



INFORMATION CAPACITY OF THE MATCHED GAUSSIAN CHANNEL WITH JAMMING

II. INFINITE-DIMENSIONAL CHANNELS

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Abstract

The additive infinite-dimensional Gaussian channel subject to jamming is modeled as a two-person zero-sum game with mutual information as the payoff function. The jammer's noise is added to the ambient Gaussian noise. The coder's signal energy is subject to a constraint given in terms of the RKHS of the covariance of the ambient noise; such a constraint is necessary in order that the capacity without feedback be finite. It is shown that use of this same RKHS constraint on the jammer's process is too strong; the jammer would then not be able to reduce capacity, regardless of the amount of jamming energy available. The constraint on the jammer is thus on the total jamming energy, without regard to its distribution relative to that of the ambient noise energy. The existence of a saddle value for the problem does not follow from the von Neuman minimax theorem in the original problem formulation. However, a solution is shown to exist. A saddle point, saddle value, and the jammer's minimax strategy are determined. The solution is a function of the problem parameters: the constraint on the coder, the constraint on the jammer, and the covariance of the ambient Gaussian noise. The essential effect of jamming is to convert the infinite-dimensional channel into a finite-dimensional channel with the same constraints, with the dimensionality depending upon the problem parameters.

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Introduction

Information capacity of the additive Gaussian channel subject to jamming is determined in [3] under the assumption that the channel is finite-dimensional. By "information capacity" we mean here a saddle point solution to a zero-sum two-person game in which mutual information is the payoff function, and the admissible strategies for coder and jammer are determined by average-energy constraints on the stochastic signals of the coder and jammer.

In this paper this problem is solved for the infinite-dimensional channel: a channel in which all sample paths belong to a real separable Hilbert space, H , with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Very substantial differences exist between this problem and that for the finite-dimensional channel. In the former, a saddle point solution is guaranteed by the von Neumann minimax theorem. Moreover, the solution can be obtained, after a preliminary development, by application of constrained optimization and the Kuhn-Tucker conditions. For the infinite-dimensional channel, it will be seen that the von Neumann theorem does not apply. Thus one does not know in advance if a saddle value exists, much less a saddle point. Moreover, constrained optimization is not available. Nevertheless, we shall prove the existence of a saddle point, obtain the saddle value, and in the process give the jammer's optimum strategy.

This problem is of interest from several viewpoints. One, of course, is that information capacity is an intrinsic measure of a channel's ability to convey information. The problem has a long history of interest for researchers in information theory, based on comments in the literature. Second, as a potentially very practical application, the information capacity is typically used to obtain coding capacity. The finite-dimensional results of [3] are sufficient for this purpose in the case of the discrete-time

channel. However, for continuous-time channels, one obtains the upper bound on coding capacity by computing the information capacity for each value of T , thus obtaining a quantity C_T , and then taking $\lim_{T \rightarrow \infty} C_T/T$. C_T is computed for the channel having sample functions in $L_2[0, T]$. Thus, in order to obtain coding capacity for continuous-time channels subject to jamming, the information capacity of the infinite-dimensional channel subject to jamming is required. Finally, in addition to applications for channels with jamming, the results obtained here can be used to obtain worst-case capacity in non-jamming applications when the channel noise is only partially known. An example that often arises is when the total noise consists of known receiver noise and additive medium noise having unknown statistical properties.

The results given here are for the general channel; there is no limitation such as stationarity, memory, or univariate nature. It is anticipated that $\lim_{T \rightarrow \infty} C_T/T$ can be given in terms of spectral densities for the stationary Gaussian channel subject to jamming. This will require limiting arguments based on the results given here, in the same way that the results of [2] have been used to obtain $\lim_T C_T/T$ of stationary Gaussian channels without jamming [4].

Mathematical Model

The channel sample paths are described by $Y = X + W + J$, where X is the coder's signal, W is the additive Gaussian noise, and J is the jammer's signal. X , W , and J are mutually independent. These quantities are described by the probabilities μ_X , μ_W , and μ_J on the Borel sets of H ; all are assumed countably additive, and second order (i.e., $\int_H \|x\|^2 d\mu(x) < \infty$). The mutual information of interest is

$$I(X,Y) = I(\mu_{XY}) = \int_{H \times H} \left[\log \frac{d\mu_{XY}}{d\mu_X \otimes \mu_Y}(x,y) \right] d\mu_{XY}(x,y)$$

where $\mu_X \otimes \mu_Y$ denotes product measure and $\mu_{XY}(A) = \mu_X \otimes \mu_W \otimes \mu_J \{(x,w,v) : (x, x+w+v) \in A\}$.

Under these assumptions, μ_X , μ_W , and μ_J have covariance operators R_X , R_W , and R_J : e.g., $\langle R_W u, v \rangle = \int_H \langle x, u \rangle \langle x, v \rangle d\mu_W(x)$. We assume WLOG that R_W is strictly positive on H and that all probabilities are zero-mean.

The assumptions imply the existence of a self-adjoint operator S , possibly unbounded, satisfying

$$R_W + R_J = R_W^{1/2} (I+S) R_W^{1/2}$$

or, equivalently,

$$R_J = R_W^{1/2} S R_W^{1/2}$$

where $\text{range}(R_W^{1/2}) \subset \mathcal{D}(S)$, the domain of S . The operator $(I+S)^{-1}$ exists, since S is non-negative; and necessarily bounded, since $\text{range}(R_W^{1/2}) \subset \text{range}(R_W + R_J)^{1/2}$.

$R_W = \sum_{n \geq 1} \lambda_n e_n \otimes e_n$, where $\lambda_n > 0$ and $\lambda_n \geq \lambda_{n+1}$ for all $n \geq 1$, $\sum_{n \geq 1} \lambda_n < \infty$, $\{e_n, n \geq 1\}$ is a CONS for H , and $(e_n \otimes e_n)v = \langle e_n, v \rangle e_n$.

The constraints on the coder are given by $\mu_X[\text{range}(R_W^{1/2})] = 1$ and $E_{\mu_X} \|x\|_W^2 \leq P_1$, where for x in $\text{range}(R_W^{1/2})$, $\|x\|_W^2 \equiv \|R_W^{-1/2} x\|^2$. Such constraints are necessary in order that the capacity without jamming be finite [1]. This constraint amounts to a RKHS (reproducing kernel Hilbert space) constraint on the coder's energy, in terms of the RKHS of the ambient channel noise covariance. As such, it limits the amount of energy that the coder can place into regions where the ambient noise energy is small.

The jammer's constraint is given by $E_{\mu_J} \|x\|^2 \leq P_2$. The stronger constraints $\mu_J[\text{range}(R_W^{1/2})] = 1$ and $E_{\mu_J} \|x\|_W^2 \leq P_2$ might be thought reasonable, since they are consistent with those imposed on the coder. However, they are too strong; the jammer would not be able to have any effect on the channel capacity under these constraints. This can be seen from the results of [2];

from Theorem 3 of that paper, if S is non-negative and has zero as the only limit point of its spectrum, then the capacity is equal to $P_1/2$. This situation would hold if one used the constraints $\mu_J[\text{range}(R_W^{1/2})] = 1$ and $E_{\mu_J} \|x\|_W^2 \leq P_2$, since the inequality (using the fact that $R_J = R_W^{1/2} S R_W^{1/2}$) is the same as $\text{Trace } S \leq P_2$. Thus, the jammer's constraint must be weaker than that of the coder.

The constraint $E_{\mu_X} \|x\|^2 \leq P_2$ places a constraint on the total jammer energy, but not on the relative jammer/noise energy. In terms of frequency ranges, this means that the jammer can use signals that have relatively large energy compared to the noise energy in appropriate frequency ranges. Such a constraint has an immediate intuitive interpretation. The jammer's optimum policy should be that of adding his available energy to the ambient noise energy in such a way that the sum provides maximum interference to the coder. The jammer will thus want to place his energy in regions available to the coder in which the noise energy is small, and this is provided by the above constraint. The use of the RKHS constraint $E_{\mu_J} \|x\|_W^2 \leq P_2$, while it limits the jammer's energy in regions that can be used by the coder in the same way that the coder's energy is limited, also limits the jammer's energy relative to the ambient noise energy, and is evidently too strong for the jammer to have an effect. This may be regarded as somewhat surprising and is one of the first differences that one sees in going from the finite-dimensional to the infinite-dimensional channel. It means that, with optimum coding-decoding by the channel users, the jammer cannot reduce the rate of data transmission if his energy limitation is subject to the same type of RKHS constraint as that applied to the coder, regardless of the value of P_2 , i.e., regardless of the total amount of energy available to the jammer. As already mentioned, the

constraint applied to the coder is implied by any constraint that gives finite capacity in the absence of jamming.

As a practical consideration, it may be noted that the total jamming energy will typically be much easier to estimate than its frequency distribution. Thus, the constraint used here is not only the appropriate constraint from the viewpoint of the jammer's capability to reduce the capacity, but is also one that can be viewed as feasible for the coder to estimate in designing his coding-decoding strategy.

In the form given above, there is no constraint on the probability distribution of the jammer's signal other than $E_{\mu_J} \|x\|^2 \leq P_2$. However, it follows from [7] that for a given R_J , the channel capacity (coder's viewpoint) will be minimized by taking μ_J to be Gaussian. Thus, the jammer should always choose μ_J to be Gaussian, and this will be assumed henceforth. With this assumption, the jamming channel is a special case of the mismatched Gaussian channel: a Gaussian channel such that the constraint covariance R_W is not the same as the noise covariance [2].

The jammer's strategy is uniquely determined by the choice of the operator S . For a given strategy, the mutual information is

$$I(\mu_{XY}) = F(z, \alpha) = \frac{1}{2} \sum_n \log \left[1 + z_n (1 + \alpha_n)^{-1} \right]$$

where (z_n) and (α_n) are defined as follows. The coder's covariance operator R_X is given by

$$R_X = \sum_n \tau_n (R_W + R_J)^{\frac{1}{2}} u_n \otimes (R_W + R_J)^{\frac{1}{2}} u_n.$$

where $\{u_n, n \geq 1\}$ is a c.o.n.s. in H , $\tau_n > 0$ for $n \geq 1$, $\sum_n \tau_n < \infty$. Then:

$$\alpha_n = \langle S U^* u_n, U^* u_n \rangle$$

$$z_n = \tau_n \|(I+S)^{\frac{1}{2}} U_n^* u_n\|^2$$

where U^* is the unitary operator satisfying $(R_W + R_J)^{\frac{1}{2}} = R_W^{\frac{1}{2}}(I+S)^{\frac{1}{2}} U^*$.

The problem to be considered is to determine if there exists a saddle value for the zero-sum two-person game with $I(\mu_{X,Y})$ as the payoff function; further, if a saddle value exists, then determine whether or not a saddle point exists; if such a point exists, then give its definition. That is, we seek to determine if

$$\sup_z \inf_\alpha F(z, \alpha) = \inf_\alpha \sup_z F(z, \alpha)$$

where the sup and inf are taken over the admissible signals for the coder and jammer, respectively. If this equality holds, then it defines the saddle value. In that case, one seeks to determine if the saddle value is actually attained by an admissible pair (z, α) ; i.e., if a saddle point exists.

The constraint on the coder is equivalent to $\sum_n z_n \leq P_1$. The constraint on the jammer is $\text{Trace } R_J = \text{Trace } R_W^{\frac{1}{2}} S R_W^{\frac{1}{2}} \leq P_2$. Since $\text{Trace } R_J = \sum_n \langle R_W^{\frac{1}{2}} S R_W^{\frac{1}{2}} u_n, u_n \rangle$, the constraint in this form cannot be effectively used, so will subsequently be given a different formulation.

θ will denote the smallest limit point of the spectrum of S . The limit points of the spectrum (\equiv the essential spectrum, $\sigma_{\text{ess}}(S)$) of S consists of all eigenvalues of infinite multiplicity, limit points of distinct eigenvalues, and points of the continuous spectrum. (τ_n) will denote the sequence of eigenvalues of S that are strictly less than θ , repeated according to their multiplicity. $\{v_n, n \geq 1\}$ will denote the corresponding eigenvectors. The following result is essential to our development.

Lemma 1 [2]: Suppose that the jammer's strategy is fixed and that $\theta < \infty$. The capacity $C_W(P_1)$ is then given as follows.

(a) If $\{\gamma_n, n \geq 1\}$ is not empty, and $\sum_n (\theta - \gamma_n) \leq P_1$, then

$$C_W(P_1) = \frac{1}{2} \sum_n \log \left[\frac{1+\theta}{1+\gamma_n} \right] + \frac{1}{2} \frac{P_1 + \sum_m (\gamma_m - \theta)}{1 + \theta}.$$

(b) If $\{\gamma_n, n \geq 1\}$ is empty, then $C_W(P_1) = \frac{P_1}{2(1+\theta)}$.

(c) In (a), the capacity can be attained if and only if $P_1 = \sum_{n \geq 1} (\theta - \gamma_n)$. In

that case, it is uniquely attained by a Gaussian μ_X with covariance

$$R_X = \sum_{n \geq 1} \tau_n R_{Nn}^{\frac{1}{2}} u_n u_n^* R_{Nn}^{\frac{1}{2}}, \text{ where } u_n = U v_n \text{ and } \tau_n = (\theta - \gamma_n)(1 + \gamma_n)^{-1}, \text{ for all}$$

$n \geq 1$ if (γ_n) is an infinite sequence; for $1 \leq n \leq K$ and $\tau_n = 0$ for $n > K$

when (γ_n) is a finite sequence with K elements. If $P_1 > \sum_{n \geq 1} (\theta - \gamma_n)$,

then capacity can be approached as closely as desired by using a

covariance R_X of the above form.

As previously noted, the above constraint on the jammer is not in a form suitable for a well-defined game theoretic problem. That is, F involves (α_n) , determined by both the coder and the jammer, defined by $\alpha_n = \langle S U^* u_n, U^* u_n \rangle$, where the c.o.n.s. $\{u_n, n \geq 1\}$ is chosen by the coder.

Lemma 2 (Energy-saving principle): When H is infinite dimensional and $\dim[\text{supp } \mu_X] = \infty$, the jammer's minimax strategy can be achieved by taking $S = \sum_{t=1}^{\infty} \tau_t e_t e_t^*$, where $\sum_{t=1}^{\infty} \lambda_t \tau_t \leq P_2$.

Proof: From Lemma 1, only the smallest limit point θ of $\sigma(S)$ and those eigenvalues γ_i strictly less than θ will affect the information capacity. One can thus assume without loss of generality that S has pure point spectrum with its eigenvalues consisting of non-negative real numbers less than θ and/or

equal to θ . We can order the eigenvalues (γ_i) of S such that $\gamma_i \leq \gamma_{i+1} \leq \theta$, all $i \geq 1$. By the definition of S , $\text{Domain}(S) \supset \text{range}(R_W^{1/2})$. All the eigenvectors of R_W lie in the domain of S ; therefore $\langle S e_n, e_n \rangle$ is well-defined for all $n \geq 1$. Let $\{v_i, i \geq 1\}$ be the CONS eigenvectors of S corresponding to the eigenvalues (γ_i) . Then consider

$$\text{Tr} R_W^{1/2} S R_W^{1/2} = \text{Tr} R_W S = \sum_{i=1}^{\infty} \langle R_W \gamma_i v_i, v_i \rangle = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \gamma_i \lambda_j \langle v_i, e_j \rangle^2$$

$$\sum_{i=1}^{\infty} \langle v_i, e_j \rangle^2 = 1,$$

$$\sum_{j=1}^{\infty} \langle v_i, e_j \rangle^2 = 1.$$

We use the facts that (λ_j) is positive and decreasing, and that γ_i is positive and increasing to θ . Let $\mu_i = \theta - \gamma_i$. Consider

$$\text{Tr} R_W^{1/2} (\theta I - S) R_W^{1/2} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_i \lambda_j \langle v_i, e_j \rangle^2$$

By Lemma 1B of [6], one has

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_i \lambda_j \langle v_i, e_j \rangle^2 \leq \sum_{i=1}^{\infty} \mu_i \lambda_i$$

Hence, $\sum_{i \geq 1} \sum_{j \geq 1} \gamma_i \lambda_j \langle v_i, e_j \rangle^2 \geq \sum_{i \geq 1} \lambda_i \gamma_i$. Thus, given any sequence (γ_n) of eigenvalues of the operator S' satisfying $R'_J = R_W^{1/2} S' R_W^{1/2}$, $S' = \sum_{n \geq 1} \gamma_n v_n \otimes v_n$, $\{v_n, n \geq 1\}$ a CONS in H , $\text{Tr} R'_J \leq P_2$, the jammer can use instead the operator $R_J = R_W^{1/2} S R_W^{1/2}$, $S = \sum_{n \geq 1} \gamma_n e_n \otimes e_n$, and then $\text{Tr} R_J \leq \text{Tr} R'_J \leq P_2$.

□

Hereafter, we define $R_J = \sum_{n \geq 1} \gamma_n \lambda_n e_n \otimes e_n$ (the set $\{e_n, n \geq 1\}$ being fixed by R_W).

Lemma 3 [2]. Suppose that $\theta < \infty$, $\{\gamma_n, n \geq 1\}$ is an infinite set, and $P_1 > 0$.

Then $P_1 + \sum_{n=1}^K \gamma_n \geq K \gamma_K$ for all $K \geq 1$ if and only if $P_1 \geq \sum_{n \geq 1} (\theta - \gamma_n)$.

Lemma 4. When $\dim(H) = \infty$, the jammer's minimax strategy requires a (γ_n) sequence such that $P_1 \geq \sum_n (\theta - \gamma_n)$. Moreover, WLOG it can be assumed that the coder chooses his covariance so that the vectors $\{u_n, n \geq 1\}$ are defined by $u_n = Ue_n$ for all $n \geq 1$.

Proof: Suppose that the jammer chooses (γ_n) such that $M\gamma_{M+1} \geq P_1 + \sum_{n=1}^M \gamma_n > M\gamma_M$ for some integer M . Then, Lemma 7 of [3] shows that such a (γ_n) sequence is not a minimax policy for the jammer.

Thus, for a minimax strategy the jammer must choose (γ_n) such that $P_1 + \sum_1^M \gamma_n \geq M\gamma_M$ for all $M \geq 1$. From Lemma 3, this implies that (γ_n) and θ satisfy $P_1 \geq \sum_{n \geq 1} (\theta - \gamma_n)$. If equality holds, then by part (c) of Lemma 1, the coder must choose the vectors $\{u_n, n \geq 1\}$ by $u_n = Ue_n$ for $n \geq 1$; if strict inequality holds, then the coder can choose these vectors in order to approach capacity as closely as desired, again by part (c) of Lemma 1. \square

We shall always assume hereafter that both coder and jammer use the $\{e_n, n \geq 1\}$ vectors. That is, they are the o.n. eigenvectors of S , $S = \sum_{n \geq 1} \gamma_n e_n \otimes e_n$, and the coder has covariance operator $R_X = \sum_{n \geq 1} \tau_n (1 + \gamma_n) \lambda_n e_n \otimes e_n$.

The objective function and the constraints now have the following form:

$$I(\mu_{XY}) = F(z, \gamma) = \frac{1}{2} \sum_{n \geq 1} \log \left[1 + z_n (1 + \gamma_n)^{-1} \right],$$

where z and γ are admissible if and only if their components are non-negative,

$\sum_{n \geq 1} z_n \leq P_1$, and $\sum_{n \geq 1} \lambda_n \gamma_n \leq P_2$. F is thus strictly concave/strictly convex.

To apply the von Neumann minimax theorem, guaranteeing the existence of a (unique) saddle point, it is necessary ([5]) that the set of admissible z constitutes a bounded and closed subset of a reflexive space H_1 , and that the set of admissible γ constitutes a bounded and closed subset of a reflexive space H . As defined, the set of admissible z constitutes a closed and

bounded subset of the non-reflexive space ℓ_1 . Of course, the admissible z constitute a bounded set in the reflexive space ℓ_2 . However, the set of admissible z is not closed in ℓ_2 , since the unit ball in ℓ_1 is not closed in the ℓ_2 norm. This can be seen by considering the sequence (f_n) in ℓ_1 defined by $f_n(k) = 1/k^2 - c_n/k$ for $k \leq n$, $f_n(k) = 0$ for $k > n$, with $c_n \rightarrow 0$ chosen so that $\sum_1^n |f_n(k)| = 1$ for $n \geq 1$. Moreover, if one considers $F(z', \gamma)$ by setting $z'_i = \sqrt{z_i}$, then F is no longer concave-convex. In summary, the von Neumann minimax theorem cannot be applied to guarantee the existence of a saddle point. We shall prove, however, that a saddle point exists, define the saddle value, and give the jammer's optimum strategy.

The above assumption on the representations of S and R_X fixes the problem in a coordinate system determined by the ambient channel noise. In view of Lemma 2, a minimax strategy for the jammer in the original problem can be obtained within this coordinate system: choosing S to have pure point spectrum, and with its eigenvectors the same as those of the channel noise covariance operator, R_W . By Lemma 4, one can then assume that R_X is defined by taking $u_n = Ue_n$ for $n \geq 1$, which gives $\{e_n, n \geq 1\}$ as CON eigenvectors for R_X . Thus, by showing that a saddle value and saddle point exist within this coordinate system, one will also show their existence for the original problem. However, this approach will not show that the saddle point is unique.

Lemma 5 [3]. Define $(\gamma_{K,t}^m)$, (z_t^m) , θ_K^m , g_K^m , and T_K^m as follows, for $K \geq 0$, $m \geq K+1$:

$$g_K^m(\theta) = \left[\sum_{i=K+1}^m \frac{\lambda_i}{P_2 + \sum_{k=K+1}^m \lambda_k + (P_1 - K\theta)\lambda_i} \right]^{-1} \quad (0 \leq \theta \leq P_1 K)$$

$$T_K^m = \frac{P_1 \lambda_{K+1}}{P_2 + \sum_{i=K+1}^m \lambda_i + K \lambda_{K+1}}$$

$$1 + \gamma_{K,i}^m = \frac{(P_2 + \sum_{k=K+1}^m \lambda_k)(1 + \theta_K^m)}{P_2 + \sum_{j=K+1}^m \lambda_j + (P_1 - K \theta_K^m) \lambda_i} \quad i \geq K+1$$

$$\gamma_{K,i}^m = 0 \quad i \leq K$$

where θ_K^m is the solution to $1 + \theta = g_K^m(\theta)$, and

$$z_i^m = \theta_K^m - \gamma_{K,i}^m \quad i \geq 1.$$

A unique solution exists for θ_K^m . If $P_1 \leq (m-1)P_2/\lambda_m$ and it is required that the coder's covariance satisfy $z_n = 0$ for all $n \geq m+1$, then the jamming problem has a saddle point (z^x, γ^x) , with $\gamma_i^x = \gamma_{K,i}^m$ for $i \leq m$ and $\gamma_i^x = 0$ for $i > m$; $z_i^x = z_i^m$ for $i \leq m$ and $z_i^x = 0$ for $i > m$; and K the smallest integer k such that $\theta_k^m > T_k^m$. This K satisfies $K \leq m-2$, and $P_1 > KP_2/\sum_{j=K+1}^m \lambda_j$

The condition $P_1 \leq (m-1)P_2/\lambda_m$ assumed in Lemma 5 will of course be satisfied for all sufficiently large m ; in fact, for all $m \geq k$, where k is such that $P_1 \leq kP_2/(\sum_{j=k+1}^{\infty} \lambda_j)$.

Lemma 6. If K is any integer ≥ 0 such that $P_1 < (K+1)P_2/\sum_{i \geq K+2} \lambda_i$, then

$$\theta_K^m > T_K^m \quad \text{for all } m \geq K+1.$$

Proof. We will show that $g_K^m(T_K^m) > 1 + T_K^m$ for all $m \geq K+1$. This will prove the statements, since otherwise one would have $\theta_K^m \leq T_K^m$ for some m ; since $g_K^m(\theta)$ is a strictly decreasing function of θ for fixed m and K , this would give

$$1 + T_K^m \geq 1 + \theta_K^m = g_K^m(\theta_K^m) \geq g_K^m(T_K^m) > 1 + T_K^m.$$

To see that $g_K^m(T_K^m) > 1 + T_K^m$, this is equivalent to

$$\sum_{i=K+1}^m \frac{\lambda_i [P_2 + \sum_{j=K+1}^m \lambda_j + K\lambda_{K+1}]}{(P_2 + \sum_{k=K+1}^m \lambda_k) [P_2 + \sum_{n=K+1}^m \lambda_n + K\lambda_{K+1} + P_1 \lambda_i]} < \frac{P_2 + \sum_{j=K+1}^m \lambda_j + K\lambda_{K+1}}{P_2 + \sum_{k=K+1}^m \lambda_k + (P_1 + K)\lambda_{K+1}}$$

or

$$\sum_{i=K+1}^m \lambda_i \left\{ 1 + \frac{P_1 (\lambda_{K+1}^{-\lambda_i})}{P_2 + \sum_{j=K+1}^m \lambda_j + K\lambda_{K+1} + P_1 \lambda_i} \right\} < P_2 + \sum_{k=K+1}^m \lambda_k.$$

This inequality holds if

$$P_1 \lambda_{K+1} \sum_{i=K+2}^m \lambda_i < P_2 [P_2 + \sum_{j=K+2}^m \lambda_j + (K+1)\lambda_{K+1}].$$

which is satisfied, since LHS $\leq P_2 \lambda_{K+1} (K+1)$. □

Now define T_K and g_K by

$$T_K = \frac{P_1 \lambda_{K+1}}{P_2 + \sum_{j=K+1}^{\infty} \lambda_j + K\lambda_{K+1}}$$

$$g_K(\theta) = \left[\sum_{i=K+1}^{\infty} \frac{\lambda_i}{P_2 + \sum_{j=K+1}^{\infty} \lambda_j + (P_1 - K\theta)\lambda_i} \right]^{-1} \quad (\theta \geq 0)$$

and let θ_K be the solution to $1 + \theta = g_K(\theta)$. It is obvious that $T_K = \lim_m T_K^m$;

we now show that the equation $1 + \theta = g_K(\theta)$ has a unique solution, θ_K ; and

that $\theta_K = \lim_m \theta_K^m$.

Lemma 7. (a) $g_K^m(\theta)$ is a strictly decreasing function of m for fixed K and θ ;

(b) θ_K^m is a strictly decreasing function of m ;

(c) θ_K is uniquely defined and $\theta_K = \lim_{m \rightarrow \infty} \theta_K^m$;

(d) $\theta_K \geq T_K$ if and only if $\sum_{i=K+1}^{\infty} \frac{P_1 (\lambda_{K+1}^{-\lambda_i}) \lambda_i}{P_2 + \sum_{j=K+1}^{\infty} \lambda_j + K\lambda_{K+1} + P_1 \lambda_i} \leq P_2$;

(e) $\theta_K \geq T_K$ if and only if $\theta_K^m > T_K^m$ for all $m \geq K+1$.

Proof. (a) It is immediate that $[g_K^{m+1}(\theta)]^{-1} > [g_K^m(\theta)]^{-1}$ if $\theta \leq P_1/K$.

(b) Since $g_K^m(\theta)$ is a strictly decreasing function of m , the solution to $1 + \theta = g_K^{m+1}(\theta)$ must be strictly less than the solution to $1 + \theta = g_K^m(\theta)$.

(c) Let $f(\theta) = 1 + \theta$. f is continuous and strictly monotone increasing. g_K is continuous and strictly monotone decreasing, and $f(0) = 1 < g_K(0)$. If $P_1/K > P_2/\sum_{j \geq K+1} \lambda_j$, then $f(P_1/K) = 1 + P_1/K > 1 + P_2/\sum_{j \geq K+1} \lambda_j = g_K(P_1/K)$. Thus a unique solution exists to $f = g_K$ if $P_1/K > P_2/\sum_{j \geq K+1} \lambda_j$. If $P_1/K \leq P_2/\sum_{j \geq K+1} \lambda_j$, then the same result holds, since then $f(P_2/\sum_{j \geq K+1} \lambda_j) > g(P_2/\sum_{j \geq K+1} \lambda_j)$.

Since (θ_K^m) is monotone decreasing as m increases, and bounded below by zero, $\lim_m \theta_K^m \equiv \theta_0$ exists. Since $g_K(\theta) = \lim_m g_K^m(\theta)$ for $\theta > 0$, $g_K(\theta_0) = \lim_m g_K^m(\theta_0) \geq \overline{\lim}_m g_K^m(\theta_0^m) = 1 + \theta_0$. Conversely, $g_K(\theta_0) = \lim_m g_K(\theta_K^m) \leq \overline{\lim}_m g_K^m(\theta_K^m) = 1 + \theta_0$. Thus, $1 + \theta_0 = g_K(\theta_0)$; since this solution is unique, $\theta_0 = \theta_K$.

(d) $g_K(T_K) \geq 1 + T_K$, since otherwise (proceeding as in the proof of part (c)) the solution to $g_K(\theta) = 1 + \theta$ will occur for $\theta < T_K$. The inequality of (d) then follows from $g_K^{-1}(T_K) \leq (1+T_K)^{-1}$.

(e) As in the proof of (d) $\theta_K^m > T_K^m$ if and only if $g_K^m(T_K^m) > 1 + T_K^m$, and this occurs if and only if

$$\sum_{i=K-1}^m \frac{P_1(\lambda_{K+1}^{-\lambda_i})\lambda_i}{P_2 + \sum_{j=K+1}^m \lambda_j + K\lambda_{K+1} + P_1\lambda_i} < P_2.$$

The LHS of this inequality is a strictly increasing function of m . Moreover,

$$g_K^{K+1}(\theta_K^{K+1}) = 1 + \theta_K^{K+1} = \left[\frac{\lambda_{K+1}}{P_2 + [P_1 - K\theta_K^{K+1} + 1]\lambda_{K+1}} \right]^{-1},$$

so that

$$\theta_K^{K+1} = \frac{P_2 + P_1\lambda_{K+1}}{(K+1)\lambda_{K+1}} > \frac{P_1\lambda_{K+1}}{P_2 + (K+1)\lambda_{K+1}} = T_K^{K+1}.$$

Thus, $\lim_{m \rightarrow \infty} \theta_K^m \geq \lim_{m \rightarrow \infty} T_K^m \Leftrightarrow \theta_K^m \geq T_K^m$ for all $m \geq K+1$. □

The following theorem is the main result of this paper. It will be seen (see Remark 2 concluding the paper) that the effect of the jammer is essentially to convert the infinite-dimensional channel into a channel of dimension $\leq K$ ($K < \infty$). This integer K is defined in the following theorem, and depends on P_1 , P_2 , and the eigenvalues (λ_n) of R_W .

Theorem.

The jamming problem has a saddle value and a saddle point. The saddle value is given by

$$I(\mu_{XY}) = \frac{1}{2} \sum_{n \geq K+1} \log \left[1 + \frac{(P_1 - K\theta_K)\lambda_n}{P_2 + \sum_{t \geq K+1} \lambda_t} \right] + \frac{K}{2} \log(1 + \theta_K)$$

where θ_K is the unique solution of

$$1 + \theta = \left[\sum_{n \geq K+1} \frac{\lambda_n}{P_2 + \sum_{j \geq K+1} \lambda_j + (P_1 - K\theta)\lambda_n} \right]^{-1}$$

and K is the smallest integer $k \geq 0$ such that

$$\sum_{i=k+1}^{\infty} \frac{P_1(\lambda_{k+1} - \lambda_i)\lambda_i}{P_2 + \sum_{j=k+1}^{\infty} \lambda_j + k\lambda_{k+1} + P_1\lambda_i} \leq P_2.$$

A saddle point is given by (z^x, τ^x) , where

$$z_i^x = \theta_K - \tau_i^x \quad \text{for all } i \geq 1$$

$$\tau_i^x = 0 \quad \text{for } i \leq K$$

$$1 + \tau_i^x = \frac{[P_2 + \sum_{j \geq K+1} \lambda_j](1 + \theta_K)}{P_2 + \sum_{t \geq K+1} \lambda_t + (P_1 - K\theta_K)\lambda_t} \quad \text{for } i \geq K+1.$$

Proof: Define $Z = \{(z_i): z_i \geq 0 \text{ for } i \geq 1 \text{ and } \sum_{i \geq 1} z_i \leq P_1\}$;

$$Z^m = \{(z_i) \in Z: z_i = 0 \text{ for } i > m\};$$

$$\Gamma = \{(\gamma_i): \gamma_i \geq 0 \text{ for } i \geq 1 \text{ and } \sum_{i \geq 1} \lambda_i \gamma_i \leq P_2\}.$$

It is sufficient [5] to show that $\sup_Z \inf_{\Gamma} F(z, \gamma) \geq \inf_{\Gamma} \sup_Z F(z, \gamma)$.

$\sup_Z \inf_{\Gamma} F(z, \gamma) \geq \sup_{Z^m} \inf_{\Gamma} F(z, \gamma) = (\text{Lemma 5}) F(z_k^m, \gamma_k^m)$, where (z_k^m, γ_k^m) is

defined in Lemma 5. Thus, $\sup_Z \inf_{\Gamma} F(z, \gamma) \geq \liminf_{m \rightarrow \infty} F(z_k^m, \gamma_k^m) =$

$\liminf_{m \rightarrow \infty} \frac{1}{2} \sum_{i \geq 1} \log[1 + \frac{z_i^m}{1 + \gamma_{k,i}^m}]$ Define $k(m)$ by $\gamma_{k(m)}^m = \gamma_k^m$. Then

$k_0 \equiv \liminf_{m \rightarrow \infty} k(m) = \liminf_{m \rightarrow \infty} \{k: \theta_k^m > \tau_k^m\} = (\text{Lemma 7}) K$, with K defined as in

the Theorem. From the definitions of γ_k^m and z_k^m , $\lim_{m \rightarrow \infty} \theta_K^m = \theta_K$ implies $\gamma_{K,i}^m \rightarrow \gamma_i^*$

and $z_i^m \rightarrow z_i^*$ for all $i \geq 1$ as $m \rightarrow \infty$. By Fatou's Lemma

$$\liminf_{m \rightarrow \infty} \frac{1}{2} \sum_{i=1}^{\infty} \log \left[1 + \frac{z_i^m}{1 + \gamma_{K,i}^m} \right] \geq \frac{1}{2} \sum_{i=1}^{\infty} \log \left[1 + \frac{z_i^*}{1 + \gamma_i^*} \right].$$

Thus, $\sup_Z \inf_{\Gamma} F(z, \gamma) \geq I(\mu_{XY})$ as given in the Theorem.

Conversely, one notes that $\gamma_i^* \nearrow \theta_K$ and that $\sum_{i=1}^{\infty} (\theta_K - \gamma_i^*) = P_1$. Thus, by

Lemma 1, $F(z^*, \gamma^*) = \sup_Z F(z, \gamma^*) \geq \inf_{\Gamma} \sup_Z F(z, \gamma)$. This shows

$\inf_{\Gamma} \sup_Z F(z, \gamma) \leq F(z^*, \gamma^*) \leq \sup_Z \inf_{\Gamma} F(z, \gamma)$, and completes the proof.

□

Remark 1. The integer K defined in the Theorem can be no larger than the smallest integer k satisfying $P_1 \leq (k+1)P_2 / \sum_{j=k+2}^{\infty} \lambda_j$. This follows from Lemma 6, and yields the following Corollary.

Corollary. The saddle value of F on Λ has the upper bound

$$F(z, \gamma) \leq \frac{K+1}{2} \log \left[1 + \frac{P_1}{K+1} \right]$$

where K is the smallest integer k satisfying $P_1 \leq (k+1)P_2 / \sum_{j=k+2}^{\infty} \lambda_j$.

Proof. From Lemma 1(a), this is the value of the channel capacity that would be obtained if the jammer used (γ_n) such that $\gamma_n = 0$ for $n \leq K+1$, and $\gamma_n = P_1/(K+1)$ for $n > K+1$. In that case, $\sum_{n=1}^{\infty} \gamma_n \lambda_n = \sum_{n=K+2}^{\infty} \gamma_n \lambda_n = (P_1/(K+1)) \sum_{n \geq K+2} \lambda_n \leq P_2$. This is thus an admissible strategy for the jammer, and the result follows by Lemma 1(a).

□

Remark 2. The capacity of the channel in the absence of jamming is $P_1/2$; see [2]. This equals $\lim_K \frac{K+1}{2} \log \left[1 + \frac{P_1}{K+1} \right]$. One thus sees that the minimum effect of jamming can be immediately gauged by determining the value of K , the largest integer such that $P_1 > KP_2 / \sum_{n=K+1}^{\infty} \lambda_n$. Since the capacity of the $(K+1)$ -dimensional channel in the absence of jamming is $\frac{K+1}{2} \log \left[1 + \frac{P_1}{K+1} \right]$, one can view the effect of jamming as converting the infinite-dimensional channel into a finite-dimensional channel. The value of this K depends on all the channel parameters: the coder's constraint P_1 , the jammer's constraint P_2 , and the covariance operator of the ambient Gaussian noise.

The statement of the Theorem can be interpreted by considering the sum of the jamming and the ambient noise. That is, if k is the largest integer such that $\gamma_k^* = 0$ (note that $k \geq K$ as defined in the Theorem and $k \leq K$ as defined in the Corollary), then corresponding to the eigenvectors $\{e_n, n \geq 1\}$ of $R_W + R_J$, the eigenvalues are given by λ_j for $j \leq k$, while for $j > k$, the eigenvalues are equal to $\lambda_j(1+\gamma_j^*)$. Thus, as $j \rightarrow \infty$, the eigenvalues are approximately equal to $\lambda_j(1+\theta)$. Since the coder will typically wish to place his energy, so far as possible, according to small eigenvalues of the total channel noise, the effect of the jamming is to increase those small eigenvalues by a factor that converges upward to $(1+\theta)$.

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