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A HIERARCHICAL GAMMA/WEIBULL REGRESSION MODEL FOR TARGET DETECTION TIMES

by

Wang Chia-Fu

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Thesis Advisor: Patricia A. Jacobs

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Combat models often involve target detection times which may vary with different observers due to characteristics of personnel , or detection systems. They may also be affected by different environmental factors such as visual levels, sea states, terrains, etc. There is often interest in quantifying the effects of different observer characteristics and environmental factors on detection times. A hierarchical gamma/Weibull regresson model is considered which can incorporate observer characteristics and environmental effects which may influence the time to detect targets. Numerical procedures for the estimation of parameters of the hierarchical gamma Weibull model based on maximum likelihood are described. Results of simulation experiments to study small sample behavior of the estimates are reported.

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A Hierarchical Gamma/Weibull Regression Model
For
Target Detection Times

by

Wang, Chia-Fu
Lieutenant , Republic of China Navy
Naval Academy,1986

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Author: Wang Chia-Fu
Wang,Chia-Fu

Approved by: Patricia A. Jacobs
Patricia A, Jacobs, Thesis Advisor

Donald P. Gaver
D.P. Gaver, Second Reader

P. Purdue
Peter Purdue, Chairman
Department of Operations Research

ABSTRACT

Combat models often involve target detection times which may vary with different observers due to characteristics of personnel, or detection systems. They may also be affected by different environmental factors such as visual levels, sea states, terrains, etc. There is often interest in quantifying the effects of different observer characteristics and environmental factors on detection times. A hierarchical gamma/Weibull regression model is considered which can incorporate observer characteristics and environmental effects which may influence the time to detect targets. Numerical procedures for the estimation of parameters of the hierarchical gamma/Weibull model based on maximum likelihood are described. Results of simulation experiments to study small sample behavior of the estimates are reported.

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- All the brothers and sisters in church for their love in Christ.

I. INTRODUCTION

A. BACKGROUND

Combat models often involve target detection times which may vary with different observers due to characteristics of personnel, or detection systems. They may also be affected by different environmental factors such as visual levels, sea states, terrains, etc. There is often interest in quantifying the effects of different observer characteristics and environmental factors on detection times. In Chapter III a hierarchical gamma/Weibull regression model [Ref. 1] is considered which can incorporate observer characteristics and environmental effects which influence the time to detect targets. Numerical procedures for the estimation of parameters of the hierarchical gamma/Weibull model based on maximum likelihood are described. Results of simulation experiments to study small sample behavior of the estimators are reported. In the remainder of this chapter, two models related to the one considered in this thesis will be described. Numerical procedures for parametric estimation for these models are studied in Bae [Ref. 2]. Experience gained concerning the numerical stability of these procedures will be described.

B. TWO RELATED MODELS

Two parametric models for the distribution of time to detect targets were considered in a thesis written by a Korean officer,Mr.Bae in Sept.,1989 [Ref. 2].

<u>Model 1</u> :[Ref.2:pp.2-14] supposes that there are M observers. Observer i has parameter A_i which characterizes the ability of the observer to detect a target. Observer i is presented with N_i targets. The parameters of all M observers, $A_1,A_2,\ldots,A_M$, are assumed independent with a common two parameter gamma

distribution. Given A_i, the detection times of observer i are conditionally independent, having Weibull distributions with known shape parameter $e^{-\xi_i}$ and known scale parameter μ_{ij} for $j=1,2,..,N_i$. Some of the detection times may be censored. Bae [Ref. 2:pp. 2-20] reports on numerical procedures to find the maximum likelihood estimates for the shape and scale parameters of the gamma distribution.

Model 2 : is a Weibull regression model [Ref.2:pp.21-30]. It assumes observer i has explanatory variables x_{ij1}, x_{ij2}, ..., x_{ijp} relating to his j^{th} target representing factors which influence his time to detection. The detection times for the observers are independent random variables having Weibull distributions. Again, some of the detection times may be censored. The scale parameter of the Weibull distribution for the detection times of the j^{th} target by observer i is of the form $\mu_{ij}=e^{x_{ij}\beta_{ij}}$ where $x_{ij}\beta=\sum_{k=1}^{p} x_{ijk}\beta_k$; the shape parameter is of the form $e^{-\xi_i}$. Bae [Ref. 2:pp. 21-36] reports on numerical procedures to find the maximum likelihood estimates of $\{\xi_i\}$ and $\{\beta_k\}$.

C. OBJECTIVE AND METHODOLOGY IN THE COMBINED MODEL

The main effort of this thesis will be to study a full hierarchical gamma/Weibull regression model which is a combination of the two previous models. Since the models considered in Bae [Ref. 2] will be special cases of the one considered in this thesis, it is expected that instabilities found in the numerical procedures to estimate the parameters in the two previous models will appear for the combined model. Two numerical instabilities that result from the numerical procedures of Bae [Ref. 2] are described below.

In Model 1, a modified Newton-Raphson method [Ref. 3] is applied to solve the nonlinear system of equations for the maximum likelihood estimates of the gamma parameters. It is found that, particularly for small sample sizes, the slope of the likelihood surface is very flat resulting in numerical instability for estimation of the scale parameter η. As a result, in the full hierarchical model presented here we have used a *single parameter gamma distribution* as the second stage model. The maximum likelihood estimate of the parameter is found by the bisection root search method [Ref. 4].

A numerical overflow phenomenon was occasionally found in the Weibull regression model while iterating the Newton procedure for $\{\xi_i\}$ in Model 2 [Ref.2:p.25]. Once again, the flatness of the relationship determining the estimate of $\{\xi_i\}$ resulted in this instability. As a result, the Newton procedure to determine $\{\xi_i\}$ in the gamma/Weibull regression model of this thesis has some checks to detect this numerically instability.

In Chapter II, a numerical procedure is presented to find the maximum likelihood estimate for a *single parameter gamma version* of Model 1. In Chapter III, the full hierarchical model is presented and numerical procedures to estimate the parameters given. In both chapters, simulation results are presented to study the small sample behavior of the estimates. All simulations were carried out on an IBM 3179 G mainframe computer at the Naval Postgraduate School using the APL GRAFSTAT random number package [Ref. 5],[Ref. 6].

We hope that the hierarchical gamma/Weibull regression model and these estimation procedures will be a useful tool to describe and predict target detection times, which are one aspect of the effect of human performance on the battlefield.

II. A SINGLE PARAMETER GAMMA HIERARCHICAL MODEL

A. MODEL DESCRIPTION

This model is very similar to Model 1 in [Ref. 2][1]. The difference is that the parameters A_i, i = 1,2,...,M, which reflect the abilities of observer i to detect targets, are assumed to be independent having a *single-parameter gamma distribution, GAM(α,α)* rather than having two parameters. The form of gamma density function used here is

$$g(\theta) = \frac{\alpha(\alpha\theta)^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\alpha\theta} \qquad for \quad \theta > 0. \tag{2.1}$$

For numerical reasons we will parametrize α as e^{η}. For convenience we will reiterate the remainder of the assumptions of Model 1 given in [Ref. 2]. Given $A_i = \theta$, the times it takes for observer i to detect target j, j = 1,2,...,N_i, denoted by U_{ij}, are assumed to be conditionally independent random variables with Weibull distributions, $WEI\left(\mu_{ij}, e^{-\xi_i}\right)$ having cumulative distribution function

$$P\left\{U_{ij} \le t \,|\, A_i = \theta\right\} = 1 - \exp\left\{-\theta(t/\mu_{ij})^{e^{-\xi_i}}\right\} \qquad t \ge 0 \tag{2.2}$$

independent of other observers. The variations of the A_i are introduced to represent the individual differences between the observers. When the i^{th} observer is presented with his j^{th} target, only an opportunity time O_{ij} is allowed for him to detect it. An observer either successfully detects the target within this time or never detects it.

---

[1] In the model of [Ref. 2], a two parameter gamma distribution is used.

Data for the i^{th} observer consist of times of detection for the successes and the lengths of opportunity times for the failures. For each $i = 1,2,...,M$, $j = 1,2,...,N_i$, let

$$Y_{ij} = \min(\ln U_{ij}, \ln O_{ij}) \tag{2.3}$$

and

$$\Delta_{ij} = \begin{cases} 1 & \text{if } U_{ij} \leq O_{ij} \\ 0 & \text{otherwise} \end{cases} . \tag{2.4}$$

The Y_{ij} are the censored ln-detection times and Δ_{ij} is an indicator of whether the ln-time to detect the j^{th} target by the i^{th} observer is censored or not. Let

$$C_i = \sum_{j=1}^{N_i} \Delta_{ij} \tag{2.5}$$

be the number of targets detected by observer i. In this chapter, we will assume the Weibull parameters $\{\mu_{ij}\}$ and $\{\xi_i\}$ are known constants, and we are only interested in estimating the gamma parameter η. In the next section, the likelihood function for η is given and the bisection method for finding the maximum likelihood estimate of η is described. In the final section, we present results of a simulation study of the behavior of the estimators for small sample sizes.

B. MAXIMUM LIKELIHOOD ESTIMATION AND THE BISECTION ROOT SEARCH

1. The Likelihood Function

Given $A_i = \theta$, the conditional likelihood function [Ref.2:p.3] for observer i using the censored ln-times Y_{ij} is

$$L_i(\mu,\xi\,|\,\theta) = \prod_{j=1}^{N_i}\left[\theta e^{(y_{ij}-\ln\mu_{ij})e^{-\xi_i}} e^{-\xi_i}\right]^{\Delta_{ij}} \exp\left[-\theta e^{(y_{ij}-\ln\mu_{ij})e^{-\xi_i}}\right]. \tag{2.6}$$

Let

$$S_i = \sum_{j=1}^{N_i} \exp\left\{(y_{ij} - \ln\mu_{ij})e^{-\xi_i}\right\} \tag{2.7}$$

and

$$K_i = \exp\left\{\sum_{j=1}^{N_i}\Delta_{ij}\left[(y_{ij}-\ln\mu_{ij})e^{-\xi_i} - \xi_i\right]\right\}. \tag{2.8}$$

Equation (2.6) can be rewritten as follows :

$$L_i(\mu,\xi\,|\,\theta) = \theta^{C_i} K_i \exp(-\theta S_i)\ . \tag{2.9}$$

From the equations (2.1) and (2.9), it follows that the unconditional likelihood function for observer i is

$$
\begin{aligned}
L_i(\mu,\xi,\alpha) &= \int_0^\infty L_i(\mu,\xi\,|\,\theta)g(\theta)d\theta \\
&= \int_0^\infty \theta^{C_i} e^{-\theta S_i} \frac{\alpha(\alpha\theta)^{\alpha-1}}{\Gamma(\alpha)} e^{-\alpha\theta} d\theta \\
&= K_i \frac{\alpha^\alpha}{(S_i+\alpha)^{C_i+\alpha}} \prod_{k=0}^{C_i-1} (\alpha+k) \ .
\end{aligned} \tag{2.10}
$$

In equation (2.10), if $C_i = 0$, then $\prod_{k=0}^{C_i-1} (\alpha+k) = 1$. Recall the parametrization $\alpha = e^\eta$. The unconditional ln-likelihood function for observer i can be rewritten as

$$
\begin{aligned}
&\ln L_i(\mu,\xi,\eta) = \\
&\ln K_i + e^\eta \eta - (C_i+e^\eta)\ln(S_i+e^\eta) + \sum_{k=0}^{C_i-1} \ln(e^\eta+k)
\end{aligned} \tag{2.11}
$$

where if $C_i = 0$, then $\sum_{k=0}^{C_i-1} \ln(e^\eta+k) = 0$. Since the observers are independent, the unconditional ln-likelihood for all M observers is

$$
\begin{aligned}
&\ln L = \sum_{i=1}^{M} \ln L_i = \\
&\sum_{i=1}^{M} \left\{ \ln K_i + e^\eta \eta - (C_i+e^\eta)\ln(S_i+e^\eta) + \sum_{k=0}^{C_i-1} \ln(e^\eta+k) \right\}.
\end{aligned} \tag{2.12}
$$

In this chapter, we will focus attention only on finding the maximum likelihood estimate of η. The derivative of the ln-likelihood with respect to η is

$$\frac{\partial \ln L}{\partial \eta} = f(\eta) =$$

$$e^{\eta} \sum_{i=1}^{M} \left\{ \eta + \frac{S_i - C_i}{S_i + e^{\eta}} - \ln(S_i + e^{\eta}) + \sum_{k=0}^{C_i - 1} \frac{1}{e^{\eta} + k} \right\}. \tag{2.13}$$

The maximum likelihood estimate for η is the solution of the equation $f(\eta) = 0$.

2. Bisection Root Search for η

Initially, the Newton procedure was used to numerically solve the equation $f(\eta) = 0$. Unfortunately, the procedure frequently either converged to an unreasonable number or encountered problems of numerical instability. Plots of $f(\eta)$ indicate that f can have multiple zeros. A representative plot of f is shown in Figure A [2] (next page). In Figure A, one $\triangledown$ symbol indicates a zero of f, while two $\triangledown$ symbols indicate the reasonable root to be used as the estimate for η.

[2] This is a typical graph for $f(\eta)$ which was generated for simulated data with O=10, M=15, N=15 and random seed = 16807. The true value of η is 1.

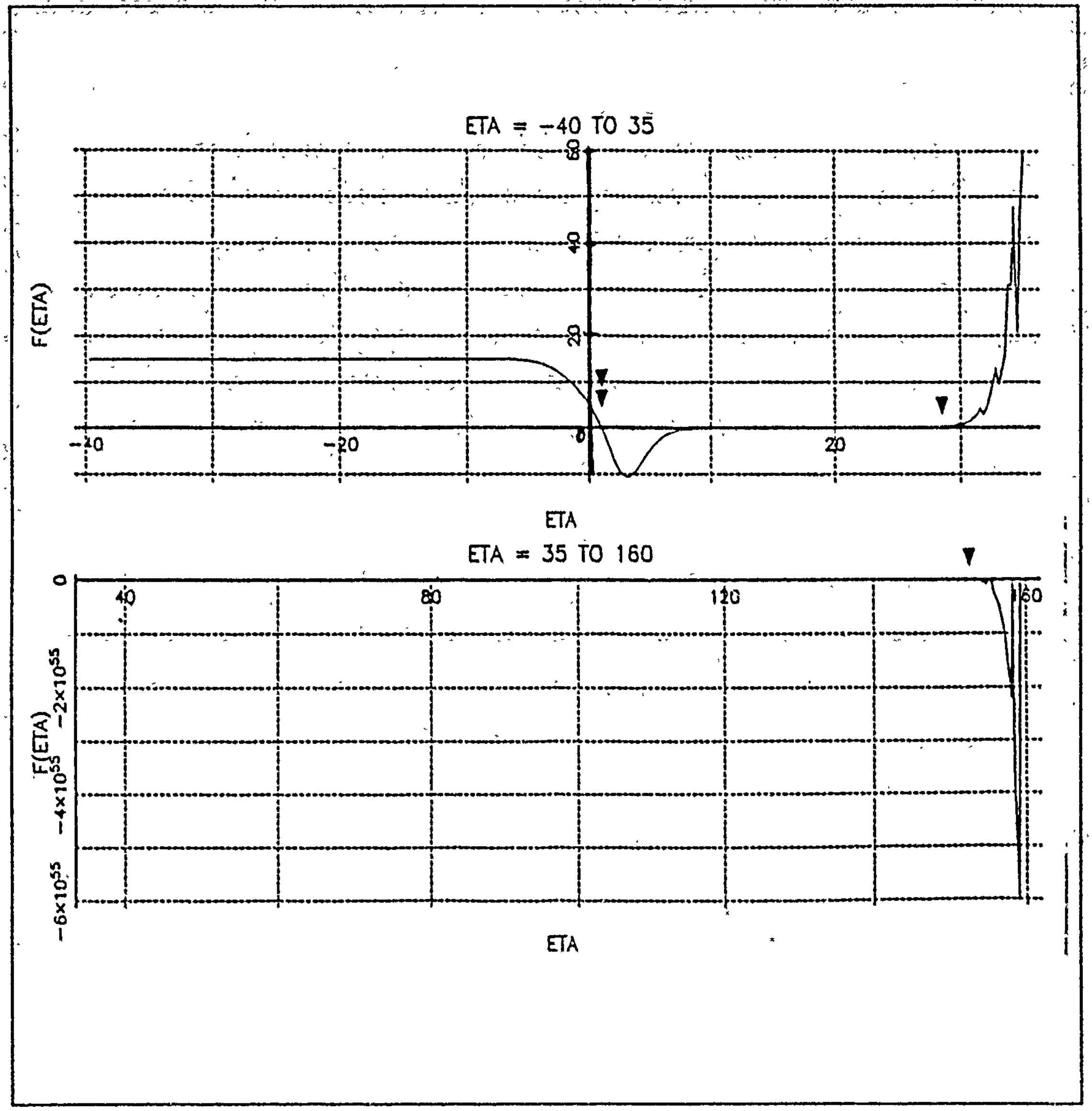

Figure A Multiple Roots of the Derivative of the ln-Likelihood Function for η

As a result of these possible multiple zeros, the bisection root search method [Ref.4] is used to solve $f(\eta)=0$. The following sub-section will detail the estimation procedure for η.

a. Initial Estimate for η

The initial estimate for η is similar to its initial estimate in the two-parameter gamma model [Ref. 2], and is obtained by setting [3] $\beta_0 = 0$ in that procedure. Consequentially, the initial estimate of η is

$$\eta^0 = -\ln\left[\hat{m}_2^2 - \hat{m}_1^2 - \frac{\pi^2}{6}\right]. \tag{2.14}$$

where

$$\hat{m}_1 = -0.5772$$

$$\hat{m}_2 = \frac{\sum_{i=1}^{M}\sum_{j=1}^{N_i} \Delta_{ij}\left[(y_{ij} - \ln\mu_{ij})e^{-\xi_i}\right]^2}{\sum_{i=1}^{M}\sum_{j=1}^{N_i} \Delta_{ij}}. \tag{2.15}$$

If $\hat{m}_2^2 - \hat{m}_1^2 - \frac{\pi^2}{6} < 0$, set $\eta^0 = 0$ [4].

b. Determination of Bounds for the Root

The purpose of this step is to bound the root that will be used as the estimate between two points, called η_L and η_R. One of these points is η^0 and the other is found as follows :

---

[3] In Mr.Bae's model [Ref.2], the regular gamma distribution is parametrized with shape parameter $\nu = e^{\eta+\beta_0}$ and scale parameter $\alpha = e^{\eta}$. Setting $\beta_0 = 0$, $\alpha = \nu = e^{\eta}$ yields the one-parameter gamma distribution.

[4] The estimate $\hat{m}_2^2 - \hat{m}_1^2 - \frac{\pi^2}{6} < 0$ suggests that the variability of the data is small. Hence, it might be better to set η^0 equal to a large positive number, such as 13. A topic for future research is to explore better initial values for η in this default case.

Set $\begin{array}{l} \eta_L=\eta^0-k\times\delta \\ \eta_R=\eta^0+k\times\delta \quad k=1,2,3...... \end{array}$ where δ is a small increment.

Compare the signs of $f(\eta_L)$ and $f(\eta_R)$ with $f(\eta^0)$ until the first different sign from $f(\eta^0)$ is found. This yields a bound on the closest root to η^0.

c. Bisection Search

After determining the left bound, η_L and right bound ,η_R, we begin the bisection search by picking the midpoint between them, called η_M, and computing the value of $f(\eta_M)$. If $f(\eta_M)$ and $f(\eta_L)$ have the same signs, the root is on the other side of η_M; thus, move the left bound η_L to midpoint η_M. If the signs are different, move the right bound to the midpoint. Iterate the bisection search until $\left|\frac{\eta_L-\eta_R}{\eta_M}\right|<0.01$. The maximum likelihood estimate of η is η_M.

C. IMPLEMENTATION OF THE SIMULATION AND RESULTS

1. Simulation

In order to study the sampling properties of the maximum likelihood estimate of η, different numbers of observers (M), targets (N_i), and opportunity times (O_{ij}) are used as inputs to the simulation. Each set of inputs (M,N_i,O_{ij}) has a simulation run of (R) 100 replications and each run begins with the same initial random seed (466801743). Statistics, of mean bias (M.B), mean square error (M.S.E) and their standard errors (S.E(M.B) and S.E(M.S.E)), are computed according to following definitions :

$$M.B = \frac{1}{R}\sum_{r=1}^{R}(\hat{\eta}_r-\eta) \qquad (2.16a)$$

$$S.E(M.B) = \sqrt{\frac{1}{R(R-2)}\sum_{r=1}^{R}\left((\hat{\eta}_r-\eta)-M.B\right)^2} \tag{2.16b}$$

$$M.S.E = \frac{1}{R}\sum_{r=1}^{R}(\hat{\eta}_r-\eta)^2 \tag{2.17a}$$

$$S.E(M.S.E) = \sqrt{\frac{1}{R(R-2)}\sum_{r=1}^{R}\left((\hat{\eta}_r-\eta)^2-M.S.E\right)^2} \tag{2.17b}$$

where $\hat{\eta}_r$ represents the point estimate of parameter η in the r^{th} replication. The fraction of detection level or the averaged uncensoring level (UC) for R (100) replications is determined by

$$UC = \frac{\sum_{r=1}^{R}\sum_{i=1}^{M}C_i(r)}{\sum_{r=1}^{R}\sum_{i=1}^{M}N_i(r)} \tag{2.18}$$

where $C_i(r)$ is the number of targets detected by observer i in the r^{th} replication and $N_i(r)$ is the number of targets presented to observer i in the r^{th} replication. Theoretically, the longer the opportunity times, the higher the UC level should be. An outline of a replication in the simulation is as follows :

a. Input to Simulation

- M the number of observers (Possible values are 15 , 25 and 35)
- N_i the number of targets presented to the ith observer , where i = 1,2....M (Possible values are 15 ,25 and 35)
- O_{ij} the opportunity time for ith observer to detect the jth target,(Possible values are 10 and 15). All of the opportunity times are the same.
- ξ_i the shape parameter of the Weibull distribution for ith observer, where i = 1,2,...M ; (Set $\xi_i = 0$ for all i)
- μ_{ij} the scale parameter of the Weibull distribution ,WEI(μ_{ij} , $e^{-\xi_i}$); (Set $\mu_{ij} = 4.2$)
- η the parameter for the single-parameter gamma distribution , GAM(α,α) , with $\alpha = e^{\eta}$; (Set the true value of $\eta = 1$)

b. Simulation of Data for ith Observer

- Generate independent single-parameter gamma r.v.'s A_i from GAM(α,α),having the density function as in equation (2.1).
- Generate independent exponential distribution r.v.'s W_{ij} with mean 1 i.e. $W_{ij} \sim$ EXP(1) for all i and j
- Compute the target detection times : $U_{ij} = \mu_{ij}\left(\frac{W_{ij}}{A_i}\right)^{e^{\xi_i}}$. [5]
- Compute the recorded ln-times Y_{ij} as in equation (2.3).
- Compute Δ_{ij} as in equation (2.4).

---

[5] $P\left\{\mu_{ij}\left(\frac{W_{ij}}{A_i}\right)^{e^{\xi_i}} \le t \,|\, A_i=\theta\right\} = P\left\{W_{ij} \le \theta\left(\frac{t}{\mu_{ij}}\right)^{e^{-\xi_i}}\right\} = 1-\exp\left\{-\theta\left(\frac{t}{\mu_{ij}}\right)^{e^{-\xi_i}}\right\}$ which is the conditional Weibull distribution of equation (2.2).

c. Initial η and Bisection Search

- Compute $\hat{m}_2$ as in equation (2.15) and determine the initial value of η as in equation (2.14).
- Perform the Bisection root search procedure for the maximum likelihood estimate of η, shown as the flowchart in Figure B.

All these procedures were written in APL codes,named "SIMULA1", "GAMMA", and "FVALUE", which are listed in Appendix C.

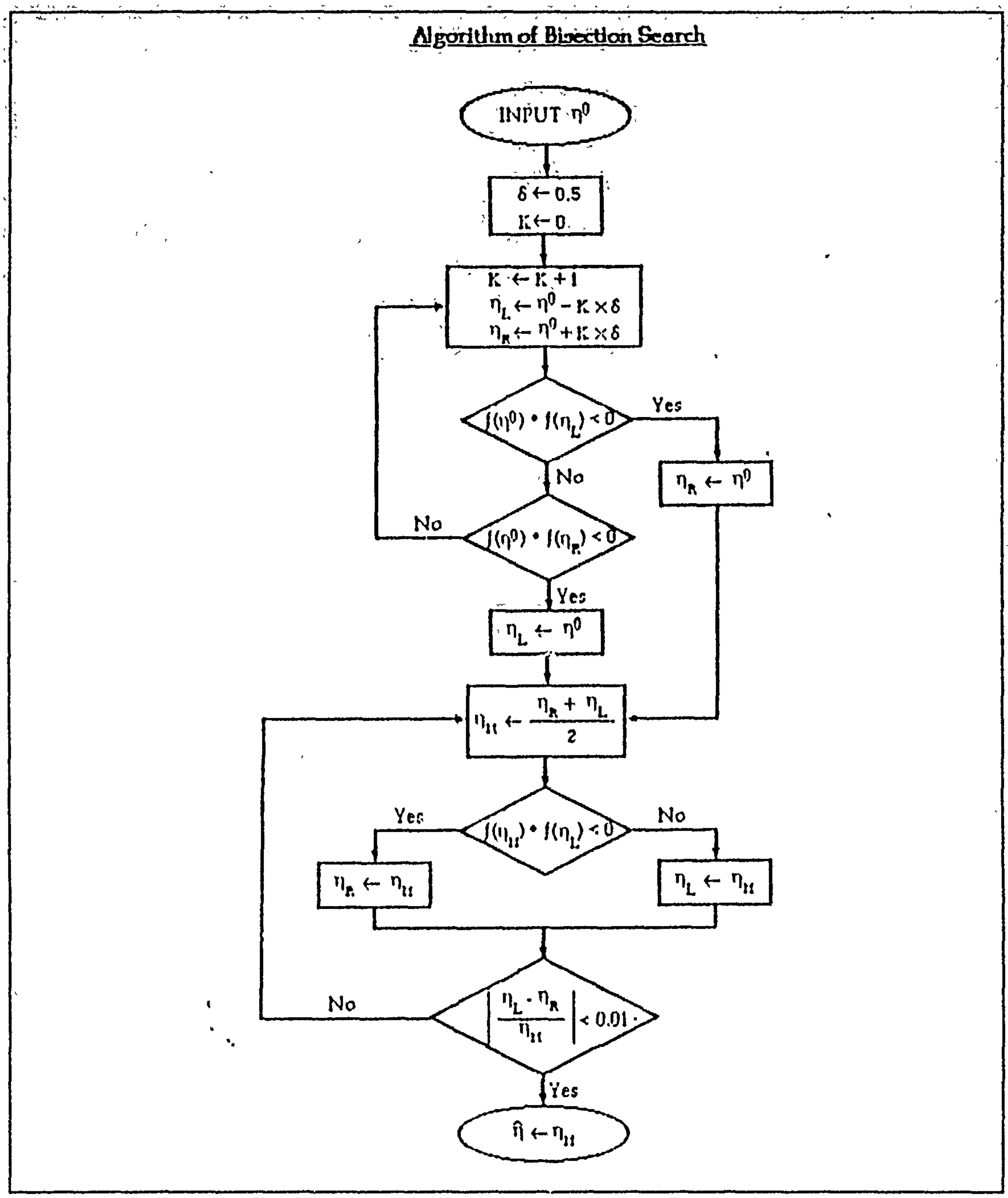

Figure B The Algorithm of Bisection Search

2. Results

The results of the simulation experiments appear in Tables and graphs in Appendix A. Tables 1 and 2 contain the general statistics for the bias of $\hat{\eta}$ at the opportunity times O = 10 and 15 respectively. Table 3 in combination with box plots, shown in Figures 1 through 3, presents statistics of the bias of the estimates of η as a function of the numbers of targets and observers in the case O = 10. Figures 4 to 6 show histograms of the bias with the number of targets fixed and the number of observers changing. Similarly, Table 4 and Figures 7 through 12 focus on the case O = 15. In summary, the simulations indicate the following :

- Opportunity time (O)
  Tables 1 and 2 indicate that a longer opportunity time is reflected in a higher uncensoring level (UC), which slightly decreases the mean bias (M.B) and mean square error (M.S.E) of η.

- The Number of Observers (M)
  Increasing the number of observers for a fixed number of targets results in greater changes in the mean bias and mean square error of η than increasing the number of targets with a fixed number of observers. All the box plots and histograms display this tendency. This phenomenon is quite reasonable, because the parameter η in gamma distribution reflects the variability in abilities of observers. Hence data with more observers will provide better estimation of η (less bias and standard error).

- The Number of Targets (N_i)
  The box plots indicate that increasing the number of targets for a fixed number of observers has a small effect on M.B and M.S.E of η with no systematic trend.

III. THE HIERARCHICAL GAMMA/WEIBULL REGRESSION MODEL

A. MODEL DESCRIPTION

Recalling the assumptions of the model, we suppose there are M observers, indexed by i. The i^{th} observer is presented with N_i targets, indexed by j. The variable U_{ij} is the time it takes for observer i to detect the target j. Let x_{ij1} , x_{ij2} ,...., x_{ijp} be the values of environmental variables (e.g. terrain, sea state , atmospheric condition etc.) which may affect U_{ij}. Given the quantified value of the ability for observer i, $A_i = \theta$, the U_{ij}'s are assumed conditionally independent random variables having Weibull distributions, $WEI(\mu_{ij}, e^{-\xi_i})$, where the scale parameter, instead of being constant, has the form

$$\mu_{ij} = \exp\{x_{ij}\beta\} = \exp\left\{\sum_{k=0}^{p} x_{ijk}\beta_k\right\}. \tag{3.1}$$

The regression coefficient β_k is the contribution to the Weibull scale parameter of the k^{th} explanatory variable. β_0 is usually the constant term in the regression. The random variables $\{A_i\}$ are assumed independent from a *single parameter gamma distribution, $GAM(\alpha,\alpha)$* with $\alpha = e^{\eta}$.

B. MAXIMUM LIKELIHOOD ESTIMATES FOR ALL PARAMETERS

The model assumptions detailed above are the same as those in Chapter II, except that the scale parameter μ_{ij} in the Weibull distribution is now a function of the explanatory variables. In this section we consider the estimation of all the parameters of η , $\{\beta_k\}$ and $\{\xi_i\}$.[6]

From equations (2.7), (2.8), (2.12) and equation (3.1), we can rewrite the unconditional ln-likelihood function for all M observers as follows :

$$\ln L =$$

$$g(\eta,\xi,\beta) = \sum_{i=1}^{M}\left\{\ln K_i + e^{\eta}\eta - (C_i+e^{\eta})\ln(S_i+e^{\eta}) + \sum_{k=0}^{C_i-1}\ln(e^{\eta}+k)\right\}$$

$$\textit{where} \quad K_i = \exp\left\{\sum_{j=1}^{N_i}\Delta_{ij}\left[(y_{ij}-x_{ij}\beta)e^{-\xi_i} - \xi_i\right]\right\} \tag{3.2}$$

$$\textit{and} \quad S_i = \sum_{j=1}^{N_i}\exp\left\{(y_{ij}-x_{ij}\beta)e^{-\xi_i}\right\};$$

Y_{ij} , Δ_{ij} , and C_i are defined in equations (2.3), (2.4) and (2.5) respectively.

In the following sub-sections, we will describe the procedures which are employed to estimate all the model parameters. The general procedure includes the initial rough estimation of $\{\beta_k\}$ and $\{\xi_i\}$ without the hierarchical gamma r.v.'s; a variability check for the need to include the hierarchical gamma parameter η; and finally, a large recursive procedure for estimating all the model parameters if a hierarchical model is necessary.

1. Initial Estimation of a Simple Weibull Regression Model

An initial model for the data is the simple Weibull regression model for the target detection times described previously as the Model 2 of Bae [Ref. 2:pp. 21-30]. The key steps of the estimation procedure are summarized as follows :

---

[6] In the Model 2 of Bae [Ref. 2], no gamma variable was involved. Only the Weibull parameters $\{\beta_k\}$ and $\{\xi_i\}$ were considered.

Step-1.1 : Initially, set[7] $\underline{x}_{ij}\underline{\beta}^0 = \ln U_{ij}$ and $\xi_i^0 = 0$.

Step-1.2 : This is an iteratively re-weighted regression step; cf. McCullagh and Nelder [Ref. 7].

Let

$$r_{ij} = \left(y_{ij} - \underline{x}_{ij}\underline{\beta}^0\right)e^{-\xi_i^0} \; ; \tag{3.3}$$

$$w_{ij} = \sqrt{\exp\{r_{ij}\}} \; ; \tag{3.4}$$

$$u_{ijh} = x_{ijh}w_{ij}e^{-\xi_i^0} \; ; \qquad \text{where } h=0,1,..,p \tag{3.5}$$

$$z_{ij} = \frac{\left(-\Delta_{ij}+w_{ij}^2\right)}{w_{ij}} + \sum_{h=0}^{p} u_{ijh}\beta_h^0 = \frac{\left(-\Delta_{ij}+w_{ij}^2\right)}{w_{ij}} + w_{ij}e^{-\xi_i^0}\underline{x}_{ij}\underline{\beta}^0 \; . \tag{3.6}$$

Regress the dependent variable z_{ij} against the independent variables u_{ijh}, for $h = 0,1,...,p$; e.g. the estimate has the form

$$\underline{B} = \left(\underline{U}^T\underline{U}\right)^{-1}\underline{U}^T\underline{Z} \tag{3.7}$$

where

---

[7] From this page on, all parameters with superscript 0 imply the current values of their estimates.

$$
Z = \begin{bmatrix} z_{11} \\ z_{12} \\ \cdot \\ \cdot \\ z_{1N_1} \\ z_{21} \\ \cdot \\ \cdot \\ \cdot \\ z_{2N_2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_{M1} \\ \cdot \\ \cdot \\ \cdot \\ z_{MN_M} \end{bmatrix} \qquad \underline{U} = \begin{bmatrix} u_{110} & \cdots & u_{11p} \\ u_{120} & \cdots & u_{12p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_{1N_10} & \cdots & u_{21p} \\ u_{210} & \cdots & u_{21p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_{2N_20} & \cdots & u_{2N_2p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_{M10} & \cdots & u_{M1p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_{MN_M0} & \cdots & u_{MN_Mp} \end{bmatrix} \qquad (3.8),(3.9)
$$

Step-1.3 : Compute the new ξ_i by an approximation based on Newton's procedure. First, update the value of

$$
r_{ij} = \left(y_{ij} - x_{ij}\underline{\beta}^0\right)e^{-\xi_i^0} \quad , \quad where \quad \underline{\beta}^0 = \underline{B} \; . \qquad (3.10)
$$

Compute

$$\xi_i = \xi_i^0 + \frac{\left\{C_i - \sum_{j=1}^{N_i} r_{ij}\left[-\Delta_{ij} + \exp(r_{ij})\right]\right\}}{\left\{-C_i - \sum_{j=1}^{N_i} r_{ij}^2 \exp(r_{ij})\right\}} \tag{3.11}$$

$$\textit{where} \quad C_i = \sum_{j=1}^{N_i} \Delta_{ij} .$$

Step-1.4 : Update ξ_i and β_k

$$\xi_i^0 = \xi_i$$
$$\beta_k^0 = \beta_k \qquad \textit{where} \quad k=0,1,\ldots,p$$

Compute the new value of $\underline{x}_{ij}\underline{\beta}^0 = \sum_{k=0}^{p} x_{ijk}\beta_k^0$ and iterate the regression, starting at *Step-1.2*, for one more time.

2. Does the Simple Weibull Regression Adequately Account for the Variability of the Data ?

The full hierarchical model will be used if the data indicate more variability than can be described by the simple Weibull regression. If the data do come from the simple Weibull model, then

$$\left(\frac{U_{ij}}{\mu_{ij}}\right)^{e^{-\xi_i}} \qquad j=1,2,\ldots,N_i$$

will have the same distribution as independent exponential random variables[8] with mean 1. To assess the variability not accounted for by the simple regression model, we compute

$$\hat{\sigma}_E^2 = \frac{1}{(N-1)}\sum_{ij}\left[\left(\frac{U_{ij}}{\mu_{ij}}\right)^{e^{-\xi_i}}-1\right]^2 \qquad (3.12)$$

$$\textit{and} \qquad N=\sum_{i=1}^{M} N_i \,.$$

Since the variance of a unit exponential is 1, if $\hat{\sigma}_E^2 \leq 1$,then we will stop the model fitting with the simple Weibull regression model. If $\hat{\sigma}_E^2 > 1$, we go on to the hierarchical model to explain the extra-variability. Unfortunately, crude estimate of $\hat{\sigma}_E^2$ uses both the observed and unobserved U_{ij}. A topic for future research is to improve the estimate of the variability not accounted for by the simple regression model.

<u>Step-1.5</u> : If $\hat{\sigma}_E^2 \leq 1$, stop and use the Weibull regression model only. Otherwise, the data will be modeled with the full hierarchical model. A recursive procedure to estimate η , $\{\beta_k\}$ and $\{\xi_i\}$ follows and is initialized with $\{\beta_k^0\}$ and $\{\xi_i^0\}$ computed above.

---

8 $P\{U_{ij} \leq t\} = P\left\{\mu_{ij}(W_{ij})^{e^{\xi_i}} \leq t\right\} = P\left\{W_{ij} \leq (t/\mu_{ij})^{e^{-\xi_i}}\right\} = 1 - e^{-(t/\mu_{ij})^{e^{-\xi_i}}}$ which is the pure Weibull distribution.

3. A Recursive Estimation Procedure for the Parameters of the Full Hierarchical Model

a. Bisection Search with Instability Check for Estimating η

The partial derivative of equation (3.2) with respect to η, $\frac{\partial \ln L}{\partial \eta}$, is the same as the derivative of the ln-likelihood with respect to η for the model of Chapter II. The derivative appears in (2.13) and once again the problem is to solve the equation $f(\eta)=0$. Using the estimates $\{\beta_k^0\}$ and $\{\xi_i^0\}$ an initial estimate for η is computed as in equation (2.14) ; that is ,

$$\eta^0 = -\ln\left[\hat{m}_2^2 - \hat{m}_1^2 - \frac{\pi^2}{6}\right] \tag{3.13}$$

where

$$\hat{m}_1 = -0.5772$$

$$\hat{m}_2 = \frac{\sum_{i=1}^{M}\sum_{j=1}^{N_i}\Delta_{ij}\left[\left(y_{ij}-x_{ij}\beta^0\right)e^{-\xi_i^0}\right]^2}{\sum_{i=1}^{M}\sum_{j=1}^{N_i}\Delta_{ij}}. \tag{3.14}$$

If $\hat{m}_2^2 - \hat{m}_1^2 - \frac{\pi^2}{6} < 0$, set $\eta^0 = 0.$ [9]

In addition to determining the bounds for a reasonable root to be the estimate of η as described in Chapter II, a check for the shape of $f(\eta)$ is also performed. This additional check is performed because two of the simulation replications exhibited a very flat $f(\eta)$. The graph of $f(\eta)$ for one of the two

[9] Same as footnote 4 on page 10.

replications appears in Figure C[10]. This unusual shape for $f(\eta)$ results in the bisection method converging to an unreasonable root (e.g. $\hat{\eta}$ =146 in the Figure C).

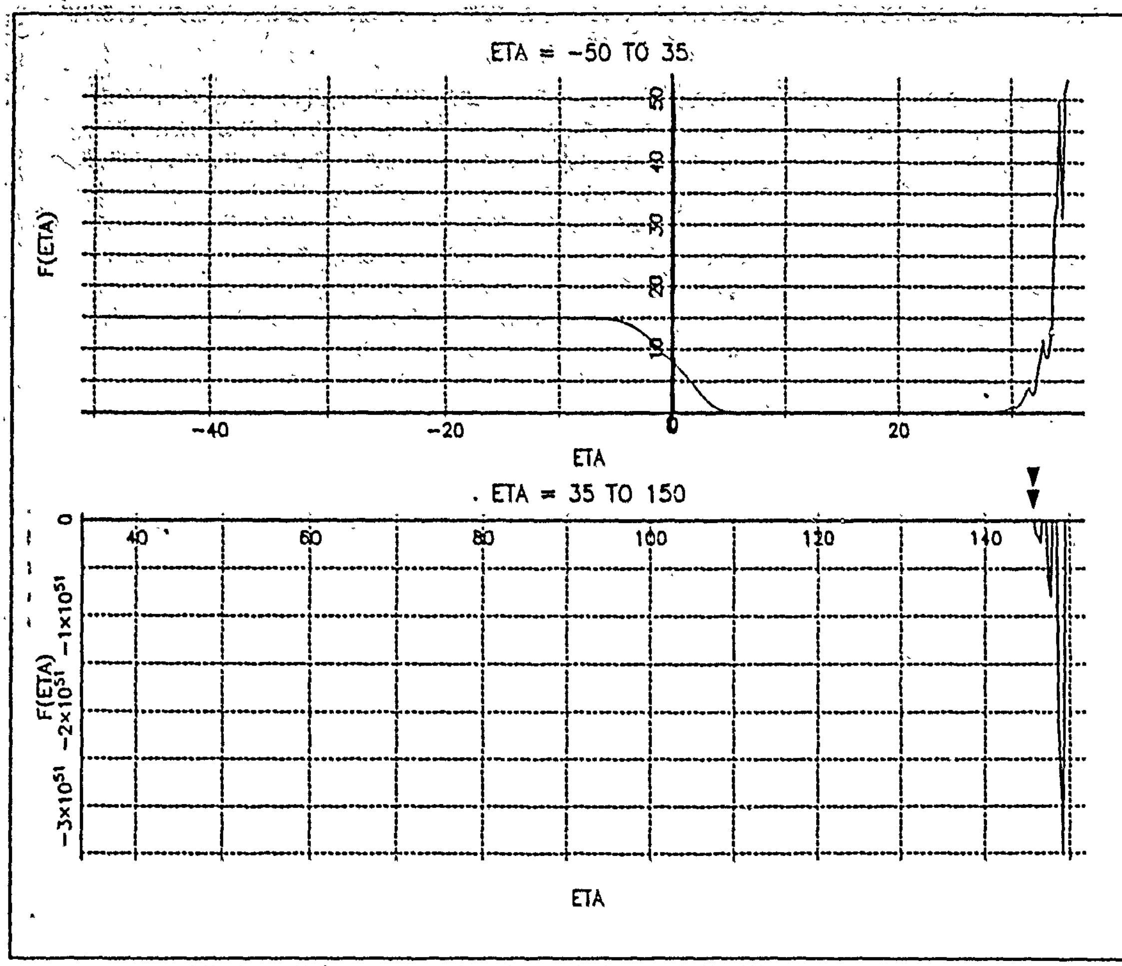

Figure C An Ill-Conditioned Derivative of the ln-Likelihood Function for η

The graph in Figure C indicates that the values of $f(\eta)$ are asymptotic to a fixed nonzero value as η tends to $-\infty$; this behavior is the same as in Figure A. However, when η is positive, f has very flat curve. The values of $f(\eta)$ are positive

[10] The graph is one of the only two ill-conditioned cases for $f(\eta)$ in the simulations. This case occurred in the gamma/Weibull regression model with O=10, M=15, N=25 and random seed = 1905665785. The other case occurred with O=15, M=15, N=25 and same random seed. The true value of η is 1.

and close to zero but the curve does not actually cross zero until η is large. The following check is done to detect this situation. If a reasonable right hand bound for a zero of $f(\eta)$ cannot be found, check the values of $f(\eta)$ and $f'(\eta)$ for η positive. If the absolute value of $f(\eta)$ is less the ϵ_1 and the absolute value of $f'(\eta)$ is less than ϵ_2, where $1 >> \epsilon_1 > \epsilon_2 > 0$, then the simulation stops for this set of random numbers and new random numbers are generated. Otherwise, perform the bisection search as described in Chapter II. In the procedure, $f'(\eta)$ for η positive is estimated as

$$\begin{aligned} f'(\eta_{R(k)}) &\approx \frac{[f(\eta_{R(k)}) - f(\eta_{R(k-1)})]}{\delta} \\ \textit{where} \quad \eta_{R(k)} &= \eta^0 + k\times\delta \\ \eta_{R(k-1)} &= \eta^0 + (k-1)\times\delta. \end{aligned} \tag{3.15}$$

The algorithm of this bisection search with the check for flatness of the curve of f will be detailed later in Figure D.

b. <u>The Least Squares Regression Procedure to Solve for $\{\beta_k\}$</u>

The previous estimates of η^0, $\{\beta_k^0\}$ and $\{\xi_i^0\}$ are input to this stage. The partial derivative of equation (3.2) with respect to β_k is

$$\frac{\partial \ln L}{\partial \beta_k} = \sum_{i=1}^{M} \left\{ \frac{\partial \ln K_i}{\partial \beta_k} + \frac{C_i + e^{\eta^0}}{S_i + e^{\eta^0}} \cdot \frac{\partial S_i}{\partial \beta_k} \right\} \tag{3.16}$$

where

$$\frac{\partial \ln K_i}{\partial \beta_k} = \sum_{j=1}^{N_i} \Delta_{ij}(-x_{ijk}) e^{-\xi_i^0}$$

$$\frac{\partial S_i}{\partial \beta_k} = \sum_{j=1}^{N_i} \exp\left[(y_{ij} - \underline{x}_{ij}\underline{\beta}) e^{-\xi_i^0}\right]\left(-x_{ijk} e^{-\xi_i^0}\right) .$$

Thus

$$\frac{\partial \ln L}{\partial \beta_k} = \sum_{i=1}^{M} \sum_{j=1}^{N_i} \left\{ -\Delta_{ij} + \frac{C_i + e^{\eta^0}}{S_i + e^{\eta^0}} \exp\left[(y_{ij} - \underline{x}_{ij}\underline{\beta}) e^{-\xi_i^0}\right] \right\} \left(x_{ijk} e^{-\xi_i^0}\right) . \tag{3.17}$$

Two terms of a Taylor expansion of equation (3.17) yields

$$0 = \frac{\partial \ln L}{\partial \beta_k} \approx \sum_{i=1}^{M} \sum_{j=1}^{N_i} \left(\left\{ -\Delta_{ij} + \frac{C_i + e^{\eta^0}}{S_i + e^{\eta^0}} \exp\left[(y_{ij} - \underline{x}_{ij}\underline{\beta}^0) e^{-\xi_i^0}\right] \right\} \right.$$
$$\left. - \left\{ \exp\left[(y_{ij} - \underline{x}_{ij}\underline{\beta}^0) e^{-\xi_i^0}\right] \sum_{h=0}^{p} x_{ijh} e^{-\xi_i^0} (\beta_h - \beta_h^0) \right\} \right) \left(x_{ijk} e^{-\xi_i^0}\right) . \tag{3.18}$$

Let

$$w_{ij} = \sqrt{\frac{C_i + e^{\eta^0}}{S_i + e^{\eta^0}} \exp\left\{(y_{ij} - \underline{x}_{ij}\underline{\beta}^0) e^{-\xi_i^0}\right\}} . \tag{3.19}$$

The equation (3.18) can be rewritten as

$$0=\frac{\partial \ln L}{\partial \beta_k}\approx\sum_{i=1}^{M}\sum_{j=1}^{N_i}\left(\frac{-\Delta_{ij}+w_{ij}^2}{w_{ij}}-\sum_{h=0}^{p}x_{ijh}e^{-\xi_i^0}\left(\beta_h-\beta_h^0\right)\right)\left(x_{ijk}e^{-\xi_i^0}\right). \tag{3.20}$$

Let

$$u_{ijh} = x_{ijh}w_{ij}e^{-\xi_i^0} \tag{3.21}$$

and

$$z_{ij} = \frac{\left(-\Delta_{ij}+w_{ij}^2\right)}{w_{ij}}+\sum_{h=0}^{p}u_{ijh}\beta_h^0 = \frac{\left(-\Delta_{ij}+w_{ij}^2\right)}{w_{ij}}+w_{ij}e^{-\xi_i^0}\underline{x}_{ij}\underline{\beta}^0 . \tag{3.22}$$

The equation (3.20) can be rewritten as

$$0=\sum_{i=1}^{M}\sum_{j=1}^{N_i}\left(z_{ij} - \sum_{h=0}^{p}u_{ijh}\beta_h\right)u_{ijk} \tag{3.23}$$

which are the normal equations for a least squares regression having dependent variable z_{ij} and independent variables u_{ijk}, $k=0,1,...,p$.
The solution of (3.23) is

$$\underline{B} = \left(\underline{U}^T\underline{U}\right)^{-1}\underline{U}^T\underline{Z} \tag{3.24}$$

where $\underline{Z}$ and $\underline{U}$ are defined as in equations (3.25) and (3.26) respectively.

$$
\mathbf{Z} = \begin{bmatrix} z_{11} \\ z_{12} \\ \cdot \\ \cdot \\ z_{1N_1} \\ z_{21} \\ \cdot \\ \cdot \\ \cdot \\ z_{2N_2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_{M1} \\ \cdot \\ \cdot \\ \cdot \\ z_{MN_M} \end{bmatrix}
\qquad
\underline{U} = \begin{bmatrix} u_{110} & \cdots & u_{11p} \\ u_{120} & \cdots & u_{12p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_{1N_10} & \cdots & u_{21p} \\ u_{210} & \cdots & u_{21p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_{2N_20} & \cdots & u_{2N_2p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_{M10} & \cdots & u_{M1p} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ u_{MN_M0} & \cdots & u_{MN_Mp} \end{bmatrix}
\qquad (3.25),(3.26)
$$

This procedure is an iteratively re-weighted least squares regression; cf. McCullagh and Nelder [Ref. 7].

c. Newton Procedure to Solve for $\{\xi_i\}$

The prior estimates of η^0, $\left\{\beta_k^0\right\}$, $\left\{\xi_i^0\right\}$ and $\left\{\xi_i^0\right\}$ are input to this stage. In order to update the values of $\{\xi_i\}$, we need to have the partial derivative of unconditional ln-likelihood function for **observer i** with respect to ξ_i. The unconditional ln-likelihood function for observer i is the same as equation (2.11).

The derivative is as follows :

$$\frac{\partial \ln L_i}{\partial \xi_i} = h(\xi_i) = \left\{\sum_{j=1}^{N_i} \Delta_{ij}\left[\left(y_{ij}-\underline{x}_{ij}\underline{\beta}^0\right)\left(e^{-\xi_i}\right)-1\right]\right\} - \frac{C_i+e^{\eta^0}}{S_i+e^{\eta^0}} \cdot \frac{\partial S_i}{\partial \xi_i}$$

$$where \quad \frac{\partial S_i}{\partial \xi_i} = \sum_{j=1}^{N_i} \exp\left\{\left(y_{ij}-\underline{x}_{ij}\underline{\beta}^0\right)e^{-\xi_i}\right\}\left(-e^{-\xi_i}\right)\left(y_{ij}-\underline{x}_{ij}\underline{\beta}^0\right) . \tag{3.27}$$

Again, let

$$r_{ij} = \left(y_{ij} - \underline{x}_{ij}\underline{\beta}^0\right)e^{-\xi_i} .$$

The equation (3.27) can be rewritten as

$$\frac{\partial \ln L_i}{\partial \xi_i} = h(\xi_i) = -C_i + \sum_{j=1}^{N_i} r_{ij}\left[-\Delta_{ij} + \frac{C_i+e^{\eta^0}}{S_i+e^{\eta^0}} \exp(r_{ij})\right] . \tag{3.28}$$

Treating $\dfrac{C_i+e^{\eta^0}}{S_i+e^{\eta^0}}$ as a constant, then

$$\frac{\partial^2 \ln L_i}{\partial \xi_i^2} = \frac{\partial h(\xi_i)}{\partial \xi_i} \approx -C_i - \frac{C_i+e^{\eta^0}}{S_i+e^{\eta^0}} \sum_{j=1}^{N_i} \exp(r_{ij})\, r_{ij}^2 \tag{3.29}$$

If $\dfrac{\partial \ln L_i}{\partial \xi_i} = h(\xi_i) = 0$, the Newton equation for ξ_i is

$$0 = \frac{\partial \ln L_i}{\partial \xi_i} = h(\xi_i) = h\left(\xi_i^0\right) + \frac{\partial}{\partial \xi_i} h\left(\xi_i^0\right)\left[\xi_i - \xi_i^0\right] . \tag{3.30}$$

Update the value of ξ_i by solving the following equation :

$$\xi_i = \xi_i^0 - \frac{h(\xi_i^0)}{\frac{\partial}{\partial \xi_i} h(\xi_i^0)} . \tag{3.31}$$

As noted before, it is possible for equation (3.31) to be badly conditioned. If $\left|h(\xi_i^0)\right| < \varepsilon_1$ and $\left|\frac{\partial}{\partial \xi_i} h(\xi_i^0)\right| < \varepsilon_2$, where $1 >> \epsilon_1 > \epsilon_2 > 0$, then equation (3.31) is badly conditioned. This behavior happens occasionally and occurs for only one ξ_i out of the M. To avoid this behavior, if $\left|h(\xi_i^0)\right| < \varepsilon_1$ and $\left|\frac{\partial}{\partial \xi_i} h(\xi_i^0)\right| < \varepsilon_2$, then put $\xi_i = \xi_i^0$; otherwise let ξ_i be the solution to equation (3.31). Return to the bisection search procedure and iterate steps 3.a to 3.c until

$$\max\left(\left|\frac{\eta_L - \eta_R}{\eta_M}\right|, \left|\frac{\beta_k - \beta_k^0}{\beta_k^0}\right|, \left|\frac{\xi_i - \xi_i^0}{\xi_i^0}\right| \right) < 0.01 \text{ for all } k=0,..,p \text{ and } i = 1,..,M.$$

C. IMPLEMENTATION OF SIMULATION

Simulation will be used to study the small sample behavior of the estimators of the five parameters of the gamma/Weibull model. Different numbers of observers (M), targets (N_i) and opportunity times (O_{ij}) are used as the input to the simulations. For each set of input, (O,M,N_i), the simulation run starts with the same random seed as in Chapter II. The simulation is run until there are 100 (R) estimates of η. A run will contain more than 100 replications if the simple Weibull regression model is found to describe the variability of the data for some of the replications or if the procedure to find the estimate of η is badly conditioned. The statistics of mean bias (M.B), mean square error (M.S.E) and their respective standard errors (S.E(M.B) and S.E(M.S.E)) for all the estimates are computed using the definitions in equations (2.16a,b) and (2.17a,b). The averaged uncensoring levels (UC) as computed using equation (2.18) are also reported. The statistics for $\{\xi_i\}$ do not include those ξ_i for which $|h(\xi_i)| < \epsilon_1$, (ϵ_1=0.1), and $|h'(\xi_i)| < \epsilon_2$, (ϵ_2=0.01).

The remainder of this section provides an outline of one replication in the simulation.

1. Input to Simulation

- M the number of observers (Possible values are 15 , 25 and 35).
- ξ_i the shape parameter of the Weibull distribution for ith observer, where i = 1,2,...M ; (Set ξ_i = 0 for all i).
- η the parameter for the single-parameter gamma distribution, GAM(α,α) , where $\alpha = e^{\eta}$; (Set η = 1).
- N_i the number of targets presented to the ith observer , where i = 1,2,...,M ; (Possible values are 15 ,25 and 35).

- x_{ijk} the k^{th} environmental factor which affects the observation time of the i^{th} observer when presented with the j^{th} target.

- O_{ij} the opportunity time for the i^{th} observer to detect the j^{th} target; (Possible values are 10 and 15).

- $\{\beta_k\}$ regression coefficients for the scale parameter of Weibull distribution ,WEI(μ_{ij} , $e^{-\xi i}$), where k = 0,1,2,...,p and

  $\mu_{ij} = \exp\left\{\sum_{k=0}^{p} x_{ijk}\beta_k\right\}$; (set p = 2 ; $\beta_0 = 0.8$, $\beta_1 = -0.2$ and $\beta_2 = 0.5$).

2. **Simulation of Data for the i^{th} Observer**

- Generate a single-parameter gamma r.v. A_i from GAM(α,α),having density function as in equation (2.1) where $\alpha = e^{\eta}$ and $\eta = 1$.

- Generate environmental factors r.v. x_{ijk} ; the x_{ijk} are independent with normal distributions having mean μ_k and variance σ_k^2 for k = 1,2,...,p; (Set p = 2 ; $\mu_1 = 1$, $\sigma_1^2 = 0.5$ and $\mu_2 = 2$, $\sigma_2^2 = 1$). Put $x_{ij0} = 1$ for all i,j.

- Compute the Weibull shape parameter μ_{ij} as in equation (3.1).

- Generate independent exponential distribution r.v.'s W_{ij} with mean 1; i.e. $W_{ij} \sim$ EXP(1) for all i and j.

- Compute the target detection times : $U_{ij} = \mu_{ij}\left(\frac{W_{ij}}{A_i}\right)^{e^{\xi_i}}$.

- Compute the recorded ln-times Y_{ij} , censoring indicator Δ_{ij} and C_i using the definitions in equation (2.3) , (2.4) and (2.5) respectively.

3. Initial Estimate of β_k's and ξ_i in the Simple Weibull Regression Model without the Hierarchical Gamma r.v.'s

Step-1.1 : Initially, set $\underline{x}_{ij}\underline{\beta}^0 = \ln U_{ij}$ and $\xi_i^0 = 0$

Step-1.2 : Iteration for regression :

- Compute r_{ij} , w_{ij} , u_{ij} and z_{ij} as in equations (3.3) to (3.6).
- Regress the dependent variable z_{ij} against the independent variables u_{ijh} for $\{\beta_k\}$ as in equations (3.7) to (3.9).

Step-1.3 : Newton's procedure for estimating ξ_i

- Update the value of r_{ij} as in equation (3.10).
- Compute the new ξ_i as in equation (3.11) with no need to check the ill condition.

Step-1.4 Put $\{\xi_i^0\} = \{\xi_i\}$ and $\{\beta_k^0\} = \{\beta_k\}$ for all i=1,2...,M and k=0,1,..,p. Compute new value of $\underline{x}_{ij}\underline{\beta}^0 = \sum_{k=0}^{p} x_{ijk}\beta_k^0$ and return to Step-1.2 for only one iteration.

Step-1.5 Check for extra-variability in the data by computing the sample variance $\hat{\sigma}_E^2$ for the times U_{ij} as in equation (3.12).

If $\hat{\sigma}_E^2 \leq 1$, stop and quote the simple Weibull regression model only with the parameters as estimated.

Otherwise, execute next the recursive procedure.

4. Recursive Procedure to Estimate the Parameters of the Full Hierarchical Gamma/Weibull Model

Input : initial β_k^0 ,k=0,1,...,p ; ξ_i^0 ,i=1,2,....,M.

Step-2.1 : Initial condition for gamma parameter η

- Update $\underline{x}_{ij}\underline{\beta}^0 = \sum_{k=0}^{p} x_{ijk}\beta_k^0$.

- Find the initial condition for η using equations (3.13) and (3.14)

  If $\hat{m}_2^2-\hat{m}_1^2-\frac{\pi^2}{6} < 0$, set $\eta^0 = 0$.

Step-2.2 Perform the bisection search for η using the algorithm shown in Figure D (next page). The algorithm includes a check for the flatness of the curve of f(η).

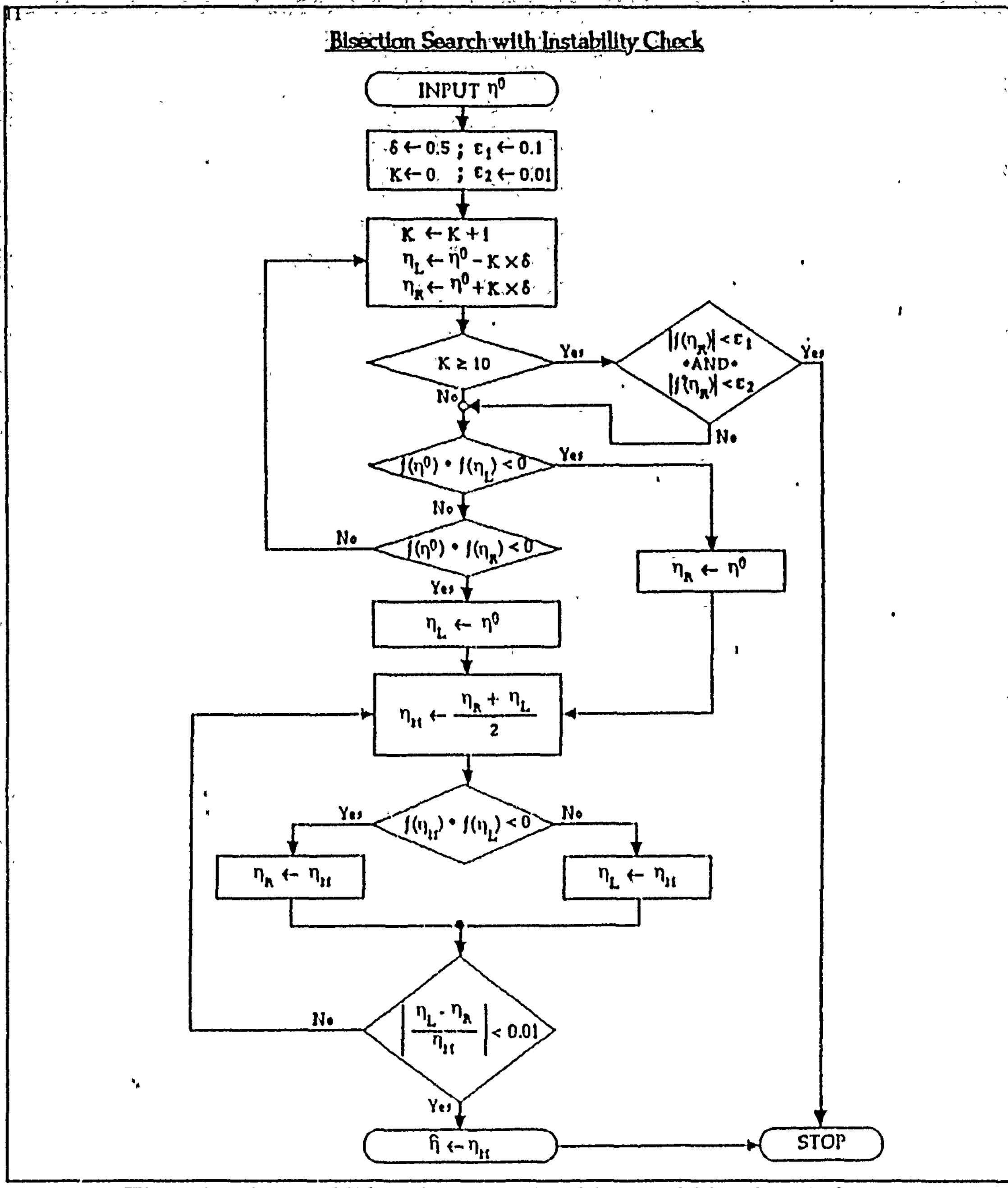

Figure D The Algorithm of Bisection Search with Instability Check for η

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11 In the Algorithm, we let $f/(\eta_R) \approx \frac{[f(\eta_{R(k)})-f(\eta_{R(k-1)})]}{\delta}$ *where* $\eta_{R(k)} = \eta^0 + k \times \delta$, $\eta_{R(k-1)} = \eta^0 + (k-1) \times \delta$,

and k is the current index of increments.

Step-3.1 : Iteration for the regression coefficients ,$\{\beta_k\}$

- Compute w_{ij} , u_{ij} and z_{ij} using equations (3.19), (3.21) and (3.22).

- Regress the dependent variable z_{ij} against the independent variables u_{ijh} using equations (3.24), (3.25) and (3.26).

Step-4.1 : Iteration for the ln-shape parameters ,$\{\xi_i\}$, of the Weibull distribution

- Update $\underline{x}_{ij}\underline{\beta}^0 = \sum_{k=0}^{p} x_{ijk}\beta_k^0$ and $r_{ij} = \left(y_{ij} - \underline{x}_{ij}\underline{\beta}^0\right)e^{-\xi_i^0}$.

- Compute $h\left(\xi_i^0\right)$ and $\frac{\partial}{\partial\xi_i}h\left(\xi_i^0\right)$ using equations (3.28) and (3.29) respectively.

- If $\left|h\left(\xi_i^0\right)\right| < \varepsilon_1$ and $\left|\frac{\partial}{\partial\xi_i}h\left(\xi_i^0\right)\right| < \varepsilon_2$, for $\epsilon_1 = 0.1$ and $\epsilon_2 = 0.01$, keep $\xi_i = \xi_i^0$. Otherwise, update ξ_i using equation (3.31).

Step-4.2 : Test for stop criteria

If $\max\left(\left|\frac{\eta_L - \eta_R}{\eta_M}\right|, \left|\frac{\beta_k - \beta_k^0}{\beta_k^0}\right|, \left|\frac{\xi_i - \xi_i^0}{\xi_i^0}\right|\right) < 0.01$ for all $k=0,..,p$ and $i=1,..,M$, then, stop and record the estimates. Otherwise, set $\eta^0 = \eta$, $\beta_k^0 = \beta_k$, $\xi_i^0 = \xi_i$ and return to *Step-2.2*.

The simulation programs for this gamma/Weibull regression model, named "SIMULA2", "GAMWEI" and "FVALUE", are detailed in Appendix D.

D. RESULTS

Results of the simulation experiments are presented in Appendix B. The results are presented in five sections. Each section shows results for one estimator; the gamma parameter η and the Weibull parameters β_0 , β_1 , β_2 and ξ_i. Each section contains tables of statistics and histograms of estimate bias for different numbers of observers (M or OBS), targets (N or TGT) and opportunity times (O).

The first two tables in each section present the statistics for different opportunity times (O=10 and O=15 in our case). The next two tables organize the information to highlight relations between the number of targets and the number of observers and the statistics of the estimates. Histograms of the biases of the estimates are also presented. All parameter estimates appearing in Appendix B are from replications for which the full hierarchical gamma/Weibull is estimated. Since $\xi_i \equiv 1$, the statistics for $\{\xi_i\}$ are for the average of the $\{\xi_i\}$ for all the observers in a replication. The average includes only those ξ_i's for which the Newton procedure converged. The replications for which $\hat{\sigma}_E^2 \leq 1$ and the replications in which η could not be found are not used to compute the statistics appearing in Appendix B; statistics concerning the numbers of these cases appear in Tables A and B below.

The statistics in Appendix B use 100 replications. Tables A and B below show the averaged uncensoring levels, the number of additional replications for which η could not be found due to the flatness of f, and the number of times a ξ_i could not be found using Newton procedure due to the smallness of $|h(\xi_i)|$ and $|h'(\xi_i)|$. The results of Tables A and B (next page) are summarized as follows :

- Uncensoring level (UC) :
  A longer opportunity time (O) is reflected in a higher average uncensoring level (UC). The opportunity time O = 10 gives about 76% UC, while O = 15 gives about 85% UC. This behavior is what we would expect.

- The variability of the data explained by the simple Weibull regression :
  Different parameter estimates in the simple Weibull regression cause the different number of cases for which $\{ \hat{\sigma}_E^2 \leq 1\}$ for the same input of number of observers and number of targets in Tables A and B. In both Tables A and B, the cases of $\{ \hat{\sigma}_E^2 \leq 1\}$ appear more frequently when the number of observers (M) is smaller. This is reasonable behavior since more observers tend to provide more evidence that a hierarchical model is needed to explain the variability of the data. The manner in which $\hat{\sigma}_E^2$ is computed prevents drawing conclusions concerning the effect of censoring on the assessment of unexplained variability.

- Instability check for η :
  There are only two replications in which η could not be found using the bisection search for the root of $f(\eta)$. The function $f(\eta)$ for one of them is plotted as in Figure C, and the function for the other is similar. Both replications used the same random seed but had a different set of inputs, (M,N_i,O_{ij}). These two replications were stopped and new random numbers were drawn. This behavior did not occur in the model of Chapter II with known parameters for the Weibull distribution.

- Instability check for ξ :
  Smaller opportunity times tend to have more replications in which the procedure to find ξ_i is badly conditioned. The maximum number for any case is 2.

TABLE A : Comparisons of Uncensoring levels, Extra Variability Checks and Checks of Numerical Instability at O = 10

Number of Observers	Number of Targets	Average UC	Number of $\{\hat{\sigma}_E^2 < 1\}$	Num. unstable $f(\eta)$	Num. unstable h and h' [12]
15	15	76.4%	11	0	1_a
	25	76.4%	5	1	0
	35	75.6%	6	0	0
25	15	75.8%	1	0	1_b
	25	75.8%	2	0	1_c
	35	76.4%	1	0	0
35	15	76.2%	0	0	2_d
	25	76.3%	1	0	1_e
	35	76.3%	0	0	0

TABLE B : Comparisons of Uncensoring levels, Extra Variability Checks and Checks of Numerical Instability at O = 15

Number of Observers	Number of Targets	Average UC	Number of $\{\hat{\sigma}_E^2 < 1\}$	Num. unstable $f(\eta)$	Num. unstable h and h'
15	15	85.0%	23	0	0
	25	85.4%	6	1	0
	35	84.6%	11	0	0
25	15	84.5%	3	0	1_f
	25	84.7%	7	0	0
	35	85.3%	5	0	0
35	15	85.2%	5	0	0
	25	84.9%	3	0	0
	35	85.1%	2	0	0

[12] All the cases of numerically unstable h and h′ were obtained while replicating the simulations starting with random seeds a:999445582, b:444977940, c:538790986, d:999445582 & 1534999080, e:1011786849 and f:444977940.

In Appendix B, the statistics for the average observer ξ_i do not include those individual ξ_i for which the Newton procedure did not converge. The statistics for β_k do not include those replications for which the simple Weibull regression model was used to describe the data. A summary of the results of the simulation experiment shown in Appendix B follows :

- The estimate of the gamma parameter, $\hat{\eta}$
  The mean bias tends to be positive. The histograms of biases indicate that the distribution is skewed to the right. Since the variance of A_i is $e^{-\eta}$, the positive mean bias of η suggests that for small sample sizes the procedure is indicating less between variability than there is in the data. Increasing the number of observers tends to decrease the mean bias and mean square error. Increasing the opportunity time has little effect on mean bias and mean square error. Increasing the number of targets has little effect on mean bias and mean square error. As noted in Chapter II, this behavior is reasonable.

- The estimates of the Weibull parameters, $\hat{\beta}_0$ through $\hat{\beta}_2$
  The first two tables in parts 2 through 4 of Appendix B indicate that all the M.B and M.S.E of $\{\beta_k\}$ are quite small compared to those for η. This is due to the larger sample size used in computing $\{\beta_k\}$. An increase in the opportunity time has little effect on the mean bias and mean square error. Increasing the number of targets or observers with a fixed number of the other yields small changes in M.B and M.S.E. However, increasing both the number of targets and the number of observers does tend to decrease the mean bias and mean square error of $\{\beta_k\}$. The M.B's are both positive and negative.

- The average estimate of the Weibull parameter, $\hat{\xi} = \frac{1}{M}\sum_{i=1}^{M}\hat{\xi}_i$

  Since $\xi_i = 1$ in the simulations, the histograms and statistics include the averaged $\hat{\xi}_i$ for all the observers for which the Newton procedure for ξ_i converged. Generally, all the magnitudes of M.B and M.S.E of $\hat{\xi}$ in the first 2 tables of part 5 in Appendix B are quite small. All the M.B of the averaged $\hat{\xi}$ are negative. The M.B. and M.S.E. are slightly smaller for longer opportunity times. Increasing the number of targets or the number of observers with the other fixed results small changes in the M.B and M.S.E. Increasing both the number of targets and the number of observers tends to decrease the mean bias and mean square error.

IV. CONCLUSIONS AND RECOMMENDATIONS

The main effort of this thesis is to study the small sample behavior of estimators for a hierarchical gamma/Weibull regression model for target detection times. This model can also be used to describe and predict times to failure of similar machines, (e.g. engines) in different environments. In this thesis, the model assumes that there are M observers. The i^{th} observer is presented with N_i targets. The variable U_{ij} is the time it takes for observer i to detect the target j. Let x_{ij1}, x_{ij2} ,...., x_{ijp} be the values of environmental variables which may affect U_{ij}. Given the quantified value of the ability for observer i, $A_i = \theta$, the U_{ij}'s are assumed conditionally independent random variables having Weibull distributions, $WEI(\mu_{ij}, e^{-\xi_i})$, where the scale parameter has the form $\mu_{ij} = \exp\{\underline{x}_{ij}\underline{\beta}\} = \exp\left\{\sum_{k=0}^{p} x_{ijk}\beta_k\right\}$. The random variables $\{A_i\}$ are assumed independent from a *single parameter gamma distribution, GAM(α,α)* with $\alpha = e^{\eta}$. A numerical procedure based on maximum likelihood is used to estimate the parameters of the model. The numerical procedure is iterative and uses a bisection root search method for estimating the gamma parameter η, least squares regression to estimate the Weibull ln-scale parameters β_0 through β_2 and an approximate Newton procedure to estimate the Weibull ln-shape parameters $\{\xi_i\}$. Simulation is used to study the behavior of the estimates for small sample sizes. Generally, the numerical estimation procedures work well. There are few numerical problems. All the mean biases and mean square errors of the estimates tend to be small compared to the theoretical inputs to the simulation.

Topics for future research include replacing the one parameter gamma distribution with the two parameter gamma distribution in the hierarchical model and exploring the use of EM methodology to estimate model parameters. The two parameter gamma distribution arises as the posterior distribution of the current model. Another area is to study using the fitted hierarchical gamma/Weibull model to predict future performance.

It is hoped that the hierarchical gamma/Weibull regression model will be a useful tool to describe and predict one aspect of the effect of human performance on the battlefield.

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APPENDIX A. STATISTICS AND GRAPHICS FOR THE ESTIMATE OF η IN THE SINGLE PARAMETER GAMMA MODEL

Table 1. Mean Bias, Mean Square Error, Std. Error and Uncensored Level for η at O = 10 in the Gamma Model

Number of Observers	Number of Targets	$\hat{\eta}$		UC
		M.B(S.E)	M.S.E(S.E)	
15	15	0.0727 (0.0497)	0.2477 (0.0693)	81%
	25	0.1027 (0.0436)	0.1967 (0.0269)	82%
	35	0.067 (0.0466)	0.2173 (0.0569)	82%
25	15	0.1003 (0.0311)	0.1051 (0.0167)	82%
	25	0.0566 (0.0304)	0.0935 (0.012)	82%
	35	0.0682 (0.0274)	0.0784 (0.0108)	82%
35	15	0.058 (0.0281)	0.0807 (0.0129)	82%
	25	0.0088 (0.0253)	0.0626 (0.0092)	82%
	35	0.0472 (0.0221)	0.05 (0.0103)	82%

Table 2. Mean Bias, Mean Square Error, Std. Error and Uncensored Level for η at O = 15 in the Gamma Model

Number of Observers	Number of Targets	$\hat{\eta}$		UC
		M.B(S.E)	M.S.E(S.E)	
15	15	0.0614 (0.0434)	0.1886 (0.0413)	89%
	25	0.0884 (0.0428)	0.1875 (0.026)	90%
	35	0.053 (0.0452)	0.2032 (0.052)	90%
25	15	0.094 (0.0305)	0.0998 (0.0151)	90%
	25	0.0517 (0.03)	0.0911 (0.0115)	90%
	35	0.0706 (0.0274)	0.0785 (0.0103)	90%
35	15	0.0519 (0.0277)	0.0781 (0.0132)	90%
	25	0.0085 (0.0246)	0.0593 (0.0086)	90%
	35	0.0471 (0.0216)	0.0481 (0.009)	90%

Table 3. Tendencies of Mean Bias and Mean Square Error for η at O = 10 in the Gamma Model

	M.B(M.S.E) of $\hat{\eta}$		
	15 TGT	25 TGT	35 TGT
15 OBS	0.073(0.248)	0.103(0.197)	0.067(0.217)
25 OBS	0.1 (0.105)	0.057(0.094)	0.068(0.078)
35 OBS	0.058(0.081)	0.009(0.063)	0.047(0.05)

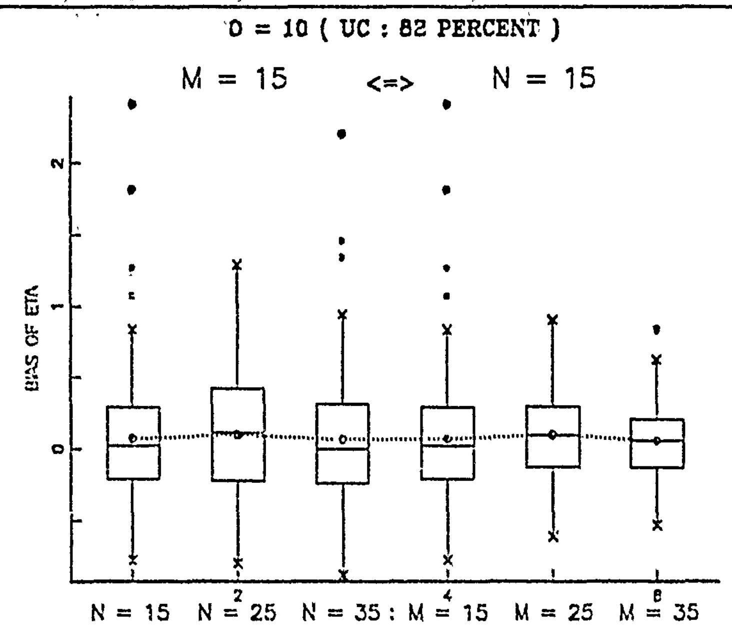

Figure 1. The Tendencies of the Bias of η with changes in OBS and TGT at O = 10 in the Gamma Model: M = number of observers ; N = number of targets.

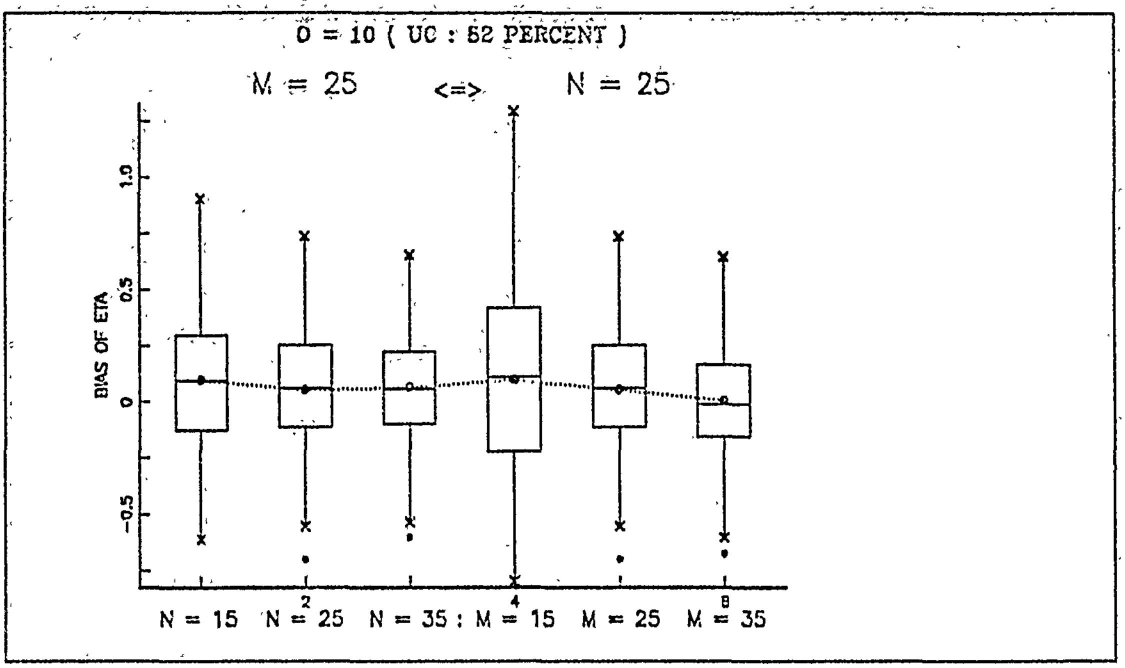

Figure 2. The Tendencies of the Bias of η with changes in OBS and TGT at O = 10 in the Gamma Model: M = number of observers ; N = number of targets.

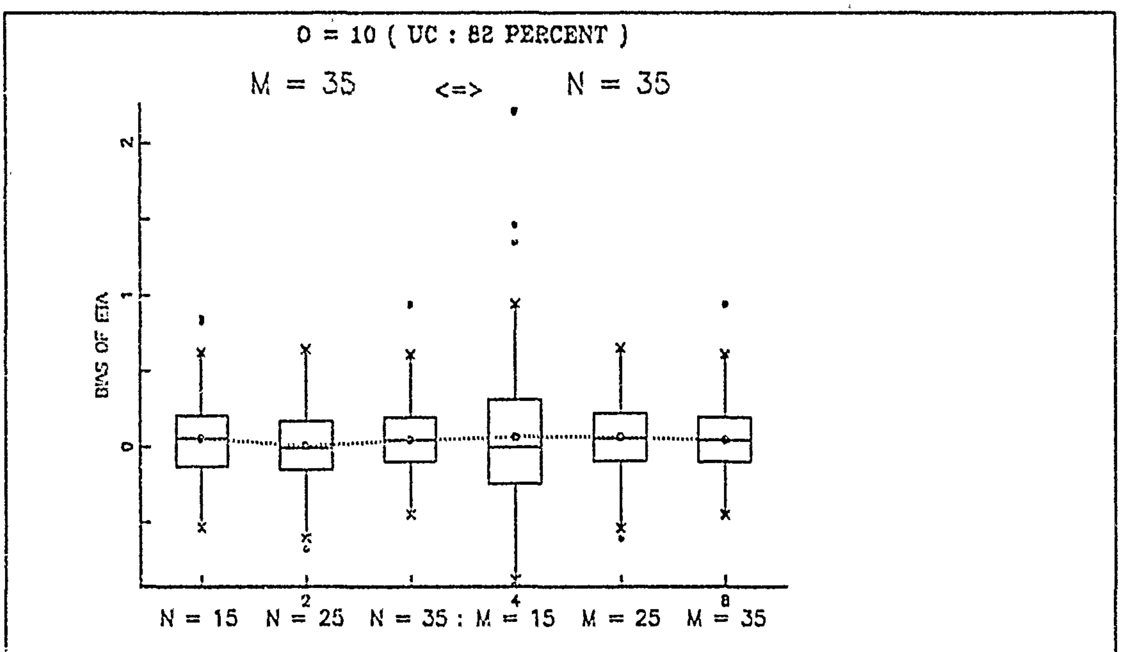

Figure 3. The Tendencies of the Bias of η with changes in OBS and TGT at O = 10 in the Gamma Model: M = number of observers ; N = number of targets.

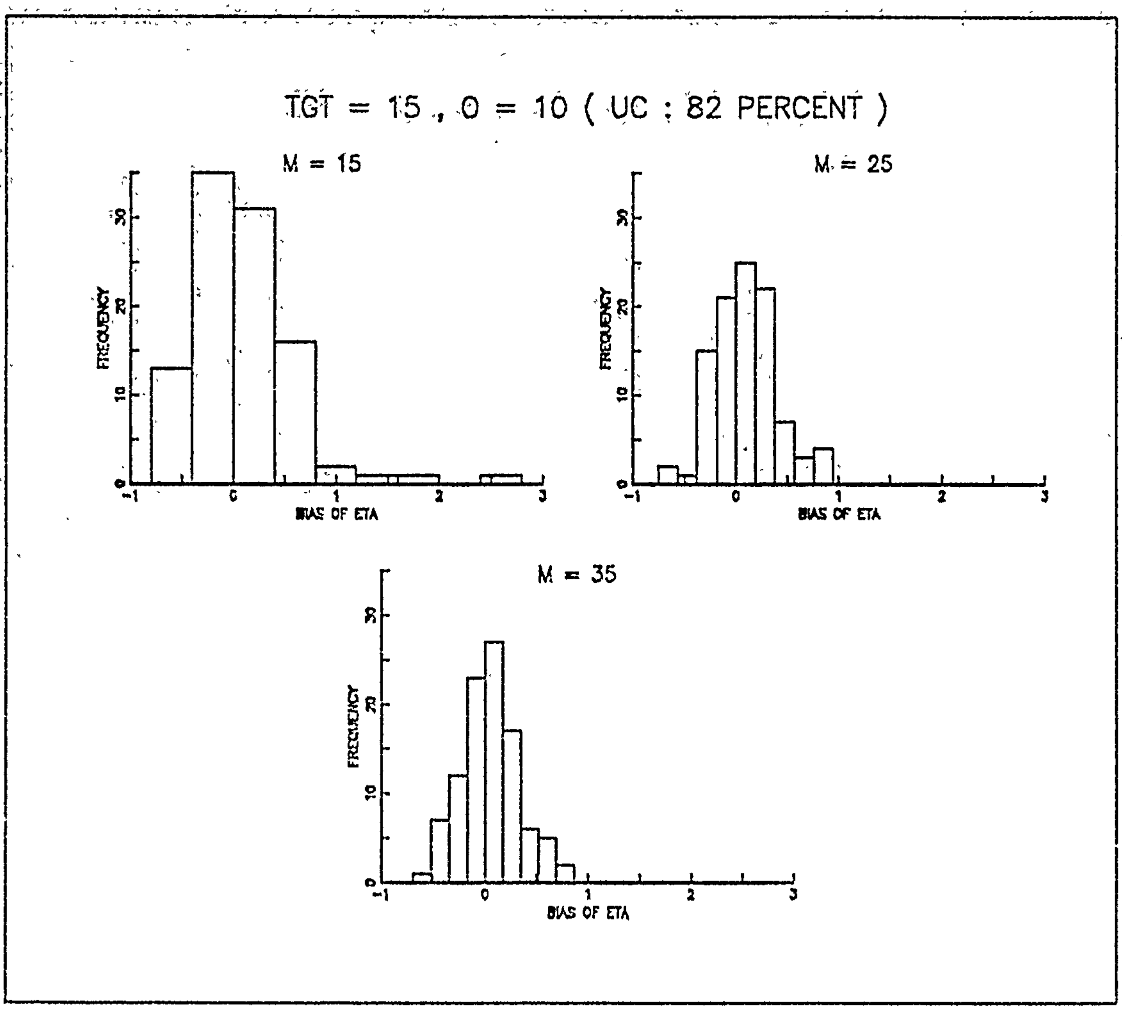

Figure 4. Comparison of the Bias of η between Different numbers of Observers with TGT = 15 at O = 10 in the Gamma Model: TGT = the number of targets.

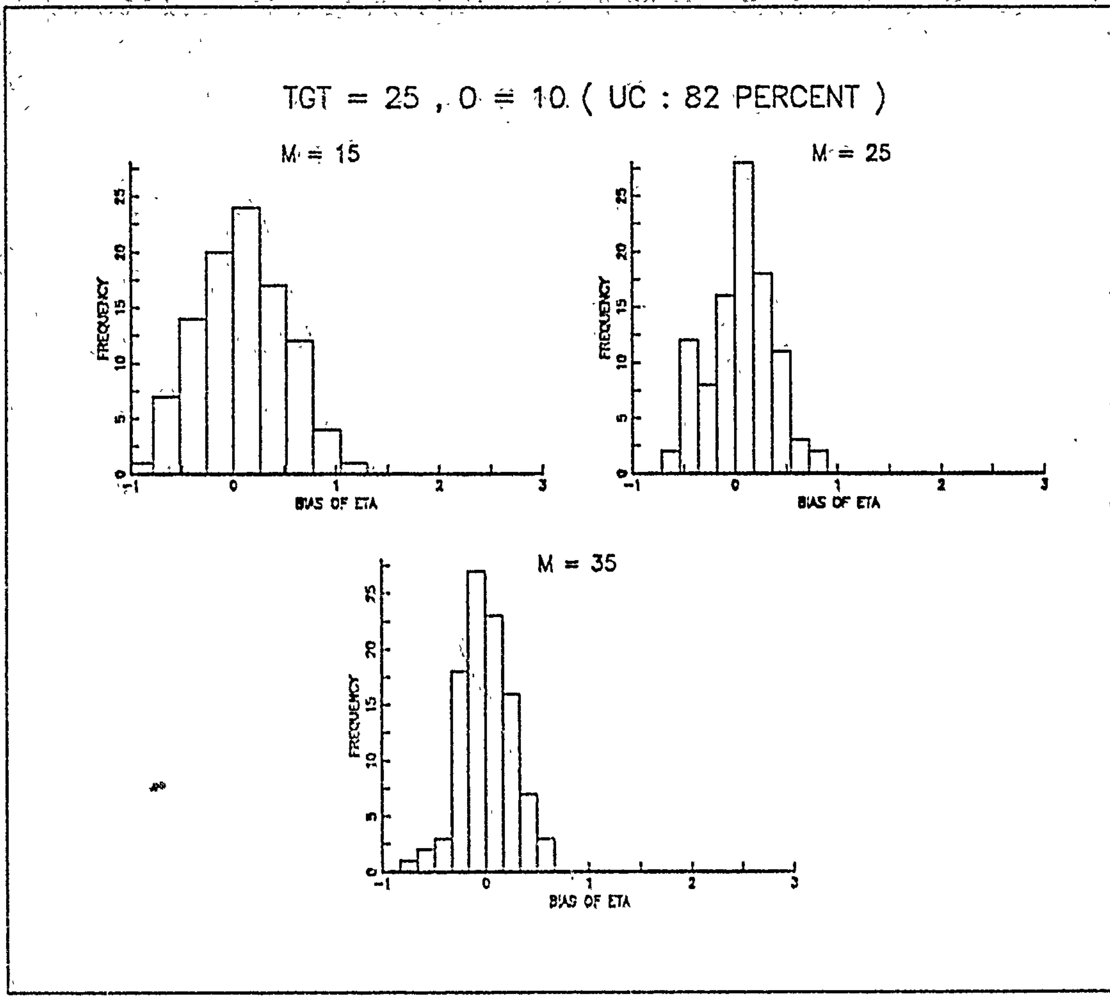

Figure 5. Comparison of the Bias of η between Different numbers of Observers with TGT = 25 at O = 10 in the Gamma Model: TGT = the number of targets.

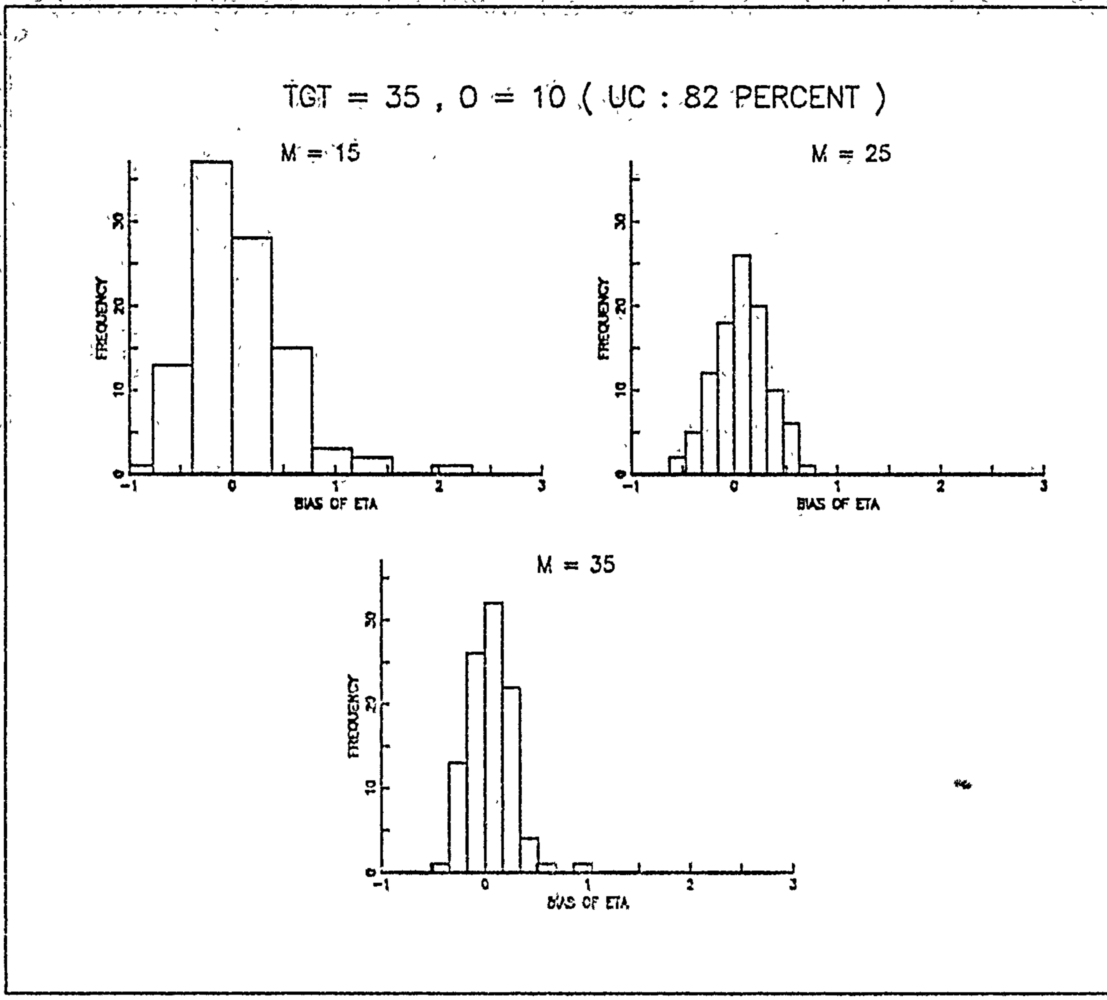

Figure 6. Comparison of the Bias of η between Different numbers of Observers with TGT = 35 at O = 10 in the Gamma Model: TGT = the number of targets.

Table 4. Tendencies of Mean Bias and Mean Square Error for η at O = 15 in the Gamma Model

	M.B(M.S.E) of $\hat{\eta}$		
	15 TGT	25 TGT	35 TGT
15 OBS	0.061(0.189)	0.088(0.188)	0.053(0.203)
25 OBS	0.094(0.1)	0.052(0.091)	0.071(0.078)
35 OBS	0.052(0.078)	0.008(0.059)	0.047(0.048)

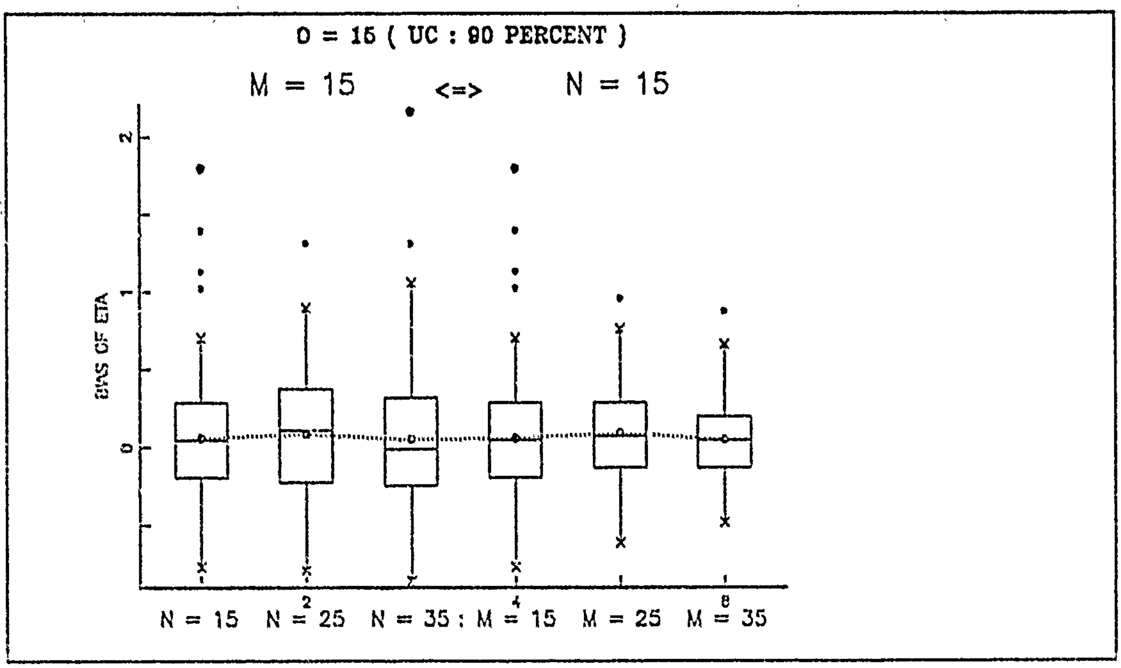

Figure 7. The Tendencies of the Bias of η with changes in OBS and TGT at O = 15 in the Gamma Model: M = number of observers ; N = number of targets.

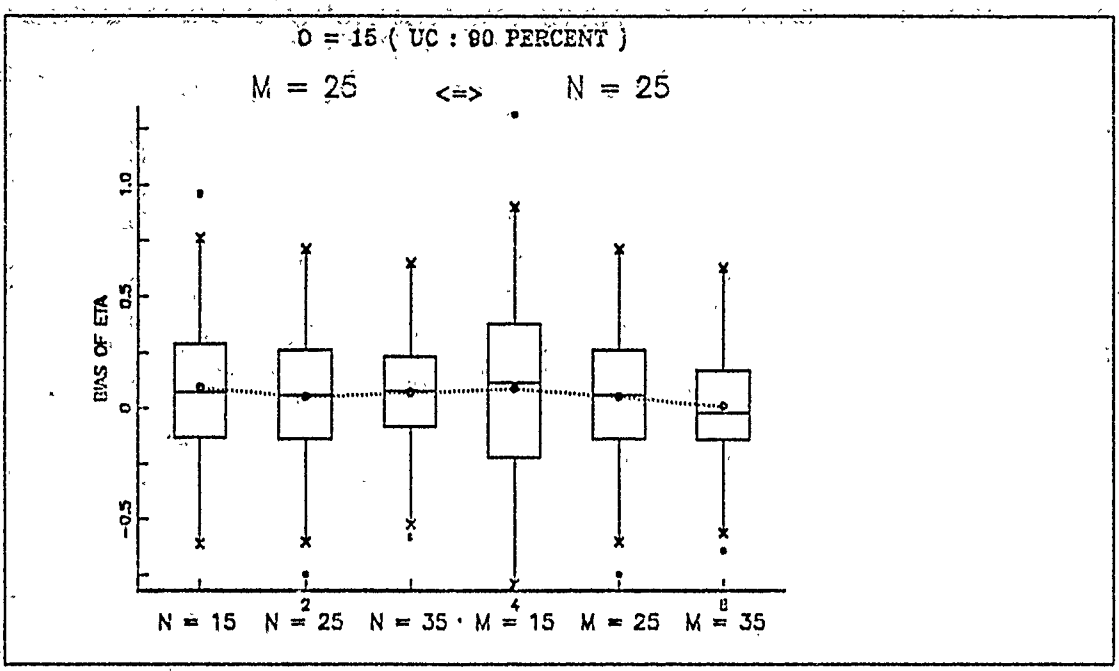

Figure 8. The Tendencies of the Bias of η with changes in OBS and TGT at O = 15 in the Gamma Model: M = number of observers ; N = number of targets.

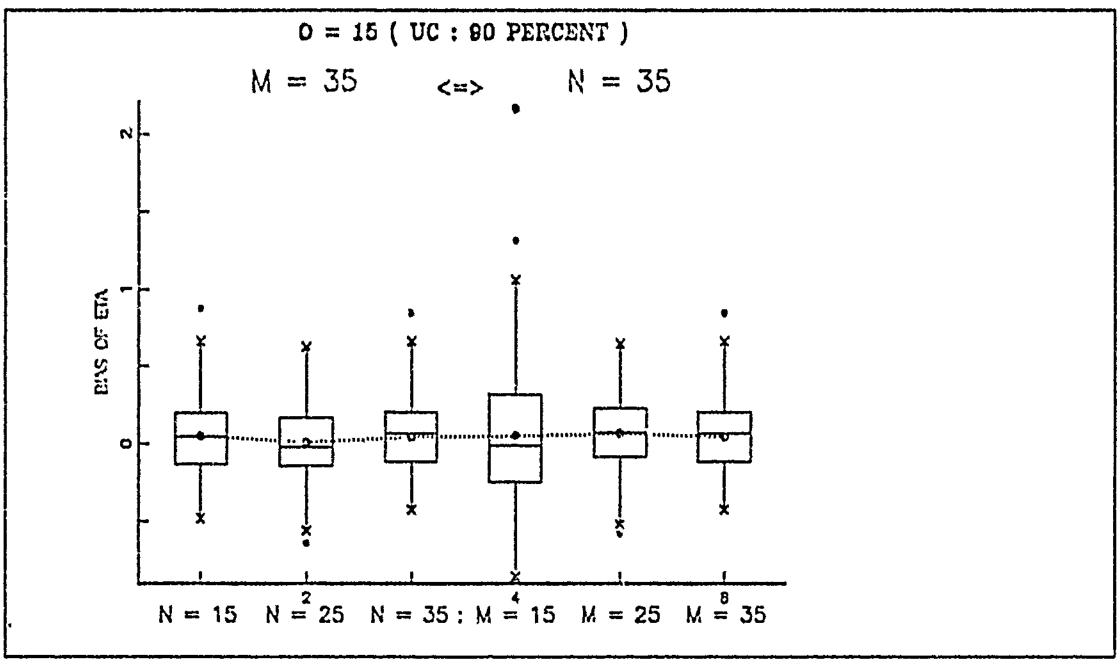

Figure 9. The Tendencies of the Bias of η with changes in OBS and TGT at O = 15 in the Gamma Model: M = number of observers ; N = number of targets.

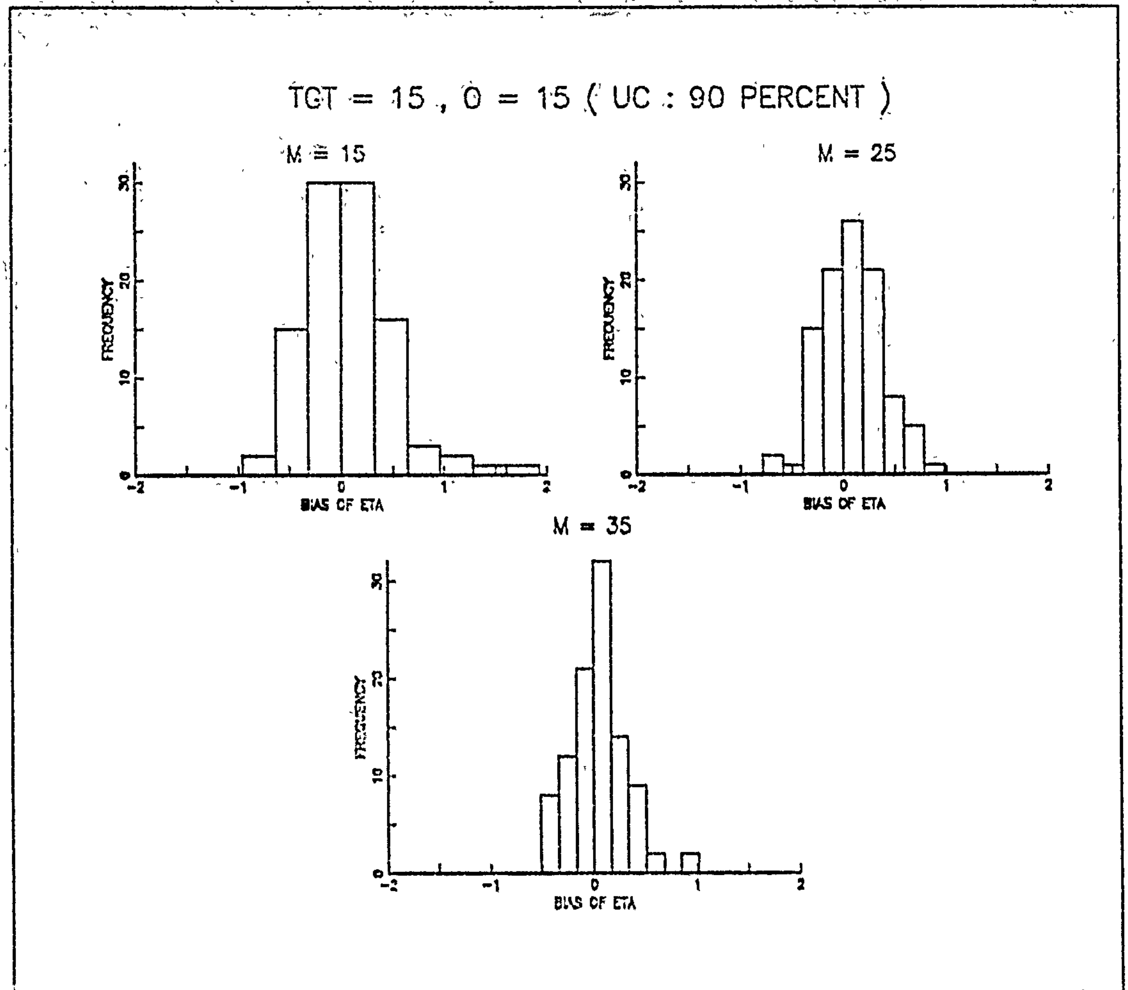

Figure 10. Comparison of the Bias of η between Different numbers of Observers with TGT = 15 at O = 15 in the Gamma Model: TGT = the number of targets.

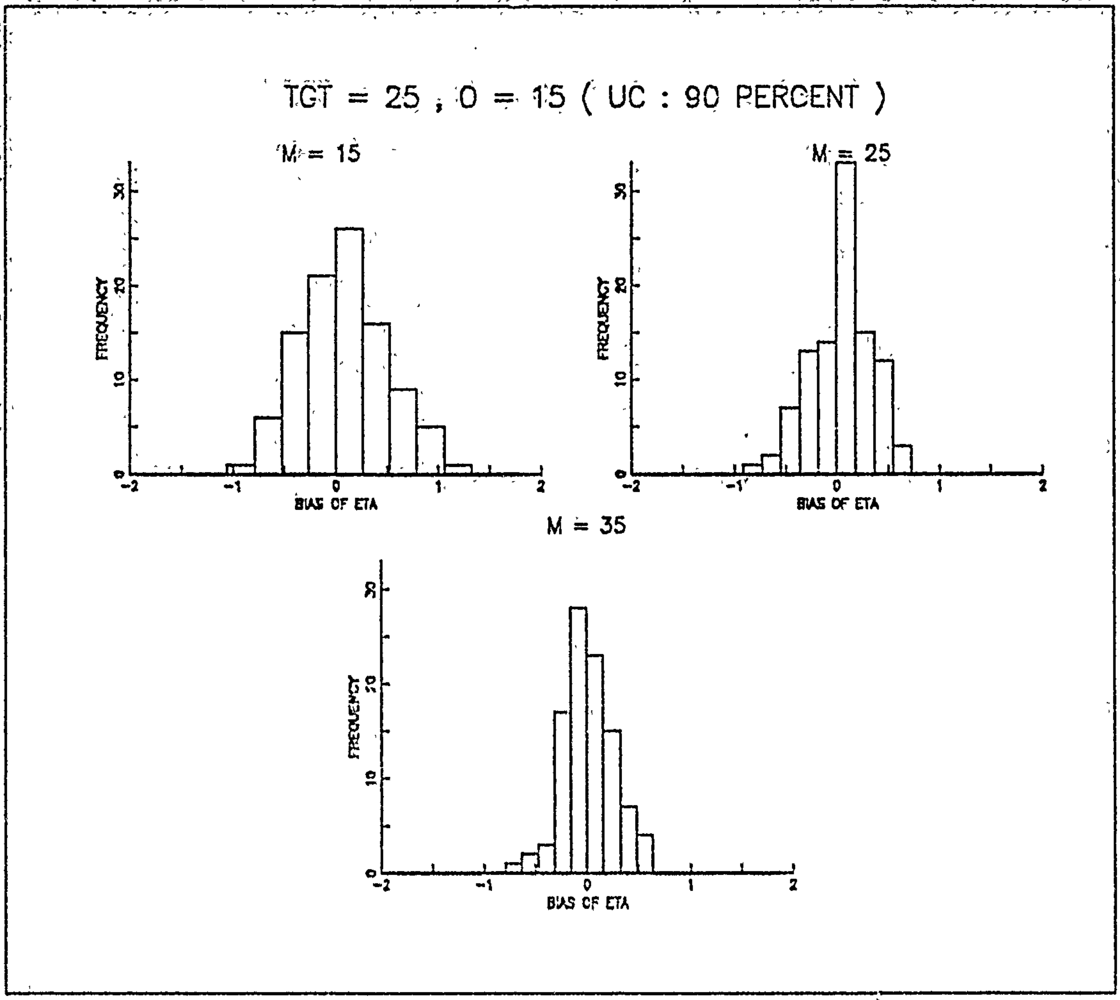

Figure 11. Comparison of the Bias of η between Different numbers of Observers with TGT = 25 at O = 15 in the Gamma Model: TGT = the number of targets.

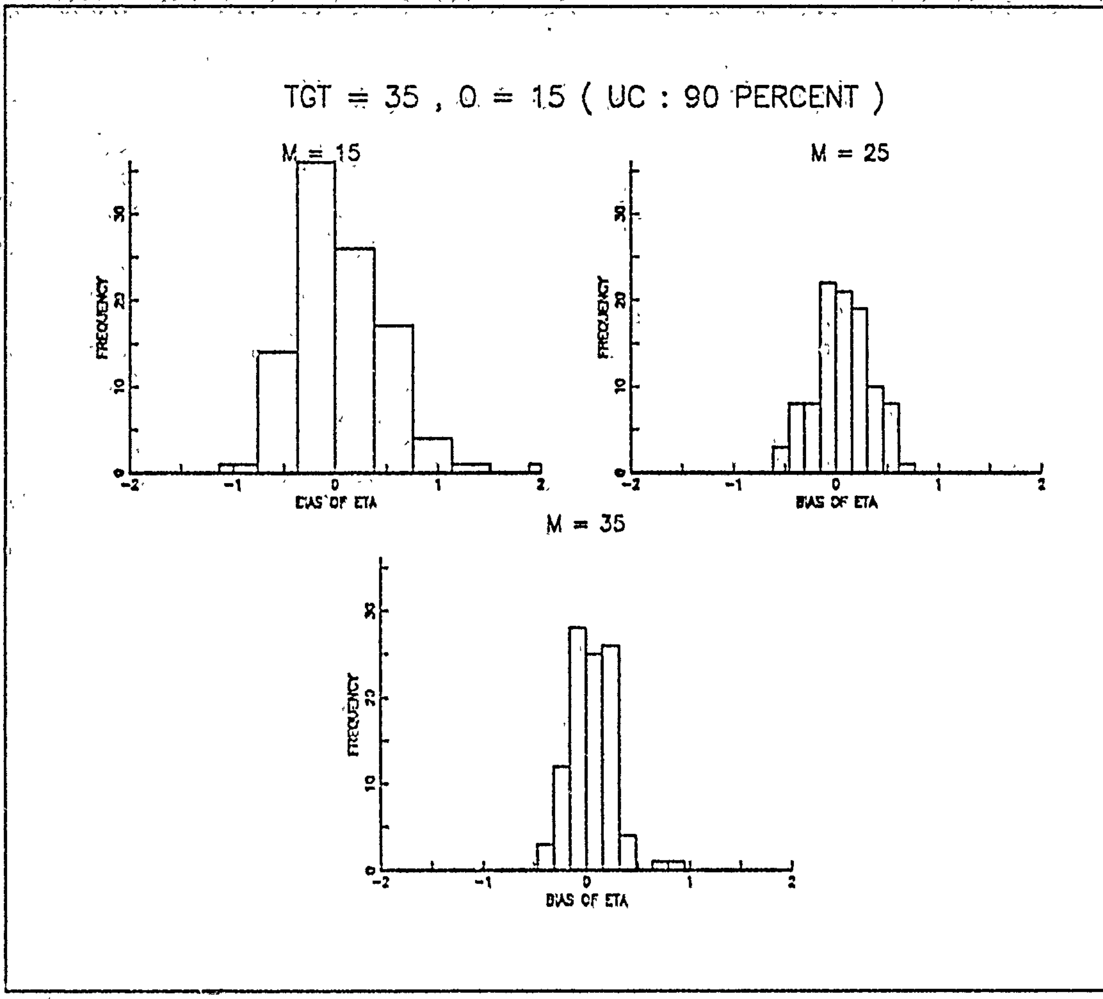

Figure 12. Comparison of the Bias of η between Different numbers of Observers with TGT = 35 at O = 15 in the Gamma Model: TGT = the number of targets.

APPENDIX B. STATISTICS AND GRAPHICS FOR ALL THE ESTIMATES IN GAMMA/WEIBULL REGRESSION MODEL

1. TABLES AND HISTOGRAMS FOR η

Table 5. Mean Bias, Mean Square Error and Std. Error for η at O = 10 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\eta}$	
		M.B(S.E)	M.S.E(S.E)
15	15	.1770(.0552)	.3299(.0892)
	25	.1573(.0455)	.2274(.0445)
	35	.0761(.0440)	.1955(.0290)
25	15	.0947(.0419)	.1811(.0420)
	25	.0571(.0315)	.1003(.0206)
	35	.1422(.0324)	.1228(.0208)
35	15	.1074(.0352)	.1328(.0216)
	25	.0658(.0307)	.0968(.0129)
	35	.0940(.0223)	.0577(.0070)

Table 6. Mean Bias, Mean Square Error and Std. Error for η at O = 15 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\eta}$	
		M.B(S.E)	M.S.E(S.E)
15	15	.2107(.0561)	.3524(.1023)
	25	.1934(.0451)	.2365(.0458)
	35	.1117(.0442)	.2038(.0319)
25	15	.1170(.0418)	.1851(.0454)
	25	.0755(.0314)	.1021(.0222)
	35	.1434(.0314)	.1174(.0197)
35	15	.1450(.0336)	.1315(.0228)
	25	.0800(.0306)	.0982(.0133)
	35	.0968(.0219)	.0566(.0072)

Table 7. Tendencies of Mean Bias and Mean Square Error for η at O = 10 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\eta}$		
	15 TGT	25 TGT	35 TGT
15 OBS	.177(.330)	.157(.227)	.076(.196)
25 OBS	.095(.181)	.057(.100)	.142(.123)
35 OBS	.107(.133)	.066(.097)	.094(.058)

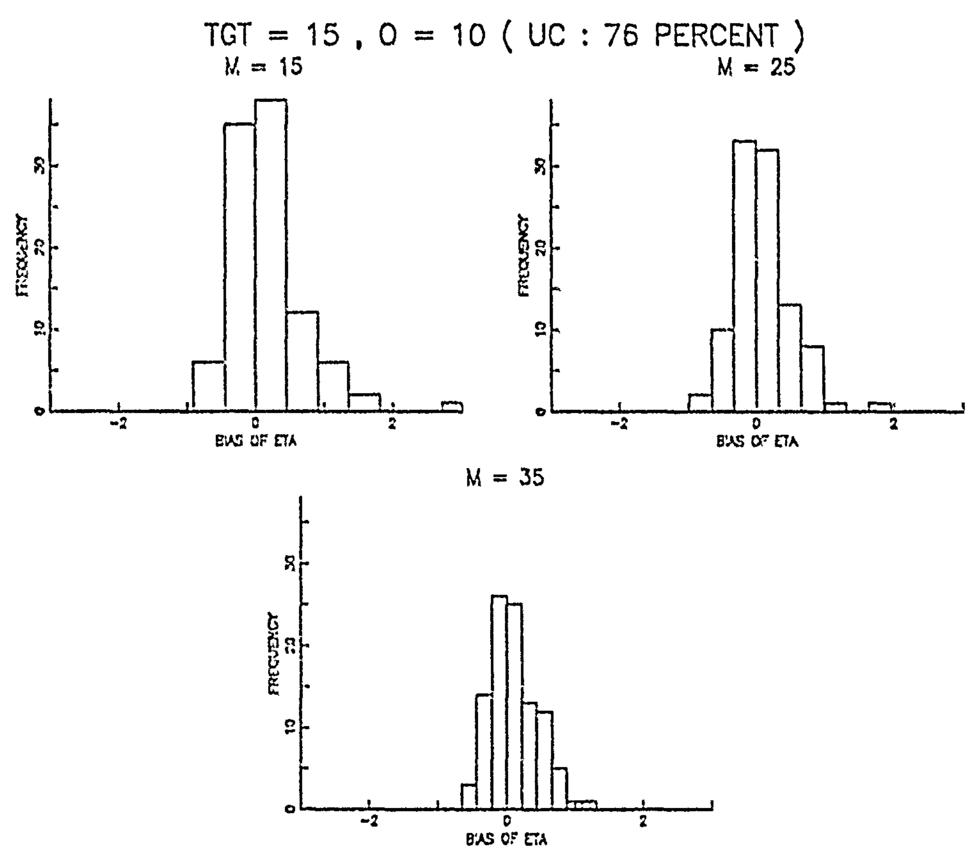

Figure 13. Comparison of the Bias of η between Different numbers of Observers with TGT= 15 at O= 10 in the GAM/WEI Regression Model: TGT = the number of targets.

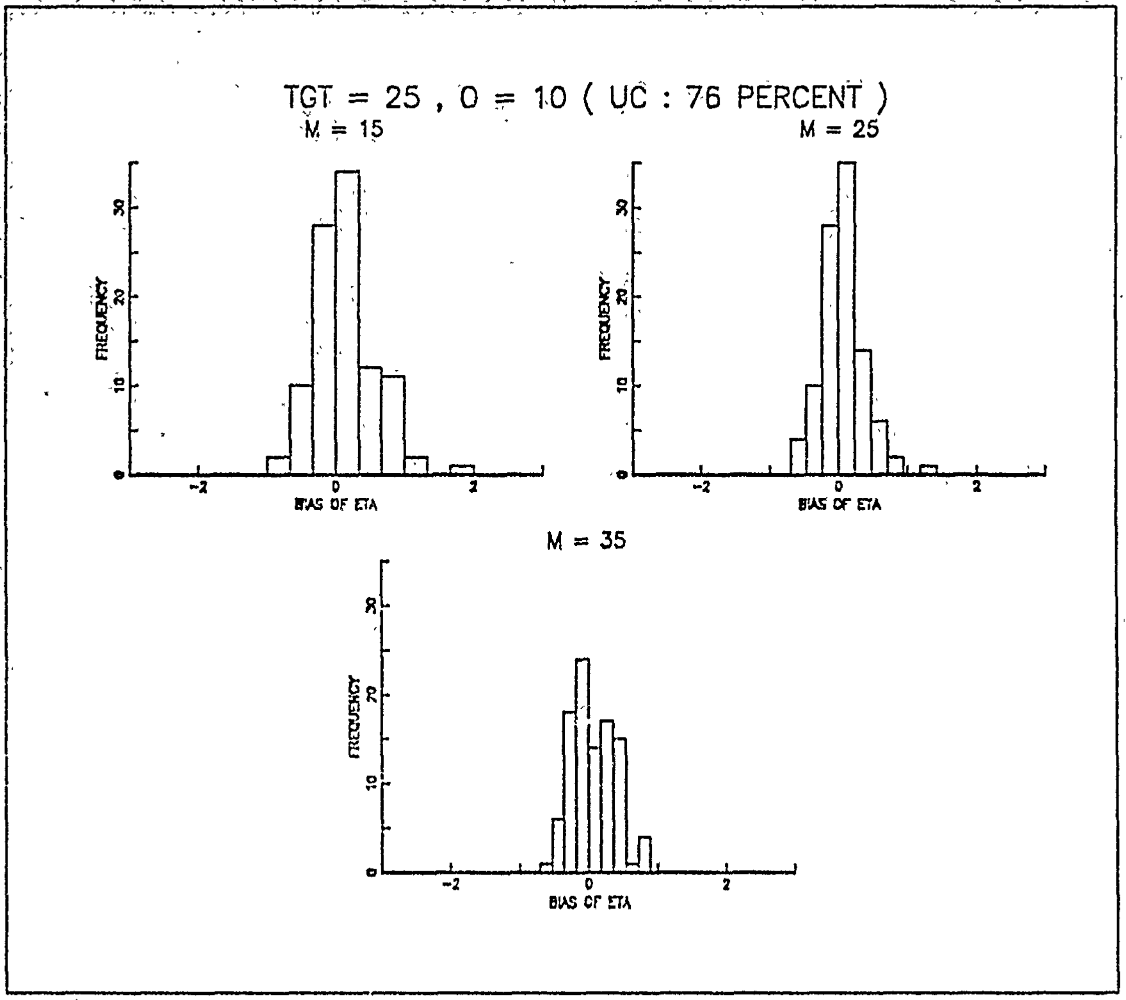

Figure 14. Comparison of the Bias of η between Different numbers of Observers with TGT=25 at O=10 in the GAM/WEI Regression Model: TGT = the number of targets.

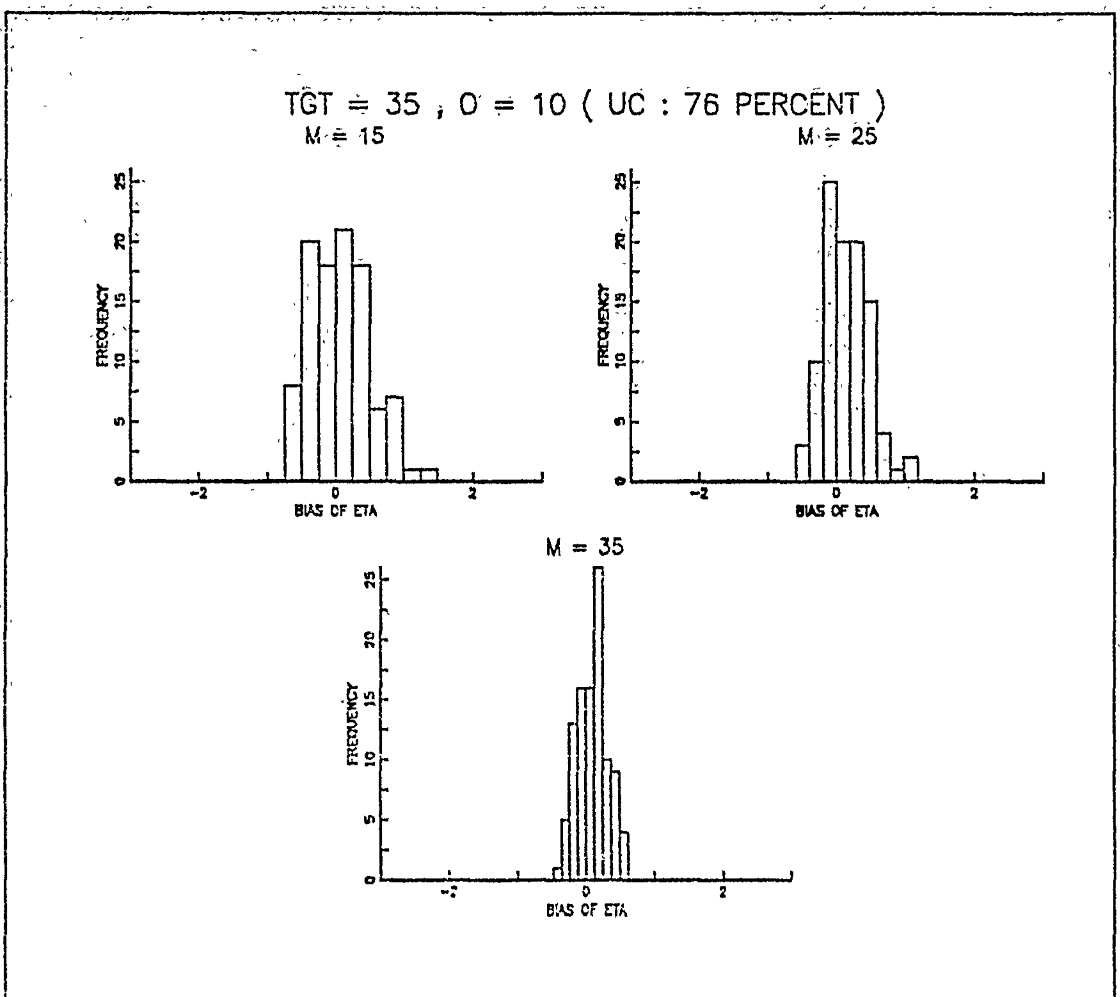

Figure 15. Comparison of the Bias of η between Different numbers of Observers with TGT=35 at O=10 in the GAM/WEI Regression Model: TGT = the number of targets.

Table 8. Tendencies of Mean Bias and Mean Square Error for η at O = 15 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\eta}$		
	15 TGT	25 TGT	35 TGT
15 OBS	.211(.352)	.193(.236)	.112(.204)
25 OBS	.117(.185)	.075(.102)	.143(.117)
35 OBS	.145(.132)	.080(.098)	.097(.057)

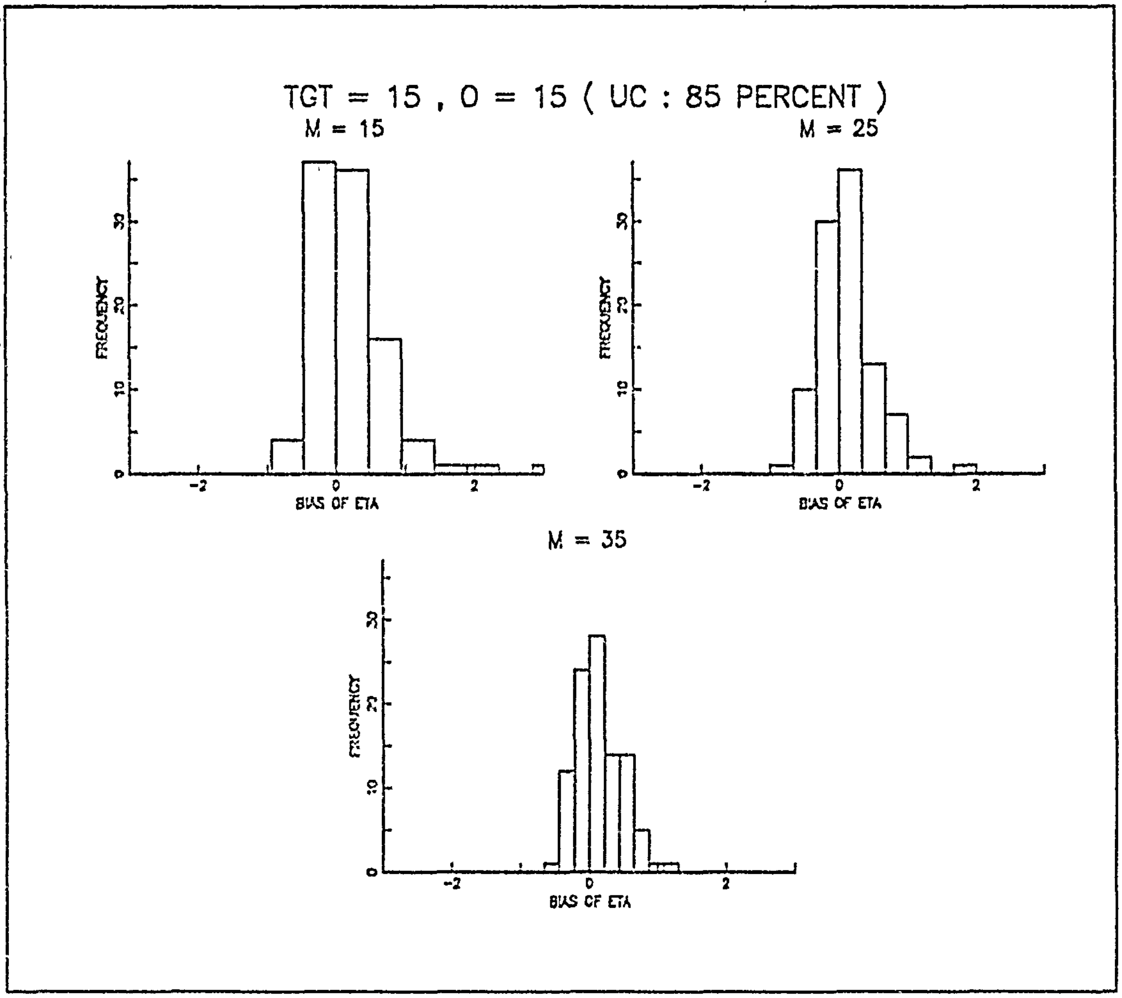

Figure 16. Comparison of the Bias of η between Different numbers of Observers with TGT = 15 at O = 15 in the GAM/WEI Regression Model: TGT = the number of targets.

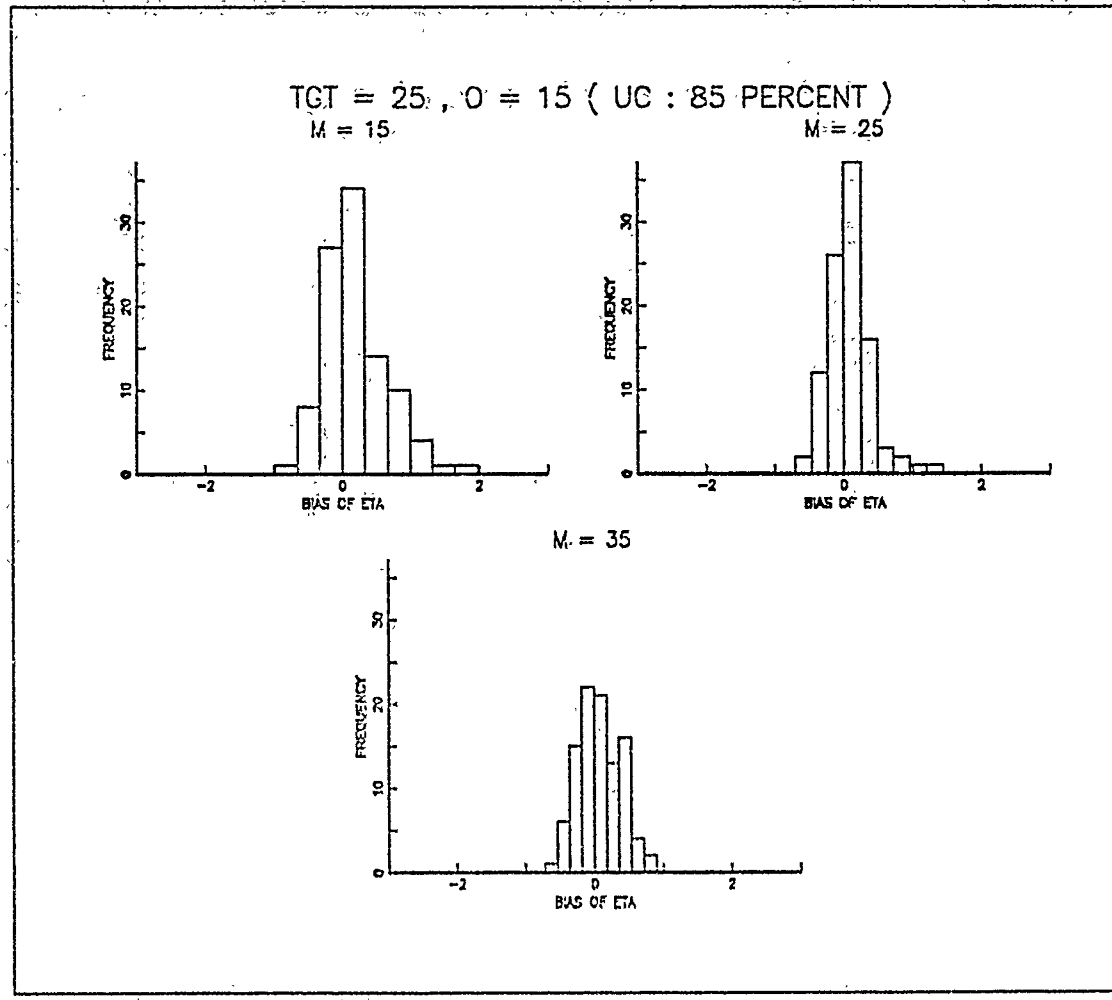

Figure 17. Comparison of the Bias of η between Different numbers of Observers with TGT = 25 at O = 15 in the GAM/WEI Regression Model: TGT = the number of targets.

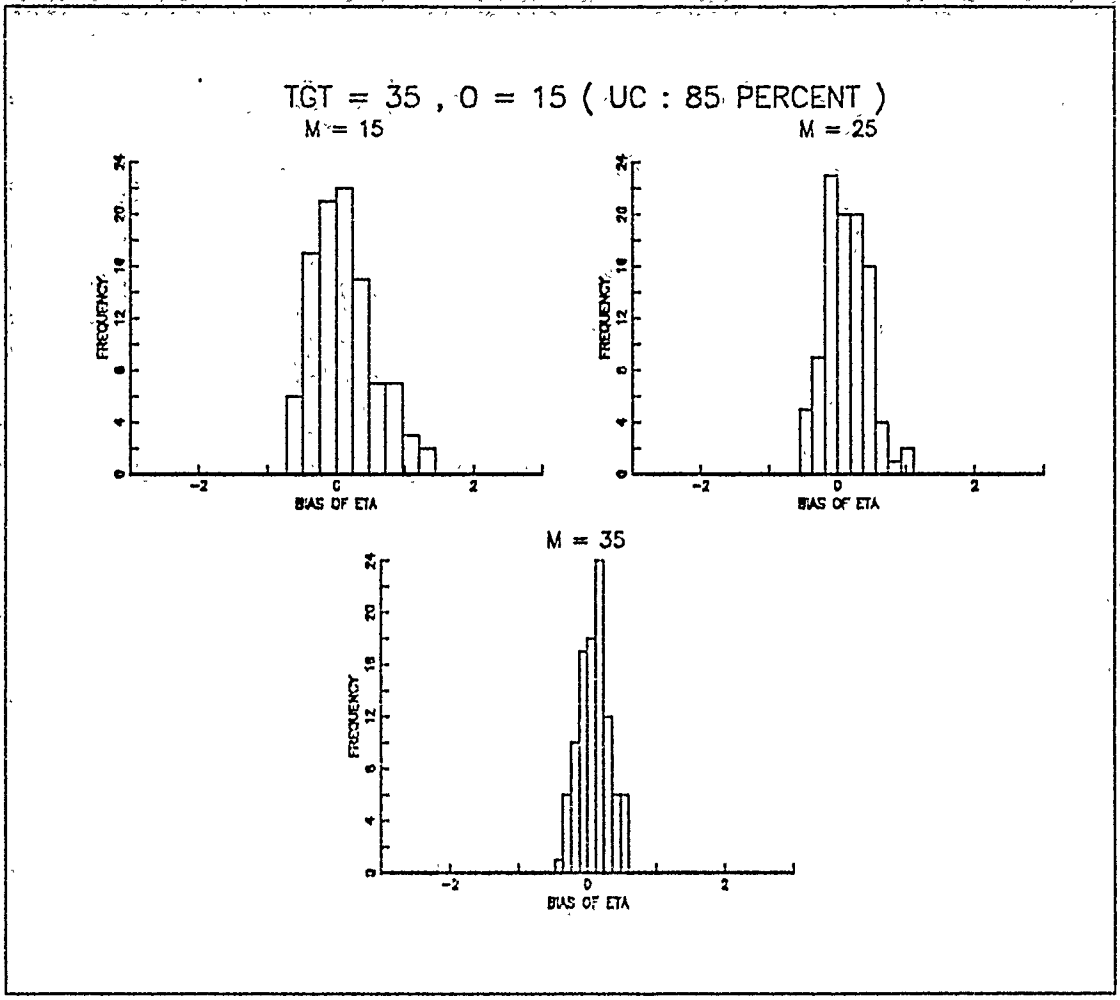

Figure 18. Comparison of the Bias of η between Different numbers of Observers with TGT = 35 at O = 15 in the GAM/WEI Regression Model: TGT = the number of targets.

2. TABLES AND HISTOGRAMS FOR $\beta 0$

Table 9. Mean Bias,Mean Square Error and Std. Error for $\beta 0$ at O = 10 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\beta}_0$ M.B(S.E)	$\hat{\beta}_0$ M.S.E(S.E)
15	15	-.0605(.0317)	.1021(.0123)
	25	-.0234(.0278)	.0764(.0109)
	35	.0304(.0223)	.0499(.0067)
25	15	.0458(.0238)	.0575(.0081)
	25	-.0167(.0222)	.0487(.0052)
	35	-.0094(.0197)	.0383(.0056)
35	15	-.0067(.0198)	.0386(.0067)
	25	-.0398(.0176)	.0319(.0047)
	35	.0062(.0152)	.0227(.0027)

Table 10. Mean Bias,Mean Square Error and Std. Error for $\beta 0$ at O = 15 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\beta}_0$ M.B(S.E)	$\hat{\beta}_0$ M.S.E(S.E)
15	15	-.0609(.0316)	.1015(.0127)
	25	-.0234(.0271)	.0725(.0095)
	35	.0263(.0224)	.0498(.0063)
25	15	.0365(.0244)	.0595(.0073)
	25	-.0292(.0210)	.0439(.0049)
	35	-.0185(.0184)	.0335(.0053)
35	15	-.0043(.0191)	.0358(.0066)
	25	-.0454(.0176)	.0324(.0041)
	35	-.0021(.0138)	.0187(.0023)

Table 11. Tendencies of Mean Bias and Mean Square Error for $\beta 0$ at O = 10 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\beta}_0$		
	15 TGT	25 TGT	35 TGT
15 OBS	-.061(.102)	-.023(.076)	.030(.050)
25 OBS	.046(.057)	-.017(.049)	-.009(.038)
35 OBS	-.007(.039)	-.040(.032)	.006(.023)

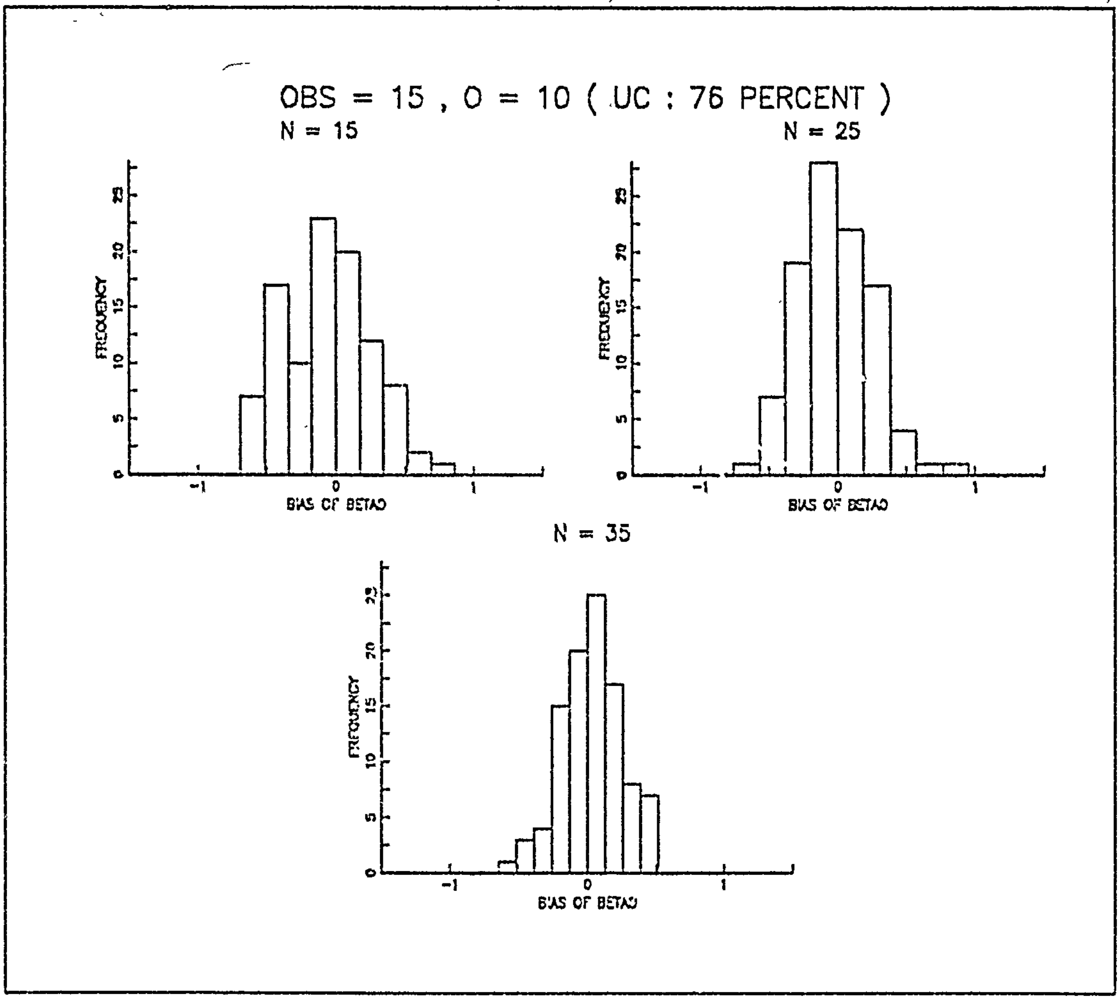

Figure 19. Comparison of the Bias of $\beta 0$ between Different numbers of targets with OBS = 15 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

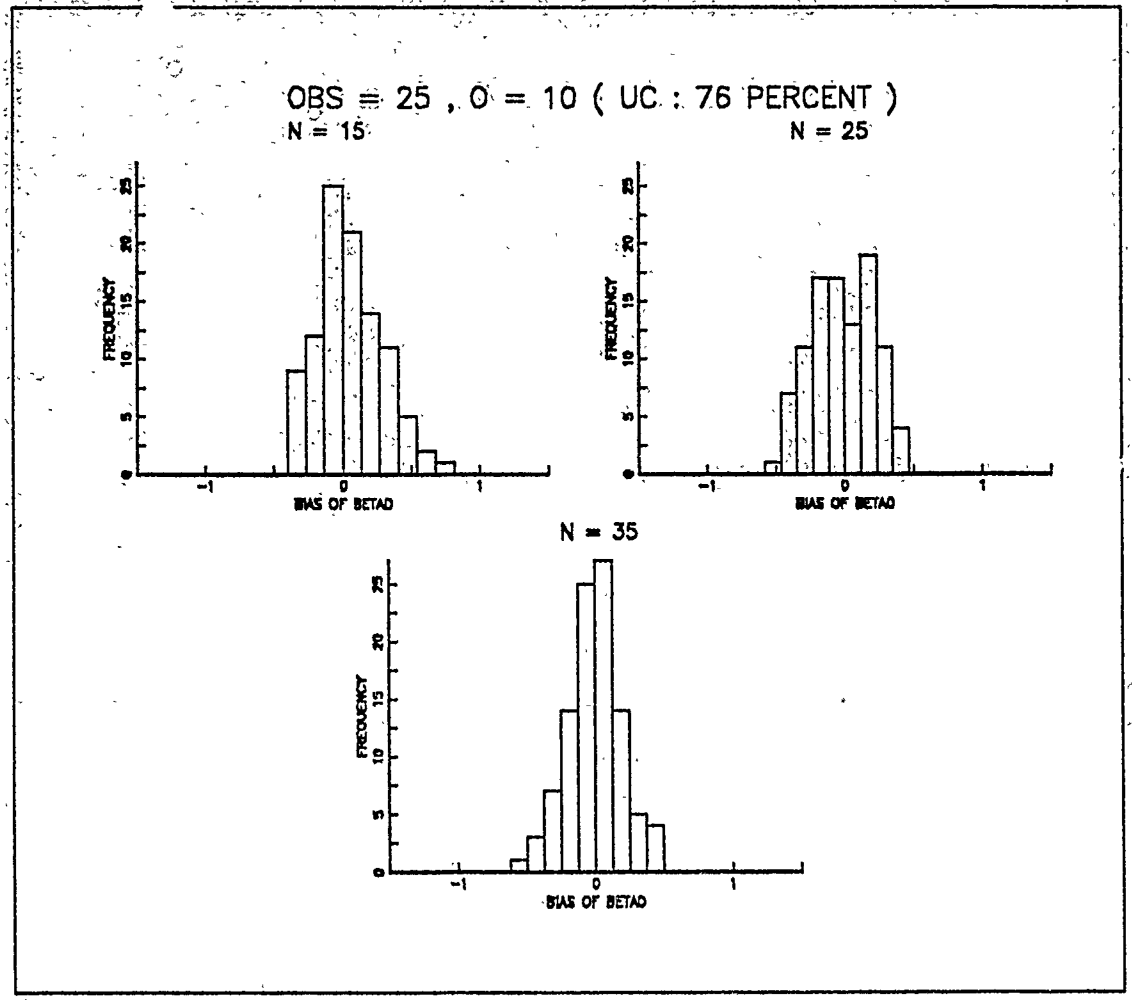

Figure 20. Comparison of the Bias of $\beta 0$ between Different numbers of targets with OBS = 25 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

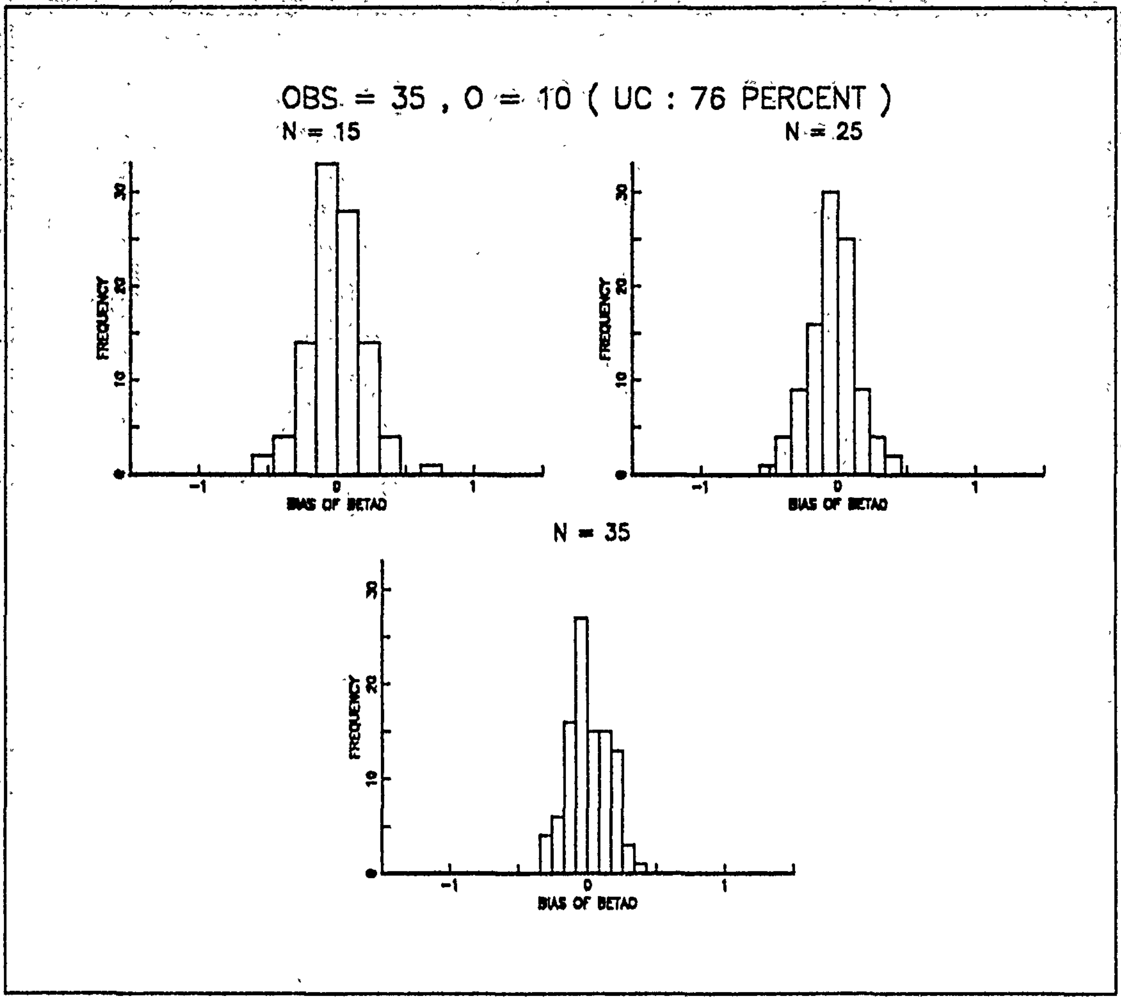

Figure 21. Comparison of the Bias of $\beta 0$ between Different numbers of targets with OBS = 35 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

Table 12. Tendencies of Mean Bias and Mean Square Error for $\beta 0$ at O = 15 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\beta}_0$		
	15 TGT	25 TGT	35 TGT
15 OBS	-.061(.102)	-.023(.073)	.026(.050)
25 OBS	.037(.060)	-.029(.044)	-.018(.033)
35 OBS	-.004(.036)	-.045(.032)	-.002(.019)

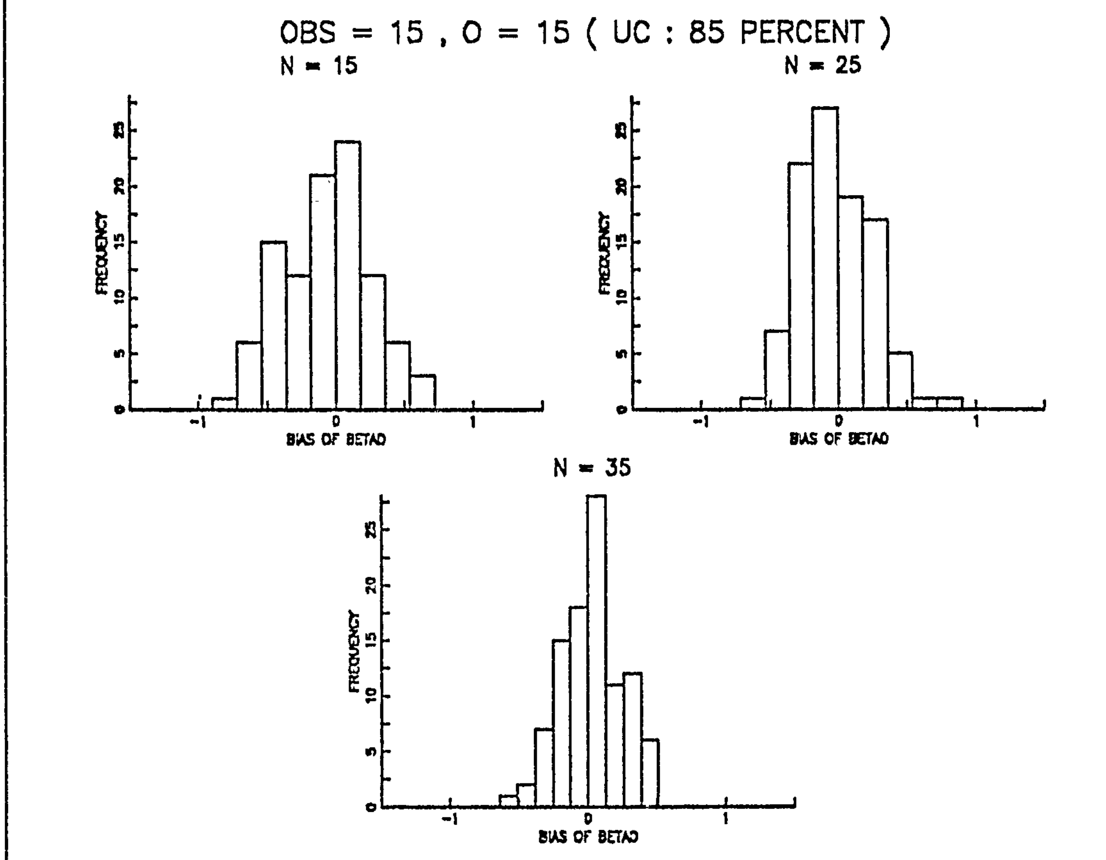

Figure 22. Comparison of the Bias of $\beta 0$ between Different numbers of targets with OBS = 15 at O = 15 in the GAM/WEI Regression Model: OBS = the number of observers

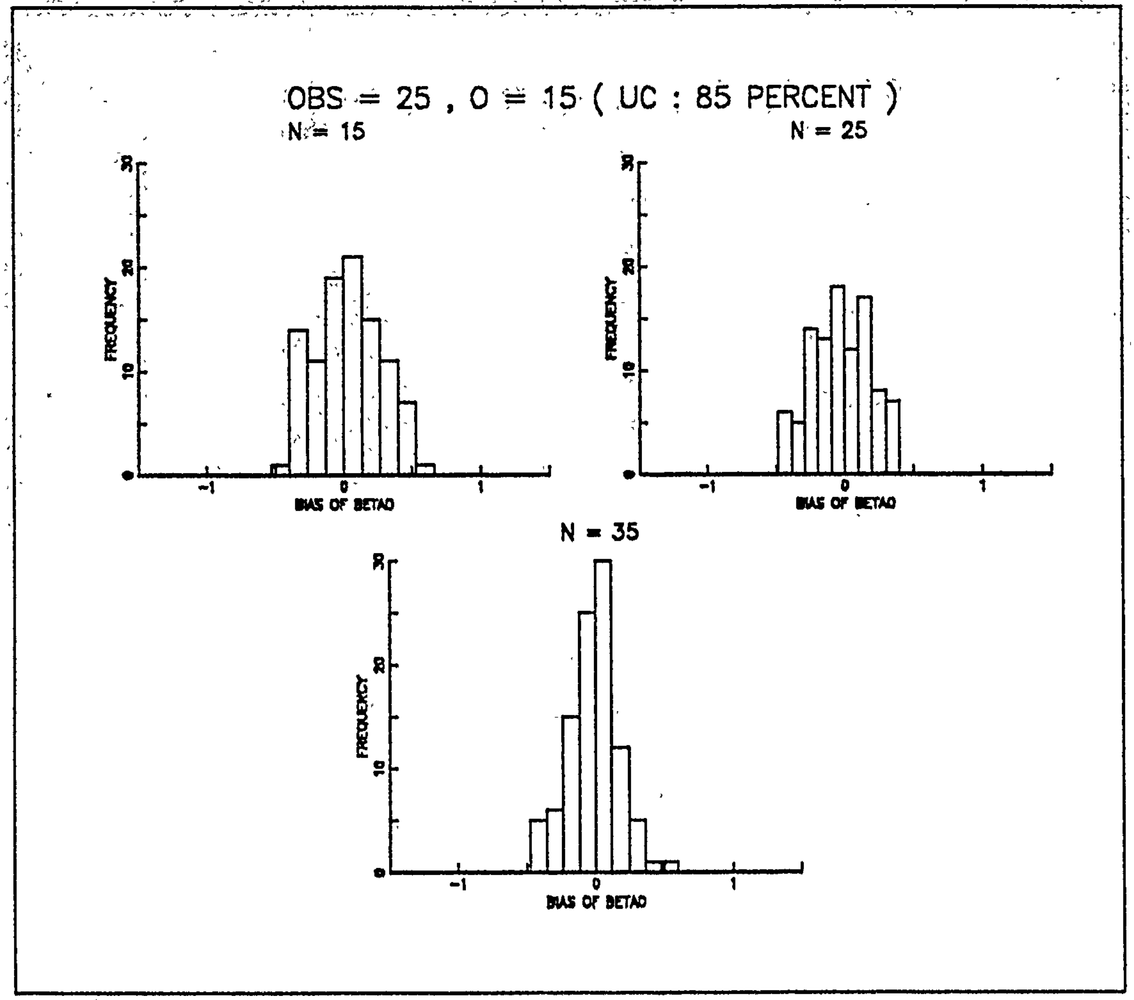

Figure 23. Comparison of the Bias of $\beta 0$ between Different numbers of targets with OBS=25 at O=15 in the GAM/WEI Regression Model: OBS = the number of observers

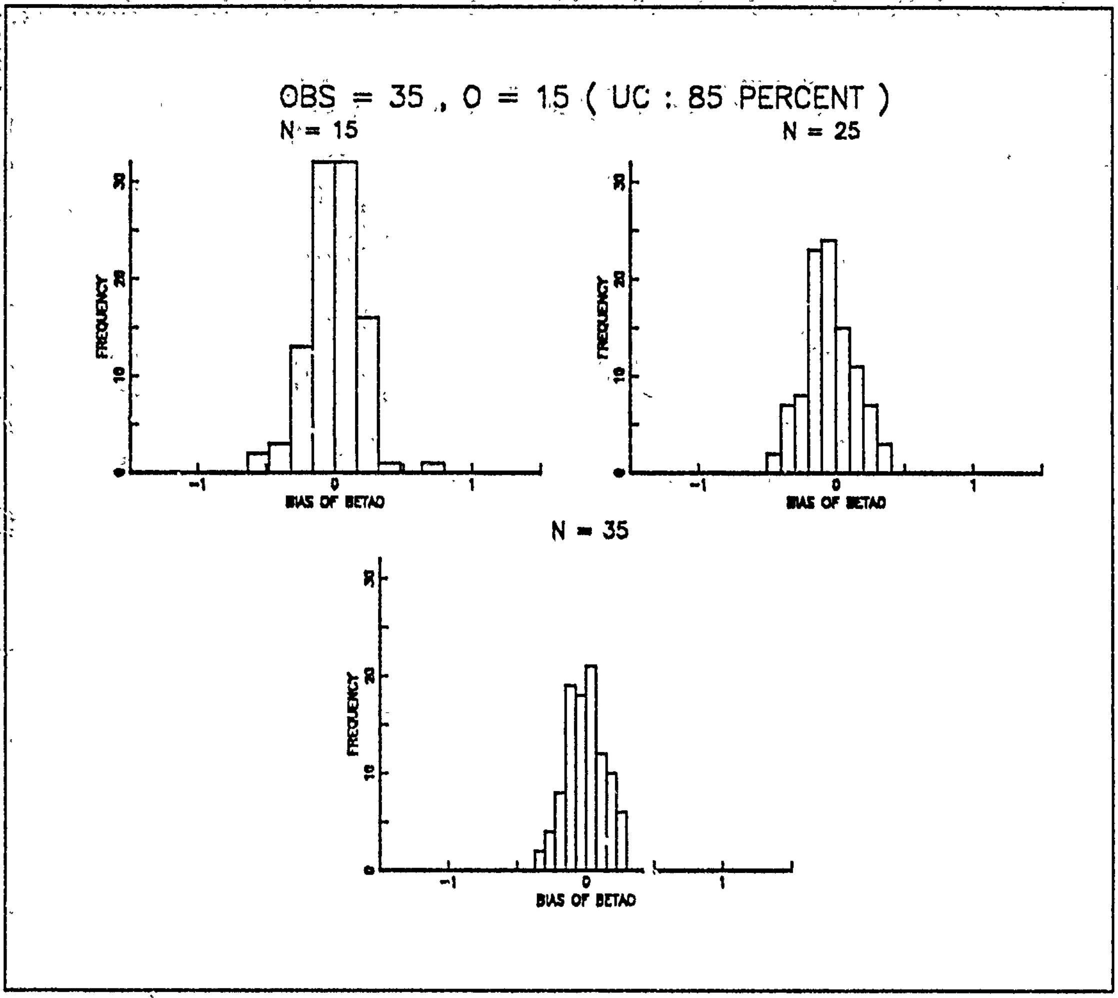

Figure 24. Comparison of the Bias of $\beta 0$ between Different numbers of targets with OBS = 35 at O = 15 in the GAM/WEI Regression Model: OBS = the number of observers

3. TABLES AND HISTOGRAMS FOR $\beta 1$

Table 13. Mean Bias,Mean Square Error and Std. Error for $\beta 1$ at O = 10 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\beta}_1$	
		M.B(S.E)	M.S.E(S.E)
15	15	.0180(.0157)	.0243(.0032)
	25	-.0002(.0140)	.0193(.0026)
	35	-.0010(.0112)	.0122(.0017)
25	15	-.0154(.0132)	.0173(.0026)
	25	-.0006(.0096)	.0091(.0014)
	35	-.0036(.0080)	.0063(.0010)
35	15	.0188(.0126)	.0160(.0020)
	25	.0094(.0096)	.0091(.0011)
	35	.0006(.0069)	.0047(.0006)

Table 14. Mean Bias,Mean Square Error and Std. Error for $\beta 1$ at O = 15 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\beta}_1$	
		M.B(S.E)	M.S.E(S.E)
15	15	.0079(.0145)	.0205(.0028)
	25	-.0060(.0127)	.0158(.0022)
	35	.0076(.0107)	.0113(.0014)
25	15	-.0214(.0127)	.0162(.0024)
	25	.0036(.0090)	.0079(.0013)
	35	-.0046(.0075)	.0055(.0009)
35	15	.0150(.0121)	.0146(.0018)
	25	.0077(.0092)	.0084(.0010)
	35	.0009(.0067)	.0043(.0005)

Table 15. Tendencies of Mean Bias and Mean Square Error for $\beta 1$ at O = 10 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\beta}_1$		
	15 TGT	25 TGT	35 TGT
15 OBS	.018(.024)	.000(.019)	-.001(.012)
25 OBS	-.015(.017)	-.001(.009)	-.004(.006)
35 OBS	.019(.016)	.009(.009)	.001(.005)

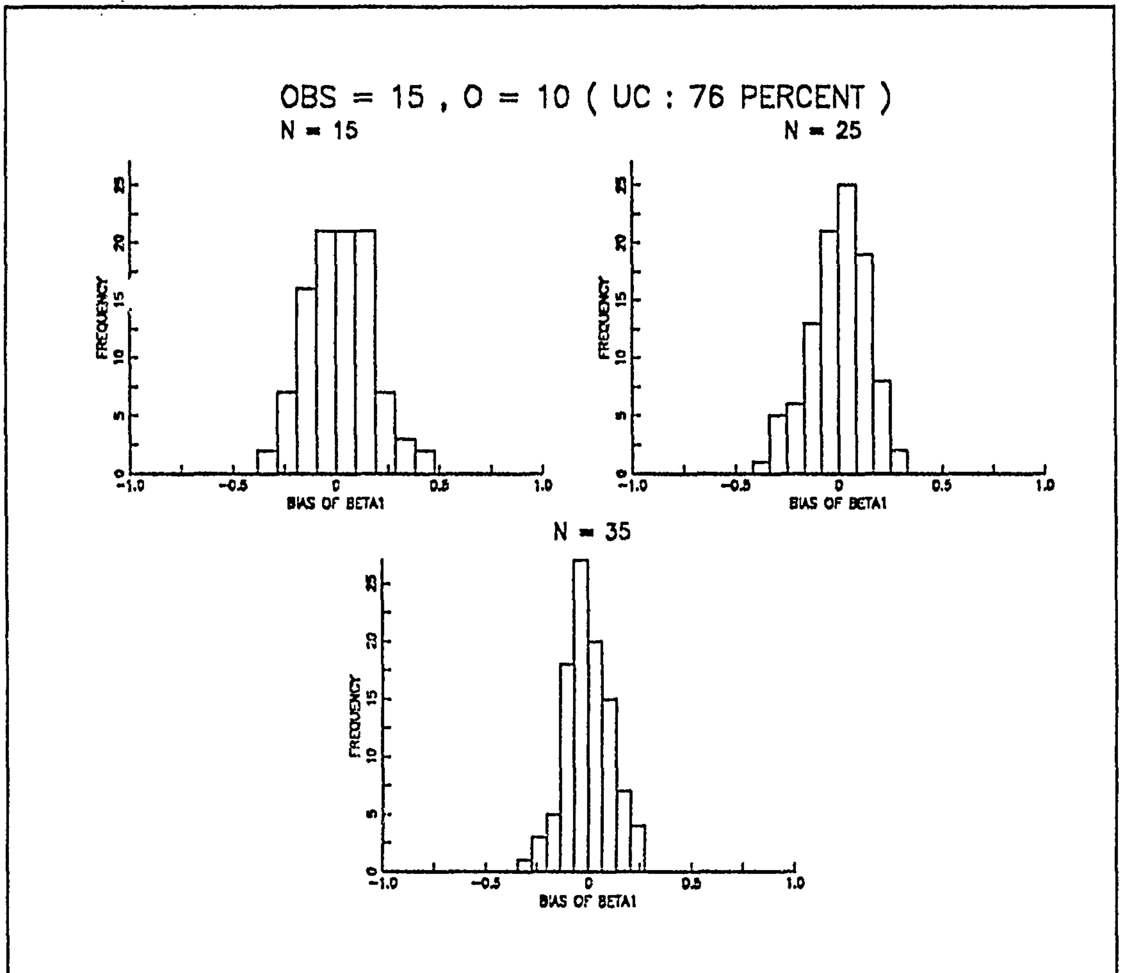

Figure 25. Comparison of the Bias of $\beta 1$ between Different numbers of targets with OBS = 15 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

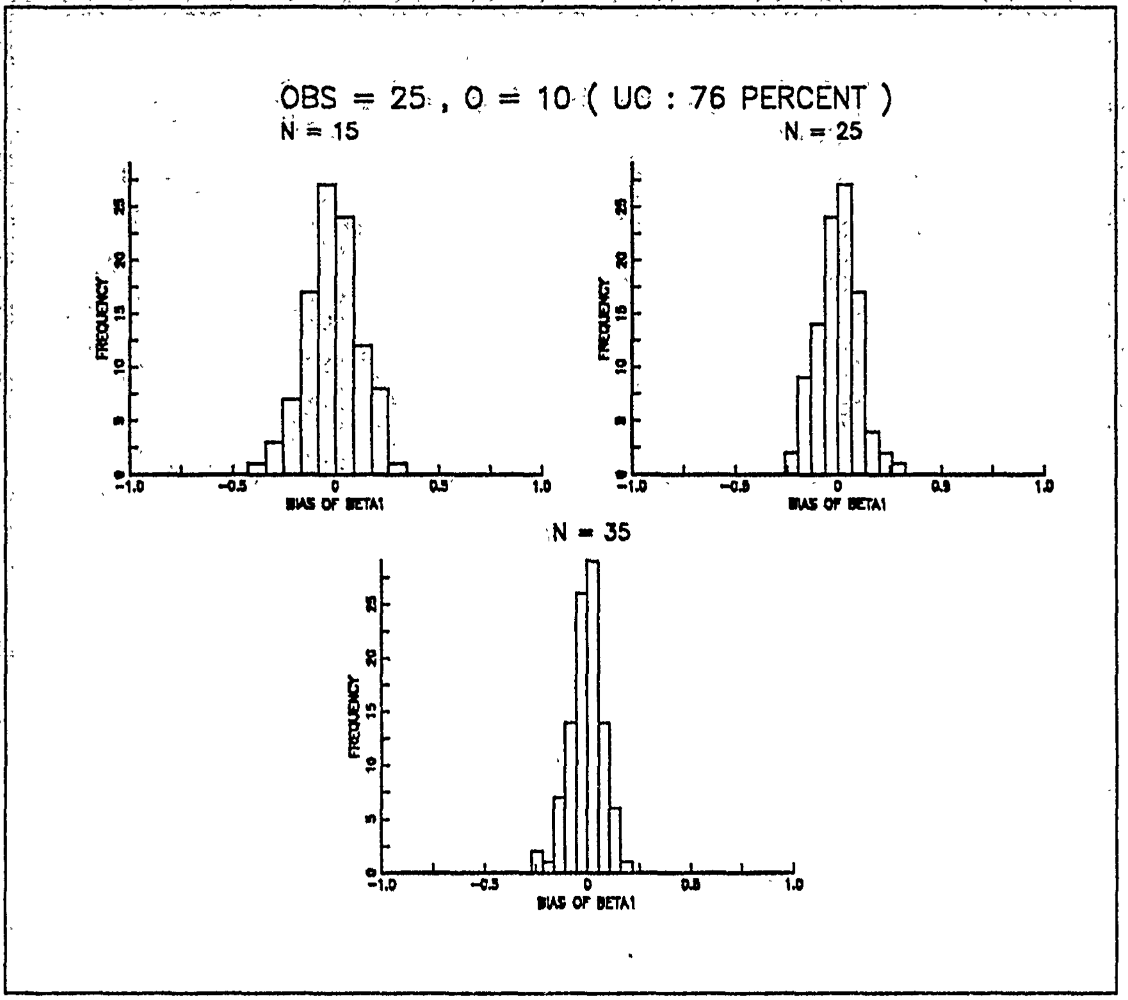

Figure 26. Comparison of the Bias of $\beta 1$ between Different numbers of targets with OBS = 25 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

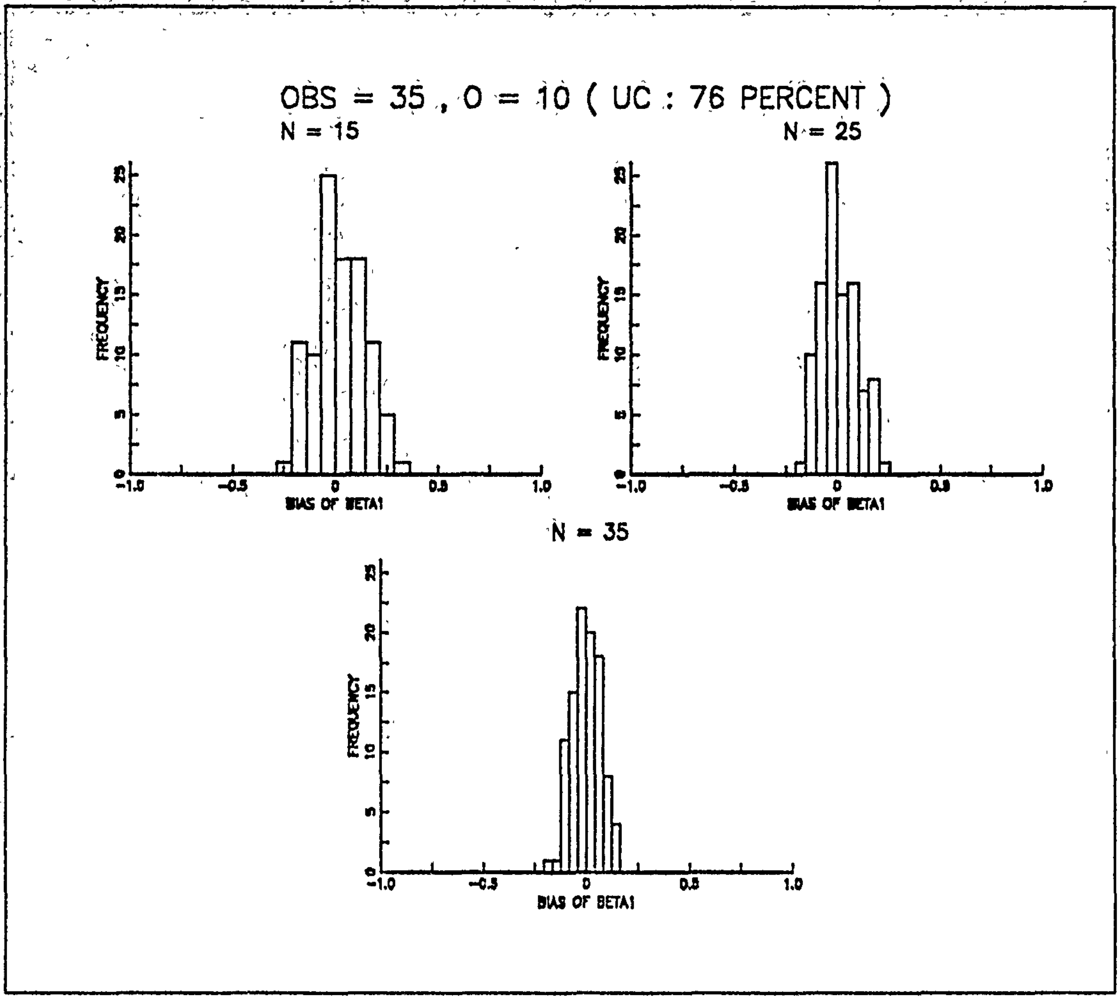

Figure 27. Comparison of the Bias of $\beta 1$ between Different numbers of targets with OBS = 35 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

Table 16. Tendencies of Mean Bias and Mean Square Error for $\beta 1$ at O = 15 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\beta}_1$		
	15 TGT	25 TGT	35 TGT
15 OBS	.008(.021)	-.006(.016)	.008(.011)
25 OBS	-.021(.016)	.004(.008)	-.005(.006)
35 OBS	.015(.015)	.008(.008)	.001(.004)

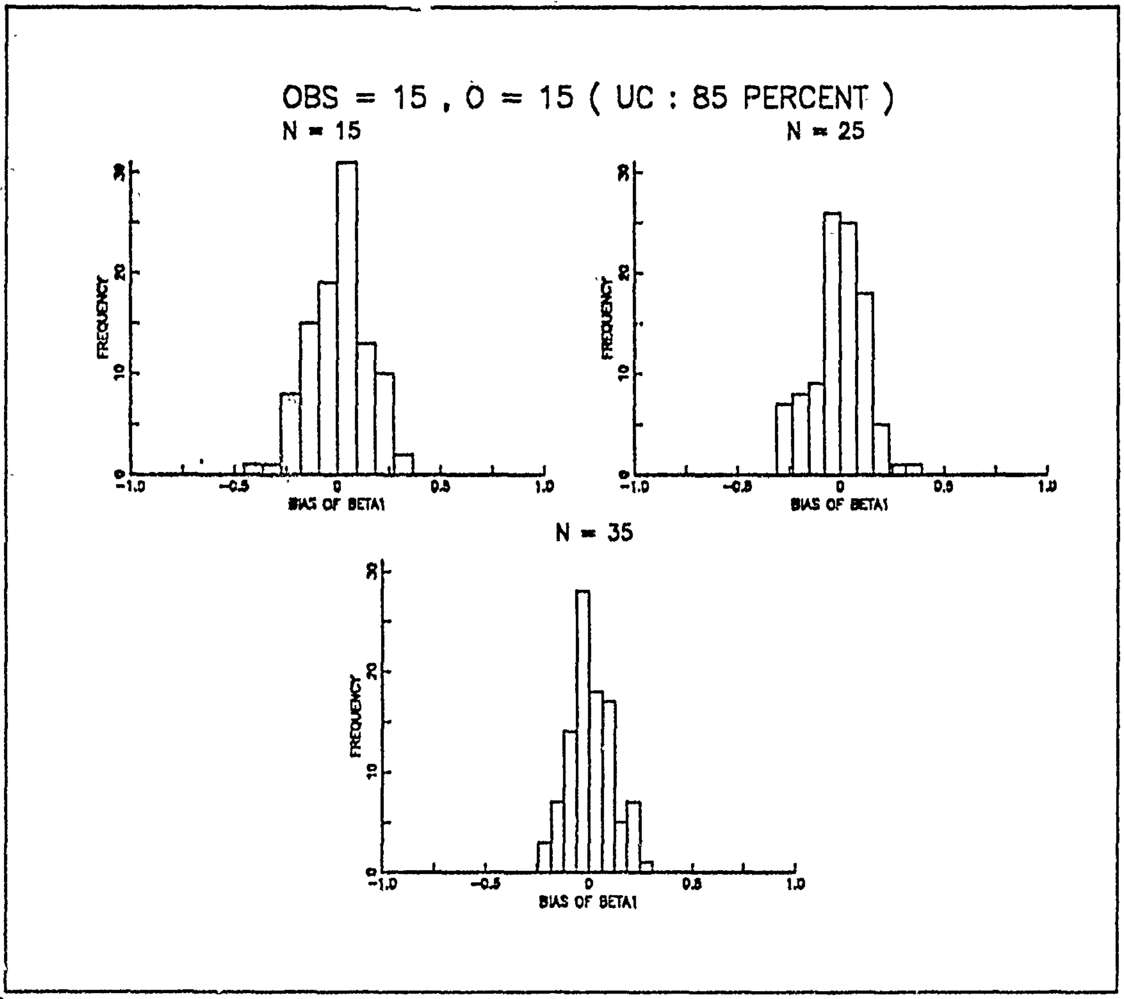

Figure 28. Comparison of the Bias of $\beta 1$ between Different numbers of targets with OBS = 15 at O = 15 in the GAM/WEI Regression Model: OBS = the number of observers

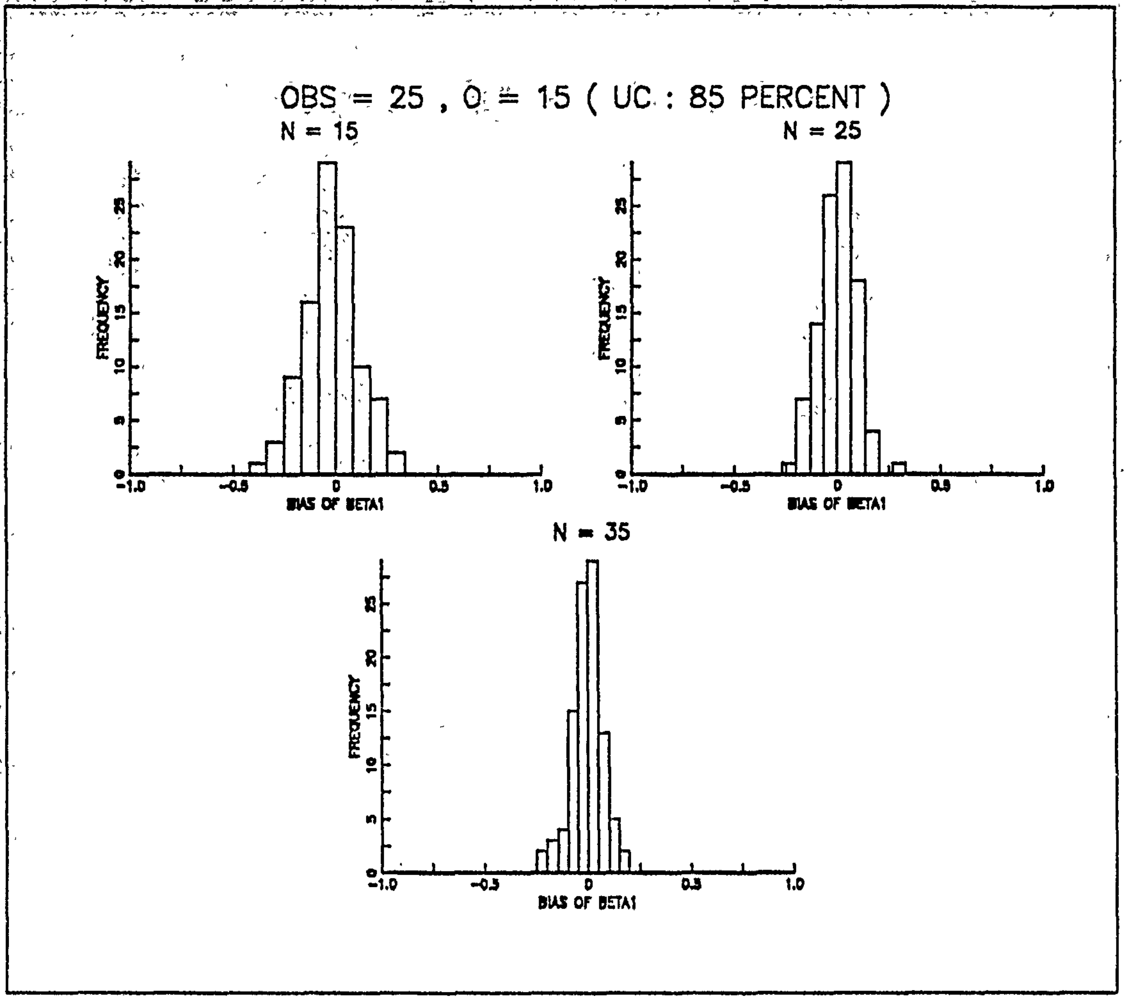

Figure 29. Comparison of the Bias of $\beta 1$ between Different numbers of targets with OBS = 25 at O = 15 in the GAM/WEI Regression Model: OBS = the number of observers

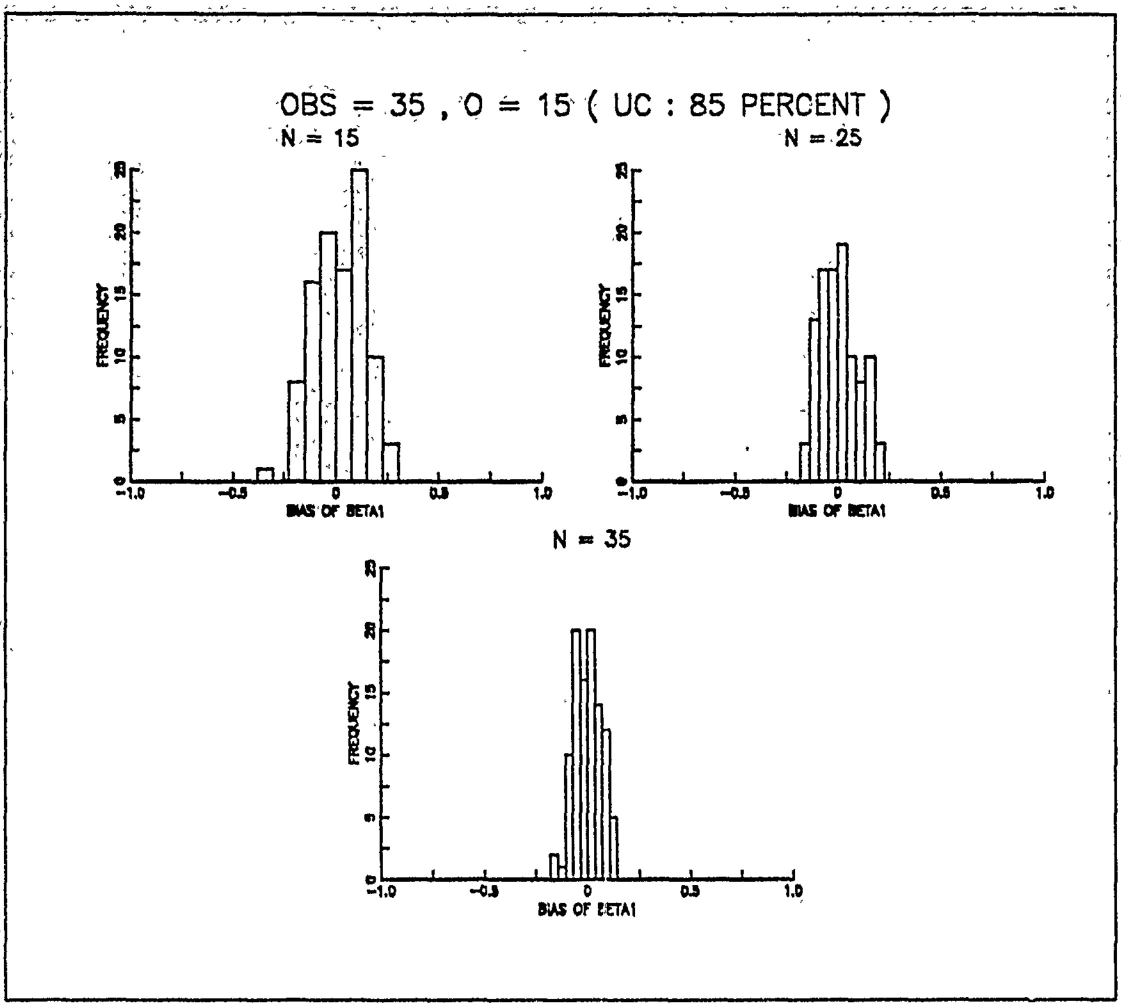

Figure 30. Comparison of the Bias of $\beta 1$ between Different numbers of targets with OBS = 35 at O = 15 in the GAM/WEI Regression Model: OBS = the number of observers

4. TABLES AND HISTOGRAMS FOR $\beta 2$

Table 17. Mean Bias,Mean Square Error and Std. Error for $\beta 2$ at O = 10 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\beta}_2$	
		M.B(S.E)	M.S.E(S.E)
15	15	.0167(.0097)	.0096(.0012)
	25	.0057(.0075)	.0056(.0008)
	35	-.0097(.0053)	.0029(.0004)
25	15	-.0051(.0073)	.0053(.0008)
	25	.0040(.0054)	.0029(.0004)
	35	.0020(.0046)	.0021(.0003)
15	15	.0002(.0062)	.0038(.0006)
	25	.0083(.0044)	.0020(.0003)
	35	-.0046(.0033)	.0011(.0001)

Table 18. Mean Bias,Mean Square Error and Std. Error for $\beta 2$ at O = 15 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\beta}_2$	
		M.B(S.E)	M.S.E(S.E)
15	15	.0185(.0091)	.0084(.0012)
	25	.0055(.0070)	.0049(.0006)
	35	-.0124(.0052)	.0028(.0004)
25	15	-.0015(.0069)	.0046(.0006)
	25	.0072(.0050)	.0025(.0003)
	35	.0024(.0042)	.0018(.0002)
35	15	-.0016(.0057)	.0032(.0005)
	25	.0086(.0045)	.0020(.0003)
	35	-.0007(.0032)	.0010(.0001)

Table 19. Tendencies of Mean Bias and Mean Square Error for $\beta 2$ at O = 10 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\beta}_2$		
	15 TGT	25 TGT	35 TGT
15 OBS	.017(.010)	.006(.006)	-.010(.003)
25 OBS	-.005(.005)	.004(.003)	.002(.002)
35 OBS	.000(.004)	.008(.002)	-.005(.001)

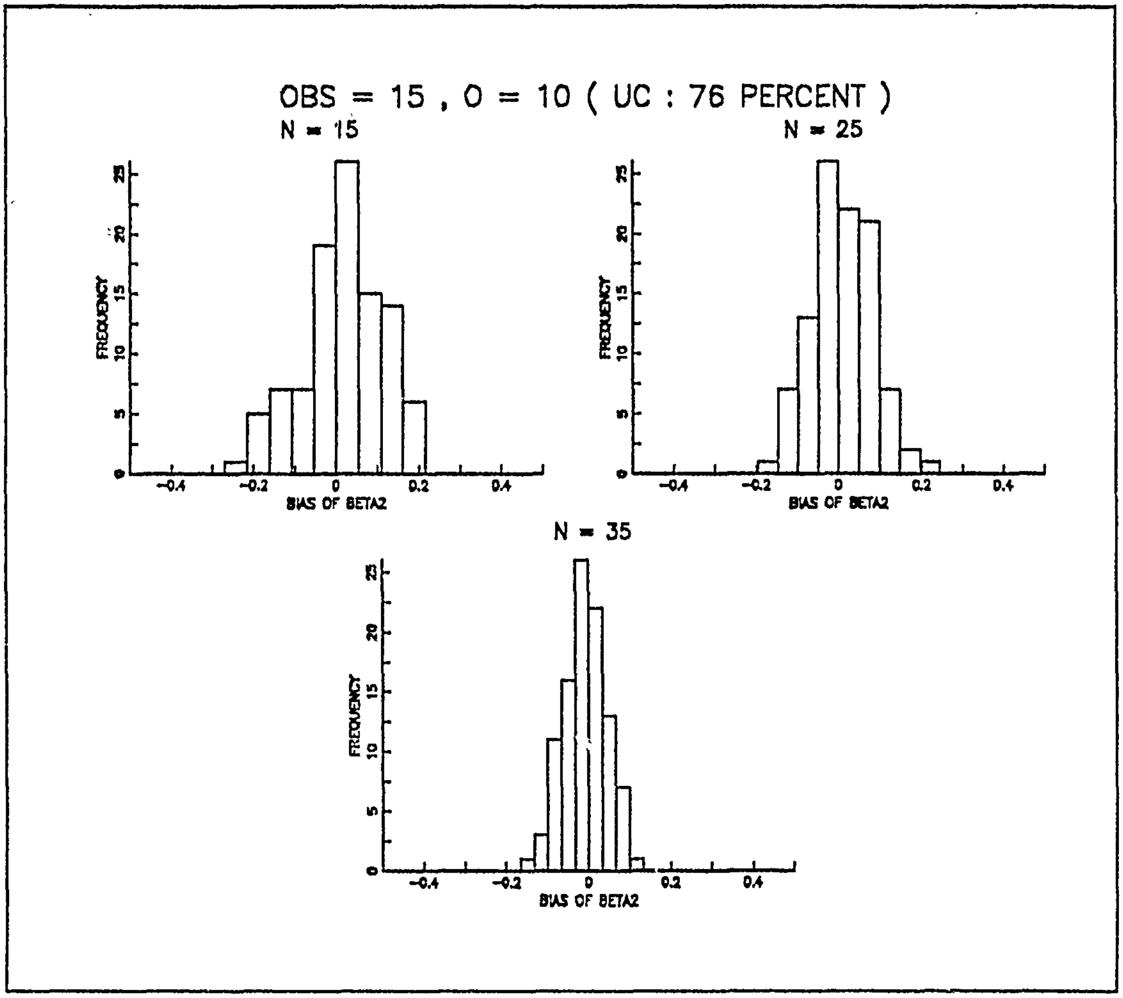

Figure 31. Comparison of the Bias of $\beta 2$ between Different numbers of targets with OBS = 15 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

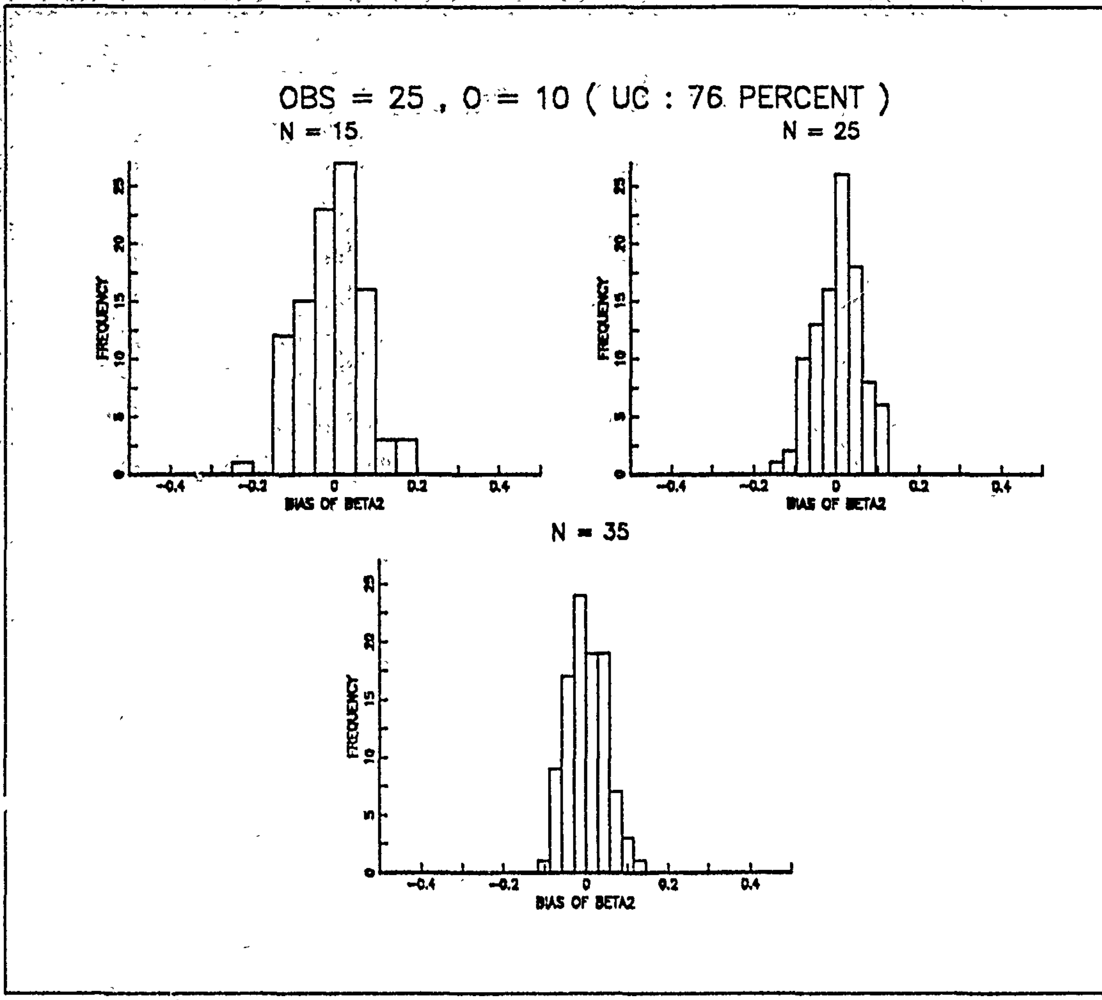

Figure 32. Comparison of the Bias of $\beta 2$ between Different numbers of targets with OBS = 25 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

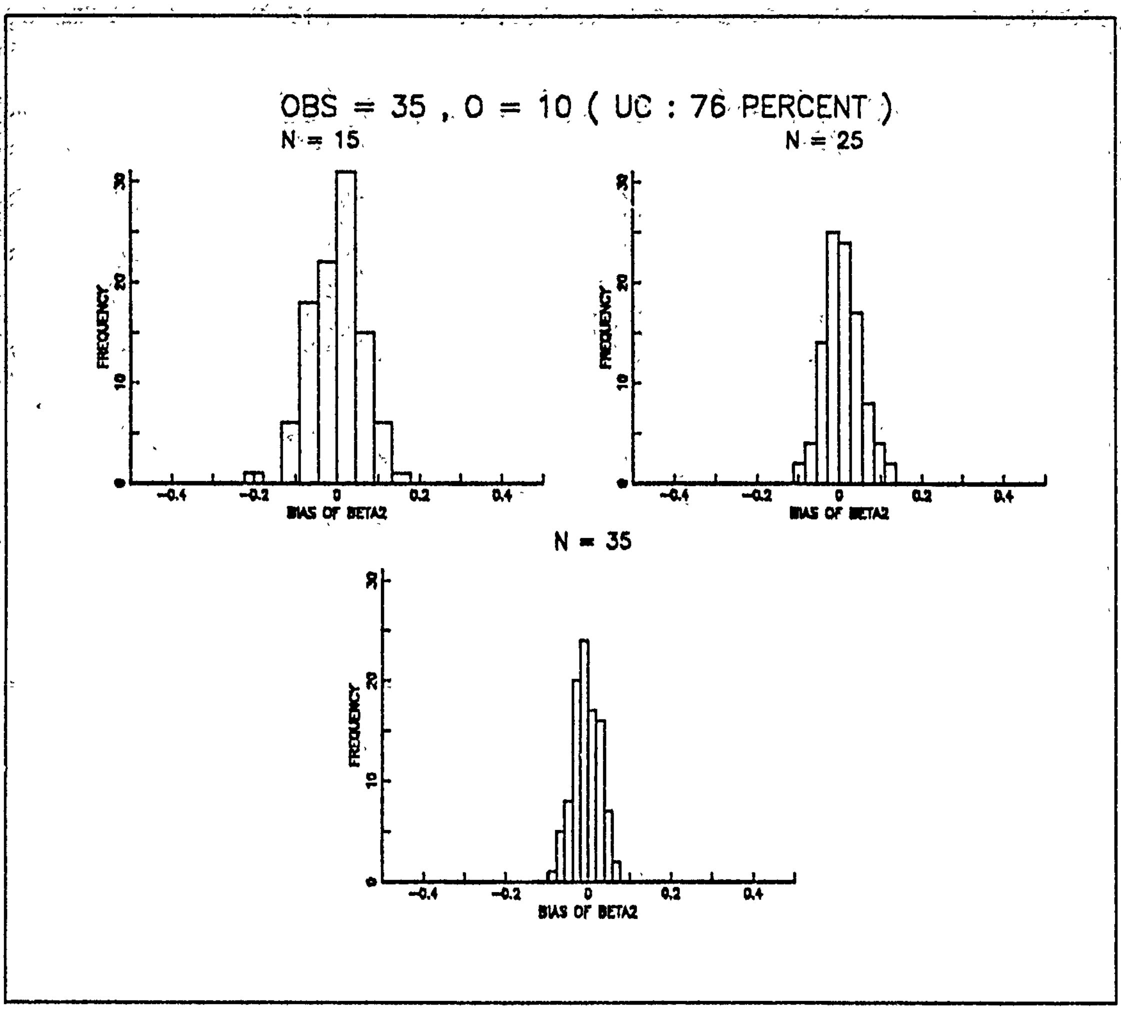

Figure 33. Comparison of the Bias of $\beta 2$ between Different numbers of targets with OBS = 35 at O = 10 in the GAM/WEI Regression Model: OBS = the number of observers

Table 20. Tendencies of Mean Bias and Mean Square Error for $\beta 2$ at O = 15 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\beta}_2$		
	15 TGT	25 TGT	35 TGT
15 OBS	.018(.008)	.006(.005)	-.012(.003)
25 OBS	-.001(.005)	.007(.002)	.002(.002)
35 OBS	-.002(.003)	.009(.002)	-.001(.001)

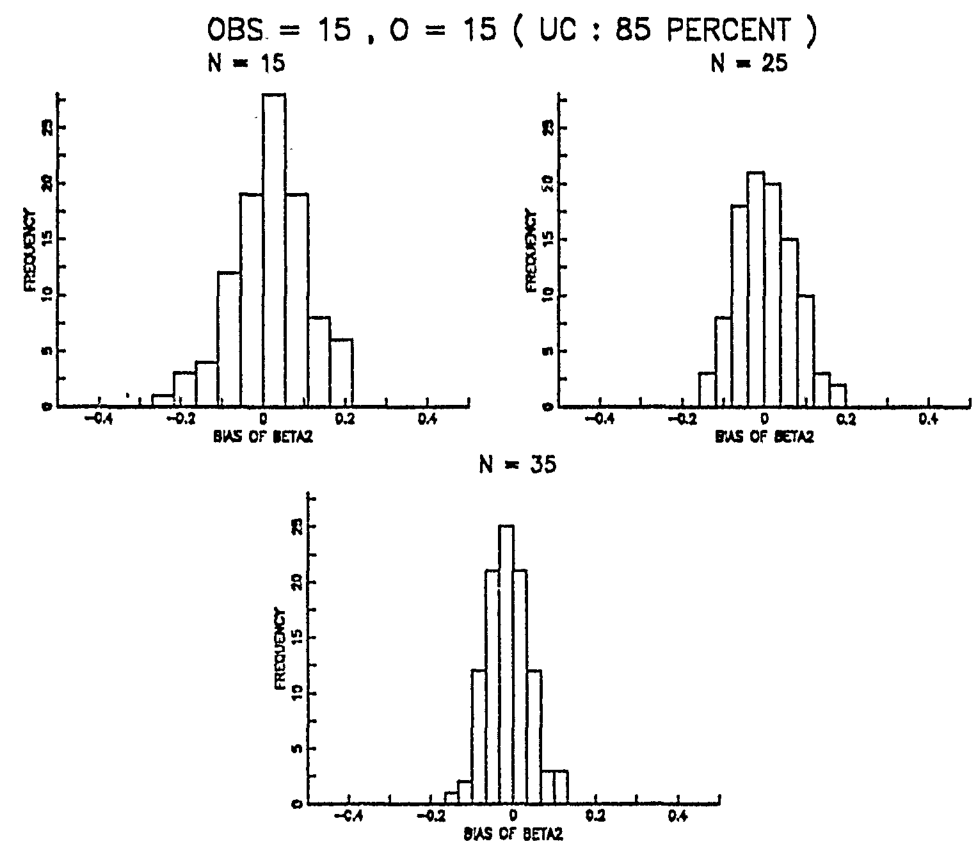

Figure 34. Comparison of the Bias of $\beta 2$ between Different numbers of targets with OBS = 15 at O = 15 in the GAM/WEI Regression Model: OBS = the number of observers

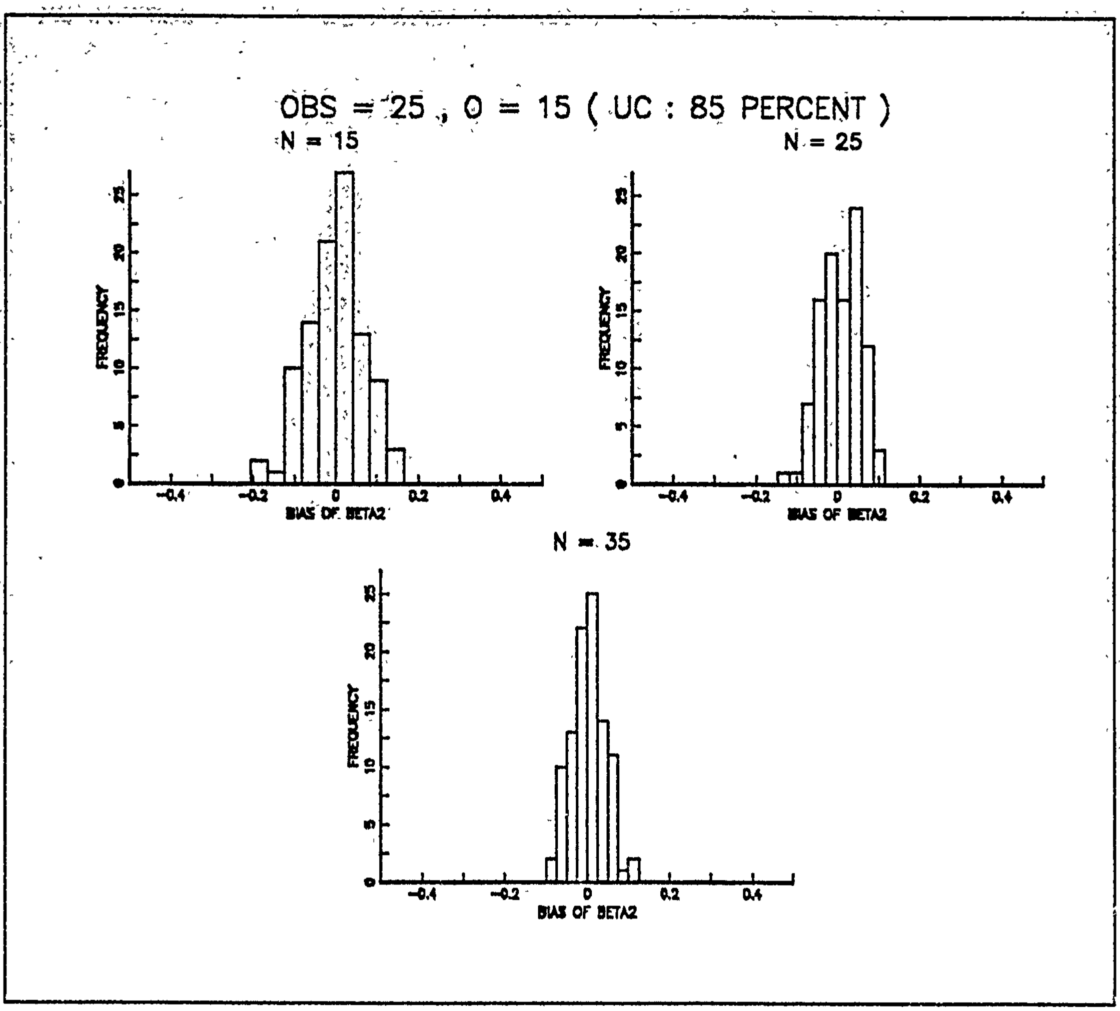

Figure 35. Comparison of the Bias of $\beta 2$ between Different numbers of targets with OBS = 25 at O = 15 in the GAM/WEI Regression Model: OBS = the number of observers

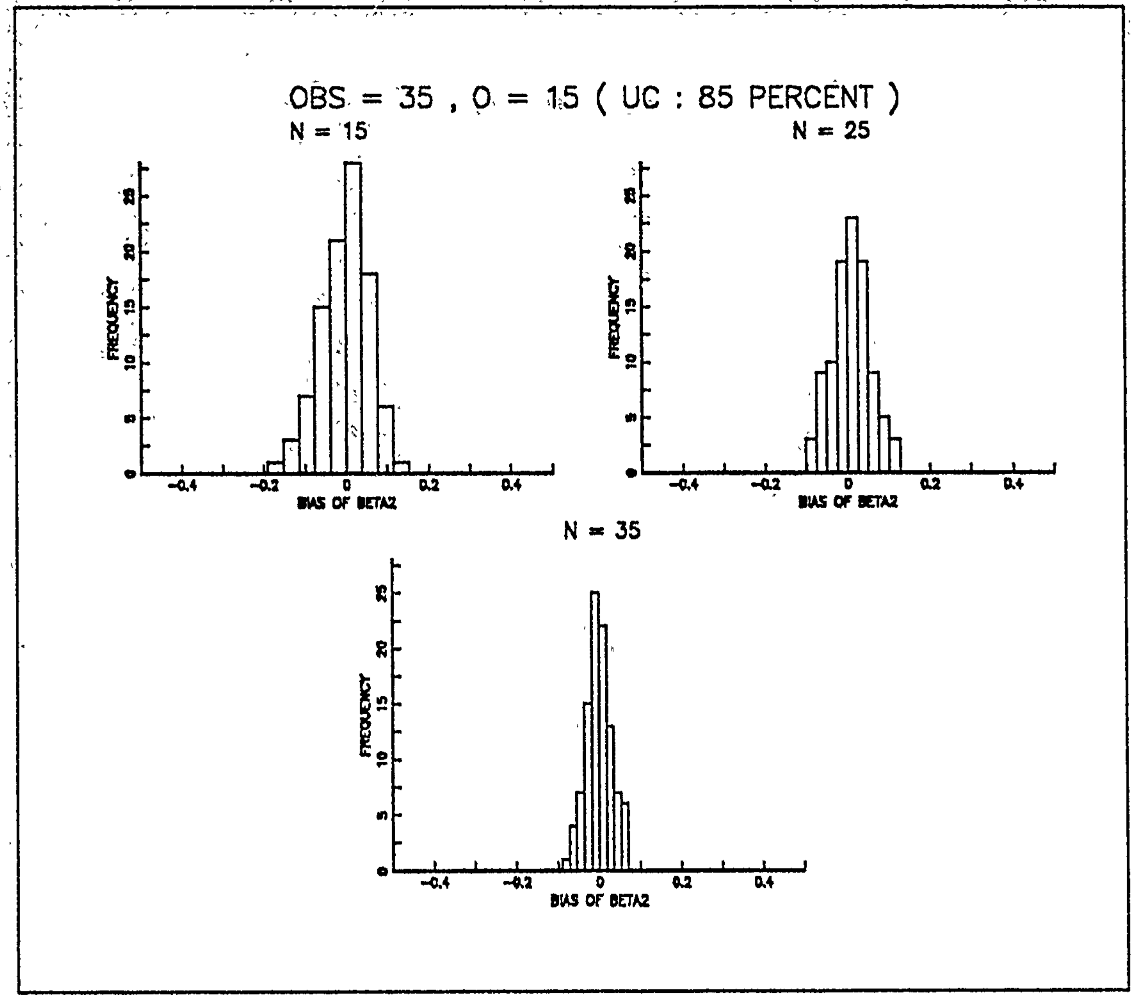

Figure 36. Comparison of the Bias of $\beta 2$ between Different numbers of targets with OBS = 35 at O = 15 in the GAM/WEI Regression Model: OBS = the number of observers

5. TABLES AND HISTOGRAMS FOR ξ

Table 21. Mean Bias, Mean Square Error and Std. Error for ξ at O = 10 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\xi}$	
		M.B(S.E)	M.S.E(S.E)
15	15	-.0318(.0067)	.0054(.0007)
	25	-.0158(.0056)	.0033(.0005)
	35	-.0131(.0048)	.0024(.0003)
25	15	-.0226(.0053)	.0033(.0005)
	25	-.0078(.0036)	.0013(.0002)
	35	-.0025(.0038)	.0014(.0002)
35	15	-.0196(.0044)	.0023(.0003)
	25	-.0065(.0036)	.0013(.0002)
	35	-.0071(.0028)	.0008(.0001)

Table 22. Mean Bias, Mean Square Error and Std. Error for ξ at O = 15 in the GAM/WEI Regression Model

Number of Observers	Number of Targets	$\hat{\xi}$	
		M.B(S.E)	M.S.E(S.E)
15	15	-.0349(.0064)	.0052(.0011)
	25	-.0147(.0051)	.0027(.0005)
	35	-.0129(.0041)	.0018(.0002)
25	15	-.0181(.0047)	.0025(.0003)
	25	-.0089(.0037)	.0014(.0002)
	35	-.0027(.0035)	.0012(.0002)
35	15	-.0192(.0041)	.0020(.0003)
	25	-.0049(.0033)	.0011(.0002)
	35	-.0083(.0025)	.0007(.0001)

Table 23. Tendencies of Mean Bias and Mean Square Error for ξ at O = 10 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\xi}$		
	15 TGT	25 TGT	35 TGT
15 OBS	-.032(.005)	-.016(.003)	-.013(.002)
25 OBS	-.023(.003)	-.008(.001)	-.003(.001)
35 OBS	-.020(.002)	-.007(.001)	-.007(.001)

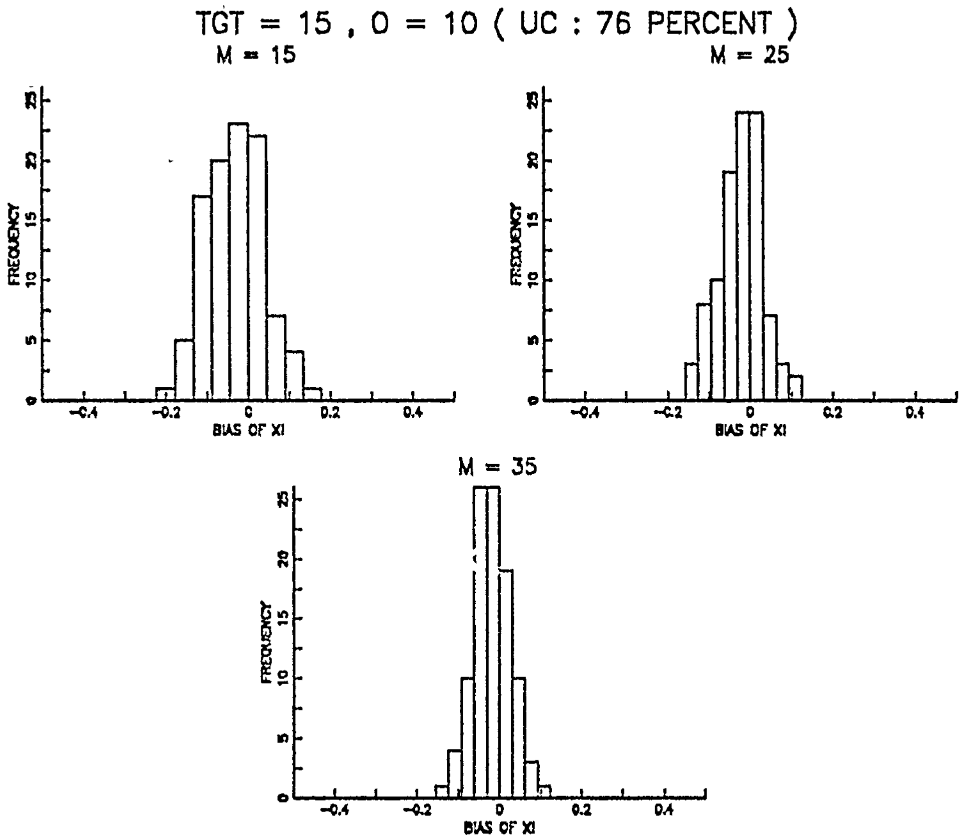

Figure 37. Comparison of the Bias of ξ between Different numbers of Obse rvers with TGT = 15 at O = 10 in the GAM/WEI Regression Model: TGT = the number of targets.

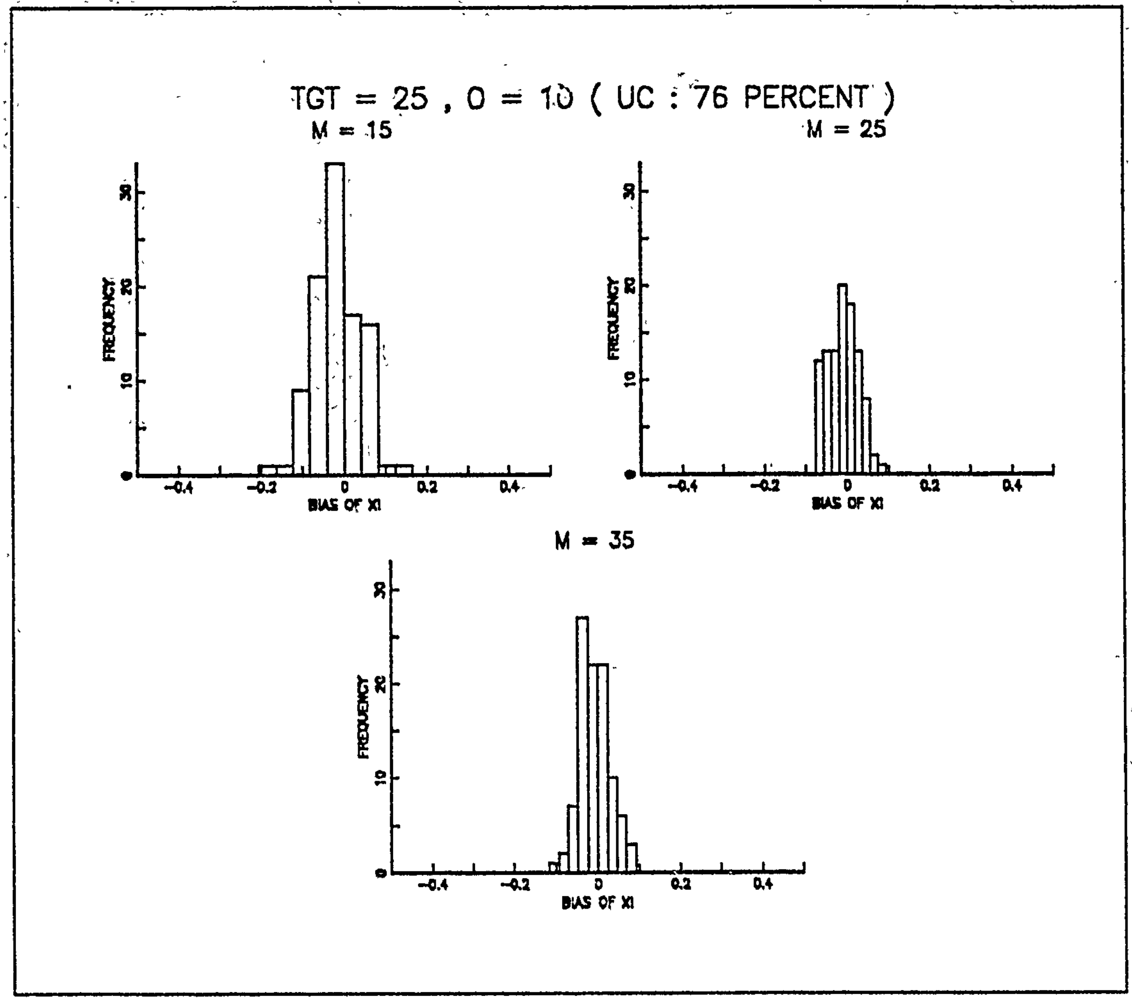

Figure 38. Comparison of the Bias of ξ between Different numbers of Observers with TGT=25 at O=10 in the GAM/WEI Regression Model: TGT = the number of targets.

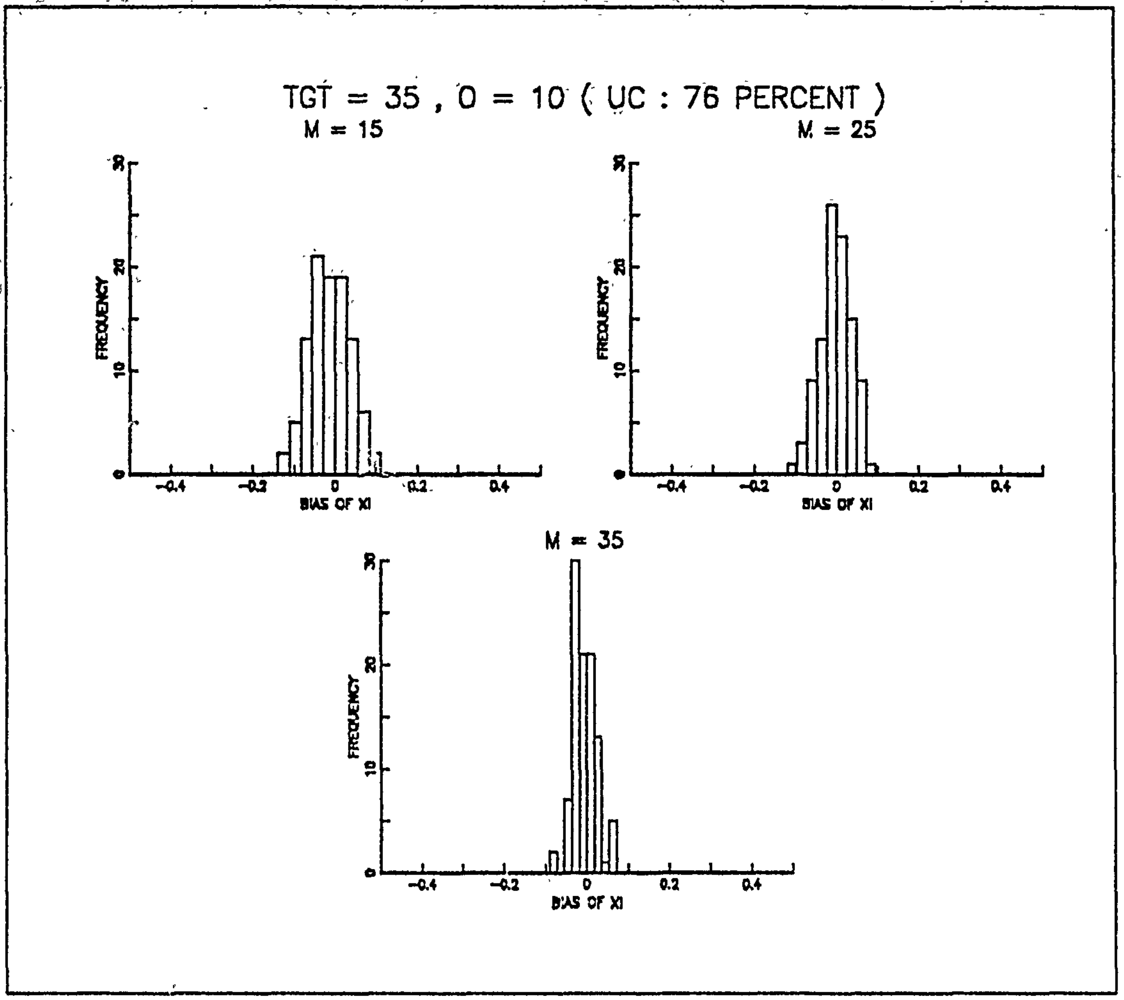

Figure 39. Comparison of the Bias of ξ between Different numbers of Obse rvers with TGT=35 at O=10 in the GAM/WEI Regression Model: TGT = the number of targets.

Table 24. Tendencies of Mean Bias and Mean Square Error for ξ at O = 15 in the GAM/WEI Regression Model

	M.B(M.S.E) of $\hat{\xi}$		
	15 TGT	25 TGT	35 TGT
15 OBS	-.035(.005)	-.015(.003)	-.013(.002)
25 OBS	-.018(.002)	-.009(.001)	-.003(.001)
35 OBS	-.019(.002)	-.005(.001)	-.008(.001)

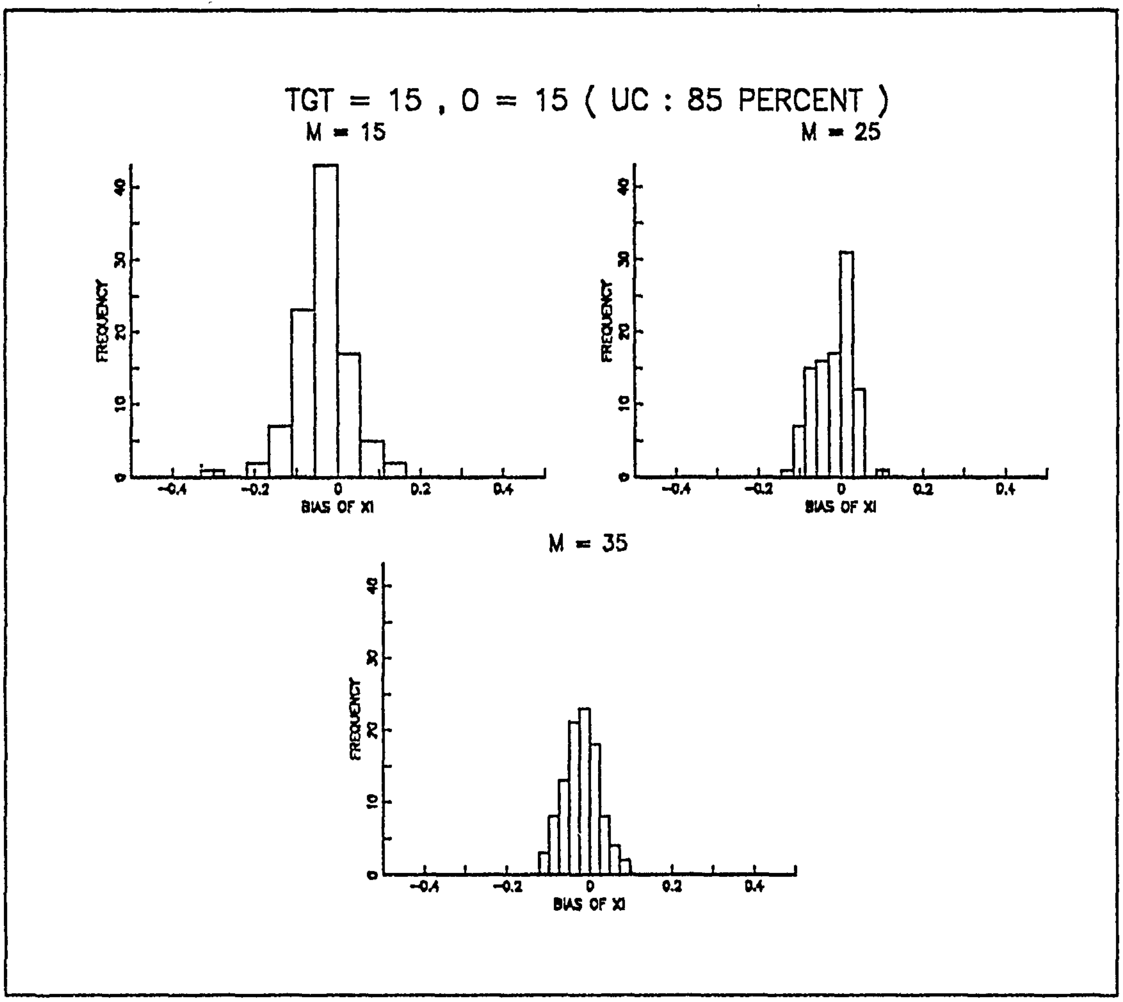

Figure 40. Comparison of the Bias of ξ between Different numbers of Observers with TGT = 15 at O = 15 in the GAM/WEI Regression Model: TGT = the number of targets.

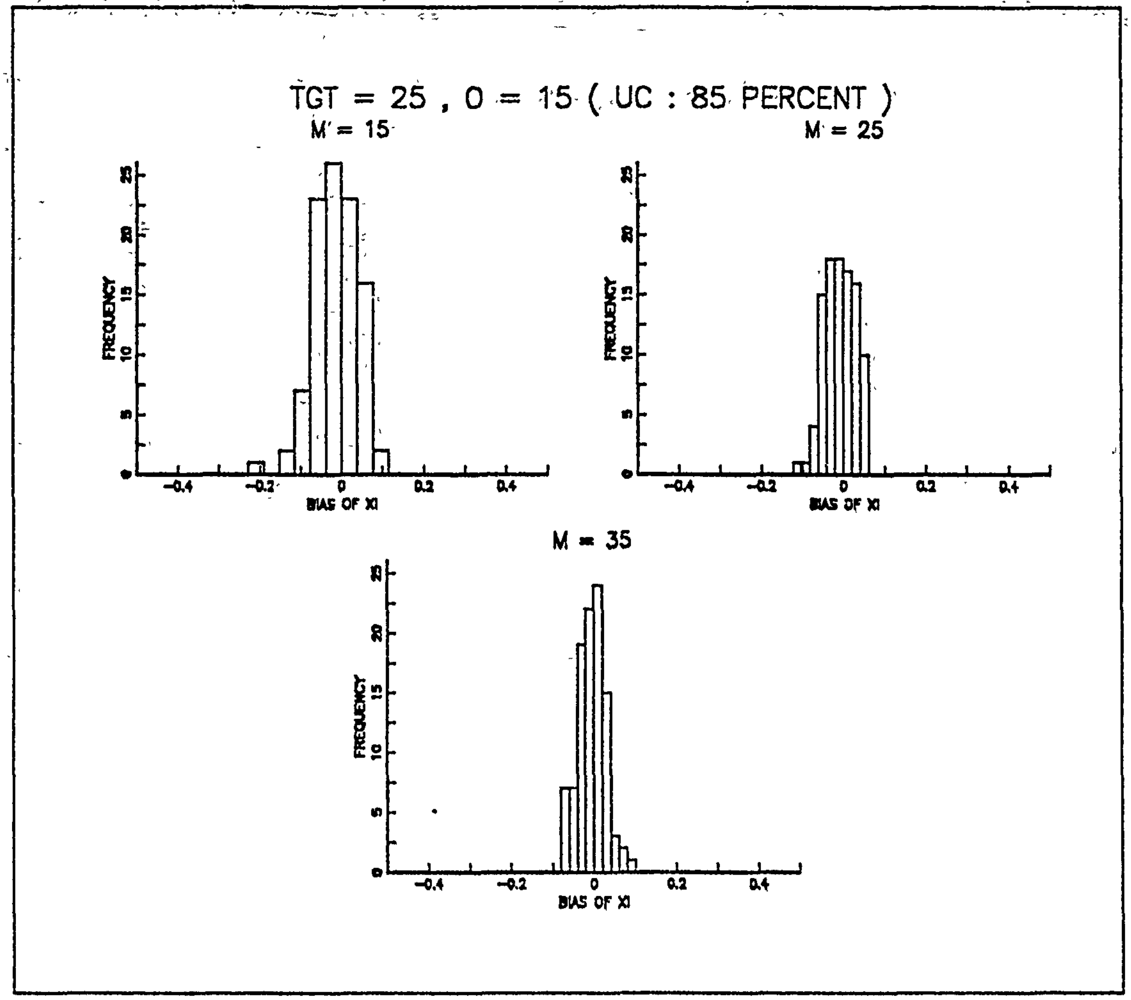

Figure 41. Comparison of the Bias of ξ between Different numbers of Observers with TGT=25 at O=15 in the GAM/WEI Regression Model: TGT = the number of targets.

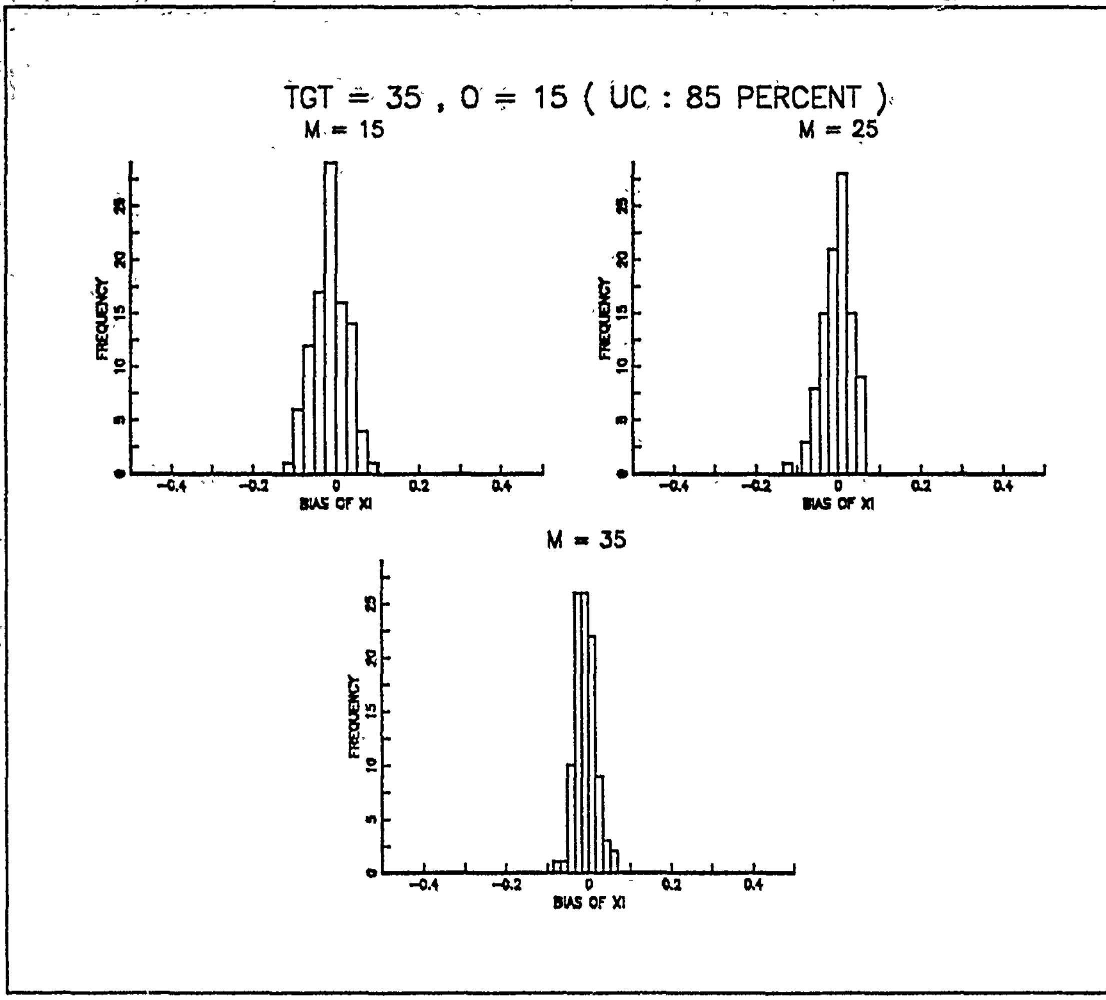

Figure 42. Comparison of the Bias of ξ between Different numbers of Observers with TGT = 35 at O = 15 in the GAM/WEI Regression Model: TGT = the number of targets.

APPENDIX C. SIMULATION PROGRAM FOR THE ESTIMATE OF η IN THE SINGLE PARAMETER GAMMA MODEL

```
     ∇ R←SIMULA1;M;N;O;I;J;K;L;ETA;OUT;PAGE;AVUCL;MB;SEMB;MSE;SEMSE
;RRR
[1]   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  KEY GLOBAL VARIABES : BOOK1 ; BOOK2 ; UCL  ⍝⍝⍝⍝⍝⍝⍝⍝⍝
[2]   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  DATA INPUT  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[3]    M← 15 25 35
[4]    N← 15 25 35
[5]    O← 10 15
[6]    ETA←1
[7]    I←J←K←L←0
[8]   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  START LOOPING THE SIMULATION 1 ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[9]   LOOPI:I←I+1
[10]  LOOPJ:J←J+1
[11]  LOOPK:K←K+1
[12]  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  RESET THE RANDOM NUMBER SEED  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[13]   ⎕←O[I],M[J],N[K]
[14]   ⎕RL←466801743
[15]  LOOPL:L←L+1
[16]  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  CALL GAMMA ESTIMATE FUNCTION  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[17]   RRR←⎕RL
[18]   OUT←GAMMA(O[I],M[J],N[K])
[19]  ⍝⍝⍝⍝⍝⍝⍝⍝  RECORD UCL AND ALL THE ESTIMATES IN REPLICATIONS  ⍝⍝⍝⍝⍝
[20]   →((L=1),L>1)/INTZ1,GO1
[21]  INTZ1:PAGE←(1,⍴OUT)⍴(OUT-0,ETA)
[22]   →LOOPL
[23]  GO1:PAGE←PAGE,[1](OUT-0,ETA)
[24]  TESTL:
[25]   →(L<100)/LOOPL
[26]  ⍝⍝⍝⍝⍝⍝⍝⍝  COMPUTE AVE. UNCENSORED LEVEL AND ALL THE STATISTICS ⍝⍝
[27]   AVUCL←((+/PAGE[;1])÷L)
[28]   MB←(+/PAGE[;2])÷L
[29]   SEMB←((+/(PAGE[;2]-MB)*2)÷(L×L-2))*0.5
[30]   MSE←(+/PAGE[;2]*2)÷L
[31]   SEMSE←((+/((PAGE[;2]*2)-MSE)*2)÷(L×L-2))*0.5
[32]   L←0
[33]  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  KEEP ALL THE INFORMATIONS FOR GRAPHICS AND TABLES ⍝⍝
[34]   →(((I×J×K)=1),(I×J×K)≠1)/INTZ2,GO2
[35]  INTZ2:
[36]   BOOK1←(1,⍴PAGE[;2])⍴PAGE[;2]
[37]   BOOK2←(1,4)⍴(MB,SEMB,MSE,SEMSE)
[38]   UCL←AVUCL
[39]   →TESTK
[40]  GO2:
[41]   BOOK1←BOOK1,[1] PAGE[;2]
[42]   BOOK2←BOOK2,[1](MB,SEMB,MSE,SEMSE)
```

```
[43]   UCL←UCL,AVUCL
[44]  TESTK:
[45]   →(K<ρN)/LOOPK
[46]   K←0
[47]   →(J<ρM)/LOOPJ
[48]   J←0
[49]   →(I<ρO)/LOOPI
       ∇
```

```
    ∇ R←GAMMA INPUT;M;N;O;ETA;ETA0;XI;XI0;A;WW;MU;U;Y;DELTA;C;XB;RR
       ;M1;M2;T;DATA;K;FL;FM;FR;LB;MP;RB;S
[1]   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  SIMULATION INPUTS AND THEORETICAL PARAMETERS  ⍝⍝⍝⍝
[2]    M←INPUT[2]
[3]    N←INPUT[3]
[4]    O←(M,N)⍴INPUT[1]
[5]    ETA←1
[6]    MU←4.2
[7]    XI0←⍉(N,M)⍴XI←M⍴0
[8]   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  RANDOM NUMBER GENERATIONS  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[9]    A←⍉(N,M)⍴M GAMRAND((*ETA),÷*ETA)
[10]   WW←(M,N)⍴(M×N) EXPRAND 1
[11]  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  DETECTION TIME AND UNCENSORING DATA ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[12]   U←MU×(WW÷A)*(*XI0)
[13]   A←WW←⍳0
[14]   Y←(⍟U)⌊(⍟O)
[15]   C←+/[2] DELTA←(⍟U)≤(⍟O)
[16]   O←⍳0
[17]   XB←⍟MU
[18]  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  INITIAL CONDITION FOR GAMMA PARAMETER ,ETA  ⍝⍝⍝⍝⍝⍝
[19]   RR←(Y-XB)×*(¯1×XI0)
[20]   M1←¯0.5772
[21]   M2←(+/+/DELTA×RR*2)÷(+/C)
[22]   ETA0←0
[23]   →(0>T←(((○1)*2)÷¯6)+(M2*2)-(M1*2))/BISECTION
[24]   ETA0←¯1×⍟T
[25]  ⍝⍝⍝⍝⍝  FIND BOUNDS AND BISECTION SEARCH FOR GAMMA PARAMETER,ETA ⍝⍝
[26]  BISECTION:K←0
[27]   DATA←(S←+/[2]*RR),C
[28]   FM←ETA0 FVALUE DATA
[29]  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  FIND THE RIGHT AND LEFT BOUND FOR ETA  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[30]  BOUND:K←K+1
[31]   FL←(LB←ETA0-K×0.5) FVALUE DATA
[32]   FR←(RB←ETA0+K×0.5) FVALUE DATA
[33]   →(((×FM)×(×FL))<0)/SETRB
[34]   →(((×FM)×(×FR))<0)/SETLB
[35]   →BOUND
[36]  SETRB:RB←ETA0
[37]   →BISECT
[38]  SETLB:LB←ETA0
[39]  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  PERFORM THE BISECTION SEARCH FOR ETA  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[40]  BISECT:
[41]   FM←(MP←(LB+RB)÷2) FVALUE DATA
[42]   FL←LB FVALUE DATA
[43]   →((RB-LB)<(|0.01×MP))/END
[44]   →((((×FM)×(×FL))<0),(((×FM)×(×FL))>0))/SWAPR,SWAPL
[45]  SWAPR:RB←MP
[46]   →BISECT
[47]  SWAPL:LB←MP
[48]   →BISECT
```

```
[49] ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  OUTPUT THE ESTIMATES  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[50] END:
[51]  ETA0←(RB+LB)÷2
[52]  R←((+/C)÷M×N),ETA0
      ∇


      ∇ R←ETA FVALUE DATA;M;S;C;SUBF;I
[1]   M←(⍴DATA)÷2
[2]   S←M↑DATA
[3]   C←M↓DATA
[4]   SUBF←0
[5]   I←0
[6]  LOOP:I←I+1
[7]   →(C[I]=0)/CHECKI
[8]   SUBF←SUBF+(+/÷(*ETA)+(⍳C[I])-1)
[9]  CHECKI:
[10]  →(I<M)/LOOP
[11]  R←(*ETA)×SUBF+(+/ETA+((S-C)÷S+*ETA)+¯1×⍟(S+*ETA))
      ∇
```

APPENDIX D. SIMULATION PROGRAM FOR THE ESTIMATES OF ALL PARAMETERS IN GAMMA/WEIBULL REGRESSION MODEL

```
    ∇ R←SIMULA2;M;N;O;I;J;K;L;X;E;ETA;B0;B1;B2;XI;OUT;PAGE;AVUCEX;MB
      ;SEMB;MSE;SEMSE;RRR
   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ KEY GLOBAL VARIABES : BOOK1 ; BOOK2 ; UCLEX ⍝⍝⍝⍝⍝⍝⍝⍝
   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝   DATA INPUT   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
    M← 15 25 35
    N← 15 25 35
    O← 10 15
    ETA←1
    B0←0.8
    B1←¯0.2
    B2←0.5
    XI←0
   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  START LOOPING THE SIMULATION  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
    I←J←K←L←E←X←0
   LOOPI:I←I+1
   LOOPJ:J←J+1
   LOOPK:K←K+1
   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  RESET THE RANDOM NUMBER SEED  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
    ⎕←O[I],M[J],N[K]
    ⎕RL←466801743
   LOOPL:L←L+1
   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  CALL GAMMA/WEIBULL REGRESSION FUNCTION  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
    RRR←⎕RL
    OUT←GAMWEI(O[I],M[J],N[K])
   ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝  SKIP THE INSUFFICIENT VARIABILITY CASE FOR GAMMA  ⍝⍝
    →(OUT[2]≠¯99999)/CHECKETA
    E←E+1
    L←L-1
    →TESTL
   ⍝⍝⍝⍝⍝⍝  RECORD THE RANDOM SEED FOR THE ILL CONDITION OF ETA  ⍝⍝⍝
   CHECKETA:
    →(OUT[2]≠¯100000)/GOGO
    ⎕←'BAD ETA'
    ⎕←RRR
    L←L-1
    →TESTL
   ⍝⍝⍝⍝⍝⍝⍝⍝⍝  RECORD UCL AND ALL THE ESTIMATES IN REPLICATIONS  ⍝⍝⍝
   GOGO:
    →((L=1),L>1)/INTZ1,GO1
   INTZ1:PAGE←(1,(ρOUT)-1)ρ((¯1↓OUT)-0,ETA,B0,B1,B2,XI)
    X←X+(¯1↑OUT)
    →LOOPL
   GO1:PAGE←PAGE,[1]((¯1↓OUT)-0,ETA,B0,B1,B2,XI)
    X←X+(¯1↑OUT)
```

```
[43] ⍝⍝⍝⍝⍝⍝⍝⍝  RECORD THE RANDOM SEED FOR THE ILL CASE OF XI  ⍝⍝⍝⍝⍝⍝⍝⍝
[44]  →((¯1↑OUT)=0)/TESTL
[45]  ⎕←'BAD XI'
[46]  ⎕←RRR
[47] ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[48] TESTL:
[49]  →(L<100)/LOOPL
[50] ⍝⍝⍝⍝⍝⍝⍝  COMPUTE AVE. UNCENSORED LEVEL AND ALL THE STATISTICS  ⍝
[51]  AVUCEX←((+/PAGE[;1])÷L),E,X
[52]  MB←(+/[1] PAGE← 0 1 ↓PAGE)÷L
[53]  SEMB←((+/[1](PAGE-(⍴PAGE)⍴MB)*2)÷(L×L-2))*0.5
[54]  MSE←(+/[1] PAGE*2)÷L
[55]  SEMSE←((+/[1]((PAGE*2)-(⍴PAGE)⍴MSE)*2)÷(L×L-2))*0.5
[56]  L←E←X←0
[57] ⍝⍝⍝⍝⍝⍝⍝⍝  KEEP ALL THE INFORMATIONS FOR GRAPHICS AND TABLES  ⍝⍝⍝
[58]  →(((I×J×K)=1),(I×J×K)≠1)/INTZ2,GO2
[59] INTZ2:
[60]  BOOK1←(1,⍴PAGE)⍴PAGE
[61]  BOOK2←(1,4,⍴MB)⍴(MB,SEMB,MSE,SEMSE)
[62]  UCLEX← 1 3 ⍴AVUCEX
[63]  →TESTK
[64] GO2:
[65]  BOOK1←BOOK1,[1] PAGE
[66]  BOOK2←BOOK2,[1]((4,⍴MB)⍴MB,SEMB,MSE,SEMSE)
[67]  UCLEX←UCLEX,[1] AVUCEX
[68] TESTK:
[69]  →(K<⍴N)/LOOPK
[70]  K←0
[71]  →(J<⍴M)/LOOPJ
[72]  J←0
[73]  →(I<⍴O)/LOOPI
     ∇
```

```
    ∇GAMWEI[⎕]∇
    ∇ R←GAMWEI INPUT;M;N;O;ETA;ETA0;B;B0;B1;B2;XI;XI0;A;WW;X1;X2
      ;MU;U;Y;DELTA;C;XB;RR;W;U0;U1;U2;UU;Z;I;VAR;M1;M2;T;DATA;K
      ;FL;FL0;FM;FR;FR0;LB;MP;RB;CS;H;DH;DIFFXI;BK;S;J;X;XX;BADXI
[1]  ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ SIMULATION INPUTS AND THEORETICAL PARAMETERS ⍝⍝⍝⍝⍝⍝⍝
[2]   M←INPUT[2]
[3]   N←INPUT[3]
[4]   O←(M,N)⍴INPUT[1]
[5]   ETA←1
[6]   B0←0.8
[7]   B1←¯0.2
[8]   B2←0.5
[9]   XI0←⍉(N,M)⍴XI←M⍴0
[10] ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ RANDOM NUMBER GENERATIONS ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[11]  A←⍉(N,M)⍴M GAMRAND((*ETA),÷*ETA)
[12]  WW←(M,N)⍴(M×N) EXPRAND 1
[13]  X1←(M,N)⍴(M×N) NORRAND 1 0.5
[14]  X2←(M,N)⍴(M×N) NORRAND 2 1
[15] ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ DETECTION TIME AND UNCENSORING DATA ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[16]  MU←*(B0+(B1×X1)+B2×X2)
[17]  U←MU×(WW÷A)*(*XI0)
[18]  A←WW←MU←⍳0
[19]  Y←(⍟U)⌊(⍟O)
[20]  C←+/[2] DELTA←(⍟U)≤(⍟O)
[21]  O←⍳0
[22] ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ INITIAL ESTIMATES WITHOUT GAMMA R.V. INVOLVED ⍝⍝⍝⍝⍝⍝
[23]  XB←⍟U
[24]  I←0
[25] INTZ:I←I+1
[26]  W←(*(Y-XB)×*(¯1×XI0))*0.5
[27]  U0←((M,N)⍴1)×W×*(¯1×XI0)
[28]  U1←X1×W×*(¯1×XI0)
[29]  U2←X2×W×*(¯1×XI0)
[30]  UU←⍉(3,M×N)⍴(,U0),(,U1),(,U2)
[31]  U0←U1←U2←⍳0
[32]  Z←⍉,(W×XB×*(¯1×XI0))+((¯1×DELTA)+W*2)÷W
[33]  W←⍳0
[34]  B←Z⌹UU
[35]  Z←UU←⍳0
[36]  BK←B
[37]  XB←B[1]+(B[2]×X1)+B[3]×X2
[38]  RR←(Y-XB)×*(¯1×XI0)
[39]  XI0←⍉(N,M)⍴XI←XI+DIFFXI←(C-+/RR×((¯1×DELTA)+*RR))÷
((¯1×C)-+/(RR*2)×*RR)
[40]  →(I<2)/INTZ
[41] ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ VARIABILITY CHECK FOR UNCENSORED TIMES ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
[42]  VAR←(+/+/(¯1+(U÷(*XB))*(*¯1×XI0))*2)÷(¯1+M×N)
[43]  U←⍳0
[44]  ETA0←¯99999
[45]  X←M⍴1
```

```
  J←0
  →(VAR<1)/END
⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ INITIAL CONDITION FOR GAMMA PARAMETER ,ETA ⍝⍝⍝⍝⍝⍝⍝
  RR←(Y-XB)×*(¯1×XI0)
  M1←¯0.5772
  M2←(+/+/DELTA×RR*2)÷(+/C)
  ETA0←0
  →(0>T←(((○1)*2)÷¯6)+(M2*2)-(M1*2))/READY
  ETA0←¯1×⍟T
READY:BADXI←0
⍝⍝ LARGE RECURSIVE LOOP INCLUDING ESTIMATE OF GAMMA PARAMETER,ETA
RECURSIVE:K←0
  J←J+1
  DATA←(S←+/[2]*RR),C
  FM←ETA0 FVALUE DATA
⍝⍝⍝⍝⍝⍝⍝ BISECTION SEARCH METHOD APPLIED TO ESTIMATE ETA ⍝⍝⍝⍝⍝⍝⍝⍝
BOUND:K←K+1
  FL←(LB←ETA0-K×0.5) FVALUE DATA
  FR←(RB←ETA0+K×0.5) FVALUE DATA
⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ DETECT ILL CONDITION FOR ETA ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
  →(K≤10)/CHECKLRB
  →(~(((|FR0-FR)÷0.5)<0.01)∧((|FR)<0.1))/CHECKLRB
  ETA0←¯100000
  →END
⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
CHECKLRB:
  →(((×FM)×(×FL))<0)/SETRB
  →(((×FM)×(×FR))<0)/SETLB
  FL0←FL
  FR0←FR
  →BOUND
SETRB:RB←ETA0
  →BISECT
SETLB:LB←ETA0
BISECT:
  FM←(MP←(LB+RB)÷2) FVALUE DATA
  FL←LB FVALUE DATA
  →((RB-LB)<(|0.01×MP))/REGRESS
  →((((×FM)×(×FL))<0),(((×FM)×(×FL))>0))/SWAPR,SWAPL
SWAPR:RB←MP
  →BISECT
SWAPL:LB←MP
  →BISECT
⍝⍝⍝⍝⍝ REGRESSION PROCEDURE FOR WEIBULL SCALE PARAMETERS,BETA'S ⍝⍝
REGRESS:
  ETA0←(RB+LB)÷2
  W←((CS←⍉(N,M)ρ((C+*ETA0)÷S+*ETA0))×*RR)*0.5
  U0←((M,N)ρ1)×W×*(¯1×XI0)
  U1←X1×W×*(¯1×XI0)
  U2←X2×W×*(¯1×XI0)
  UU←⍉(3,M×N)ρ(,U0),(,U1),(,U2)
```

```
U0←U1←U2←⍳0
Z←⍉,(W×XB×*(¯1×XI0))+((¯1×DELTA)+W*2)÷W
B←Z⌹UU
UU←Z←⍳0
⍝⍝⍝⍝⍝⍝⍝⍝⍝ NEWTON PROCEDURE FOR WEIBULL SHAPE PARAMETER,XI'S ⍝⍝⍝
XB←B[1]+(B[2]×X1)+B[3]×X2
RR←(Y-XB)×*(¯1×XI0)
H←(¯1×C)+(+/[2] RR×((¯1×DELTA)+CS×*RR))
DH←(¯1×C)-(+/[2] CS×(*RR)×RR*2)
⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ DETECT THE ILL CONDITION OF XI'S ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
XX←~((|H)<0.1)∧((|DH)<0.01)
→(~((×/XX)=0)∧BADXI=0)/NEWTONXI
BADXI←1
X←XX
NEWTONXI:
XI0←⍉(N,M)⍴XI←XI+DIFFXI←XX×(¯1×H)÷DH
⍝⍝ TEST FOR STOPPING CRITERIA AND UPDATE THE VALUE OF VARIABLES
→((×/(|(DIFFXI÷XI),(BK-B)÷B)<0.01)∨(J≥50))/END
BK←B
RR←(Y-XB)×*(¯1×XI0)
→RECURSIVE
⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝ OUTPUT UNCENSORED LEVEL AND ALL ESTIMATES ⍝⍝⍝⍝⍝⍝⍝⍝⍝⍝
END:
R←((+/C)÷M×N),ETA0,B[1],B[2],B[3],((+/XI×X)÷(+/X)),+/~X
∇
```

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9. Wang Chia-Fu
No. 9, Aly. 30, Ln. 185, Central Rd. Sec. 4 2
Tu-Cheng 23604, Taipei County
Taiwan R.O.C

10. Won Wen-I
SMC 1830, NPS 1
Monterey, CA 93943