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SPACETIME GEODESY

ARKADY KHEYFETS



July 1991

Final Report

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Air Force Systems Command
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I. Introduction

Proper operating of the satellite-based Global Positioning System (GPS) imposes increasingly demanding requirements on global clock synchronization. Such synchronization account for both special relativistic effects and the effects caused by the earth's gravitational field. A consistent treatment of both kinds of effects can be performed only within the framework of general relativity.

My original involvement in the research on relativistic effects in GPS, in the summer 1988, was related to some apparent confusion concerning the formulation of the global clock synchronization problem. Using the 4-dimensional geometric approach in a systematic fashion I pinpointed the source of the confusion and resolved the question once for all.

The 4-Geodesy Section (and, later the Spacetime Physics Group) of the Advanced Concept Branch of the USAF Weapons Laboratory at Kirtland Air Force Base recognized that general relativistic approach and the use of the 4-geometry language was a necessity in formulating and solving the GPS problems.

Furthermore, the idea of the satellite based Spacetime Common Grid was suggested as a means to formulate the GPS problems in a proper general relativistic fashion. The need in a 4-geometric mathematical formulation of the Spacetime Common Grid idea was recognized.

My research interests have been in the area of applications of modern mathematical methods in classical and quantum gravity. I have a very strong background in general relativity, both in foundational aspects and in applications. I owe it to the University of Texas at Austin (where I received my Ph. D. degree), and, in particular, to Prof. John A. Wheeler (with whom I worked for more than four years). My particular strength is the ability to see clearly the geometric content of general relativistic problems. This geometric insight played a key role in performing this research.

II. Objectives of the Research Effort

The requirements on global clock synchronization are becoming increasingly demanding in GPS operations. Furthermore, when more satellites are added to the GPS constellation to form a spacetime common grid (especially with cross-link ranging between satellites), it is believed that the precision requirements will become crucial for the coherent functioning of the system as a whole.

The GPS constellation will someday contain 24 clocks (with 3 active spares) moving with respect to each other. The GPS satellites will have almost circular orbits of 4 earth radii, with 12-hour periods, which means that the satellite velocity will be ~ 8 times the velocity of the surface station observer originating from earth rotation. Therefore, the required precision of the clocks synchronization necessitates taking into account special relativistic effects on the rate of the clocks. In addition, all of the activity of the GPS constellation occurs in the earth's gravitational field with clocks placed in positions with different gravitational potentials. The gravitational influence of the field on a clock's rate is determined by the parameter $\frac{M_0}{r}$ which produces an effect of the same order of magnitude as the second order special relativistic effects. A consistent treatment of both effects together can be done only within the framework of general relativity.

The need for a general relativistic treatment of the GPS clock synchronization problem was recognized prior to the summer 1988¹ and stressed again during that summer². Investigation of the general relativistic effects on clock rates was performed on several occasions¹⁶. The results have been implemented partially into the ranging procedure. However, by the summer 1988 the matter became a subject of controversy^{3,4,5,6} and confusion. Although the results of the most of researchers indicated that the treatment of the clock rates implemented in the GPS was correct, the source of the confusion was not clearly pinpointed.

Working within the 1988 Air Force Summer Faculty Research Program (May

- July, 1988) I used the covariant technique of the tensor series expansion (1) to find the correct general relativistic expression for the Doppler shift in the earth's gravity field up to the second order and to give the results physical interpretation, and (2) to formulate and solve the problem of global clock synchronization in the earth's gravitational field.

In my work, I utilized maximally the 4-geometric language in formulating the problems and pictorial demonstrations of the problems peculiarities, having as a goal to avoid confusion in interpretation of the results in the future. This allowed me to pinpoint the source of the confusion and to resolve the problem once for all.

The analysis performed by me during the 1988 Air Force Summer Faculty Research Program (cf. my 1988 SFRP Final Report¹⁷) was done only for the simplest cases. For instance, I restricted my analysis to the case of circular orbits when comparing the rates of the clock of the ground observer and the satellite clock, and to the case the equatorial ground observer and equatorial plane of the satellite orbit when considering the initial clock synchronization (in mathematical language, when evaluating the constants of integration). The simplifications were necessary to stress the physics of the problems and to get rid of the details that did not have the relativistic origin. They allowed me to achieve a clear understanding of the key relativistic features involved in GPS time transfer.

The main results of the 1988 SFRP¹⁷ research are used extensively in sections III, and IV of this report. Section V analyses a misconception that lead to the controversy mentioned above.

It is worth mentioning here that there was at least one more occasion when a confusion occurred due to neglecting the basic geometric interpretation of the problem. It was associated to the Sagnac effect which was considered for a while in some GPS related discussions¹⁰ as a separate one from the ordinary general relativistic time dialation. If this point of view was accepted, a part of the second order relativistic correction would be counted twice over. The geometric analysis of

this phenomena can be found in section VI of this report.

It so happened that amidst the controversy of 1988 the question of evaluating the constants of integration in the time transfer relations was put forward by H. Fligel of the Aerospace Corporation. The question turned out to be not related in any way to the controversy. It concerned much deeper issues, such as the initial clock synchronization and evaluating the GPS time transfer and the ranging procedure errors. A discussion of this problem can be found in section IV and X of this report.

A part of my objectives for the research efforts within the URRP were dictated by the necessity to learn how the basic relativistic effects described previously in the simplest situations look in more realistic setting:

- (1) Although the orbits of the GPS satellites ideally should be circular, it is clear that they cannot be perfectly circular. Consequently, the question is how the expressions comparing the rates of clocks must be modified if the orbits are slightly noncircular (if they indeed should be modified).
- (2) The initial clock synchronization procedure was considered by us previously for the highly idealized case of the equatorially placed ground observer and equatorial satellite orbits. It was shown that the procedure allowed to initially set the clocks in such a way that the constant of integration in the formula relating the ground observer and the satellite clock time became equal to zero with a nonaccumulating error of the order of $M_{\odot}/r$ (cf. section IV of this report). However the GPS satellite orbits planes are inclined with respect to the equatorial plane. Also, the ground observer ethalon clock is not placed on the equator. The question is whether the initial synchronization procedure can be made to work with the same precision in this more realistic setting.
- (3) The basic relation between the range data and time data standardly used

in GPS is the special relativistic expression

$$d_{AB} = c(t_B - t_A),$$

where c is the speed of light, or, in the system of units with $c = 1$ (commonly accepted in relativity ^{8,9,17})

$$d_{AB} = t_B - t_A$$

Here A and B are two events in spacetime related by a light signal (or, in mathematical language, connected by a null geodesic), d_{AB} is the range between A and B , and t_A, t_B are the times of events A and B , respectively. The range d_{AB} and the times t_A, t_B in special relativity are measured in the frame of the same (but arbitrary) global inertial observer, or, to put it in different language, in the same global orthonormal coordinate system in spacetime. Neither the concept of a global inertial frame nor the concept of a global orthonormal coordinate system makes sense in curved spacetime for the general gravitational field. This makes the interpretation of the range-time-data relation ambiguous in the general situation. However in the case of the weak, static, spherically symmetric gravitational field (which is the case in all GPS problems) one might hope to find a global interpretation of the relation. The relation then will become an approximate one, and the question is whether it can be kept precise enough to meet the requirements of the GPS.

- (4) The approximations used in the relativistic treatment of the GPS leave the residual errors of higher order in the procedures of the time transfer and of the ranging. Some errors accumulate with time, others do not. It is very important to be able to estimate these errors and, for the errors accumulating with time, to find out how long can the system function without special maintenance to support a given level of precision (say,

1 ns). The latter question becomes especially important if the goal of making the GPS autonomous is considered.

- (5) The history of the GPS and of the relativistic effects treatment in it demonstrates clearly a need for a general relativistic language describing functioning of the GPS and its possible modifications. The idea of the Spacetime Common Grid suggests such a language. It expresses all the GPS procedures in terms of the clock readings and propagating light (radio) signals. To make the Spacetime Common Grid language precise it is necessary to give its mathematical formulation in terms of 4-geometry. It is relatively straightforward to express all elementary procedures of the GPS in the language of the timelike curves lengths and null geodesics in 4-dimensional spacetime. It is putting all the measurements in agreement (globalization) that presents a problem. It cannot be done in relativity in general case. Only the symmetries of the gravitational field can enable one to achieve such a globalization. One of the goals of this research was to outline a complete mathematical formulation of the Spacetime Common Grid on the levels of both elementary operations and functioning of the system as a whole.
- (6) Even if the idea of the Spacetime Common Grid is formulated mathematically and applied to the problems of global positioning its geometric clarity and simplicity can be lost if inadequate computational techniques are used to solve the GPS problems. It is necessary, consequently, to evaluate different mathematical formalisms to find the one that is the most appropriate to keep the ideas of the Spacetime Common Grid clear in applications. The geometric description of the Spacetime Common Grid together with such an adequate mathematical machinery for solving both local and global GPS problems will provide a complete covariant description of the GPS.

In sections VII-X we analyze systematically the relativity produced errors in

ranging. Our results provide a firm theoretical basis for making estimates of the accuracy of the existing GPS procedures as well as evaluating any future changes in it.

It should be noted that the report does not provide a complete and fully realistic picture of the global time transfer and global positioning. Some practically important points have been omitted. For instance, the fact that the reference clocks are on geoid as well as the contribution of the earth's quadrupole moment are not discussed. One can be referred to an excellent report by N. Ashby²³ for such a discussion. These effects could be included but the way it stands at present time the techniques that are used in our report do not seem to provide essential advantages over the techniques used by N. Ashby.

In section XI we give an outline of a complete geometric description of the Spacetime Common Grid and evaluate two different techniques as candidates one of which (or some combination of both) can be used as a computational tool in a complete theory of Spacetime Common Grid.

III. Doppler Shift in a Schwarzschild Field.

We use the Schwarzschild geometry as the model of the earth's gravitational field. In doing, so we neglect contribution of the earth rotation in the gravitational field. An enhanced model would involve the Kerr metric. However, the evaluation of the Kerr model parameters shows that the produced effect of the inertial frames dragging would be of higher order than the effects caused by the parameters coming from the Schwarzschild model⁷. Meanwhile, the estimate of the Schwarzschild model parameters shows that the effect of $M_{\odot}/r$ (here $M_{\odot}$ is the earth mass and r is the Schwarzschild radial coordinate^{8,9}) is of the same order as effects of the squares of the relevant velocities. This is obvious for the satellite orbits. Indeed, the Kepler's law for circular orbits (and it is well known that the Kepler's law is satisfied exactly for circular orbits even in general relativity⁵ when expressed in Schwarzschild coordinates) reads $M_{\odot} = v^2 r$ so that $M_{\odot}/r = v^2$. For the ground observer v^2 is less than $M_{\odot}/r$. Nevertheless, for all the situations considered in GPS one can consider that $v^2 \sim M_{\odot}/r$.

Thus, to discuss Doppler shift up to the second order, we can use Schwarzschild geometry as a model of the earth's gravitational field and, provided that in all approximations terms proportional to $M_{\odot}/r$ are retained (we can neglect higher powers of $M_{\odot}/r$), we obtain a satisfactory expression for the Doppler correction up to the second order with respect to the velocities involved in the picture. We can also say that all the relations below are satisfied up to the second order with respect to v or $\sqrt{M_{\odot}/r}$. In general, throughout this report the abbreviated expression "the relation is satisfied up to the n^{th} order" means that it is satisfied up to the n^{th} order in v and $\sqrt{M_{\odot}/r}$.

The Schwarzschild geometry is a static spherically symmetric geometry. Its metric in Schwarzschild coordinates is given by the expression⁸

$$ds^2 = - \left(1 - \frac{2M_{\odot}}{r} \right) dt^2 + \left(1 - \frac{2M_{\odot}}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where t , r , θ , and ϕ are Schwarzschild coordinates*, and $M_\odot$ is the mass of the earth.

In the geometric picture (cf. Fig. 1) describing the Doppler shift of an electromagnetic signal sent from the transmitting satellite to the surface station observer, the free falling satellite has a geodesic world line, whereas the observer, being attached to the earth, has a world line with all three of the curvatures nonzero⁹. A Doppler shift arises since the 4-velocity of the satellite (at the moment of signal transmission) and the observer (at the moment of receiving) are not parallel. More precisely, the result of the parallel transport of the satellite 4-velocity along the null geodesic connecting the event of transmitting and the event of receiving does not coincide with the 4-velocity of the observer.

The frequency shift can be expressed in terms of the 4-velocities of the satellite and the observer and the 4-momentum of the photon traveling from the satellite to the observer

$$\mathcal{D} = \frac{\nu_s - \nu_o}{\nu_s} = \frac{sP_\mu V_s^\mu - oP_\mu V_o^\mu}{sP_\mu V_s^\mu}, \quad (2)$$

where V_s^μ , V_o^μ are the 4-velocities of the satellite and the observer and sP_μ , oP_μ are the photon 4-momentum at the event of transmitting and the event of receiving, respectively. The 4-momentum of the photon is parallel transported along the null geodesic connecting the events of transmitting and receiving and is tangent to the null geodesic at all times.

We have used for calculation of the Doppler shift (up to the second order) the technique of the tensor series expansion of the world function⁹ developed by J. L. Synge. Here we only describe and explain the result. Following J. L. Synge we introduce new coordinates $(x^\mu)_{\mu=0,1,2,3}$ related to the Schwarzschild coordinates as

* We use throughout this report the system of units commonly accepted in general relativity with both the velocity of light and the gravitational constant equal to unity

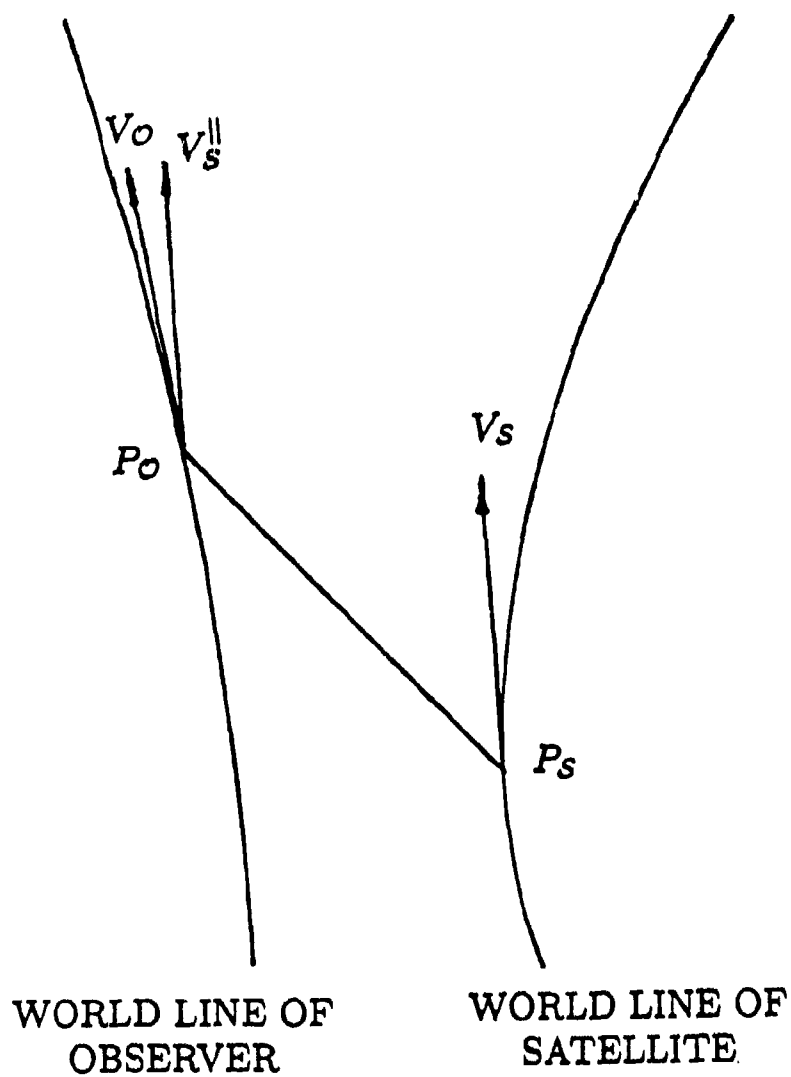


Fig. 1. Geometry of the Doppler shift. The vector $V_S^{\parallel}$ is the result of the parallel transport of V_S along the null geodesic $P_S P_O$. The Doppler shift is caused by $V_O \neq V_S^{\parallel}$.

follows

$$x^0 = t, \quad x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \theta. \quad (3)$$

The metric tensor in these coordinates can be expressed as the sum $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $\gamma_{\mu\nu}$ are small and static ($\gamma_{\mu\nu,0} = 0$). The coordinates (x^μ) are very convenient for a pictorial representation of the Doppler shift. In Fig. 2 these coordinates are used as coordinates of a Euclidean space. Of course, in this space the geodesics of the original Schwarzschild space do not always look like straight lines. The world line of the satellite in this picture is geodesic but looks curved. The vertical straight lines are the integral lines of the timelike Killing vector field of the Schwarzschild metric (described by the equations $x^i = \text{const}$, $i = 1, 2, 3$). The satellite and the observer are moving with respect to Schwarzschild coordinates, so that the 4-velocities V_S , V_O are not parallel to the Killing vectors $\frac{\partial}{\partial t} = \frac{\partial}{\partial x^0}$. The angles between V_S , V_O and the Killing vectors $\frac{\partial}{\partial t}$ (directed upward) are different and determined by the satellite and the observer orbital velocities. If the satellite and the observer were at rest with respect to Schwarzschild coordinates (in which case their world lines would be pictured as vertical straight lines), we would get for the Doppler shift

$$\mathcal{D} = \frac{M_\otimes}{R_S} - \frac{M_\otimes}{R_O}. \quad (4)$$

The right hand side of Eqn. (4) is often called the gravitational Doppler shift. It is of second order in magnitude and should be expected to appear as one of the terms in the final result.

In fact, the final result for the Doppler shift up to the second order is

$$\mathcal{D} = (V_O^i - V_S^i) \frac{\Delta x^i}{\Delta t} + (V_O^i - V_S^i) \frac{\Delta x^i}{\Delta t} V_S^k \frac{\Delta x^k}{\Delta t} + \left(\frac{M_\otimes}{R_S} - \frac{M_\otimes}{R_O} \right) + \frac{1}{2} (V_S^i V_S^i - V_O^i V_O^i) \quad (5)$$

where $i, k = 1, 2, 3$, $\Delta x^i = x_O^i - x_S^i$, $\Delta t = x_O^0 - x_S^0$, and the summation over repeating indices is assumed.

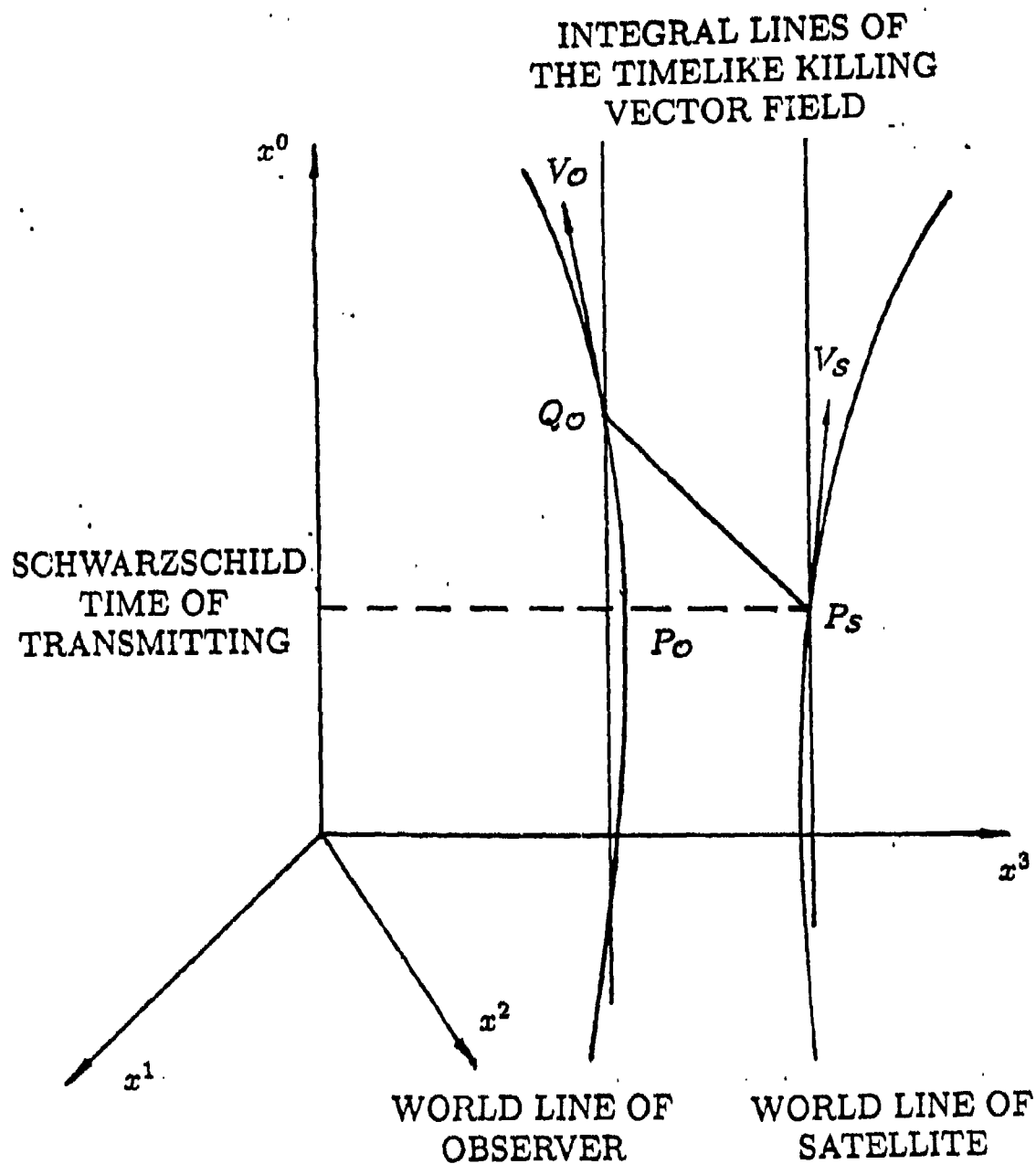


Fig. 2. The Doppler shift and its main contributing factors as viewed by observers resting with respect to the Schwarzschild coordinates.

The first two terms in this expression are the first order Doppler shift and the second order correction to the first order term (note that the second term is not symmetric with respect to V_O , V_S ; it is related to the nonsymmetry of expression (2) with respect to ν_O , ν_S). The third term is the gravitational Doppler shift (cf. Eqn. (4)). The physical origin of the last term is the motion of the satellite and the observer with respect to Schwarzschild coordinates.

We want to point out that in the case of circular orbits only the first two terms contain information about time delay between transmitting and receiving, and only these two terms are time dependent.

IV. Global Clock Synchronization in Schwarzschild Field.

The relation between the clock rates and the Doppler shift is established^{8,9} via the relation (cf. Fig. 3)

$$\mathcal{D} = \frac{\nu_S - \nu_O}{\nu_S} = 1 - \frac{\nu_O}{\nu_S} = 1 - \frac{d\tau_S}{d\tau_O}, \quad (6)$$

or

$$d\tau_S = (1 - \mathcal{D})d\tau_O \quad (7)$$

However, a closer look at this formula and at Fig. 3 makes it obvious that the infinitesimal interval of the satellite and the observer proper times ($d\tau_S$, $d\tau_O$) are measured at different Schwarzschild times. A more precise form of (7) would be

$$(d\tau_S)_{t_i} = (1 - \mathcal{D})(d\tau_O)_{t_r}, \quad (8)$$

where t_i and t_r are the Schwarzschild times of transmission and reception of the signal. The retardation of t_r compared to t_i is reflected in Eqn. (5) by the structure of the first order term and the second order correction to the first order term. This circumstance was obviously the prime concern of H. Fligel² of Aerospace Corporation.

The procedure, described above, of comparing the proper time rates of two clocks in general relativity is the only one (up to equivalence) that is correct for arbitrary gravitational fields. However, generally speaking, it will work only for two clocks (and under some reasonable conditions). In a general gravitational field (with no symmetries) it will not provide global synchronization for more than two clocks. It is not a drawback of this particular procedure. It is well known that in general relativity global synchronization of clocks in gravitational field with no symmetries is impossible in principle.

However, our model gravitational field of the earth (Schwarzschild field) is very symmetric (static, spherically symmetric). One can convince himself easily that in

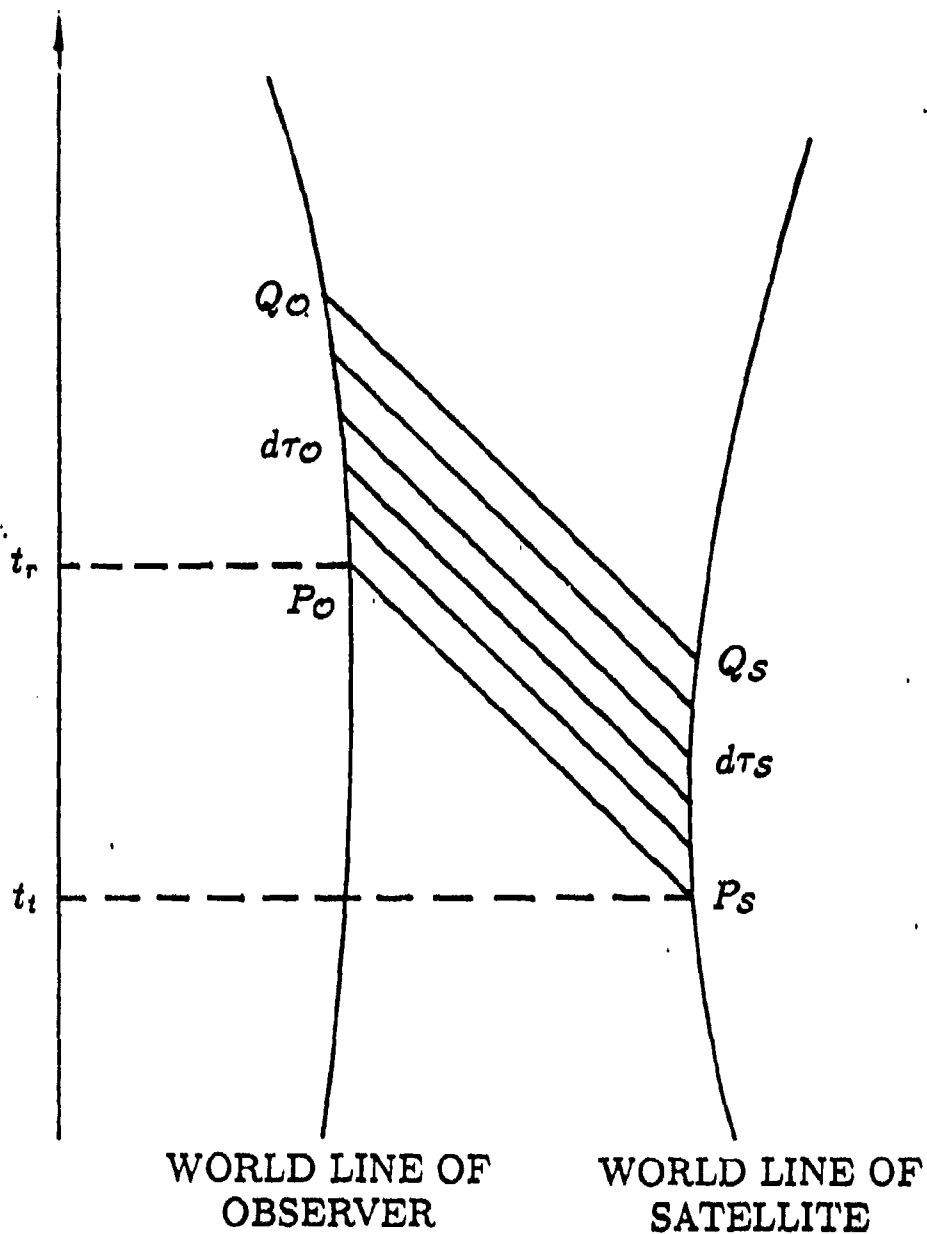


Fig. 3. Relation between the Doppler shift and the rates of the moving clocks. Shown is the set of null geodesics joining the world lines of the satellite and the observer. Each geodesic represents a wave crest. If there are n such crests and $d\tau_S$, $d\tau_O$ are the clock-measures of $P_S Q_S$ and $P_O Q_O$ respectively, then $n = \nu_S d\tau_S = \nu_O d\tau_O$

this particular case our procedure will do the job. But so will many others. The task is to find the simplest one. For instance, one would like to minimize the participation in the procedure of time dependent contributions like the first two terms of Eqn. (5). It would be a good idea to make all the clocks to display Schwarzschild coordinate time, i. e. the time of an observer placed at spatial infinity and resting with respect to the Schwarzschild coordinates. Schwarzschild coordinate time is the closest possible analog of the time of the ECI frame (the special relativistic limit of the Schwarzschild coordinate frame coincides with the ECI frame).

The first step in this direction is to compare the rates of the clocks of the satellite and the observer with Schwarzschild clocks simultaneously with respect to Schwarzschild time (cf. Fig. 4). Elementary calculations show that up to the second order

$$(d\tau_S)_t = \left(1 - \frac{2M_\otimes}{R_S}\right)^{\frac{1}{2}} (1 - V_{S\perp}^2)^{\frac{1}{2}} dt \quad (9)$$

$$(d\tau_O)_t = \left(1 - \frac{2M_\otimes}{R_O}\right)^{\frac{1}{2}} (1 - V_{O\perp}^2)^{\frac{1}{2}} dt \quad (10)$$

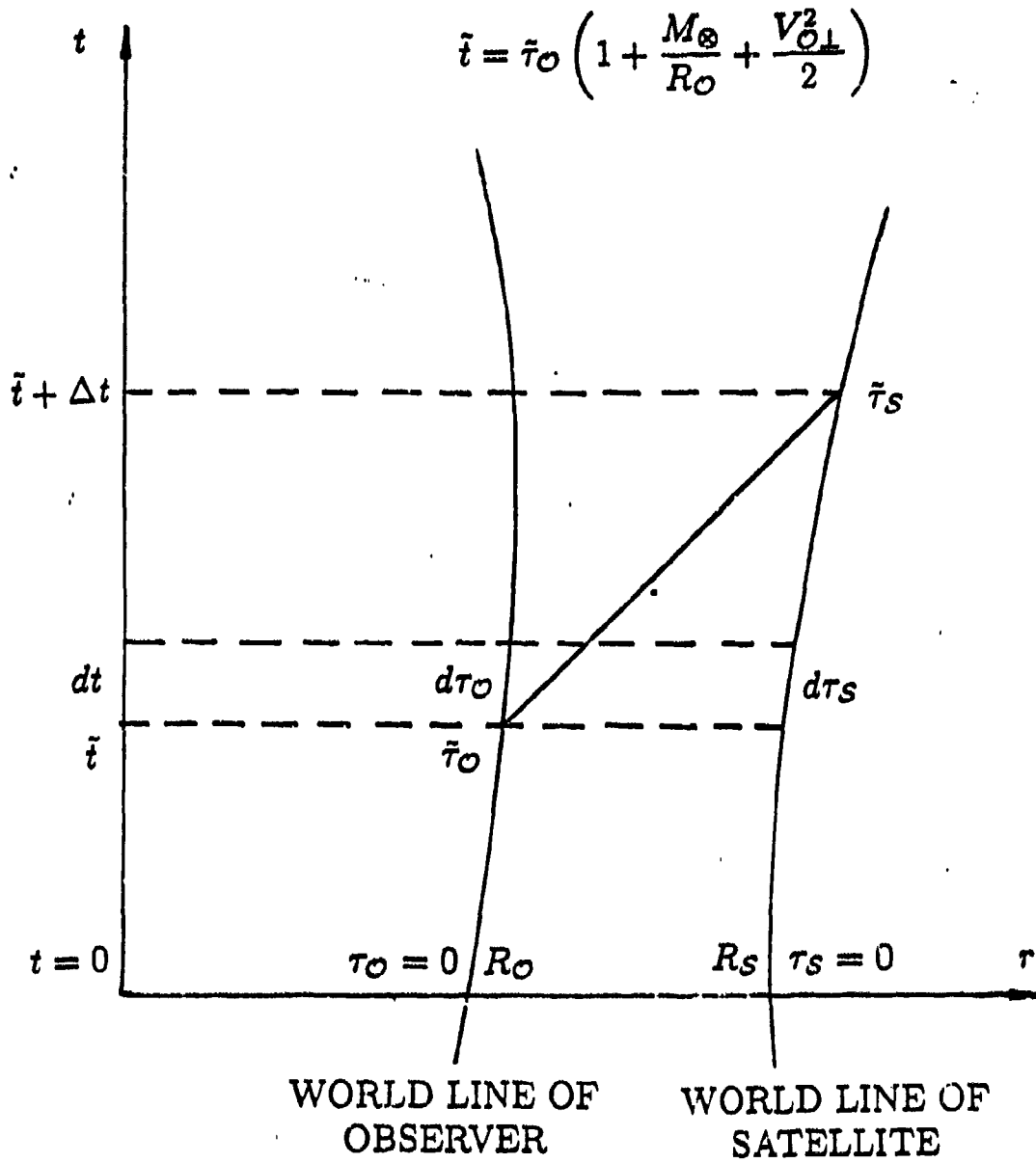
where $V_{O\perp}$ ($V_{S\perp}$) is the component of the observer's (satellite's) 4-velocity V_O (V_S) orthogonal to the timelike Killing vector field of the Schwarzschild metric.

We will perform the rest of our calculation in this section in a highly idealized fashion. Namely, we assume that R_O , $V_{O\perp}^2$, R_S , $V_{S\perp}^2$ are constant (the purpose of this idealization is to get rid of all the details of nonrelativistic origin). In this case both $d\tau_O$ and $d\tau_S$ are proportional to dt with constant proportionality coefficients. Thus one can take any of them as fundamental (the different choices are equivalent to the different choices of time units). Dividing (9) by (10), we obtain

$$(d\tau_S)_t = \frac{\left(1 - \frac{2M_\otimes}{R_S}\right)^{\frac{1}{2}} (1 - V_{S\perp}^2)^{\frac{1}{2}}}{\left(1 - \frac{2M_\otimes}{R_O}\right)^{\frac{1}{2}} (1 - V_{O\perp}^2)^{\frac{1}{2}}} (d\tau_O)_t, \quad (11)$$

or, in the usual second order approximation,

$$(d\tau_S)_t = \left[1 - \left(\frac{M_\otimes}{R_S} - \frac{M_\otimes}{R_O}\right) - \frac{1}{2} (V_{S\perp}^2 - V_{O\perp}^2)\right] (d\tau_O)_t, \quad (12)$$



$$\Delta t = (R_S - R_O) + 2M_\odot \ln \frac{R_S - 2M_\odot}{R_O - 2M_\odot}$$

Fig. 4. The relation between the satellite and the observer clock rates in a Schwarzschild simultaneity band and the initial clock synchronization procedure.

which is interesting to compare with Eqn. (5) (note the loss of the terms related to the time delay).

Integration of Eqn. (12) yields

$$\tau_S = \left[1 - \left(\frac{M_\odot}{R_S} - \frac{M_\odot}{R_O} \right) - \frac{1}{2} (V_{S\perp}^2 - V_{O\perp}^2) \right] \tau_O + C. \quad (13)$$

The constant of integration C can be made equal to zero by employing an appropriate choice of the origin for τ_O , τ_S . In this section we show one way to do it for the particular case when the observer is placed on equator, the plane of the satellite orbit is equatorial, and the orbit period is shorter than the period of the earth rotation. The general case is considered in section IX.

Let us suppose now that the ground observer is sending messages of his clock time continuously in the upward direction (we assume here that the aberration problem is properly taken care of), so that the satellite receiver knows that the signals are propagated along the radial null geodesics. For such signals expression (1) for the Schwarzschild metric implies (with $ds^2 = 0$, $d\theta = d\phi = 0$)

$$dt = \left(1 - \frac{M_\odot}{r} \right)^{-1} dr. \quad (14)$$

Integrating (14) we come up with the expression for the Schwarzschild travel time of the signal

$$\Delta t = R_S - R_O + 2M_\odot \ln \frac{R_S - 2M_\odot}{R_O - 2M_\odot}. \quad (15)$$

Thus, if the satellite receives the ground station message sent at $\tau_O = \bar{\tau}_O$ and, at the moment of receiving, sets on its clock time to

$$\bar{\tau}_S = \left(1 - \frac{M_\odot}{R_S} - \frac{V_{S\perp}^2}{2} \right) \left[\left(1 + \frac{M_\odot}{R_O} + \frac{V_{O\perp}^2}{2} \right) \bar{\tau}_O + \Delta t \right], \quad (16)$$

then the event on the world line of the satellite at $\tau_S = 0$ and on the world line of the observer at $\tau_O = 0$ become simultaneous with respect to Schwarzschild time, and, if we choose as $t = 0$ the Schwarzschild time hypersurface passing through

both events, then at any Schwarzschild moment of time t the clock of the observer and the satellite will display

$$\tau_O = \left(1 - \frac{M_\otimes}{R_O} - \frac{V_{O\perp}^2}{2}\right) t, \quad (17)$$

and

$$\tau_S = \left(1 - \frac{M_\otimes}{R_S} - \frac{V_{S\perp}^2}{2}\right) t, \quad (18)$$

making it possible to tell Schwarzschild time by looking at any of the clocks.

The constant C in Eqn. (13) thus becomes equal to zero. It is clear that the described procedure allows one to synchronize as many clocks as he wishes to Schwarzschild time and the procedure can be generalized to any placement of the observer on the earth and any satellite orbit inclination. In so doing, one might expect the expression for Δt to become more complicated.

Therefore, it is worth to take another look of the Eqn. (15) and to evaluate the last term in it since this term is the one that has a tendency to become more complicated. Let us rewrite Eqn. (15) as follows

$$\Delta t = R_S - R_O + K R_O \quad (19)$$

where

$$K = \frac{2M_\otimes}{R_O} \left(\ln \frac{R_S}{R_O} + \ln \frac{1 - 2M_\otimes/R_S}{1 - 2M_\otimes/R_O} \right) \quad (20)$$

Using the Maclaurin series expansion and dropping all the powers of $M_\otimes/r$ higher than the first, we estimate (20) as follows

$$K \approx \frac{2M_\otimes}{R_O} \left(\ln \frac{R_S}{R_O} - \frac{2M_\otimes}{R_S} + \frac{2M_\otimes}{R_O} \right) \approx \frac{2M_\otimes}{R_O} \ln \frac{R_S}{R_O} \quad (21)$$

Taking into account that for the GPS satellites $R_S \approx 2R_O$, and using the values of $M_\otimes$ and R_O , we come up with $K \sim 10^{-9}$. This means that if we replace the exact expression (19) for an approximate one

$$\Delta t \approx R_S - R_O \quad (22)$$

we introduce an error in ranging of the order of 1 cm (more precisely, 0.08 ns). The error is introduced in the constant of integration and, consequently, does not accumulate. It is clear, therefore, that for any practical purpose we can neglect the last term and use the approximate range-time-data relation (22) instead of exact equation (19).

It is interesting to note here that Eqn. (22) can be also written as

$$\Delta t = \Delta r(1 + O_2) \quad (23)$$

This gives us a hint that the range-data relation might admit the general relativistic interpretation and that it is satisfied only up to the first order. This subject will be developed in a general context and in more detail in section VIII.

However, we can make a conjecture that, most probably, the initial synchronization procedure can use the special relativistic range-time data relation in present-day GPS. The conjecture, as any conjecture, is formulated in a rather vague fashion. We will turn it into a precise statement in subsequent sections.

V. Geometric Approach and the "Relativity of Simultaneity".

As it is well known^{8,9} the geometric approach is a key feature of modern relativity and its applications. This approach helps one to see clearly the physics of relativistic procedures at all steps of calculations and, typically, it considerably reduces the amount of calculations necessary to describe relativistic effects. We are going to demonstrate the power of the approach by analyzing the proposal to change the relativistic treatment of the GPS time transfer by means of the "taking into account" of the so-called "relativity of simultaneity"⁶ (we will use the abbreviated expression "RS-proposal" for it in the rest of the discussion). This proposal led to an incredible waste of time and effort of the researchers in 1987–1988. It was based heavily on the special relativistic concept of simultaneity with respect to an inertial frame of reference applied mistakenly to non-inertial frames. This concept has no analogues in general relativity. The techniques of calculations used in this approach was that of the pre-Minkowski epoch thus leading to many pages of calculations and creating ample opportunity for producing mistakes.

The geometry of the RS-proposal, as we show below, is primitive. If the geometric approach had been used, the main results of the RS-proposal⁶ could have been obtained in two lines (cf. Eqns. (29)–(30) below), together with a clear physical interpretation of what actually had been done.

To compare the general relativistic calculations of section IV with the special relativistic calculations of the RS-proposal we consider the case $M_{\odot} = 0$ in the equations of section IV. This means that we neglect the gravitational correction. Thus the Schwarzschild frame transforms into the ECI frame. Therefore,

$$\Delta t = R_S - R_O, \quad (24)$$

$$d\tau_O = (1 - \vec{V}_O^2)^{\frac{1}{2}}, \quad (25)$$

and

$$d\tau_S = (1 - \vec{V}_S^2)^{\frac{1}{2}}. \quad (26)$$

(we have replaced $V_{\perp}$ of section IV for $\vec{V}$; $V_{\perp}$ and $\vec{V}$ coincide up to the second order). Let us try to reconstruct now the geometry of the RS-proposal. It can be expressed as follows

The RS-proposal suggests that we compare the intervals of proper times $d\tau_S$ and $d\tau_O$ simultaneous in the frame of the observer (cf. Fig. 5),

$$d\tau_O = -(V_O \cdot V_S) d\tau_S, \quad (27)$$

where the components of the 4-velocities V_O and V_S , as represented in the ECI frame basis, are given by (up to the second order),

$$V_O = \left((1 - \vec{V}_O^2)^{-\frac{1}{2}}, \vec{V}_O \right), \quad V_S = \left((1 - \vec{V}_S^2)^{-\frac{1}{2}}, \vec{V}_S \right). \quad (28)$$

Here $\vec{V}_O$ and $\vec{V}_S$ are the 3-velocities of the observer and the satellite, respectively, in the ECI frame. Substituting (28) into (27) we obtain (up to the second order),

$$(d\tau_O)_{i_2} = \left((1 - \vec{V}_O^2)^{-\frac{1}{2}} (1 - \vec{V}_S^2)^{-\frac{1}{2}} - \vec{V}_O \cdot \vec{V}_S \right) (d\tau_S)_{i_1}, \quad (29)$$

or, otherwise,

$$(d\tau_S)_{i_1} = \left(1 - \frac{1}{2} (\vec{V}_O^2 + \vec{V}_S^2) + \vec{V}_O \cdot \vec{V}_S \right) (d\tau_O)_{i_2}. \quad (30)$$

It is instructive to look now at the geometry of the RS-proposal. What happens is that we are using for the ordering in time the alleged slicing of spacetime by the family of proper spaces of the observer, i. e. by planes $\tau_O = \text{const}$, instead of the slicing by proper spaces of ECI frame. But the family of the proper spaces of the observer does not provide a slicing (cf. Fig. 6). These proper spaces intersect each other. As a result, the global synchronization has not been achieved (and cannot be achieved) in this way. Also, one should notice that $\vec{V}_O$ and $\vec{V}_S$ are not measured at the same moment of Schwarzschild time, so that the scalar product in equations (29) and (30) actually should be written as $(\vec{V}_O)_{i_2} \cdot (\vec{V}_S)_{i_1}$ (one should perform

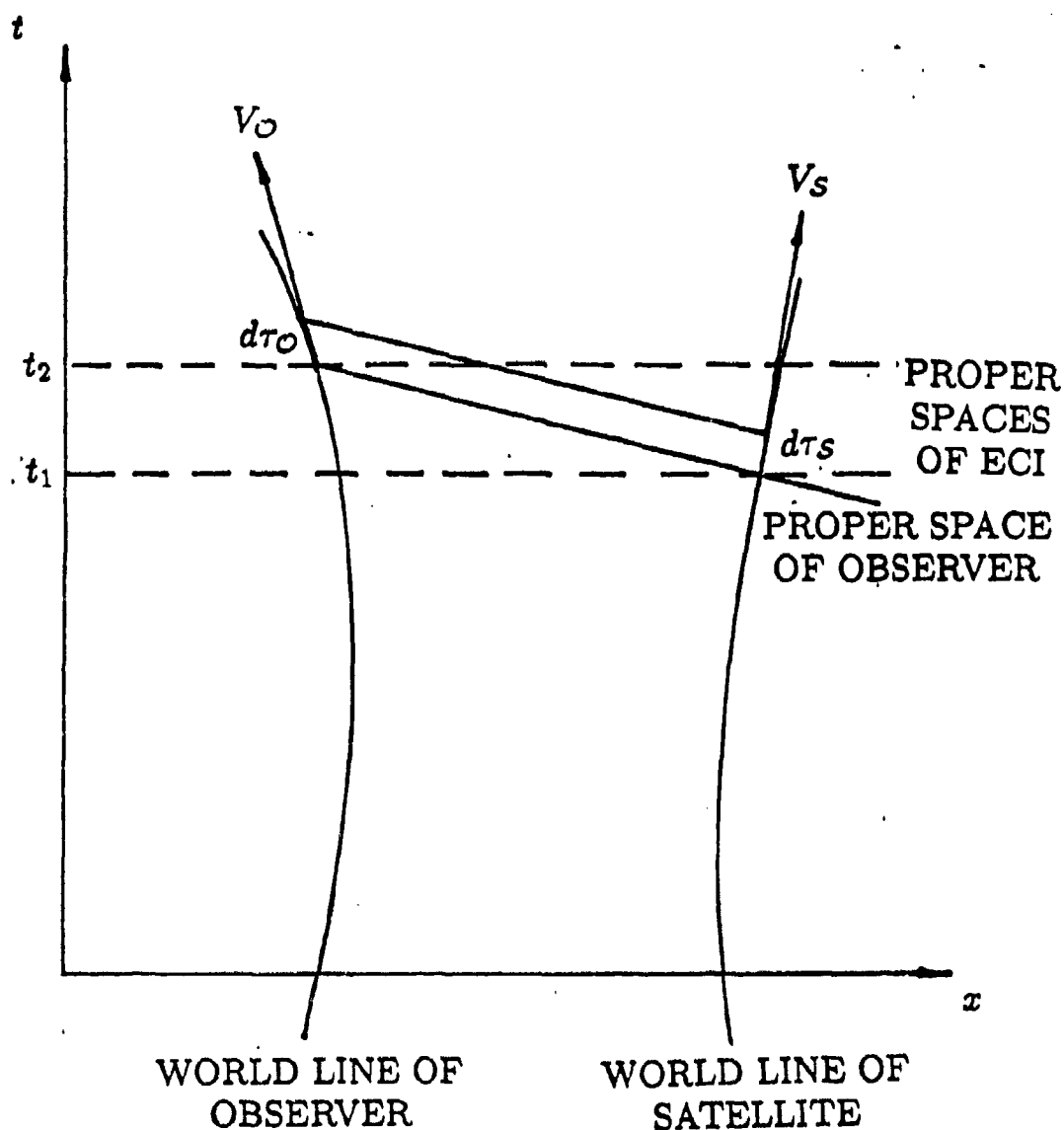


Fig. 5. The geometry of the "relativity of simultaneity" proposal. The intervals of the satellite and the observer proper time are compared not in the Schwarzschild simultaneity band but rather in the simultaneity band of the instantaneous comoving frame of the observer. Such a construction cannot be defined in curved spacetime.

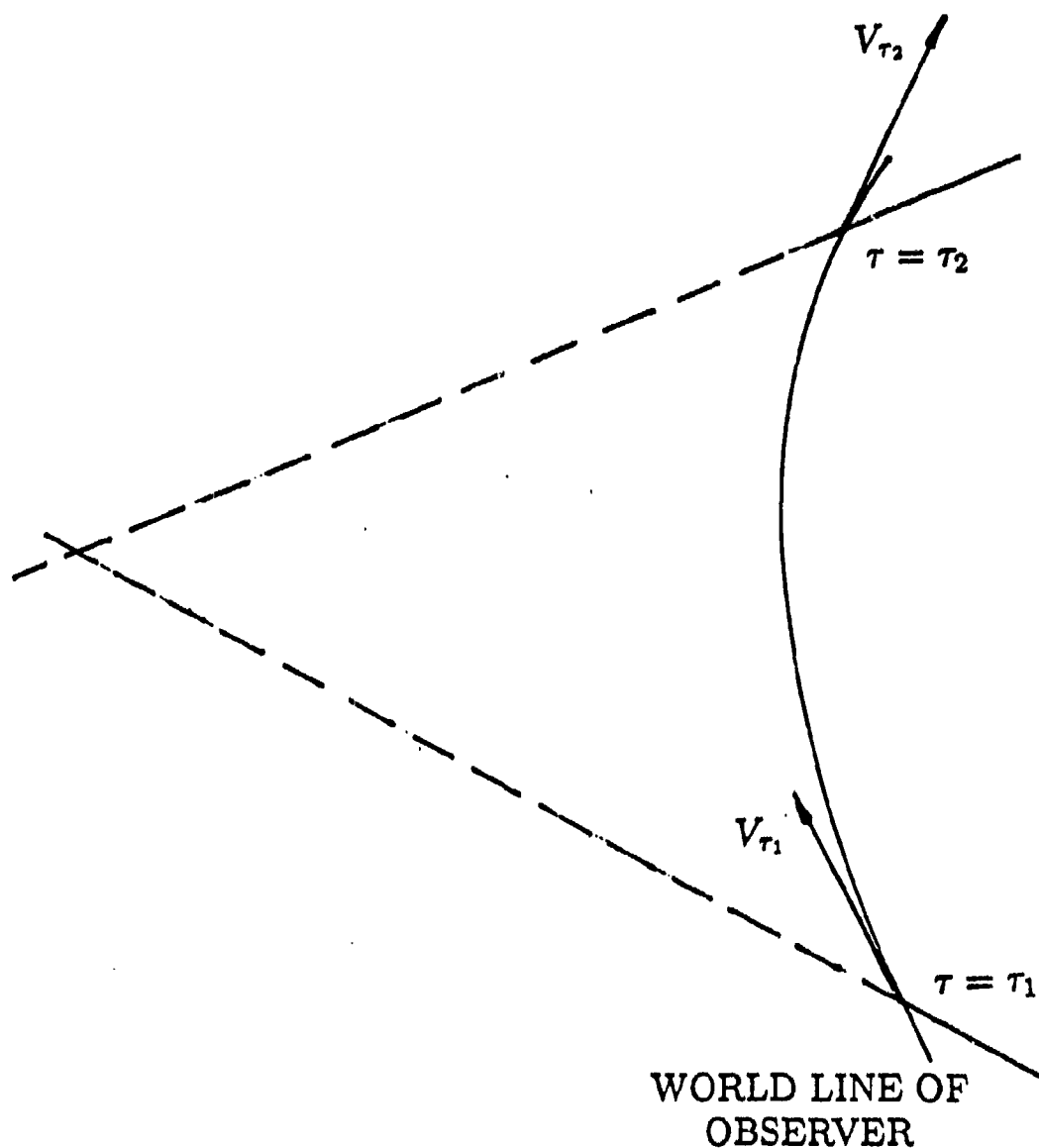


Fig. 6. Even in special relativity proper spaces of the instantaneous comoving frame of the observer are not capable of supporting global synchronization. Acceleration of the observer causes the proper spaces of the instantaneous comoving frames of the observer corresponding to two different moments of time to intersect each other. The simultaneity bands are not well defined in this approach.

similar corrections in other terms; however, in the case of circular orbits $\vec{V}_O^2$ and $\vec{V}_S^2$ are constant, which implies $(\vec{V}_O^2)_{t_1} = (\vec{V}_O^2)_{t_2}$ and $(\vec{V}_S^2)_{t_1} = (\vec{V}_S^2)_{t_2}$.

Expressions (29) and (30) are not useful for comparing proper times τ_O and τ_S with the ECI frame time t . Even if we decided to do so (in a close neighborhood of the observer world line), it would require taking into account (consisting of bulky calculations) the difference between t_1 and t_2 at each moment when integrating, and not just for the initial synchronization.

The equations themselves are correct. Nevertheless, they have nothing to do with synchronization. One can use them (close to the observer world line) if one finds out the relation between dt_1 and dt_2 , which is not hard to do. The result is,

$$dt_2 = \left[(1 - \vec{V}_O^2)^{-\frac{1}{2}} (1 - \vec{V}_S^2)^{-\frac{1}{2}} - \vec{V}_O \cdot \vec{V}_S \right]^{-1} \frac{(1 - \vec{V}_S^2)^{\frac{1}{2}}}{(1 - \vec{V}_O^2)^{\frac{1}{2}}} dt_1, \quad (31)$$

which, when used together with Eqns. (29)–(30), gives

$$(d\tau_S)_{t_1} = \frac{(1 - \vec{V}_S^2)^{\frac{1}{2}}}{(1 - \vec{V}_O^2)^{\frac{1}{2}}} (d\tau_O)_{t_1} = \left(1 - \frac{1}{2} (\vec{V}_S^2 - \vec{V}_O^2) \right) (d\tau_O)_{t_1}. \quad (32)$$

This is identical to the analysis presented in section IV. The difficulties of the RS-proposal are related mainly to the missing piece of information, namely (cf. Fig. 7),

$$dt_1 \neq dt_2 \neq dt_3. \quad (33)$$

Otherwise, this analysis would work as well as any other correct procedure, although it is not clear why one should put himself through all of this to get simple results. Of course, it wipes out the effect of the cross-term (the key result of the RS-approach).

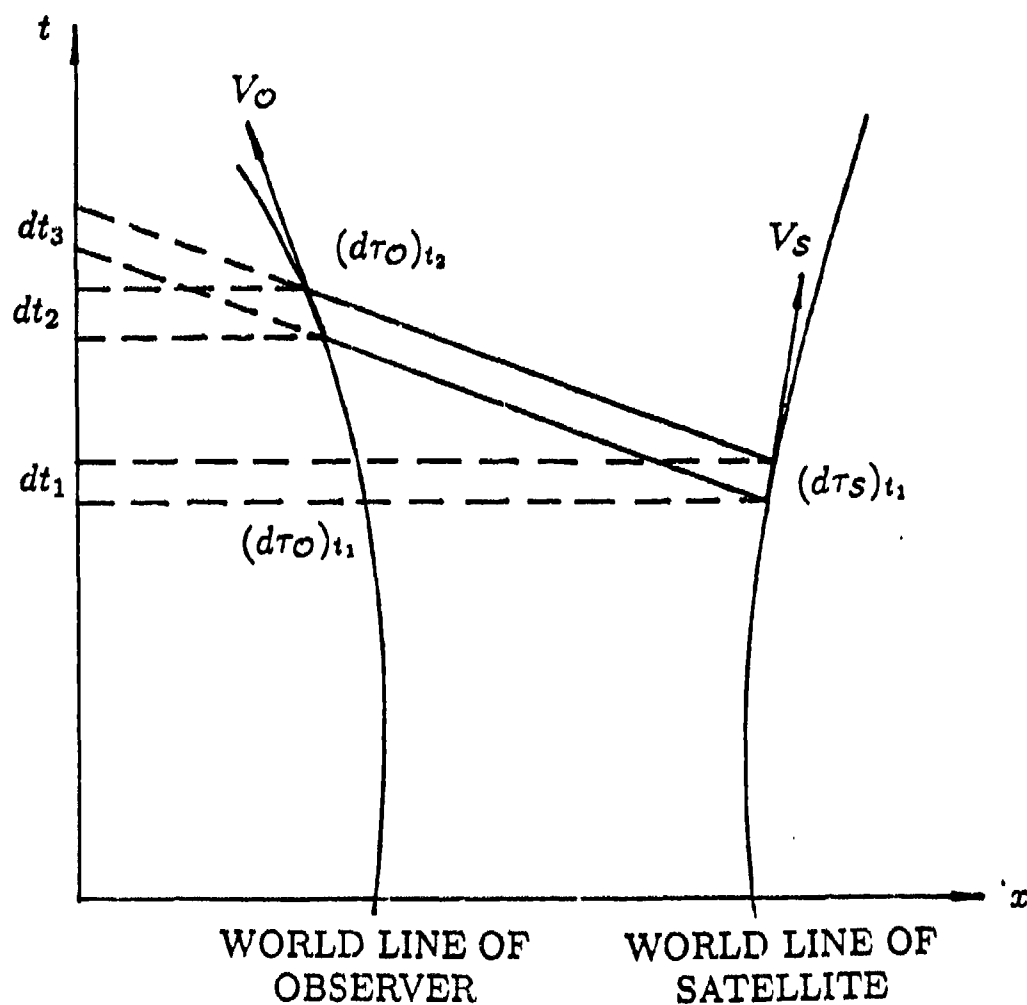


Fig. 7. The time intervals dt_1 , dt_2 , and dt_3 are not equal. After taking into account the correct relation between them, we come up with the clock synchronization scheme equivalent to the synchronization in the Schwarzschild coordinate system.

VI. Sagnac Effect.

In section V we have demonstrated the power of the geometric approach in analyzing a manifestly erroneous proposal and uncovering the source of the error in it.

In this section we are going to consider the so-called Sagnac effect. There was a tendency for a while, even in serious discussions¹⁰ related to the Global Positioning, to treat this effect as something additional to the basic relativistic time dilation effect. Such a treatment of the Sagnac effect is, at least, misleading, and if a proper care is not taken, might lead to the wrong account of relativistic corrections in the global time transfer. We are going to analyze this effect from the geometric point of view and demonstrate that, as a matter of fact, it represents a mere special case of the ordinary relativistic time dilation.

To simplify our discussion we consider the Sagnac effect in special relativity. Let us introduce an inertial frame of reference and the global Lorentz coordinate system associated to it. If an observer **A** is at rest with respect to this frame then his proper time coincides with the coordinate time of the described above Lorentz coordinate system.

$$\tau_A = t \tag{34}$$

Let us consider now another observer **B** moving around the center of the Lorentz coordinate system at a speed V along a circle of radius r . The time to complete the full circle with respect to the observer **A** is

$$t = \tau_A = \frac{2\pi r}{V} \tag{35}$$

(we can think of the observer **A** as sitting at rest with respect to the inertial frame at some point of the orbit of the observer **B**).

The interval of proper time of the observer **B** corresponding to the completion

of the full circle is (up to the second order)

$$\tau_B = t \left(1 - \frac{V^2}{2} \right) \quad (36)$$

The difference between the readings of the clocks of those two observers upon the completion of the full circle will be

$$\tau_A - \tau_B = t - t \left(1 - \frac{V^2}{2} \right) = \pi r V \quad (37)$$

Let us suppose now that the observer **A** is not at rest with respect to the original inertial frame but, instead, is moving along the same circle as the observer **B** so that the angular velocity of this motion is ω . We choose the sign of ω in such a way that $\omega > 0$ if **A** and **B** are moving in the same direction, and $\omega < 0$ if they are moving in opposite directions. The interval of coordinate time necessary for **B** to complete the circle with respect to the observer **A** is now

$$\tilde{t} = \frac{2\pi r}{V - \omega r} \quad (38)$$

The interval of proper time of observer **B** corresponding to $\tilde{t}$ is

$$\tilde{\tau}_B = \tilde{t} \left(1 - \frac{V^2}{2} \right) \quad (39)$$

and of observer **A** corresponding to $\tilde{t}$

$$\tilde{\tau}_A = \tilde{t} \left(1 - \frac{\omega^2 r^2}{2} \right) \quad (40)$$

The difference between the readings of the clocks of those two observers upon the completion of the full circle (as registered by the observer **A** will be

$$\tilde{\tau}_A - \tilde{\tau}_B = \frac{\tilde{t}}{2} (V^2 - \omega^2 r^2) = \frac{\tilde{t}}{2} (V + \omega r)(V - \omega r) \quad (41)$$

which, after substitution of the expression (38) for $\tilde{t}$ and slight transformations, yields

$$\tilde{\tau}_A - \tilde{\tau}_B = (\tau_A - \tau_B) + A\omega \quad (42)$$

where

$$A = \pi r^2 \quad (43)$$

is the area circumvented by the circular orbit of the observer **B** (in the inertial frame of reference).

The additional term $A\omega$ in Eqn. (42) is caused by the circular motion of the observer **A** (previously resting with respect to the original inertial frame) and is sometimes called the Sagnac effect. Its origin is rather obvious geometrically (cf. Fig. 8). In global positioning practice the circular motion of the observer **A** is due to the rotation of the earth with respect to the ECI frame (which can be considered as an inertial frame in special relativity). Very frequently the Sagnac effect is identified in the literature with the expression for $\hat{\tau}_A - \hat{\tau}_B$ in the limiting case when V approaches ωr . Such a limiting expression can be obtained from either (42) or (41) by substituting (37) in (42) or (38) in (41) and then taking the limit as V approaches ωr . The result is

$$\hat{\tau}_A - \hat{\tau}_B = 2A\omega$$

We want to stress here the difference between the described above Sagnac effect and the optical Sagnac effect mentioned frequently^{9,11} in the literature on the relativity theory. In addition to the special relativistic Sagnac effect, in a gravitational field one would encounter also a gravitational Sagnac effect having the general relativistic nature. We are not going to describe it here in detail because this effect is of a higher order than the special relativistic one.

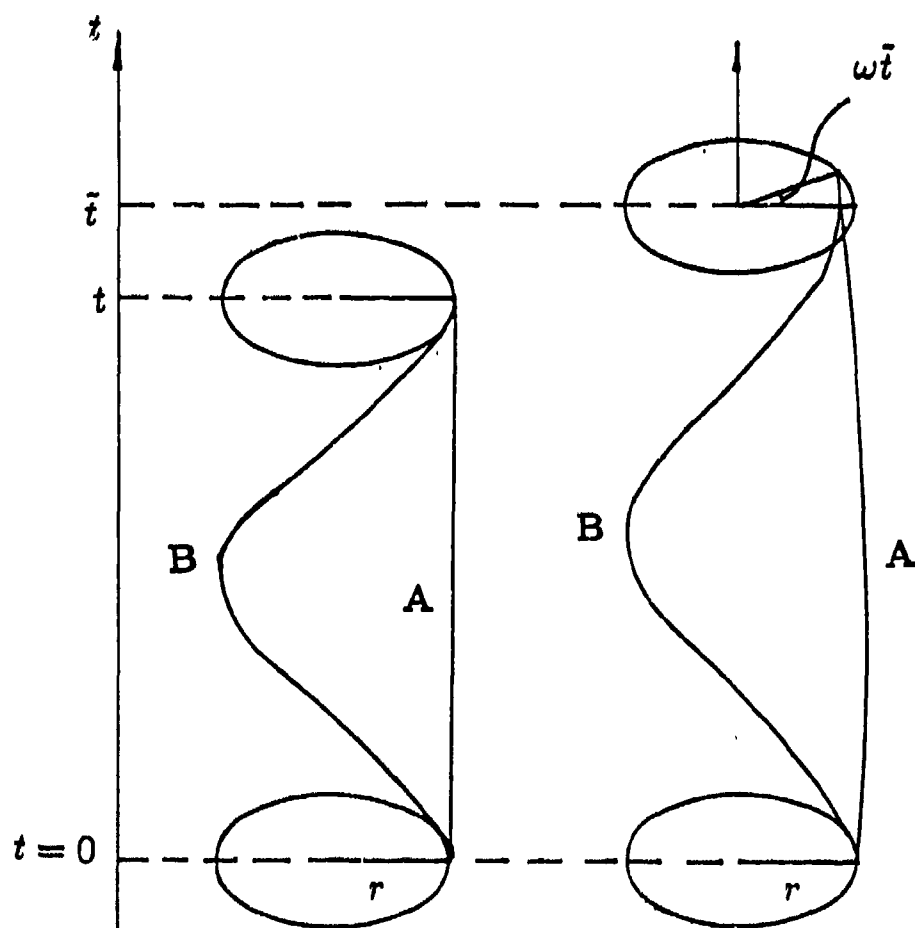


Fig. 8. Spacetime diagram of the Sagnac effect. Shown on the left is the case when observer A is at rest and observer B is moving along a circle of radius r at a speed V . On the right is the case when observer A is moving along the circle at a speed ωr . One full rotation (2π) of B with respect to A corresponds to rotation of B equal to $2\pi + \omega \bar{t}$ with respect to the inertial observer.

VII. Relation between the Range Data and the Time Data in a Schwarzschild Field.

We are going to consider now the relation between the range and time data in a Schwarzschild field for the light (or radio) signal traveling between the ground observer and the satellite. The placement of the ground observer is not restricted anymore. He can be placed anywhere at the ground surface and may even be in motion. The satellite orbital plane can be inclined with respect to the equatorial plane and the orbit does not have to be circular. Actually, the analysis that we are going to undertake can be applied to a pair of satellites or even to a pair of spaceships which are accelerating while the ranging is being performed.

Mathematically the problem can be formulated as follows. First we introduce the coordinates (x^0, x^1, x^2, x^3)

$$x^0 = t, \quad x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \theta \quad (44)$$

the same way as we did in section III.

The Schwarzschild metric reexpressed in these coordinates is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = - \left(1 - \frac{2M_\odot}{r} \right) dx^{0^2} + dx^k dx^k + \frac{2M_\odot}{r^3} \left(1 - \frac{2M_\odot}{r} \right)^{-1} (x^k dx^k)^2 \quad (45)$$

where $k = 1, 2, 3$, and the summation over repeating indices is assumed. It is interesting to note here that the Schwarzschild radial coordinate r can be expressed as

$$r^2 = x^k x^k \quad (46)$$

Up to second order we have

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \quad (47)$$

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \quad (48)$$

$$\gamma_{00} = \frac{2M_{\odot}}{r} \quad (49)$$

$$\gamma_{0k} = 0 \quad (50)$$

$$\gamma_{ik} = \frac{2M_{\odot}x^i x^k}{r^3} = 2M_{\odot}(r^{-1}\delta_{ik} - r_{,ik}) \quad (51)$$

where δ_{ik} is the Kronecker delta and $r_{,ik} = \frac{\partial^2 r}{\partial x^i \partial x^k}$.

We want to stress that in the space of these coordinates (pictured as if they were Cartesian coordinates) the geodesic lines and, particularly, the null geodesic lines do not usually look like straight lines.

Let us suppose (cf. Fig. 9) that in this space $\mathcal{O}$ is the event of emitting the light (or radio) signal by the ground observer, and $\mathcal{S}$ is the event of receiving the signal by the satellite. The solid (curved) line $\mathcal{OS}$ represents the null geodesic world line of the photon traveling from $\mathcal{O}$ to $\mathcal{S}$. The vertical dashed lines represent the integral lines of the timelike Killing vector field of the Schwarzschild metric passing through the events $\mathcal{O}$ and $\mathcal{S}$. In general

$$\Delta t \neq (\Delta x^k \Delta x^k)^{\frac{1}{2}} \quad (52)$$

If there was no gravitational field, i. e. if the spacetime was flat the world line of the photon received at $\mathcal{S}$ would be the straight inclined dashed line $\mathcal{O}'\mathcal{S}$ intersecting the integral line of the timelike Killing vector field passing through $\mathcal{O}$ at a point $\mathcal{O}'$ not coinciding with $\mathcal{O}$. Then we would have

$$(\Delta t)_f = (\Delta x^k \Delta x^k)^{\frac{1}{2}} \quad (53)$$

The straight line segment $\mathcal{OO}'$ pictures the difference between Δt and $(\Delta t)_f$, and it is this difference

$$\Delta t - (\Delta t)_f = \Delta t - (\Delta x^k \Delta x^k)^{\frac{1}{2}} \quad (54)$$

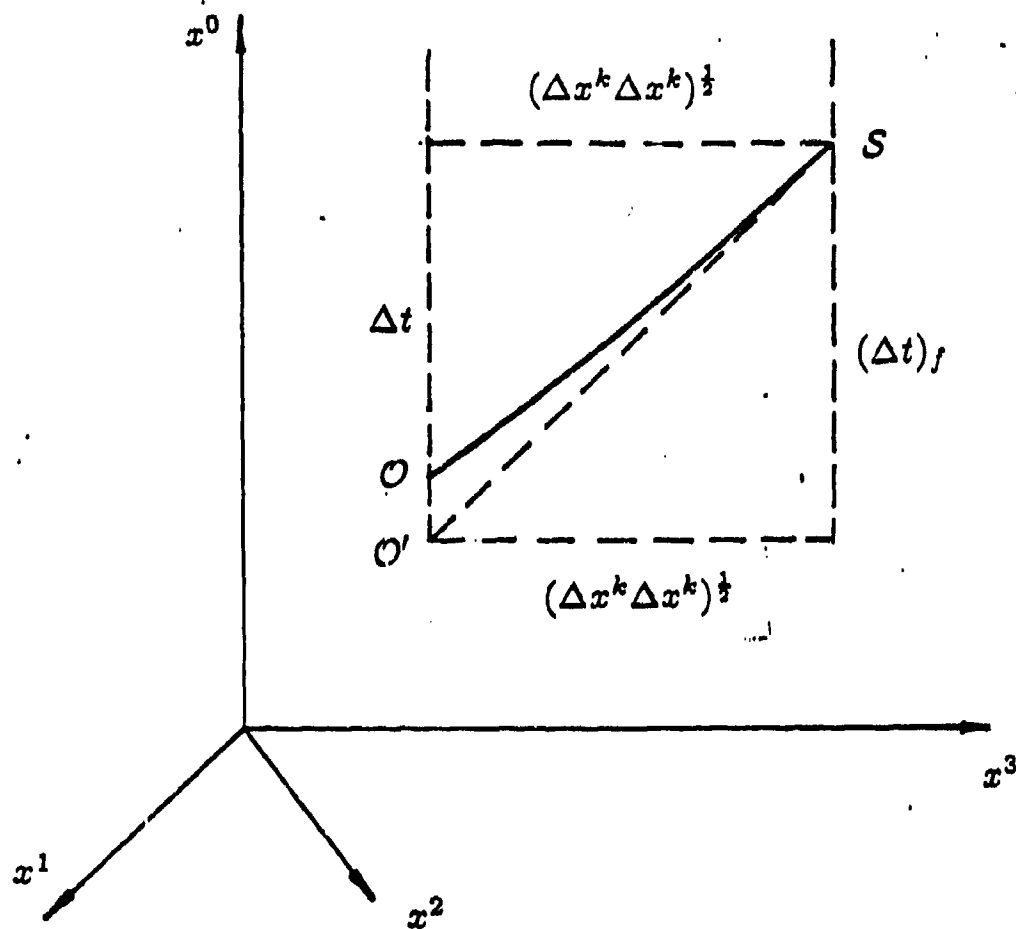


Fig. 9. An estimate of the difference between the general relativistic and special relativistic range-time relation. The special relativistic relation can be used, but it introduces an error of the second order with respect to $\sqrt{M_\odot/r}$.

that we want to estimate.

Let us recall that the line $\mathcal{OS}$ is a null geodesic. Its equations can be written as

$$\frac{d^2 x^\mu}{dw^2} + \Gamma_{\lambda\nu}^\mu \frac{dx^\lambda}{dw} \frac{dx^\nu}{dw} = 0 \quad (55)$$

where $\Gamma_{\lambda\nu}^\mu$ are Kristoffel symbols and w is an affine parameter such that $w = 0$ at $\mathcal{O}$ and $w = 1$ at $\mathcal{S}$. Calculations using the covariant tensor series expansion (J. L. Synge's world function⁹) lead us to the equation (precise up to the second order)

$$\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = Q \quad (56)$$

where

$$Q = s\gamma_{\mu\nu} \Delta x^\mu \Delta x^\nu - 2\Delta x^\mu \Delta x^\nu \int_0^1 \gamma_{\mu\nu} dw - \Delta x^\mu \Delta x^\nu \Delta x^\lambda \int_0^1 \gamma_{\mu\nu,\lambda} w dw \quad (57)$$

Hence

$$\Delta t^2 = \Delta x^k \Delta x^k - Q \quad (58)$$

$$\Delta t = (\Delta x^k \Delta x^k)^{\frac{1}{2}} - \frac{1}{2} Q (\Delta x^k \Delta x^k)^{-\frac{1}{2}} \quad (59)$$

The last term in (59) is small, so that in calculation of Q we may substitute $\Delta t = (\Delta x^k \Delta x^k)^{\frac{1}{2}}$. Eqn. (59) can be rewritten in form

$$\Delta t = (\Delta x^k \Delta x^k)^{\frac{1}{2}} \left[1 - \frac{1}{2} Q (\Delta x^k \Delta x^k)^{-1} \right] \quad (60)$$

or, introducing notation

$$F = -\frac{1}{2} Q (\Delta x^k \Delta x^k)^{-1} \quad (61)$$

reduced to

$$\Delta t = (\Delta x^k \Delta x^k)^{\frac{1}{2}} [1 + F] \quad (62)$$

The expression (61) can be evaluated but it is not an easy task. However the order of magnitude of F can be estimated rather easily for the case $\Delta x^k \Delta x^k \sim r^2$ which

is the case in all GPS problems. Just by looking at Eqns. (62), (57), and (49)–(51) one can make an obvious conclusion

$$F \sim \frac{M_{\otimes}}{r} \quad (63)$$

This solves clearly the objectives (2) and (3) of section II. The standard GPS range – time – data relation takes form

$$\Delta t = (\Delta x^k \Delta x^k)^{\frac{1}{2}} \quad (64)$$

The relation indeed can be interpreted globally in the region of curved Schwarzschild spacetime where $M_{\otimes}/r$ is small, provided that t is Schwarzschild time and the coordinates x^k are related to the Schwarzschild coordinates as in Eqns. (44). The relation (64) is approximate. It is satisfied only up to the first order (with respect to $\sqrt{M_{\otimes}/r}$).

This result does not depend in any way on 4-velocities and accelerations of the ground observer and the satellite, so that it remains true also for a pair of satellites or even for a pair of spaceships with arbitrary accelerations.

It is also clear that the procedure of initial synchronization described at the end of section IV will work in the general case of the observer placement and an arbitrary choice of the satellite orbit. All that one needs to do is to relax the requirement that the signal should be sent vertically (one can send it any way he wants), and to use our new relation (64) instead of the range–time–data relation (22). In this way, he will be able to perform the initial clock synchronization (setting the constant of integration in the Eqn. (13) to zero) with an error of the first order. Since the error does not accumulate in time this synchronization is quite sufficient for any practical purpose of modern GPS.

VIII. Noncircular Orbits in General Relativity.

In section IV we obtained the expression relating the time of the ground observer clock and the satellite clock for the case of the satellite circular orbit (Eqn. (13)). One can notice, however, that all the analysis preceding the Eqn. (13) does not use the assumption of a circular orbit. Eqn. (12) remains unchanged if we relax this requirement. It is the procedure of integration of the Eqn. (12) leading to the Eqn. (13) that uses the assumptions $R_s = \text{const}$ and $V_{s\perp}^2 = \text{const}$ (which are equivalent to assuming the satellite orbit being circular).

Although the orbits of the GPS satellites are meant to be circular in reality they can never be perfectly circular. We want to know now what happens if the satellite orbit is slightly noncircular.

In Newtonian mechanics the orbits slightly deviating from circular are elliptic. The simplest way to analyze the satellite motion on such an elliptic orbit is to use the Hamilton-Jacobi method⁸. The main results are as follows

- (1) The orbit of the satellite is planar. Consequently, one can introduce spherical coordinates (r, θ, ϕ) in such a way that

$$\theta = \frac{\pi}{2} = \text{const} \quad (65)$$

- (2) The total energy of the satellite and its angular momentum are conserved and so are the total energy per unit mass of the satellite $\tilde{\epsilon}$ and its angular momentum per unit mass $\tilde{L}$

$$\tilde{\epsilon} = -\frac{M_\odot}{r} + \frac{V^2}{2} \quad (66)$$

$$\tilde{L} = r^2 \dot{\phi} \quad (67)$$

- (3) The orbit of the satellite is elliptic and it is defined by the equation

$$\theta = \int \frac{\tilde{L} \left[2(\tilde{\epsilon} + M_\odot/r - \tilde{L}^2/2r^2) \right]^{-\frac{1}{2}} dr}{r^2} \quad (68)$$

or, after integration,

$$r = \frac{\tilde{L}^2/M_{\odot}}{1 + e \cos \phi} \quad (69)$$

where e is the eccentricity of the orbit

$$e = \left(1 + \frac{2\tilde{\epsilon}\tilde{L}^2}{M_{\odot}^2}\right)^{\frac{1}{2}} \quad (70)$$

The semimajor axis of the orbit a is

$$a = \frac{r_{\max} + r_{\min}}{2} = \frac{\tilde{L}^2/M_{\odot}}{1 - e^2} = \frac{M_{\odot}}{(-2\tilde{\epsilon})} \quad (71)$$

The constant of integration in (68) has been picked up in such a way that the position of the closest approach (periastron) is achieved when $\phi = 0$. The satellite returns to the periastron position at $\phi = 2k\pi$ for any integer k .

(4) Time as correlated with position is given by

$$t = \int \left[2(\tilde{\epsilon} + M_{\odot}/r - \tilde{L}^2/2r^2)\right]^{-\frac{1}{2}} dr \quad (72)$$

To simplify the integration it is common practice to introduce a new parameter u so that

$$r = \frac{M_{\odot}}{(-2\tilde{\epsilon})^{3/2}}(1 - e \cos u) = a(1 - \cos u) \quad (73)$$

The parameter u is the so-called "mean eccentric anomaly", or, otherwise, Bessel's time parameter. Substitution of (73) into (72) and subsequent integration gives

$$t = \frac{M_{\odot}}{(-2\tilde{\epsilon})^{3/2}}(u - e \sin u) \quad (74)$$

where the constant of integration is chosen so that at $t = 0$, $u = 0$.

Bessel's time parameter is related to the angle coordinate ϕ as follows

$$\sin u = \frac{(1 - e^2)^{\frac{1}{2}} \sin \phi}{1 + e \cos \phi} \quad (75)$$

$$\cos u = \frac{\cos \phi + e}{1 + e \cos \phi} \quad (76)$$

$$\cos \phi = \frac{\cos u - e}{1 - e \cos u} \quad (77)$$

$$\sin \phi = \frac{(1 - e^2)^{\frac{1}{2}} \sin u}{1 - e \cos u} \quad (78)$$

Although u has the same period as ϕ , the difference between them is very essential, apart from the case when $e = 0$ (circular orbit) which gives $u = \phi$. We will be interested later in the relation between u and ϕ for $e \neq 1$ when ϕ and u are small. Then (74) yields (up to the first order)

$$u \approx \frac{(1 - e^2)^{\frac{1}{2}} \phi}{1 + e} = \sqrt{\frac{1 - e}{1 + e}} \phi \quad (79)$$

which means that for small ϕ the parameter u is of the same or higher order of smallness as ϕ

$$u \sim \phi \quad (80)$$

In general relativity the satellite motion for a noncircular orbit is more complicated. In general, the motion is nonperiodic. The physical reason for the difference stems from the fact that, in general relativity, the period of the radial motion does not coincide with the period of the angular motion of the satellite. For the orbit of small eccentricity the difference between the classical and relativistic motion can be pictured as the periastron shift.

For a nearly circular orbit of the radius r_0 , the angle swept between two successive periastrons is

$$\Delta \phi = 2\pi \left(1 - \frac{6M_{\odot}}{r_0}\right)^{-\frac{1}{2}} \quad (81)$$

or, up to the first order,

$$\Delta \phi \approx 2\pi + 6\pi \frac{M_{\odot}}{r_0} \quad (82)$$

i.e., the angle variable acquires an additional shift

$$\delta\phi = 6\pi \frac{M_{\odot}}{r_0} \quad (83)$$

per one period of radial motion compared to the classic case (when $\delta\phi = 0$).

As in classical mechanics a complete description of the satellite motion can be produced using the Hamilton-Jacobi method. The main results can be presented in the following way.

- (1) The orbit of the satellite is planar, as in the case of classical mechanics, so that the Schwarzschild coordinates (t, r, θ, ϕ) can be chose in such a way that

$$\theta = \text{const} = \frac{\pi}{2} \quad (84)$$

- (2) The total energy per unit mass $\tilde{E}$ and the angular momentum per unit mass $\tilde{L}$ are conserved. $\tilde{E}$ and $\tilde{L}$ are called the energy (per unit mass) and the angular momentum (per unit mass) at infinity (for a Schwarzschild observer at infinity). We want to note that the Eqn. (67) is not satisfied exactly in general relativity but is still correct up to the second order.

- (3) The orbit of the satellite is defined by the equation

$$\phi = \int \frac{\tilde{L} [\tilde{E}^2 - (1 - 2M_{\odot}/r)(1 + \tilde{L}^2/r^2)]^{-\frac{1}{2}} dr}{r^2} \quad (85)$$

- (4) Time, as correlated with the position of the satellite, is given by

$$t = \int \tilde{E} [\tilde{E}^2 - (1 - 2M_{\odot}/r)(1 + \tilde{L}^2/r^2)]^{-\frac{1}{2}} \frac{dr}{(1 - 2M_{\odot}/r)} \quad (86)$$

It is obvious that to perform calculations using relations (85), (86) is much harder than for the classical formulae (68), (72). This circumstance motivates us to estimate first the difference in predictions from (85), (86) compared to those from (68), (72).

The detailed analysis shows that the difference between them is of second order. One of the ways to handle the situation is to modify the parameters. We will consider the details only for the particular case that we will use in this report, i. e. only for the Eqns. (72), (73). To keep these equation unchanged we modify the argument u for $u + \delta u$ where $\delta u \sim \frac{M_{\odot}}{r}$, i. e.

$$r = a[1 - e \cos(u + \delta u)] \quad (87)$$

$$\frac{M_{\odot}^{1/2}}{a^{3/2}} t = u + \delta u - e \sin(u + \delta u) \quad (88)$$

These equations coincide with the classical equations only up to the first order.

IX. The Rates of Clocks for Noncircular Orbits.

As we mentioned above the differential relations (9)–(12) of section IV are correct for an arbitrary orbit. It is only the integrated relation (13) that uses the assumption of the orbit being circular. Using the results of section VII we are going to modify the results of section IV to include the case of a slightly noncircular orbit. Let us rewrite Eqn. (9) in the form

$$(d\tau)_t = \left[1 - \frac{M_\odot}{r} - \frac{V^2}{2} \right] dt \quad (89)$$

We have dropped the index S everywhere because we are discussing now only the satellite motion. We will restore it whenever it becomes necessary.

Let us modify now Eqn. (89) to the form more suitable for integrating in the case of noncircular orbits. First, we use the fact that the Newtonian equation of the total energy conservation per unit mass is also true in general relativity up to the second order

$$-\frac{M_\odot}{r} + \frac{V^2}{2} = \tilde{\epsilon} = -\frac{M_\odot}{2a} = \text{const} \quad (90)$$

This allows us to eliminate $V^2/2$ from Eqn. (89)

$$(d\tau)_t = \left[1 - 2M_\odot \left(\frac{1}{r} - \frac{1}{4a} \right) \right] dt \quad (91)$$

Now we can use Eqn. (87), from which it follows that up to the first order

$$\frac{1}{r} = \frac{1}{a} \frac{1}{1 - e \cos u} \quad (92)$$

Substitution of (92) in (91) produces a (correct up to second order) expression

$$(d\tau)_t = \left[1 - \frac{3}{2} \frac{M_\odot}{a} - \frac{2M_\odot e}{a} \frac{\cos u}{(1 - e \cos u)} \right] dt \quad (93)$$

where the Bessel's time parameter u is a function of t . Integrating Eqn. (93) we come up with

$$\tau = t - \frac{3}{2} \frac{M_\odot}{a} t - \frac{2M_\odot e}{a} \int \frac{\cos u}{1 - e \cos u} dt + C \quad (94)$$

To evaluate the term

$$\frac{2M_{\odot}e}{a} \int \frac{\cos u}{1 - e \cos u} dt \quad (95)$$

we use the fact that differentiation of Eqn. (86) yields, up to the first order

$$dt = \frac{a^{3/2}}{M_{\odot}^{1/2}} (1 - e \cos u) du \quad (96)$$

Thus, up to second order

$$\frac{2M_{\odot}e}{a} \int \frac{\cos u}{1 - e \cos u} dt = 2M_{\odot}^{1/2} a^{1/2} e \int \cos u du = 2M_{\odot}^{1/2} a^{1/2} e (\sin u - \sin u_0) \quad (97)$$

so that, finally, up to the second order, we have

$$\tau_S = t - \frac{3}{2} \frac{M_{\odot}}{a} t - 2M_{\odot}^{1/2} a^{1/2} e \sin u + C_S \quad (98)$$

For the ground observer the expression for τ_O does not change

$$\tau_O = t - \frac{M_{\odot}}{R_O} t - \frac{V_{O\perp}^2}{2} t + C_O \quad (99)$$

The initial synchronization procedure described at the end of section IV should be modified as follows. First, we define the global Schwarzschild coordinate time t so that $t = 0$ when $\tau_O = 0$. This defines $C_O = 0$, so that Eqn. (99) turns into

$$\tau_O = \left(1 - \frac{M_{\odot}}{R_O} - \frac{V_{O\perp}^2}{2} \right) t \quad (100)$$

Inverting it we obtain

$$t = \left(1 + \frac{M_{\odot}}{R_O} + \frac{V_{O\perp}^2}{2} \right) \tau_O \quad (101)$$

Substitution of (101) into (98) produces the relation between τ_S and τ_O

$$\begin{aligned} \tau_S = & \left(1 + \frac{M_{\odot}}{R_O} + \frac{V_{O\perp}^2}{2} - \frac{3}{2} \frac{M_{\odot}}{a} \right) \tau_O \\ & - 2M_{\odot}^{1/2} a^{1/2} e \sin \left\{ u \left[\left(1 + \frac{M_{\odot}}{R_O} + \frac{V_{O\perp}^2}{2} \right) \tau_O \right] \right\} + C_S \end{aligned} \quad (102)$$

To make $\tau_S = 0$ at $t = 0$, the same procedure as the one described at the end of section IV should be used. But now, taking into account the results of section

VII, the direction of the synchronizing signal does not need to be restricted to the radial one anymore. Such a procedure will allow us to define the constant C_S . As it has been shown in section VII, we can use the approximation of the expression for the retardation

$$\Delta t = |\vec{r}_S - \vec{r}_O| \quad (103)$$

Let us suppose now that the synchronizing signal was sent by the ground observer at $\tau_O = \tilde{\tau}_O$. To achieve $\tau_S = 0$ at $t = 0$ the satellite clock should be set to

$$\begin{aligned} \tilde{\tau}_S = & \left(1 - \frac{3}{2} \frac{M_\odot}{a}\right) \left[\left(1 - \frac{M_\odot}{R_O} - \frac{V_{O\perp}^2}{2}\right) \tau_O + \Delta t \right] \\ & - 2M_\odot^{\frac{1}{2}} a^{\frac{1}{2}} e \sin u \left[\left(1 - \frac{M_\odot}{R_O} - \frac{V_{O\perp}^2}{2}\right) \tau_O + \Delta t \right] \end{aligned} \quad (104)$$

The formulae (98)-(104) provide a complete account of the relation between the rates of clocks of an observer placed at arbitrary latitude on the ground surface and a satellite in a noncircular orbit (in general, with the plane of its orbit inclined with respect to the equatorial plane), and of the procedure of initial synchronization of the clocks¹².

The formulae contain second order non-accumulating error in the constant of integration (initial synchronization) and fourth order accumulating with time error.

X. Summary of Relativistic Corrections and the Residual Errors in Global Positioning.

Relativistic effects are described (up to the second order) in the following way (cf. Sect. IV, IX):

- (a) The differential relation between the proper time of a moving clock and the Schwarzschild coordinate time is

$$d\tau = \left[1 - \frac{M_{\odot}}{r} - \frac{V^2}{2} \right] dt \quad (105)$$

t — Schwarzschild time;

τ — proper time of a moving clock;

r — Schwarzschild radial coordinate of the clock;

V — clock velocity with respect to the Schwarzschild frame;

$M_{\odot}$ — earth mass.

- (2) Integrated relations are different for the satellite and for the ground observer.

- (a) For the satellite

$$\tau_S = t - \underbrace{\frac{3}{2} \frac{M_{\odot}}{a} t}_{\text{II}} - \underbrace{2 M_{\odot}^{\frac{1}{2}} a^{\frac{1}{2}} e \sin u}_{\text{III}} + \underbrace{C_S}_{\text{IV}} \quad (106)$$

a — major semiaxis of the satellite orbit ($a \approx R_S$);

e — satellite orbit eccentricity;

u — Bessel's time parameter.

- (b) For the ground observer

$$\tau_O = t - \frac{M_{\odot}}{R_O} t - \frac{V_O^2}{2} t + C_O \quad (107)$$

R_O — radius of the earth at the observer location;

V_O — observer velocity due to the earth rotation.

The magnitudes of the relativistic corrections and the residual errors is determined by the value of the clock's speed V with respect to the Schwarzschild frame and the value of the gravitational potential $M_{\odot}/r$. In all GPS applications

$$V \ll 1, \quad \frac{M_{\odot}}{r} \ll 1, \quad \text{and} \quad V^2 \sim \frac{M_{\odot}}{r} \quad (107)$$

We say (cf. Sect. II) that a term in an equation is of the first order of magnitude if it is of the same order as $V \sim \sqrt{M_{\odot}/r}$, of the second order if it is of the same order as $V^2 \sim M_{\odot}/r$, and so on.

In the system of units commonly accepted in relativity masses and distances are measured in seconds. The values of the parameters relevant to our estimates are as follows

$$\begin{aligned} M_{\odot} &= 1.479 \times 10^{-11} \text{ sec}, & e &= 0.005 \\ R_{\odot} &= 2.125 \times 10^{-2} \text{ sec}, & a \approx R_S &= 8.5 \times 10^{-2} \text{ sec} \end{aligned} \quad (109)$$

In the formula for the satellite, terms II and III are the second order relativistic corrections that have been already implemented in the GPS. Term II has a value $0.261 \times 10^{-9} t \text{ sec}$. It accumulates with time and exceeds 1 nanosecond after $\approx 3.83 \text{ sec}$. Term III is due to the ellipticity of the satellite orbit. It is periodic, nonaccumulating, and bounded by $1.21 \times 10^{-8} \text{ sec} = 12.1 \text{ nsec}$. The term needs to be taken into account if 1 nsec precision is required or if the satellite orbits have larger eccentricities than our value (0.005).

The errors of relativistic origin arise due to the following reasons

- (a) Errors in the constants of integration are caused by neglecting the general relativistic effect of the light rays bending (the relation between the time data and the range data used currently in the GPS is special relativistic). The synchronization procedure allows to make all the constants of integration equal to zero with an error

$$\delta C_S \approx 2M_{\odot} \ln \frac{R_S}{R_{\odot}} \approx 2M_{\odot} \ln 4 \quad (110)$$

if the synchronization is performed when the satellite is right over the ground observer, and

$$\delta C_S \approx 2M_\odot \ln(4 + \sqrt{15}) \quad (111)$$

if the satellite is on the horizon. In any case the error is ≈ 0.08 nsec and does not accumulate.

- (b) Errors are caused by neglecting the higher order terms in relativistic corrections. In case of the currently accepted model (Schwarzschild field) for the GPS, the errors are of the fourth order. They can be estimated via multiplying the value of the corresponding term by the factor $M_\odot/R_S$, which gives for term II $0.454 \times 10^{-19} t$ sec.

The error accumulates with time but it takes over 100 years to exceed 1 nsec threshold. The error in term III is periodic, nonaccumulating and bounded by 0.21×10^{-17} sec.

- (c) Errors arise due to the light bending. If the GPS is used for measuring the major semiaxis of the orbit then the value of the major semiaxis is of the same order of magnitude as the error in the constant of integration (5.916×10^{-11} sec) which leads to the error in term II, equal to

$$\frac{3}{2} M_\odot \delta \left(\frac{1}{a} \right) t \approx 1.814 \times 10^{-19} t \quad (112)$$

that accumulates with time. However, it takes over 100 years for it to reach 1 nsec.

The error in term III, equal to

$$2M_\odot^{\frac{1}{2}} e \delta(a^{\frac{1}{2}}) \sin u \quad (113)$$

is periodic, nonaccumulating, and bounded by 1.104×10^{-18} sec.

For the sake of convenience we summarize all the information concerning the relativistic corrections and the residual relativistic errors in the Table (cf. Table 1).

Other errors occur in global positioning due to the effects of nonrelativistic nature, such as imperfections in the clock technology (especially for space qualified

$$\tau_S = t - \underbrace{\frac{3}{2} \frac{M_\odot}{a} t}_{\text{II}} - \underbrace{2 M_\odot^{\frac{1}{2}} a^{\frac{1}{2}} e \sin u}_{\text{III}} + \underbrace{C_S}_{\text{IV}}$$

Term	II	III	IV
Analytical Expression	$-\frac{3}{2} \frac{M_\odot}{a} t$	$-2 M_\odot^{\frac{1}{2}} a^{\frac{1}{2}} e \sin u$	C_S
Value	$0.261 \times 10^{-9} t \text{ sec}$	12.1 nsec	0
Does it accumulate?	Yes	No	No
Error due to neglecting the higher order terms	Value $\times \frac{M_\odot}{a}$ $.454 \times 10^{-19} t \text{ sec}$	Value $\times \frac{M_\odot}{a}$ $.21 \times 10^{-17} \text{ sec}$	Value $\times \frac{M_\odot}{a}$ 0
Error due to neglecting the light ray bending	$\frac{3}{2} M_\odot \delta \left(\frac{1}{a} \right) t$ $1.814 \times 10^{-19} t \text{ sec}$	$2 M_\odot^{\frac{1}{2}} e \delta (a^{\frac{1}{2}}) \sin u$ $1.104 \times 10^{-18} \text{ sec}$	$2 M_\odot \ln(4 + \sqrt{15})$ 0.08 nsec
Does the error accumulate?	Yes	No	No

Table 1. Relativistic effects in global positioning.

clocks), difficulties in the separation of the satellite position (ephemeris) errors from the clock errors, the atmospheric effect on the signals propagation, and so on. We give here a brief (and by no means complete) discussion of these errors and compare them with relativistic errors¹⁰.

The separation of satellite position or ephemeris errors from clock error is an inherent problem in a purely passive monitor system. Such a high precision separation is vital for autonomous operation in the event of interruption of communication with the Master Control Stations, or of use of a more stable clock. The read-out resolution in the current system ("Kalman Filter") does not satisfy the use of more stable clocks (they cannot be read with a precision commensurate with their intrinsic capability). When the satellite ephemeris is determined using the pseudorange measurements methods they are monitored whenever they are above a fixed chosen elevation angle at the monitor station. Real time ephemeris accuracy is believed to be ≈ 2 m radial, ≈ 7 m intrack, ≈ 5 m crosstrack. It can be improved by a factor of 2 with post-fitting. Satellite clock error is of the order of the radial error (≈ 10 nsec).

The satellite ephemeris error can be significantly reduced and separated from the clock error if one employs methods to determine the GPS satellite orbits independent from the use of pseudorange measurements. There are several such methods, for example

- (a) Radar tracking (not sensitive enough).
- (b) Laser ranging (accuracy of 1 cm (0.03 nsec) should be routine soon, the current L-band ranging accuracy is 30 cm (0.1 nsec))

Any attempt to operate a global navigation and the time transfer system at or below the 1 nsec level will operationally require the clear separation of satellite position errors from clock errors.

The methods under consideration are:

(a) Use of existing GPS signals.

1. Improved use of pseudorange measurements
2. Doppler tracking of the carrier phase
3. Carrier phase ranging by resolving ambiguities
4. Very long baseline interferometry

(b) Two-way ranging

1. Satellite transponder
2. Laser ranging

(possibly with additional on-board event timer for clock read-out)

(c) Ranging between GPS satellites (crosslink ranging)

The concept of crosslink ranging was originally developed to facilitate eventual system autonomy.

The present system depends on daily update of satellite ephemeris and clock parameters from the master control station. Without it the clock phase error would grow $\approx 10 \text{ nsec/day}$, the satellite ephemeris errors also would grow (intrack - 2 km in 100 days, 10km in 200 days, radial and crosstrack would experience daily oscillations with growing amplitude: 200 m for radial, and 70 m for crosstrack after 200 days)

Crosslink ranging would allow in principle to maintain original navigation consistency among GPS users with the satellites left unattended, although accuracy with respect to earth coordinates may be seriously degraded.

The effects of the propagation of the signal through the earth's atmosphere are of two kinds:

- (a) Systematic time and frequency shifts
- (b) Random effects which can be represented statistically.

The random effect results from fluctuations in signal propagation through the earth's troposphere. This noise degrades the cesium performance by a factor of 10,

and that of hydrogen masers by a factor of 100.

The long-term propagation effects owing to barometric pressure variations impose a limit $\Delta f/f \approx 10^{-14}$ for time intervals of one day. Tropospheric noise and systematics are independent of the radio signal frequency, except for molecular absorption bands (5 mm and 2.5 mm for oxygen, and 1.35 cm and 1.6 mm for water vapor). The tropospheric effects limit the accuracy in all electromagnetic one-way propagation systems.

Ionospheric propagation effects are highly affected by the frequency of the signals; both the path delay and the path itself are affected.

Both the systematic and the random noise depend of the number of electrons per unit area over the distance of the ray path (columnar electron density), which varies greatly with the level of solar activity and the time of the day. This quantity is very unpredictable.

Since the ionosphere's columnar electron density affects the speed of light in a well understood way that depends on frequency, one can measure the density by simultaneously receiving signals transmitted on different frequencies.

For troposphere one might try to introduce corrections using barometric pressure and humidity data taken at the ground station, and vapor radiometer data taken in the line of sight to the spacecraft.

Thus far the best technique for subnanosecond time transfer, or for frequency comparisons at or below 10^{-14} level, is to measure the round trip delay, divide the result by two, and use this value to correct the data for spaceborne clock.

The final conclusion is that the technology of the foreseeable future provides all the means for precise time transfer and global positioning on the level of 1 ns. Relativistic effects of the second order play an important role in the procedures of global positioning. They are treated correctly in the current GPS as far as pairwise time transfer (master clock - satellite clock, satellite clock - satellite clock, etc.) is concerned.

XI. Spacetime Common Grid and the Covariant Formulation of the Global Positioning Problems.

As it has been stated in the previous section, proper care can be taken of the relativistic corrections in global time transfer and global positioning. Our estimate of errors, both accumulating and nonaccumulating with time, shows that they can be kept within 1 ns for many years without any need of resynchronization as far as relativistic effects are concerned. Rapid progress in the clocks technology, study of the atmospheric phenomena influencing the travel time of the time transfer and positioning signals, and improvements in aerospace technology make global positioning at the level of 1 ns the matter of a quite foreseeable future.

One can imagine that the improvements in clock technology, particularly the development of precise space qualified clocks, will make unnecessary the presence of the ground placed master clock in the system, so important at present. Or, the system might contain a whole class of preferred clocks (part of them in space) synchronized with each other via the ground - satellite and satellite - satellite cross - linking. Any such development would turn the Global Positioning System into a system capable of upkeeping itself and functioning quite independently for prolonged intervals of time. Computer simulations¹⁰ indicate that a system based on cross - linked satellites indeed can be kept self consistent for long time. However, the capability of such a system in performing the task of global positioning on the ground surface or in measuring the location of space vehicles with respect to the ground surface deteriorates rapidly due to irregularities of the earth's rotation with respect to the Schwarzschild frame. The latter is virtually unaffected by those irregularities. To make such an autonomous system effective in global positioning while maintaining its independence a study of the earth's rotation irregularities should be undertaken. The US Air Force involvement in the "LAGEOS-3" project is the first and the most important step in this direction. Variations in the earth's

rotation do not influence in any essential way the clocks synchronization. It is not a resynchronization which is required because of the earth's rotation irregularities but rather recoordination.

To keep such a big Global Positioning System consistent and efficient in performing its tasks there is a need for an adequate mathematical model that allows to track the positions and maintain timekeeping of all GPS satellites and ground stations (no matter whether the system is based on one master clock, a whole class of preferred clocks, or treats all the clocks on equal footing). It is our opinion that such a model should be based on the general relativistic approach.

The idea of such a relativistically based model — the Spacetime Common Grid — suggested by Capt Warner A. Miller seems to be the most appropriate one. It treats the GPS system as a construction consisting of the clocks and null rays (propagating light or radio signals) in 4-dimensional curved spacetime. The use of the Spacetime Common Grid will exclude in the future any mistakes in treatment and upkeeping of the GPS. At the same time it is maximally flexible, so that it stimulates incorporation in the system of new ideas and technological advances in the easiest possible way.

Mathematically, the Spacetime Common Grid model pictures clocks as time-like world lines in 4-dimensional spacetime that are geodesic for the satellite clocks and curved for the ground stations clocks. The time transfer in the Spacetime Common Grid is pictured as a reparametrization of these timelike world lines conducted in a consistent fashion. The possibility of such a consistent procedure of parametrization (synchronization) is based on the 4-dimensional symmetries of the earth's gravitational field (cf. Section IV). The world lines of the clocks are marked by the events of transmitting signals containing information of the clock readings as well as their positions with respect to a global coordinate system (for instance, the Schwarzschild coordinate system or very popular in astronomical observation isotropic coordinates). An event on the world line of a clock is, thereby, identified

as the intersection of the timelike world line of a clock and a null geodesic.

The positioning procedure is reduced in this picture to finding the time and coordinate data of an event on the world line of a user from the fact that this event can be considered as the intersection of several null geodesics connecting this event with some marked events on the world lines of the GPS clocks (cf. Fig. 10).

The Spacetime Common Grid model is very simple and clear conceptually, but, at the same time, it is powerful and flexible enough to be equally capable of describing the systems with complete "democracy" of clocks, the systems containing a preferred class of clocks, or, the present day system with one master clock.

As we have mentioned above, the Spacetime Common Grid model makes an assumption that the symmetries of the gravitational field are such that the region of its activities in spacetime can be covered by one coordinate system (x^0, x^1, x^2, x^3) such that the coordinate x^0 is timelike (i. e., vectors tangent to the lines determined by the equations $x^1 = \text{const}$, $x^2 = \text{const}$, $x^3 = \text{const}$, are timelike) whereas all three remaining coordinates are spacelike. In this case, the time coordinate $t = x^0$ provides a universal time ordering of all events and assures that the time transfer between the GPS clocks can be organized in a coordinated way provided that the motion of these clocks with respect to the global coordinate system is known. In the GPS practice the global coordinate system is determined by the fact that the earth's gravitational field is, in a first approximation, static and centrally symmetric. A typical example of such a system is the Schwarzschild coordinate system used throughout this report and prevalent in investigations on theoretical problems concerning static spherically symmetric fields. Another example is the isotropic coordinate system ⁸ frequently used in describing astronomical and cosmological observations. In practice we also use extensively the fact that the earth's gravitational field is weak. This allows us to use approximations instead of performing exact calculations of the relativistic effects in GPS. The approximate equations can possess additional symmetries and, thereby, extend further the choice of an appro-

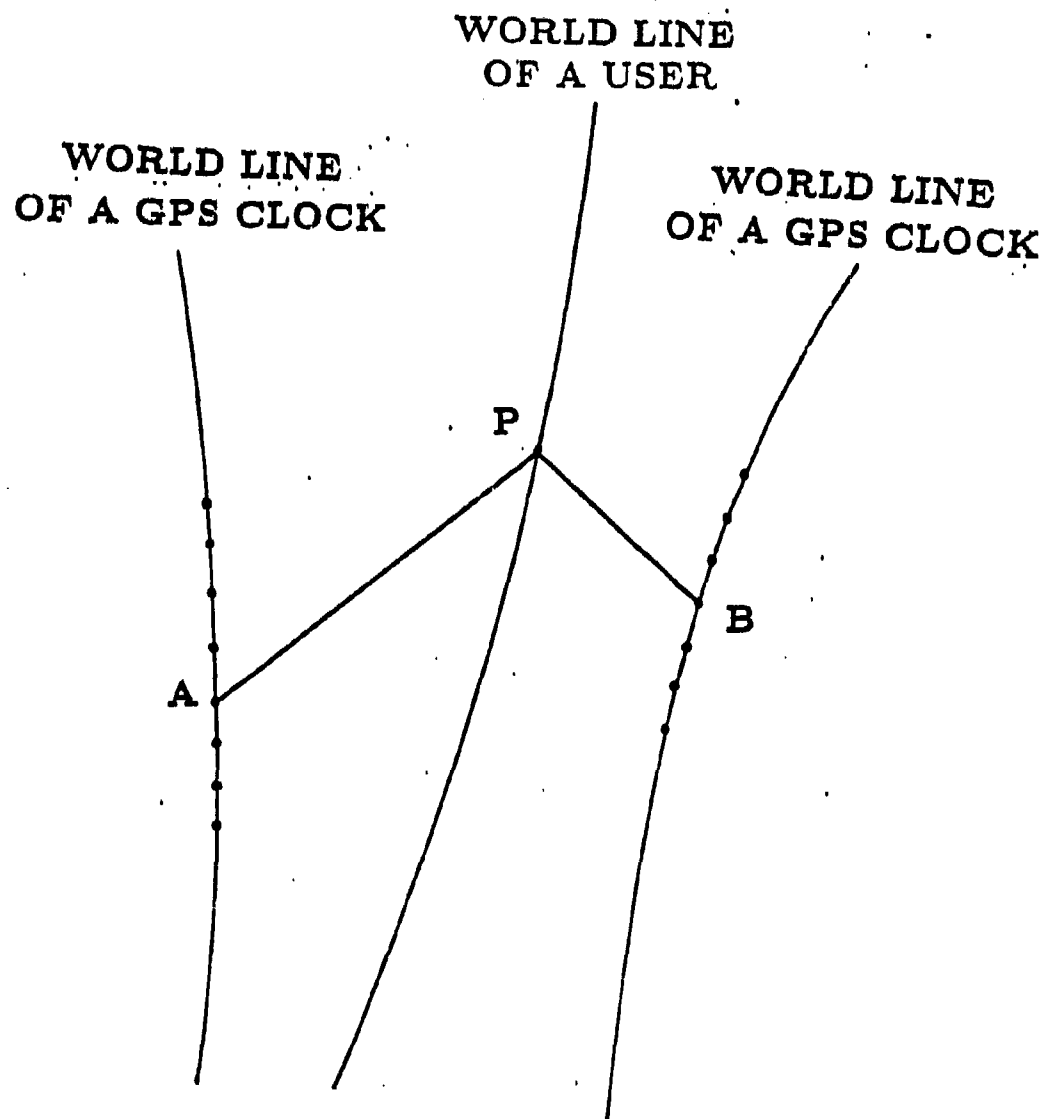


Fig. 10. Spacetime diagram of positioning an event P on the world line of a user according to the Spacetime Common Grid picture. An event P on the world line of the user is represented as the intersection of the null geodesics AP and BP connecting the event P to the marked events A and B on the world lines of the GPS clocks.

priate global coordinate systems (which is indeed the case, as one can see when considering the PPN formalism⁸).

The first problem is to describe the motion of the GPS clocks with respect to the global coordinate system, and the clock rates compared to the global coordinate time. The issue of the clock rates is discussed in sections IV - X of this report in a rather detailed fashion. There is very little that can be added to this description except for the technical details. As about the description of the clocks motion, the issue is more involved due to the importance of nongravitational effects in the case of the satellite motion¹⁰, and the irregularities of the earth's rotation, tidal effects, etc., in the case of ground stations. A complete account of the problems involved is definitely beyond the scope of this report. We want only to note here that the problem is very multifaceted, involves a large number of factors and should be taken very seriously in view of the extreme sensitivity of the global positioning precision to it.

The second problem is positioning of the user with respect to global coordinates at a moment of time chosen by this user by means of the signals from the GPS clocks (the so-called pseudorange procedure). Theoretically, the problem can be reduced to solving of geodesic triangles in curved spacetime. There are several techniques for that. We are going to discuss two of them. However, in practice the problem is complicated by atmospheric effects on propagation of radio signals¹⁰.

Upon the completion of the user positioning with respect to global coordinates, the corrections taking into account irregularities of the earth rotation should be made if one wishes to position the user with respect to ground placed objects.

The Spacetime Common Grid is an essentially general relativistic model of the Global Positioning System. As we have demonstrated above, the Spacetime Common Grid model expresses all the steps of global positioning in terms of geometric concepts in the curved spacetime of general relativity. The idea of the Spacetime Common Grid supplies, thereby, a relativistic covariant formulation of the global

positioning problems. It is very important to keep clear geometric interpretation all the way from the formulation of the problems to the stage when their final solutions are obtained. It happened too often in the course of the short history of GPS that the geometric interpretation was lost or ignored and errors were made that could be avoided.

A way to achieve a total geometric clarity throughout all the GPS is to choose carefully computational techniques used to obtain solutions. We want to mention here two of these techniques.

- (1) The formalism of the world function developed by J. L. Synge^{9,14,15}, who turned it in a geometrically transparent, extremely powerful, and versatile computational method ideally suitable for solving 4-dimensional chrono-geometric problems (the GPS problems represent one particular class of such problems).
- (2) The PPN formalism developed primarily for comparison and testing different gravitation theories and subsequently used for describing the gravitational experiments within the solar system (for example, the development of the ideas related to the "LAGEOS-3" experiment was performed entirely in the language of the PPN formalism).

Although the world function approach and the PPN formalism are very similar in some respects, they differ considerably in many other ways. Both of them can be characterized as series expansions techniques.

The world function formalism develops a sophisticated technique for an approximate calculations of a function of two points P_1, P_2 in spacetime. The value of this world function $\Omega(P_1, P_2)$ is equal to

$$\Omega(P_1, P_2) = \frac{1}{2} L_{P_1 P_2}^2 \quad (114)$$

where $L_{P_1 P_2}$ is the distance between points P_1 and P_2 in spacetime (in GPS the value of this function is frequently related to the readings of the clocks). The deriva-

tives of the world function with respect to the positions of points P_1 and P_2 provide a vector quantity presenting an extension for curved spacetime of the concept of radius vector, so useful in standard vector analysis. It also turns out that in the particular case when P_1 and P_2 lie on a null geodesic, the scalar product of the derivative of the $\Omega(P_1, P_2)$ at point, say P_2 and the 4-velocity of an observer at the point P_2 is proportional to the frequency of the light signal (propagation of which is depicted by the null geodesic $P_1 P_2$) as measured by this observer. This provides a description of the general relativistic Doppler effect in terms of the world function. The formalism of the world function defines, further, the whole sequence of the higher covariant derivatives and endows them with a transparent physical interpretation. After that, the world function formalism develops the method for approximating the world function and its covariant derivatives by covariant Taylor series expansions (the coefficients are determined in terms of covariant derivatives rather than partial derivatives) for the cases when the spacetime curvature is small, or the points P_1 and P_2 are placed close to each other, or both. Approximations can be performed in any order and all approximate expressions are manifestly generally covariant. The covariance of the theory is not violated when a local observer is introduced because it is described in the theory by its 4-vector of velocity, acceleration, etc., and by invariant parameters such as three curvatures of the observer's world line. It is only when an attempt of globalization of observations is undertaken that the covariance gets broken. The geometric interpretation of the results, however, does not get lost even then. The reason for this is that in generic cases globalization of the observations in general relativity is impossible due to the fundamental nonintegrability. It is only in cases when some additional symmetries are imposed on spacetime that globalization becomes possible. These additional symmetries can be usually described in geometric terms as a special kind of vector or tensor fields – Killing fields. The geometry of global observations includes then the geometry of these Killing fields. Special coordinate systems can be formed as con-

structions including these fields and the usual geometric devices such as geodesics, etc. In practice it is not always necessary to follow this route. It is important, however, that in the formalism of the world function, one always can recover the whole picture of globalized observations (such as global positioning) entirely in terms of the geometry of geodesics and the geometry of Killing fields related to the spacetime symmetries.

We cannot go into more detail concerning the use of the world function formalism here. A study of the world function formalism applications to GPS is a subject of a separate, although related, research which is under way now¹⁸.

The PPN formalism (Parametrized Post-Newtonian formalism) from the very beginning is heavily based on the assumption (which is very good for all the phenomena in our solar system) of weakness of gravitational field. It makes use of this assumption to obtain a "second" approximation of the general relativistic equations (considering as the first approximation the standard Newtonian theory). The formalism was developed by the efforts of Eddington¹⁹, Robertson²⁰, Schiff²¹, and, in modern form, of Will and Nordtvedt²². The PPN formalism considers the general relativistic description as an extension of the Newtonian picture. Because of that the assumptions about the gravitational field weakness and the parameters of the gravitational field sources are formulated in Newtonian terms.

More exactly, it is assumed that the system under consideration has weak gravity

$$|\Phi| = |\text{Newtonian potential}| \ll 1, \quad (115)$$

the matter that generates this gravitational field moves slowly

$$v^2 \leq |\Phi| \ll 1, \quad (116)$$

and has small stress and internal energy

$$\frac{|T_{jk}|}{\rho_0} = (\text{stress per unit baryon "mass density"}) \approx |\Phi| \ll 1 \quad (117)$$

$$\Pi = (\rho - \rho_0) = \left(\frac{\text{internal energy density per}}{\text{unit baryon "mass density"}} \right) \approx |\Phi| \ll 1 \quad (118)$$

The idea of the PPN formalism is to analyze the system by a simultaneous expansion in the small parameters $|\Phi|$, v^2 , $|T_{jk}|/\rho_0$, and Π . Such a "weak-field, slow-motion expansion" gives: (1) flat empty spacetime in "zero order"; (2) the Newtonian treatment of the system in "first order"; and (3) Post-Newtonian corrections in "second order". Such a PPN expansion can be developed not only for general relativity but for almost any metric gravitation theory. Moreover, all the expansions can be given a universal expression via introduction of parameters taking different values for different theories⁸. We are going to consider here only the PPN approximation of general relativity.

The PPN formalism assumes that the analyzed system is covered with assumes that the analyzed system is covered with coordinates $(t, x_j) = (t, x^j)$ that are as nearly globally Lorentz as possible

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \gamma_{\alpha\beta}; \quad |\gamma_{\alpha\beta}| \leq |\Phi| \ll 1 \quad (119)$$

In GPS problems

$$|\Phi| \sim \frac{M_\odot}{r} \quad (120)$$

The velocity of the coordinate system (i. e. the 4-velocity of its spatial origin) is so chosen that the system as a whole is approximately at rest in these coordinates.

For any gravitationally bound system (which is the case with GPS) the Newtonian approximation imposes limits on the values of various dimensionless physical quantities

$$\begin{aligned} \epsilon^2 &\equiv \text{maximum value of Newtonian potential } U \\ &\geq \text{values anywhere of } U, v^2, |T_{ij}|/\rho_0, \Pi \end{aligned} \quad (121)$$

where

$$U = -\Phi = \int \frac{\rho_0(x') d^3x'}{|x'_x|} \quad (122)$$

Moreover, changes of all quantities at fixed x_j are due primarily to the motion of the matter. As a result, time derivatives are small by $O(\epsilon)$ compared to space

derivatives

$$\left| \frac{\partial A / \partial t}{\partial A / \partial x_j} \right| \sim |v_j| \leq \varepsilon \quad \text{for any quantity } A \quad (123)$$

Conditions (121) suggest to expand metric coefficients in powers in the small parameter ε treating U , v^2 , $|T_{ik}|/\rho_0$, and Π as though they were all of $O(\varepsilon^2)$ and treating time derivatives as though they are $O(\varepsilon)$ smaller than space derivatives.

In such "Post-Newtonian" expansion terms odd in ε (for instance, such as

$$\int \frac{\rho_0(x', t) v_j(x', t)}{|x - x'|} d^3 x' \quad (124)$$

who's total number of v 's and $\partial/\partial t$'s is odd) change sign under time reversal, whereas terms even in ε do not.

Time reversal also changes the sign of g_{0i} , but leaves g_{00} and g_{ij} unchanged. Therefore, g_{0j} must contain only terms odd in ε ; whereas g_{00} and g_{ij} must contain only even terms.

The form of expansion through Newtonian order is obtained when one demands that

$$\begin{aligned} g_{00} &= -1 + 2U + [\text{terms} \leq \varepsilon^4] \\ g_{0j} &= [\text{terms} \ll \varepsilon^3] \\ g_{ij} &= \delta_{ij} + [\text{terms} \ll \varepsilon^2] \end{aligned} \quad (125)$$

The stated limits in the higher order corrections are dictated by demanding that the space components of the geodesic equation agree with the Newtonian equation of motion

$$\begin{aligned} \frac{d^2 x_j}{dt^2} &\approx \frac{d^2 x_j}{d\tau^2} = -\Gamma_{\alpha\beta}^j \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \approx -\Gamma_{\alpha\beta}^j \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \\ &= -\Gamma_{00}^j - 2\Gamma_{0k}^j v_k - \Gamma_{kl}^j v_k v_l \\ &= U_{,j} + \text{terms of order } \{\varepsilon g_{0k,j}; \varepsilon^2 g_{kl,j}\} \end{aligned} \quad (126)$$

One would get wrong Newtonian limit if g_{0k} were $O(\varepsilon)$ or greater, and if $g_{kl} - \delta_{kl}$ where $O(1)$ or greater.

The above pattern continues to all orders in the expansion, i. e. g_{00} always goes hand in hand with εg_{0k} and $\varepsilon^2 g_{jk}$. We are not going to write down the expression

for post-Newtonian corrections in metric coefficients. They can be found in the standard literature on the subject. We want only to note that for general relativity the PPN formalism is invariant under Lorentz transformations of the PPN coordinates (combined with some infinitesimal coordinate transformations introduced to keep the expressions for metric coefficients in their simplest form).

The PPN formalism makes the whole theory to look more like a Newtonian theory with rather complicated corrections. It introduces globalization of measurements simultaneously with series expansion. The relation of globalization to the spacetime symmetries becomes rather remote, and the geometric interpretations frequently become not so obvious, although they can be recovered.

The advantages of the PPN formalism are that the formalism is very well developed and, consequently, is a very powerful calculational tool for the conditions it was developed originally, i. e. for gravitationally bound systems with weak gravity, slowly moving sources and susceptible to globalization (its advantages are especially obvious in cases where the minor deviations from symmetries or nongravitational corrections are involved). However, the PPN formalism, as it could be seen from our brief description does much worse than the world function formalism in carrying the spirit of the idea of the Spacetime Common Grid.

XII. Recommendations.

a. The general relativistic analysis of the Doppler shift and the global synchronization problems in the Schwarzschild field demonstrates clearly the following important features:

- (1) The only experimentally measured parameters are the frequencies and the readings of the ground observer clocks and the satellite clocks (proper time);
- (2) These parameters are determined at the events (points) of 4-dimensional space-time, and not at points of the 3-dimensional proper space of a reference frame. Anything else depends on the point of view and the suggested synchronization scheme. It is a matter of interpretation and should be treated as such;
- (3) For the Schwarzschild model of the earth's gravitational field, more than one global synchronization scheme, different in the degree of complexity, can be suggested. The synchronization scheme of section IV is much simpler than the one based on the Doppler shift in section III. This simplicity is achieved by the maximal utilization of the Schwarzschild metric symmetries and by the elimination of the terms in the Doppler shift caused by time delay. The scheme is the result of careful analysis of the physical origin of each term in the Doppler shift expression;
- (4) Not every interpretation leads to a global synchronization scheme. For instance, an attempt to interpret all the observations in the frame of reference of the ground observer⁶ for the purpose of global synchronization is wrong (cf. section V). The world line of the ground observer is curved in such a way that the proper spaces of its frame, corresponding to different moments of his proper time, intersect each other, and consequently cannot be used for global synchronization, even in the special relativistic limit. It is recommended to formulate the problems in four dimensions to avoid mistakes of this kind. In relativity only events are real, and not 3-dimensional positions;

- (5) The analysis of the relativistic corrections in the ranging procedure performed in this report shows that the relativistic corrections to the rates of clocks used in modern GPS lead to the fourth order errors in ranging accumulating with time and that the commonly used procedure of initial synchronization (determining the constants of integration) causes an error of second order that does not accumulate. The standard special relativistic time - to - range conversion formula also produces the second order error. This estimate has been completed for the first time in the present report. It implies that the relativistic contribution to the GPS ranging errors can be kept within few centimeters (1-10 cm) for many years without the resynchronization of the GPS satellite clocks. The analysis was done for an arbitrary latitude of the ground observer and a non-circular orbit of the satellite with an arbitrary orbital plane inclination;
- (6) The problem of cross - linking based on the satellite - to - satellite ranging can be solved using the techniques of the sections IV-VIII of the present report. The problem is not any harder (in any respect) than the observer - satellite problem.
- (7) It should be noted that by now basic relativistic effects involved in synchronization of a pair of clocks or any single procedure of positioning are rather well understood^{16,17,23} and properly incorporated in the GPS operations. However a development of a global relativistic mathematical model of functioning the GPS as a whole is a matter of the future. The idea of Spacetime Common Grid is a first step in this direction.
- (8) The idea of the satellite-based Spacetime Common Grid provides a firm ground for a general relativistic covariant formulation of all the GPS problems, local as well as global. Mathematical description of the Spacetime Common Grid, as we have shown in section XI of this report is based on a combination of the geometry of timelike curves and null geodesics plus the geometry of Killing fields expressing the symmetries of 4-dimensional spacetime considered in the

GPS problems.

It is recommended that the use of the mathematical formulation of the Spacetime Common Grid idea be used on everyday basis in formulating and solving the GPS problems. Reformulation of the GPS problems in the language of the Spacetime Common Grid will eliminate in the future misunderstandings related to implementing relativistic corrections in the current GPS, as well as in all its possible future modifications. The development of mathematical techniques adequate for the Spacetime Common Grid description of GPS, procedures is necessary. Our comparison of the world function formalism versus the PPN formalism indicates that the world function formalism agrees better with the basic ideas of the Spacetime Common Grid. However, the very well developed PPN formalism is very powerful in the problems arising due to slight violations of spacetime symmetries. It is also very responsive with respect to implementing nongravitational corrections of all kinds. This feature is very important in the GPS practice. The world function formalism might show to be as powerful in this respect as as the PPN formalism, but it is unknown at this time. It is recommended to investigate this question. A successful outcome of this research could provide a complete theory of the Spacetime Common Grid.

We want to stress that development of the Spacetime Common Grid theory as the foundation GPS operations will not merely provide a new computational and interpretational capabilities. It will put the whole issue on a different, higher level.

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