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EXPONENTIALLY STABLE APPROXIMATIONS OF WEAKLY DAMPED WAVE EQUATIONS[1]

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ABSTRACT

We consider wave equations with damping in the boundary conditions. Techniques to ascertain the uniform preservation under approximation of exponential stability are presented. Several schemes for which preservation can be guaranteed are analyzed. Numerical results that demonstrate the lack of stability under approximation for several popular schemes (including standard finite difference and finite element schemes) are given.

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1 Introduction

In this paper, we consider approximation methods for the boundary damped normalized (i.e., wave speed $c = 1$) wave equation system

$$\partial_t^2 u(t,x) = \triangle_x u(t,x), \qquad t > 0, \quad x \in \Omega \subset R^n, \tag{1.1}$$

$$u(t,x) = 0, \qquad t > 0, \quad x \in \Gamma_0 \subset \partial\Omega, \tag{1.2}$$

$$\partial_\eta u(t,x) + \alpha\partial_t u(t,x) = 0, \qquad t > 0, \quad x \in \Gamma_1 \subset \partial\Omega, \tag{1.3}$$

where the domain Ω is an open bounded subset of R^n, Γ_0 is a relatively closed subset of the boundary $\partial\Omega$ of the domain Ω, and Γ_1 is the complementary subset of Γ_0 in the boundary $\partial\Omega$. The symbol ∂_η represents the directional derivative operator in the outward normal direction of the boundary. Since the boundary condition (1.2) is often referred to as a reflecting boundary condition, Γ_0 is called the reflecting boundary. Similarly, Γ_1 is called a partially absorbing boundary.

The system (1.1)–(1.3) arises in many important models for distributed parameter control problems. In particular, in the model of a vibrating flexible membrane, the solution $u(t,x)$ represents the transverse displacement of the membrane, and in models for acoustic pressure fields, the solution $u(t,x)$ represents the fluid pressure (see, [BKS], [BKSW], [BPS], [Li], for more detailed examples). It can be shown that for a given initial state $u(0,x) = u_0(x)$ and $u_t(0,x) = v_0(x)$, the solution of (1.1)–(1.3) decays exponentially in time and the decay rate is uniform for all initial states (u_0, v_0) in a certain function space (see [C], [L]). The stability of the solutions of (1.1)–(1.3) plays an essential role in several control theoretical issues ([BK],[BW]). In this paper, we are interested in approximation methods that uniformly preserve the exponential stability of the solutions of (1.1)–(1.3) for the approximate solutions.

The equations (1.1)–(1.3) can be written abstractly as a differential equation

$$\frac{d}{dt}w(t) = Aw(t), \quad t > 0, w(t) \in \mathcal{H},$$

in an infinite dimensional function space $\mathcal{H}$. In this context, we consider a linear control system given by

$$\frac{d}{dt}w(t) = Aw(t) + Bh(t), \quad h(t) \in R^m, \tag{1.4}$$

where h is a control input and B is a linear operator from R^m into $\mathcal{H}$. The most common approach for the approximation of a control problem involving (1.4) is to formulate a sequence of finite dimensional control systems of the form

$$\dot{w}^N(t) = A^N w^N(t) + B^N h(t), \quad t > 0, \quad w^N(t) \in \mathcal{H}^N, \tag{1.5}$$

where the dimension of the space $\mathcal{H}^N$ increases toward infinity as N tends to infinity. In general, equation (1.5) is derived from (1.4) using space discretization techniques such as finite difference, finite elements or spectral methods developed for the approximation of the solutions of (1.1)–(1.3). A control strategy is then designed for the finite dimensional control problem involving (1.5). This control is used as an approximation to the desired control function for the control problem involving (1.4) (for example, see [G], [BK], [BW]). One of the most practical conditions to assure the well-posedness of the finite dimensional control problems, as well as the convergence of the approximate controls to the desired control for the infinite-dimensional system is that the solutions of (1.5) for $h \equiv 0$ preserve the exponential decay of the solutions of (1.1)–(1.3).

From a stability analysis of the solutions of (1.1)–(1.3), it is easy to see that the energy dissipation in (1.1)–(1.3) comes exclusively from the boundary condition (1.3). Since no medium damping exists in (1.1), we refer to (1.1)–(1.3) as a weakly damped wave equation. Although many approximation techniques can provide convergent approximations for the solutions of (1.1)–(1.3) by the solutions of (1.5) with $h \equiv 0$, the nature of the dissipation in (1.1)–(1.3) makes it very difficult to preserve the uniform exponential decay of the solutions of (1.1)–(1.3). Numerical results from our investigations reveal that most of the popular discretization techniques can not maintain a uniform decay rate in the solutions of (1.5) with $h \equiv 0$ as the dimension of the approximate system (1.5) increases.

Although the preservation of the stability or the stabilizability of hyperbolic type control systems is well-known to be a delicate approximation problem, there exist, to our knowledge, only a few analytical results and these are for the approximation of "hyperbolic" delay equations (see [IK]). In this paper, we present a general approach for the analysis of uniform preservation of exponential stability of approximation systems for weakly damped wave equations. This approach is later used to show that a particular mixed finite element method and polynomial based Galerkin methods preserve a uniform exponential decay rate in the approximate solutions as the dimension of the approximate system increases to infinity.

An outline of the paper is as follows. In Section 2, we specify the class of approximation methods for the weakly damped wave equation considered in this paper. Then we present a general approach for the analysis of decay rate in the solutions of these approximate wave equations. In Section 3, the approach developed in the proceeding section is used to prove that a mixed finite element method for the 1-dimensional problem preserves uniformly an exponential decay rate in the approximate solutions. In Section 4, polynomial based Galerkin approximations of weakly damped wave equations in hypercubical domains are analyzed. These are also shown to preserve uniformly an exponential decay rate. In Section 5, several popular discretization methods are investigated numerically; a sharp distinction between the methods analyzed in the earlier sections and the other popular methods emerges from our numerical findings. In Section 6, our concluding remarks offer a perspective of the results presented here.

We finish this introduction by observing that interest in the weakly damped wave equation is also strongly motivated by the question of exact controllability of the wave equation via partial boundary control. As it was pointed out by Datko (see [D]), exact controllability or stabilizability via boundary control may be extremely sensitive to perturbations. However, we view equations (1.1)–(1.3) as the model of a physical system, where the control input is a nonhomogeneous term in (1.1) as it is formulated in ([BFS], [BKSW], [BKS], etc.). Thus, our control input does not introduce changes in the boundary condition (1.3). More discussion on the significance of our results for the preservation of boundary stabilizability of the wave equation is given in Section 6.

2 Estimation of decay rate for approximate solutions of the wave equation: A general approach

In many analytical studies of control problems, the system (1.1)-(1.3) is taken in the sense of distributions and one seeks mild or weak solutions. As a second order equation, the state space for the mild solution $(u(t), u_t(t))$ of (1.1)-(1.3) is taken to be $\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, where the Hilbert space $H^1_{\Gamma_0}(\Omega)$ is defined by

$$H^1_{\Gamma_0}(\Omega) = \{u(\cdot) \in H^1(\Omega) | u(x) = 0, x \in \Gamma_0\}.$$

For simplicity of analysis, we assume that Γ_0 has a non empty relative interior; thus an inner product in $H^1_{\Gamma_0}(\Omega)$ that is equivalent to the usual $H^1(\Omega)$ inner product can be defined by

$$< u, v >_1 = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in H^1_{\Gamma_0}(\Omega).$$

The infinitesimal generator for the mild solution semigroup of (1.1)-(1.3) is defined by

$$\mathcal{D}(A) = \left\{ \begin{array}{c} (u,v) \in \mathcal{H} | v \in H^1_{\Gamma_0}(\Omega), \triangle u \in L^2(\Omega), \\ \partial_\eta u(x) + \alpha v(x) = 0, x \in \Gamma_1 \end{array} \right\}$$

$$A : \mathcal{D}(A) \subset \mathcal{H} \mapsto \mathcal{H}, \quad A(u,v) = (v, \triangle_x u).$$

It is easy to verify that A generates a C_0-contraction semigroup $S(t)$ in $\mathcal{H}$. It is shown in [C] and [L] that there exist constants $M \geq 1, \omega > 0$ such that

$$\|S(t)\|_{L(\mathcal{H})} \leq Me^{-\omega t}, \quad \text{for } t \geq 0.$$

Alternatively and equivalently, for the initial state $(u_0, v_0) \in \mathcal{D}(A)$, the solution $(u(t), v(t)) = S(t)(u_0, v_0)$ satisfies the following variational equality,

$$\text{(2.1)} \qquad \left\{ \begin{array}{rcl} \dfrac{d}{dt} < u(t), f >_1 & = & < v(t), f >_1 \\ \dfrac{d}{dt} < v(t), g >_{L^2(\Omega)} & = & - < u(t), g >_1 - \displaystyle\int_{\Gamma_1} \alpha v(t,x) g(x) dx \end{array} \right.$$

for all $(f, g) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$. Equation (2.1) is often referred to as a weak formulation of (1.1)-(1.3) and it is a natural formulation in which to consider approximation methods.

Consider two finite dimensional spaces H_1^N and H_2^N of functions defined on Ω given by

$$H_1^N = \text{span}\{\phi_k^N\}_{k=1}^N, \quad H_2^N = \text{span}\{\psi_k^N\}_{k=1}^N.$$

We consider approximations to the solutions of (1.1)-(1.3) in the form

$$u^N(t,x) = \sum_{k=1}^{N} u_k^N(t)\phi_k^N(x), \quad v^N(t,x) = \sum_{k=1}^{N} v_k^N(t)\psi_k^N(x).$$

In most cases, H_1^N and H_2^N are chosen to be subspaces of $H^1_{\Gamma_0}(\Omega)$ and $L^2(\Omega)$, respectively. However, this condition is not essential in the analysis presented below. Let us denote the column vectors of the coefficients $u_k^N(t), v_k^N(t)$ by

$$\bar{u}^N(t) = \begin{pmatrix} u_1^N(t) \\ \vdots \\ u_N^N(t) \end{pmatrix}, \bar{v}^N(t) = \begin{pmatrix} v_1^N(t) \\ \vdots \\ v_N^N(t) \end{pmatrix}.$$

Then system (1.1)-(1.3) is approximated by an ordinary differential equation of the form

$$\begin{bmatrix} K^N & 0 \\ 0 & M^N \end{bmatrix} \begin{pmatrix} \dot{\bar{u}}^N(t) \\ \dot{\bar{v}}^N(t) \end{pmatrix} = \begin{bmatrix} 0 & K^N \\ -K^N & -B^N \end{bmatrix} \begin{pmatrix} \bar{u}^N(t) \\ \bar{v}^N(t) \end{pmatrix}. \tag{2.2}$$

The matices K^N and M^N are symmetric and positive definite by construction. The matrix B^N is symmetric nonnegative definite. It arises from the boundary integral term in (2.1). In general, mappings $q^N : H_2^N \mapsto H_1^N$ are chosen [IK]. The approximate solution $(u^N(t), v^N(t))$ is required to satisfy the following variational system which is analoguous to (2.1):

$$\begin{aligned} \frac{d}{dt} < u^N(t), \phi_k^N >_1 &= < q^N(v^N(t)), \phi_k^N >_1 \\ \frac{d}{dt} < v^N(t), \psi_j^N >_{L^2(\Omega)} &= - < u^N(t), q^N(\psi_j^N) >_1 - \int_{\Gamma_1} \alpha q^N(v^N(t)) q^N(\psi_j^N) dx. \end{aligned}$$

In particular, if the mappings q^N are defined as $q^N(\psi_j^N) = \phi_j^N, j = 1, \ldots, N$, then, the matrices K^N, M^N and B^N are defined by

$$\begin{aligned} K_{ij}^N &= \int_\Omega \nabla\phi_i^N(x) \cdot \nabla\phi_j^N(x) dx, \\ M_{ij}^N &= \int_\Omega \psi_i^N(x)\psi_j^N(x) dx, \\ B_{ij}^N &= \int_{\Gamma_1} \alpha\phi_i^N(x)\phi_j^N(x) dx. \end{aligned}$$

In the case of Galerkin type of methods, i.e., finite elements, spline based Galerkin methods, polynomial based Galerkin methods, $H_1^N = H_2^N$, and therefore, $q^N = I$ the identity operator.

We are interested in the exponential decay rate of the approximate solutions (u^N, v^N). First, we need to assume that the appropriate norm for the approximate solutions is used. In general, the convergence properties of the approximation schemes can be stated as follows: Let i^N be a mapping from $\mathcal{H}^N = H_1^N \times H_2^N$ into $\mathcal{H}$. Then (u^N, v^N) is said to converge to (u, v), if

$$\|i^N(u^N, v^N) - (u, v)\|_{\mathcal{H}} \to 0,$$

as N tends to infinity.

Definition 2.1 *(Uniform exponentially stable approximation) A given approximation method is said to preserve uniformly the exponential stability of the solutions of* (1.1)-(1.3)*, if there exist constants M and $\alpha > 0$ independent of N such that for any initial state (u_0, v_0) the corresponding aproximate solutions satisfy*

$$\|i^N(u^N(t), v^N(t))\|_{\mathcal{H}} \leq Me^{-\alpha t}\|(u_0, v_0)\|_{\mathcal{H}}, N = 1, 2, \ldots.$$

It follows from the positivity of the matrices K^N and M^N that a norm is defined on $H_1^N \times H_2^N$ by

$$\|(u^N, v^N)\|_{H_1^N \times H_2^N} =< K^N \bar{u}^N, \bar{u}^N >_{R^N} + < M^N \bar{v}^N, \bar{v}^N >_{R^N},$$

where $\bar{u}^N$ and $\bar{v}^N$ are vector representations of u^N and v^N with respect to the bases of H_1^N and H_2^N, respectively. We require the following condition to hold.

Condition 2.1 *(Norm compatibility) There exist constants c_1, c_2 independent of N such that for all $(u^N, v^N) \in H_1^N \times H_2^N$,*

$$c_1\|(u^N, v^N)\|_{H_1^N \times H_2^N} \leq \|i^N(u^N, v^N)\|_{\mathcal{H}} \leq c_2\|(u^N, v^N)\|_{H_1^N \times H_2^N}.$$

We note that in the case of Galerkin type of methods mentioned previously, the above condition holds with $c_1 = c_2 = 1$. Under the above condition, in order to establish preservation of exponential stability, it is sufficient to show that the approximate solutions $(u^N(t), v^N(t))$ have a uniform exponential decay rate under the norm $\|\cdot\|_{H_1^N \times H_2^N}$.

In the analysis of the stability of solutions of (1.1)-(1.3), a generalized Lyaponov function $Q(t)$ is used in ([C], [L]). It is defined by

$$Q(t) = \frac{t}{2}(\|(u(t), v(t))\|_{\mathcal{H}}^2) + \int_\Omega \left\{2v(t)(\ell \cdot \nabla u(t)) + \left(\sum_{i=1}^n \ell_{i,i} - 1\right) u(t)v(t)\right\} dx$$

where ℓ is a vector field defined on Ω that represents certain characteristics of the domain Ω and its boundary, $\ell_{i,j}(x) = \partial_{x_j}\ell_i(x)$, and the term $E(t) \equiv \|(u(t), v(t)\|_{\mathcal{H}}^2$ represents the energy associated with a solution $(u(t), v(t))$ at time t. The main steps of the analysis consist in establishing the following results:

(L1) There exists a constant $T_1 > 0$ such that for all $t \geq T_1$,

$$Q(t) \geq \frac{t}{4}E(t).$$

(L2) There exists a constant T_2, such that for all $t \geq T_2$, $\dot{Q}(t) \leq 0$.
(L3) For any given $T > 0$, there exists a constant M such that for all $t \leq T$, $|Q(t)| \leq ME(t)$.

From (L1)-(L3), we can find a constant C such that for all initial conditions and for all $t \geq T = \max\{T_1, T_2\}$ we have

$$E(t) \leq \frac{4ME(T)}{t} \leq C\frac{E(0)}{t}.$$

Using the semigroup property of $S(t)$, e.g., see [P, p.116], we conclude that $S(t)$ is exponentially stable as t tends to infinity.

Since (2.2) is an approximation of (1.1)-(1.3), it is natural to attempt to follow a similar idea to analyze the stability of the solutions of (2.2). Thus, we introduce a function $Q^N(t)$ defined by

$$\begin{aligned} Q^N(t) &= \frac{t}{2}(< K^N\bar{u}^N(t), \bar{u}^N(t) >_{R^N} + < M^N\bar{v}^N(t), \bar{v}^N(t) >_{R^N}) \\ &\quad + < W^N\bar{v}^N(t), \bar{u}^N(t) >_{R^N} \end{aligned}$$

where W^N is an $N \times N$ matrix. Obviously, W^N plays a role analoguous to that of the integral term in $Q(t)$ involving the vector field $\ell(\cdot)$. Then if we take $E^N(t) = \|(u^N(t), v^N(t))\|^2_{H_1^N \times H_2^N}$, by establishing (L1)-(L3) for some constants T_1, T_2, and M, independent of N, we can obtain the uniform exponential stability of the approximate solutions following an argument similar to that outlined previously for the solutions of (1.1)-(1.3). The following condition is required if (L1) and (L3) hold with constants T_1 and M independent of N.

Condition 2.2 *(Boundedness of W^N) There exist constants β_1 and β_2 independent of N such that that for all $f^N, g^N \in R^N$,*

$$| < W^N f^N, g^N >_{R^N} | \leq \beta_1 < K^N g^N, g^N >_{R^N} + \beta_2 < M^N f^N, f^N >_{R^N} .$$

Lemma 2.1 *Under Condition 2.2, there exists a constant T_1 independent of N such that for all $t \geq T_1$,*

$$Q^N(t) \geq \frac{t}{4} E^N(t).$$

Moreover, for any given $T > 0$, there exists a constant M independent of N such that for all $t \leq T$,

$$|Q^N(t)| \leq M E^N(t).$$

Proof: It is sufficient to take $T_1 = 4\max\{\beta_1, \beta_2\}$ and $M \geq T/2 + \max\{\beta_1, \beta_2\}$.

□

As it is in the case of the analysis of the decay rate for solutions of (1.1)-(1.3), condition (L2) is the most difficult to establish (see [C], [L]). In fact, by using (2.2), we can compute the derivative of Q^N as follows

$$\begin{aligned} \dot{Q}^N(t) &= \frac{1}{2}(< K^N \bar{u}^N, \bar{u}^N >_{R^N} + < M^N \bar{v}^N, \bar{v}^N >_{R^N}) \\ &\quad -t < B^N \bar{v}^N, \bar{v}^N >_{R^N} + < W^N \bar{v}^N, \bar{v}^N >_{R^N} \\ &\quad - < W^N (M^N)^{-1} (K^N \bar{u}^N + B^N \bar{v}^N), \bar{u}^N >_{R^N} . \end{aligned}$$

The above derivation involves use of the expressions

$$\dot{\bar{u}}^N(t) = \bar{v}^N(t), \quad \dot{\bar{v}}^N(t) = -(M^N)^{-1}(K^N \bar{u}^N(t) + B^N \bar{v}^N(t)).$$

obtained from (2.2). By adding and subtracting

$$(< K^N \bar{u}^N, \bar{u}^N >_{R^N} + < M^N \bar{v}^N, \bar{v}^N >_{R^N})/2,$$

we obtain

$$\begin{aligned} (2.3) \quad \dot{Q}^N(t) &= -\frac{1}{2}\left(< K^N \bar{u}^N(t), \bar{u}^N(t) >_{R^N} + < M^N \bar{v}^N(t), \bar{v}^N(t) >_{R^N}\right) \\ &\quad + < K^N \bar{u}^N(t), \bar{u}^N(t) >_{R^N} - < K^N \bar{u}^N(t), C^N \bar{u}^N(t) >_{R^N} \\ &\quad + < M^N \bar{v}^N(t), \bar{v}^N(t) >_{R^N} + < M^N \bar{v}^N(t), C^N \bar{v}^N(t) >_{R^N} \\ &\quad -t < B^N \bar{v}^N(t), \bar{v}^N(t) >_{R^N} - < B^N \bar{v}^N(t), C^N \bar{u}^N(t) >_{R^N}, \end{aligned}$$

where the matrix C^N is defined by $(C^N)^T = W^N (M^N)^{-1}$.

The following conditions can be of practical use in establishing (L2).

Condition 2.3 *There exists constants a_1, a_2, a_3, μ, and δ, with $(\mu + \delta) < \frac{1}{2}$, each independent of N such that*

(S1) For any $f^N \in R^N$,

$$< M^N f^N, f^N >_{R^N} + < M^N f^N, C^N f^N >_{R^N} \leq (a_1 + \frac{a_2}{\mu}) < B^N f^N, f^N >_{R^N} + \mu < M^N f^N, f^N >_{R^N};$$

(S2) For any $f^N, g^N \in R^N$,

$$< K^N g^N, g^N >_{R^N} - < K^N g^N, C^N g^N >_{R^N} - < B^N f^N, C^N g^N >_{R^N} \leq (\delta + \mu) < K^N g^N, g^N >_{R^N} + \frac{a_3}{\mu} < B^N f^N, f^N >_{R^N} .$$

We note that a necessary (but not suffic[illegible]) condition that (S2) hold (take $(f^N = 0)$ is

(S3) For any $g^N \in R^N$,

$$< K^N g^N, g^N >_{R^N} - < K^N g^N, C^N g^N >_{R^N} \leq \delta < K^N g^N, g^N >_{R^N} .$$

If a matrix W^N can be found such that Condition 2.3 holds, we can show that (L2) holds immediately.

Lemma 2.2 *Under Condition 2.3 there exists a constant T_2 independent of N such that $\dot{Q}^N(t) \leq 0$ for all $t \geq T_2$.*

Proof: Let μ be chosen such that $\delta + \mu \leq 1/2$, then by taking $T_2 = a_1 + a_2/\mu + a_3/\mu$, using (2.3), we have $\dot{Q}^N(t) \leq 0$ for all $t \geq T_2$.

□

In order to obtain a uniform exponential decay rate for the approximate solution semigroups $S^N(\cdot)$, where $(u^N(t), v^N(t)) = S^N(t)(u^N(0), v^N(0))$, we also need the following Lax-stability condition for the approximation scheme.

Condition 2.4 *There exist constants $C \geq 1$ and $\omega > 0$ independent of N such that*

$$\|S^N(t)\|_{L(H_1^N \times H_2^N)} \leq C e^{\omega t}$$

for all $t \geq 0$.

The above condition is automatically satisfied if the approximation is convergent. Finally, we state the main result of this section.

Theorem 2.1 *Suppose that a matrix W^N can be found for each N such that Conditions 2.2-2.4 hold. Then, there exists constants $L \geq 1$ and $\alpha > 0$ such that*

$$\|S^N(t)\|_{L(H_1^N \times H_2^N)} \leq Le^{-\alpha t}$$

for all $t \geq 0$ and for all N.

Proof: By Lemma 2.1 and 2.2, we can find constants T and M independent of N such that for all $t \geq T$, the following statements hold.

(a) $Q^N(t) \geq tE^N(t)/4$.
(b) $\dot{Q}^N(t) \leq 0$.
(c) $Q^N(T) \leq ME^N(T)$.

By Condition 2.4 and (c), we obtain

$$Q^N(T) \leq ME^N(T) \leq C^2 Me^{2\omega T} E^N(0).$$

Combining the above inequality with (a) and (b), we obtain

$$E^N(t) \leq \frac{4}{t} C^2 Me^{2\omega T} E^N(0).$$

If we let $\lambda^2 = 4C^2 Me^{2\omega T}$, the above inequality can be rewritten as

$$\|S^N(t)(u^N(0), v^N(0))\|^2_{H_1^N \times H_2^N} \leq \frac{\lambda^2}{t} \|(u^N(0), v^N(0))\|^2_{H_1^N \times H_2^N},$$

for all $(u^N(0), v^N(0)) \in H_1^N \times H_2^N$ and $t \geq T$. As a consequence, for $\tau = 4\lambda^2$, we have

$$\|S^N(\tau)\|_{L(H_1^N \times H_2^N)} \leq \frac{1}{2},$$

for all N. Using the semigroup property of $S^N(\cdot)$ and Condition 2.4, we obtain the inequality

$$\|S^N(t)\|_{L(H_1^N \times H_2^N)} \leq L \cdot e^{-\alpha t}$$

by taking $\alpha = \ln 2/4\lambda^2$ and L is a constant independent of N.

□

It is clear now that the difficult part of the analysis for a given approximation is to estabish Conditions 2.1–2.4. In the subsequence sections, we will specify the required matrix W^N for two different types of approximation methods and verify that Conditions 2.1–2.4 hold.

3 Mixed finite element methods

In the most common implementations of finite element methods, the two components of the mild solutions of (1.1)-(1.3) are approximated by functions $u^N(t,x)$ and $v^N(t,x)$ with the same smoothness in spatial variable x, or more precisely, the approximation spaces H_1^N and H_2^N are often chosen to be identical. However, both the analysis for the existence of the solutions of (1.1)-(1.3) and nature of the wave propagation suggest that u and v have different smoothness in x. The methods analyzed here use different approximation spaces H_1^N and H_2^N which give different smoothness in x to the functions u^N and v^N. The term "mixed finite element method" is used here to indicate that two different types of approximation elements are involved in these schemes.

We first consider the case of a one-dimensional wave equation. Let $\Omega = (0,1)$, so that the equations (1.1)-(1.3) can be written for a typical set of boundary conditions (i.e. $\Gamma_0 = \{0\}, \Gamma_1 = \{1\}$)

$$\frac{\partial^2}{\partial t^2}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x), \quad t>0, x\in(0,1), \tag{3.1}$$

$$u(t,0) = 0, \quad t>0, \tag{3.2}$$

$$\frac{\partial}{\partial x}u(t,1) = -\alpha\frac{\partial}{\partial t}u(t,1), \quad t>0. \tag{3.3}$$

The following approximation method has been proposed by Ito and Kappel in [IK]. The basis elements ϕ_k^N and ψ_k^N for spaces H_1^N and H_2^N, respectively are defined by

$$\phi_k^N(x) = \begin{cases} 1 - N|x - x_k|, & x \in [x_{k-1}, x_{k+1}] \cap [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\psi_k^N(x) = \begin{cases} 1, & x \in [x_{k-1}, x_{k+1}] \cap [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

where $k = 1, \cdots, N$ and $x_k = k/N$. It is easy to see that $\{\phi_k^N\}_{k=1}^N$ forms a basis for the set of continuous piecewise linear polynomials (i.e. linear splines) on $[0,1]$ corresponding to the uniform mesh $\{k/N\}_{k=0}^N$ with function value equal to zero at $x=0$. The ψ_k^N's are piecewise constant functions with the same support as the corresponding ϕ_k^N's. Therefore, $H_1^N \subset H_L^1(\Omega) = \{\phi \in H^1(0,1) | \phi(0) = 0\}$ and $H_2^N = \text{span}\{\psi_k^N\}_{k=1}^N \subset L^2(\Omega)$. The matrices M^N, K^N, and B^N of the previous section are defined by

$$M_{ij}^N = \int_0^1 \psi_i^N(x)\psi_j^N(x)dx,$$

$$
\begin{aligned}
K_{ij}^N &= \int_0^1 \frac{d}{dx}\phi_i^N(x)\frac{d}{dx}\phi_j^N(x)dx, \\
B_{ij}^N &= \alpha\phi_i^N(1)\phi_j^N(1).
\end{aligned}
$$

Before choosing the matrix W^N, we first establish the following useful equalities.

Lemma 3.1 *For any $f^N \in R^N$, the following equalities hold.*

(i) *Let $g^N = M^N f^N$, then the component g_k^N is given by*

$$
g_k^N = \begin{cases} (2f_1^N + f_2^N)/N, & k = 1 \\ (f_{k-1}^N + 2f_k^N + f_{k+1}^N)/N, & k = 2, \ldots, N-1, \\ (f_N^N + f_{N-1}^N)/N, & k = N. \end{cases}
$$

(ii) *Let $g^N = K^N f^N$, then the component g_k^N is given by*

$$
g_k^N = \begin{cases} N(2f_1^N - f_2^N), & k = 1 \\ N(-f_{k-1}^N + 2f_k^N - f_{k+1}^N), & k = 2, \ldots, N-1, \\ N(f_N^N - f_{N-1}^N), & k = N. \end{cases}
$$

(iii) *For any $f^N \in R^N$, we have*

$$
< M^N f^N, f^N >_{R^N} = \frac{1}{N}(f_1^N)^2 + \frac{1}{N}\sum_{k=2}^{N}(f_k^N + f_{k-1}^N)^2.
$$

(iv) *For any $f^N \in R^N$, we have*

$$
< K^N f^N, f^N >_{R^N} = N(f_1^N)^2 + N\sum_{k=2}^{N}(f_k^N - f_{k-1}^N)^2.
$$

Proof: Equalities (i) and (ii) follow directly from the definitions of the matrices M^N and K^N, respectively. The following equalities are used frequently in this section, and in particular, are useful in establishing (iii) and (iv). Let a_k, b_k be given numbers for $k = 1, \ldots, N$. Then

$$
a_2 b_1 + a_N b_N + \sum_{k=2}^{N-1}(a_k + a_{k+1})b_k = \sum_{k=2}^{N} a_k(b_k + b_{k-1}), \tag{3.4}
$$

$$
-a_2 b_1 + a_N b_N + \sum_{k=2}^{n-1}(a_k - a_{k+1})b_k = \sum_{k=2}^{N} a_k(b_k - b_{k-1}). \tag{3.5}
$$

From (i), we have

$$\begin{aligned}< M^N f^N, f^N >_{R^N} &= \frac{1}{N}(2f_1^N + f_2^N)f_1^N + \frac{1}{N}(f_N^N + f_{N-1}^N)f_N^N \\ &+\frac{1}{N}\sum_{k=2}^{N-1}((f_k^N + f_{k-1}^N) + (f_{k+1}^N + f_k^N))f_k^N.\end{aligned}$$

By taking $a_k = f_k^N + f_{k-1}^N, b_k = f_k^N$, and applying (3.4), we obtain

$$< M^N f^N, f^N >_{R^N} = \frac{1}{N}(f_1^N)^2 + \frac{1}{N}\sum_{k=2}^{N}(f_k^N + f_{k-1}^N)^2.$$

From (ii), we have

$$\begin{aligned}< K^N f^N, f^N >_{R^N} &= N(2f_1^N - f_2^N)f_1^N + N(f_N^N - f_{N-1}^N)f_N^N \\ &+N\sum_{k=2}^{N-1}((f_k^N - f_{k-1}^N) - (f_{k+1}^N - f_k^N))f_k^N.\end{aligned}$$

By taking $a_k = f_k^N - f_{k-1}^N$, and $b_k = f_k^N$, and applying (3.5), we obtain

$$< K^N f^N, f^N >_{R^N} = N(f_1^N)^2 + N\sum_{k=2}^{N}(f_k^N - f_{k-1}^N)^2.$$

□

Now we can define a matrix C^N (and hence the matrix $W^N = (C^N)^T M^N$) as follows. For any vector $f^N \in R^N$, the vector $g^N = C^N f^N$ is given by

$$g_k^N = \begin{cases} f_2^N, & k = 1, \\ k(f_{k+1}^N - f_{k-1}^N), & k = 2, \ldots, N-1, \\ 2N(f_N^N - f_{N-1}^N), & k = N. \end{cases}$$

The definition of C^N can be seen as a discrete approximation of the term in the multiplier $Q(t)$ involving the vector field ℓ. In fact, in our case, we can take $\ell(x) = 2x$. In fact, for any function $f^N(x) = \sum_{k=1}^N f_k^N \phi_k^N(x) \in H_1^N$, we approximate $2\ell(x)df^N(x)/dx$ by

$$e^N(x) \equiv \psi^N(x)^T \cdot C^N f^N.$$

where $\psi^N(x)^T$ is a row vector valued function given by $\psi^N(x)^T = (\psi_1^N(x), \ldots, \psi_N^N(x))$. One can verify from the definition of C^N, ψ_k^N and ϕ_k^N, that on each sub-interval $[x_{k-1}, x_k]$, for $k = 3, \ldots, N-1$, we have

$$e^N(x) = 2\left(x_{k-1}\frac{f^N(x_k) - f^N(x_{k-2})}{x_k - x_{k-2}} + x_k\frac{f^N(x_{k+1}) - f^N(x_{k-1})}{x_{k+1} - x_{k-1}}\right),$$

which can be regarded as a discrete approximation to the first integral in the definition of $Q(t)$.

Thus, for any $v^N \in H_2^N$ with

$$v^N(x) = \sum_{k=1}^{N} v_k^N \psi_k^N(x),$$

and e^N as defined above, the following equality holds

$$\int_0^1 e^N(x) v^N(x) dx = < W^N v^N, f^N >_{R^N} .$$

The Lax-stability for this approximation scheme is given by the convergence analysis in [IK], and the norm compatibility condition holds with $c_1 = c_2 = 1$. We need to establish Conditions 2.2 and 2.3.

Lemma 3.2 *For any vector $f^N, g^N \in R^N$, the following inequality holds*

$$| < M^N f^N, C^N g^N >_{R^N} | \leq 2 < M^N f^N, f^N >_{R^N} + 2 < K^N g^N, g^N >_{R^N} .$$

Proof: By definition of the matrices M^N and C^N, we have

$$\begin{aligned} < M^N f^N, C^N g^N >_{R^N} = & \frac{1}{N}(2f_1^N + f_2^N) g_2^N + 2(f_N^N + f_{N-1}^N)(g_N^N - g_{N-1}^N) \\ & + \frac{1}{N} \sum_{k=2}^{N-1} k \left((f_{k+1}^N + f_k^N) + (f_k^N + f_{k-1}^N) \right) \left((g_{k+1}^N - g_k^N) + (g_k^N - g_{k-1}^N) \right) . \end{aligned}$$

Using the inequality $|ab| \leq (a^2/N + Nb^2)/2$, we obtain

$$|(a_1 + a_2)(b_1 + b_2)| \leq \frac{1}{N}(a_1^2 + a_2^2) + N(b_1^2 + b_2^2). \tag{3.6}$$

By (3.6), we obtain

$$\begin{aligned} | < M^N f^N, C^N g^N >_{R^N} | \leq & \frac{1}{N} \left((f_1^N)^2 + (f_1^N + f_2^N)^2 + (f_N^N + f_{N-1}^N)^2 \right) \\ & + \frac{1}{N} \sum_{k=2}^{N-1} \left((f_{k+1}^N + f_k^N)^2 + (f_k^N + f_{k-1}^N)^2 \right) \\ & + N \left((g_1^N)^2 + (g_1^N - g_2^N)^2 + (g_N^N - g_{N-1}^N)^2 \right) \\ & + N \sum_{k=2}^{N-1} \left((g_{k+1}^N - g_k^N)^2 + (g_k^N - g_{k-1}^N)^2 \right) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{2}{N}\left((f_1^N)^2+\sum_{k=2}^{N}(f_k^N+f_{k-1}^N)^2\right)\\
&\quad +2N\left((g_1^N)^2+\sum_{k=2}^{N}(g_k^N-g_{k-1}^N)^2\right)
\end{aligned}
$$

□

Lemma 3.3 *For any $f^N \in R^N$, the following equalities hold.*

$$
\begin{aligned}
(3.7)\qquad &< M^N f^N, C^N f^N >_{R^N} = -< M^N f^N, f^N >_{R^N}\\
&\qquad +4(f_N^N+f_{N-1}^N)f_N^N-(f_N^N+f_{N-1}^N)^2,\\
(3.8)\qquad &< K^N f^N, C^N f^N >_{R^N} = < K^N f^N, f^N >_{R^N} +N^2(f_N^N-f_{N-1}^N)^2.
\end{aligned}
$$

Proof: First consider (3.7) and observe, as in the previous proof, that we have

$$
\begin{aligned}
&< M^N f^N, C^N f^N >_{R^N} = \frac{1}{N}(2f_1^N+f_2^N)f_2^N+2(f_N^N+f_{N-1}^N)(f_N^N-f_{N-1}^N)\\
&\quad +\frac{1}{N}\sum_{k=2}^{N-1}k\left((f_{k+1}^N+f_k^N)+(f_k^N+f_{k-1}^N)\right)\left((f_{k+1}^N+f_k^N)-(f_k^N+f_{k-1}^N)\right).
\end{aligned}
$$

The sum $\sum_{k=2}^{N-1}$ on the right hand side can be written as

$$
\begin{aligned}
&\frac{1}{N}\sum_{k=2}^{N-1}\left[(k+1)(f_{k+1}^N+f_k^N)^2-k(f_k^N+f_{k-1}^N)^2-(f_{k+1}^N+f_k^N)^2\right]\\
&= (f_N^N+f_{N-1}^N)^2-\frac{2}{N}(f_2^N+f_1^N)^2-\frac{1}{N}\sum_{k=3}^{N}(f_k^N+f_{k-1}^N)^2.
\end{aligned}
$$

Therefore, we obtain

$$
< M^N f^N, C^N f^N >_{R^N} = -\frac{1}{N}(f_1^N)^2-\frac{1}{N}\sum_{k=2}^{N}(f_k^N+f_{k-1}^N)^2+4(f_N^N+f_{N-1}^N)f_N^N-(f_N^N+f_{N-1}^N)^2.
$$

Thus using Lemma 3.1 (iii), we find

$$
< M^N f^N, C^N f^N >_{R^N} = -< M^N f^N, f^N >_{R^N} +4(f_N^N+f_{N-1}^N)f_N^N-(f_N^N+f_{N-1}^N)^2.
$$

Next considering (3.8), we have

$$
\begin{aligned}
&< K^N f^N, C^N f^N >_{R^N} = N(2f_1^N-f_2^N)f_2^N+2N^2(f_N^N-f_{N-1}^N)^2\\
&\quad +N\sum_{k=2}^{N-1}k\left((f_k^N-f_{k-1}^N)-(f_{k+1}^N-f_k^N)\right)\left((f_k^N-f_{k-1}^N)+(f_{k+1}^N-f_k^N)\right).
\end{aligned}
$$

In a manner similar to that above, the sum $\sum_{k=2}^{N-1}$ can be rewritten as

$$N\sum_{k=2}^{N-1}\left[(k-1)(f_k^N-f_{k-1}^N)^2-k(f_{k+1}^N-f_k^N)^2+(f_k^N-f_{k-1}^N)^2\right]$$
$$=-N(N-1)(f_N^N-f_{N-1}^N)^2+N(f_2^N-f_1^N)^2+N\sum_{k=2}^{N-1}(f_k^N-f_{k-1}^N)^2.$$

Using the above equality, we have

$$< K^N f^N, C^N f^N >_{R^N}= N\left((f_1^N)^2+\sum_{k=2}^{N}(f_k^N-f_{k-1}^N)^2\right)+N^2(f_N^N-f_{N-1}^N)^2.$$

Using Lemma 3.1, we obtain (3.8).

□

By the definition of the matrix B^N we have for any $f^N, g^N \in R^N$,

$$< B^N f^N, f^N >_{R^N}= \alpha(f_N^N)^2, \quad < B^N f^N, C^N g^N >_{R^N}= 2\alpha N f_N^N(g_N^N-g_{N-1}^N).$$

Combining the results in Lemma 3.3, we can obtain Condition 2.3.

Lemma 3.4 *Condition 2.3 holds.*

Proof: Consider

$$\begin{aligned} < M^N f^N, f^N >_{R^N} + < M^N f^N, C^N f^N >_{R^N} &= -(f_N^N+f_{N-1}^N)^2+4f_N^N(f_N^N+f_{N-1}^N) \\ &\leq 4(f_N^N)^2=\frac{4}{\alpha} < B^N f^N, f^N >_{R^N} . \end{aligned}$$

Also, we have

$$\begin{aligned} & < K^N g^N, g^N >_{R^N} - < K^N g^N, C^N g^N >_{R^N} - < B^N f^N, C^N g^N >_{R^N} \\ & = -N^2(g_N^N-g_{N-1}^N)^2-2\alpha N f_N^N(g_N^N-g_{N-1}^N) \\ & \leq \alpha^2(f_N^N)^2=\alpha < B^N f^N, f^N >_{R^N} . \end{aligned}$$

□

We can thus invoke Theorem 2.1 to conclude that the method described above preserves uniformly exponential stability of the solutions of equations (3.1) - (3.3).

In the case of a two-dimensional wave equation on a rectangular domain $\Omega = \{(x,y) : 0 < x < 1, 0 < y < 1\}$, we may consider the following special case of (1.1)-(1.3):

$$
\begin{aligned}
(3.9) \qquad \frac{\partial^2}{\partial t^2} u(t,x,y) &= \frac{\partial^2}{\partial x^2} u(t,x,y) + \frac{\partial^2}{\partial y^2} u(t,x,y), \quad (x,y) \in \Omega, t > 0, \\
(3.10) \qquad u(t,x,y) &= 0, \quad (x,y) \in \Gamma_0, \quad t > 0, \\
(3.11) \qquad \partial_\eta u(t,x,y) &= -\alpha \frac{\partial}{\partial t} u(t,x,y), \quad (x,y) \in \Gamma_1, t > 0,
\end{aligned}
$$

where $\Gamma_0 = \{(x,y) \in \bar{\Omega}, x = 0, \text{ or } y = 0\}$ and $\Gamma_1 = \Omega / \Gamma_0$. The mixed finite element method is defined as follows. For each given integer N, let (x_i, y_j) be the grid points defined by $x_i = i/N, y_j = j/N$, for $i,j = 0,1,\ldots,N$. For each pair of indices (i,j) with $1 \le i,j \le N$, a neighborhood of the grid point (x_i, y_j) is defined as

$$D_{ij}^N = \{(x,y) \in \Omega : |x - x_i| \le \frac{1}{N}, |y - y_j| \le \frac{1}{N}, |x - y - x_i + y_j| \le \frac{1}{N}\}.$$

Two families of functions are defined by

$$
\begin{aligned}
\phi_{ij}^N(x,y) &= \begin{cases} 1 - N \max\{|x - x_i|, |y - y_j|, |x - y - x_i - y_j|\}, & (x,y) \in D_{ij}^N, \\ 0, & \text{otherwise}, \end{cases} \\
\psi_{ij}^N(x,y) &= \begin{cases} \gamma_{ij}, & (x,y) \in D_{ij}^N, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}
$$

where the constants γ_{ij} are chosen such that we have the equality

$$\int_\Omega \phi_{ij}^N(x,y)\,dxdy = \int_\Omega \psi_{ij}^N(x,y)\,dxdy.$$

We then choose $H_1^N = \text{span}\{\phi_{ij}^N\}_{i,j=1}^N$, $H_2^N = \text{span}\{\psi_{ij}^N\}_{i,j=1}^N$. We prefer, for simplicity of presentation, to denote an element of an $N^2 \times N^2$ matrix A^N by $A_{(ij),(kl)}^N$. Using this notation, the matrices M^N, K^N, and B^N are defined by

$$
\begin{aligned}
M_{(ij)(kl)}^N &= \int_\Omega \psi_{ij}^N(x,y) \psi_{kl}^N(x,y)\,dxdy, \\
K_{(ij),(kl)}^N &= \int_\Omega \nabla \phi_{ij}^N(x,y) \cdot \nabla \phi_{kl}^N(x,y)\,dxdy, \\
B_{(ij),(kl)}^N &= \int_{\Gamma_1} \alpha \phi_{ij}^N(x,y) \phi_{kl}^N(x,y)\,d\Gamma_1.
\end{aligned}
$$

Although our numerical results indicate that this method preserves uniformly exponential stability of the solutions of (3.9)-(3.11), we are, to date, unable to confirm it analytically. We have attempted to use arguments analogous to the one-dimensional case, but the number of terms to compute and the associated tedium quickly become overwhelming.

4 Polynomial based Galerkin approximation

In this section, we present a stability analysis for a general class of Galerkin type approximations of the wave equation in R^n. Several technical assumptions on the domain Ω and the approximation spaces H_1^N, H_2^N are needed for exponential stability of the approximate solutions. In particular, hypercubic domains in R^n with all reflecting sides sharing at least one common corner satisfy these conditions. The requirements on the approximation spaces can be satisfied if we take $H_1^N = H_2^N = H^N$, and H^N is a carefully defined subspace spanned by the tensor products of polynomial functions.

Since the arguments developed here can also be useful for the analysis of other approximation methods, the assumptions are made in a general form. We will indicate in each case how polynomial based Galerkin methods satisfy these assumptions.

Now consider the wave equation given by (1.1)-(1.3) where we make the following assumptions on the domain Ω.

Assumption 4.1 *There exists a constant $r > 0$ such that for any $\epsilon > 0$, there exists a vector field $\ell(\cdot) \in C^4(\bar{\Omega}; R^n)$, with the properties:*

(V1) The matrix $L(x)$ defined by

$$L_{ij}(x) = \frac{1}{2}\left(\frac{\partial \ell_i(x)}{\partial x_j} + \frac{\partial \ell_j(x)}{\partial x_i}\right) = \frac{1}{2}(\ell_{i,j}(x) + \ell_{j,i}(x))$$

satisfies $L(x) - I \geq 0$ on the domain Ω;

(V2) For $x \in \partial\Omega$, the outward normal unit vector $\eta(x)$ satisfies

$$\ell(x) \cdot \eta(x) \leq 0, \quad \textit{for } x \in \Gamma_0, \quad \ell(x) \cdot \eta(x) \geq r, \quad \textit{for } x \in \Gamma_1;$$

(V3) The following inequalities hold

$$|\sum_{i,j=1}^{n} \ell_{i,ij}| \leq \epsilon, \quad |\sum_{i,j=1}^{n} \ell_{i,ijj}| \leq \epsilon.$$

where

$$\ell_{i,ij} = \frac{\partial^2}{\partial x_i \partial x_j}\ell_i(x), \quad \ell_{i,ijj} = \frac{\partial^3}{\partial x_i \partial^2 x_j}\ell_i(x).$$

We note that (V1)-(V3) are exactly the same conditions as these required in [C] where the author treats the exponential decay of the solution of the wave equation. For a given domain Ω, if there exists a point $x_0 \in R^n$ such that $(x - x_0) \cdot \eta(x) \leq 0$ on Γ_0 and $(x - x_0) \cdot \eta(x) \geq r$, on Γ_1, by taking $\ell(x) = x - x_0$, the conditions (V1)-(V3) are clearly satisfied. In particular, consider the hypercube Ω given by $\Omega = \{x \in R^n, 0 < x_0 < 1\}$; if Γ_0 is the union of a collection of sides that share at least one common point x_0, then one can verify that $\ell(x) = x - x_0$ satisfies (V1)-(V3). In fact, in the case of a hypercube, if Γ_0 is the union of a collection of sides without any common point, it can be shown that the solutions of the partial differential equation (1.1)-(1.3) are not exponentially stable.

Now consider subspaces $H_1^N = H_2^N = H^N \subset H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ given by $H^N = \text{span}\{\phi_k^N\}_{k=1}^N$, where the ϕ_k^N are a general family of approximation elements. The matrices M^N, K^N, B^N of Section 2 are defined by

$$
\begin{aligned}
M_{ij}^N &= \int_\Omega \phi_i^N(x)\phi_j^N(x)dx, \\
K_{ij}^N &= \int_\Omega \nabla\phi_i^N(x) \cdot \nabla\phi_j^N(x)dx \\
B_{ij}^N &= \int_{\Gamma_1} \alpha\phi_i^N(x)\phi_j^N(x)dx.
\end{aligned}
$$

Before we introduce the multiplier matrix C^N, we present a few useful special properties of the matrices M^N, K^N, B^N.

Lemma 4.1 *Consider a sequence $\{\theta_k^N(\cdot)\}_{k=1}^N$ of functions in H^N and let V^N be the matrix defined by*

$$V_{ij}^N = \int_\Omega \phi_i^N(x)\theta_j^N(x)dx.$$

Then, the matrix $D^N \equiv (V^N)^T(M^N)^{-1}B^N$ can also be given by

$$D_{ij}^N = \int_{\Gamma_1} \alpha\theta_i^N(x)\phi_j^N(x)dx.$$

Proof: Since the $\theta_k^N(\cdot)$'s are elements of H^N, there exist constants θ_{jk}^N such that

$$\theta_k^N(x) = \sum_{j=1}^N \phi_j^N(x)\theta_{jk}^N.$$

Let Θ^N be the matrix with entries θ_{ij}^N; then we have $V^N = M^N\Theta^N$. In fact,

$$V_{ij}^N = \sum_{l=1}^N \left(\int_\Omega \phi_i^N(x)\phi_l^N(x)dx\right)\theta_{lj}^N = \sum_{l=1}^N M_{il}^N\theta_{lj}^N.$$

Therefore, we obtain $D^N = (\Theta^N)^T B^N$ and hence

$$D_{ij}^N = \sum_{l=1}^{N} \theta_{li}^N B_{lj} = \int_{\Gamma_1} \sum_{l=1}^{N} \alpha\phi_l^N(x)\phi_j^N(x)\theta_{li}^N dx = \int_{\Gamma_1} \alpha\theta_i^N(x)\phi_j^N(x)dx.$$

$\square$

Lemma 4.2 *Let V^N be the same matrix as defined above. Then, the matrix $F^N \equiv (V^N)^T(M^N)^{-1}K^N$ is also given by*

$$F_{ij}^N = \int_{\Gamma_1} \theta_i^N(x)\frac{\partial}{\partial\eta}\phi_j^N(x)dx - \int_\Omega \theta_i^N(x)\triangle\phi_j^N(x)dx.$$

Proof: We only need to consider $F^N = (\Theta^N)^T K^N$ where the matrix Θ^N is the same as before. We have

$$\begin{aligned} F_{ij}^N &= \int_\Omega \sum_{l=1}^{N} \theta_{li}\nabla\phi_l^N \cdot \nabla\phi_j^N(x)dx \\ &= \int_{\Gamma_1} \sum_{l=1}^{N} \theta_{li}\phi_l^N(x)\frac{\partial}{\partial\eta}\phi_j^N(x)dx - \int_\Omega \sum_{l=1}^{N} \theta_{li}\phi_l^N(x)\triangle\phi_j^N(x)dx. \end{aligned}$$

Therefore, we obtain

$$F_{ij}^N = \int_{\Gamma_1} \theta_i^N(x)\frac{\partial}{\partial\eta}\phi_j^N(x)dx - \int_\Omega \theta_i^N(x)\cdot\triangle\phi_j^N(x)dx.$$

$\square$

Now, by imitating the multiplier used in Section 2 for $Q(t)$, we define the matrix $W^N = (C^N)^T M^N$ as

$$W_{ij}^N = \int_\Omega \{2(\ell^N \cdot \nabla\phi_i^N)\phi_j^N(x) + (\sum_{k=1}^{n} \ell_{k,k}^N - 1)\phi_i^N(x)\phi_j^N(x)\}dx$$

where $\ell^N(\cdot)$ is a vector field which satisfies Assumption 4.1. In order to be able to use the previous matrix equalities, we need further technical assumptions on ℓ^N.

Assumption 4.2 *There exist constants $r > 0$ and $M > 0$ independent of N such that there exists a vector field $\ell^N(\cdot) \in C^4(\bar{\Omega}; R^n)$ which satisfies Assumption 4.1 and furthermore*

$$2\ell^N \cdot \nabla\phi_k^N + \sum_{i=1}^{n} \ell_{i,i}^N\phi_k^N \in H^N, \quad \textit{with} \quad \|\ell^N(x)\|_{R^n} \le M, \quad |\sum_{i=1}^{n} \ell_{i,i}^N(x)| \le M,$$

for all $x \in \bar{\Omega}$ and $k = 1, \cdots, N$.

The above assumption is satisfied if there exists $x_0 \in R^n$ such that $\ell^N(x) = x - x_0$ satisfies Assumption 4.1 and if H^N is defined by

$$H^N = \left\{ \begin{array}{l} f(x_1, \ldots, x_n) = \prod_{k=1}^{n} p_k(x_k) \in H^1_{\Gamma_0}(\Omega), \text{ where} \\ p_k(\cdot) \text{ are polynomials of degree less than or equal to } m_k. \end{array} \right\}.$$

In the case of a hypercube, one can easily construct H^N by taking $p_k \in P_{m_k}(0,1)$, where $P_{m_k}(0,1)$ are polynomials of degree less than equal to m_k which satisfy the appropriate Dirichlet boundary condition.

The following lemma will establish Condition 2.2.

Lemma 4.3 *Under Assumption 4.1, there exists constants c_1, c_2 depending only on the domain Ω such that*

$$\begin{aligned} | < M^N f^N, C^N g^N >_{R^N} | &= | < W^N f^N, g^N >_{R^N} | \\ &\leq c_1 < K^N g^N, g^N >_{R^N} + c_2 < M^N f^N, f^N >_{R^N}, \end{aligned}$$

for all $f^N, g^N \in R^N$.

Proof: Let us associate each vector $f^N \in R^N$ with an element $f^N(\cdot) \in H^N$ given by $f^N(x) = \sum_{k=1}^N f_k^N \phi_k^N(x)$. It is easy to verify that for any vectors $f^N, g^N \in R^N$

$$\begin{aligned} < K^N g^N, g^N >_{R^N} &= \int_\Omega \|\nabla g^N(x)\|^2_{R^n} dx = \|g^N(\cdot)\|_1^2, \\ < M^N f^N, f^N >_{R^N} &= \|f^N(\cdot)\|^2_{L^2(\Omega)}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &< M^N f^N, C^N g^N >_{R^N} = < W^N f^N, g^N >_{R^N} \\ &= \int_\Omega \left[2 f^N(x) \left(\ell^N(x) \cdot \nabla g^N(x) \right) + \left(\sum_{i=1}^n \ell_{ii}^N(x) - 1 \right) f^N(x) g^N(x) \right] dx. \end{aligned}$$

From standard results, there exists a constant c depending only on Ω such that for all $u \in H^1_{\Gamma_0}(\Omega)$, $\|u\|^2_{L^2(\Omega)} \leq c\|u\|_1^2$. By Assumption 4.2, we obtain

$$\begin{aligned} | < M^N f^N, C^N g^N >_{R^N} | &\leq \frac{3}{2} \|f^N(\cdot)\|^2_{L^2(\Omega)} \\ &\quad + \left(\frac{c}{2} \left\| \sum_{i=1}^n \ell_{i,i}^N(\cdot) - 1 \right\|_{L^\infty(\Omega)} + n\|\ell^N(\cdot)\|_{L^\infty(\Omega)} \right) \|g^N(\cdot)\|_1^2 \\ &= c_2 < M^N f^N, f^N >_{R^N} + c_1 < K^N g^N, g^N >_{R^N}, \end{aligned}$$

where $c_2 = 3/2, c_1 = n\|\ell^N(\cdot)\|_{L^\infty(\Omega)} + c/2 \| \sum_{i=1}^n \ell_{i,i}^N(\cdot) - 1\|_{L^\infty(\Omega)}$. □

The following two lemmas are useful in establishing Condition 2.3.

Lemma 4.4 *Let Assumption 4.2 hold. For any* $f^N \in R^N$*, let* $f^N(\cdot)$ *be the function in* H^N *given by* $f^N(x) = \sum_{k=1}^N f_k^N \phi_k^N(x)$*. Then*

$$\begin{aligned} < M^N f^N, C^N f^N >_{R^N} &= < W^N f^N, f^N >_{R^N} \\ &= - < M^N f^N, f^N >_{R^N} + \int_{\Gamma_1} f^N(x)^2 \ell^N(x) \cdot \eta(x) dx. \end{aligned}$$

Proof: Let $G^N = W^N + M^N$; then

$$G_{ij}^N = \int_\Omega \left\{ 2(\ell^N \cdot \nabla \phi_i^N) \phi_j^N(x) + (\sum_{k=1}^n \ell_{k,k}^N) \phi_i^N(x) \phi_j^N(x) \right\} dx.$$

Then, we have

$$\begin{aligned} & < G^N f^N, f^N >_{R^N} \\ = & \int_\Omega \sum_{ij=1}^N \left\{ 2(\ell^N \cdot \nabla \phi_i^N f_i^N) \phi_j^N(x) f_j^N + \left(\sum_{k=1}^n \ell_{k,k}^N(x) \right) \phi_i^N(x) f_i^N \phi_j^N(x) f_j^N \right\} dx \\ = & \int_\Omega \left\{ 2(\ell^N \cdot \nabla f^N(x)) f^N(x) + \sum_{k=1}^n \ell_{k,k}^N(x) f^N(x)^2 \right\} dx \\ = & \int_\Omega \operatorname{div}(f^N(x)^2 \ell^N(x)) dx \\ = & \int_{\Gamma_1} f^N(x)^2 \ell^N(x) \cdot \eta(x) dx, \end{aligned}$$

because $f^N(x) = 0$ on Γ_0. □

Let us define the matrix $L^N(x)$ by (recall condition (V1) of Assumption 4.1)

$$L_{ij}^N(x) = \frac{1}{2} \left(\frac{\partial \ell_i^N(x)}{\partial x_j} + \frac{\partial \ell_j^N(x)}{\partial x_i} \right).$$

Lemma 4.5 *Under Assumption 4.2, for any* $g^N \in R^N$*, let* $g^N(\cdot)$ *be the function in* H^N *given by* $g^N(x) = \sum_{k=1}^N g_k \phi_k^N(x)$*. Then we have*

$$\begin{aligned} & < K^N g^N, C^N g^N >_{R^N} - < K^N g^N, g^N >_{R^N} \\ = & \ 2 \int_\Omega (\nabla g^N)^T (L^N(x) - I) \nabla g^N(x) dx - \frac{1}{2} \int_\Omega (\sum_{i,j=1}^n \ell_{i,ijj}^N) g^N(x)^2 dx \\ & + \int_{\Gamma_1} \frac{1}{2} (\sum_{i,j=1}^n \ell_{i,ij}^N \eta_j) g^N(x)^2 dx + \int_{\Gamma_1} |\nabla g^N|^2 \ell^N \cdot \eta dx. \end{aligned}$$

Proof: We have $< K^N g^N, C^N g^N >_{R^N} = < W^N (M^N)^{-1} K^N g^N, g^N >_{R^N}$. By taking

$$\theta_i^N(x) = 2\ell^N \cdot \nabla \phi_i^N(x) + \left(\sum_{k=1}^n \ell_{k,k}^N(x) - 1\right) \phi_i^N(x),$$

we have

$$W_{ij}^N = \int_\Omega \theta_i^N(x) \phi_j^N(x) dx = V_{ji}^N,$$

where V^N is as defined in Lemmas 4.1 and 4.2. By Lemma 4.2, we have

$$< K^N g^N, C^N g^N >_{R^N} = < F^N g^N, g^N >_{R^N},$$

where the matrix F^N is given by

$$F_{ij}^N = \int_{\Gamma_1} \theta_i^N(x) \frac{\partial}{\partial \eta} \phi_j^N(x) dx - \int_\Omega \theta_i^N(x) \cdot \triangle \phi_j^N(x) dx.$$

As a consequence, we obtain

$$\begin{aligned}
(4.1) \qquad & < K^N g^N, C^N g^N >_{R^N} \\
= & -\int_\Omega (2\ell^N \cdot \nabla g^N(x) + (\sum_{k=1}^n \ell_{k,k}^N - 1) g^N(x)) \triangle g^N(x) dx \\
& + \int_{\Gamma_1} \left(2\ell^N \cdot \nabla g^N(x) + (\sum_{k=1}^n \ell_{k,k}^N - 1) g^N(x)\right) \frac{\partial}{\partial \eta} g^N(x) dx.
\end{aligned}$$

Consider first the term

$$I^N = \int_\Omega \left(2\ell^N \cdot \nabla g^N(x) + (\sum_{k=1}^n \ell_{k,k}^N - 1) g^N(x)\right) \triangle g^N(x) dx.$$

For all functions $\phi \in H^2(\Omega)$

$$\begin{aligned}
& 2\triangle\phi(\ell^N \cdot \nabla\phi) + \left(\sum_{k=1}^n \ell_{k,k}^N\right) \phi \triangle \phi \\
= & \operatorname{div}\left\{2\nabla\phi(\ell^N \cdot \nabla\phi) - |\nabla\phi|^2 \ell^N + \phi(\sum_{i=1}^n \ell_{i,i}^N)\nabla\phi\right\} \\
& -2(\nabla\phi)^T L^N \nabla\phi - \sum_{i,j=1}^n \ell_{i,ij}^N \phi \frac{\partial \phi}{\partial x_j}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} I^N &= -\int_\Omega 2\left\{\nabla g^N(x)\cdot L^N\nabla g^N(x) + \sum_{i,j=1}^n \ell^N_{i,ij} g^N \frac{\partial g^N}{\partial x_j} + g^N \triangle g^N(x)\right\} dx \\ &\quad + \int_{\Gamma_1}\left[\left(2(\ell^N\cdot\nabla g^N) + (\sum_{k=1}^n \ell^N_{k,k})g^N\right)\frac{\partial g^N}{\partial \eta} - |\nabla g^N|^2 \ell^N\cdot\eta\right] dx \\ &\quad - \int_{\Gamma_0} |\nabla g^N|^2 \ell^N\cdot\eta\, dx, \end{aligned}$$

where we have used Assumption 4.2 to observe that $2(\ell^N\cdot\nabla g^N) + \sum_{k=1}^n \ell_{k,k} g^N$ is in H^N and hence vanishes on Γ_0. Since g^N vanishes on Γ_0, we have that also $2(\ell^N\cdot\nabla g^N)$ must vanish on Γ_0. Moreover, since $g^N(x) = 0$ on Γ_0, $\nabla g^N(x) = \pm|\nabla g^N|\eta$ on Γ_0 and hence $|\nabla g^N|^2\ell^N\cdot\eta = \pm|\nabla g^N|\ell^N\cdot\nabla g^N = 0$ on Γ_0. As a consequence, we obtain

$$\begin{aligned} I^N &= -\int_\Omega \left\{2 < L^N\nabla g^N, \nabla g^N >_{R^N} + \sum_{i,j=1}^n \ell^N_{i,ij}\frac{\partial g^N}{\partial x_j} g^N + g^N\triangle g^N\right\} dx \\ &\quad + \int_{\Gamma_1}\left[\left(2(\ell^N\cdot\nabla g^N) + (\sum_{k=1}^n \ell^N_{k,k})g^N\right)\frac{\partial g^N}{\partial\eta} - |\nabla g^N|^2\ell^N\cdot\eta\right] dx. \end{aligned}$$

Integrating the terms

$$\sum_{i,j=1}^n \ell^N_{i,ij}\frac{\partial g^N}{\partial x_j} g^N \quad \text{and} \quad g^N\triangle g^N(x)$$

by parts once, we obtain

$$\begin{aligned} I^N &= -\int_\Omega\{2 < L^N\nabla g^N, \nabla g^N >_{R^n} - |\nabla g^N|^2 - (\sum_{i,j=1}^n \ell_{i,ijj})\frac{(g^N)^2}{2}\} dx \\ &\quad + \int_{\Gamma_1}\left(2\ell^N\cdot\nabla g^N + (\sum_{k=1}^n \ell^N_{k,k} - 1)g^N\right)\frac{\partial g^N}{\partial\eta} dx \\ &\quad - \int_{\Gamma_1}\left\{|\nabla g^N|^2\ell^N\cdot\eta + (\sum_{i,j=1}^n \ell^N_{i,ij}\eta_j)\frac{(g^N)^2}{2}\right\} dx. \end{aligned}$$

Finally combining the above result with (4.1), we obtain

$$\begin{aligned} &< K^N g^N, C^N g^N >_{R^N} \\ &= \int_\Omega\left[2 < L^N\nabla g^N, \nabla g^N >_{R^n} - |\nabla g^N|^2 - \sum_{i,j=1}^n \frac{1}{2}\ell^N_{i,ijj}(g^N)^2\right] dx \\ &\quad + \int_{\Gamma_1}\left\{|\nabla g^N|^2\ell^N\cdot\eta + \frac{1}{2}(\sum_{i,j=1}^n \ell^N_{i,ij}\eta_j)(g^N)^2\right\} dx. \end{aligned}$$

This least equality is equivalent to

$$
\begin{aligned}
& < K^N g^N, C^N g^N >_{R^N} - < K^N g^N, g^N >_{R^N} \\
= & \int_\Omega 2\left(< L^N \nabla g^N, \nabla g^N >_{R^n} - |\nabla g^N|^2\right) dx \\
& -\frac{1}{2}\int_\Omega (\sum_{ij=1}^{n} \ell^N_{i,ijj})(g^N)^2 dx + \int_{\Gamma_1} |\nabla g^N|^2 \ell^N \cdot \eta dx \\
& +\frac{1}{2}\int_{\Gamma_1} (\sum_{j=1}^{n} \ell^N_{i,ijj}\eta_j)(g^N)^2 dx.
\end{aligned}
$$

□

By using Lemma 4.1 and the same notation as for Lemma 4.4, 4.5, we find

$$
\begin{aligned}
< B^N f^N, C^N g^N >_{R^N} &= \int_{\Gamma_1} \alpha \left(2(\ell^N \cdot \nabla g^N + (\sum_{k=1}^{n} \ell^N_{k,k} - 1) g^N\right) f^N dx, \\
< B^N f^N, f^N >_{R^N} &= \int_{\Gamma_1} \alpha (f^N)^2 dx.
\end{aligned}
$$

Lemma 4.6 *Under Assumption 4.2, Condition 2.3 holds.*

Proof: (i) By the boundedness of ℓ^N and Lemma 4.4, (S1) holds. (ii) By taking ϵ sufficiently small in Assumption 4.1 and 4.2, we have

$$
\int_{\Gamma_1} \left\{ |\nabla g^N(x)|^2 \ell^N(x) \cdot \eta(x) + \frac{1}{2}(\sum_{i,j=1}^{n} \ell^N_{i,ij}\eta_j) g^N(x)^2 \right\} dx \geq 0,
$$

and

$$
\frac{1}{2}\int_\Omega (\sum_{ij=1}^{n} \ell^N_{i,ijj}) g^N(x)^2 \leq \delta \int_\Omega |\nabla g^N(x)|^2 dx
$$

for δ arbitarily small. Therefore (S3) following Condition 2.3 holds. (iii) By the Schwartz inequality, $< B^N f^N, C^N g^N >_{R^N}$ can be bounded by

$$
| < B^N f^N, C^N g^N >_{R^N} | \leq \frac{c_1}{\nu} \int_{\Gamma_1} |f^N(x)|^2 dx + c_2 \nu \int_{\Gamma_1} |\nabla g^N(x)|^2 d\Gamma
$$

where the constants c_1, c_2, and ν are independent of N and ν can be arbitrarily chosen. Therefore, (S2) of Condition 2.3 holds.

As a consequence of Lemma 4.3 and 4.6, on the domain Ω and the subspace H^N, we conclude that under Assumption 4.2 these approximation methods uniformly preserve the exponential stability of the weakly damped wave equation.

5 Numerical studies of approximation methods

As has been demonstrated in the previous three sections, establishing a uniform decay rate estimate for a given approximation method can be complex and tedious. In fact the general approach outlined in Section 2 offers only a broad direction toward the selection of the appropriate multiplier Q^N. On the other hand, for any given approximation method, by computing the eigenvalues of the matrices A^N in (1.5), one can observe the trends in the location of the eigenvalues as N increasess. In several cases presented below, the numerical results clearly indicate that some of the eigenvalues of A^N are approaching the imaginary axis as N increases, and it is therefore unlikely that a uniform decay rate can be preserved.

The example used for demonstration in this section is the two-dimensional wave equation (3.9)-(3.11) where the parameter α is taken to be 1. The construction of the matrix A^N is presented for each approximation method. The eigenvalues are then computed using the subroutine F02AFF of the FORTRAN software library NAG. The results were compared with the results using the IMSL library, and no visible difference can be observed. All computations were carried out on a IBM 3081 computer.

5.1 Polynomial based Galerkin approximation methods

Let $\{Q_k(\cdot)\}_{k=0}^{N}$ be the Legendre polynomials of degree less than or equal to N on the interval $[-1,1]$. We define a sequence of polynomials P_k by

$$P_k(s) = Q_k(s) + (-1)^k Q_0(s)$$

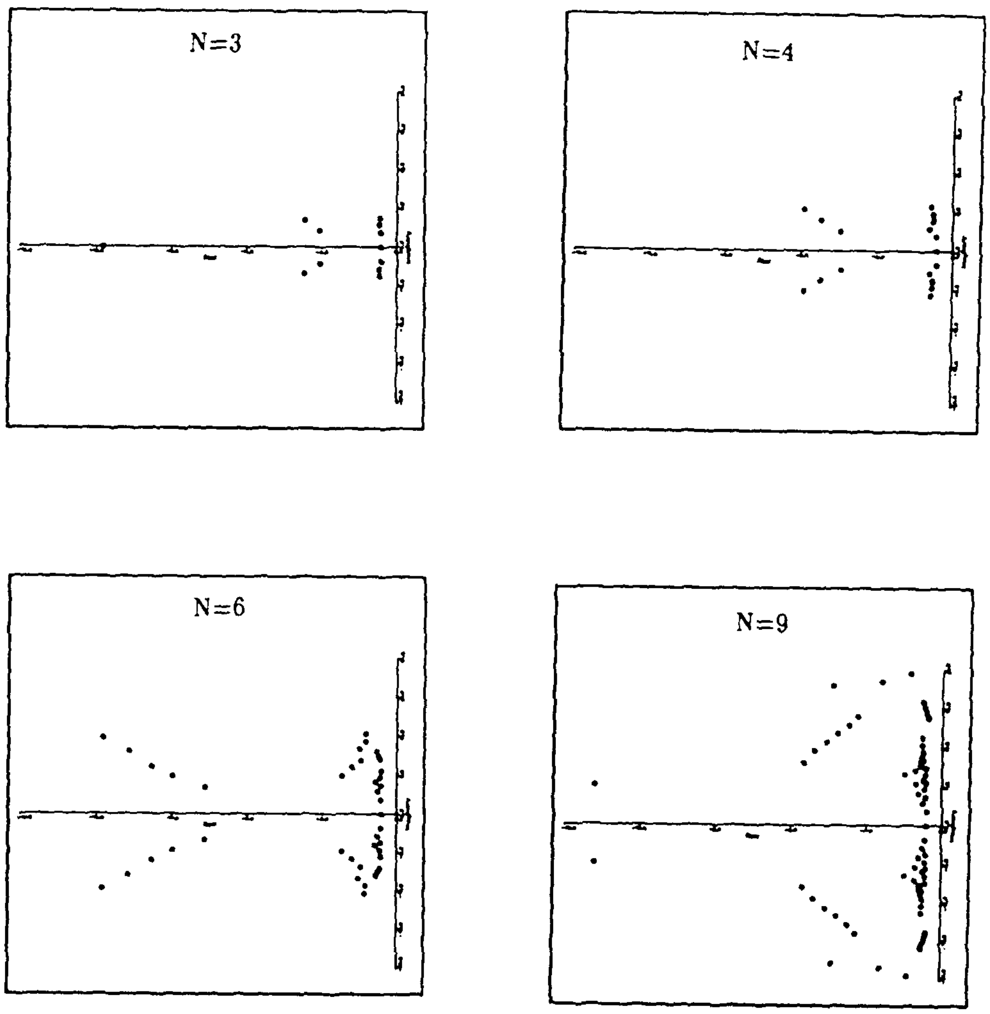
for $k = 1, \ldots, N$.

It is easy to see that $P^N = \text{span}\{P_k\}_{k=1}^{N}$ is the set of polynomials of degree less than or equal to N vanishing at $s = -1$. Let us define matrices M_s^N, K_s^N as follows

$$\begin{aligned} [M_s^N]_{ij} &= \int_{-1}^{+1} P_i(s)P_j(s)ds, \\ [K_s^N]_{ij} &= \int_{-1}^{+1} \frac{d}{ds}P_i(s)\frac{d}{ds}P_j(s)ds. \end{aligned}$$

The matrices M_s^N, K_s^N can be easily computed using the recursive properties and the orthogonality of the Legendre polynomials. Now let $\phi_{ij}^N(x,y) = \psi_{ij}^N(x,y) = P_i(x)P_j(y)$. We assume a scaling of the domain Ω onto $[-1,1] \times [-1,1]$ is made.

Figure 5.1: Locations of the eigenvalues of the matrix A^N for the polynomial based Galerkin method.



The matrices M^N, K^N, and B^N defined in the previous section can be computed as follows:

$$
\begin{aligned}
[M^N]_{(ij),(kl)} &= [M_s^N]_{ik}[M_s^N]_{jl}, \\
[K^N]_{(ij),(kl)} &= [K_s^N]_{ik}[M_s^N]_{jl} + [M_s^N]_{ik}[K_s^N]_{jl}, \\
[B^N]_{(ij),(kl)} &= \alpha[M_s^N]_{ik}P_j(1)P_l(1) + \alpha P_i(1)P_k(1)[M_s^N]_{jl}.
\end{aligned}
$$

The matrix A^N is given by

$$
A^N = \begin{pmatrix} 0 & I \\ -(M^N)^{-1}K^N & -(M^N)^{-1}B^N \end{pmatrix}.
$$

The dimension of A^N is N^2. For $N = 3,4,6,,9$, the locations of eigenvalues are displayed in Figure 5.1. We note that as would be predicted from the analysis of Section 4, a uniform margin between the eigenvalues of A^N and the imaginary axis is maintained for all N. In the Table 5.1, we list the margin of stability for each N.

Table 5.1: Margin between the eigenvalues of the matrix A^N and the imaginary axis.

N	$\max\{\mathrm{Re}\lambda, \lambda \in \sigma(A^N)\}$
4	-0.6230
5	-0.6215
6	-0.6232
7	-0.6097
8	-0.5815
9	-0.5628

5.2 Polynomial spline based Galerkin approximation methods

For N a given integer, the interval $[0,1]$ is divided into equal sized subintervals $[x_{k-1}, x_k]$ with $x_k = k/N$ for $k = 1,\ldots,N$. Let $B^{N,m}$ be the set of polynomial splines of order m corresponding to the grid $\{x_k\}$ that vanish at $x = 0$. Consider a basis $\{B_k^{N,m}\}_{k=1}^{N+m-1}$ for $B^{N,m}$ on $\Omega = [0,1] \times [0,1]$. The matrices M_s^N, K_s^N are defined as

$$
[M_s^N]_{ij} = \int_0^1 B_i^{N,m}(x)B_j^{N,m}(x)dx, \quad i,j = 1,\ldots,N+m-1,
$$

Figure 5.2: Locations of the eigenvalues of the matrix A^N for the linear spline based Galerkin method.

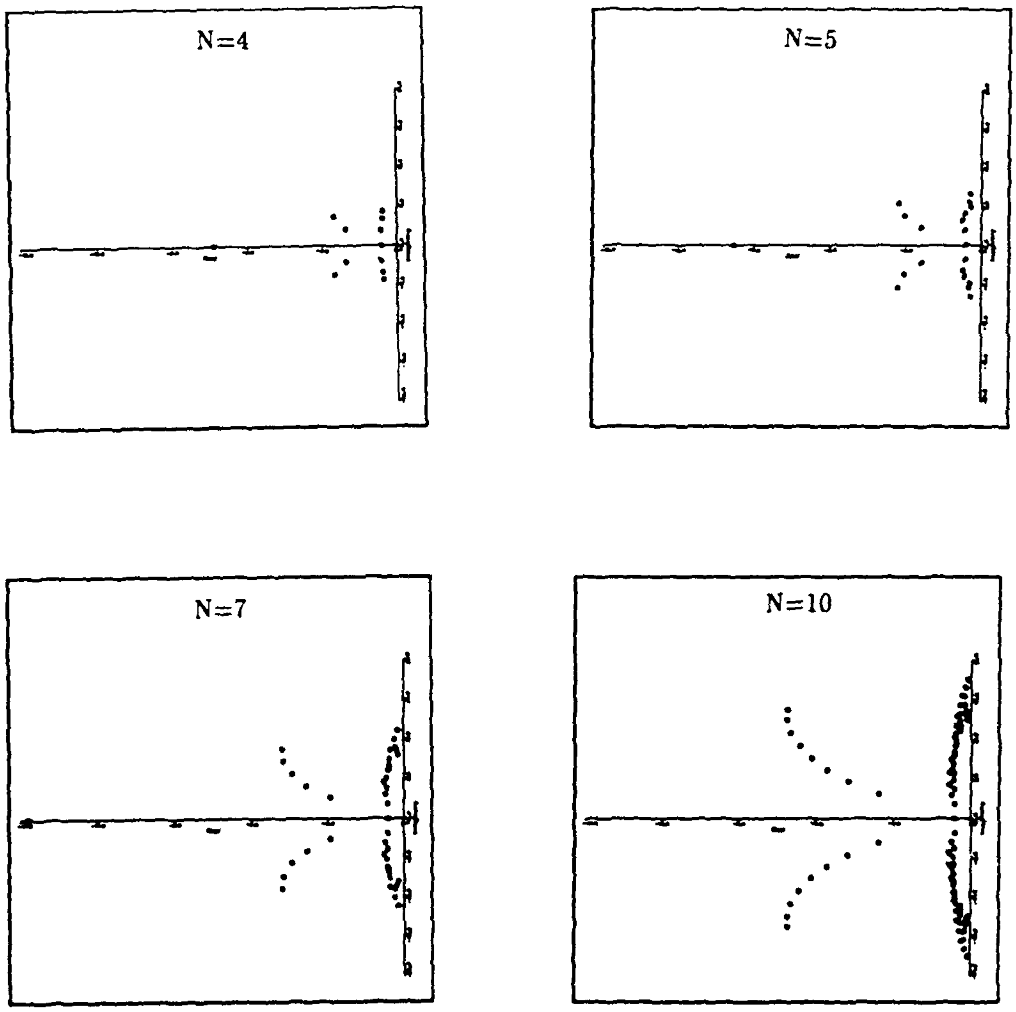
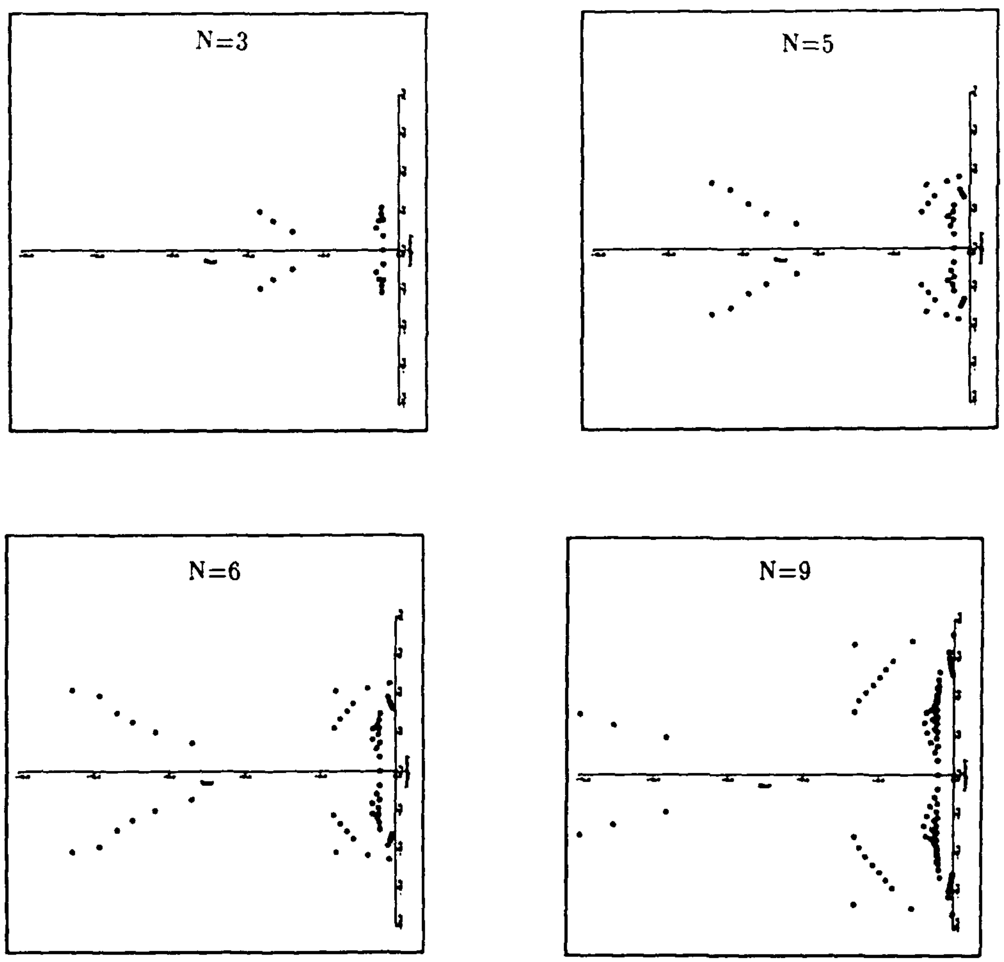


Figure 5.3: Locations of the eigenvalues of the matrix A^N for the cubic spline based Galerkin method.



$$[K_s^N]_{ij} = \int_0^1 \frac{d}{dx}B_i^{N,m}(x)\frac{d}{dx}B_j^{N,m}(x)dx, \quad i,j = 1,\ldots,N+m-1.$$

The matrices M_s^N, K_s^N can be easily evaluated. We define $\phi_{ij}^N(x,y) = \psi_{ij}^N(x,y) = B_i^{N,m}(x)B_j^{N,m}(y)$. The the matrices M^N, K^N and B^N are defined as

$$\begin{aligned}
[M^N]_{(ij),(kl)} &= \int_\Omega \phi_{ij}^N(x,y)\phi_{kl}^N(x,y)dxdy, \\
[K^N]_{(ij),(kl)} &= \int_\Omega \nabla\phi_{ij}^N(x,y)\cdot\nabla\phi_{kl}^N(x,y)dxdy \\
[B^N]_{(ij),(kl)} &= \int_{\Gamma_1} \alpha\phi_{ij}^N(x,y)\phi_{kl}^N(x,y)d\Gamma_1.
\end{aligned}$$

Again, the matrices M^N, K^N, and B^N can be computed using M_s^N and K_s^N. Since only the first derivative of the spline function is required, either linear spline or cubic spline functions can be used. In Figures 5.2 and 5.3, the locations of the eigenvalues of the matrix A^N using linear and cubic spline functions, respectively, are depicted. We note the similarity between the locations of the eigenvalues with small imaginary part and the locations of the eigenvalues of the matrix A^N using polynomials depicted in Figure 5.1. However, the eigenvalues with large imaginary part tend toward the imaginary axis. Another interesting observation is that unlike the eigenvalues of A^N using polynomials and cubic splines, the eigenvalues of A^N using linear splines do not have exceedingly large negative real parts; further discussion concerning this observation will be given in the final section.

5.3 Finite element method

The classical finite element method requires that we subdivide the domain Ω into triangles where functions linear on each triangle are used for the approximation. More precisely, for any $1 \le i \le N$ and $1 \le j \le N$, a function ϕ_{ij}^N is defined as

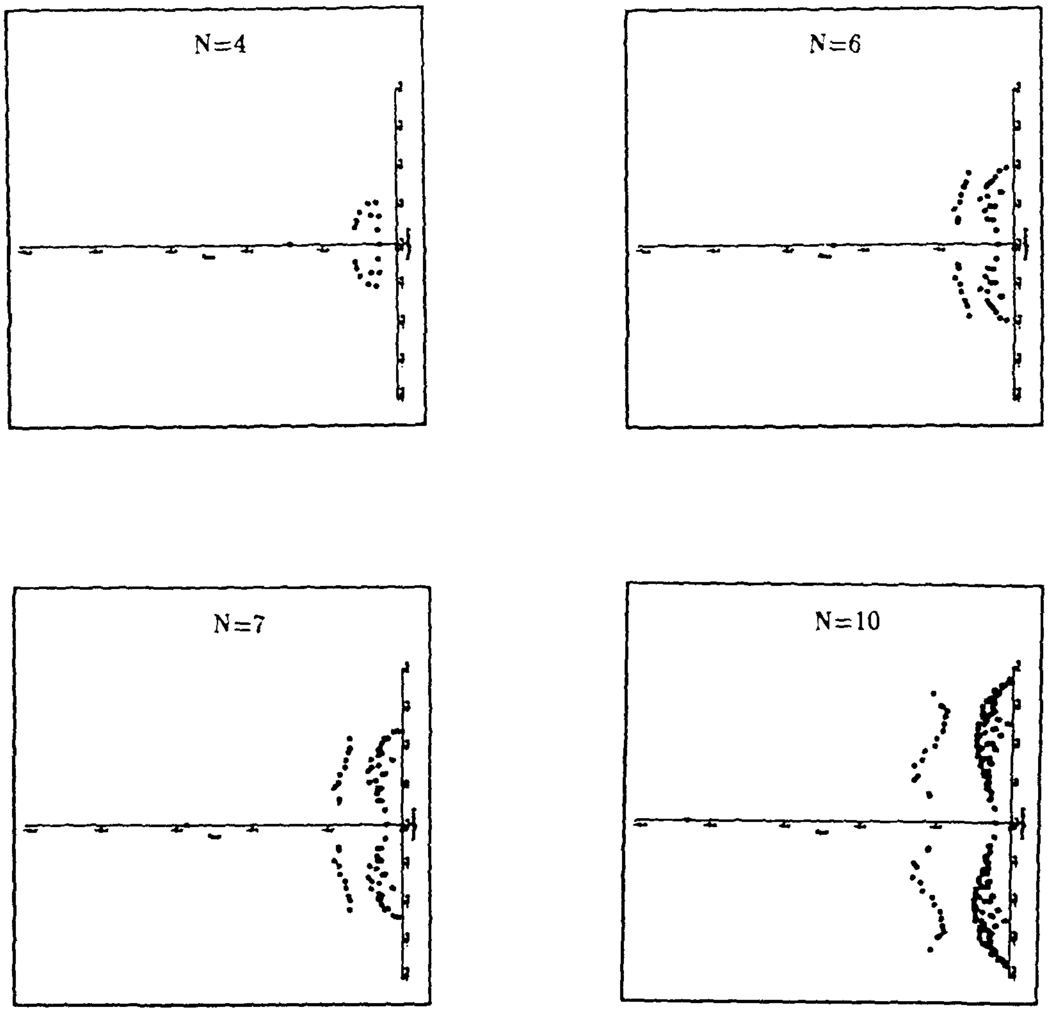
$$\phi_{ij}^N(x,y) = \begin{cases} 1 - N\max\{|x-x_i|, |y-y_j|, |x-y-x_i+y_j|\}, & (x,y)\in D_{ij}^N, \\ 0, \quad \text{otherwise}, \end{cases}$$

where the support D_{ij}^N is defined by

$$D_{ij}^N = \{(x,y)\in\bar{\Omega} : |x-x_i| \le \frac{1}{N}, |y-y_j| \le \frac{1}{N}, |x-y-x_i+y_j| \le \frac{1}{N}\}$$

with $x_i = i/N, y_j = j/N$. We choose the functions $\psi_{ij}^N(x,y) = \phi_{ij}^N(x,y)$ and the matrices M^N, K^N, and B^N are defined as in the other Galerkin methods.

Figure 5.4: Loaction of the eigenvalues of the matrix A^N for the finite element method.



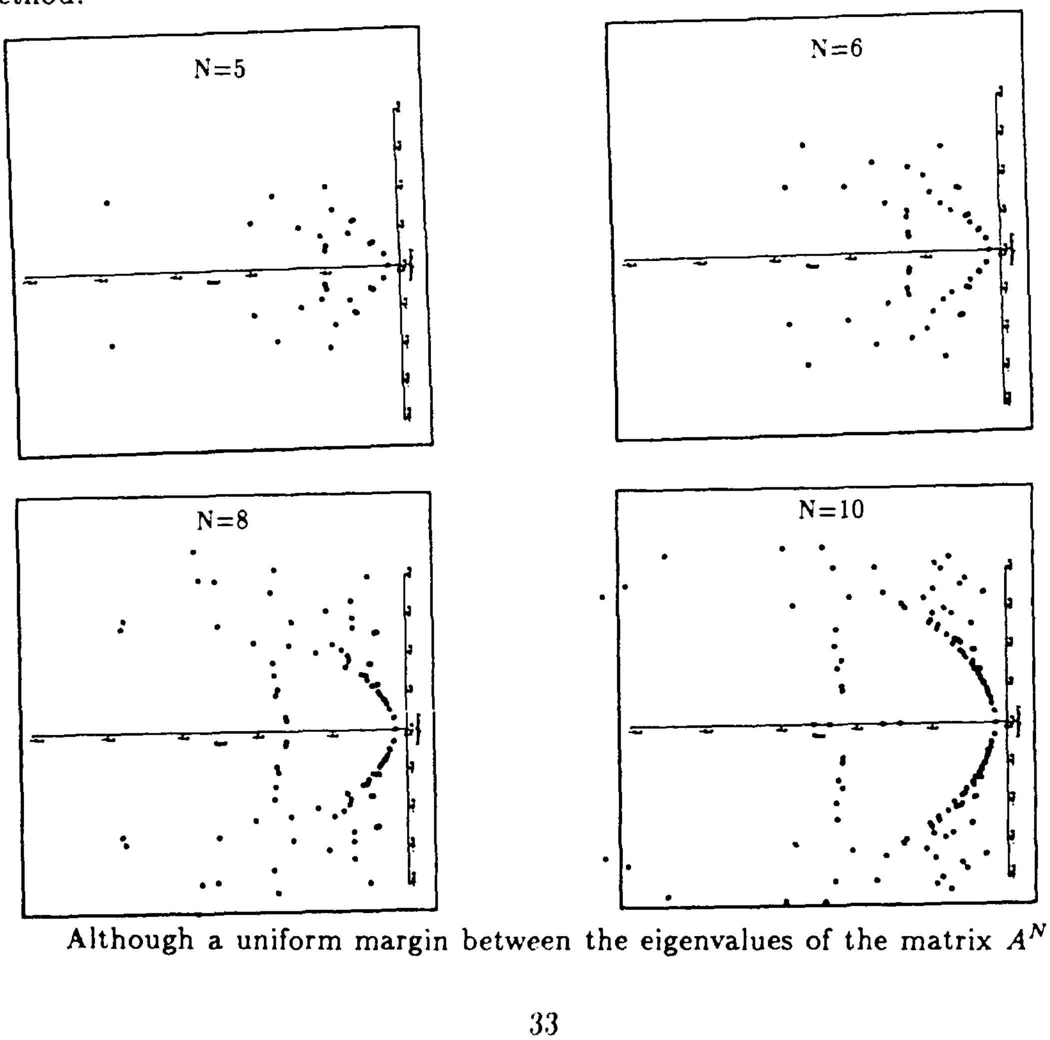
The computation of the matrices M^N, K^N, and B^N uses finite element techniques based on elementary matrices.

The location of the eigenvalues are depicted in Figure 5.4. We note that some eigenvalues with large imaginary part tend toward the imaginary axis which suggest that no uniform decay rate of solutions is preserved. We also observe that only a few eigenvalues have large negative real part.

5.4 Mixed finite element method.

The mixed finite element method presented in Section 3 was also implemented. In Figure 5.5, the location of eigenvalues are depicted.

Figure 5.5: Location of the eigenvalues of the matrix A^N for the mixed finite element method.



Although a uniform margin between the eigenvalues of the matrix A^N and the imaginary axis is maintained, the locations of the eigenvalues are dramatically different from the previous approximation schemes at lower values of N. This might suggest a slower convergence rate for the approximation scheme.

In Table 5.2, the size of the margin between the eigenvalues of A^N and the imaginary axis is reported.

Table 5.2: Margin between the eigenvalues of the matrix A^N and the imaginary axis for the mixed finite-element method.

N	$\max\{\mathrm{Re}\lambda, \lambda \in \sigma(A^N)\}$
4	-0.4689
5	-0.4596
6	-0.4552
7	-0.4527
8	-0.4512
9	-0.4969
10	-0.4496
15	-0.4632

5.5 Finite difference approximation method

Unlike the other approximation methods presented so far, in the finite difference method one attempts to approximate the solution of the wave equation at the grid points directly. In fact, for any given integer N, let $u_{ij}^N(t), v_{ij}^N(t)$ be the approximation of the solution values $u(t, x_i, y_j), v(t, x_i y_j) = u_t(t, x_i, y_j)$ of (3.9)-(3.11). The wave equation is approximated by

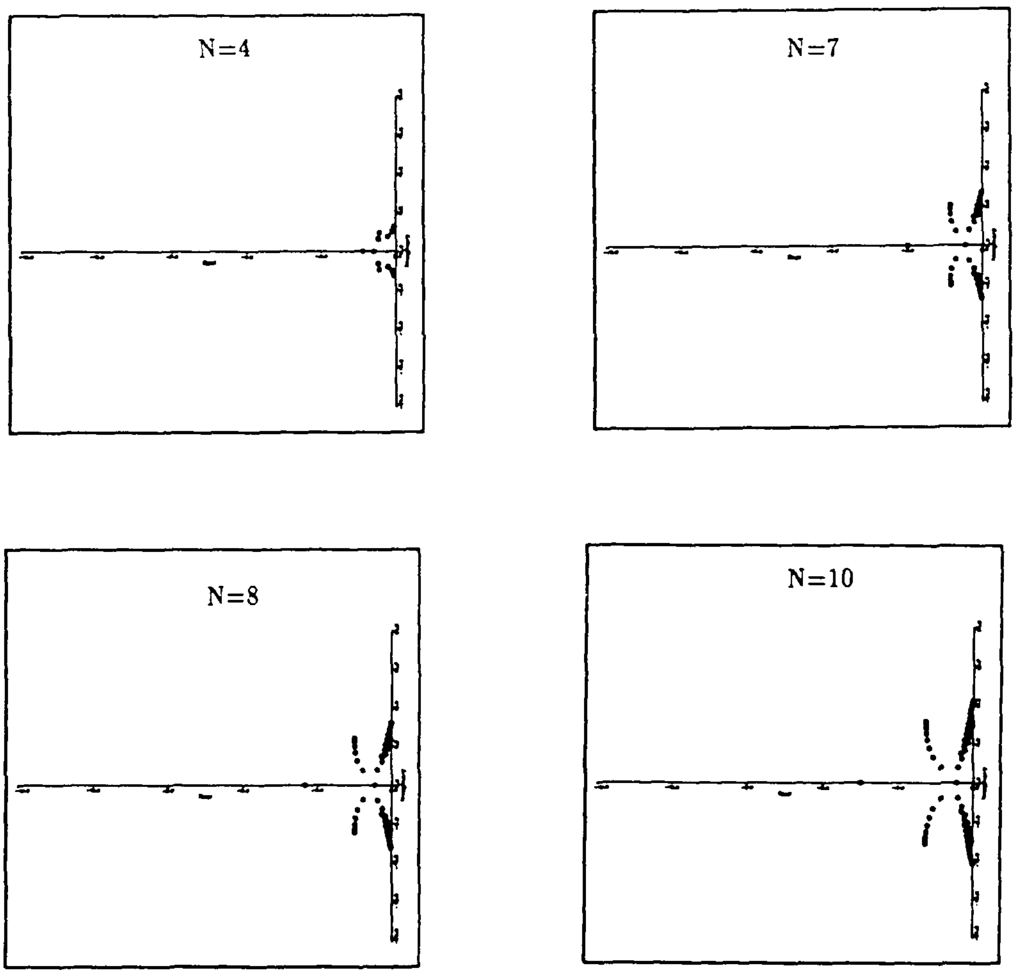
$$(5.1) \quad \frac{d}{dt} u_{i,j}^N(t) = v_{i,j}^N(t),$$

$$(5.2) \quad \frac{d}{dt} v_{i,j}^N(t) = N^2 \left(-4u_{i,j}^N(t) + u_{i-1,j}^N(t) + u_{1+i,j}^N(t) + u_{i,j-1}^N + u_{i,j+1}^N(t)\right),$$

for $i = 2, \ldots, N-1$ and $j = 2, \ldots, N-1$.

At the boundary $x = 0$ or $y = 0$, by taking $u_{0,j}^N = u_{i,0}^N = 0$, the equation (5.2) still holds. However, at the boundary $x = 1$ or $y = 1$, the second derivatives are

Figure 5.6: Locations of the eigenvalues of the matrix A^N for the finite-difference method.



approximated by one sided finite differences. In particular, we take

$$\begin{aligned}
\frac{d}{dt} v_{i,N}^N &= N^2 \left(-2u_{i,N}^N(t) + u_{i-1,N}^N(t) + u_{i+1,N}^N(t) \right) \\
&\quad + N \left(\alpha v_{i,N}^N(t) - N(u_{i,N}^N(t) - u_{i,N-1}^N(t)) \right) \\
\frac{d}{dt} v_{N,j}^N &= N^2 \left(-2u_{N,j}^N(t) + u_{N,j-1}^N(t) + u_{N,j+1}^N(t) \right) \\
&\quad + N \left(\alpha v_{N,j}^N(t) - N(u_{N,j}^N(t) - u_{N-1,j}^N(t)) \right)
\end{aligned}$$

for $i = 1, \ldots, N-1, j = 1, \ldots, N-1$. At the corner (1,1), we take

$$\begin{aligned}
\frac{d}{dt} v_{N,N}^N &= N \left(\alpha v_{N,N}^N(t) - N(u_{N,N}^N(t) - u_{N,N-1}^N(t)) \right) \\
&\quad + N \left(\alpha v_{N,N}^N(t) - N(u_{N,N}^N(t) - u_{N-1,N}^N(t)) \right) .
\end{aligned}$$

The locations of the eigenvalues of the matrix A^N are depicted in Figure 5.6.

It was a surprise to us that the eigenvalues of A^N for this finite difference approximation are so different from those of the other schemes. Many variations of the finite difference methods exist; we believe that an adjustment of the method used here may well produce different locations for eigenvalues. However, our experience with the above standard finite difference approximation method suggests that no uniform margin between the eigenvalues of A^N and imaginary axis is maintained for all N.

6 Concluding remarks

Even in view of the analysis for the methods given in Sections 3 and 4, one might ask why these two methods succeed while several other popular approximation schemes fail. One possible explanation is the following. As approximation methods of a second order system, many approximation methods approximate the second component v which has the same smoothness as ∇u theoretically by functions with equal order of smoothness as the approximation for u. In another words, if $u_0^N \in H_1^N$ and $v_0^N \in H_2^N$ are taken to be the initial values of the equations (1.1)-(1.3), the solution v is much less smooth than elements in H_2^N. This is not the case in the mixed finite element method and the polynomial based Galerkin methods.

We should also comment on the implications of the present results with respect to the boundary control problem. If the boundary condition (1.3) is considered as a boundary feedback control, with an appropriate approximation of the input operator, the results reported here can be used to construct finite dimensional approximate control systems that are uniformly exponentially stabilizable. However, this does not guarantee that the infinite dimensional system under the approximate control is exponentially stable. The inherent sensitivity of the conservative wave equation with respect to perturbations of the boundary conditions remains one major drawback of this type of model.

Finally, we observe that in both polynomial based Galerkin methods and the mixed finite element method, a large number of extraneous eigenvalues with large negative real parts are introduced. Therefore, the resulting finite dimensional ordinary differential equation generated by these methods can be very stiff. This can cause problems in integration algorithms. Since our main concern is to solve linear quadratic control problems using these approximation schemes, this observation should not substantially diminish the attractiveness of these two methods.

References

[BFS] H.T. Banks, W. Fang and R.J. Silcox, *Active control of noise: A time domain approach*, in Proc. Conf. on Recent Advances in Active Control of Sound and Vibration (VPISU, Blacksburg) April, 1991, pp. S43-S47.

[BK] H.T. Banks and K. Kunisch, *The linear regulator problem for parabolic systems*, SIAM J. Control and Opt. 22(1984), pp.684-698.

[BKS] H.T. Banks, S.L. Keeling and R.J. Silcox, *Optimal control techniques for active noise suppression*, Proc. 27th IEEE Conf. on Dec. and Control, Austin, TX, 1988, pp. 2006-2011.

[BKSW] H.T. Banks, S.L. Keeling, R.J. Silcox and C. Wang, *Linear quadratic tracking problem in Hilbert space: Application to optimal active noise suppression*, Proc. 5th IFAC Symp. on Control of DPS (A. El-Jai, M. Amouroux, eds.), Perpignan, France, June, 1989, pp. 17-22.

[BPS] H.T. Banks, G. Propst and R.J. Silcox, *A comparison of time domain boundary conditions for acoustic waves in wave guides*, ICASE Rep No. 91-27,1991, NASA Langly Res. Ctr., Hampton, VA.

[BW] H.T. Banks and C. Wang, *Optimal feedback control of infinite-dimensional parabolic evolution systems: approximation techniques*, SIAM J. Control and Optim., 27(1989), pp. 1182-1219.

[C] G. Chen, *Energy decay estimates and exact boundary value controllability for wave equation in a bounded domain*, J. Math. Pures Appl., 5(1979), pp. 249-274.

[D] R. Datko, *Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedback*, SIAM J. Control and Opt. 26 (1988), pp.697-713.

[G] J.S. Gibson, *The Riccati integral equations for optimal control problems on Hilbert spaces*, SIAM J. Control Optim., 17(1979), pp. 537-565.

[IK] K. Ito and F. Kappel, *On variational formulations of the Trotter-Kato theorem*, CAMS Rep. #91-7, Univ. of Southern California, April, 1991.

[L] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Diff. Eqns., 50(1983), pp. 163-182.

[Li] J.L. Lions, *Exact controllability, stabilization and perturbation for distributed systems*, SIAM Review, 30(1988), pp. 1-68.

[P] A. Pazy, *Semigroups of Linear Operator and Applications to Partial Differential Equations*, Springer, 1983.

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16. Abstract

We consider wave equations with damping in the boundary conditions. Techniques to ascertain the uniform preservation under approximation of exponential stability ar. presented. Several schemes for which preservation can be guaranteed are analyzed. Numerical results that demonstrate the lack of stability under approximation for several popular schemes (including standard finite difference and finite element schemes) are given.

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