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Exact Result for the Grazing Angle of Specular Reflection from a Sphere

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An exact formula is given for the grazing angle of specular reflection from a sphere by displaying the zeros of a certain self-inversive quartic polynomial.

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EXACT RESULT FOR THE GRAZING ANGLE OF SPECULAR REFLECTION FROM A SPHERE

1. Introduction

When the angle of incidence and the angle of reflection of a wave or signal are equal, we say that we have specular reflection. The angle of specular reflection or grazing angle from a perfectly flat surface is easy to compute, but this same angle from a perfect sphere is computed with some difficulty and no exact formula for it is presently known. The problem for the sphere goes back almost half a century to W.T. Fishback [1] who gave an approximation for the grazing angle.

Let r be the radius of a sphere, h_1 and h_2 respectively the perpendicular heights of an observer and source above the sphere and ϕ the angle (measured from the center of the sphere) between the observer and source. If we trace a wave from source to observer (or vice versa), let ψ be the specular angle of reflection from the sphere (see Figure 1). Miller and Vegh [2] gave an easy iterative method for computing the grazing angle to any degree of accuracy. They also showed that in principle a formula for the grazing angle can be found by computing the roots of the self-inversive quartic polynomial equation:

$$\alpha z^4 + \beta z^3 + \gamma z^2 + \bar{\beta} z + \bar{\alpha} = 0 \tag{1}$$

where

$$\alpha = e^{i\phi}(e^{i\phi} - k_1 k_2)$$

$$\beta = k_1^2 + k_2^2 - 2k_1 k_2 e^{i\phi}$$

$$\gamma = 2(k_1^2 + k_2^2 - k_1 k_2 \cos\phi - 1) \ ;$$

and

$$k_i = r/(r + h_i) \ , \quad i = 1,2$$

$$0 \le \phi \le \cos^{-1}(k_1) + \cos^{-1}(k_2) \le \pi \ .$$

If the latter inequality does not hold, reflection cannot occur and no grazing angle exists.

Further, if z_* designates any of the roots of equation (1) and

$$\psi_* = \frac{1}{2}\cos^{-1}(\mathrm{Re}\ z_*)$$

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they showed that the unique value ψ of ψ_* that satisfies

$$\phi + 2\psi = \cos^{-1}(k_1 \cos\psi) + \cos^{-1}(k_2 \cos \psi) \tag{2}$$

is the grazing angle.

Recently, Secrest proved the conjecture of Miller [3] that the roots of equation (1) have absolute value one. Moreover, Secrest showed that for

$$0 < k_1, k_2 < 1 \ , \quad k_1 \neq k_2 \ , \quad 0 < \phi < \pi \tag{3}$$

four distinct roots of equation (1) exist. It was noted there also that the roots themselves are unknown in general.

In this paper, we shall use the fact that equation (1) has distinct roots of absolute value one when the inequalities (3) hold to factor the quartic polynomial, thus giving an explicit formula for the grazing angle. When $k \equiv k_1 = k_2$ the grazing angle is given by

$$\psi = \tan^{-1}(\cot \phi/2 - k \operatorname{cosec} \phi/2) \ .$$

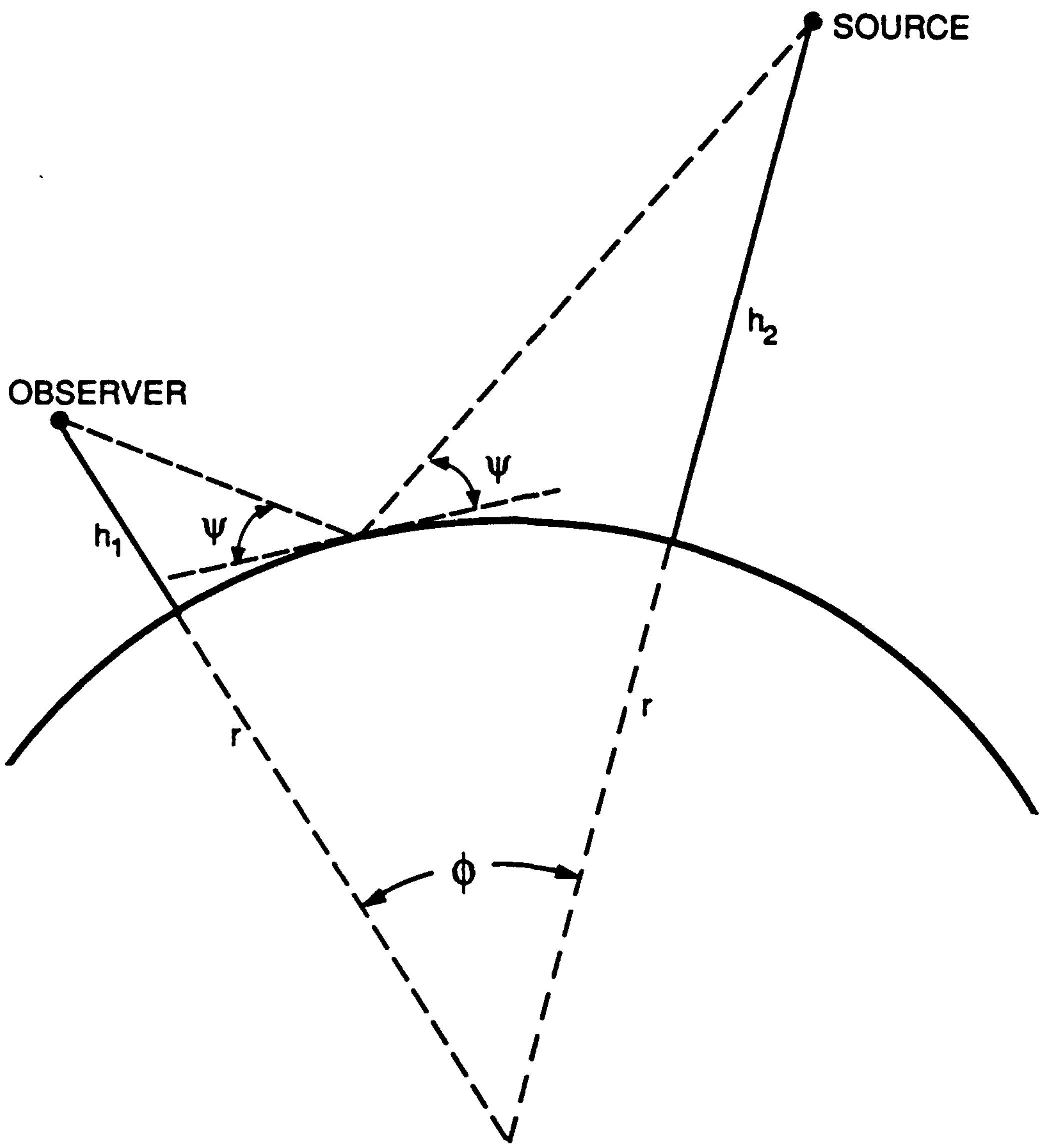

Figure 1. Geometry of spherical specular reflection.

2. Preliminaries

In equation (1) we make the transformation

$$z = (\bar{\alpha}/\alpha)^{1/4} s = e^{-(1/2)i \arg \alpha} s \tag{4}$$

thereby rotating the roots thru an angle of $(-1/2)\ \arg\alpha$. Thus we arrive at

$$s^4 + as^3 + bs^2 + \bar{a}s + 1 = 0 \tag{5}$$

where

$$a = \beta(|\alpha|\alpha)^{-1/2}$$

$$b = \gamma/|\alpha| \ .$$

Since equation (1) has distinct roots of absolute value one, then so does equation (5) which we factor into two quadratics:

$$(s^2 + u_1 s + e^{2i\theta})(s^2 + u_2 s + e^{-2i\theta}) = 0 \tag{6}$$

where u_1, u_2, θ are to be determined. Writing equation (6)

$$(e^{-i\theta}s^2 + u_1 e^{-i\theta}s + e^{i\theta})(e^{i\theta}s^2 + u_2 e^{i\theta}s + e^{-i\theta}) = 0$$

since each quadratic factor has zeros on the unit circle, we must have

$$u_1 = \mu_1 e^{i\theta} \ , \quad u_2 = \mu_2 e^{-i\theta} \tag{7}$$

where μ_1, μ_2 are real and $|\mu_1| \leq 2$, $|\mu_2| \leq 2$. Now comparing the coefficients of like powers of s in equations (5) and (6) and using equations (7) we obtain

$$a = \mu_1 e^{i\theta} + \mu_2 e^{-i\theta} \tag{8}$$

$$b = e^{2i\theta} + e^{-2i\theta} + \mu_1\mu_2 \ . \tag{9}$$

Solving equation (8) for μ_1 and μ_2 we deduce

$$\mu_1 = \frac{1}{2}\left(\frac{\mathrm{Re}\ a}{\cos\theta} + \frac{\mathrm{Im}\ a}{\sin\theta}\right) \tag{10}$$

$$\mu_2 = \frac{1}{2}\left(\frac{\operatorname{Re} a}{\cos\theta} - \frac{\operatorname{Im} a}{\sin\theta}\right) ; \tag{11}$$

and defining

$$t \equiv \cos^2\theta , \qquad 0 < \theta < \pi/2$$

we find from equation (9) that θ is determined by the zeros of the cubic monic polynomial

$$f(t) \equiv t^3 - \frac{b+6}{4} t^2 + \frac{8+4b+|a|^2}{16} t - \frac{\operatorname{Re}^2 a}{16} .$$

Assuming that Re a and Im a are different from zero we readily see that

$$f(0) = -\frac{\operatorname{Re}^2 a}{16} < 0$$

$$f(1) = \frac{\operatorname{Im}^2 a}{16} > 0$$

so that $f(t)$ has at least one zero in the interval $0 < t < 1$. In fact, $f(t)$ has three distinct zeros in the interval $0 < t < 1$. To see this, let s_1, s_2, s_3, s_4 be the distinct roots of equation (5) which we may write

$$[(s-s_1)(s-s_2)][(s-s_3)(s-s_4)] = 0$$

$$[(s-s_1)(s-s_3)][(s-s_2)(s-s_4)] = 0$$

$$[(s-s_1)(s-s_4)][(s-s_2)(s-s_3)] = 0$$

Hence we may factor equation (5) into two quadratics in three different ways, so that in equation (6) we would expect three distinct triples (u_1, u_2, θ). Further, since the cubic equation $f(t) = 0$ has three distinct real roots, its discriminant is positive and we can easily write formulas for these three roots (see for example [4, p. 125]). Thus if we let T be any zero of $f(t)$ we have from equations (10) and (11)

$$\mu_1 = \frac{1}{2}\left(\frac{\operatorname{Re} a}{\sqrt{T}} + \frac{\operatorname{Im} a}{\sqrt{1-T}}\right) \tag{12}$$

$$\mu_2 = \frac{1}{2}\left(\frac{\operatorname{Re} a}{\sqrt{T}} - \frac{\operatorname{Im} a}{\sqrt{1-T}}\right) \tag{13}$$

where $0 < T < 1$. In particular, setting

$$c_2 = -\frac{b+6}{4}, \quad c_1 = \frac{8+4b+|a|^2}{16}, \quad c_0 = -\frac{\mathrm{Re}^2 a}{16}$$

$$p = c_1 - c_2^2/3, \quad q = (2c_2^3 - 9c_1c_2 + 27c_0)/27$$

$$\omega = \frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)$$

the zeros of $f(t)$ are

$$2\sqrt{\frac{-p}{3}}\cos\omega, \quad 2\sqrt{\frac{-p}{3}}\cos(\omega + \frac{2}{3}\pi), \quad 2\sqrt{\frac{-p}{3}}\cos(\omega + \frac{4}{3}\pi)$$

and we may choose T to be any one of the latter expressions.

3. The Roots of the Quartic

Noting equation (7) we write the four roots of equation (6):

$$
\begin{aligned}
s_1 &= \frac{1}{2}\left(-\mu_1\cos\theta - \sqrt{4-\mu_1^2}\,\sin\theta\right) + \frac{i}{2}\left(-\mu_1\sin\theta + \sqrt{4-\mu_1^2}\,\cos\theta\right)\\
s_2 &= \frac{1}{2}\left(-\mu_1\cos\theta + \sqrt{4-\mu_1^2}\,\sin\theta\right) + \frac{i}{2}\left(-\mu_1\sin\theta - \sqrt{4-\mu_1^2}\,\cos\theta\right)\\
s_3 &= \frac{1}{2}\left(-\mu_2\cos\theta + \sqrt{4-\mu_2^2}\,\sin\theta\right) + \frac{i}{2}\left(\mu_2\sin\theta + \sqrt{4-\mu_2^2}\,\cos\theta\right)\\
s_4 &= \frac{1}{2}\left(-\mu_2\cos\theta - \sqrt{4-\mu_2^2}\,\sin\theta\right) + \frac{i}{2}\left(\mu_2\sin\theta - \sqrt{4-\mu_2^2}\,\cos\theta\right);
\end{aligned}
$$

and now applying the rotation equation (4) we obtain the four distinct roots of equation (1):

$$z_k = \exp\left[i(\arg\, s_k - \frac{1}{2}\arg\,\alpha)\right]\ , \quad (k = 1,2,3,4) \tag{14}$$

where

$$\arg\,\alpha = \tan^{-1}\left(\frac{\sin 2\phi - k_1k_2\sin\phi}{\cos 2\phi - k_1k_2\cos\phi}\right) \tag{15}$$

and

$$\arg\, s_1 = \tan^{-1}\left(\frac{-\mu_1\sqrt{1-T} + \sqrt{4-\mu_1^2}\,\sqrt{T}}{-\mu_1\sqrt{T} - \sqrt{4-\mu_1^2}\,\sqrt{1-T}}\right) \tag{16.1}$$

$$\arg\, s_2 = \tan^{-1}\left(\frac{-\mu_1\sqrt{1-T} - \sqrt{4-\mu_1^2}\,\sqrt{T}}{-\mu_1\sqrt{T} + \sqrt{4-\mu_1^2}\,\sqrt{1-T}}\right) \tag{16.2}$$

$$\arg\, s_3 = \tan^{-1}\left(\frac{\mu_2\sqrt{1-T} + \sqrt{4-\mu_2^2}\,\sqrt{T}}{-\mu_2\sqrt{T} + \sqrt{4-\mu_2^2}\,\sqrt{1-T}}\right) \tag{16.3}$$

$$\arg\, s_4 = \tan^{-1}\left(\frac{\mu_2\sqrt{1-T} - \sqrt{4-\mu_2^2}\,\sqrt{T}}{-\mu_2\sqrt{T} - \sqrt{4-\mu_2^2}\,\sqrt{1-T}}\right) \tag{16.4}$$

In equations (16) μ_1 and μ_2 are given respectively by equations (12) and (13) and T $(0 < T < 1)$ is any zero of the cubic polynomial $f(t)$. We remark that the signs of the numerator and denominator of the argument of the inverse tangent function in equations (15) and (16) must not be disturbed by, for example, simplification of these equations.

4. The Grazing Angle

From equation (14) we now find the four angles:

$$\psi_1 = \frac{1}{2}\left(\arg s_1 - \frac{1}{2}\arg\alpha\right)$$

$$\psi_2 = \frac{1}{2}\left(\arg s_2 - \frac{1}{2}\arg\alpha\right)$$

$$\psi_3 = \frac{1}{2}\left(\arg s_3 - \frac{1}{2}\arg\alpha\right)$$

$$\psi_4 = \frac{1}{2}\left(\arg s_4 - \frac{1}{2}\arg\alpha\right)$$

and the unique value of ψ_i that satisfies equation (2) is the grazing angle ψ. We observe that since ψ must be in the interval $(0, \pi/2)$, we can reject immediately any of the ψ_i not in this interval.

It is interesting to note that if we choose the zero of $f(t)$ to be

$$T = 2\sqrt{\frac{-p}{3}}\cos\left(\omega + \frac{4}{3}\pi\right)$$

extensive numerical computations indicate that the grazing angle is given by

$$\psi = \begin{cases} \frac{1}{2}(\arg s_1 - \frac{1}{2}\arg\alpha), & 0 < \phi < \pi/2 \\ \frac{1}{2}(\arg s_2 - \frac{1}{2}\arg\alpha), & \pi/2 \le \phi < \pi\ . \end{cases}$$

Although the authors cannot prove this conjecture analytically, it is certainly true; and applied workers should not hesitate to use it, since it is a simple matter to check numerically by substitution whether ψ is in fact the unique root of equation (2).

5. Special Cases

Since

$$\text{Re } a = \left|\frac{\beta}{\alpha}\right| \cos(\arg\beta - \frac{1}{2}\arg\alpha)$$

$$\text{Im } a = \left|\frac{\beta}{\alpha}\right| \sin(\arg\beta - \frac{1}{2}\arg\alpha)$$

we have

$$\text{Re } a \text{ Im } a = \left|\frac{\beta}{\alpha}\right|^2 \sin(2\arg\beta - \arg\alpha)$$

so that Re a or Im a vanish provided that

$$2\arg\beta = \arg\alpha + n\pi\ , \quad (n = 0, \pm 1, \pm 2, \ldots)\ .$$

Taking the tangent of both sides of this equation gives

$$\frac{\text{Im }\alpha}{\text{Re }\alpha} = 2\,\frac{\text{Re }\beta \text{ Im }\beta}{\text{Re}^2\,\beta - \text{Im}^2\,\beta}$$

which yields after a little computation the result that Re a or Im a vanish provided that k_1, k_2, ϕ satisfy the equation:

$$\cos\phi = \frac{1}{2}\,k_1 k_2 \left[\left(\frac{k_1^2 - k_2^2}{k_1^2 + k_2^2}\right)^2 + \frac{4}{k_1^2 + k_2^2}\right]\ . \tag{17}$$

For $0 < k_1, k_2 < 1$, $k_1 \neq k_2$ the right side of equation (17) is bounded by one so that ϕ always exists in the interval $(0, \pi/2)$.

Numerical computations show that when equation (17) holds, then Re $a = 0$ and Im $a < 0$; it appears then that Im a never vanishes. Nevertheless, for completeness, we shall discuss below the two special cases Re $a = 0$ and Im $a = 0$, since we have not proved the latter assertion but only indicated that its truth is supported by numerical evidence.

In the case when a is real, then it is easy to show from equations (6)-(9) that the roots of equation (5) are given by

$$\frac{1}{2}(-\mu_1 + i\,\sqrt{4 - \mu_1^2})\ ,$$

$$\frac{1}{2}(-\mu_1 - i\,\sqrt{4 - \mu_1^2})\ ,$$

$$\frac{1}{2}(-\mu_2 + i\sqrt{4-\mu_2^2}) \; ,$$
$$\frac{1}{2}(-\mu_2 - i\sqrt{4-\mu_2^2}) \; ,$$

where

$$\mu_1 = \frac{1}{2}(a + \sqrt{a^2 - 4b + 8}) \tag{18.1}$$

$$\mu_2 = \frac{1}{2}(a - \sqrt{a^2 - 4b + 8}) \; ; \tag{18.2}$$

and in the case a $(= ic,\ c$ real) is purely imaginary the roots of equation (5) are

$$\frac{1}{2}(-\mu_1 i + \sqrt{4-\mu_1^2}) \; ,$$
$$\frac{1}{2}(-\mu_1 i - \sqrt{4-\mu_1^2}) \; ,$$
$$\frac{1}{2}(-\mu_2 i + \sqrt{4-\mu_2^2}) \; ,$$
$$\frac{1}{2}(-\mu_2 i - \sqrt{4-\mu_2^2})$$

where now

$$\mu_1 = \frac{1}{2}(c + \sqrt{c^2 + 4b + 8}) \tag{19.1}$$

$$\mu_2 = \frac{1}{2}(c - \sqrt{c^2 + 4b + 8}) \; . \tag{19.2}$$

Therefore, if $0 < k_1, k_2 < 1$, $k_1 \neq k_2$, $0 < \phi < \pi$ are such that equation (17) holds, then in the case a is real we have

$$\psi_1 = \frac{1}{2}\left[\tan^{-1}\left(\frac{\sqrt{4-\mu_1^2}}{-\mu_1}\right) - \frac{1}{2}\arg\alpha\right]$$

$$\psi_2 = \frac{1}{2}\left[\tan^{-1}\left(\frac{-\sqrt{4-\mu_1^2}}{-\mu_1}\right) - \frac{1}{2}\arg\alpha\right]$$

$$\psi_3 = \frac{1}{2}\left[\tan^{-1}\left(\frac{\sqrt{4-\mu_2^2}}{-\mu_2}\right) - \frac{1}{2}\arg\alpha\right]$$

$$\psi_4 = \frac{1}{2}\left[\tan^{-1}\left(\frac{-\sqrt{4-\mu_2^2}}{-\mu_2}\right) - \frac{1}{2}\arg\alpha\right]$$

where $\arg\alpha$ and μ_1, μ_2 are given respectively by equations (15) and (18); and in the case where $a = ic$ (c real) we have

$$\psi_1 = \frac{1}{2}\left[\tan^{-1}\left(\frac{-\mu_1}{\sqrt{4-\mu_1^2}}\right) - \frac{1}{2}\arg\alpha\right]$$

$$\psi_2 = \frac{1}{2}\left[\tan^{-1}\left(\frac{-\mu_1}{-\sqrt{4-\mu_1^2}}\right) - \frac{1}{2}\arg\alpha\right]$$

$$\psi_3 = \frac{1}{2}\left[\tan^{-1}\left(\frac{-\mu_2}{\sqrt{4-\mu_2^2}}\right) - \frac{1}{2}\arg\alpha\right]$$

$$\psi_4 = \frac{1}{2}\left[\tan^{-1}\left(\frac{-\mu_2}{-\sqrt{4-\mu_2^2}}\right) - \frac{1}{2}\arg\alpha\right]$$

where μ_1 and μ_2 are given by equations (19).

6. Conclusions

In the present paper we have shown that it is possible to exhibit the grazing angle of specular reflection as one of four angles derived from the zeros of a certain self-inversive quartic polynomial whose coefficients are complex. Thus the grazing angle may now be found deterministically, whereas heretofore it had to be computed either by approximations or numerical procedures.

The preceeding analysis shows further that factoring, in a useful way, a quartic polynomial whose complex coefficients depend on even a small number of parameters, is not always a trivial matter, even though the methods of Cardan and Ferrari have been known for centuries.

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