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by

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Exact Solution of a Quantum Forced Time-Dependent Harmonic Oscillator

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The Schrödinger equation is used to exactly evaluate the propagator, wave function, energy expectation values, uncertainty values and coherent state for a harmonic oscillator with a time-dependent frequency and an external driving time-dependent force. These quantities represent the solution of the classical equation of motion for the time-dependent harmonic oscillator.

I. Introduction

It is well known that an exact solution of the Schrödinger equation is possible only for special cases. For this reason, approximate methods are needed. Exact solutions provide important tests for these approximate methods and for various models of physical phenomena. In general, the solution of the Schrödinger equation for explicit time-dependent systems has met with limited success because of analytical difficulties, although progress has been made during the past three decades.[1-5] Camiz et al[6] have obtained the wave functions of a time-dependent harmonic oscillator perturbed by an inverse quadratic potential, using the Schrödinger formalism and a generating function. Further, Khandekar and Lawande[7] have evaluated the exact propagator and wave function for a time-dependent harmonic oscillator, both with and without an inverse quadratic potential, using Feynman path integrals. In addition, Jannussis et al[8] have calculated the propagator for several quantum mechanical systems with friction.

In a previous paper,[9] we have evaluated the propagator, wave function, energy expectation values, uncertainty values and transition amplitudes for a quantum damped driven harmonic oscillator by using path integral methods. Also, we have obtained the coherent state for the damped harmonic oscillator[10] and calculated the propagator for coupled driven harmonic oscillators.[11]

In this paper we discuss the exact quantum theory of a forced harmonic oscillator with a time-dependent frequency. In Sec. II we evaluate the propagator using the Schrödinger equation and path integral methods, and in Sec. III we calculate the wave functions using the propagator. In Sec. IV we define the energy operator and calculate energy expectation values. In Sec. V we obtain the uncertainty values. In Sec. IV we determine the coherent state and its properties. Finally, in Sec. VII we present results and a discussion.

II. Propagator

We consider a system whose classical Hamiltonian is of the form

$$H = \frac{1}{2M} p^2 + \frac{M}{2} \omega^2(t)\, x^2 - f(t)x \quad , \tag{2.1}$$

where x is a canonical coordinate, p is its conjugate momentum, $\omega(t)$ is a frequency as a function of time, M is a positive real mass, and f(t) is an external driving force. The Lagrangian corresponding to the Hamiltonian (2.1) is

$$L = \frac{1}{2} M\dot{x}^2 - \frac{1}{2} M\omega^2(t)x^2 + f(t)x \quad . \tag{2.2}$$

Here, the Hamiltonian H and Lagrangian L depend on time. The classical equation of motion for our system is

$$\frac{d^2}{dt^2} x + \omega^2(t)x = \frac{1}{M} f(t) \quad . \tag{2.3}$$

For the case where $\omega(t) = \omega_o$ (constant), the solution of Eq. (2.3) represents harmonic motion; otherwise, it is difficult to evaluate the exact solution.

The path integral formulation of Feynman provides an alternate approach to solving dynamical problems in quantum mechanics.[12] In this approach, the usual Schrödinger equation is replaced by the integral equation

$$\psi(x,t) = \int dx'\, K(x,t;x't')\, \psi(x',t') \qquad (t > t') \tag{2.4}$$

with the initial condition $\psi(x,t) = \psi(x',t)$. Here, $\psi(x,t)$ is a wave function and $K(x,t;\, x',t')$ is a propagator. The propagator $K(x,t;\, x',t')$ is defined by the path integral[12]

$$K(x,t;\, x',t') = \lim \int_{(x',t')}^{(x,t)} \prod_{j=1}^{N-1} dx_j\, \exp[\frac{i}{\hbar} S(x,t;\, x',t')] \quad , \tag{2.5}$$

where the integration is over all possible paths from the point (x',t') to the point (x,t), and $S(x,t;\, x',t')$ is the action defined as

$$S(x,t;\, x',t') = \int_{t'}^{t} d\tau\; L(\dot{x},x,\tau) \quad . \tag{2.6}$$

For a short time interval ϵ, substitution of Eqs. (2.2) and (2.6) into Eq. (2.5) gives the normalizing factor A_j and the usual Schrödinger equation:

$$A_j = (2i\pi\epsilon\hbar/M)^{\frac{1}{2}} \tag{2.7}$$

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}M\omega^2(t)\,x^2\,\psi - f(t)\,x\,\psi \quad . \tag{2.8}$$

Since $K(x,t;\, x't')$ can be thought of as a function of the variables (x,t) or of (x',t'), it is a special wave function, and it satisfies Eq. (2.8):

$$i\hbar \frac{\partial}{\partial t} K(x,t;\, x',t') = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} K(x,t;x',t') + \frac{1}{2}M\omega^2(t)\,x^2\,K(x,t;\, x',t')$$

$$- f(t)\,x\,K(x,t;x',t'), \qquad (t > t') \tag{2.9}$$

$$-i\hbar \frac{\partial}{\partial t'} K(x,t;x',t') = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x'^2} K(x,t;\, x't') + \frac{1}{2}M\omega(t')\,x'^2$$

$$\times\; K(x,t;x',t') - f(t')\,x'\,K(x,t;x',t') \quad ,$$

$$(t' > t) \quad . \tag{2.10}$$

Because the Lagrangian is quadratic, the propagator has the form[12,13]

$$K(x,t;\, x't') = \exp[a(t,t')x^2 + b(t,t')xx' + c(t,t')x'^2 + g(t,t')x$$

$$+ h(t,t')x' + d(t,t')] \quad , \tag{2.11}$$

where from Eqs. (2.9) and (2.10) we can easily deduce that the coefficient of the third and higher powers in x is zero.

Substituting Eq. (2.11) into Eqs. (2.9) and (2.10), we obtain the differential equations

$$\frac{d}{dt}\, a = \frac{2i\hbar}{M}\, a^2 + \frac{M}{2i\hbar}\, \omega^2(t) \tag{2.12}$$

$$\frac{d}{dt}\, (bx'+g) = \frac{2i\hbar}{M}\, a(bx'+g) + \frac{i}{\hbar}\, f(t) \tag{2.13}$$

$$\frac{d}{dt}\, (cx'^2+hx'+d) = \frac{i\hbar}{M}\, a + \frac{i\hbar}{2M}\, (bx'+g)^2 \tag{2.14}$$

$$\frac{d}{dt'}\, c = \frac{2\hbar}{Mi}\, c^2 + \frac{iM}{2\hbar}\, \omega(t')^2 \tag{2.15}$$

$$\frac{d}{dt'}\, (bx+h) = \frac{2\hbar}{Mi}\, c\, (bx+h) + \frac{f(t')}{i\hbar} \tag{2.16}$$

$$\frac{d}{dt'}\, (ax^2+gx+d) = \frac{\hbar}{2iM}\, (bx+h)^2 + \frac{\hbar}{Mi}\, c \quad . \tag{2.17}$$

Equations (2.12) and (2.15) are nonlinear equations. For the case where $\omega(t) = \omega_o$, a solution is easily found, but in other cases it is difficult to find an exact solution. If $q(t)$ obeys the differential equation

$$\frac{d^2}{dt^2}\, q(t) + \omega^2(t)\, q(t) = 0 \quad , \tag{2.18}$$

then the solutions of Eqs. (2.12)-(2.14) are

$$a(t) = \frac{iM}{2\hbar}\, \frac{\dot{q}(t)}{q(t)} \tag{2.19}$$

$$b(t)x' + g(t) = \frac{i}{\hbar}\, \frac{1}{q}\, [\int^t ds\; f(s)q(s) + b_o] \tag{2.20}$$

$$c(t)x'^2 + h(t)x' + d(t) = \ln q^{-\frac{1}{2}} + \frac{b_o^2}{2iM\hbar} \int^t \frac{ds}{q(s)^2} + \frac{b_o}{iM\hbar} \int^t \frac{ds}{q(s)^2}$$

$$\times \int^s dp\; f(p)q(p) + \frac{1}{2i\hbar M} \int^t \frac{ds}{q(s)^2} \int^s dp\; f(p)q(p) \int^s dr\; f(r)q(r) + d_o \quad , \tag{2.21}$$

where b_o and d_o are constants of integration and do not depend on t, and the solutions of Eqs. (2.15)-(2.17) are

$$c(t') = \frac{M}{2i\hbar}\, \frac{\dot{q}(t')}{q(t')} \tag{2.22}$$

$$b(t')x + h(t') = \frac{1}{i\hbar q}\, [\int^{t'} ds\; f(s)q(s) + b'_o] \tag{2.23}$$

$$a(t')x^2 + g(t')x + d(t') = \ln q(t')^{-\frac{1}{2}} + \frac{ib_o'^2}{2\hbar M}\int^{t'}\frac{ds}{q(s)^2} + \frac{ib_o'}{M\hbar}\int^{t}\frac{ds}{q(s)^2}$$

$$\times \int^{s} dp\ f(p)q(p) + \frac{1}{2iM\hbar}\int^{t'}\frac{ds}{q(s)^2}\int^{s} dp\ f(p)q(p)\int^{s} dr\ f(r)q(r) + d_o' \quad , \qquad (2.24)$$

where b_o' and d_o' are constants of integration and independent of t'. Since only t is a variable in Eqs. (2.19)-(2.21), we have suppressed t' in $a(t,t')$, $b(t,t')$, etc., and we have similarly suppressed t in Eqs. (2.22)-(2.24).

In polar form we may write

$$q(t) = \eta(t)e^{i\gamma(t)} \quad , \qquad (2.25)$$

where $\eta(t)$ and $\gamma(t)$ are real quantities. From Eqs. (2.18) and (2.25) we note that

$$\ddot{\eta}(t) - \eta(t)\ \dot{\gamma}^2(t) + \omega^2(t)\ \eta(t) = 0 \qquad (2.26)$$

$$2\dot{\eta}(t)\dot{\gamma}(t) + \eta(t)\ \ddot{\gamma}(t) = 0 \qquad (2.27)$$

$$\eta^2(t)\dot{\gamma}(t) = \Omega \qquad (2.27')$$

where the constant Ω is a time-variant quantity. From Eqs. (2.26) and (2.27), we find another form for the solution of Eq. (2.18) as

$$q(t) = \eta(t)\ \sin(\gamma-\gamma') \qquad (2.28)$$

$$q(t') = \eta(t')\ \sin(\gamma-\gamma') \quad , \qquad (2.29)$$

where $\gamma = \gamma(t)$ and $\gamma' = \gamma(t')$.

Substitution of Eq. (2.28) into Eqs. (2.19) and (2.21) gives

$$a(t) = \frac{iM}{2\hbar}\left[\frac{\dot{\eta}}{\eta} + \dot{\gamma}\cot(\gamma-\gamma')\right] \qquad (2.30)$$

$$b(t)x' + g(t) = \frac{ib_o}{\hbar\eta\sin(\gamma-\gamma)} + \frac{i}{\hbar\eta\sin(\gamma-\gamma')}\int^{t} ds\ \eta(s)f(s)\ \sin[\gamma(s)-\gamma'] \qquad (2.31)$$

$$c(t)x'^2 + h(t)x' + d(t) = \ln[\eta^{-\frac{1}{2}}\ \sin^{-\frac{1}{2}}(\gamma-\gamma')] + \frac{ib_o^2}{2\hbar\Omega M}\cot(\gamma-\gamma')$$

$$+\frac{b_o}{i\hbar\Omega M\,\sin(\gamma-\gamma')}\int^t ds\; f(s)\eta(s)\;\sin[\gamma(s)-\gamma')]$$

$$+\frac{1}{2i\hbar\Omega M\,\sin(\gamma-\gamma')}\int^t ds\int^t dp\; f(s)f(p)\eta(s)\eta(p)$$

$$\times\,\sin[\gamma(s)-\gamma']\sin[\gamma(p)-\gamma') \tag{2.32}$$

Furthermore, substitution of Eq. (2.29) into Eqs. (2.22) and (2.24) gives

$$c(t') = \frac{iM}{2\hbar}\,[-\frac{\dot{\eta}'}{\eta'}+\dot{\gamma}'\,\cot(\gamma-\gamma')] \tag{2.33}$$

$$b(t')x + h(t') = \frac{b'_o}{i\hbar\eta'\sin(\gamma-\gamma')}+\frac{1}{i\hbar\eta\sin(\gamma-\gamma')}\int^{t'} ds\; f(s)\eta(s)\sin[\gamma-\gamma(s)] \tag{2.34}$$

$$a(t')x^2 + g(t')x + d(t') = ln[\eta'^{-\frac{1}{2}}\,\sin^{-\frac{1}{2}}(\gamma-\gamma')] + \frac{ib'^{\,2}_o}{2\hbar\Omega M}\cot(\gamma-\gamma')$$

$$-\frac{ib'_o}{\hbar\Omega M\sin(\gamma-\gamma')}\int^{t'} ds\; f(s)\eta(s)\;\sin[\gamma-\gamma(s)]$$

$$+\frac{1}{2i\Omega\hbar M\sin(\gamma-\gamma')}\int^{t'} ds\int^{t'} dp\; f(s)f(p)\eta(s)\eta(p)$$

$$\times\,\sin[\gamma(s)-\gamma]\sin[\gamma(p)-\gamma]\quad . \tag{2.35}$$

From Eqs. (2.31), (2.32), (2.34) and (2.35), we deduce that the constants b_o and b'_o are given as

$$b_o = -M\dot{\gamma}'\eta'x' \tag{2.36}$$

$$b'_o = M\dot{\gamma}\eta x \quad . \tag{2.37}$$

Also, from the normalization condition,

$$d_0 = ln(\frac{M}{2\pi\hbar})^{\frac{1}{2}} \quad . \tag{2.38}$$

From Eqs. (2.30)-(2.37) and (2.27'), we find that

$$a(t,t') = \frac{iM}{2\hbar}\left[\frac{\dot{\eta}}{\eta} + \dot{\gamma}\cot(\gamma-\gamma')\right] \tag{2.39}$$

$$c(t,t') = \frac{iM}{2\hbar}\left[-\frac{\dot{\eta}'}{\eta'} + \dot{\gamma}'\cot(\gamma-\gamma')\right] \tag{2.40}$$

$$b(t,t') = -\frac{iM}{\hbar}\frac{\sqrt{\dot{\gamma}\dot{\gamma}'}}{\sin(\gamma-\gamma')} \tag{2.41}$$

$$g(t,t') = \frac{i\sqrt{\dot{\gamma}}}{\hbar\sin(\gamma-\gamma')}\int_{t'}^{t} ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma(s)-\gamma] \tag{2.42}$$

$$h(t,t') = \frac{i\sqrt{\dot{\gamma}'}}{\hbar\sin(\gamma-\gamma')}\int_{t'}^{t} ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma-\gamma(s)] \tag{2.43}$$

$$d(t,t') = -\frac{i}{2M\hbar\sin(\gamma-\gamma')}\int_{t'}^{t} ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma-\gamma(s)]\int_{t'}^{t} dp\,\frac{f(p)}{\sqrt{\dot{\gamma}(p)}} \times \sin[\gamma(p)-\gamma'] \tag{2.44}$$

Inserting Eqs. (2.39)-(2.44) in Eq. (2.11), we obtain the propagator for the forced time-dependent harmonic oscillator as

$$\begin{aligned} K(x,t;x't') = {} & \left[\frac{M\,(\dot{\gamma}'\dot{\gamma})^{\frac{1}{2}}}{2\pi i\hbar\sin(\gamma-\gamma')}\right] \\ & \times \exp\left[\frac{iM}{2\hbar}\left(\frac{\dot{\eta}}{\eta}x^2 - \frac{\dot{\eta}'}{\eta'}x'^2\right)\right] \\ & \times \exp\left\{\frac{iM}{2\hbar\sin(\gamma-\gamma')}\left[(\dot{\gamma}x^2 + \dot{\gamma}'x'^2)\cos(\gamma-\gamma') - 2\sqrt{\dot{\gamma}\dot{\gamma}'}\,xx'\right.\right. \\ & + \frac{2\sqrt{\dot{\gamma}}}{M}x\int_{t'}^{t} ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma(s)-\gamma'] \\ & + \frac{2\sqrt{\dot{\gamma}'}}{M}x'\int_{t'}^{t} ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma-\gamma(s)] \\ & \left.\left. - \frac{1}{M^2}\int_{t'}^{t} ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma-\gamma(s)]\int_{t'}^{t} dp\,\frac{f(p)}{\sqrt{\dot{\gamma}(p)}}\sin[\gamma(p)-\gamma']\right]\right\} , \end{aligned} \tag{2.45}$$

where the unprimed and the primed variables denote the quantities which are functions of time t and t', respectively. It may be easily verified that for the case where $\omega(t)$ is a real positive constant ω_o, we have $\eta(t) = 1$ and $\gamma(t) = \omega_o t$, and the propagator of Eq. (2.45) reduces to the usual expression for a forced harmonic oscillator.[12]

III: Wave function

We now rewrite the propagator in another form in order to derive the wave function:

$$
\begin{aligned}
K(x,t\colon xt) = &\left[\frac{M\,(\dot\gamma'\dot\gamma)^{1/2}}{2\pi i\hbar\,\sin(\gamma-\gamma')}\right]^{1/2} \\
&\times \exp\left[\frac{iM}{2\hbar}\left\{\left[\frac{\dot\eta}{\eta}x^2 + \frac{2}{M}\sqrt{\dot\gamma}\,x\int^{t} ds\,\frac{f(s)}{\sqrt{\dot\gamma(s)}}\cos[\gamma-\gamma(s)]\right]\right.\right. \\
&\left.\left. - \left[\frac{\dot\eta'}{\eta'}x'^2 + \frac{2}{M}\sqrt{\dot\gamma'}\,x'\int^{t'} ds\,\frac{f(s)}{\sqrt{\dot\gamma(s)}}\cos[\gamma'-\gamma(s)]\right]\right\}\right] \\
&\times \exp\left[\frac{iM}{2\hbar}\cot(\gamma-\gamma')\left\{\left[\sqrt{\dot\gamma}\,x - \frac{1}{M}\int^{t} ds\,\frac{f(s)}{\sqrt{\dot\gamma(s)}}\sin[\gamma-\gamma(s)]\right]^2\right.\right. \\
&\left. + \left[\sqrt{\dot\gamma'}\,x' - \frac{1}{M}\int^{t'} ds\,\frac{f(s)}{\sqrt{\dot\gamma(s)}}\sin[\gamma'-\gamma(s)]\right]^2\right\} \\
&- \frac{iM}{\hbar}\frac{1}{\sin(\gamma-\gamma')}\left\{\left[\sqrt{\dot\gamma}\,x - \frac{1}{M}\int^{t} ds\,\frac{f(s)}{\sqrt{\dot\gamma(s)}}\sin[\gamma-\gamma(s)]\right]\right. \\
&\left.\left.\times\left[\sqrt{\dot\gamma'}\,x' - \frac{1}{M}\int^{t'} ds\,\frac{f(s)}{\sqrt{\dot\gamma(s)}}\sin[\gamma'-\gamma(s)]\right]\right\}\right] \\
&\times \exp\left[\frac{i}{2\hbar M}\left\{\cot(\gamma-\gamma')\left[\int^{t} ds\,\frac{f(s)}{\sqrt{\dot\gamma(s)}}\sin[\gamma-\gamma(s)]\right]^2\right.\right. \\
&+ \cot(\gamma-\gamma')\left[\int^{t'} dp\,\frac{f(p)}{\sqrt{\dot\gamma(p)}}\sin[\gamma'-\gamma(p)]\right]^2 \\
&+ \frac{2}{\sin(\gamma-\gamma')}\int^{t} ds\,\sin(\gamma-\gamma_s)\int^{t'} dp\,\frac{f(p)}{\sqrt{\dot\gamma(p)}}\sin[\gamma'-\gamma(p)] \\
&- \frac{1}{\sin(\gamma-\gamma')}\int_{t'}^{t} ds\,\frac{f(s)}{\sqrt{\dot\gamma(s)}}\sin[\gamma-\gamma(s)]\int_{t'}^{t} dp\,\frac{f(p)}{\sqrt{\dot\gamma(p)}} \\
&\left.\left.\times \sin[\gamma(p)-\gamma']\right\}\right]
\end{aligned}
\tag{3.1}
$$

$$= \left(\frac{M}{\pi\hbar}\right)^{1/2} (\dot{\gamma}')^{1/4}(\dot{\gamma})^{1/4} \frac{e^{-i(\gamma-\gamma')}}{1 - e^{-2i(\gamma-\gamma')}}$$

$$\times \exp\left[\frac{iM}{2\hbar}\left\{\left[\frac{\dot{\eta}}{\eta} x^2 + \frac{2}{M}\sqrt{\dot{\gamma}}\, x \int^t ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \cos[\gamma-\gamma(s)]\right]\right.\right.$$

$$\left.\left. - \left[\frac{\dot{\eta}'}{\eta'} x'^2 + \frac{2}{M}\sqrt{\dot{\gamma}'}x' \int^t ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \cos[\gamma'-\gamma(s)]\right]\right\}\right]$$

$$\times \exp\left[\frac{M}{2\hbar}\left\{\left[\sqrt{\dot{\gamma}}\, x - \frac{1}{M}\int^t ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma-\gamma(s)]\right]^2\right.\right.$$

$$\left.\left. + \left[\sqrt{\dot{\gamma}'}x' - \frac{1}{M}\int^{t'} ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma'-\gamma(s)]\right]^2\right\}\right]$$

$$\times \exp\left[\frac{-M}{\hbar[1-e^{-2i(\gamma-\gamma')}]}\right]\left\{\left[\sqrt{\dot{\gamma}}\, x - \frac{1}{M}\int^t ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma-\gamma(s)]\right]^2\right.$$

$$+ \left[\sqrt{\dot{\gamma}'}x' - \frac{1}{M}\int^{t'} ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[(\gamma-\gamma(s)]\right]^2$$

$$- 2\left[\sqrt{\dot{\gamma}}\, x - \frac{1}{M}\int^t ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma-\gamma(s)]\right]\left[\sqrt{\dot{\gamma}'}x'\right.$$

$$\left.\left. - \frac{1}{M}\int^{t'} ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma'-\gamma(s)]\right]\right\}\Bigg] \; e^{-i\theta(t)}e^{i\theta(t')} \quad , \qquad (3.2)$$

where

$$\theta(t') - \theta(t) = \frac{1}{2\hbar M}\left\{\cot(\gamma-\gamma')\left[\int^t ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma-\gamma(s)]\right]^2\right.$$

$$+ \cot(\gamma-\gamma')\left[\int^{t'} ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma'-\gamma(s)]\right]^2 + 2\,\frac{1}{\sin(\gamma-\gamma')}$$

$$\times \int^t ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma-\gamma(s)] \int^{t'} dp \frac{f(p)}{\sqrt{\dot{\gamma}(p)}} \sin[\gamma'-\gamma(p)]$$

$$\left. - \int_{t'}^t ds \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma-\gamma(s)] \int_{t'}^t dp \frac{f(p)}{\sqrt{\dot{\gamma}(p)}} \sin[\gamma(p)-\gamma'] \right. \quad . \qquad (3.3)$$

Let us introduce Mehler's formula,[14]

$$\frac{\exp[-(X^2+Y^2-2XY)/(1-Z^2)]}{\sqrt{1-Z^2}} = e^{-(X^2+Y^2)}\sum_{n=0}^{\infty}\frac{Z^n}{2^n n!}H_n(X)H_n(Y) \tag{3.4}$$

where

$$X = \sqrt{\frac{M}{\hbar}}\left[\sqrt{\dot{\gamma}}\; x + \frac{1}{M}\int_0^t ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma-\gamma(s)]\right] \tag{3.5}$$

$$Y = \sqrt{\frac{M}{\hbar}}\left[\sqrt{\dot{\gamma}'}x' - \frac{1}{M}\int^{t'} ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma'-\gamma(s)]\right] \quad . \tag{3.6}$$

$$Z = e^{-i(\gamma-\gamma')} \tag{3.7}$$

Substituting Eqs. (3.4)-(3.7) in Eq. (3.2), we obtain

$$K(x,t;\,x',t) = \sum_{n=0}^{\infty}\psi_n^*(x,t)\;\psi_n(x',t') \quad , \tag{3.8}$$

where

$$\begin{aligned}\psi_n(x,t) = {} & \left[\frac{1}{2^n n!}\left(\frac{M\dot{\gamma}}{\pi\hbar}\right)^{1/2}\right]^{1/2}\exp\left\{\frac{iM}{2\hbar}\left[\frac{\dot{\eta}}{\eta}x^2 - \frac{2}{M}\sqrt{\dot{\gamma}}\;x\int^t ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\cos[\gamma-\gamma(s)]\right]\right\} \\ & \times\exp\left\{\frac{M}{2\hbar}\left[\sqrt{\dot{\gamma}}\;x - \frac{1}{M}\int^t ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma-\gamma(s)]\right]^2\right\} \\ & \times H_n\left\{\sqrt{\frac{M\dot{\gamma}}{\hbar}}\left[x - \frac{1}{M\sqrt{\dot{\gamma}}}\int^t ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\sin[\gamma-\gamma(s]\right]\right\} \\ & \times e^{i[\theta(t)-(n+\frac{1}{2})\gamma(t)]} \quad .\end{aligned} \tag{3.9}$$

Moreover, we may write

$$\psi_n(x,t) = \exp\{i[\theta(t) - (n+\tfrac{1}{2})\gamma(t)]\}\;\phi_n(x,t) \quad , \tag{3.10}$$

where

$$\phi_n(x,t) = \left[\frac{1}{2^n n!}\left(\frac{M\dot{\gamma}}{\pi\hbar}\right)^{1/2}\right]^{1/2}\exp\left\{\frac{iM}{2\hbar}\left[\frac{\dot{\eta}}{\eta}x - \frac{2}{M}\sqrt{\dot{\gamma}x}\int^t ds\,\frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\cos[\gamma-\gamma(s)]\right]\right\}$$

$$\times \exp\left\{\frac{M}{2\hbar}\left[\sqrt{\dot{\gamma}}\ x - \frac{1}{M}\int^{t} ds\ \frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\ \sin[\gamma-\gamma(s)]\right]^2\right\}$$

$$\times H_n\left[\sqrt{\frac{M}{\hbar}}\left\{\sqrt{\dot{\gamma}}x - \frac{1}{M}\int^{t} ds\ \frac{f(s)}{\sqrt{\dot{\gamma}(s)}}\ \sin[\gamma-\gamma(s)]\right\}\right] \quad . \tag{3.11}$$

In Eq. (3.10), the wave function $\psi_n(x,t)$ is merely a unitary transformation of $\phi_n(x,t)$, and thus $\phi_n(x,t)$ satisfies all the properties associated with $\psi_n(x,t)$:

$$\int dx\ \psi_n^*\psi_n = \langle m|n\rangle = \int dx\ \phi_n^*\phi_n = \delta_{m,n} \quad . \tag{3.12}$$

The expectation value of a given operator O is

$$\langle m|O|n\rangle = \int dx\ \psi_n^*\ O\psi_n = \int dx\ \phi_n^* O\phi_n \quad . \tag{3.13}$$

IV. Energy expectation values

For the forced time-dependent harmonic oscillator system, both the Hamiltonian and Lagrangian have the units of energy but depend on time. We must therefore find a time-invariant energy operator. If $\beta(t)$ is a particular solution of Eq. (2.3), we have

$$\frac{d^2}{dt^2}\ (x-\beta) + \omega^2(t)\ (x-\beta) = 0 \quad , \tag{4.1}$$

and from Eqs. (2.26) and (2.27′) we note that

$$\ddot{\eta} + \omega^2(t)\ \eta = \Omega^2/\eta^3 \quad . \tag{4.2}$$

From Eqs. (4.1) and (4.2), we get the following expression for the energy:

$$E = \frac{1}{2M}\ (\eta p - M\dot{\eta}x)^2 + (M\dot{\eta}x - \eta p)(\beta\dot{\eta} - \eta\dot{\beta}) + \frac{M}{2}(\beta\dot{\eta} - \eta\dot{\beta})^2 + \frac{M}{2}\ \Omega^2\left(\frac{x-\beta}{\eta}\right)^2 \quad . \tag{4.3}$$

Because Eq. (4.3) is time invariant, we can use it for the quantum mechanical energy operator,

$$E_{op} = -\frac{\hbar^2\eta^2}{2M}\ \frac{\partial^2}{\partial x^2} + \frac{M}{2}(\dot{\eta}^2 + \eta\dot{\gamma}^2)x^2 - \frac{\eta\dot{\eta}}{2}\ \left(2x\frac{\partial}{\partial x} + 1\right)$$

$$+ (\beta\dot{\eta} - \eta\dot{\beta})\left(i\hbar\frac{\partial}{\partial x} + M\dot{\eta}x\right) + M\eta\dot{\gamma}^2\beta x + \frac{M}{2}\ \eta\dot{\gamma}^2\beta^2 + \frac{M}{2}(\beta\dot{\eta} - \eta\dot{\beta})^2 \quad . \tag{4.4}$$

Equation (3.11) now simplifies to the expression

$$\phi_n(x,t) = \left(\frac{\delta}{2^n\ n!\sqrt{\pi}}\right)^{\frac{1}{2}} e^{i\mu x^2+i\lambda x}\ e^{-\frac{1}{2}\delta^2(x-\beta)^2}\ H_n[\delta(x-\beta)]$$

$$= \left(\frac{\delta}{2^n n!\sqrt{\pi}}\right)^{\frac{1}{2}} e^{Ax^2+Bx}\ H_n[\delta(x-\beta)] \quad , \tag{4.5}$$

where

$$\delta = (M\gamma/\hbar)^{\frac{1}{2}} \tag{4.6}$$

$$\mu(t) = \frac{M}{2\hbar}\ \frac{\dot{\eta}}{\eta} \tag{4.7}$$

$$\lambda(t) = \frac{1}{\hbar}\sqrt{\dot{\gamma}} \int^t ds\ \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \cos[\gamma-\gamma(s)] \tag{4.8}$$

$$\beta(t) = \frac{1}{M\sqrt{\dot{\gamma}}} \int^t ds\ \frac{f(s)}{\sqrt{\dot{\gamma}(s)}} \sin[\gamma-\gamma(s)] \tag{4.9}$$

$$A = i\mu - \delta^2/2 \tag{4.10}$$

$$B = i\lambda + \beta\delta^2 \quad . \tag{4.11}$$

Here, $\beta(t)$ is a particular solution of Eq. (2.3).

In order to evaluate the energy expectation $E_{m,n} = \langle m|E_{op}|n\rangle$, we perform the following calculations:

$$x|n\rangle = \frac{1}{\sqrt{2}\delta}\ [\sqrt{n+1}|n+1\rangle + \sqrt{n}|n-1\rangle] \tag{4.12}$$

$$x^2|n\rangle = \frac{1}{2\delta^2}\ [\sqrt{(n+2)(n+1)}|n+2\rangle + (2n+1)|n\rangle + \sqrt{n(n-1)}|n-2\rangle] \tag{4.13}$$

$$p|n\rangle = \frac{\hbar A}{i\delta}\ 2(n+1)|n\rangle + \frac{\hbar}{i}\ B|n\rangle + \frac{\hbar}{i}(\delta+\frac{A}{\delta})\ \sqrt{2n}|n-1\rangle \tag{4.14}$$

$$p^2|n\rangle = -\hbar^2\ [\frac{2A^2}{\delta^2}\ \sqrt{(n+2)(n+1)}|n+2\rangle + 2\sqrt{2}\ \frac{AB}{\delta}\ \sqrt{n+1}|n\rangle + 2(A+\frac{A^2}{\delta^2})(2n+1)|n\rangle$$

$$+ 2\sqrt{2}\ \frac{AB}{\delta} + \delta B)\sqrt{n}|n-1\rangle + 2(\frac{A^2}{\delta^2}+2A+\delta^2)\sqrt{n(n-1)}|n\rangle] \tag{4.15}$$

$$xp|n\rangle = \frac{\hbar}{i}\{\frac{A}{\delta^2}\sqrt{(n+2)(n+1)}|n+2\rangle + \frac{B}{\sqrt{2}\delta}\sqrt{n+1}|n+1\rangle + [\frac{A}{\delta^2}(2n+1)+n]|n\rangle$$

$$+ \frac{B}{\sqrt{2}\delta}\sqrt{n}|n-1\rangle + (\frac{A}{\delta^2}+1)\sqrt{n(n+1)}|n-2\rangle\} \tag{4.16}$$

$$px|n\rangle = xp|n\rangle + |n\rangle \quad . \tag{4.17}$$

Substituting Eqs. (4.6)-(4.17) into Eq. (4.4), we directly obtain the energy expectation values as

$$E_{n,n} = E_n = \frac{\hbar}{2}(\eta^2\dot{\gamma})(2n+1)$$

$$= \Omega\hbar(n+\frac{1}{2}) \quad . \tag{4.18}$$

This energy expectation value is a time-invariant quantity.

V. Uncertainty values

The uncertainty product defined as

$$(\Delta x\Delta p)_{m,n} = \{[(\langle m|x^2|n\rangle - \langle m|x|n\rangle^2)^*(\langle m|x^2|n\rangle - \langle m|x|n\rangle^2)]^{\frac{1}{2}}$$

$$\times [(\langle m|p^2|n\rangle - \langle m|p|n\rangle^2)^*(\langle m|p^2|n\rangle - \langle m|p|n\rangle^2)]^{\frac{1}{2}}\}^{\frac{1}{2}} \quad . \tag{5.1}$$

Inserting Eqs. (4.12)-(4.15) into Eq. (5.1), we obtain

$$(\Delta x\Delta p)_{n,n} = (1 + \frac{\dot{\eta}^2}{\dot{\gamma}^2\eta^2})^{\frac{1}{2}}\,(n+\frac{1}{2})\hbar \tag{5.2}$$

$$(\Delta x\Delta p)_{n+2,n} = (1 + \frac{\dot{\eta}^2}{\dot{\gamma}^2\eta^2})^{\frac{1}{2}}\,\sqrt{(n+2)(n+1)}\,\hbar \tag{5.3}$$

$$(\Delta x\Delta p)_{n,n+2} = (1 + \frac{\eta^2}{\dot{\gamma}^2\eta^2})^{\frac{1}{2}}\,\sqrt{n(n-1)}\,\hbar \quad . \tag{5.4}$$

VI. Coherent states of the time-dependent harmonic oscillator

First, we construct the creation operator $a^\dagger$ and destruction operator a. For a forced time-dependent harmonic oscillator, it is not possible to construct a and $a^\dagger$, but we can construct a and $a^\dagger$ for the time dependent harmonic oscillator. From Eqs. (4.12) and (4.14), we obtain

$$a^{\dagger} = \left(\frac{M\dot{\gamma}}{2\hbar}\right)^{\frac{1}{2}} \left[\left(1 + \frac{\dot{\eta}}{\eta\dot{\gamma}}\right) x - i\frac{p}{M\dot{\gamma}}\right] \tag{6.1}$$

$$a = \left(\frac{M\dot{\gamma}}{2\hbar}\right)^{\frac{1}{2}} \left[\left(1 - i\frac{\dot{\eta}}{\eta\dot{\gamma}}\right) x + i\frac{p}{M\dot{\gamma}}\right] \tag{6.2}$$

From Eqs. (6.1) and (6.2), we can represent (x,p) in terms of $(a^{\dagger},a)$ as

$$x = \left(\frac{\hbar}{2M\dot{\gamma}}\right)^{\frac{1}{2}} (a^{\dagger}+a) \tag{6.3}$$

$$p = \left(\frac{\hbar M\dot{\gamma}}{2}\right)^{\frac{1}{2}} \left[\left(\frac{\dot{\eta}}{\eta\dot{\gamma}} + i\right) a^{\dagger} + \left(\frac{\dot{\eta}}{\eta\dot{\gamma}} - i\right) a\right] \quad . \tag{6.4}$$

Also from Eqs. (6.1) and (6.2), if $[x,p] = i\hbar$ we see that

$$[a^{\dagger},a] = 1 \quad . \tag{6.5}$$

Conversely, from Eqs. (6.3) and (6.4), if $[a^{\dagger},a] = 1$ we note that $[x,p] = i\hbar$.

The *coherent state* can be defined by the eigenstate of the nonhermition operator a,[15]

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad . \tag{6.6}$$

Let us find the coordinate representation of the coherent state. From Eqs. (6.2) and (6.3), we have

$$\left(\frac{M\dot{\gamma}}{2\hbar}\right)^{\frac{1}{2}} \left[\left(1 - i\frac{\dot{\eta}}{\eta\dot{\gamma}}\right) x + \frac{\hbar}{M\dot{\gamma}} \frac{\partial}{\partial x'}\right] \langle x'|\alpha\rangle = \alpha\langle x'|\alpha\rangle \quad . \tag{6.7}$$

We solve this equation and change the variable x′ into x for convenience,

$$\langle x|\alpha\rangle = N \exp \left[\frac{M\dot{\gamma}}{2\hbar}\left(-1 + i\frac{\dot{\eta}}{\eta\dot{\gamma}}\right) x^2 + \left(\frac{2M\dot{\gamma}}{\hbar}\right)^{\frac{1}{2}} \alpha\, x\right] \quad . \tag{6.8}$$

We choose the constant of integration N such that

$$\int dx\ |\langle x|\alpha\rangle|^2 = 1 \quad . \tag{6.9}$$

Then, we find the eigenvector of the operator a in the coordinate representation $|x\rangle$ as

$$<x|\alpha> = (\frac{M\dot{\gamma}}{\pi\hbar})^{\frac{1}{4}} \exp[\frac{M\dot{\gamma}}{2\hbar}(-1 + i\frac{\dot{\eta}}{\eta\dot{\gamma}})\ x^2 + (\frac{2M\dot{\gamma}}{\hbar})^{\frac{1}{2}}\ \alpha\ x - \frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^2] \quad . \tag{6.10}$$

Next, we show that a coherent state is a minimum uncertainty state. From Eqs. (6.3), (6.4) and (6.6) and their adjoints, we evaluate the expectation values of x, p, x^2 and p^2 in the state $|\alpha>$:

$$<\alpha|x|\alpha> = (\frac{\hbar}{2M\dot{\gamma}})^{\frac{1}{2}}\ (\alpha^*+\alpha) \tag{6.11}$$

$$<\alpha|p|\alpha> = (\frac{M\hbar\dot{\gamma}}{2})^{\frac{1}{2}}\ [(\frac{i}{\eta\dot{\gamma}} + i)\alpha^* + (\frac{i}{\eta\dot{\gamma}} - i)\alpha] \tag{6.12}$$

$$<\alpha|x^2|\alpha> = \frac{\hbar}{2M\dot{\gamma}}\ <\alpha|a^{+2}+a^2+aa^++a^+a|\alpha>$$

$$= \frac{\hbar}{2M\dot{\gamma}}\ (\alpha^{*2}+\alpha^2+2\alpha\alpha^*+1) \tag{6.13}$$

$$<\alpha|p^2|\alpha> = \frac{\hbar M\dot{\gamma}}{2}\ \{(\frac{\dot{\eta}}{\eta\dot{\gamma}} + i)^2\ \alpha^{*2} + (\frac{\dot{\eta}}{\eta\dot{\gamma}} - i)^2\alpha^2$$

$$+ [(\frac{\dot{\eta}}{\eta\dot{\gamma}})^2 + 1](2\alpha\alpha^*+1)\} \quad . \tag{6.14}$$

The uncertainty value is

$$\Delta x\Delta p = [(<\alpha|x^2|\alpha> - <\alpha|x|\alpha>)(<\alpha|p^2|\alpha> - <\alpha|p|\alpha>)]^{\frac{1}{2}}$$

$$= \hbar/2\ [1+(\frac{\dot{\eta}}{\eta\dot{\gamma}})^2]^{\frac{1}{2}} \quad , \tag{6.15}$$

which is the minimum value allowed by Eq. (5.2).

VII. Results and discussion

In the previous sections, we have obtained the propagator, wave function, energy expectation values, uncertainty values and coherent state for a quantum forced time-dependent harmonic oscillator. These quantities represent the solution of the classical equation of motion for the time-dependent harmonic oscillator. If we set f(t) equal to zero, then our solution is correct for the time-dependent harmonic oscillator. Setting $\omega(t) = \omega_o$ gives results for the forced harmonic oscillator. For the case where $f(t) = 0$ and $\omega(t) = \omega_0$, our results are those of the simple harmonic oscillator.

For the explicit time-dependent system, we need to consider the quantum mechanical operator. In our work, the Hamiltonian, Lagrangian and mechanical energy have the units of energy, but these are not time invariant. Yet, in order to solve macroscopic physical problems, we use time-invariant operators. For this reason, we have derived the energy operator from the classical equation of motion and used it to calculate energy expectation values. Our energy operator is similar to the Ermakov-Lewis invariant operator.[1,2] Our quantum energy expectation vlaues are time-independent quantities, and our uncertainty values are consistent with Heisenberg's uncertainty principle. Yet, our uncertainty values are time dependent, in contrast to those of time-independent systems.

Since it is not possible to construct a coherent state for the forced time-dependent harmonic oscillator, we have constructed it for the time-dependent harmonic oscillator. In general, the coherent state is a minimum uncertainty state, which is also true for our system.

Time-dependent systems are observed in various physical experiments. Two general types of such systems are: that which is formed through its own environmental conditions, and that which is formed when external forces are added. In regard to the second type, various experiments are being carried out to see how an applied, time-dependent electric, magnetic or other field can alter the physical properties of materials such as semiconductors and superconductors. Experiments show that a system becomes time dependent when a time-dependent electric or magnetic field (such as a.c.) is applied. However, obtaining the quantum mechanical solution by a direct method is not easy mathematically. One way of obtaining a solution is to use the propagator method as indicated in this paper, where the relevant equations are those of a time-dependent harmonic oscillator.

Our results, which are exact for one dimension, can be extended to two or more dimensions, and they can also be applied to time-dependent macroscopic systems. One example of an extension to two dimensions would be to solve the motion of a quantum electron in a time-dependent magnetic field.

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