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SEQUENTIAL ESTIMATION OF OPTIMAL AGE REPLACEMENT POLICIES

by

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Under an age replacement policy a system is replaced at a fixed age φ or at failure whichever comes first. If the cost of replacing the system before failure is less than the cost of replacing it at failure, this type of maintenance policy can lead to considerable savings. An often used criterion for finding an "optimal" replacement age φ, is to minimize the long run expected cost per unit time of a policy with replacement age φ. This cost function clearly depends on the underlying distribution of the system lifetimes. When this distribution is unknown, the cost function and hence φ^* need to be estimated.

In this thesis, we study the large and small sample properties of a procedure which estimates φ^*. In particular, we study sequential maximum likelihood estimators of φ^* which are updated at each replacement based on the replacement history of the system so far. In this sequential procedure each system is subject to the age replacement policy with estimated φ^* based on all the data gathered so far. This type of procedure should control the actual cost per unit time while gathering data needed to estimate φ^*.

This thesis contains a detailed description of the sequential estimation procedure when the underlying system life times have a Weibull distribution and a Gamma distribution. Monte-Carlo methods are then used to study the behavior of the estimated optimal age replacement times and more importantly the actual costs per unit time for different sample sizes, costs and choices of the underlying Weibull and Gamma distributions.

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Sequential Estimation
of
Optimal Age Replacement Policies

by

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ABSTRACT

Under an age replacement policy a system is replaced at a fixed age φ or at failure whichever comes first. If the cost of replacing the system before failure is less than the cost of replacing it at failure, this type of maintenance policy can lead to considerable savings. An often used criterion for finding an "optimal" replacement age φ, is to minimize the long run expected cost per unit time of a policy with replacement age φ. This cost function clearly depends on the underlying distribution of the system lifetimes. When this distribution is unknown, the cost function and hence $\varphi^\star$ need to be estimated.

In this thesis, we study the large and small sample properties of a procedure which estimates $\varphi^\star$. In particular, we study sequential maximum likelihood estimators of $\varphi^\star$ which are updated at each replacement based on the replacement history of the system so far. In this sequential procedure each system is subject to the age replacement policy with estimated $\varphi^\star$ based on all the data gathered so far. This type of procedure should control the actual cost per unit time while gathering data needed to estimate $\varphi^\star$.

This thesis contains a detailed description of the sequential estimation procedure when the underlying system life times have a Weibull distribution and a Gamma distribution. Monte-Carlo methods are then used to study the behavior of the estimated optimal age replacement times and more importantly the actual costs per unit time for different sample sizes, costs and choices of the underlying Weibull and Gamma distributions.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

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I. INTRODUCTION

A. BACKGROUND

Optimal maintenance policies are designed to reduce the number of system failures and minimize the cost of repair by scheduling planned replacements. By far, most of the research in this area has been from the modeling stand-point. Even in the most basic scenario, where the underlying system lifetimes are assumed to be independent and identically distributed, the problem of updating the maintenance policy using the past maintenance history has not been adequately solved. In this thesis, Monte-Carlo methods are used to study a particular parametric procedure which updates estimates of an optimal age replacement policy after each replacement.

B. MAINTENANCE AND MAINTENANCE POLICIES

Maintenance is a combination of actions carried out to retain a unit in, or restore it to, an acceptable condition. There are two forms of maintenance. These are *preventive* and *corrective* maintenance. Corrective maintenance is carried out to restore (including minor adjustment and repair) a unit which has ceased to meet an acceptable condition. Preventive maintenance is intented to reduce the likelihood of a unit not meeting an acceptable condition [Ref. 1: pp. 1-8].

Maintenance policies reduce the number of system failures and maintenance costs by adopting a schedule of planned maintenance. For instance, when the failure of a unit during actual operation is dangerous or costly (i.e., the cost C_1 of unscheduled maintenance due to system failure is more than the cost C_2 of scheduled maintenance before system failure) and the unit is characterized by a failure rate that increases with age, it may be wise to maintain it before it has aged too greatly [Ref. 2: p. 46]. On the other hand, performing maintenance actions too frequently can also be costly. Thus, choosing the preventive maintenance policy that schedules maintenance to best control costs will be influenced by the relative costs of failure and preventive maintenance and the stochastic properties of the lifetime of the unit [Ref. 3: pp. 19-24]. We are interested in determining the sequence of times at which preventive maintenance should occur. In particular, we are actually interested in determining an *optimal* maintenance policies which minimizes long run expected maintenance cost per unit time.

Preventive maintenance is the total of all service functions aimed at maintaining and improving reliability performance characteristics and concerns itself with such activities as the replacement and repair of systems, inspections, testing and checking of working parts during their operation. In this thesis, we will only consider maintenance actions which involve replacement of a system. It will be assumed that the replacement action returns the equipment to the **as new** condition, thus providing the same services as the equipment which has just been replaced. By making this assumption, we are implying that the distribution of time to failure of the new system is the same as that of the system which was replaced. In addition, we assume that system lifetimes are independent. The unscheduled and scheduled replacement costs C_1 and C_2 remain constant where $C_1 > C_2$.

An important replacement policy is the policy based on age (*age replacement*). Such a policy is in force if a unit is always replaced at the time of failure or φ units of time after its installation, whichever comes first. Under a *block replacement* policy the unit is replaced at times $k\varphi$ ($k = 1, 2, \ldots$), and at failure. This replacement policy derives its name from the commonly employed practice of replacing a block or group of units in a system at prescribed times $k\varphi$ ($k = 1, 2, \ldots$) independent of the failure history of the system [Ref. 2: p. 46].

" Age replacement is administratively more difficult to implement, since the age of the unit must be recorded. But block replacement, although simpler to administer since the age of the system need not be recorded, leads to more frequent replacement of relatively new items " [Ref. 4: p. 158]. In this thesis we only consider age replacement policies.

An optimal age replacement policy is the age replacement policy which yields the smallest long run expected replacement costs per unit time. To find an optimal age replacement policy we require explicit knowledge about the system's lifetime distribution. When we don't know the lifetime distribution explicitly, the systems life distribution and the optimal age replacement policy needs to be estimated. Estimation based on a fixed sample of independent and identically distributed (*i.i.d.*) system life times has been examined in detail [Refs. 5,6,7,8]. In order to collect such *i.i.d.* data, experimental systems must be left in service until failure. When observation of complete system lifetimes is not available (because it either takes too much time or is too costly), the most cost effective approach is to start with an initial estimate of the optimal replacement age and then update this estimate after each system replacement. After each replacement, the next system is subject to a replacement policy that is close to the best estimated policy so far [Ref. 9: pp. 2-3]. This procedure is described in detail in Chapter II. We will use simulation to study this sequential estimation procedure when the underlying life distribution comes from a parametric family. In particular, in Chapter III we study this procedure with an underlying Weibull distributions and in Chapter IV we consider Gamma life distributions. Conclusions and recommendations are given in Chapter V.

II. THE SEQUENTIAL ESTIMATION PROCEDURE

A. THE OPTIMAL REPLACEMENT AGE

Consider a system or component which is replaced upon failure. When the cost of a replacement that is planned in advance is less than the cost of an unplanned replacement, a simple age replacement policy can lead to considerable savings. In this type of policy, an age φ is specified; items that are still functioning at that age are replaced (these are *planned* replacements); items which fail and are thus replaced prior to φ are the *unplanned* replacements.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (*i.i.d.*) positive random variables with common distribution function F_θ belonging to a family of distributions parameterized by θ. The sequence $X_1, X_2, \ldots$ represents the sequence of system lifetimes that would be observable if the systems were replaced at failure. Let C_1, C_2, ($C_1 > C_2$) be the respective costs of planned replacement (before system failure) and unplanned replacement (at system failure). The observed durations between replacement, $\min(X_i, \varphi)$ i = 1, 2, ..., form a renewal processes [Ref. 2: p. 87]. Therefore, the long run expected cost per unit time with age replacement at φ is

$$C(\varphi) = \frac{C_1 \times F_\theta(\varphi) + C_2 \times \overline{F}_\theta(\varphi)}{E[\min\ (X_i, \varphi)]} = \frac{C_1 \times F_\theta(\varphi) + C_2 \times \overline{F}_\theta(\varphi)}{\int_0^\varphi \overline{F}_\theta(x)\, dx}, \tag{2.1}$$

where $\overline{F}_\theta(x) = 1 - F_\theta(x)$ is the survival function. In equation (2.1) the numerator is the expected cost of one replacement under the age replacement policy, and the denominator is the expected time between replacement.

Under reasonable conditions, there exists a unique and finite time $\varphi^\star < \infty$ where $C(\varphi)$ attains a global minimum [Ref. 10:, pp. 161-168]. For example, a sufficient condition for the existence of $\varphi^\star$ is that F_θ has a failure rate $\lambda(t)$ that strictly increases to infinity.

B. THE SEQUENTIAL ESTIMATION PROCEDURE

Finding the optimal planned replacement age $\varphi^\star$ requires knowledge of the underlying distribution function F_θ. If the underlying lifetime distribution F_θ is completely specified, then finding $\varphi^\star$ can be done either explicitly or numerically using equation (2.1). However, if the parameter θ is unknown, we need to estimate $\varphi^\star$. If full lifetimes $X_1, X_2, ..., X_n$ are available, then $C(\varphi)$ and thus $\varphi^\star$ can be estimated by replacing θ in equation (2.1) with the usual parametric estimators of θ. This approach has the obvious disadvantage that it requires several systems to operate until failure to be observed before a replacement policy is implemented. In addition, subsequent data is not used to update the policy. A more practical and potentially more cost effective approach is to update estimate $\varphi^\star$ after each replacement and implement the updated age replacement policy (through estimates of $\varphi^\star$) after each replacement.

Let $\hat{\varphi}_1, \hat{\varphi}_2, ...$ be a sequence of estimators of φ where $\hat{\varphi}_n$ depends on the first n replacement ages. A procedure to compute the estimators $\hat{\varphi}_n$ is developed as follows:

1. At the nth replacement observe the system lifetime X_n or $\hat{\varphi}_{n-1}$ whichever comes first. Let $Z_n = \min(X_n, \hat{\varphi}_{n-1})$ and $\delta_n = I(X_n \le \hat{\varphi}_{n-1})$ where $I(A)$ is an indicator function of the set A. In other words, if the unit is replaced before failure, then the replacement time $Z_n = \hat{\varphi}_{n-1}$ and $\delta_n = 1$, otherwise $Z_n = X_n$ and $\delta_n = 0$.
2. The data available to estimate $\hat{\varphi}_n$ are the pairs (Z_i, δ_i) i = 1, 2, ..., n (Throughout, without loss of generality we take special case as $Z_1 = X_1, \delta_1 = 1$). The maximum likelihood estimator *(MLE)* $\hat{\theta}_n$ of θ is computed from this right censored data.
3. $\hat{\varphi}_n$ is then found to minimize equation (2.1) with $\hat{\theta}_n$ taking the place of the unknown parameter θ.

The procedure is then repeated. The technical problem of using this data to estimate φ at each stage is that the pairs (Z_i, δ_i) i = 1, 2, ..., n are clearly not *i.i.d.* Thus, the usual methods for studying the properties of the sequence of estimators $\{\hat{\varphi}_n\}$ from the right censored data are not immediately applicable. Thus, we will use simulation to study both the large and small sample properties of this sequential estimation procedure.

The replacement cost for the *i*th system is C_1 if $X_i < \hat{\varphi}_{i-1}$, otherwise the replacement cost is C_2. With this sequential estimation procedure the actual total replacement cost for the first n systems that are observed is given in equation (2.2)

$$c_n = \sum_{i=1}^{n} \{C_1 \times \delta_i + C_2 \times (1 - \delta_i)\}, \tag{2.2}$$

and equation (2.3) is the total operating time for the n systems,

$$t_n = \sum_{i=1}^{n} \min(X_i, \hat{\varphi}_{i-1}) = \sum_{i=1}^{n} Z_i. \tag{2.3}$$

When studying the properties of this sequential estimation procedure it is important to see if and how fast the actual cost per unit time $\frac{c_n}{t_n}$ converges to optimal cost $C(\varphi^*)$ [Ref. 11].

In this thesis we simulate the sequential estimation procedure for two parametric families of distributions considered: In Chapter III, F belongs to a Weibull family of distributions with the shape parameter $\alpha > 1$ and unknown scale parameter (λ), and in Chapter IV, F belongs to a Gamma family of distributions with shape parameter $p > 1$ and unknown scale parameter (θ). Both the Weibull and Gamma distributions were chosen for the simulation because they are commonly used to model system lifetimes, they have increasing failure rates when their shape parameters > 1, and because estimation of the unknown parameters and minimization of the estimated cost function are numerically tractable.

III. SIMULATION SETTING FOR THE WEIBULL DISTRIBUTION

A. UNDERLYING LIFE DISTRIBUTION

This chapter is concerned with the estimation of the optimal replacement time when it is known that the underlying lifetime distribution is a member of the two parameter Weibull family with shape parameter α and scale parameter $\lambda > 0$, where the density is given by

$$f(t) = \alpha \lambda (\lambda t)^{\alpha - 1} e^{-(\lambda t)^{\alpha}} \qquad for \ t > 0. \tag{3.1}$$

The Weibull distribution has failure rate

$$\lambda(t) = \alpha \lambda (\lambda t)^{\alpha - 1}, \qquad t \geq 0. \tag{3.2}$$

When $\alpha > 1.0$, the failure rate in equation (3.2) is strictly increasing to infinity. Thus, for Weibull distributions, with $\alpha > 1.0$, a unique and finite optimal replacement age $\varphi^{\star}$ exists. To compare the results with [Ref. 9: p. 12], the same ten different Weibull distributions used in the simulation have α values 1.1, 1.2, ..., 1.9, 2.0. This selection of α values gives us a range of distributions which become more like the exponential distribution as α decreases from 2.0 to 1.1. The Weibull distribution with shape parameter $\alpha = 2.0$ is called the Rayleigh distribution and has a linearly increasing failure rate. To make fair comparisons between Weibull distributions, the scale parameter λ was chosen so that the expected system lifetime $E(X_i) = 2.0$. In our figures we have selected only certain values ($\alpha = 1.2, 1.4, 1.6, 1.8, 2.0$) for the scale parameter α. This was done to make the figures more readable. See Figures 1 and 2, for plots of the Weibull densities and corresponding failure rates.

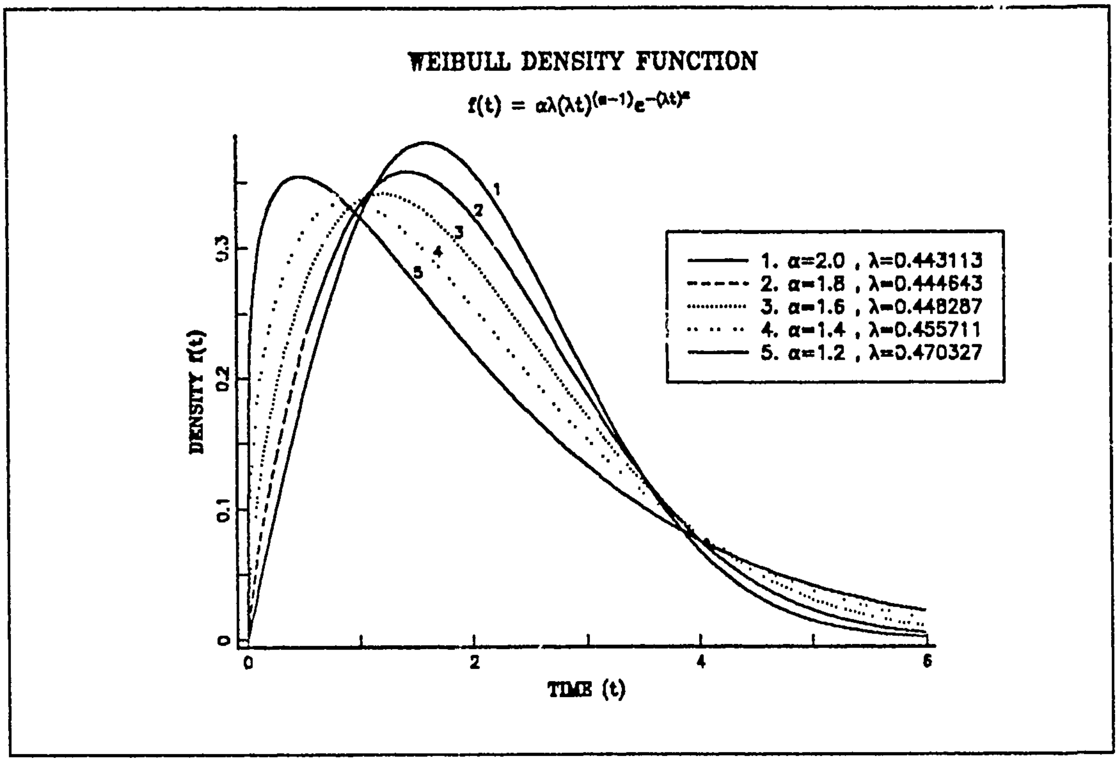

Figure 1. The Weibull Density Function f(t) with $E(X_i) = 2.0$

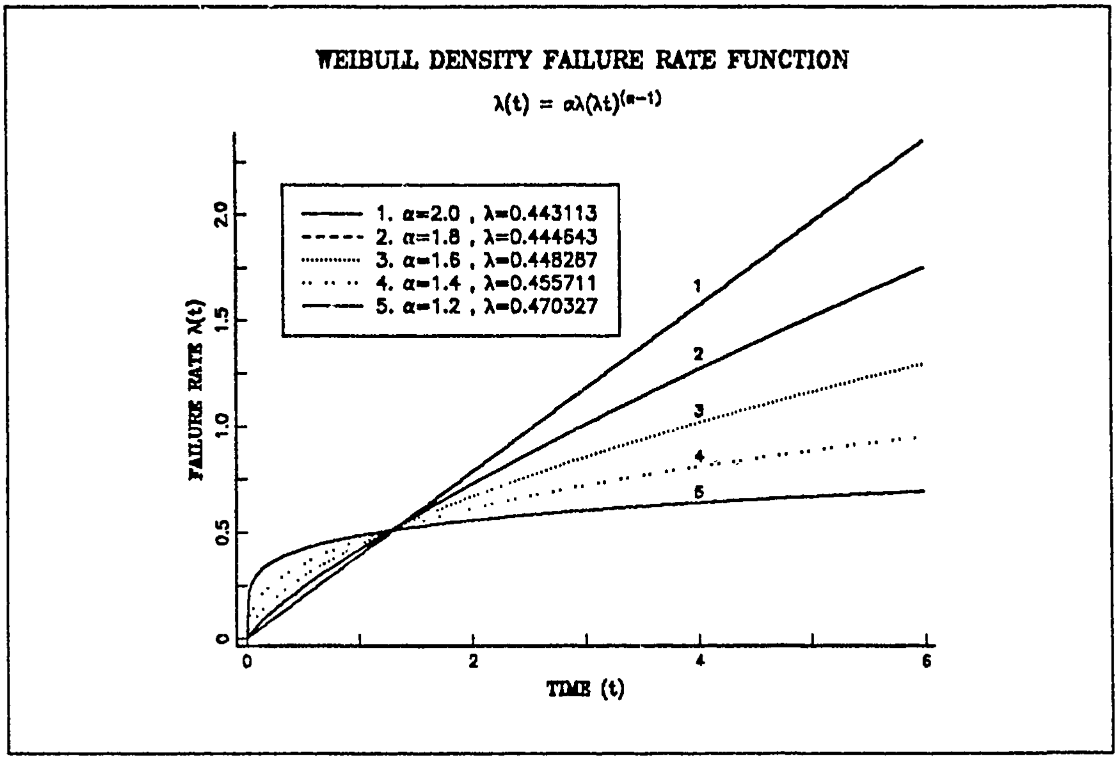

Figure 2. The Failure Rate of The Weibull Distribution with $E(X_i) = 2.0$

B. OPTIMAL REPLACEMENT TIME

When the distribution of the system lifetime is Weibull, the reliability (survival) function defined by

$$\bar{F}(t) = P(X > t)$$
$$= \begin{cases} e^{-(\lambda t)^{\alpha}}, & t > 0,\ \lambda > 0, \\ 1, & t \leq 0. \end{cases} \tag{3.3}$$

The long run expected cost per unit time under a simple age replacement policy with scheduled replacement at age φ, by using equation (2.1) is given by

$$C_{\lambda,\alpha}(\varphi) = \frac{C_1\,(1 - e^{-(\lambda\varphi)^\alpha}) + C_2\, e^{-(\lambda\varphi)^\alpha}}{\int_0^\varphi e^{-(\lambda x)^\alpha}\, dx}. \tag{3.4}$$

See Figure 3, for plot of the long run expected average cost function when the underlying life distribution is Weibull with shape parameter α varying from 1.1 to 2.0. For each curve on Figure 3 the optimal replacement time $\varphi^\star$ can be located on the x-axis at the minimum point.

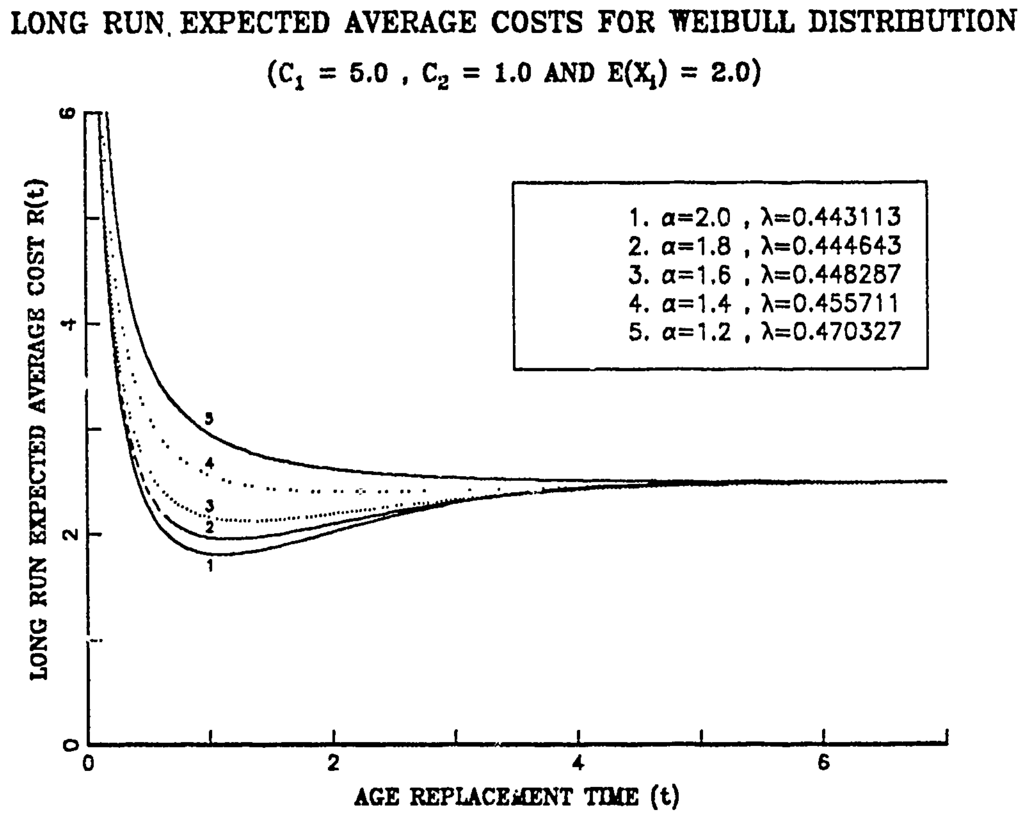

Figure 3. The Long Run Expected Average Cost Curves with $E(X_i) = 2.0$

C. SEQUENTIAL ESTIMATION PROCEDURE OF THE WEIBULL DISTRIBUTION

We will use the sequential estimation procedure which we discussed in general in Chapter II, with the known shape parameter $\alpha > 1.0$ and the unknown scale parameter λ.

Let $\hat{\lambda}_n$ be the MLE of λ after n replacements. The corresponding estimator $\hat{\varphi}_n$ of $\varphi^\star$ which minimizes $C_{\hat{\lambda}_n,\alpha}(\varphi)$ needs to be found numerically. The following result, simplifies this procedure considerably.

Lemma 1. Let $\varphi^\star$ minimize $C_{\lambda,\alpha}(\varphi)$ then $\varphi^\star = \lambda x^\star$ where $x^\star$ minimizes $C_{1,\alpha}(x)$.

PROOF. By using equation (2.1), the cost function at age t with scale parameter λ becomes,

$$C_{\lambda,\alpha}(t) = \frac{C_1 \int_0^t f(x)\, dx + C_2 \int_t^\infty f(x)\, dx}{\int_0^t \int_u^\infty f(v)\, dv\, du}. \tag{3.5}$$

If we substitute the Weibull density into the equation (3.5), the numerator can then be shown to be

$$C_1 \int_0^t \lambda^\alpha \alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}\, dx + C_2 \int_t^\infty \lambda^\alpha \alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}\, dx,$$

with the change of variable $y = \lambda x$, we obtain

$$\frac{\lambda^{\alpha}}{\lambda}\left\{C_1\int_0^{\lambda t}\alpha(\frac{y}{\lambda})^{\alpha-1}e^{-y^{\alpha}}\,dy + C_2\int_{\lambda t}^{\infty}\alpha(\frac{y}{\lambda})^{\alpha-1}e^{-y^{\alpha}}\,dy\right\}. \tag{3.6}$$

If we factor the constant term $(\frac{1}{\lambda})^{\alpha-1}$ from (3.6), we get

$$\frac{\lambda^{\alpha}}{\lambda\lambda^{\alpha-1}}\left\{C_1\int_0^{\lambda t}\alpha y^{\alpha-1}e^{-y^{\alpha}}\,dy + C_2\int_{\lambda t}^{\infty}\alpha y^{\alpha-1}e^{-y^{\alpha}}\,dy\right\}. \tag{3.7}$$

In expression (3.7) the constant terms cancel, and the integrands are just the density of the Weibull distribution with $\lambda = 1$. Thus (3.7) is

$$C_1F_{\lambda=1}(\lambda t) + C_2\overline{F}_{\lambda=1}(\lambda t). \tag{3.8}$$

With the change of variables $y = \lambda v$ the integrand in the denominator of equation (3.5) becomes,

$$\int_{\lambda u}^{\infty}\alpha y^{\alpha-1}e^{-y^{\alpha}}\,dy = \overline{F}_{\lambda=1}(\lambda t). \tag{3.9}$$

From (3.8) and (3.9), the cost function may be written as

$$C_{\lambda,\alpha}(t) = \frac{C_1F_{\lambda=1}(\lambda t) + C_2\overline{F}_{\lambda=1}(\lambda t)}{\int_0^t \overline{F}_{\lambda=1}(\lambda u)\,du}.$$

Set $y = \lambda u$, then

$$C_{\lambda,\alpha}(t) = \lambda\left\{\frac{C_1F_{\lambda=1}(\lambda t) + C_2\overline{F}_{\lambda=1}(\lambda t)}{\int_0^{\lambda t}\overline{F}_{\lambda=1}(y)\,dy}\right\} \tag{3.10}$$
$$= \lambda C_{\lambda=1,\alpha}(\lambda t).$$

Therefore, if $x\star$ minimizes, $C_{\lambda=1,\alpha}(t)$ and $\varphi^\star$ minimizes $C_{\lambda,\alpha}(t)$ then $\varphi^\star = \lambda x^\star$. ∎

To estimate λ (unknown scale parameter) from the right censored data; recall that, the available data to estimate $\hat{\lambda}_n$ are the pairs of (Z_i, δ_i) i = 1, 2, ..., n where $Z_1 = X_1$, and $\delta_1 = 0$. By using the *maximum likelihood estimation* (MLE) *procedure* we can estimate $\hat{\lambda}_1$ from the first observation X_1. The MLE of λ can be found by differentiating the likelihood (or the loglikelihood) with respect to λ, setting the results equal to zero, and solving for λ.

$$L(\lambda, \alpha|Z) = \alpha\lambda^{\alpha}(x_1)^{\alpha-1}e^{-(\lambda x_1)^{\alpha}},$$

$$l(\lambda, \alpha|Z) = \ln\alpha + \alpha\ln\lambda + (\alpha - 1)\ln x_1 - \lambda^{\alpha}x_1^{\alpha},$$

$$\frac{\partial l(\lambda, \alpha|Z)}{\partial\lambda} = \frac{\alpha}{\lambda} - \alpha\lambda^{\alpha-1}x_1^{\alpha},$$

$$\frac{\partial l(\lambda, \alpha|Z)}{\partial\lambda} = 0 \Rightarrow \hat{\lambda}_1 = \frac{1}{x_1}.$$

By Lemma 1, we can estimate $\hat{\varphi}_1^\star$ such that $\hat{\varphi}_1^\star = \hat{\lambda}_1 x^\star$ where $\hat{\varphi}_1^\star$ minimizes $C_{\hat{\lambda}_1,\alpha}(\varphi)$ and $x^\star$ minimizes $C_{1,\alpha}(x)$.

After the second observation, we set $Z_2 = \min(X_2, \hat{\varphi}_1^\star)$ and

$$\delta_2 = \begin{cases} 1 & X_2 \le \hat{\varphi}_1^\star \\ 0 & \textit{otherwise.} \end{cases}$$

After n observations, data will be (Z_1, δ_1), (Z_2, δ_2), ..., (Z_n, δ_n) where $Z_i = \min(X_i, \hat{\varphi}^{\star}_{i-1})$, X_i is the ith lifetime, and $\hat{\varphi}^{\star}_{i-1}$ is the best estimation of our optimal policy so far. In general, if we repeat the sequential estimation procedure n times, we obtain

$$L(\lambda, \alpha | Z) = f(z_1)\,[f(z_2)]^{\delta_2}\,[\bar{F}(z_2)]^{1-\delta_2} \ldots [f(z_n)]^{\delta_n}\,[\bar{F}(z_n)]^{1-\delta_n},$$

$$l(\lambda, \alpha | Z) = \alpha(1 + \delta_2 + \ldots + \delta_n)\ln \lambda - \lambda^{\alpha}(z_1^{\alpha} + z_2^{\alpha} + \ldots + z_n^{\alpha}),$$

$$\frac{\partial l(\lambda, \alpha | Z)}{\partial \lambda} = \frac{\alpha(1 + \delta_2 + \ldots + \delta_n)}{\lambda} - \alpha\lambda^{\alpha-1}(z_1^{\alpha} + z_2^{\alpha} + \ldots + z_n^{\alpha}),$$

$$\frac{\partial l(\lambda, \alpha | Z)}{\partial \lambda} = 0 \Rightarrow \hat{\lambda}_n = \left[\frac{\delta_1 + \delta_2 + \ldots + \delta_n}{z_1^{\alpha} + z_2^{\alpha} + \ldots + z_n^{\alpha}}\right]^{\frac{1}{\alpha}},$$

$$\hat{\lambda}_n = \left[\frac{\sum_{i=1}^{n}\delta_i}{\sum_{i=1}^{n} z_i^{\alpha}}\right]^{\frac{1}{\alpha}}. \tag{3.11}$$

By Lemma 1, we can find the MLE $\hat{\varphi}_n$ of $\varphi^{\star}$ explicitly from the MLE $\hat{\lambda}_n$ of λ based on the first n replacements using

$$\hat{\varphi}_n^{\star} = \hat{\lambda}_n \times x^{\star}. \tag{3.12}$$

D. SIMULATION RESULTS FOR THE WEIBULL DISTRIBUTION

1. Finding of The Scale Parameter λ

As mentioned before, the ten different Weibull distributions used in the simulation have α values 1.1, 1.2, ..., 1.9, 2.0 and the scale parameter λ was chosen so that the expected system lifetimes $E(X_i) = 2.0$. The mean of the Weibull distribution is

$$E[X] = \frac{\Gamma(\frac{1}{\alpha})}{\alpha\lambda}. \tag{3.13}$$

In equation (3.13), the scale parameter λ value can be obtained easily since the scale parameter α and $E[X]$ are known. The APL program Weibull in Appendix A calculates the scale parameter λ values for given α and $E[X]$. The results are given in Table 1 for the shape parameters and corresponding scale parameters.

Table 1. SCALE PARAMETER λ VALUES FOR $E(X_i) = 2.0$

Shape parameter α	$\alpha = 2.0$	$\alpha = 1.9$	$\alpha = 1.8$	$\alpha = 1.7$	$\alpha = 1.6$
Scale parameter λ	0.44311346	0.44368166	0.44464337	0.44612225	0.44828714
Shape parameter α	$\alpha = 1.5$	$\alpha = 1.4$	$\alpha = 1.3$	$\alpha = 1.2$	$\alpha = 1.1$
Scale parameter λ	0.45137265	0.45571117	0.46178836	0.47032793	0.48245624

2. **Finding of The Actual Optimal Age Replacement Time and Corresponding Actual Cost**

The actual age replacement time $\varphi^\star$ can be located in Figure 3 on the x-axis at the minimum point of the cost function $C_{\lambda,\alpha}(\varphi)$ (3.4). In order to find the unique minimum point $\varphi^\star$, it hard to solve equation 3.4 analytically. Therefore, the actual age replacement time $\varphi^\star$ found by simulating the cost function (3.4) for different φ where the simulated cost function is given by

$$\frac{C_1 \times \left\{ \frac{\sum I(X_i \leq \varphi)}{n} \right\} + C_2 \times \left\{ 1 - \left[\frac{\sum I(X_i \leq \varphi)}{n} \right] \right\}}{\left\{ \frac{\sum_{i=1}^{n} \min(X_i, \varphi)}{n} \right\}}, \qquad (3.14)$$

where X_i (i= 1, 2, ..., n) are the simulated *i.i.d.* Weibull random variables with the specific α and λ. $\varphi^\star$ is the minimum of $C_{\lambda,\alpha}(\varphi)$. Since the optimal replacement time $\varphi^\star$ comes from the simulation result, it varies slightly with the number of pseudo random variables used and the seed numbers used to generate them, thus we choose $n = 15 \times 10^5$.

3. **Finding of The Average Age Replacement Times and The Average Costs**

In order to find the average of the estimated optimal age replacement times and the average actual costs based on lifetimes with a Weibull distribution, we wrote the Fortran simulation program Averweib in Appendix C. From Lemma 1, we know that $x^\star$ minimizes $C_{\lambda=1,\alpha}(t)$ and $\varphi^\star$ minimizes $C_{\lambda,\alpha}(t)$ when $\varphi^\star = \lambda x^\star$. Thus, $x^\star$ can be found dividing the actual optimal age replacement times by the scale parameter λ, where the actual age replacement time is taken from the simulation program Sim and the λ values are taken from the Table 1 for the specific value of the shape parameter α. The unplanned and planned replacement costs C_1 and C_2 (to calculate the cost function); the scale parameter λ (to calculate the x^* value); the actual age replacement time (from program Sim to calculate the x^* value and to find MSE for each estimated age replacement time); the actual cost value (from program Sim to find MSE value for each estimated cost); and the shape parameter α must be given by the user in the initialization part of the simulation program Awerweib. In much of the simulation, we considered the sample sizes $N = 10$, 50, 250 and 1000. Other sample sizes are also considered, but in much less detail. Each repetition of the simulation is based on generating N system lifetimes. These system lifetimes (X_i) are used one at a time for the sequential estimation procedure. At the first observation, $Z_1 = X_1$, and $\delta_1 = 1$. The unknown scale parameter λ values are estimated by using the equation (3.11). After finding $\hat{\lambda}$, the estimated age replacement time values are calculated by using the equation (3.12) in the simulation. By using N generated system lifetimes, we determine N estimated age replacement times and N estimated costs in each simulation. We repeat this simulation 1000 times (NREP = 1000) and then we find the average value for both age replacement times and replacement costs.

Let C_{jN}, $j = 1, 2, \ldots, 1000$ be the actual cost per unit time for the first N replacements of the jth repetition of a simulation (where each C_{jN} is computed using (2.2) and (2.3)). The average actual cost per unit time over the 1000 repetitions is

$$\overline{C}_N = \sum_{j=1}^{1000} \frac{C_{jN}}{1000} . \tag{3.15}$$

Let $\varphi^{\star}_{jN}, j = 1, 2, ..., 1000$ be the estimated optimal replacement time for the first N replacements of the jth repetition of a simulation. The average age replacement time over the 1000 repetitions is

$$\overline{\varphi}^{\star}_{N} = \sum_{j=1}^{1000} \frac{\varphi^{\star}_{jN}}{1000}. \tag{3.16}$$

For each simulation, we also calculated

$$\overline{\text{MSECOS}} = \sum_{j=1}^{1000} \frac{(C_{jN} - C(\varphi^{\star}))^2}{1000}, \tag{3.17}$$

$$\overline{\text{MSEAGE}} = \sum_{j=1}^{1000} \frac{(\varphi^{\star}_{jN} - \varphi^{\star})^2}{1000}, \tag{3.18}$$

where $\overline{\text{MSECOS}}$ is the average squared difference of the actual replacement cost per unit time from the estimated minimum long run expected replacement cost per unit time and $\overline{\text{MSEAGE}}$ is the average squared difference of the actual optimal age replacement time from the estimated optimal age replacement time. These MSE values are calculated in the simulation in order to see if the sequential estimation procedure is converging the actual cost and the actual optimal age replacement time. If we get the MSE values close to the zero, then we can say that this procedure is working well.

Tables 2, 3, 4 and 5 summarize the simulation results of the optimal age replacement times and the minimum long run expected replacement costs per unit time for different values of α with $N = 1000$ for fixed costs $C_1 = 2.0$, $C_1 = 5.0$, $C_1 = 8.0$, $C_1 = 10.0$ while holding the C_2 fixed at 1.0. Tables 6, 7 and 8 summarize the simulation results using the three sample sizes of 10, 50 and 250 as small, moderate and large sample sizes for different values of α with fixed costs $C_1 = 2.0$ and $C_1 = 1.0$. Included in Tables 2~8, is the probability that a system will fail before the time φ under the optimal age replacement policy,

$$P(X_i < \varphi^{\star}) = 1 - e^{-(\lambda \varphi^{\star})^{\alpha}}. \tag{3.19}$$

We have also plotted the results of the average age replacement times (Fig. 4) and the average costs (Fig. 5) for $\alpha = 1.2, 1.4, 1.6, 1.8, 2.0$ when fixed costs $C_1 = 5.0$ and $C_2 = 1.0$. From these plots and the results from Tables 2~8, we observe that in general when α decreases from 2.0 (i.e., the underlying life distribution is becomes more exponential) the optimal age replacement time $\varphi^{\star}$, the long run expected optimal replacement cost and MSE values increase. For values of α close to 1.0, very little is gained by replacing the system before failure at the higher cost C_1. When the ratio of the unplanned replacement cost C_1 to the planned replacement cost C_2 increases the optimal replacement time for a specific α decreases. The larger values of $\varphi^{\star}$, insure that a small percentage of replacement will be made before failure, which is what we desire if the system's life distribution is close to exponential.

By looking at the tables for different sample sizes, we also observe that the average cost per unit time up to the Nth replacement decreases with N, the number of replacement. This result is promising because the ultimate goal of the sequential estimation procedure is to decrease costs while sampling. Even though as $N \to \infty$, the long run average cost per unit time will approach the optimal replacement cost $C(\varphi^{\star})$

with probability 1.0 [Ref. 11], there is no guarantee that the average replacement cost will decrease for the first observations.

Table 2. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE WEIBULL MODEL WITH $C_1 = 2.0$, $C_2 = 1.0$, $E(X_i) = 2.0$ AND $N = 1000$

Shape parameter α	$x^* = \frac{\varphi^*}{\lambda}$	Optimal replacement time φ^*	Average φ^*	MSE of φ^*	Long run expected optimal replacement cost $C(\varphi^*)$	Average $C(\varphi^*)$	MSE of $C(\varphi^*)$	$P(X_i \le \varphi^*)$
2.0	5.60421	2.18330	2.18282	0.00234	0.96281	0.96734	0.00038	0.70205
1.9	6.02887	2.67490	2.67455	0.00281	0.97122	0.97579	0.00040	0.74957
1.8	6.60912	2.93870	2.93934	0.00341	0.97892	0.98376	0.00041	0.80179
1.7	7.43092	3.31510	3.31622	0.00457	0.98522	0.99047	0.00044	0.85701
1.6	8.68907	3.89520	3.89690	0.00660	0.99061	0.99567	0.00047	0.91281
1.5	10.80659	4.87780	4.88050	0.01127	0.99379	0.99901	0.00051	0.96187
1.4	15.01938	6.84450	6.85008	0.02502	0.99505	1.00053	0.00057	0.99267
1.3	25.61541	11.82890	11.83988	0.08617	0.99517	1.00080	0.00065	0.99988
1.2	41.08195	19.32199	19.34289	0.26980	0.99509	1.00095	0.00076	0.99998
1.1	41.40083	19.97409	19.99884	0.34320	0.99500	1.00115	0.00089	0.99999

Table 3. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE WEIBULL MODEL WITH $C_1 = 5.0$, $C_2 = 1.0$, $E(X_i) = 2.0$ AND N = 1000

Shape parameter α	$x^* = \frac{\varphi^*}{\lambda}$	Optimal replacement time φ^*	Average φ^*	MSE of φ^*	Long run expected optimal replacement cost $C(\varphi^*)$	Average $C(\varphi^*)$	MSE of $C(\varphi^*)$	$P(X_i \leq \varphi^*)$
2.0	2.62077	1.16130	1.15795	0.00159	1.80786	1.81501	0.00371	0.23264
1.9	2.67016	1.18470	1.18135	0.00165	1.88493	1.88502	0.00396	0.25520
1.8	2.75630	1.22557	1.22344	0.00169	1.96297	1.96297	0.00420	0.28487
1.7	2.88351	1.28640	1.28392	0.00183	2.03792	2.04504	0.00457	0.32230
1.6	3.06203	1.37267	1.37036	0.00199	2.13370	2.13386	0.00476	0.36861
1.5	3.35217	1.51308	1.51018	0.00231	2.22635	2.22651	0.00497	0.43131
1.4	3.86012	1.75910	1.75702	0.00310	2.21461	2.22242	0.00547	0.51991
1.3	4.84161	2.23580	2.23507	0.00483	2.10207	2.11223	0.00457	0.64740
1.2	7.37698	3.46960	3.46567	0.01026	2.16672	2.18019	0.00546	0.83466
1.1	22.50028	10.85540	10.86805	0.10160	2.18746	2.19997	0.00563	0.99793

Table 4. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE WEIBULL MODEL WITH $C_1 = 8.0$, $C_2 = 1.0$, $E(X_i) = 2.0$ AND N = 1000

Shape parameter α	$x^* = \frac{\varphi^*}{\lambda}$	Optimal replacement time φ^*	Average φ^*	MSE of φ^*	Long run expected optimal replacement cost $C(\varphi^*)$	Average $C(\varphi^*)$	MSE of $C(\varphi^*)$	$P(X_i \le \varphi^*)$
2.0	1.94894	0.86360	0.85912	0.00153	2.27209	2.28580	0.01041	0.13622
1.9	1.95546	0.86760	0.86381	0.00147	2.19619	2.51185	0.01149	0.15043
1.8	1.97709	0.87910	0.87523	0.00152	2.64187	2.65269	0.01293	0.16837
1.7	2.01985	0.90110	0.89736	0.00159	2.80216	2.81123	0.01457	0.19137
1.6	2.08973	0.93680	0.93345	0.00170	2.98188	2.98973	0.01634	0.22083
1.5	2.24914	1.01520	1.01248	0.00182	3.18180	3.18835	0.01764	0.26669
1.4	2.18293	1.13150	1.12910	0.00205	3.39835	3.40519	0.01847	0.32674
1.3	2.94984	1.36220	1.35973	0.00258	3.62432	3.63179	0.01848	0.42154
1.2	4.07120	1.91480	1.91378	0.00446	3.83556	3.84932	0.01833	0.58601
1.1	9.07875	4.38010	4.38316	0.00180	3.96953	3.99155	0.01547	0.89745

Table 5. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE WEIBULL MODEL WITH $C_1 = 10.0$, $C_2 = 1.0$, $E(X_i) = 2.0$ AND N = 1000

Shape parameter α	$x^* = \frac{\varphi^*}{\lambda}$	Optimal replacement time φ^*	Average φ^*	MSE of φ^*	Long run expected optimal replacement cost $C(\varphi^*)$	Average $C(\varphi^*)$	MSE of $C(\varphi^*)$	$P(X_i \leq \varphi^*)$
2.0	1.71333	0.75920	0.75485	0.00153	2.68251	2.70471	0.01680	0.10700
1.9	1.70798	0.75780	0.75306	0.00155	2.84430	2.86303	0.01943	0.11844
1.8	1.71261	0.76150	0.75684	0.00155	3.02798	3.04391	0.02169	0.13270
1.7	1.73181	0.77260	0.76893	0.00145	3.23710	3.25166	0.02399	0.15086
1.6	1.77431	0.79540	0.79170	0.00151	3.47530	3.48676	0.02713	0.17474
1.5	1.85612	0.83780	0.83471	0.00165	3.74569	3.75593	0.03124	0.20749
1.4	2.01202	0.91690	0.91362	0.00181	4.04892	4.05337	0.03377	0.25526
1.3	2.28854	1.10300	1.10075	0.00213	4.37636	4.38241	0.03354	0.34034
1.2	3.19756	1.50390	1.50147	0.00324	4.69973	4.70893	0.03210	0.65938
1.1	6.55541	3.16270	3.16431	0.01073	4.94021	4.96496	0.02718	0.79643

Table 6. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE WEIBULL MODEL WITH $C_1 = 2.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 10)

Shape parameter α	$x^* = \frac{\varphi^*}{\lambda}$	Optimal replacement time φ^*	Average φ^*	MSE of φ^*	Long run expected optimal replacement cost $C(\varphi^*)$	Average $C(\varphi^*)$	MSE of $C(\varphi^*)$	$P(X_i \le \varphi^*)$
2.0	5.60421	2.18330	2.12870	0.24731	0.96281	0.99736	0.03285	0.70205
1.9	6.02887	2.67490	2.65634	0.35938	0.97122	1.00179	0.03856	0.74957
1.8	6.60912	2.93870	2.94467	0.45415	0.97892	1.01056	0.04185	0.80179
1.7	7.43092	3.31510	3.34668	0.61503	0.98522	1.01731	0.04637	0.85701
1.6	8.68907	3.89520	3.98598	0.89009	0.99061	1.02701	0.05129	0.91281
1.5	10.80659	4.87780	5.06132	1.49859	0.99379	1.03651	0.05788	0.96187
1.4	15.01938	6.84450	7.18304	3.29738	0.99505	1.04428	0.06690	0.99267
1.3	25.61541	11.82890	12.54384	11.25283	0.99517	1.05299	0.07893	0.99988
1.2	41.08195	19.32199	20.65824	36.29063	0.99509	1.06284	0.09586	0.99998
1.1	41.40083	19.97409	21.56398	48.08590	0.99500	1.07589	0.11946	0.99999

Table 7. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE WEIBULL MODEL WITH $C_1 = 2.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 50)

Shape parameter α	$x^* = \frac{\varphi^*}{\lambda}$	Optimal replacement time φ^*	Average φ^*	MSE of φ^*	Long run expected optimal replacement cost $C(\varphi^*)$	Average $C(\varphi^*)$	MSE of $C(\varphi^*)$	$P(X_i \le \varphi^*)$
2.0	5.60421	2.18330	2.16275	0.04749	0.96281	0.98115	0.00627	0.70205
1.9	6.02887	2.67490	2.66781	0.06037	0.97122	0.98343	0.00722	0.74957
1.8	6.60912	2.93870	2.93602	0.07367	0.97892	0.99064	0.00771	0.80179
1.7	7.43092	3.31510	3.32412	0.09603	0.98522	0.99804	0.00831	0.85701
1.6	8.68907	3.89520	3.92008	0.13930	0.99061	1.00408	0.00895	0.91281
1.5	10.80659	4.87780	4.92133	0.23771	0.99379	1.00780	0.00988	0.96187
1.4	15.01938	6.84450	6.92258	0.53115	0.99505	1.01007	0.01130	0.99267
1.3	25.61541	11.82890	11.99438	1.81361	0.99517	1.01211	0.01307	0.99988
1.2	41.08195	19.32199	19.62654	5.72521	0.99509	1.01419	0.01542	0.99998
1.1	41.40083	19.97409	20.33311	7.33616	0.99500	1.01706	0.01848	0.99999

Table 8. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE WEIBULL MODEL WITH $C_1 = 2.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 250)

Shape parameter α	$x^* = \frac{\varphi^*}{\lambda}$	Optimal replacement time φ^*	Average φ^*	MSE of φ^*	Long run expected optimal replacement cost $C(\varphi^*)$	Average $C(\varphi^*)$	MSE of $C(\varphi^*)$	$P(X_i \leq \varphi^*)$
2.0	5.60421	2.18330	2.17942	0.00788	0.96281	0.97330	0.00129	0.70205
1.9	6.02887	2.67490	2.67173	0.01039	0.97122	0.97674	0.00138	0.74957
1.8	6.60912	2.93870	2.93645	0.01328	0.97892	0.98440	0.00149	0.80179
1.7	7.43092	3.31510	3.31576	0.01778	0.98522	0.99140	0.00161	0.85701
1.6	8.68907	3.89520	3.89856	0.02670	0.99061	0.99669	0.00176	0.91281
1.5	10.80659	4.87780	4.88247	0.04566	0.99379	0.99967	0.00192	0.96187
1.4	15.01938	6.84450	6.85578	0.10078	0.99505	1.00126	0.00217	0.99267
1.3	25.61541	11.82890	11.85452	0.34589	0.99517	1.00179	0.00249	0.99988
1.2	41.08195	19.32199	19.37074	1.08303	0.99509	1.00218	0.00292	0.99998
1.1	41.40083	19.97409	20.03218	1.37837	0.99500	1.00270	0.00348	0.99999

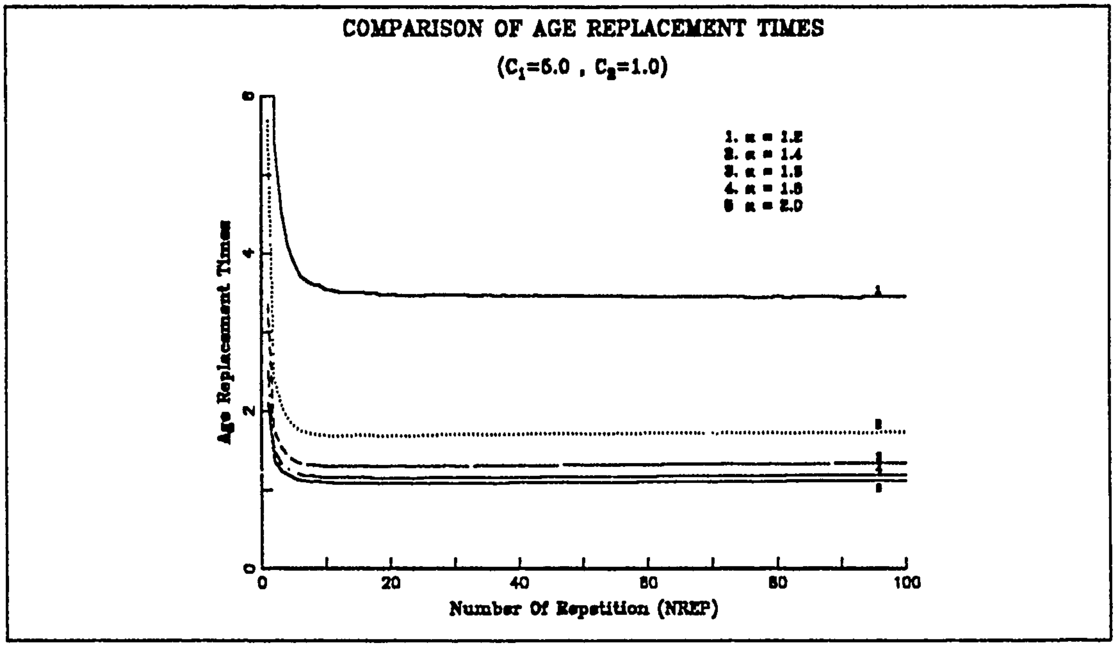

Figure 4. The Average Age Replacement Times For The Weibull Model

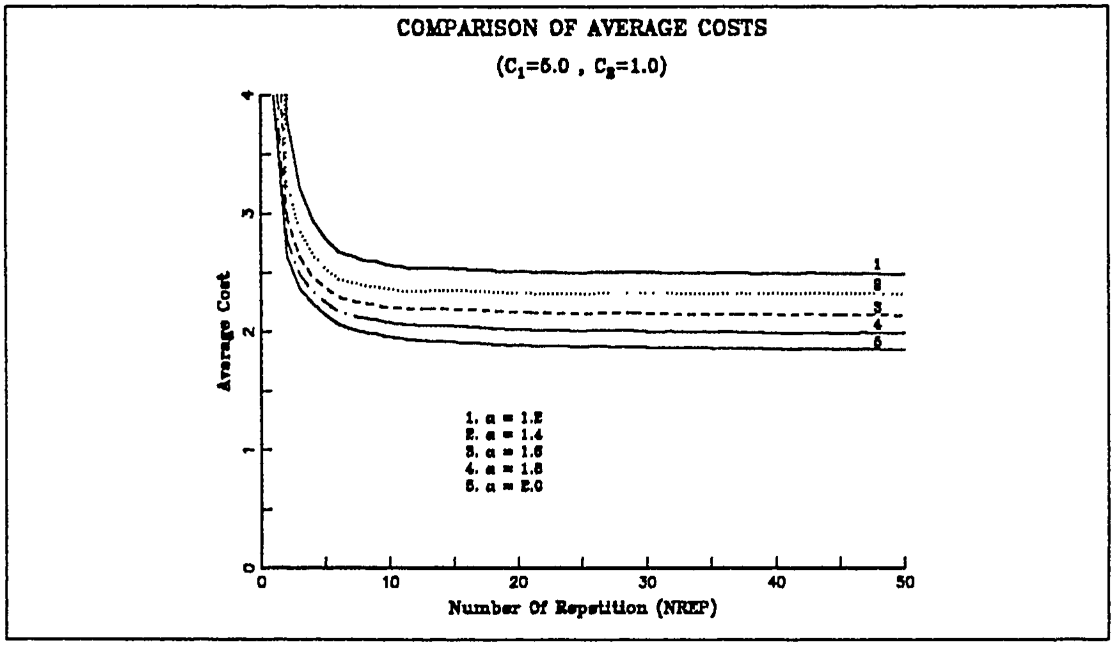

Figure 5. The Average Costs For The Weibull Model

IV. SIMULATION SETTING FOR THE GAMMA DISTRIBUTION

A. UNDERLYING LIFE DISTRIBUTION

This chapter is concerned with the estimation of the optimal replacement time when it is known that the underlying lifetime distribution is a member of the two parameter Gamma family with shape parameter $p > 1$ and scale parameter $\theta > 0$, where the density is given by

$$f(t) = \frac{t^{p-1} e^{-\frac{t}{\theta}}}{\Gamma(p)\,\theta^{p}} \qquad for \quad t > 0. \tag{4.1}$$

If $p = 1.0$, the Gamma density with scale parameter θ reduces to

$$f(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}} \qquad t \geq 0. \tag{4.2}$$

In equation (4.2), we have an exponential density with parameter θ. The reciprocal of the failure rate of the Gamma distribution is given in equation (4.3)

$$\frac{1}{r(t)} = \int_{t}^{\infty} (\frac{x}{t})^{p-1} e^{-\frac{(x-t)}{\theta}} dx, \qquad t \geq 0. \tag{4.3}$$

By solving equation (4.3) analytically for different values of p, we get useful functional forms of $r(t)$ in Table 9 on page 31. The failure rate for the Gamma random variable with $p > 1.0$ is a strictly increasing continuous function and is bounded above by $(\frac{1}{\theta})$. A unique and finite optimum replacement policy $\varphi^{\star}$ exists and will be finite if and only if $(p - 1)$ is strictly greater than $\frac{C_2}{C_1 - C_2}$, where C_1 and C_2, are unplanned and planned replacement costs ($C_1 > C_2$). Similar to the Weibull case, this unique minimum of $C(\varphi)$ occurs at the minimum point where the first derivative is zero.

Table 9. FUNCTIONAL FORM OF $r(t)$

Shape parameter p	Scale parameter θ	Functional form
2.0	1.0	$\frac{t}{t+1}$
3.0	0.666667	$\frac{27t^2}{18t^2 + 24t + 16}$
4.0	0.5	$\frac{8t^3}{4t^3 + 6t^2 + 6t + 3}$
5.0	0.4	$\frac{3125t^4}{1310t^4 + 2000t^3 + 2400t^2 + 1920t + 768}$

In this chapter cost C_1, C_2 are chosen so that φ^* is finite and unique. We used shape parameter p = 2.0, 3.0, 4.0, 5.0 ($p > 1$) and scale parameter θ = 2/2, 2/3, 2/4, 2/5 in our simulation, respectively. This selection of p values gives us a range of distributions which become more like the exponential as p decreases from 5.0 to 2.0. To make fair comparisons between Gamma distributions, the scale parameter θ was chosen so that the expected system lifetime $E(X_i) = 2.0$. See Figures 6 and 7, for plots of the Gamma densities and corresponding failure rates.

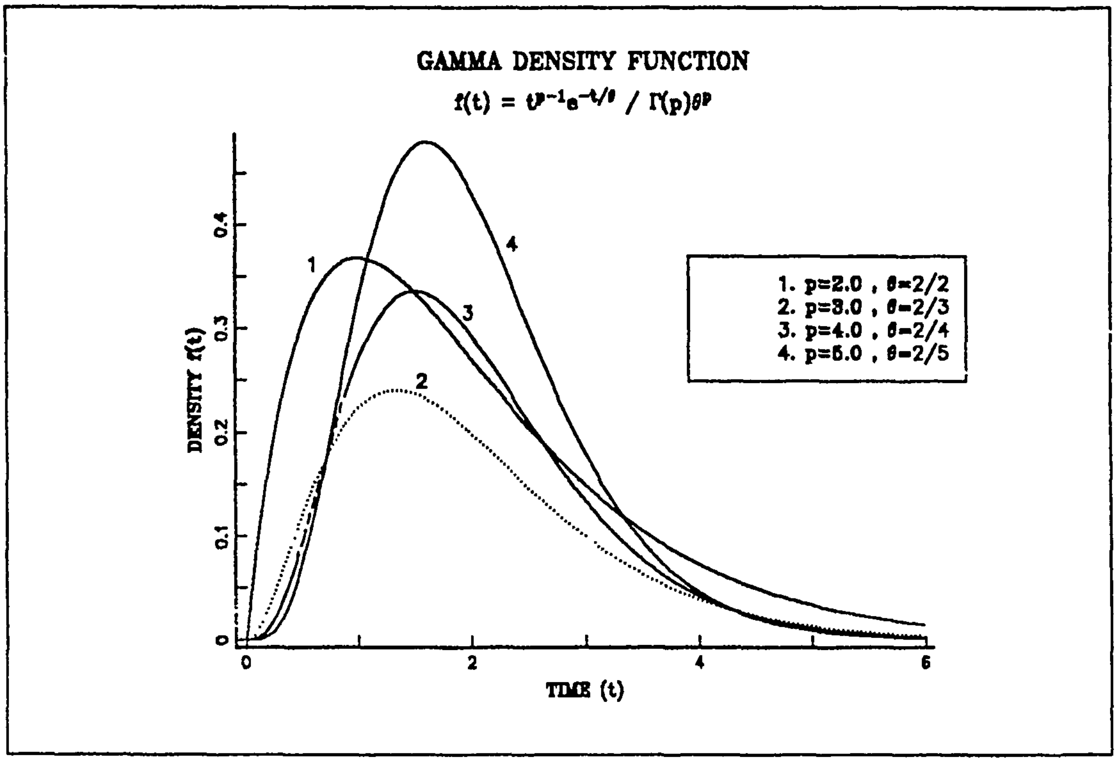

Figure 6. The Gamma Density Function f(t) with $E(X_i) = 2.0$

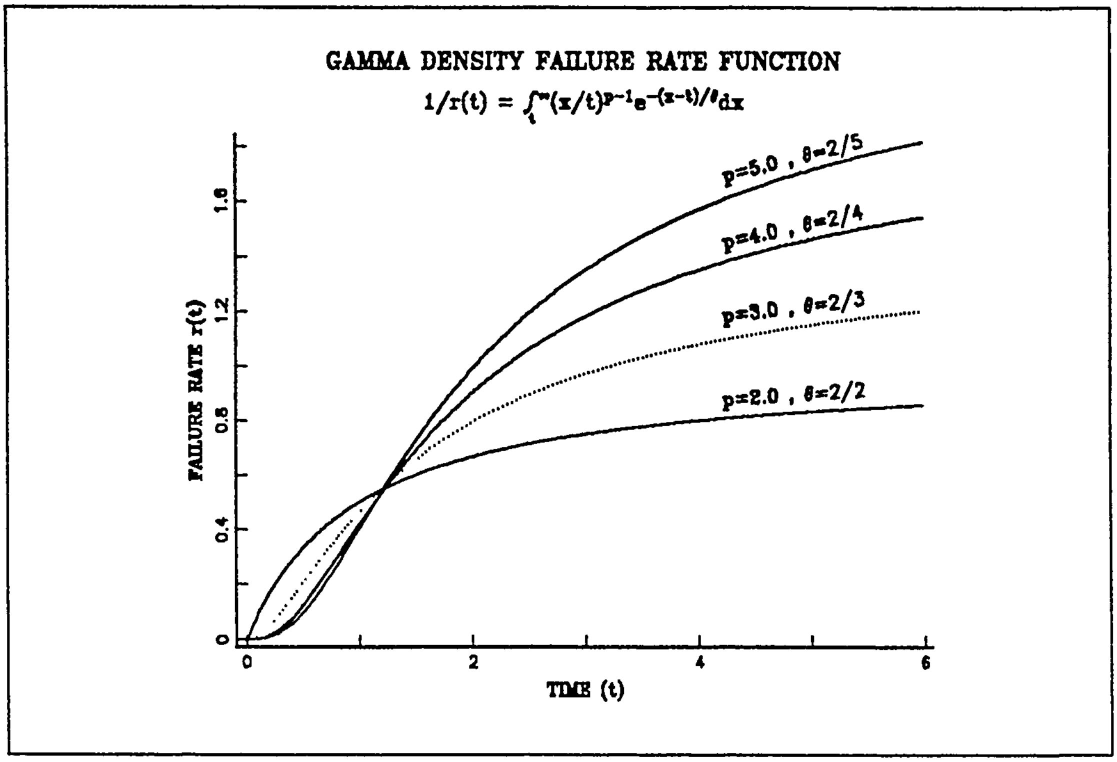

Figure 7. The Failure Rate of The Gamma Distribution with $E(X_i) = 2.0$

B. OPTIMAL REPLACEMENT TIME

When the distribution of a system lifetime is Gamma, the expected long run average cost per unit time under a simple age replacement policy with scheduled replacement at age t, by using equation (2.1) is given by

$$C_{\theta,p}(t) = \frac{C_1 \displaystyle\int_0^t \frac{x^{p-1} e^{-\frac{x}{\theta}}}{\Gamma(p)\theta^p}\, dx + C_2 \displaystyle\int_t^\infty \frac{x^{p-1} e^{-\frac{x}{\theta}}}{\Gamma(p)\theta^p}\, dx}{\displaystyle\int_0^t \overline{F}(x)\, dx}. \tag{4.4}$$

In equation (4.4), the corresponding survival function $\overline{F}(t)$ is,

$$\overline{F}(t) = \int_t^{\infty} \frac{x^{p-1} e^{-\frac{x}{\theta}}}{\Gamma(p)\theta^p}\, dx, \qquad t \geq 0. \tag{4.5}$$

In this thesis, we obtained the numerical values for $\overline{F}(t)$, by solving equation (4.5) analytically for some specific values of p. Table 10 gives the useful functional form of $\overline{F}(t)$. See Figure 8 for plot of the $C_{\theta,p}(t)$ when the underlying life distribution is Gamma with shape parameter p varying from 2.0 to 5.0. For each curve on Figure 8 the optimal replacement time $\varphi^{\star}$ can be located on the x-axis at the minimum point.

Table 10. FUNCTIONAL FORM OF $\overline{F}(T)$

Shape parameter p	Scale parameter θ	Functional form
2.0	1.0	$\frac{t+1}{e^t}$
3.0	0.666667	$\frac{2.25t^2 + 3t + 2}{2e^{1.5t}}$
4.0	0.5	$\frac{4t^3 + 6t^2 + 6t + 3}{3e^{2.0t}}$
5.0	0.4	$\frac{1310t^4 + 2000t^3 + 2400t^2 + 1920t + 768}{768e^{2.5t}}$

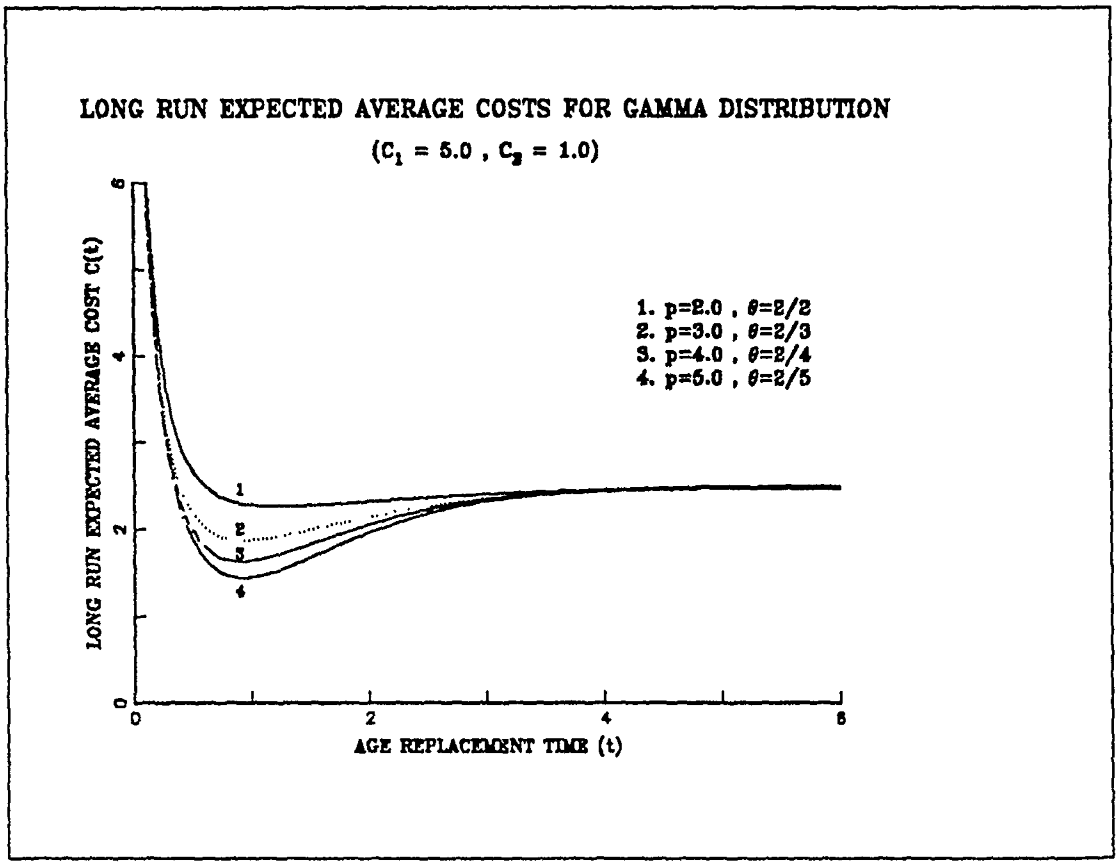

Figure 8. The Long Run Expected Average Cost Curves with $E(X_i) = 2.0$

C. SEQUENTIAL ESTIMATION PROCEDURE OF THE GAMMA DISTRIBUTION.

For the Gamma distribution, we will use our parametric sequential estimation procedure again which we mentioned in Chapter II, with the known shape parameter $p > 1.0$ and the unknown scale parameter θ.

As in the Weibull case, minimizing $C_{\hat{\theta}_n,p}(t)$ (where $\hat{\theta}_n$ is the MLE of θ after n replacements) can be simplified with the following result:

Lemma 2. Let $\varphi^\star$ minimize $C_{\theta,p}(\varphi)$ then $\varphi^\star = \frac{x^\star}{\theta}$ where $x^\star$ minimizes $C_{1,p}(x)$.

PROOF. From the cost function $C_{\theta,p}(\varphi)$ for the Gamma distribution in equation (4.4), after making the substitution $y = \frac{x}{\theta}$, the numerator may be written as

$$C_1 \int_0^{\frac{t}{\theta}} \frac{y^{p-1}\theta^p e^{-y}}{\theta\Gamma(p)\theta^p}\,\theta\, dy + C_2 \int_{\frac{t}{\theta}}^{\infty} \frac{y^{p-1}\theta^p e^{-y}}{\theta\Gamma(p)\theta^p}\,\theta\, dy. \tag{4.6}$$

Canceling the scale parameters (θ) we obtain

$$C_1 \int_0^{\frac{t}{\theta}} \frac{y^{p-1}e^{-y}}{\Gamma(p)}\, dy + C_2 \int_{\frac{t}{\theta}}^{\infty} \frac{y^{p-1}e^{-y}}{\Gamma(p)}\, dy. \tag{4.7}$$

The integrands of equation (4.7) are the density of the Gamma distribution with $\theta = 1$. Finally the numerator of (4.4) becomes

$$C_1 F_{\theta=1}\left(\frac{t}{\theta}\right) + C_2 \overline{F}_{\theta=1}\left(\frac{t}{\theta}\right). \tag{4.8}$$

In equation (4.4) the denominator may be written as

$$\int_0^{\infty}\int_u^{\infty} f(v)\, dv\, du. \tag{4.9}$$

If we first solve the inner integral by changing variable y with $\frac{v}{\theta}$, we get

$$\int_u^\infty f(v)\, dv = \int_u^\infty \frac{v^{p-1} e^{-\frac{v}{\theta}}}{\Gamma(p)\theta^p}\, dv,$$

$$\int_{\frac{u}{\theta}}^\infty \frac{(y\theta)^{p-1} e^{-y}}{\Gamma(p)\theta^p}\, \theta\, dy.$$

After cancellation, the denominator is

$$\int_{\frac{u}{\theta}}^\infty \frac{y^{p-1} e^{-y}}{\Gamma(p)}\, dy. \tag{4.10}$$

Again the integrand of equation (4.10) is the density of the Gamma distribution with $\theta = 1$. From (4.8) and (4.10), the cost function may be written as

$$C_{\theta,p}(t) = \frac{C_1 F_{\theta=1}(\frac{t}{\theta}) + C_2 \overline{F}_{\theta=1}(\frac{t}{\theta})}{\int_0^t \overline{F}_{\theta=1}(\frac{u}{\theta})\, du}. \tag{4.11}$$

If we let the $y = \frac{u}{\theta}$, equation (4.11) becomes,

$$C_{\theta,p}(t) = \frac{1}{\theta}\left\{ \frac{C_1 F_{\theta=1}(\frac{t}{\theta}) + C_2 \bar{F}_{\theta=1}(\frac{t}{\theta})}{\int_0^{\frac{t}{\theta}} \bar{F}_{\theta=1}(y)\, dy} \right\} \tag{4.12}$$

$$= \frac{1}{\theta} C_{\theta=1,p}(\frac{t}{\theta}).$$

Therefore, if $x^\star$ minimizes, $C_{\theta=1,p}(t)$ and $\varphi^\star$ minimizes $C_{\theta,p}(t)$ then $\varphi^\star = \frac{x^\star}{\theta}$. ■

There is no closed form solution for the MLE of θ based on right censored data. Thus, we use the EM algorithm to find the MLE numerically.

1. EM Algorithm

" The *Expectation-Maximization* (EM) algorithm is an iterative algorithm used to compute the MLE in incomplete-data problems. The algorithm is applicable to more general missing-data patterns, but usually involves more computations than methods designed for special patterns. The iterations of the EM algorithm are

1. Replace missing values by estimated values.
2. Estimate parameters.
3. Reestimate the missing values assuming the new parameter estimates are correct.
4. Reestimate parameters.

and so forth, iterating until convergence " [Ref. 12: pp. 127-141]. Since each iteration of the algorithm consist of an expectation step (E) followed by a maximization step (M) it calls the EM algorithm.

The M step is performed by maximizing the likelihood as if there were no missing data. Thus, the M step of EM uses the identical computational methods as MLE from $l(\theta, p|X)$ [Ref. 12: pp. 127-141]. With the shape parameter p assumed known, the maximum likelihood estimator of θ based on n uncensored observations $X_1, ..., X_n$ is

$$\hat{\theta}_n = \frac{\sum_{i=1}^{n} X_i}{np}. \tag{4.13}$$

The E step finds the conditional expectation of the missing (censored) data given the observed data and current estimated parameters, and then substitutes these expectations for the missing (censored) data [Ref. 12: pp. 127-141]. Thus at the ith iteration of the EM algorithm the MLE $\hat{\theta}_n$ based on n replacements is approximated by

$$\hat{\theta}_{n,i} = \frac{\sum_{i=1}^{n} [\delta_i Z_i + (1 - \delta_i) E(X|X > Z_i)]}{np}, \tag{4.14}$$

where $E(X|X > Z_i)$ is the conditional expectation of the random variable $X \sim$ Gamma($\theta = \hat{\theta}_{n,i-1}, p$). The functional forms of this conditional expectation for the specific shape parameter p are given in Table 11 on page 40.

As in the Weibull case, we have no information about the age replacement time φ at the first observation. Thus, the Z_1 value equals the first observation X_1. We find $\hat{\theta}_1$ (by using equation (4.13)), we get

$$\hat{\theta}_1 = \frac{X_1}{p}.$$

From the Lemma 2, we can estimate $\hat{\varphi}_1^{\star}$ such that $\hat{\varphi}_1^{\star} = \frac{x^{\star}}{\hat{\theta}_1}$, where $\hat{\varphi}_1^{\star}$ minimizes $C_{\hat{\theta}_1, p}(\varphi)$, and $x^{\star}$ minimizes $C_{1,p}(x)$.

At the second observation, we have two cases. If X_2 is less than $\hat{\varphi}_1^{\star}$, then $\hat{\theta}_2$ is

Table 11. FUNCTIONAL FORM OF $E(X|X > \varphi)$ WHERE $X \sim GAM(\theta, p)$

Shape parameter p	Scale parameter θ	Functional form
2.0	1.0	$\frac{\varphi^2 + 2\theta\varphi + 2\theta^2}{\varphi + \theta}$
3.0	0.666667	$\frac{\varphi^3 + 3\theta(\varphi^2 + 2\theta\varphi + 2\theta^2)}{\varphi^2 + 2\theta\varphi + 2\theta^2}$
4.0	0.5	$\frac{\varphi^4 + 4\theta(\varphi^3 + 3\theta(\varphi^2 + 2\theta\varphi + 2\theta^2))}{\varphi^3 + 3\theta(\varphi^2 + 2\theta\varphi + 2\theta^2)}$
5.0	0.4	$\frac{\varphi^5 + 5\theta(\varphi^4 + 4\theta(\varphi^3 + 3\theta(\varphi^2 + 2\theta\varphi + 2\theta^2)))}{\varphi^4 + 4\theta(\varphi^3 + 3\theta(\varphi^2 + 2\theta\varphi + 2\theta^2))}$

$$\hat{\theta}_2 = \frac{X_1 + X_2}{2p}.$$

Otherwise, the observation is censored, and we use the equation (4.14) for the E (expectation) step of the EM algorithm until $\hat{\theta}_2$ convergence. These iterations are

$$\hat{\theta}_{2,1} = \frac{Z_1 + E_{\hat{\theta}_1}[X_2 | X_2 > \hat{\varphi}_1^{\star}]}{2p},$$

$$\hat{\theta}_{2,2} = \frac{Z_1 + E_{\hat{\theta}_{2,1}}[X_2 | X_2 > \hat{\varphi}_1^{\star}]}{2p},$$

and so on. Finally, these iterations converge $\hat{\theta}_{2,n}$. When the difference of the absolute value of $\hat{\theta}_{2,n}$, and $\hat{\theta}_{2,n-1}$ is small, the stopping criteria is satisfied. At that point, we can replace the $\hat{\theta}_{2,n}$ with $\hat{\theta}_2$. Again, we can estimate $\hat{\varphi}_2^\star$ such that $\hat{\varphi}_2^\star = \frac{x^\star}{\hat{\theta}_2}$.

The procedure is then repeated. After determining $\hat{\theta}$ values for each replacement, we can apply this estimated θ values to the Lemma 2. The estimated optimal age replacement time can be expressed by,

$$\hat{\varphi}_n^\star = \frac{x^\star}{\hat{\theta}_n}. \tag{4.15}$$

For large n, the estimated optimal age replacement time $\hat{\varphi}_n^\star$ converges to an optimal age replacement time $\varphi^\star$.

D. SIMULATION RESULTS FOR THE GAMMA DISTRIBUTION

1. Finding of The Scale Parameter θ

The four different Gamma distributions used in the simulation have p values 2.0, 3.0, 4.0, 5.0, and the scale parameter θ was chosen so that the expected system lifetimes $E(X_i) = 2.0$. The mean of the Gamma distribution is $E(X) = p\theta$. The θ value can be obtain since both E(X) and p are known. For example, if $p = 3.0$, then $\theta = 0.666667$.

2. Finding of The Actual Optimal Age Replacement Time and Corresponding Actual Cost.

Similar to the Weibull case, we found the minimum value of the cost function (4.4) and corresponding actual age replacement time $\varphi^\star$ for different costs C_1, C_2 and parameters (θ, p) by simulation.

3. Finding of The Average Age Replacement Times and The Average Costs

As in the Weibull case, in order to find the average of the estimated optimal age replacement times and the average actual costs based on lifetimes with a Gamma distribution, we wrote the Fortran simulation program Avergam in Appendix D. From Lemma 2, $x^{\star} = \varphi^{\star}\theta$ where the actual age replacement time $\varphi^{\star}$ is taken from the simulation program Sim and the θ values are calculated from the equation $E(X) = p\theta$ for the specific value of the shape parameter p. The unplanned and planned replacement costs C_1 and C_2 (to calculate the cost function); the scale parameter θ (to calculate the $x^{\star}$ value); the actual age replacement time (from program Sim to calculate the $x^{\star}$ value and to find MSE for each estimated age replacement time); the actual cost value (from program Sim to find MSE value for each estimated cost); and the shape parameter p must be given by the user in the initialization part of the simulation program. In much of the simulation, we considered the sample sizes N = 10, 50, 250 and 1000. The other sample sizes are also considered, but in much less detail. Each simulation is based on generating 1000 sequences of N system lifetimes. We used these system lifetimes (X_i) one at a time for the sequential estimation procedure. At the first observation $Z_1 = X_1$, and $\delta_1 = 1$. As mentioned before, the scale parameter θ values are estimated by the EM algorithm. After finding $\hat{\theta}$, the estimated age replacement time values are calculated by using the equation (4.15) in the simulation. Similar to the Weibull case, by using N generated system lifetimes, we determine N estimated age replacement times and N estimated costs in each simulation. We repeat this simulation 1000 times (NREP = 1000) and then we find the average value for both age replacement times and replacement costs. See the equations (3.15) and (3.16) of Chapter III. At each simulation we also calculated the MSE values for both the age replacement times and the long run expected average costs by using the equations (3.17) and (3.18).

Tables 12, 13, 14 and 15 summarize the simulation results of the optimal age replacement times and the minimum long run expected replacement costs per unit time for different values of p with $N = 1000$ for fixed costs $C_1 = 2.0$, $C_1 = 5.0$, $C_1 = 8.0$,

$C_1 = 10.0$ while holding the C_2 fixed at 1.0. Tables 16, 17 and 18 summarize the simulation results using the three sample sizes of 10, 50 and 250 with different values of the shape parameter p for fixed costs $C_1 = 2.0$ and $C_2 = 1.0$. Included in Tables 12~18, is the probability that a system will fail before the time φ under the optimal age replacement policy (3.19).

We have also plotted the results of the average age replacement times and the average costs for different shape parameter p. See Figure 9, for plot of the average age replacement times for $p = 3.0$, 4.0, 5.0 ($p = 2.0$ was not selected because its average age replacement times were high acording to the others) and Figure 10, for the average costs for $p = 2.0$, 3.0, 4.0, 5.0 when the fixed costs $C_1 = 2.0$ and $C_2 = 1.0$.

From these plots and the results from Tables 12~18, we observe that when p decreases from 5.0 to 2.0 (i.e., the underlying life distribution is becomes more exponential) the long run expected optimal replacement costs increase. When the ratio of the unplanned replacement cost C_1 to the planned replacement cost C_2 increases the optimal replacement time for a specific p decreases. The larger values of $\varphi^{\star}$, insure that a small percentage of replacement will be made before failure, which is what we desire if the system's life distribution is close to exponential.

As in the Weibull case, by looking the tables for different sample sizes, we observe that the average cost per unit time up to the Nth replacement decreases with N, the number of replacement. Thi result is promising because the ultimate goal of the sequential estimation procedure is to decrease costs while sampling. Even though as $N \to \infty$, the long run average cost per unit time will approach the optimal replacement cost $C(\varphi^{\star})$ with probability 1.0 [Ref. 11], there is no guarantee that the average replacement cost will decrease for the first observations. For large sample sizes, our estimated average $\varphi^{\star}$ and the average $C(\varphi^{\star})$ values are too close to their actual values.

Table 12. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE GAMMA MODEL WITH $C_1 = 2.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 1000)

Shape parameter p	p = 2.0	p = 3.0	p = 4.0	p = 5.0
Scale parameter θ	1.0	0.666667	0.5	0.4
$x^\star = \theta \times \varphi^\star$	35.35100	2.11607	1.10275	0.75876
Optimal replacement time $\varphi^\star$	35.35100	3.17410	2.20550	1.89690
Average $\varphi^\star$	35.37980	3.17659	2.20480	1.89694
MSE of $\varphi^\star$	0.62197	0.00371	0.00168	0.00106
Long run expected optimal replacement cost $C(\varphi^\star)$	1.00000	0.99456	0.97167	0 93374
Average $C(\varphi^\star)$	1.00081	0.99552	0.97201	0.94614
MSE of $C(\varphi^\star)$	0.00049	0.00036	0.00032	0.00041
$P(X_i \leq \varphi^\star)$	1.00000	0.61467	0.18173	0.03994

Table 13. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE GAMMA MODEL WITH $C_1 = 5.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 1000)

Shape parameter p	p = 2.0	p = 3.0	p = 4.0	p = 5.0
Scale parameter θ	1.0	0.666667	0.5	0.4
$x^\star = \theta \times \varphi^\star$	1.30500	0.67220	0.48920	0.39952
Optimal replacement time $\varphi^\star$	1.30500	1.00830	0.97840	0.99880
Average $\varphi^\star$	1.30421	1.00727	0.97494	0.99402
MSE of $\varphi^\star$	0.00177	0.00107	0.00106	0.00202
Long run expected optimal replacement cost $C(\varphi^\star)$	2.26480	1.87695	1.63237	1.44237
Average $C(\varphi^\star)$	2.26830	1.88581	1.64236	1.48797
MSE of $C(\varphi^\star)$	0.00528	0.00367	0.00307	0.01029
$P(X_i \leq \varphi^\star)$	0.37495	0.19428	0.13517	0.09291

Table 14. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE GAMMA MODEL WITH $C_1 = 8.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 1000)

Shape parameter p	p = 2.0	p = 3.0	p = 4.0	p = 5.0
Scale parameter θ	1.0	0.666667	0.5	0.4
$x^\star = \theta \times \varphi^\star$	0.81890	0.49540	0.38620	0.32956
Optimal replacement time $\varphi^\star$	0.81890	0.74310	0.77240	0.82390
Average $\varphi^\star$	0.81677	0.73945	0.76560	0.81510
MSE of $\varphi^\star$	0.00124	0.00112	0.00172	0.00334
Long run expected optimal replacement cost $C(\varphi^\star)$	3.15166	2.28405	1.97650	1.68813
Average $C(\varphi^\star)$	3.16146	2.10281	2.00406	1.77106
MSE of $C(\varphi^\star)$	0.01767	0.01076	0.01375	0.03062
$P(X_i \leq \varphi^\star)$	0.19803	0.03960	0.00806	0.00120

Table 15. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE GAMMA MODEL WITH $C_1 = 10.0, C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 1000)

Shape parameter p	p = 2.0	p = 3.0	p = 4.0	p = 5.0
Scale parameter θ	1.0	0.666667	0.5	0.4
$x^{\star} = \theta \times \varphi^{\star}$	0.68010	0.43693	0.35020	0.30424
Optimal replacement time $\varphi^{\star}$	0.68010	0.65540	0.70040	0.76060
Average $\varphi^{\star}$	0.67754	0.65103	0.68916	0.74911
MSE of $\varphi^{\star}$	0.00113	0.00111	0.00263	0.00359
Long run expected optimal replacement cost $C(\varphi^{\star})$	3.64327	2.64540	2.14725	1.80642
Average $C(\varphi^{\star})$	3.66032	2.67401	2.19022	1.90982
MSE of $C(\varphi^{\star})$	0.02993	0.01724	0.02745	0.04275
$P(X_i \leq \varphi^{\star})$	0.14892	0.02894	0.00576	0.00082

Table 16. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE GAMMA MODEL WITH $C_1 = 2.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 10)

Shape parameter p	p = 2.0	p = 3.0	p = 4.0	p = 5.0
Scale parameter θ	1.0	0.666667	0.5	0.4
$x^\star = \theta \times \varphi^\star$	35.35100	2.11607	1.10275	0.75876
Optimal replacement time $\varphi^\star$	35.35100	3.17410	2.20550	1.89690
Average $\varphi^\star$	37.57612	3.23432	2.18015	1.82749
MSE of $\varphi^\star$	93.99036	0.51523	0.24643	0.16467
Long run expected optimal replacement cost $C(\varphi^\star)$	1.00000	0.99456	0.97167	0.93374
Average $C(\varphi^\star)$	1.06294	1.02846	1.00494	0.97408
MSE of $C(\varphi^\star)$	0.07521	0.04433	0.03426	0.02609
$P(X_i \leq \varphi^\star)$	1.00000	0.61467	0.18173	0.03994

Table 17. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE GAMMA MODEL WITH $C_1 = 2.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 50)

Shape parameter p	p = 2.0	p = 3.0	p = 4.0	p = 5.0
Scale parameter θ	1.0	0.666667	0.5	0.4
$x^\star = \theta \times \varphi^\star$	35.35100	2.11607	1.10275	0.75876
Optimal replacement time $\varphi^\star$	35.35100	3.17410	2.20550	1.89690
Average $\varphi^\star$	35.84104	3.19201	2.20190	1.87172
MSE of $\varphi^\star$	13.43787	0.07847	0.03622	0.03245
Long run expected optimal replacement cost $C(\varphi^\star)$	1.00000	0.99456	0.97167	0.93374
Average $C(\varphi^\star)$	1.01386	1.00393	0.98294	0.95588
MSE of $C(\varphi^\star)$	0.01075	0.00731	0.00583	0.00545
$P(X_i \leq \varphi^\star)$	1.00000	0.61467	0.18173	0.03994

Table 18. ESTIMATED OPTIMAL REPLACEMENT TIMES OF THE GAMMA MODEL WITH $C_1 = 2.0$, $C_2 = 1.0$, AND $E(X_i) = 2.0$ (N = 250)

Shape parameter p	p = 2.0	p = 3.0	p = 4.0	p = 5.0
Scale parameter θ	1.0	0.666667	0.5	0.4
$x^* = \theta \times \varphi^*$	35.35100	2.11607	1.10275	0.75876
Optimal replacement time φ^*	35.35100	3.17410	2.20550	1.89690
Average φ^*	35.47148	3.18530	2.20761	1.89861
MSE of φ^*	2.53150	0.01452	0.00649	0.00436
Long run expected optimal replacement cost $C(\varphi^*)$	1.00000	0.99456	0.97167	0.93374
Average $C(\varphi^*)$	1.00340	0.99883	0.97499	0.94996
MSE of $C(\varphi^*)$	0.00202	0.00141	0.00124	0.00127
$P(X_i \leq \varphi^*)$	1.00000	0.61467	0.18173	0.03994

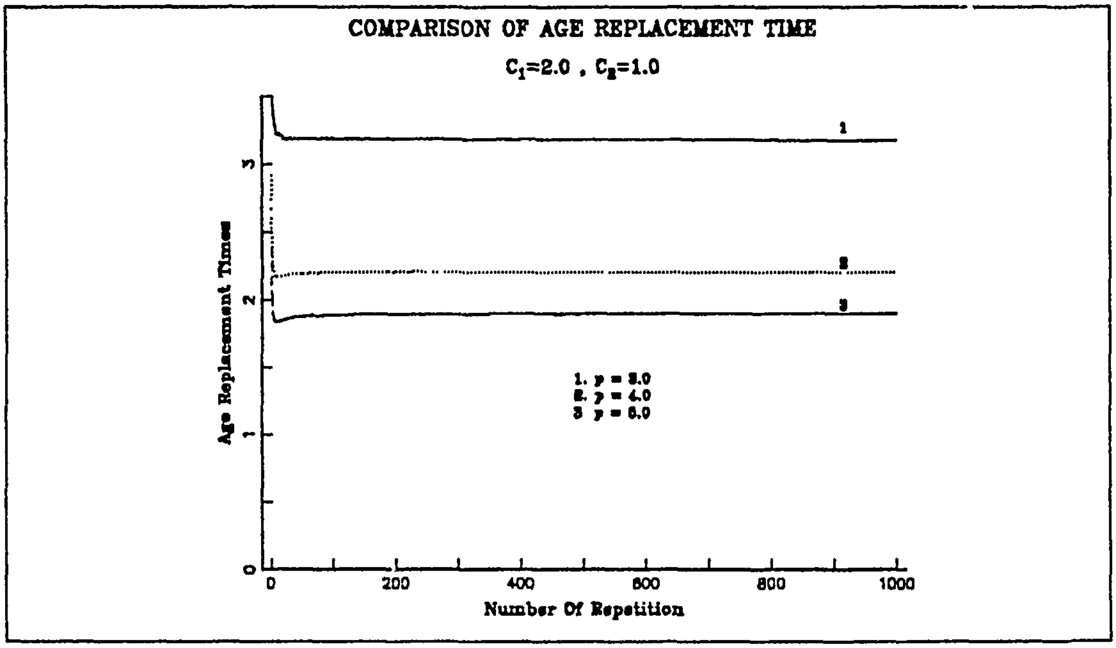

Figure 9. The Average Age Replacement Times For The Gamma Model

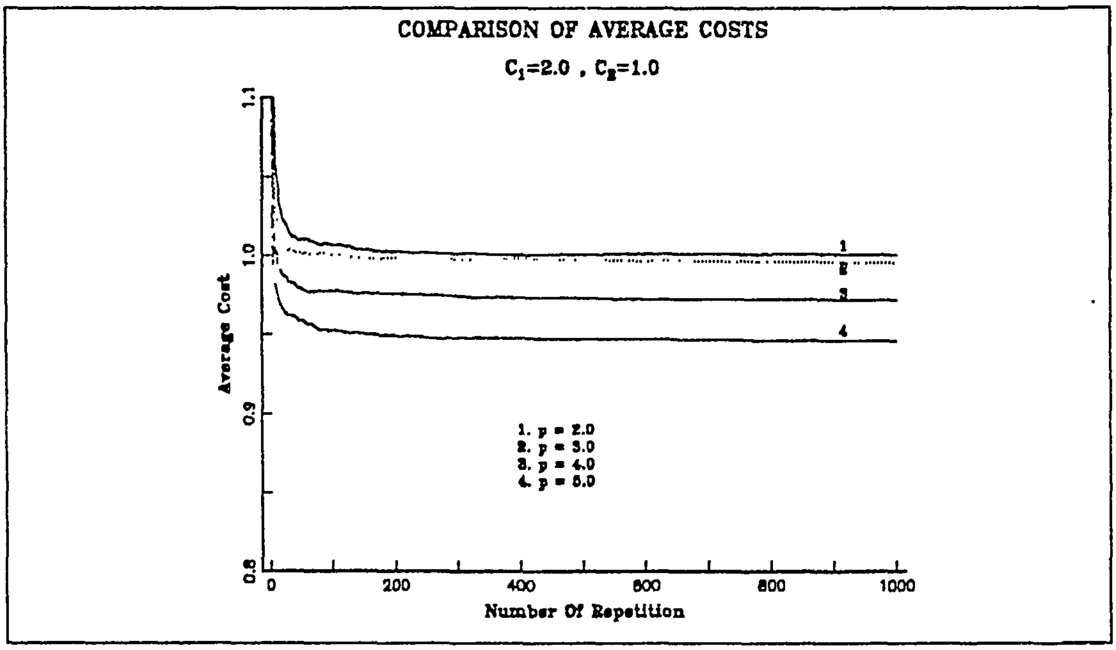

Figure 10. The Average Costs For The Gamma Model

V. CONCLUSIONS AND RECOMMENDATIONS

Throughout this thesis, we have considered age replacement policies. In such policies a unit is always replaced at the time of failure or φ units of time after its installation, whichever comes first. The time φ is called the scheduled replacement time or the scheduled censoring time. The optimal replacement time φ^* achieves the minimum long run expected maintenance cost per unit time. The results of the simulations show that the parametric estimators work well under the conditions for which they were intended and by using the sequential estimation procedure significant cost and time savings can be effected.

In most cases, the estimated optimal age replacement times and the actual costs got close to the true optimal age replacement time and the minimum expected cost per unit time respectively, even with moderate sample sizes. Therefore, we conclude that our sequential estimation procedure of the age replacement policies to minimize the long run expected cost can be applied to the real problem.

An important part of our analysis, which would be very difficult to prove theoretically, showed that the average actual cost per unit time decreased monotonically as the sample size increased. Such a decrease makes intuitive sense, i.e. an age replacement policy using an estimated φ^* should do better as more data is collected. This result is desirable for a sequential estimation procedure.

Directions for Future Research

It is our hope the ideas and the sequential estimation procedure described in this thesis represent a plateau for the development of the optimum age replacement policies to minimize the long run expected maintenance costs. The minimization problem is in principle difficult and future developments will exploit fundamentally new ideas. Here, we briefly describe directions in which future research might be pursued.

- We have described the scenario in which the underlying lifetimes are *i.i.d.*, along with a straight forward age replacement policy. In many situations this model is not adequate. For example, if an item is repaired at failure rather than replaced, the *i.i.d.* assumption is equivalent to requiring that the repaired item function as well as a new one. Clearly, modeling times between failure of a repairable system as *i.i.d.* is inappropriate. More realistic models incorporate the possibility that repairs are less than perfect. It is also possible that the quality of planned maintenance varies from time to time. If an imperfect repair model is permitted how should the sequential estimation procedure be changed to fit new situation.

- Sequential estimation when the underlying life distribution F comes from the normal or the modified extreme value distributions which have increasing failure rate.

- The minimization of the long run expected costs is not appropriate under all circumstances because in this model the planned and unplanned costs are fixed. Other cost functions need to be considered. For example, costs can be modeled to change with time.

- How to change our estimation procedure if we have different systems such that serial, parallel or bridge systems with same or different lifetimes rather than one unit and one lifetime.

APPENDIX A. APL CODE TO CALCULATE LAMBDA OR MEAN FOR THE WEIBULL DISTRIBUTION

```
    ∇ WEIBULL;A;E;L;X
  ⍝ THIS APL PROGRAM CALCULATES THE SCALE PARAMETER
  ⍝ LAMBDA AND THE E[X] FOR THE WEIBULL DISTRIBUTION.
  ⍝ THE SHAPE PARAMETER α AND THE OTHER UNKNOWN
  ⍝ MUST BE GIVEN BY THE USER.
   LUCK←0
  START:'PLEASE TYPE 1 IF YOU WISH TO FIND LAMBDA'
   'PLEASE TYPE 2 IF YOU WISH TO FIND E[X]'
   TYPE1←⍞
   →((TYPE1≠'1')∧(TYPE1≠'2'))/WARNING1
   →(1↑TYPE1='2')/EXPECVA
  LAMBDA:'PLEASE ENTER THE α VALUE'
   A←⎕
   'PLEASE ENTER THE EXPECTED VALUE E[X]'
   E←⎕
  ⍝ LAMBDA = GAMMA(1/α) / (α × E[X])
   X←1÷A
   (!(X-1))÷(A×E)
   →END
  EXPECVA:'PLEASE ENTER THE α VALUE'
   A←⎕
   'PLEASE ENTER THE LAMBDA VALUE'
   L←⎕
  ⍝ E[X] = GAMMA(1/α) / (α × LAMBDA)
   X←1÷A
   (!(X-1))÷(A×L)
   →END
  WARNING1:
```

```
[28]  LUCK←LUCK+1
[29]  →(LUCK≥2)/CHEW1
[30]  'YOU CAN ONLY ENTER THE NUMBERS 1 OR 2'
[31]  →START
[32] CHEW1:'JUST TYPE 1 OR 2'
[33]  →START
[34] WARNING2:
[35]  LUCK←LUCK+1
[36]  →(LUCK≥2)/CHEW2
[37]  'YOU CAN ONLY ENTER THE LETTERS Y OR N'
[38]  →END
[39] CHEW2:'JUST TYPE Y OR N'
[40]  →END
[41] END:'DO YOU WANT TO RUN THE PROGRAM AGAIN?(Y/N)'
[42]  TYPE2←⍞
[43]  →((1↑TYPE2)='Y')/START
[44]  →(1↑TYPE2≠'Y')∧(1↑TYPE2≠N)/WARNING2
[45]  'BYE BYE'
      ∇
```

APPENDIX B. APL CODE TO CALCULATE THE ACTUAL AGE REPLACEMENT TIME AND CORRESPONDING THE ACTUAL MINIMUM COST

```
    ∇ SIM;C1;C2;I;J;FX;X;T;XA;YA;C;D;XMIN;YMIN
  ⍝ THIS PROGRAM SIMULATES THE COST FUNCTION (EQ. 2.1) TO
  ⍝ FIND THE MINIMUM VALUE (YMIN) OF THE COST FUNCTION AND
  ⍝ CORRESPONDING AGE REPLACEMENT TIME (XMIN) FOR THAT
  ⍝ POINT.  AFTER FINDING MINIMUM VALUES INSIDE THE LOOP1
  ⍝ IT REPEATS THE PROCEDURE 300 TIMES INSIDE THE LOOP2.
  ⍝ FINALLY, THE PROGRAM GIVES THE AVERAGE VALUES FOR
  ⍝ BOTH MINIMUM POINT AS AXST AND ACST.
  ⍝
   T←(⍳5000)÷100
  ⍝ THIS GIVES US A VECTOR OF T(0.01, 0.02, ..., 50)
  ⍝ TO CALCULATE FIRST C(0.01) AND THEN C(0.02) UP TO
  ⍝ C(50) OF 5000 COST VECTOR.
  ⍝ INITIALIZATION...
  ⍝ UNPLANNED AND PLANNED REPLACEMENT COSTS MUST BE
  ⍝ GIVEN BY THE USER.
   C1←5
   C2←1
   XA←⍳0
   YA←⍳0
   J←0
  ⍝ J IS THE INCREMENT OF THE LOOP2 J=1, 2, ..., 300
  ⍝ MODEL...
  LOOP2:
   X←5000 WEIRAND 2 2.2567587
  ⍝ PREVIOUS LINE, GENERATES 5000 SYSTEM LIFETIMES FROM
  ⍝ WEI(ALPHA=2.0,BETA=2.2567587) AS VECTOR X.HERE BETA
```

```
[27] ⍝ VALUE REPRESENTS 1 OVER LAMBDA=1÷0.44311346
[28] ⍝ FOR GAMMA DISTRIBUTION LINE 24 CAN BE SWITCH WITH
[29] ⍝ X←5000 GAMRAND 4 0.5 GAM( P=4 ,THETA=0.5 ).
[30]  J←J+1
[31]  C←⍳0
[32]  I←0
[33] ⍝ I IS THE INCREMENT OF THE INNER LOOP I=1, 2, ..., 5000
[34] LOOP1:
[35]  I←I+1
[36] ⍝ C IS THE SIMULATED COST FUNCTION
[37] ⍝ D IS THE DENUMERATOR OF THE COST FUNCTION
[38]  D←((+/(X⌊T[I]))÷5000)
[39]  C←C,(((C2×(1-FX))+(C1×(FX←((+/X≤T[I])÷5000))))÷D)
[40] ⍝ IN THE FIRST LOOP C VECTORS OBTAIN FOR EACH T
[41]  →(I<5000)/LOOP1
[42]  YMIN←⌊/C
[43]  XMIN←T[1↑⍋C]
[44] ⍝ YMIN: THE MINIMUM VALUE OF THE COST FUNCTION
[45] ⍝       FOR SPECIFIC T
[46] ⍝ XMIN: THE CORRESPONDING AGE REPLACEMENTTIME (T)
[47]  XA←XA,XMIN
[48]  YA←YA,YMIN
[49] ⍝ XA: THE VECTOR OF THE AGE REPLACEMENT TIMES (300)
[50] ⍝ YA: THE VECTOR OF THE YMIN (300)
[51]  →(J<200)/LOOP2
[52]  AXST←(+/XA)÷⍴XA
[53]  ACST←(+/YA)÷⍴YA
[54] ⍝ AXST: THE AVERAGE VALUE OF THE AGE REPLACEMENT
[55] ⍝       TIMES AFTER 300 REPETITIONS.
[56] ⍝ ACST: THE AVERAGE VALUE OF THE YMIN
[57] ⍝       AFTER 300 REPETITIONS.
     ∇
```

APPENDIX C. PROGRAM AVERWEIB

```
C      PROGRAM AVERWEIB
C      ***************************************************************
C      *** PURPOSE : This program calculates the average costs and ***
C      *** corresponding average replacement times by using the    ***
C      *** sequential estimating procedure for the weibull distri- ***
C      *** bution. The program also calculates the mean square     ***
C      *** error values for the average costs and the average rep- ***
C      *** lacement times at the each run.                         ***
C      ***************************************************************
       PARAMETER (N=1000, NREP=1000)
       INTEGER I, J, K, L, M, SEED, DEL(N)
       REAL*8 X(N), LAMHAT(N), Z(N), FISTAR(N), XSTAR, SUMZ(N), NUM(N)
       REAL*8 SUM(N), SUM2(N), AVG(N), FSTAR, MSEAGE(N), C1, C2, COSTST
       REAL*8 DEN(N), ACOST(N), SUM3(N), SUM4(N), AVACOS(N), MSECOS(N)
C      *** The following real numbers are not double precision     ***
C      *** because the random number generator subroutine usee     ***
C      *** single precision only.                                  ***
       REAL U,LAM,AL
C      ***************************************************************
C      ***    KEY VARIABLES :                                      ***
C      ***    ISEED          : The random number seed              ***
C      ***    X              : Generated system lifetimes          ***
C      ***    LAM            : The scale parameter                 ***
C      ***    AL             : The shape parameter                 ***
C      ***    C1             : Unplanned replacement cost          ***
C      ***    C2             : Planned replacement cost            ***
C      ***    XSTAR          : A constant which minimizes the cost ***
C      ***                     function when the scale parameter   ***
C      ***                     equals 1                            ***
```

```
C    ***   FSTAR         : The actual age replacement time (from ***
C    ***                   the APL program Sim)                   ***
C    ***   COSTST        : The actual cost (from the APL program  ***
C    ***                   Sim)                                   ***
C    ***   U             : The return generated random number     ***
C    ***   N             : The number of the run                  ***
C    ***   NREP          : The number of the repetition           ***
C    ***   DEL           : Indicates the summation of the delta   ***
C    ***                   values to calculate the average costs  ***
C    ***   NUM           : The numerator of the cost function     ***
C    ***                   which given in the Equation (2.6)      ***
C    ***   DEN           : The denominator of the cost function   ***
C    ***                   which given in the Equation (2.7)      ***
C    ***   ACOST         : The average cost                       ***
C    ***   FISTAR        : The average age replacement time       ***
C    ***   Z             : The minimum value of the X(i) (genera-***
C    ***                   ted system lifetimes) or the age rep- ***
C    ***                   lacement time                          ***
C    ***   LAMHAT        : The estimated scale parameter          ***
C    ***   AVG           : The average value of the age replace- ***
C    ***                   ment times after NREP repetition       ***
C    ***   MSEAGE        : Mean Square Error for the average of   ***
C    ***                   the age replacement times              ***
C    ***   AVACOS        : The average value of the costs after   ***
C    ***                   NREP repetition                        ***
C    ***   MSECOS        : Mean Square Error for the average of   ***
C    ***                   the costs                              ***
C    ****************************************************************
C    ***   INITIALIZATION                                         ***
C    ****************************************************************
     SEED=16807
     LAM=0.4557111695
     AL=1.4D0
     C1=10.0D0
```

```
      C2=1.0D0
      FSTAR=0.91690D0
C     *** From the main lemma xstar=fstar/lambda
      XSTAR=FSTAR/LAM
      COSTST=4.048919D0
      DO 2 I=1,N
         SUM(I)=0.0D0
         AVG(I)=0.0D0
         MSE(I)=0.0D0
         SUM2(I)=0.0D0
         SUM3(I)=0.0D0
         SUM4(I)=0.0D0
  2   CONTINUE
C     ************************************************************
C     ***     CALCULATIONS                                    ***
C     ************************************************************
      DO 3 L=1,NREP
C
         CALL EXCMS ('FILEDEF 11 DISK AL14 OUTPUT A')
         CALL RANNUM (1,SEED,0.0,1.0,0,U)
C
         X(1)=((-1*LOG(1-U))/(LAM**AL))**(1/AL)
         Z(1)=X(1)
         LAMHAT(1)=1/X(1)
         FISTAR(1)=LAMHAT(1)*XSTAR
         SUMZ(1)=X(1)**AL
C     *** First delta value is 1 (Failure occur before the time     ***
C     *** and at the first observation there is no comparison for   ***
C     *** the minimums.  Z(1) = X (1)
         DEL(1)=1
C     *** Calculation of the average cost for the first observation***
         NUM(1)=C1
         DEN(1)=Z(1)
         ACOST(1)=NUM(1)/DEN(1)
```

```
      DO 5 J=2,N
C
        CALL RANNUM (1,SEED,0.0,1.0,0,U)
C
        X(J)=((-1*LOG(1-U))/(LAM**AL))**(1/AL)
C    *** Comparisons to find the Z and delta values                  ***
        IF(FISTAR(J-1) .LT. X(J)) THEN
          SUMZ(J)=SUMZ(J-1)+(FISTAR(J-1)**AL)
          DEL(J)=DEL(J-1)
          Z(J)=FISTAR(J-1)
        ELSE
          SUMZ(J)=SUMZ(J-1)+(X(J)**AL)
          DEL(J)=DEL(J-1)+1
          Z(J)=X(J)
        END IF
        LAMHAT(J)=(DEL(J)/SUMZ(J))**(1/AL)
        FISTAR(J)=LAMHAT(J)*XSTAR
        NUM(J)=(C1*DEL(J))+(C2*(J-DEL(J)))
        DEN(J)=DEN(J-1)+Z(J)
        ACOST(J)=NUM(J)/DEN(J)
 5    CONTINUE
      DO 10 K=1,N
        SUM(K)=SUM(K)+FISTAR(K)
        SUM2(K)=SUM2(K)+(FISTAR(K)-FSTAR)**2
        SUM3(K)=SUM3(K)+ACOST(K)
        SUM4(K)=SUM4(K)+(ACOST(K)-COSTST)**2
 10   CONTINUE
  3  CONTINUE
     DO 20 I=1,N
       AVG(I)=SUM(I)/NREP
       MSEAGE(I)=SUM2(I)/NREP
       AVACOS(I)=SUM3(I)/NREP
       MSECOS(I)=SUM4(I)/NREP
       WRITE(11,*) AVG(I),AVACOS(I),MSEAGE(I),MSECOS(I)
```

```
 20   CONTINUE
      STOP
      END
C*************************************************************************
      SUBROUTINE RANNUM(DISTN, SEED, RPARM1, RPARM2, IPARM, X)
C*************************************************************************
C     *** THIS SUBROUTINE PROVIDES AN INTERFACE WITH THE LLRANDOMII***
C     *** ROUTINES PROVIDED IN THE NONIMSL LIBRARY.  THE PARAMETER ***
C     *** REQUIREMENTS AND CALLING PROCEDURES ARE AS FOLLOWS:      ***
C     ***                                                          ***
C     *** DISTN = DISTRIBUTION TYPE YOU WANT TO SELECT             ***
C     ***         AN INTEGER BETWEEN 1 AND 7                       ***
C     *** SEED = THE RANDOM NUMBER SEED YOU WISH TO USE            ***
C     *** RPARM1, RPARM2, AND IPARM ARE REAL AND INTEGER PARAMETERS***
C     *** PASSED TO THE ROUTINE WITH MEANINGS WHICH VARY WITH THE  ***
C     *** TYPE OF DISTRIBUTION YOU DESIRE                          ***
C     *** X = THE RETURNED RANDONM NUMBER, IT IS ALWAYS REAL       ***
C                                                                  ***
C     *** DISTRIBUTION NUMBERS AND THE ASSOCIATED PARM DEFINITIONS ***
C                                                                  ***
C     *** 1--UNIFORM ON THE INTERVAL RPARM1 TO RPARM2              ***
C     *** 2--NORMAL WITH MEAN RPARM1 AND VARIANCE RPARM2           ***
C     *** 3--EXPONENTIAL WITH RATE RPARM1                          ***
C     *** 4--COUCHY WITH A = RPARM1 AND B = RPARM2                 ***
C     *** 5--GAMMA WITH SHAPE RPARM2 AND RATE RPARM1               ***
C     *** 6--POISSON WITH RATE RPARM1                              ***
C     *** 7--GEOMETRIC WITH P = RPARM1                             ***
C     ***                                                          ***
      REAL RPARM1, RPARM2, X
      INTEGER DISTN, SEED, IPARM, N
C     ***                                                          ***
      REAL TEMP, VARIAT(1)
      IF (DISTN.LE.0.OR.DISTN.GT.8) THEN
         WRITE(10, *) 'ILLEGAL CALL TO RANNUM, BAD DISTN'
```

```
      STOP
   ENDIF
C  ***                                                       ***
   GOTO (10, 20, 30, 40, 50, 60, 70), DISTN
C  ***                                                       ***
C  *** GENERATE A UNIFORM BETWEEN RPARM1 AND RPARM2          ***
10 CONTINUE
C  ***                                                       ***
   IF (RPARM1 - RPARM2.EQ.0) THEN
      WRITE(10, *) 'ILLEGAL EQUAL RPARMS IN RANNUM'
      STOP
   ENDIF
   IF (RPARM1.GT.RPARM2) THEN
      TEMP = RPARM1
      RPARM1 = RPARM2
      RPARM2 = TEMP
   ENDIF
   CALL LRND(SEED, VARIAT, 1, 1, 0)
   VARIAT(1) = RPARM1 + (RPARM2 - RPARM1) * VARIAT(1)
   GOTO 99
C  ***                                                       ***
C  GENERATE A NORMAL WITH MEAN RPARM1 AND STDDEV RPARM2      ***
20 CALL LNORM(SEED, VARIAT, 1, 1, 0)
   WRITE(*, *) 'NORMAL (0, 1) ', VARIAT(1)
   VARIAT(1) = (VARIAT(1) * RPARM2) + RPARM1
   GOTO 99
C  ***                                                       ***
C  *** GENERATE AN EXPONENTIAL WITH RATE (1/MEAN) RPARM1     ***
30 CONTINUE
   IF (RPARM1.EQ.0) THEN
      WRITE(10, *) 'ILLEGAL ZERO RATE IN RANNUM'
      STOP
   ENDIF
   CALL LEXPN(SEED, VARIAT, 1, 1, 0)
```

```
      VARIAT(1) = VARIAT(1) / RPARM1
      GOTO 99
C     ***                                                           ***
C     ***GENERATE A COUCHY WITH A = RPARM1 AND B = RPARM2           ***
40    CONTINUE
      IF (RPARM2.LE.0) THEN
         WRITE(10, *) 'ILLEGAL COUCHY SPREAD IN RANNUM, B = ',RPARM2
         STOP
      ENDIF
      CALL LCCHY(SEED, VARIAT, 1, 1, 0)
      VARIAT(1) = (VARIAT(1) * RPARM2) + RPARM1
      GOTO 99
50    CONTINUE
      IF (RPARM1.LE.0) THEN
         WRITE(10, *) 'ILLEGAL NONPOSITIVE GAMMA RATE IN RANNUM'
         STOP
      ENDIF
      IF (RPARM2.LE.0) THEN
         WRITE(10, *) 'ILLEGAL SHAPE PARAMETER IN RANNUM'
         STOP
      ENDIF
      CALL LGAMA(SEED, VARIAT, 1, 1, 0, RPARM2)
      VARIAT(1) = VARIAT(1) * (1.0 / RPARM1)
      GOTO 99
60    CONTINUE
      IF (RPARM1.LE.0) THEN
         WRITE(10, *) 'ILLEGAL POISSON RATE IN RANNUM'
         STOP
      ENDIF
      CALL LPOIS(SEED, VARIAT, 1, 1, 0, RPARM1)
      GOTO 99
70    CONTINUE
      IF (RPARM1.LE.0) THEN
         WRITE(10, *) 'ILLEGAL GEOM PROB IN RANNUM'
```

```
         STOP
      ENDIF
      CALL LGEOM(SEED, VARIAT, 1, 1, 0, RPARM1)
      GOTO 99
C     ***                                                        ***
99    CONTINUE
      X = VARIAT(1)
      END
```

APPENDIX D. PROGRAM AVERGAM

```
C     PROGRAM AVERGAM
C     ******************************************************************
C     *** PURPOSE : This program calculates the average costs and ***
C     *** corresponding average replacement times by using the    ***
C     *** sequential estimating procedure for the gamma distribu- ***
C     *** tion.  The program also calculates the mean square error***
C     *** values for the average costs and the average replacement***
C     *** times at the each run.                                  ***
C     ******************************************************************
      PARAMETER (N=1000, NREP=1000)
      INTEGER I, J, K, L, M, ISEED, DEL(N), DELTA(N)
      REAL*8 X(N), TEHAT(N), Z(N), FISTAR(N), XSTAR, COSTST
      REAL*8 SUM(N), SUM2(N), AVG(N), FSTAR, MSEAGE(N), C1, C2
      REAL*8 ACOST(N), SUM3(N), SUM4(N), AVACOS(N), MSECOS(N)
      REAL*8 NUMCO(N), DENCO(N), T1, T, NUM, A, B, C, D
C     *** The following Real numbers are not Double precision     ***
C     *** the number generator subroutine uses single precision   ***
      REAL U,TETA,P,RATE
C     ***     KEY VARIABLES :                                     ***
C     ***     ISEED       : The random number seed                ***
C     ***     X           : Generated system lifetimes            ***
C     ***     TETA        : The scale parameter                   ***
C     ***     P           : The shape parameter                   ***
C     ***     RATE        : Reciprocal of the scale parameter to  ***
C     ***                   use the subroutine Rannum             ***
C     ***     C1          : Unplanned replacement cost            ***
C     ***     C2          : Planned replacement cost              ***
C     ***     XSTAR       : A constant which minimizes the cost   ***
C     ***                   function when the scale parameter     ***
C     ***                   equals 1                              ***
```

```
C     ***   FSTAR         : The actual age replacement time (from ***
C     ***                   the APL program Sim)                  ***
C     ***   COSTST        : The actual cost (from the APL program ***
C     ***                   Sim)                                  ***
C     ***   U             : The return generated random number    ***
C     ***   N             : The number of the run                 ***
C     ***   NREP          : The number of the repetition          ***
C     ***   DELTA         : Indicates 0 or 1. If failure occurs   ***
C     ***                   before the time age, then equals 0.   ***
C     ***                   Otherwise equals 1                    ***
C     ***   DEL           : Indicates the summation of the delta  ***
C     ***                   values to calculate the average costs ***
C     ***   NUMCO         : The numerator of the cost function    ***
C     ***                   which given in the Equation (2.6)     ***
C     ***   DENCO         : The denominator of the cost function  ***
C     ***                   which given in the Equation (2.7)     ***
C     ***   ACOST         : The average cost                      ***
C     ***   FISTAR        : The average age replacement time      ***
C     ***   Z             : The minimum value of the X(i) (genera-***
C     ***                   ted system lifetimes) or the age rep- ***
C     ***                   lacement time                         ***
C     ***   TEHAT         : The estimated scale parameter         ***
C     ***   T1            : The converged scale parameter value   ***
C     ***                   at the E step of the EM algorithm     ***
C     ***   T             : The value of the scale parameter at   ***
C     ***                   the previous calculation              ***
C     ***   NUM           : The numerator of the Equation 4.14    ***
C     ***   A, B, C, D    : To determine the conditional expecta- ***
C     ***                   tion value from Table 8               ***
C     ***   AVG           : The average value of the age replace- ***
C     ***                   ment times after NREP repetition      ***
C     ***   MSEAGE        : Mean Square Error for the average of  ***
C     ***                   the age replacement times             ***
C     ***   AVACOS        : The average value of the costs after  ***
```

```
C     ***                     NREP repetition                        ***
C     ***   MSECOS          : Mean Square Error for the average of   ***
C     ***                     the costs                              ***
C     ****************************************************************
C     ***   INITIALIZATION                                           ***
C     ****************************************************************
      ISEED=16807
      TETA=2.0/5.0
      RATE=1.0/TETA
      P=5.0D0
      C1=2.0D0
      C2=1.0D0
      FSTAR=1.89690D0
C     *** From the main lemma xstar=fstar*theta                      ***
      XSTAR=FSTAR*TETA
      COSTST=0.933742D0
      DO 2 I=1,N
         SUM(I)=0.0D0
         AVG(I)=0.0D0
         MSE(I)=0.0D0
         SUM2(I)=0.0D0
         SUM3(I)=0.0D0
         SUM4(I)=0.0D0
  2   CONTINUE
C     ****************************************************************
C     ***   CALCULATIONS                                             ***
C     ****************************************************************
      DO 3 M=1,NREP
C
         CALL EXCMS ('FILEDEF 15 DISK P5C2 OUTPUT A')
         CALL RANNUM (5,ISEED,RATE,P,0,U)
C
         X(1)=DBLE(U)
```

```
      Z(1)=X(1)
C     *** First theta value from the Maximum Likelihood Estimation ***
      TEHAT(1)=Z(1)/P
C     *** Use the main lemma to find the estimated age replac. time***
      FISTAR(1)=XSTAR/TEHAT(1)
C     *** First DELTA value is 1 (Failure occur before time age)   ***
C     *** and at the first observation there is no comparison       ***
C     *** Z(1)=X(1)                                                 ***
      DEL(1)=1
      DELTA(1)=1
C     *** Calculation of the average cost for the first observation***
      NUMCO(1)=C1
      DENCO(1)=Z(1)
      ACOST(1)=NUMCO(1)/DENCO(1)
      DO 5 J=2,N
C
         CALL RANNUM (5,ISEED,RATE,P,0,U)
C
         X(J)=DBLE(U)
C     *** Comparison to find the Z and DELTA values                 ***
         IF(FISTAR(J-1) .LT. X(J)) THEN
            Z(J)=FISTAR(J-1)
            DEL(J)=DEL(J-1)
            DELTA(J)=0
         ELSE
            Z(J)=X(J)
            DEL(J)=DEL(J-1)+1
            DELTA(J)=1
         END IF
         T=TEHAT(J-1)
90       CONTINUE
         NUM=Z(1)
         DO 100 L=2,J
```

```
              IF(DELTA(L) .EQ. 1) THEN
C     *** Observation is uncensored
                 NUM=NUM+Z(L)
              ELSE
C     *** Observation is censored
C     *** Calculate the conditional Expectation E(X|X>Z) values    ***
C     *** by using Table 8
                 A=(Z(L)*Z(L))+(2*Z(L)*T)+(2*T*T)
                 B=(Z(L)*Z(L)*Z(L))+(3*T*A)
                 C=(Z(L)*Z(L)*Z(L)*Z(L))+(4*T*B)
                 D=(Z(L)*Z(L)*Z(L)*Z(L))+(5*T*C)
                 NUM=NUM+(D/C)
              END IF
100         CONTINUE
            T1 = NUM / (J*P)
            IF (ABS(T1-T) .LT. 0.001) GO TO 200
C     *** This if statement is to check stopping criteria          ***
C     *** If this satisfied we can accept T1 converged to the theta***
            T=T1
            GO TO 90
200         CONTINUE
            TEHAT(J)=T1
            FISTAR(J)=XSTAR/TEHAT(J)
            NUMCO(J)=(C1*DEL(J))+(C2*(J-DEL(J)))
            DENCO(J)=DENCO(J-1)+Z(J)
            ACOST(J)=NUMCO(J)/DENCO(J)
 5        CONTINUE
          DO 10 K=1,N
            SUM(K)=SUM(K)+FISTAR(K)
            SUM2(K)=SUM2(K)+(FISTAR(K)-FSTAR)**2
            SUM3(K)=SUM3(K)+ACOST(K)
            SUM4(K)=SUM4(K)+(ACOST(K)-COSTST)**2
 10       CONTINUE
  3     CONTINUE
```

```
      DO 30 I=1,N
         AVG(I)=SUM(I)/NREP
         MSEAGE(I)=SUM2(I)/NREP
         AVACOS(I)=SUM3(I)/NREP
         MSECOS(I)=SUM4(I)/NREP
         WRITE(15,300) AVG(I),AVACOS(I),MSEAGE(I),MSECOS(I)
300      FORMAT (F10.7,'      ',F10.7,'      ',F10.7,'      ',F10.7)
 30   CONTINUE
      STOP
      END
```

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