DTIC

June 1991

MTR11172

M. M. Weiner

Lagrange's Equations for a Dissipative Uniform Transmission Line

CONTRACT SPONSOR MSR
CONTRACT NO. N A
PROJECT NO. 91260
DEPT D085

Approved for public release; distribution unlimited.

MITRE

The MITRE Corporation
Bedford, Massachusetts

Department Approval: R. Wm. Bush

MITRE Project Approval: Melvin Weiner

ABSTRACT

Complex constitutive system parameters are used to derive the form of Lagrange's equations for a non-conservative continuous system with a steady-state sinusoidal excitation. The resulting form of Lagrange's equations, for the case of a dissipative uniform transmission line, is shown to yield the well-known results derivable from the telegrapher's equations. The concept of time-averaged Lagrangian density is introduced, including the correct form for the time-averaged energy density dissipation function.

A-1

PREFACE

High-frequency (HF) antenna elements for radar and communications systems often consist of some form of vertical monopole element with a ground plane resting on or in close proximity to the earth. The ground plane is usually made as large as is economically feasible to alleviate, at large ranges, the reduction (caused by the earth's multipath pattern null on the horizon) of low-angle directive gain, and the reduction (caused by power absorption and dielectric discontinuity at the earth interface) of the fraction of available transmitter power that is radiated into the hemisphere above the ground.

The objective of MITRE sponsored research project 91260 is to develop computer programs and accurate models for predicting the element pattern, radiation efficiency, and input impedance of HF monopole elements on ground planes of various forms that rest on or are in close proximity to earth. The results will provide better analytical capability to optimize the performance and minimize the cost of the antenna elements and their associated ground planes for HF antenna arrays. The results could also support a subsequent experimental program investigating the peformance of monopole elements under nonideal operating and environmental conditions.

A ground plane of interest is a wire-mesh screen of arbitrary grid geometry. One approach to determining the steady-state wave propagation along a mesh screen above earth is a Floquet mode analysis as illustrated by Hill and Wait for a rectangular bonded wire mesh geometry [1]. Another approach is to consider the ground screen and earth as a continuous, dissipative non-uniform transmission line with the mesh screen as one conductor and the earth as the return conductor. The line's series inductance per unit length, series resistivity, shunt capacitance per unit length, and shunt conductivity, as a function of radial position along the screen, are usually determinable from the mesh geometry and complex permittivity of the earth.

For the latter approach of a dissipative non-uniform transmission line, the propagation constant may be determined either from Maxwell's equations (together with Kirchhoff's voltage and current laws) or from Lagrange's equations. Maxwell's equations are convenient if a suitable coordinate system can be found to express the fields. Lagrange's equations are convenient if the stored energy densities and energy dissipation function are readily expressible independently of the coordinate system.

For example, it has been suggested that Lagrange's equations might be advantageous for a distributed transmission line in which there is mutual coupling among elements such as in an infinite array of closely-packed monopole elements on a mesh groundscreen above earth. However, before Lagrange's equations can be applied to such a case, one must determine the appropriate form of Lagrange's equations. Although the forms of Lagrange's equations for dissipative discrete systems and non-dissipative continuous systems are well known, the form of Lagrange's equations for dissipative continuous systems are not readily found in the literature. Furthermore, most of the applications of Lagrange's equations are for transient excitations whereas the form of Lagrange's equations for steady-state excitations is not as readily found in the literature. It is also not clear whether the use of Lagrange's equations for such an application will offer any particular advantage over approaches based on Maxwell's equations.

This paper attempts to address these issues by applying Lagrange's equations to the case of a continuous, dissipative uniform transmission line with a steady-state excitation.

ACKNOWLEDGMENT

The author is grateful to T. R. Foster, J. D. R. Kramer, and J. E. Kriegel of The MITRE Corporation for helpful conversations on the use and manipulation of Lagrange's equations.

TABLE OF CONTENTS

SECTION 1

INTRODUCTION

Continuously-distributed dissipative uniform transmission lines have been employed in communication and radar systems since the earliest conception of such systems. Examples of such lines are the parallel two-wire line and the coaxial line. Such lines are usually analyzed, for sinusoidal steady-state excitation and for the condition that the radiation from the lines is negligible, by use of the telegrapher's equations.

The purpose of this paper is to present a form of Lagrange's equations and the associated Lagrangian density, for a dissipative uniform transmission line with steady-state excitation, that yield identical results as the telegrapher's equations. One motivation for doing so is to provide a foundation for future effort in applying Lagrange's equations to more complex transmission lines, as discussed in the Preface. A second motivation is to evaluate whether Lagrange's equations offer any particular advantage for such an application.

The forms of Lagrange's equations, for <u>transient</u> excitation of <u>dissipative</u> <u>discrete</u> systems and <u>non-dissipative</u> <u>continuous</u> systems are well known. However the form of Lagrange's equations for <u>sinusoidal steady-state</u> excitation of <u>dissipative</u> <u>continuous</u> systems is not readily found in the literature. This paper attempts to address this deficiency.

Some of the concepts associated with Lagrange's equations that are addressed in this paper are:

(1) the form of Lagrange's equations for distributed vs. discrete systems;

(2) the time-averaged Lagrangian density for a continuous system with a force field derivable from a generalized potential comprising scalar and vector potentials;

(3) the time-averaged energy density dissipation function for a non-conservative electromagnetic system with series and shunt ohmic losses;

(4) the relationship of Lagrange's equations to Hamilton's principle and the calculus of variations; and

(5) the definition in the calculus of variations of the partial derivative of a non-analytic complex function.

The well known results, for the propagation constant of a continuously-distributed dissipative uniform transmission line with sinusoidal steady-state excitation, are derived from the telegrapher's equations in section 2 and from Lagrange's equations in section 3 supported by an appendix. The conclusions are given in section 4.

SECTION 2

TELEGRAPHER'S EQUATIONS

Consider an increment dx of a dissipative uniform transmission line (see figure 1) with distributed series inductance L dx (henries), shunt capacitance C dx (farads) series resistance R dx (ohms) and shunt conductance G dx (Siemens) along the x axis. For a steady-state sinusoidal wave of radian frequency ω (rad/s), the voltage v(x,t) at point x and time t may be expressed in the form

$$v(\mathrm{x},t) = \mathrm{Re}[V(\mathrm{x})e^{j\omega t}] \tag{2-1}$$

$$i(\mathrm{x},t) = \mathrm{Re}[I(\mathrm{x})e^{j\omega t}] \tag{2-2}$$

It is assumed that the power radiated by the transmission line into the surrounding free-space medium is negligible.

The incremental voltage dv and incremental current di are given by [2].

$$\begin{aligned} dv(\mathrm{x},t) &= v(\mathrm{x}+\mathrm{dx},t) - v(\mathrm{x},t) \\ &= -R\,\mathrm{dx}\,i(\mathrm{x},t) - L\,\mathrm{dx}\frac{\partial i(\mathrm{x},t)}{\partial t} \\ &= \mathrm{Re}\left[-(R + j\omega L)I(\mathrm{x})\,\mathrm{dx}\,e^{j\omega t}\right] \\ &= \mathrm{Re}\left[dV(\mathrm{x})e^{j\omega t}\right] \end{aligned} \tag{2-3}$$

IN676

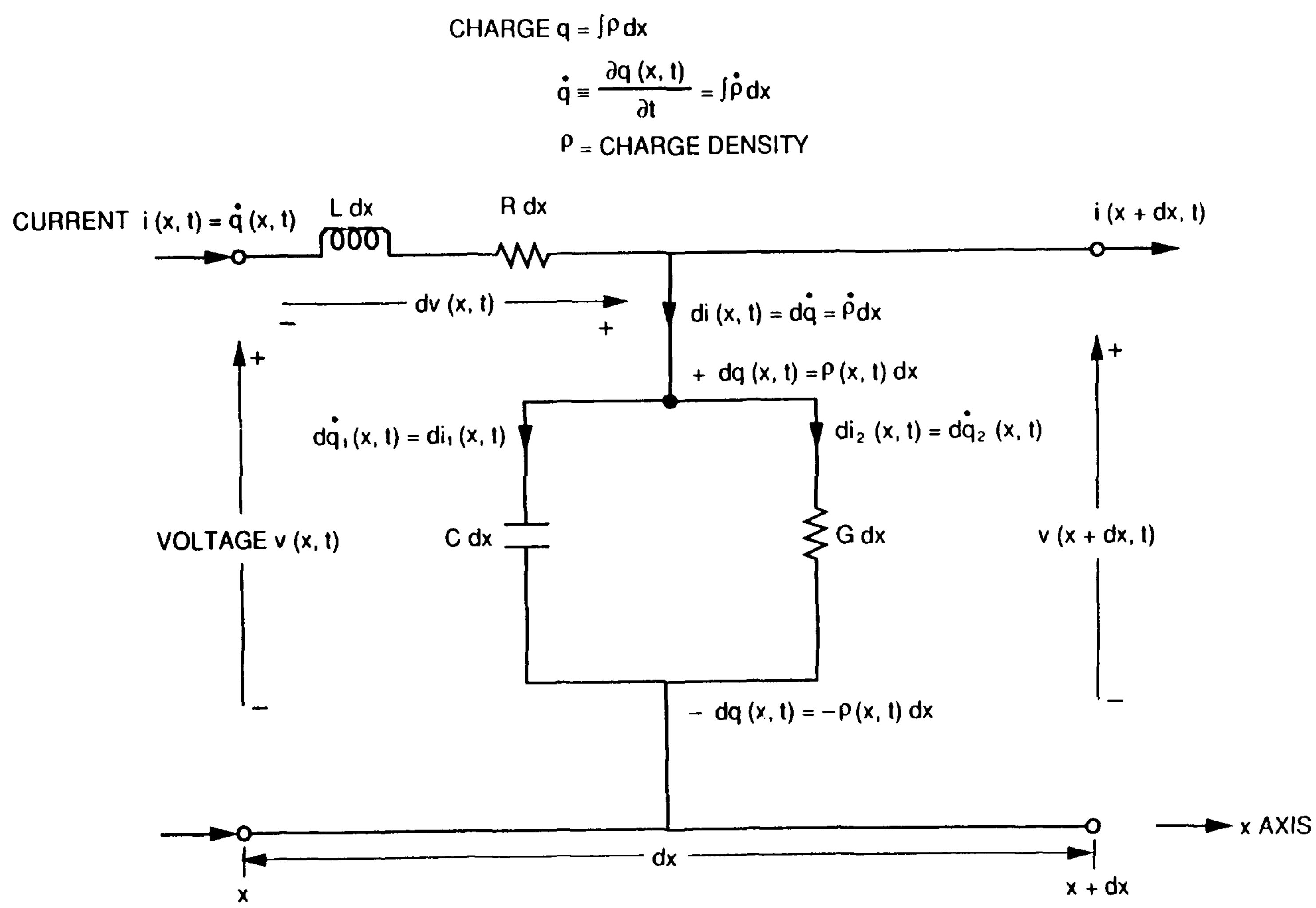

Figure 1. Increment of Dissipative Uniform Transmission Line

$$
\begin{aligned}
di(\mathrm{x},t) &= i(\mathrm{x}+\mathrm{dx},t) - i(\mathrm{x},t) \\
&= G\,\mathrm{dx}\,v(x,t) - C\,\mathrm{dx}\frac{\partial v(\mathrm{x},t)}{\partial t} \\
&= \mathrm{Re}\left[-(G+j\omega C)V(\mathrm{x})\,dx\,e^{j\omega t}\right] \\
&= \mathrm{Re}\left[dI(\mathrm{x})e^{j\omega t}\right]
\end{aligned} \tag{2-4}
$$

Equations (2-3) and (2-4), after cancelling the common real part designation ano the common factor $e^{j\omega t}$, yield the well-known telegrapher's equations:

$$
\frac{dV(\mathrm{x})}{\mathrm{dx}} = -(R+j\omega L)I(\mathrm{x}) \tag{2-5}
$$

$$
\frac{dI(\mathrm{x})}{\mathrm{dx}} = -(G+j\omega C)V(\mathrm{x}) \tag{2-6}
$$

Differentiating equation (2-5) with respect to x and then substituting dI/dx from equation (2-6),

$$
\frac{d^2V(\mathrm{x})}{\mathrm{dx}^2} - (R+j\omega L)(G+j\omega C)V(\mathrm{x}) = 0 \tag{2-7}
$$

Noting that $(R + j\omega L)(G + j\omega C)$ is a complex constant, equation (2-7) has the general solution

$$
V(\mathrm{x}) = V_+ \exp(-\gamma \mathrm{x}) + V_- \exp(\gamma \mathrm{x}) \tag{2-8}
$$

where V_+ and V_- are arbitrary complex constants and

$$
\gamma^2 \equiv (R+j\omega L)(G+j\omega C) \tag{2-9}
$$

Differentiating equation (2-8) with respect to x,

$$\frac{dV(\mathrm{x})}{d\mathrm{x}} = -\gamma V_+ \exp(-\gamma \mathrm{x}) + \gamma V_- \exp(\gamma \mathrm{x}) \tag{2-10}$$

Substituting equation (2-10) into equation (2-5),

$$I(\mathrm{x}) = -\frac{dV(\mathrm{x})\ d\mathrm{x}}{R + j\omega L} = \frac{\gamma}{R + j\omega L}\left[V_+ \exp(-\gamma x) - V_- \exp(\gamma \mathrm{x})\right]$$

$$= \left(\frac{G + j\omega C}{R + j\omega L}\right)^{1/2}\left[V_+ \exp(-\gamma x) - V_- \exp(\gamma \mathrm{x})\right]$$

$$= (1 / Z_o)\left[V_+ \exp(-\gamma x) - V_- \exp(\gamma \mathrm{x})\right] \tag{2-11}$$

where

$Z_o \equiv [(R + j\omega L) / (G + j\omega C)]^{1/2}$ = characteristic impedance of the transmission line.

The quantity γ in equations (2-8) through (2-11) is defined as the complex wave propagation constant (or simply "propagation constant"). Equation (2-9) may be rewritten as

$$\gamma^2 - (R + j\omega L)(G + j\omega C) = 0 \tag{2-12}$$

which may be thought of as the characteristic equation for the propagation constant γ. Equation (2-12) is a quadratic equation with two roots for γ. Since the transmission line of figure 1 is a passive medium, the propagation constant γ is that root of equation (2-12) for which $\mathrm{Re}\,\gamma \geq 0$ and $\mathrm{Im}\,\gamma \geq 0$ for $\omega > 0$. The propagation constant $\gamma = \alpha + j\beta$ is therefore given by

$$\gamma = [(R + j\omega L)(G + j\omega C)]^{1/2}; \quad \alpha \geq 0, \ \beta \geq 0 \ \text{for} \ \omega > 0 \tag{2-13}$$

where

$\alpha = \mathrm{Re}\,\gamma =$ attenuation constant (m^{-1})

$\beta = \mathrm{Im}\,\gamma =$ phase constant (m^{-1})

SECTION 3

LAGRANGE'S EQUATIONS

3.1 Currents, Charges, Charge Densities

For the sinusoidal steady-state the total charge q(x,t) (coulombs) in the incremental circuit of figure 1 may be expressed in the form

$$q(\mathrm{x},t) = \mathrm{Re}[Q(\mathrm{x})e^{j\omega t}] = \mathrm{Re}[Z(\mathrm{x},t)] = (1/2)[Z(\mathrm{x},t) + \overline{Z}(\mathrm{x},t)] \tag{3-1}$$

where

$$\mathrm{Z}(\mathrm{x},t) \equiv Q(\mathrm{x})e^{j\omega t} = \text{ complex total charge}$$

$$\overline{\mathrm{Z}}(\mathrm{x},t) = \overline{Q}(\mathrm{x})e^{-j\omega t} = \text{ complex conjugate total charge}$$

and the superscript — denotes complex conjugate. The charge density ρ(x,t) (coulombs/m) is related to the charge q(x,t) by

$$q = \int \rho \ \mathrm{dx}$$

or (3-2)

$$\rho = \partial q / \partial \mathrm{x} \equiv q_{\mathrm{x}}$$

The current i(x,t) (amperes) is related to the charge q(x,t) by

$$i = \partial q / \partial t = \dot{q} \tag{3-3}$$

where the superscript • denotes $\partial / \partial t$. Differentiating equation (3-20) with respect to t,

$$\dot{\rho} = \dot{q}_x = i_x = \partial \dot{i} / \partial x \qquad (3\text{-}4)$$

Equation (3-4) is the continuity equation for charge and current. Equation (3-4) differs by a minus sign from the conventional form of the continuity equation because in figure 1 the positive charge at node "a" of the shunt circuit is identified with positive current flowing <u>into</u> the node. In the conventional derivation of the continuity equation, positive charge is identified with positive current flowing <u>out of</u> the node [3] (corresponding to positive current density at a surface in the direction of the <u>outward</u> normal to the surface).

The differential current di, in the shunt circuit consisting of the capacitance/length C in parallel with the conductance/length G, is found from equations (3-2) through (3-4) to be

$$di = d\dot{q} = \dot{\rho}\, dx = \dot{q}_x dx = i_x d\, x \qquad (3\text{-}5)$$

3.2 Current and Voltage Constraints

Conservation of charge and current requires that the sum of the charges or currents into a node must equal the sum of the charges or currents out of the node. Accordingly, the charge density ρ, charge dq, and current di into node "a" are related to the charge densities, charges, and currents in arms 1 and 2 of the shunt circuit by

$$\rho = \rho_1 + \rho_2 \qquad (3\text{-}6)$$

$$di = di_1 + di_2 \qquad (3\text{-}7)$$

$$dq = dq_1 + dq_2 \qquad (3\text{-}8)$$

From equation (3-6) it follows that

$$\dot{\rho} = \dot{\rho}_1 + \dot{\rho}_2 \tag{3-9}$$

The voltage across the unit capacitance C must equal the voltage across the unit conductance G. Accordingly, from equation (3-5) it follows that

$$v(x + dx, t) = \frac{dq_1(x,t)}{Cdx} = \frac{\rho_1 dx}{Cdx} = \frac{\rho_1}{C} = \frac{q_{1x}}{C} = \frac{d\dot{q}_2(x,t)}{Gdx} = \frac{\dot{\rho}_2}{G} \tag{3-10}$$

Therefore,

$$\dot{\rho}_2(x,t) = (G / C)\rho_1(x,t) \tag{3-11}$$

Substituting equation (3-10) into equation (3-9),

$$\dot{\rho} = \dot{\rho}_1 + (G / C)\rho_1 = \dot{q}_{1x} + (G / C)q_{1x} \tag{3-12}$$

Multiplying both sides of equation (3-12) by dx and taking note of equation (3-5), equation (3-12) reduces to

$$\dot{q} = \dot{q}_1 + (G / C)q_1 \tag{3-13}$$

3.3 Time-Averaged Lagrangian Density in the Absence of Ohmic (Dissipative) Losses

For a steady-state sinusoidal waveform, the time-averaged Lagrangian density $\langle \mathcal{L} \rangle$ (J/m) for the circuit of figure 1, in the absence of ohmic losses (R=G=0), is given (with reference to equation (A-8) of the appendix) by

$$\left\langle \mathcal{L}(\dot{Z}, Z_x, Z, \dot{\bar{Z}}, \bar{Z}_x, \bar{Z}, x, t) \right\rangle = \langle m \rangle - \langle e \rangle, \quad R = G = 0 \tag{3-14}$$

where

$\left\langle m(\dot{Z}, Z, \dot{\bar{Z}}, \bar{Z}, x) \right\rangle$ = time-averaged stored magnetic (kinetic) energy density (J/m) for R=G=0.

$\left\langle e(Z_x Z, \bar{Z}_x, \bar{Z}, x) \right\rangle$ = time-averaged stored electric (potential) energy density (J/m) for R=G=0.

Z(x,t) = complex total charge defined by equation (14).

The time-averaged stored magnetic and electric energy densities for R=G=O are given by:

$$\langle m \rangle = \left\langle \frac{1}{2} L i^2 (x, t) \right\rangle = \left\langle \frac{1}{2} L \dot{q}^2 \right\rangle = (1/4) L \dot{Z} \dot{\bar{Z}} = (1/4) L \dot{Z}_1 \dot{\bar{Z}}_1, \quad R = G = O) \tag{3-15}$$

$$\langle e \rangle = \left\langle 1/2) C v^2 (x + dx, t) \right\rangle = \left\langle (1/2) C (q_{1x} / C)^2 \right\rangle = (1/4)(1/C) Z_{1x} \bar{Z}_{1x}, \quad R = G = 0 \tag{3-16}$$

where

$i, \dot{q}, v$ are given by equations (3-5), (3-13), and (3-10), respectively.

$Z_1(x, t) = Q_1(x) e^{j\omega t}$ = complex charge at the distributed capacitance C such that $q_1(x, t) = \mathrm{Re}[Z_1(x, t)]$ (3-17)

Lagrange's equation for the circuit of figure 1 with R=G=O is given by equation (A-10) of the appendix as:

$$\frac{d}{dt}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\dot{\bar{Z}}_1}\right)+\frac{d}{dx}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_{1x}}\right)-\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_1}=0 \tag{3-18}$$

where the partial derivatives of $\langle\mathcal{L}\rangle$ are defined in the appendix from the calculus of variations by treating $\dot{Z}, Z_x, Z, \dot{\bar{Z}}, \bar{Z}_x, \bar{Z}$ as independent variables. The time-averaged Lagrangian density $\langle\mathcal{L}\rangle$ is found from equations (3-14) through (3-17) to be

$$\langle\mathcal{L}\rangle=(1/4)L\dot{Z}_1\dot{\bar{Z}}_1\ -\ (1/4)(1/C)Z_{1x}\bar{Z}_{1x},\quad R=G=O \tag{3-19}$$

It follows that:

$$\frac{\partial\langle\mathcal{L}\rangle}{\partial\dot{\bar{Z}}_1}=(1/4)L\dot{Z}_1=(1/4)j\omega LZ_1 \tag{3-20}$$

$$\frac{d}{dt}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\dot{\bar{Z}}_1}\right)=(1/4)L\ddot{Z}_1=-(1/4)\omega^2LZ_1=-(1/4)\omega^2LQ_1{}^{j\omega t} \tag{3-21}$$

$$\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_{1x}}=-(1/4)(1/C)Z_{1x} \tag{3-22}$$

$$\frac{d}{dx}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_{1x}}\right)=-(1/4)(1/C)Z_{1xx}=-(1/4)(1/C)Q_{1xx}e^{i\omega t} \tag{3-23}$$

$$\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_1}=0 \tag{3-24}$$

Substituting equations (3-20) through (3-24) into equation (3-18),

$$-(1/4)\omega^2 L Q_1 e^{i\omega t} - (1/4)(1/C) Q_{1xx} e^{i\omega t} = 0, \quad R = G = O$$

or

$$Q_{1xx} + \omega^2 LCQ_1 = 0, \quad R = G = O \qquad (3\text{-}25)$$

Designating $\gamma^2 = -\omega^2$ LC, equation (3-25) may be written as:

$$Q_{1xx} - \gamma^2 Q_1 = 0, \quad R = G = 0 \qquad (3\text{-}26)$$

Equation (3-26) has the general solution:

$$Q_1(x) = Q_+ e^{-\gamma x} + Q_- e^{\gamma x}, \quad R = G = O \qquad (3\text{-}27)$$

where:

$\gamma = \alpha + j\beta$ = propagation constant = $j\omega\,(LC)^{1/2}$, R=G=0 (3-28)

α= attenuation constant = 0, R=G=0 (3-29)

β = phase constant = $\omega\,(LC)^{1/2}$, R=G=0 (3-30)

Q_+, Q_- = complex constants

The charge q(x,t) = Re[Z_1(x,t)] is therefore given by

$$\begin{aligned} q_1(x,t) &= \mathrm{Re}\left[\left(Q_+ e^{-\gamma x} + Q_- e^{\gamma x}\right) e^{i\omega t}\right] \\ &= \mathrm{Re}\left[Q_+ e^{j\omega t - \gamma x} - Q_- e^{-j\omega t + \gamma x}\right] \\ &= \mathrm{Re}\left[Q_+ e^{j(\omega t - \beta x)} - Q_- e^{j(-\omega t + \beta x)}\right] \\ &= \mathrm{Re}\left\{Q_+ e^{j\omega\left[t - (x/v_p)\right]} - Q_- e^{-j\omega\left[t + (v_p)\right]}\right\}, \quad R = G = O \end{aligned} \qquad (3\text{-}31)$$

The complex charge $Z_1(x,t)$ in a non-dissipative uniform transmission line consists of two waves travelling with a phase velocity $v_p = (\omega/\beta) = (LC)^{-1/2}$ -- one wave of constant amplitude Q_+ travelling in the +x direction and the other wave of constant amplitude Q_- travelling in the -x direction.

3.4 Time-Averaged Lagrangian Density in the Presence of Ohmic (Dissipative) Losses

The transmission line of figure 1 is a continuous nonconservative electromagnetic system with stored conservative electromagnetic fields and dissipative electromagnetic forces acting on moving charges. The total force field (the so-called "Lorentz force") is derivable from a generalized potential that is a combination of scalar and vector magnetic potentials [4,5]. Lagrange's equation for this total force field may be written in the same form as equation (3-18) provided that one substitutes for the time-averaged Lagrangian density $\langle \mathcal{L} \rangle$, given by equation (3-19), a modified time-averaged Lagrangian density $\langle \mathcal{L}' \rangle$ (the prime indicates "modified") that is a function of the generalized potential [4,5].

The modified time-averaged Lagrangian density $\langle \mathcal{L}' \rangle$ may be expressed as:

$$\langle \mathcal{L}' \rangle = \langle \mathcal{L} \rangle + \langle \mathcal{F} \rangle \tag{3-32}$$

where:

$\langle \mathcal{L} \rangle$ = is given by equation (3-19)

$\langle \mathcal{F} \rangle$ = time-averaged energy density dissipation function (J/m)

The time-averaged energy density dissipation function $\langle \mathcal{F} \rangle$ is generally a complex quantity [6]. The dissipation function $\langle \mathcal{F} \rangle$ is readily determined by defining a complex inductance/length, L', and a complex capacitance/length, C', by:

$$L' = L + (1/j\omega)R = L - j\,(R/\omega) \tag{3-33}$$

$$C' = C + (1/j\omega)G = C - j\,(G/\omega) \tag{3-34}$$

It should be noted that L' and C' have the dimensions of complex permability and complex permittivity, respectively, since [L] = [henries/m] = [permeability], [C] = farads/m] = [permittivity], [R] = [ohms/m] = [resistivity], and [G] = [siemens/m] = [conductivity]. Substituting L' for L and C' for C in equation (3-19):

$$\begin{aligned}\langle \mathcal{L}' \rangle = \langle m' \rangle - \langle e' \rangle &= (1/4)L'\dot{Z}_1\dot{\bar{Z}}_1 - (1/4)(1/C')Z_{1x}\bar{Z}_{1x} \\ &= (1/4)L\dot{Z}_1\dot{\bar{Z}}_1 - (1/4)j(R/\omega)L\dot{Z}_1\dot{\bar{Z}}_1 - (1/4)[C - j(G/\omega)]^{-1}Z_{1x}\bar{Z}_{1x}\end{aligned} \tag{3-35}$$

where:

$\langle m' \rangle$ = time-averaged complex magnetic energy density

$$= (1/4)L'\dot{Z}_1\dot{\bar{Z}}_1 = (1/4)L\dot{Z}_1\dot{\bar{Z}}_1 - (1/4)j(R/\omega)L\dot{Z}_1\dot{\bar{Z}}_1 \tag{3-36}$$

$\langle e' \rangle$ = time-averaged complex potential energy density

$$= (1/4)(1/C')Z_{1x}\bar{Z}_{1x} = (1/4)[C - j(G/\omega]^{-1}Z_{1x}\bar{Z}_{1x} \tag{3-37}$$

Substituting equations (3-35) and (3-19) into equation (3-32), the time-averaged energy density dissipation function $\langle \mathcal{F} \rangle$ is given by

$$\langle \mathcal{F} \rangle = (1/4)(L' - L)\dot{Z}_1\dot{\bar{Z}}_1 - (1/4)[(1/C') - (1/C)]Z_{1x}\bar{Z}_{1x}$$

$$= (1/4)(R/j\omega)\dot{Z}_1\dot{\bar{Z}}_1 + (1/4)[(G/C)/(j\omega C + G)]Z_{1x}\bar{Z}_{1x} \qquad (3\text{-}38)$$

Substituting $\langle \mathcal{L}' \rangle$ for $\langle \mathcal{L} \rangle$ in equation (3-18), Lagrange's equation for the system of figure 1 becomes:

$$\frac{d}{dt}\left(\frac{\partial\langle \mathcal{L}' \rangle}{\partial\dot{\bar{Z}}_1}\right) + \frac{d}{dx}\left(\frac{\partial\langle \mathcal{L}' \rangle}{\partial\bar{Z}_{1x}}\right) - \frac{\partial\langle \mathcal{L}' \rangle}{\partial\bar{Z}_1} = 0 \qquad (3\text{-}39)$$

where $\langle \mathcal{L} \rangle$ is given by equation (3-35). It follows from equations (3-20) through (3-25) that equation (3-39) reduces to:

$$Q_{1xx} + \omega^2 L' C' Q_1 = 0 \qquad (3\text{-}40)$$

which has the general solution

$$Q_1(x) = Q'_+ \exp(-\gamma' x) + Q'_- \exp(\gamma' x) \qquad (3\text{-}41)$$

where:

$$\gamma' = \gamma' + j\beta' = \text{propagation constant} = j\omega(L'C')^{1/2}$$
$$= j\omega[L + (R/j\omega)]^{1/2}[C + (G/j\omega)]^{1/2} = [(j\omega L + R)(j\omega C + G)]^{1/2}$$
$$= j\omega(LC)^{1/2}\{[1 - (jR/\omega L)][1 - (jG/\omega C)]\}^{1/2} \qquad (3\text{-}42)$$

$$\alpha' = \text{attenuation constant} = \sqrt{2}\left\{\left[\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)\right]^{1/2}+RG-\omega^2LC\right\}^{1/2} \tag{3-43}$$

$$\beta' = \text{ phase constant} = \sqrt{2}\left\{\left[\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)\right]^{1/2}+RG+\omega^2LC\right\}^{1/2} \tag{3-44}$$

$Q_+^{\cdot}, Q_-^{\cdot}$ = complex constants

The charge $q_1(x,t) = \text{Re}\,[Z_1(x,t)]$ is therefore given by:

$$\begin{aligned} q_1(x,t) &= \text{Re}\left\{\left[Q_+^{\cdot}\ \exp\left(-\gamma' x\right)\ +\ Q_-^{\cdot}\ \exp\left(\gamma' x\right)\right]\right\} \\ &= \text{Re}\ \left[Q_+^{\cdot}\ \exp\left(j\omega t - \gamma' x\right)\ -\ Q_-^{\cdot}\ \exp\left(-j\omega t + \gamma' x\right)\right] \\ &= \text{Re}\ \left[Q_+^{\cdot} e^{-\alpha' x} e^{j(\gamma'\omega t - \beta' x)} - Q_-^{\cdot} e^{\alpha' x} e^{j(-\gamma'\omega t + \beta' x)}\right] \\ &= \text{Re}\ \left[Q_+^{\cdot} e^{-\alpha' x} e^{j\omega\left[t-\left(x/v'_p\right)\right]} - Q_-^{\cdot} e^{\alpha' x} e^{-j\omega\left[t+\left(x/v'_p\right)\right]}\right] \end{aligned} \tag{3-45}$$

The complex charge $Z_1(x,t)$ in a dissipative uniform tranmission line consists of two waves travelling with a phase velocity

$$v_p^{\cdot} = (\omega / \beta') = (LC)^{-1/2}\ \left\{\left[1-(jR/\omega L)\right]\left[1-(jG/\omega C)\right]\right\}^{-1/2}$$

-- one wave of decreasing amplitude $Q_+^{\cdot}\ e^{-\alpha' x}$ travelling in the +x direction and the other wave of decreasing amplitude $Q_-^{\cdot}\ e^{-\alpha' x}$ travelling in the -x direction.

It should be noted that the propagation constants given by equations (3-42) and (2-13) are identical. Therefore, the results obtained from Lagrange's equations agree with those obtained from the telegrapher's equations, as should be the case.

SECTION 4

CONCLUSIONS

Complex constitutive system parameters are used to derive the form of Lagrange's equations for a non-conservative continuous system with a steady-state sinusoidal excitation. Confirmation is demonstrated by considering the case of a uniform transmission line with series and shunt ohmic losses, sinusoidal steady-state excitation, and wave propagation in a transverse electromagnetic (TEM) mode. The resulting voltages, currents, propagation constant, and characteristic impedance are in agreement with the well-known results derivable from the telegrapher's equations.

The forms of Lagrange's equations for sinusoidal steady-state excitation and for dissipative continuous systems are not readily found in the literature. (The forms of Lagrange's equations for transient excitations, dissipative discrete systems, and non-dissipative continuous systems are well known.)

In particular, some of the concepts that are addressed in this paper are:

(1) The form of Lagrange's equations for distributed versus discrete systems;

(2) The time-averaged Lagrangian density for a continuous system with a force derivable from a generalized potential comprising scalar and vector potentials; and

(3) The time-averaged energy density dissipative function for a non-conservative electromagnetic system with series and shunt ohmic losses.

For the case of a dissipative uniform transmission line, Lagrange's equations offer no advantages over that of the telegrapher's equations or equivalent equations based on Kirchhoff's voltage and current laws or Poynting's complex power theorems. The reason is that, for this case, at least as much effort is required to determine the stored energy densities and dissipation function as is required to express the fields directly.

However, having confirmed the basic soundness of the Lagrangian formulation for the analysis of dissipative transmission lines and having established the form of Lagrange's equations for the simplest case of a uniform transmission line, this paper is expected to facilitate the use of Lagrange's equations for transmission lines with more complex geometries, particularly geometries for which the telegrapher's equations or Maxwell's equations may not be tractable. A Lagrangian formulation is superior in cases where the fields are not readily expressible in a suitable coordinate system but where the stored energy densities and energy density dissipation functions are expressible without reference to a particular set of generalized coordinates[4].

A suitable candidate application might be a distributed transmission line in which there is mutual coupling among several elements such as in an infinite array of closely-packed monopole antenna elements on a mesh groundscreen above earth. The forms of Lagrange's equations developed in this paper should be applicable to such a transmission line but the detailed application for this case remains to be investigated. It is unclear, however, based on the present analysis, that Lagrange's equations will offer any particular advantage for that application.

LIST OF REFERENCES

1. Hill, D. A. and J. R. Wait, September-October 1978, "Surface Wave Propagation on a Rectangular Bonded Wire Mesh Located Over the Ground," *Radio Science*, Vol. 13, No. 5 pp. 793-799.

2. Adler, R. B., L. J. Chu, and R. M. Fano, 1960, *Electromagnetic Energy Transmission and Radiation* (John Wiley and Sons, New York), pp. 60-62, 180

3. Frank, N. H., 1950, *Introduction to Electricity and Optics* (McGraw-Hill, N.Y.), p. 186.

4. Goldstein, H., 1951, *Classical Mechanics* (Addison-Wesley, Cambridge, MA, 1st edition), pp. 19-22, 38-44.

5. Panofsky, W. K., and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Cambridge, MA, 1955), pp. 369-371.

6. op. cit. 5, equations (24) through (34).

APPENDIX

LAGRANGE'S EQUATIONS FOR A CONTINUOUS SYSTEM WITH A STEADY-STATE SINUSOIDAL WAVEFORM

Consider a continuous system that is conservative with a force field derivable from a scalar potential or is nonconservative with a force field derivable from a generalized potential comprising scalar and vector potentials. For such a system with n coupled elements, the Lagrangian density L is defined as [1,3].

$$\mathcal{L} = m - e = \mathcal{L}(\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, q_{1x}, q_{2x}, \ldots q_{nx}; q_1, q_2, \ldots q_n, \mathrm{x}, t) \qquad \text{(A-1)}$$

where:

m = kinetic (magnetic) energy density of the system

e = potential (electric) energy density of the system

q_i = position (charge) of i^{th} element that is a function of spatial and time coordinates x and t, respectively

superscript • denotes $\partial/\partial t$

subscript x denotes $\partial/\partial x$

Hamilton's principle states that the motion (field) of the system from point P_1 (x_1,t_1) to point P_2 (x_2,t_2) is such that the line integral Φ is given by [2,3]

$$\Phi = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathcal{L} \; dx \, dt = \text{ extremum for the path of motion} \qquad \text{(A-2)}$$

i.e., the system will travel only that path for which the integral of the Lagrangian density from point P_1 to point P_2 is either a minimum or maximum.

Equivalently equation (A-2) states that the variation $\delta\Phi$ of the line integral Φ for a fixed t_1, t_2, x_1, x_2 is zero [2,3], i.e.,

$$\delta\Phi = \delta\left[\int_{t_1}^{t_2}\int_{x_1}^{x_2} \mathcal{L}\, dx\, dt = 0\right] \tag{A-3}$$

With the substitution of equation (A-1) into equation (A-3) and the utilization of the calculus of variations [1,2,3], equation (A-3) reduces to

$$\delta\Phi = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left\{\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial q_{ix}}\right) - \frac{\partial \mathcal{L}}{\partial q_i}\right]\right\} dx\, dt = 0 \tag{A-4}$$

where the partial derivatives are defined in the calculus of variations by treating q_i, q_{ix}, q_i as independent variables. Since the q_i variables are independent $\delta\Phi = 0$ if and only if the coefficients of dq_i separately vanish [1,3]. Accordingly,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial q_{ix}}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0; \qquad i = 1, 2, \ldots.. n \tag{A-5}$$

The n equations given by equation (A-5) are Lagrange's equations for a continuous system.

For continuous systems with a steady-state sinusoidal waveform, the <u>time-averaged</u> Lagrangian density may be more convenient to compute than the <u>instantaneous</u> Lagrangian density given by equation (A-4). Lagrange's equations that are a function of a time-averaged Lagrangian density may be derived by dividing both sides of eq (A-3) by $t_2 - t_1$. Accordingly,

$$\frac{1}{t_2-t_1}\delta\Phi = \frac{1}{t_2-t_1}\delta\left[\int_{t_1}^{t_2}\int_{x_1}^{x_2}\mathcal{L}\ \ \mathrm{dx}\, dt\right] = \delta\left[\left(\frac{1}{t_2-t_1}\right)\int_{t_1}^{t_2}\int_{x_1}^{x_2}\mathcal{L}\ \ \mathrm{dx}dt\right] = \delta\int_{x_1}^{x_2}\langle\mathcal{L}\rangle\, d\mathrm{x} \qquad \text{(A-6)}$$

$$\text{where } \langle\mathcal{L}\rangle = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\mathcal{L}\ \mathrm{dt} = \text{ time - averaged Lagrangian density}$$

and the brackets $\langle\ \rangle$ indicate time-averaged Lagrangian density.

For a steady-state sinusoidal waveform (t_2-$t_1 \rightarrow \infty$) the time-averaged Lagrangian density is in general a function of $\dot{Z}_i\dot{\bar{Z}}_i$, $Z_{ix}\bar{Z}_{ix}$, $Z_i\bar{Z}_i$ where

$$Z_i(x,t) = Q_i(\mathrm{x})\ e^{j\omega t} \text{ with } \mathrm{Re}\left[Z_i(\mathrm{x},t)\right] = q_i(\mathrm{x},t) \qquad \text{(A-7)}$$

For such a waveform the time-averaged Lagrangian density $\langle\mathcal{L}\rangle$ may be written in the functional form

$$\langle\mathcal{L}\rangle = \left\langle\mathcal{L}\left(\sum_{i=1}^{n}\left[\dot{Z}_i,\ Z_{ix},\ Z_i,\ \dot{\bar{Z}}_i,\ \bar{Z}_{ix},\ \bar{Z}_i\right],\ \mathrm{x},t\right)\right\rangle = \sum_{i=1}^{n} a_i\dot{Z}_i\dot{\bar{Z}}_i + b_i Z_{ix}\bar{Z}_{ix} + c_i Z_i\bar{Z}_j \qquad \text{(A-8)}$$

where a_i, b_i, c_i are complex constants. Equations (A-8) and (A-6) are analogous to equations (A-1) and (A-3), respectively, Lagrange's equations that are a function of the time-averaged Lagrangian given by equation (A-8) are exactly analogous to equations (A-5). Accordingly, the corresponding Lagrange's equations are given by:

$$\frac{d}{dt}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\dot{Z}_i}\right) + \frac{d}{d\mathrm{x}}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial Z_{ix}}\right) - \frac{\partial\langle\mathcal{L}\rangle}{\partial Z_i} = 0;\quad i = 1,2,\ldots n \qquad \text{(A-9a)}$$

$$\frac{d}{dt}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\dot{\bar{Z}}_i}\right)+\frac{d}{dx}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_{ix}}\right)-\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_i}=0;\quad i=1,2,\dots n \tag{A-9b}$$

It should be noted in equation (A-9) that $\langle\mathcal{L}\rangle$ is not an analytic function of the complex parameters $Z,\bar{Z}$ because $\langle\mathcal{L}\rangle$ given by equation (A-8) does not satisfy the Cauchy-Riemann equations. Therefore, the derivative of $\langle\mathcal{L}\rangle$ with respect to either Z or $\bar{Z}$ is not defined (lim $\Delta\langle\mathcal{L}\rangle/\Delta Z$ as $\Delta Z\rightarrow 0$ is dependent upon the path taken in the Z plane). However, the partial derivatives in equation (A-9) are defined in the exact analogous manner as the partial derivatives in equation (A-4), i.e., by treating $\dot{Z}_i\dot{\bar{Z}}_i$, $Z_{ix}\bar{Z}_{ix}$, $Z_i\bar{Z}_i$ as independent variables.

Equation (A-9a) leads to a solution for $\bar{Z}_i$ whereas equations (A-9b) leads to a solution for Z_i. Both sets of equations give the same result for the propagation constant. Since $\bar{Z}_i$ is derivable from Z_i and vice versa, equations (A-9a) and (A-9b) are redundant. Only one set of equations need be considered. Selecting equation (A-9b), a *non-redundant set* of Lagrange's equations are given by:

$$\frac{d}{dt}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\dot{\bar{Z}}_i}\right)+\frac{d}{dx}\left(\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_{ix}}\right)-\frac{\partial\langle\mathcal{L}\rangle}{\partial\bar{Z}_i}=0;\quad i=1,2,\dots n \tag{A-10}$$

A more restricted form of Lagrange's equations for a steady-state sinusoidal waveform may be obtained by considering only the space-dependent complex amplitude Q(x) of the variable q(x,t) in equation (A-7). The corresponding Lagrangian density $\mathcal{L}$ is then of the form

$$\mathcal{L}=\mathcal{L}\left(\sum_{i=1}^{n}[Q_{ix},Q_i,\bar{Q}_{ix},\bar{Q}_i],\ x\right) \tag{A-11}$$

Hamilton's principle states that the variation $\delta\Phi$ of the line integral

$\Phi = \int_{x_1}^{x_2} \mathcal{L}\, dx$ for a fixed x_1 and x_2 is zero, i.e.

$$\delta\Phi = \delta\int_{x_1}^{x_2} \mathcal{L}\, dx = 0 \tag{A-12}$$

The calculus of variations yields in a manner exactly analogous to that of equation (A-4), a variation $\delta\Phi$ given by

$$\delta\Phi = \int_{x_1}^{x_2}\left\{\sum_{i=1}^{n}\left[\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial Q_{ix}}\right)-\frac{\partial \mathcal{L}}{\partial Q_i}\right]\delta Q_i + \sum_{i=1}^{n}\left[\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \bar{Q}_{ix}}\right)-\frac{\partial \mathcal{L}}{\partial \bar{Q}_i}\right]\delta\bar{Q}_i\right\}dx = 0 \tag{A-13}$$

where $Q_{ix}, \bar{Q}_{ix}, Q_i, \bar{Q}_i$ are treated as independent variables. Since Q_i and $\bar{Q}_i$ are independent, $\delta\Phi = 0$ if, and only if, the coefficients of δQ_i and $\delta\bar{Q}_i$ separately vanish. Accordingly,

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial Q_{ix}}\right)-\frac{\partial \mathcal{L}}{\partial Q_i} = 0 \tag{A-14}$$

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \bar{Q}_{ix}}\right)-\frac{\partial \mathcal{L}}{\partial \bar{Q}_i} = 0 \tag{A-15}$$

Since Q_i is derivable from $\bar{Q}_i$ and vice versa, equations (A-14) and (A-15) are redundant. Therefore, either set of equations (A-14) or (A-15) is sufficient.

LIST OF REFERENCES

APPENDIX

1. Goldstein, H., 1951, *Classical Mechanics* (Addison-Wesley, Cambridge, MA, 1951, 1st edition), pp. 350-355. For continuous systems, unlike discrete systems, the Lagrangian density is a function of q_{ix} and Lagrange's equations contains the additional term $\frac{d}{dx}\left(\frac{\partial L}{\partial q_{ix}}\right)$. [Compare equations (2-16) and (1-53) of Reference 1.)

2. op. cit. 1, pp. 30-38.

3. Panofsky, W. K. H., and M. Phillips, 1955, *Classical Electricity and Magnetism* (Addison-Wesley, Cambridge, MA), pp. 364-366. Please note that in equation (24-10) the total derivatives d/dx of $\partial\ /\partial h_x$ is intended (even though the partial derivative $\partial/\partial x$ is denoted) because what is intended is the derivative of $\partial\ /\partial h_x$ with respect to its explicit and implicit dependence upon x. For exampie, if $\partial\ /\partial h_x = f\ (h_x, h, x)$ then the derivative of f with respect to its explicit and implicit dependence upon x is

$$\frac{df}{dx} = \frac{\partial f}{\partial \eta_x}\frac{d\eta_x}{dx} + \frac{\partial f}{\partial \eta}\frac{d\eta}{dx} + \frac{\partial f}{\partial x}$$

whereas the derivative of f with respect to only its explicit dependence upon x is simply $\partial f/\partial x$. Similarly, in equation (24-10), the total derivative d/dt of $\partial / \partial \dot{\eta}$ is intended (even though the partial derivative $\partial/\partial t$ is denoted).