N}}{c} v^{TAN}_{k|N} \cos(b_{N|N} - b_{k|N}) \frac{\partial R_{N|N}}{\partial \bullet}$$
$$+ \frac{R_{N|N}}{R_{k|N}} \frac{f_{0j|N}}{c} v^{TAN}_{k|N} \sin(b_{N|N} - b_{k|N}) \left(\frac{\partial b_{N|N}}{\partial \bullet} - \frac{\partial b_{k|N}}{\partial \bullet}\right) \tag{4.32}$$
$$- \frac{R_{N|N}}{R_{k|N}} \frac{f_{0j|N}}{c} \cos(b_{N|N} - b_{k|N}) \frac{\partial v^{TAN}_{k|N}}{\partial \bullet}$$
$$- \frac{R_{N|N}}{R_{k|N}} \frac{v^{TAN}_{k|N}}{c} \cos(b_{N|N} - b_{k|N}) \frac{\partial f_{0j|N}}{\partial \bullet}$$

By using the first order derivative expressions of Table 4.2 we get:

$$\frac{\partial^2 f_{kj|N}}{\partial \bullet\ \partial b_{N|N}} = \frac{\partial f_{kj|N}}{\partial b_{N|N}} \left(\frac{1}{R_{N|N}} \frac{\partial R_{N|N}}{\partial \bullet} - \frac{1}{R_{k|N}} \frac{\partial R_{k|N}}{\partial \bullet} + \frac{1}{f_{0j|N}} \frac{\partial f_{0j|N}}{\partial \bullet} + \frac{1}{v^{TAN}_{k|N}} \frac{\partial v^{TAN}_{k|N}}{\partial \bullet}\right)$$
$$+ R_{N|N} \frac{\partial f_{kj|N}}{\partial R_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial b_{N|N}}{\partial \bullet}\right) \tag{4.33}$$

The derivative $\partial v^{TAN}_{k|N}/\partial \bullet$ can be obtained from (4.13). The elements of $\partial^2 f_{kj|N}/\partial y_{N|N} \partial R_{N|N}$ are obtained in a similar way:

$$\frac{\partial^2 f_{kj|N}}{\partial \bullet\ \partial R_{N|N}} = \frac{\partial f_{kj|N}}{\partial R_{N|N}} \left(- \frac{1}{R_{k|N}} \frac{\partial R_{k|N}}{\partial \bullet} + \frac{1}{f_{0j|N}} \frac{\partial f_{0j|N}}{\partial \bullet} + \frac{1}{v^{TAN}_{k|N}} \frac{\partial v^{TAN}_{k|N}}{\partial \bullet}\right)$$
$$- \frac{1}{R_{N|N}} \frac{\partial f_{kj|N}}{\partial b_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial b_{N|N}}{\partial \bullet}\right) \tag{4.34}$$

The elements of $\partial^2 f_{kj|N}/\partial y_{N|N} \partial C^{(i)}_{N|N}$ are determined as follows:

$$\frac{\partial^2 f_{kj|N}}{\partial\bullet\ \partial C^{(i)}_{N|N}} = \frac{1}{f_{0j|N}}\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}}\frac{\partial f_{0j|N}}{\partial\bullet} + \frac{1}{V^{(i)}_{N|N}}\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}}\frac{\partial v^{(i)}_{N|N}}{\partial\bullet} - \frac{f_{0j|N}}{c}\frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}}\frac{\partial v^{TAN}_{k|N}}{\partial\bullet}$$

$$+ V^{(i)}_{N|N}\frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}}\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial\bullet}\right) + \frac{f_{0j|N}}{c}\frac{v^{TAN}_{k|N}}{R_{k|N}}\frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}}\frac{\partial R_{k|N}}{\partial\bullet} \qquad (4.35)$$

Similar, $\partial^2 f_{kj|N}/\partial y_{N|N}\partial v^{(i)}_{N|N}$ is stated below:

$$\frac{\partial^2 f_{kj|N}}{\partial\bullet\ \partial v^{(i)}_{N|N}} = \frac{1}{f_{0j|N}}\frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}}\frac{\partial f_{0j|N}}{\partial\bullet} - \frac{1}{v^{(i)}_{N|N}}\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}}\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial\bullet}\right)$$

$$+ \frac{f_{0j|N}}{c}\frac{v^{TAN}_{k|N}}{R_{k|N}}\frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}}\frac{\partial R_{k|N}}{\partial\bullet} - \frac{f_{0j|N}}{c}\frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}}\frac{\partial v^{TAN}_{k|N}}{\partial\bullet} \qquad (4.36)$$

The derivatives $\partial^2 f_{kj|N}/\partial y_{N|N}\partial f_{0j|N}$ are obtained straight forward from the range-rate derivatives (4.13). The result is summarized in Table 4.7.

$$\frac{\partial^2 f_{kj|N}}{\partial \bullet\, \partial b_{N|N}} = \frac{\partial f_{kj|N}}{\partial b_{N|N}} \left(\frac{1}{R_{N|N}} \frac{\partial R_{N|N}}{\partial \bullet} - \frac{1}{R_{k|N}} \frac{\partial R_{k|N}}{\partial \bullet} + \frac{1}{f_{0j|N}} \frac{\partial f_{0j|N}}{\partial \bullet} + \frac{1}{v^{TAN}_{k|N}} \frac{\partial v^{TAN}_{k|N}}{\partial \bullet} \right)$$

$$+ R_{N|N} \frac{\partial f_{kj|N}}{\partial R_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial b_{N|N}}{\partial \bullet} \right)$$

$$\frac{\partial^2 f_{kj|N}}{\partial \bullet\, \partial R_{N|N}} = \frac{\partial f_{kj|N}}{\partial R_{N|N}} \left(- \frac{1}{R_{k|N}} \frac{\partial R_{k|N}}{\partial \bullet} + \frac{1}{f_{0j|N}} \frac{\partial f_{0j|N}}{\partial \bullet} + \frac{1}{v^{TAN}_{k|N}} \frac{\partial v^{TAN}_{k|N}}{\partial \bullet} \right)$$

$$- \frac{1}{R_{N|N}} \frac{\partial f_{kj|N}}{\partial b_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial b_{N|N}}{\partial \bullet} \right)$$

$$\frac{\partial^2 f_{kj|N}}{\partial \bullet\, \partial C^{(i)}_{N|N}} = \frac{1}{f_{0j|N}} \frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} \frac{\partial f_{0j|N}}{\partial \bullet} + \frac{1}{v^{(i)}_{N|N}} \frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} \frac{\partial v^{(i)}_{N|N}}{\partial \bullet}$$

$$- \frac{f_{0j|N}}{c} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} \frac{\partial v^{TAN}_{k|N}}{\partial \bullet} + v^{(i)}_{N|N} \frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial \bullet} \right)$$

$$+ \frac{f_{0j|N}}{c} \frac{v^{TAN}_{k|N}}{R_{k|N}} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} \frac{\partial R_{k|N}}{\partial \bullet}$$

$$\frac{\partial^2 f_{kj|N}}{\partial \bullet\, \partial v^{(i)}_{N|N}} = \frac{1}{f_{0j|N}} \frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}} \frac{\partial f_{0j|N}}{\partial \bullet} - \frac{1}{v^{(i)}_{N|N}} \frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial \bullet} \right)$$

$$+ \frac{f_{0j|N}}{c} \frac{v^{TAN}_{k|N}}{R_{k|N}} \frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} \frac{\partial R_{k|N}}{\partial \bullet} - \frac{f_{0j|N}}{c} \frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} \frac{\partial v^{TAN}_{k|N}}{\partial \bullet}$$

$$\frac{\partial^2 f_{kj|N}}{\partial \bullet\, \partial f_{0j|N}} = - \frac{1}{c} \frac{\partial \dot{r}_{k|N}}{\partial \bullet}$$

Note: $\bullet$ is an element of $\mathbf{y}_{N|N}$

Table 4.7 *Second Derivatives of Direct-Path frequency.*

Since all bottom-bounce expressions are range dependent it is more convenient to determine the 2nd derivatives of the range $R_{k|N}$ first. The result is stated in Table 4.8.

$$\frac{\partial^2 R_{k|N}}{\partial\bullet\,\partial b_{N|N}} = \frac{\partial R_{N|N}}{\partial\bullet}\sin(b_{k|N} - b_{N|N}) + R_{N|N}\cos(b_{k|N} - b_{N|N})\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial b_{N|N}}{\partial\bullet}\right)$$

$$\frac{\partial^2 R_{k|N}}{\partial\bullet\,\partial R_{N|N}} = -\sin(b_{k|N} - b_{N|N})\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial b_{N|N}}{\partial\bullet}\right)$$

$$\frac{\partial^2 R_{k|N}}{\partial\bullet\,\partial C^{(i)}_{N|N}} = T^{(i)}_{k,N}\sin(b_{k|N} - C^{(i)}_{N|N})\frac{\partial v^{(i)}_{N|N}}{\partial\bullet} + T^{(i)}_{k,N}v^{(i)}_{N|N}\cos(b_{k|N} - C^{(i)}_{N|N})\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial\bullet}\right)$$

$$\frac{\partial^2 R_{k|N}}{\partial\bullet\,\partial v^{(i)}_{N|N}} = -T^{(i)}_{k,N}\sin(b_{k|N} - C^{(i)}_{N|N})\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial\bullet}\right)$$

$$\frac{\partial^2 R_{k|N}}{\partial\bullet\partial f_{0j|N}} = 0$$

Table 4.8 *Second order Range derivatives.*

The bottom-bounce bearing derivatives are obtained by differentiation of the expression (4.23):

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial y_{N|N}} = \frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial b_{k|N}}\frac{\partial b_{k|N}}{\partial y_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}}\frac{\partial^2 b_{k|N}}{\partial\bullet\partial y_{N|N}} + \frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\partial\alpha_{k|N}}\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\frac{\partial R_{k|N}}{\partial y_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial\alpha_{k|N}}\frac{\partial^2\alpha_{k|N}}{\partial\bullet\partial R_{k|N}}\frac{\partial R_{k|N}}{\partial y_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial\alpha_{k|N}}\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\frac{\partial^2 R_{k|N}}{\partial\bullet\,\partial y_{N|N}} \qquad (4.38)$$

We memorize that the first order derivatives in (4.38) are already known from the Tables 4.1, 4.3 and 4.4. The derivatives $\partial^2 b^{BB}_{k|N}/\partial\bullet\partial b_{k|N}$ and $\partial^2 b^{BB}_{k|N}/\partial\bullet\partial\alpha_{k|N}$ can be found from:

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\ \partial b_{k|N}} = \left(\frac{\partial}{\partial\bullet}\ (1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{-1/2}\ \right)\alpha_{k|N}\mathrm{SIN}(\beta_{k|N})\mathrm{SIGN}(\beta_{k|N})$$

$$+\ (1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{-1/2}\ \frac{\partial}{\partial\bullet}(\alpha_{k|N}\ \mathrm{SIN}(\beta_{k|N})\ \mathrm{SIGN}(\beta_{k|N})) \tag{4.39}$$

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\ \partial \alpha_{k|N}} = -\left(\frac{\partial}{\partial\bullet}\ (1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{-1/2}\ \right)\bullet\cos(\beta_{k|N})$$

$$+\ (1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{-1/2}\ \mathrm{SIN}(\beta_{k|N})\frac{\partial b_{k|N}}{\partial\bullet} \tag{4.40}$$

It is obvious that

$$\frac{\partial}{\partial\bullet}\ (1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{-1/2} = \frac{\alpha_{k|N}\cos^2\beta_{k|N}}{(1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{+3/2}}\ \frac{\partial\alpha_{k|N}}{\partial\bullet}$$

$$+\ \frac{-\alpha^2_{k|N}\cos\beta_{k|N}\mathrm{SIN}\beta_{k|N}}{(1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{+3/2}}\ \frac{\partial b_{k|N}}{\partial\bullet} \tag{4.41}$$

Substitution in (4.39) leads to

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\ \partial b_{k|N}} = \frac{\mathrm{SIGN}(\beta_{k|N})}{(1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{3/2}}\ \{\ (\alpha^2_{k|N}\mathrm{SIN}(\beta_{k|N})\cos^2\beta_{k|N}$$

$$+\ (1-\alpha^2_{k|N}\cos^2\beta_{k|N})\mathrm{SIN}(\beta_{k|N}))\frac{\partial\alpha_{k|N}}{\partial\bullet}\ +\ (-\alpha^3_{k|N}\cos(\beta_{k|N})\mathrm{SIN}^2(\beta_{k|N})$$

$$+\ (1-\alpha^2_{k|N}\cos^2\beta_{k|N})\alpha_{k|N}\cos(\beta_{k|N}))\frac{\partial b_{k|N}}{\partial\bullet}\ \} \tag{4.42}$$

$$= \frac{\mathrm{SIGN}(\beta_{k|N})}{(1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{3/2}}\ \{\mathrm{SIN}(\beta_{k|N})\frac{\partial\alpha_{k|N}}{\partial\bullet} + \alpha_{k|N}\cos\beta_{k|N}(1-\alpha^2_{k|N})\frac{\partial b_{k|N}}{\partial\bullet}\ \}$$

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial\alpha_{k|N}} = (1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{-3/2}(\mathrm{SIN}(\beta_{k|N})\frac{\partial b_{k|N}}{\partial\bullet} - \alpha_{k|N}\bullet\cos^3(\beta_{k|N})\frac{\partial\alpha_{k|N}}{\partial\bullet}) \tag{4.43}$$

The 2nd derivative $\partial^2\alpha_{k|N}/\partial\bullet\partial R_{k|N}$ in the 4th term of (4.38) is equal to

$$\frac{\partial^2\alpha_{k|N}}{\partial\bullet\,\partial R_{k|N}} = \frac{-3D^2_{BB}\;R_{k|N}}{(R^2_{k|N}+D^2_{BB})^{5/2}}\;\frac{\partial R_{k|N}}{\partial\bullet} \tag{4.44}$$

The result is recapitulized in Table 4.9.

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial b_{k|N}} = \frac{\mathrm{SIGN}(\beta_{k|N})}{(1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{3/2}}\;(\mathrm{SIN}(\beta_{k|N})\;\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\;\frac{\partial R_{k|N}}{\partial\bullet}$$

$$+\;\alpha_{k|N}\cos\beta_{k|N}(1-\alpha^2_{k|N})\frac{\partial b_{k|N}}{\partial\bullet})$$

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial\alpha_{k|N}} = (1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{-3/2}(\mathrm{SIN}(\beta_{k|N})\frac{\partial b_{k|N}}{\partial\bullet} - \alpha_{k|N}\bullet\cos^3(\beta_{k|N})\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\;\frac{\partial R_{k|N}}{\partial\bullet})$$

$$\frac{\partial^2\alpha_{k|N}}{\partial\bullet\,\partial R_{k|N}} = \frac{-3D^2_{BB}\;R_{k|N}}{(R^2_{k|N}+D^2_{BB})^{5/2}}\;\frac{\partial R_{k|N}}{\partial\bullet}$$

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial y_{N|N}} = \frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial b_{k|N}}\;\frac{\partial b_{k|N}}{\partial y_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}}\;\frac{\partial^2 b_{k|N}}{\partial\bullet\partial y_{N|N}} + \frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\partial\alpha_{k|N}}\;\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\;\frac{\partial R_{k|N}}{\partial y_{N|N}}$$

$$+\;\frac{\partial b^{BB}_{k|N}}{\partial\alpha_{k|N}}\;\frac{\partial^2\alpha_{k|N}}{\partial\bullet\;\partial R_{k|N}}\;\frac{\partial R_{k|N}}{\partial y_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial\alpha_{k|N}}\;\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\;\frac{\partial^2 R_{k|N}}{\partial\bullet\;\partial y_{N|N}}$$

Table 4.9 *Second order bottom-bounce bearing derivatives.*

The bottom-bounce frequency derivatives are determined from (4.26). By partial differentiation of $f^{BB}_{kj|N}$ with respect to $f_{0j|N}$, $f_{kj|N}$ and $R_{k|N}$ we obtain

$$\frac{\partial^2 f^{BB}_{kj|N}}{\partial\cdot\ \partial y_{N|N}} = \frac{\partial^2 f^{BB}_{kj|N}}{\partial\cdot\partial f_{kj|N}} \frac{\partial f_{kj|N}}{\partial y_{N|N}} + \frac{\partial f^{BB}_{kj|N}}{\partial f_{kj|N}} \frac{\partial^2 f_{kj|N}}{\partial\cdot\partial y_{N|N}} + \frac{\partial^2 f^{BB}_{kj|N}}{\partial\cdot\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial y_{N|N}}$$

$$+ \frac{\partial f^{BB}_{kj|N}}{\partial R_{k|N}} \frac{\partial^2 R_{k|N}}{\partial\cdot\partial y_{N|N}} + \frac{\partial^2 f^{BB}_{kj|N}}{\partial\cdot\partial f_{0j|N}} \frac{\partial f_{0j|N}}{\partial y_{N|N}} + \frac{\partial f^{BB}_{kj|N}}{\partial f_{0j|N}} \frac{\partial^2 f_{0j|N}}{\partial\cdot\partial y_{N|N}} \tag{4.45}$$

The first derivatives in (4.45) are known from the Tables 4.2, 4.3. and 4.5. Moreover $\partial^2 f_{kj|N}/\partial\cdot\partial y_{N|N}$ and $\partial^2 R_{N|N}/\partial\cdot\partial y_{N|N}$ are obtained from the Tables 4.7 and 4.8. The remaining factors to be derived are:

$$\frac{\partial f^{BB}_{kj|N}}{\partial f_{kj|N}} = \alpha_{k|N}$$

$$\frac{\partial^2 f^{BB}_{kj|N}}{\partial\cdot\ \partial f_{kj|N}} = \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial\cdot} \tag{4.46}$$

$$\frac{\partial^2 f_{0j|N}}{\partial\cdot\partial y_{N|N}} = 0$$

and

$$\frac{\partial^2 f^{BB}_{kj|N}}{\partial\cdot\ \partial R_{k|N}} = \left(\frac{\partial f_{kj|N}}{\partial\cdot} - \frac{\partial f_{0j|N}}{\partial\cdot}\right)\frac{\partial \alpha_{k|N}}{\partial R_{k|N}} - f_{0j|N} \frac{\dot{r}_{k|N}}{c} \frac{\partial^2 \alpha_{k|N}}{\partial\cdot\ \partial R_{k|N}} \tag{4.47}$$

Note that terms with $\partial f_{kj|N}$ / $\partial R_{k|N}$ become zero since we use the partial derivatives of (4.26) with respect to $f_{kj|N}$ and $R_{k|N}$ to obtain (4.45). The result is stated in Table 4.10.

$$
\begin{aligned}
\frac{\partial^2 f^{BB}_{kj|N}}{\partial \bullet\, \partial y_{N|N}} = {} & \overset{\substack{\text{Table 4.4}\\ \downarrow}}{\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}}\, \overset{\substack{\text{Table 4.3}\\ \downarrow}}{\frac{\partial R_{k|N}}{\partial \bullet}}\, \overset{\substack{\text{Table 4.2}\\ \downarrow}}{\frac{\partial f_{kj|N}}{\partial y_{N|N}}} + \alpha_{k|N}\, \overset{\substack{\text{Table 4.7}\\ \downarrow}}{\frac{\partial^2 f_{kj|N}}{\partial \bullet\, \partial y_{N|N}}} \\
& + \Big(\overset{\substack{\text{Table 4.2}\\ \downarrow}}{\frac{\partial f_{kj|N}}{\partial \bullet}} - \frac{\partial f_{0j|N}}{\partial \bullet}\Big) \overset{\substack{\text{Table 4.4}\\ \downarrow}}{\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}}\, \overset{\substack{\text{Table 4.3}\\ \downarrow}}{\frac{\partial R_{k|N}}{\partial y_{N|N}}} \\
& - f_{0j|N}\, \frac{\dot{r}_{k|N}}{c} \Big(\overset{\substack{\text{Table 4.9}\\ \downarrow}}{\frac{\partial^2 \alpha_{k|N}}{\partial \bullet\, \partial R_{k|N}}}\, \overset{\substack{\text{Table 4.3}\\ \downarrow}}{\frac{\partial R_{k|N}}{\partial y_{N|N}}} - \overset{\substack{\text{Table 4.4}\\ \downarrow}}{\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}}\, \overset{\substack{\text{Table 4.8}\\ \downarrow}}{\frac{\partial^2 R_{k|N}}{\partial \bullet\, \partial y_{N|N}}} \Big) \\
& - \frac{\partial \alpha_{k|N}}{\partial R_{k|N}}\, \frac{\partial R_{k|N}}{\partial \bullet}\; e_{2(m+1)+j}
\end{aligned}
$$

Table 4.10 *Second order bottom-bounce frequency derivatives.*

5 CONCLUSIONS

In this report the outline of the MLE method for TMA is described. The method requires some kind of optimization procedure. Since TMA problems are often ill-conditioned a robust numeric optimization method is required. In [Gmelig Meyling] a Newton-type method is proposed by which TMA problems can be solved very efficiently. This report describes how numerical experiments can be checked by using the Cramer-Rao bound and moreover that the MLE may not produce unbiased estimates. Newton-type methods require first order derivatives of the function to be optimized. Near the optimum also second order derivative information can be used to improve the convergence of the method. Although derivative information may be obtained by using finite differences much computation time is saved by using the analytic expressions which have been derived in chapter 4. The experimental results and specific details about the Newton-type method which we prefer can be found in two related reports [Gmelig Meyling, Gmelig Meyling and de Vlieger].

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SOME MATRIX DIFFERENTIATION RULES

Suppose x, y, z and m are arbitrary column vectors, M is a matrix, and f is a scalar function, then the following theorems hold:

(1) If $y = Mx$ then
$dy = Mdx$ implies $dy/dx = M'$ and conversely

(2) If $f(x) = m'x$ then
$df = dm'x = m'dx$ implies $df/dx = m$ and conversely

(3) If $x = mf$ then
$dx = mdf$ implies $dx/df = m'$ and conversely

(4) $dz/dx = dy/dx \, . \, dz/dy$

(5) If $f(x) = x'Mx$ then
$df = (dx)'Mx + x'Mdx$ implies $df/dx = (M+M')x$
and conversely.

(6) $df/dxdy = d(df/dy)/dx$

(7) If $f = x'My$ then
$df = (dx)'My + x'Mdy$ implies
$df/dy = M'x$ and $d^2f/dxdy = M$

(8) If $f = x'Mx$ then

$$d^2f/dx^2 = M' + M$$

TABLES OF CHAPTER 4

In this appendix the tables with first and second order derivatives are repeated as a quick look facility for those readers who are only interested in the results of Chapter 4.

$$\frac{\partial b_{k|N}}{\partial b_{N|N}} = \frac{R_{N|N}}{R_{k|N}} \cos(b_{N|N} - b_{k|N})$$

$$\frac{\partial b_{k|N}}{\partial R_{N|N}} = \frac{1}{R_{k|N}} \sin(b_{N|N} - b_{k|N})$$

$$\frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} = \frac{T^{(i)}_{k,N}}{R_{k|N}} v^{(i)}_{N|N} \cos(b_{k|N} - C^{(i)}_{N|N}) \qquad i = 1, \ldots, m$$

$$\frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} = - \frac{T^{(i)}_{k,N}}{R_{k|N}} \sin(b_{k|N} - C^{(i)}_{N|N}) \qquad i = 1, \ldots, m$$

$$\frac{\partial b_{k|N}}{\partial f_{0j|N}} = 0$$

Table 4.1 *First derivatives of Direct-path bearing estimate* $b_{k|N}$.

$$\frac{\partial f_{kj|N}}{\partial b_{N|N}} = \frac{-f_{0j|N}}{c} v^{TAN}_{k|N} \frac{R_{N|N}}{R_{k|N}} \cos(b_{N|N} - b_{k|N})$$

$$\frac{\partial f_{kj|N}}{\partial R_{N|N}} = \frac{-f_{0j|N}}{c} v^{TAN}_{k|N} \frac{1}{R_{k|N}} \sin(b_{N|N} - b_{k|N})$$

$$\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} = \frac{-f_{0j|N}}{c} v^{(i)}_{N|N} \sin(b_{k|N} - C^{(i)}_{N|N})$$

$$t^{(i-1)} \le t_k < t^{(i)} \text{ or } t_k = t^{(m)}$$

$$- \frac{f_{0j|N}}{c} v^{(i)}_{N|N} \frac{v^{TAN}_{k|N}\ T^{(i)}_{k,N}}{R_{k|N}} \cos(b_{k|N} - C^{(i)}_{N|N})$$

$$\frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}} = \frac{-f_{0j|N}}{c} \cos(b_{k|N} - C^{(i)}_{N|N})$$

$$t^{(i-1)} \le t_k < t^{(i)} \text{ or } t_k = t^{(m)}$$

$$+ \frac{f_{0j|N}}{c} \frac{v^{TAN}_{k|N}\ T^{(i)}_{k,N}}{R_{k|N}} \sin(b_{k|N} - C^{(i)}_{N|N})$$

$$\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} = - \frac{f_{0j|N}}{c} v^{(i)}_{N|N} \frac{v^{TAN}_{k|N}\ T^{(i)}_{k,N}}{R_{k|N}} \cos(b_{k|N} - C^{(i)}_{N|N})$$

$$t_k \le t^{(i-1)} \text{ or } t_k > t^{(i)}$$

$$\frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}} = \frac{f_{0j|N}}{c} \frac{v^{TAN}_{k|N} T^{(i)}_{k,N}}{R_{k|N}} \sin(b_{k|N} - C^{(i)}_{N|N})$$

$$t_k \le t^{(i-1)} \text{ or } t_k > t^{(i)}$$

$$\frac{\partial f_{kj|N}}{\partial f_{0j|N}} = 1 - \frac{\dot{r}_{k|N}}{c}$$

$$\frac{\partial f_{kj|N}}{\partial f_{0i|N}} = 0 \qquad i \ne j$$

Table 4.2 *First derivatives of direct-path frequency estimate* $f_{kj|N}$.

$$\frac{\partial R_{k|N}}{\partial b_{N|N}} = -R_{N|N}\sin(b_{N|N} - b_{k|N})$$

$$\frac{\partial R_{k|N}}{\partial R_{N|N}} = \cos(b_{N|N} - b_{k|N})$$

$$\frac{\partial R_{k|N}}{\partial C^{(i)}_{N|N}} = T^{(i)}_{k,N}.v^{(i)}_{N|N}.\sin(b_{k|N} - C^{(i)}_{N|N}) \qquad i = 1, \ldots, m$$

$$\frac{\partial R_{k|N}}{\partial v^{(i)}_{N|N}} = T^{(i)}_{k,N}\cos(b_{k|N} - C^{(i)}_{N|N}) \qquad i = 1, \ldots, m$$

$$\frac{\partial R_{k|N}}{\partial f_{0j|N}} = 0$$

Table 4.3 *First derivatives of Range estimate* $R_{k|N}$.

$$\beta_{k|N} = b_{k|N} - C_k^{OS}$$

$$\frac{\partial b_{k|N}^{BB}}{\partial b_{k|N}} = \frac{\alpha_{k|N}\text{SIN}(\beta_{k|N}).\text{SIGN}(\beta_{k|N})}{(1-\alpha_{k|N}^2\cos^2(\beta_{k|N}))^{1/2}}$$

$$\frac{\partial b_{k|N}^{BB}}{\partial \alpha_{k|N}} = -\frac{\cos(\beta_{k|N})}{(1-\alpha_{k|N}^2.\cos^2(\beta_{k|N}))^{1/2}}$$

$$\frac{\partial \alpha_{k|N}}{\partial R_{k|N}} = \frac{4d^2}{(R_{k|N}^2 + 4d^2)^{3/2}}$$

Table 4.1 ↓ Table 4.3 ↓

$$\frac{\partial b_{k|N}^{BB}}{\partial b_{N|N}} = \frac{\partial b_{k|N}^{BB}}{\partial b_{k|N}}\frac{\partial b_{k|N}}{\partial b_{N|N}} + \frac{\partial b_{k|N}^{BB}}{\partial \alpha_{k|N}}\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}\frac{\partial R_{k|N}}{\partial b_{N|N}}$$

$$\frac{\partial b_{k|N}^{BB}}{\partial R_{N|N}} = \frac{\partial b_{k|N}^{BB}}{\partial b_{k|N}}\frac{\partial b_{k|N}}{\partial R_{N|N}} + \frac{\partial b_{k|N}^{BB}}{\partial \alpha_{k|N}}\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}\frac{\partial R_{k|N}}{\partial R_{N|N}}$$

$$\frac{\partial b_{k|N}^{BB}}{\partial C_{N|N}^{(i)}} = \frac{\partial b_{k|N}^{BB}}{\partial b_{k|N}}\frac{\partial b_{k|N}}{\partial C_{N|N}^{(i)}} + \frac{\partial b_{k|N}^{BB}}{\partial \alpha_{k|N}}\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}\frac{\partial R_{k|N}}{\partial C_{N|N}^{(i)}}$$

$$i = 1, \ldots, m$$

$$\frac{\partial b_{k|N}^{BB}}{\partial v_{N|N}^{(i)}} = \frac{\partial b_{k|N}^{BB}}{\partial b_{k|N}}\frac{\partial b_{k|N}}{\partial v_{N|N}^{(i)}} + \frac{\partial b_{k|N}^{BB}}{\partial \alpha_{k|N}}\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}\frac{\partial R_{k|N}}{\partial v_{N|N}^{(i)}}$$

$$\frac{\partial b_{k|N}^{BB}}{\partial f_{0j|N}} = 0$$

Table 4.4 *First Derivatives of Bottom-bounce bearing.*

$$\frac{\partial f^{BB}_{kj|N}}{\partial b_{N|N}} = \alpha_{k|N} \overset{\text{Table 4.2}}{\overbrace{\frac{\partial f_{kj|N}}{\partial b_{N|N}}}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \overset{\text{Table 4.4}}{\overbrace{\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}}} \overset{\text{Table 4.3}}{\overbrace{\frac{\partial R_{k|N}}{\partial b_{N|N}}}}$$

$$\frac{\partial f^{BB}_{kj|N}}{\partial R_{N|N}} = \alpha_{k|N} \frac{\partial f_{kj|N}}{\partial R_{N|N}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial R_{N|N}}$$

$$\frac{\partial f^{BB}_{kj|N}}{\partial C^{(i)}_{N|N}} = \alpha_{k|N} \frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial C^{(i)}_{N|N}}$$

$$i = 1, \ldots, m$$

$$\frac{\partial f^{BB}_{kj|N}}{\partial v^{(i)}_{N|N}} = \alpha_{k|N} \frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}} - \frac{f_{0j|N}\dot{r}_{k|N}}{c} \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial v^{(i)}_{N|N}}$$

$$\frac{\partial f^{BB}_{kj|N}}{\partial f_{0j|N}} = \left(1 - \alpha_{k|N}\frac{\dot{r}_{k|N}}{c}\right)$$

Table 4.5 *First Derivatives of Bottom-bounce frequency.*

$$\frac{\partial^2 b_{k|N}}{\partial \bullet\, \partial b_{N|N}} = -\frac{1}{R_{k|N}} \frac{\partial b_{k|N}}{\partial b_{N|N}} \frac{\partial R_{k|N}}{\partial \bullet} + \frac{1}{R_{N|N}} \frac{\partial b_{k|N}}{\partial b_{N|N}} \frac{\partial R_{N|N}}{\partial \bullet}$$

$$+ R_{N|N} \frac{\partial b_{k|N}}{\partial R_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial b_{N|N}}{\partial \bullet}\right)$$

$$\frac{\partial^2 b_{k|N}}{\partial \bullet\, \partial R_{N|N}} = -\frac{1}{R_{k|N}} \frac{\partial b_{k|N}}{\partial R_{N|N}} \frac{\partial R_{k|N}}{\partial \bullet} - \frac{1}{R_{N|N}} \frac{\partial b_{k|N}}{\partial b_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial b_{N|N}}{\partial \bullet}\right)$$

$$\frac{\partial^2 b_{k|N}}{\partial \bullet\, \partial C^{(i)}_{N|N}} = -\frac{1}{R_{k|N}} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} \frac{\partial R_{k|N}}{\partial \bullet} + \frac{1}{v^{(i)}_{N|N}} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} \frac{\partial v^{(i)}_{N|N}}{\partial \bullet}$$

$$+ v^{(i)}_{N|N} \frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial \bullet}\right)$$

$$i = 1, \ldots, m$$

$$\frac{\partial^2 b_{k|N}}{\partial \bullet\, \partial v^{(i)}_{N|N}} = -\frac{1}{R_{k|N}} \frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}} \frac{\partial R_{k|N}}{\partial \bullet}$$

$$- \frac{1}{v^{(i)}_{N|N}} \frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}} \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial \bullet}\right)$$

$$i = 1, \ldots, m$$

$$\frac{\partial^2 b_{k|N}}{\partial \bullet\, \partial f_{0j}} = 0$$

Note: $\bullet$ is an element of $\mathbf{y}_{N|N}$

Table 4.6 *Second Derivatives of Direct-Path bearing.*

$$\frac{\partial^2 f_{kj|N}}{\partial\bullet\,\partial b_{N|N}} = \frac{\partial f_{kj|N}}{\partial b_{N|N}}\left(\frac{1}{R_{N|N}}\frac{\partial R_{N|N}}{\partial\bullet} - \frac{1}{R_{k|N}}\frac{\partial R_{k|N}}{\partial\bullet} + \frac{1}{f_{0j|N}}\frac{\partial f_{0j|N}}{\partial\bullet} + \frac{1}{v^{TAN}_{k|N}}\frac{\partial v^{TAN}_{k|N}}{\partial\bullet}\right)$$

$$+ R_{N|N}\frac{\partial f_{kj|N}}{\partial R_{N|N}}\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial b_{N|N}}{\partial\bullet}\right)$$

$$\frac{\partial^2 f_{kj|N}}{\partial\bullet\,\partial R_{N|N}} = \frac{\partial f_{kj|N}}{\partial R_{N|N}}\left(-\frac{1}{R_{k|N}}\frac{\partial R_{k|N}}{\partial\bullet} + \frac{1}{f_{0j|N}}\frac{\partial f_{0j|N}}{\partial\bullet} + \frac{1}{v^{TAN}_{k|N}}\frac{\partial v^{TAN}_{k|N}}{\partial\bullet}\right)$$

$$- \frac{1}{R_{N|N}}\frac{\partial f_{kj|N}}{\partial b_{N|N}}\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial b_{N|N}}{\partial\bullet}\right)$$

$$\frac{\partial^2 f_{kj|N}}{\partial\bullet\,\partial C^{(i)}_{N|N}} = \frac{1}{f_{0j|N}}\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}}\frac{\partial f_{0j|N}}{\partial\bullet} + \frac{1}{v^{(i)}_{N|N}}\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}}\frac{\partial v^{(i)}_{N|N}}{\partial\bullet}$$

$$- \frac{f_{0j|N}}{c}\frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}}\frac{\partial v^{TAN}_{k|N}}{\partial\bullet} + v^{(i)}_{N|N}\frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}}\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial\bullet}\right)$$

$$+ \frac{f_{0j|N}}{c}\frac{v^{TAN}_{k|N}}{R_{k|N}}\frac{\partial b_{k|N}}{\partial C^{(i)}_{N|N}}\frac{\partial R_{k|N}}{\partial\bullet}$$

$$\frac{\partial^2 f_{kj|N}}{\partial\bullet\,\partial v^{(i)}_{N|N}} = \frac{1}{f_{0j|N}}\frac{\partial f_{kj|N}}{\partial v^{(i)}_{N|N}}\frac{\partial f_{0j|N}}{\partial\bullet} - \frac{1}{v^{(i)}_{N|N}}\frac{\partial f_{kj|N}}{\partial C^{(i)}_{N|N}}\left(\frac{\partial b_{k|N}}{\partial\bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial\bullet}\right)$$

$$+ \frac{f_{0j|N}}{c}\frac{v^{TAN}_{k|N}}{R_{k|N}}\frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}}\frac{\partial R_{k|N}}{\partial\bullet} - \frac{f_{0j|N}}{c}\frac{\partial b_{k|N}}{\partial v^{(i)}_{N|N}}\frac{\partial v^{TAN}_{k|N}}{\partial\bullet}$$

$$\frac{\partial^2 f_{kj|N}}{\partial\bullet\,\partial f_{0j|N}} = -\frac{1}{c}\frac{\partial \dot{r}_{k|N}}{\partial\bullet}$$

Note: $\bullet$ is an element of $\mathbf{y}_{N|N}$

Table 4.7 ***Second Derivatives of Direct-Path frequency.***

$$\frac{\partial^2 R_{k|N}}{\partial \bullet\, \partial b_{N|N}} = \frac{\partial R_{N|N}}{\partial \bullet} \sin(b_{k|N} - b_{N|N}) + R_{N|N} \cos(b_{k|N} - b_{N|N}) \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial b_{N|N}}{\partial \bullet}\right)$$

$$\frac{\partial^2 R_{k|N}}{\partial \bullet\, \partial R_{N|N}} = - \sin(b_{k|N} - b_{N|N}) \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial b_{N|N}}{\partial \bullet}\right)$$

$$\frac{\partial^2 R_{k|N}}{\partial \bullet\, \partial C^{(i)}_{N|N}} = T^{(i)}_{k,N} \sin(b_{k|N} - C^{(i)}_{N|N}) \frac{\partial v^{(i)}_{N|N}}{\partial \bullet}$$

$$+ T^{(i)}_{k,N} v^{(i)}_{N|N} \cos(b_{k|N} - C^{(i)}_{N|N}) \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial \bullet}\right)$$

$$\frac{\partial^2 R_{k|N}}{\partial \bullet\, \partial v^{(i)}_{N|N}} = -T^{(i)}_{k,N} \sin(b_{k|N} - C^{(i)}_{N|N}) \left(\frac{\partial b_{k|N}}{\partial \bullet} - \frac{\partial C^{(i)}_{N|N}}{\partial \bullet}\right)$$

$$\frac{\partial^2 R_{k|N}}{\partial \bullet \partial f_{0j|N}} = 0$$

Table 4.8 *Second order Range derivatives.*

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial b_{k|N}} = \frac{\text{SIGN}(\beta_{k|N})}{(1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{3/2}}\,\{\text{SIN}(\beta_{k|N})\,\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\,\frac{\partial R_{k|N}}{\partial\bullet}$$

$$+\ \alpha_{k|N}\cos\beta_{k|N}(1-\alpha^2_{k|N})\frac{\partial b_{k|N}}{\partial\bullet}\}$$

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial\alpha_{k|N}} = (1-\alpha^2_{k|N}\cos^2\beta_{k|N})^{-3/2}(\text{SIN}(\beta_{k|N})\frac{\partial b_{k|N}}{\partial\bullet} - \alpha_{k|N}\bullet\cos^3(\beta_{k|N})\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\,\frac{\partial R_{k|N}}{\partial\bullet}\,)$$

$$\frac{\partial^2\alpha_{k|N}}{\partial\bullet\,\partial R_{k|N}} = \frac{-3D^2_{BB}\ R_{k|N}}{(R^2_{k|N}+D^2_{BB})^{5/2}}\,\frac{\partial R_{k|N}}{\partial\bullet}$$

$$\frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial y_{N|N}} = \frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial b_{k|N}}\,\frac{\partial b_{k|N}}{\partial y_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial b_{k|N}}\,\frac{\partial^2 b_{k|N}}{\partial\bullet\,\partial y_{N|N}} + \frac{\partial^2 b^{BB}_{k|N}}{\partial\bullet\,\partial\alpha_{k|N}}\,\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\,\frac{\partial R_{k|N}}{\partial y_{N|N}}$$

$$+\ \frac{\partial b^{BB}_{k|N}}{\partial\alpha_{k|N}}\,\frac{\partial^2\alpha_{k|N}}{\partial\bullet\,\partial R_{k|N}}\,\frac{\partial R_{k|N}}{\partial y_{N|N}} + \frac{\partial b^{BB}_{k|N}}{\partial\alpha_{k|N}}\,\frac{\partial\alpha_{k|N}}{\partial R_{k|N}}\,\frac{\partial^2 R_{k|N}}{\partial\bullet\,\partial y_{N|N}}$$

Table 4.9 *Second order bottom-bounce bearing derivatives.*

$$
\begin{aligned}
\frac{\partial^2 f^{BB}_{kj|N}}{\partial \bullet \, \partial y_{N|N}} = {} & \overset{\text{Table 4.4}}{\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}} \overset{\text{Table 4.3}}{\frac{\partial R_{k|N}}{\partial \bullet}} \overset{\text{Table 4.2}}{\frac{\partial f_{kj|N}}{\partial y_{N|N}}} + \alpha_{k|N} \overset{\text{Table 4.7}}{\frac{\partial^2 f_{kj|N}}{\partial \bullet \, \partial y_{N|N}}} \\
& + \left(\overset{\text{Table 4.2}}{\frac{\partial f_{kj|N}}{\partial \bullet}} - \frac{\partial f_{0j|N}}{\partial \bullet} \right) \overset{\text{Table 4.4}}{\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}} \overset{\text{Table 4.3}}{\frac{\partial R_{k|N}}{\partial y_{N|N}}} \\
& - f_{0j|N} \frac{\dot{r}_{k|N}}{c} \left(\overset{\text{Table 4.9}}{\frac{\partial^2 \alpha_{k|N}}{\partial \bullet \, \partial R_{k|N}}} \overset{\text{Table 4.3}}{\frac{\partial R_{k|N}}{\partial y_{N|N}}} - \overset{\text{Table 4.4}}{\frac{\partial \alpha_{k|N}}{\partial R_{k|N}}} \overset{\text{Table 4.8}}{\frac{\partial^2 R_{k|N}}{\partial \bullet \partial y_{N|N}}} \right) \\
& - \frac{\partial \alpha_{k|N}}{\partial R_{k|N}} \frac{\partial R_{k|N}}{\partial \bullet} \quad e_{2(m+1)+j}
\end{aligned}
$$

Table 4.10 *Second order bottom-bounce frequency derivatives.*

This theorem can be proven by defining the vector X as

$$X = \begin{bmatrix} e(Z) - \lambda(y_N) \\ \hdashline \dfrac{\partial \ln(p(Z|y_N))}{\partial y_N} \end{bmatrix} \tag{C.1}$$

The covariance matrix of X must be non-negative definite:

$$E(XX^T|y_N) = \begin{bmatrix} \Lambda_e & I_n + \dfrac{\partial \lambda(y_N)}{\partial y_N} \\ I_n + (\dfrac{\partial \lambda(y_N)}{\partial y_N})^T & M \end{bmatrix} \geq 0 \tag{C.2}$$

Hence, for any vectors x and y the following quadratic form must be nonnegative:

$$x^T \Lambda_e x + x^T (I_n + \frac{\partial \lambda(y_N)}{\partial y_N}) y + y^T (I_n + \frac{\partial \lambda(y_N)}{\partial y_N})^T x + y^T M y \geq 0 \tag{C.3}$$

Assuming that both Λ_e and M are positive definite the minimum of the left expression is attained for:

$$y = - M^{-1} (I_n + \frac{\partial \lambda(y_N)}{\partial y_N})^T x \tag{C.4}$$

Substituting (C.4) into (C.3) leads to the following condition:

$$x^T[\ \Lambda_e - (I_n + \frac{\partial\lambda(y_N)}{\partial y_N})M^{-1}(I_n + \frac{\partial\lambda(y_N)}{\partial y_N})^T\ ]x \geq 0 \tag{C.5}$$

Hence, when the estimate has a bias $\lambda(y_N)$ the Cramer-Rao theorem changes into:

$$\Lambda_e \geq (I_n + \frac{\partial\lambda(y_N)}{\partial y_N})M^{-1}(I_n + \frac{\partial\lambda(y_N)}{\partial y_N})^T \tag{C.6}$$

If $\lambda(y_N)$ is zero, condition (C.6) reduces to (3.28).▪

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THE TRACKING PROBLEM OF UNKNOWN MARINE PLATFORMS USING PASSIVE SONAR MEASUREMENTS IS GENERALLY REFERRED TO AS *TARGET MOTION ANALYSIS* (TMA). THIS REPORT DESCRIBES THE APPLICATION OF A *MAXIMUM LIKELIHOOD ESTIMATION* (MLE) METHOD TO OBTAIN POSITION AND VELOCITY ESTIMATES OF A PLATFORM WHEN BEARING ANGLE AND FREQUENCY MEASUREMENTS ARE AVAILABLE FROM A PASSIVE SONAR SYSTEM. THE FREQUENCY MEASUREMENTS ARE RELATED TO ONE OR MORE CARDINAL FREQUENCY PEAKS FROM THE RADIATED FREQUENCY SPECTRUM BY THE PLATFORM.
THE MLE METHOD OFFERS THE OPPORTUNITY TO USE A MULTI-LEG MODEL, I.E. THE PLATFORM TO BE LOCALIZED IS ASSUMED TO MOVE ACCORDING TO A PIECEWISE LINEAR TRACK, WHERE EACH PART IS REFERRED TO AS A LEG. ON EACH LEG CONSTANT COURSE AND SPEED IS ASSUMED. ADDITIONALLY, BEARING AND FREQUENCY MEASUREMENTS RELATED TO BOTTOM REFLECTIONS OF THE ACOUSTICAL SIGNALS CAN BE USED.
IN REF. [GMELIG MEYLING, 1989-82] A NEWTON-TYPE OPTIMIZATION METHOD USING FIRST AND SECOND ORDER DERIVATIVE INFORMATION OF THE RESIDUAL FUNCTIONS IS PROPOSED. BY USING THE ANALYTIC EXPRESSIONS OF THE DERIVATIVES AS DESCRIBED IN THIS REPORT A MAJOR REDUCTION OF COMPUTATION TIME IS ACCOMPLISHED.

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