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Analysis of the Efficiency of an A-Posteriori Error Estimator for Linear Triangular Finite Elements

by

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This paper adresses the problem of determining upper and lower bounds for the effectivity index on the a-posteriori estimate of the error in the finite element method. These bounds are given explicitly for a certain concrete estimator for linear elements and unstructured triangular meshes. They depend strongly on the geometry of the triangles and (relatively weakly) on the smoothness of the solution. An example shows that the bounds are not over pessimistic. In [4] detailed numerical experimentation is given.

ANALYSIS OF THE EFFICIENCY OF AN A–POSTERIORI ERROR ESTIMATOR FOR LINEAR TRIANGULAR FINITE ELEMENTS

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Abstract: This paper adresses the problem of determining upper and lower bounds for the effectivity index on the a–posteriori estimate of the error in the finite element method. These bounds are given explicitly for a certain concrete estimator for linear elements and unstructured triangular meshes. They depend strongly on the geometry of the triangles and (relatively weakly) on the smoothness of the solution. An example shows that the bounds are not over pessimistic. In [4] detailed numerical experimentation is given.

1. Introduction. Since the first papers by Babuška and Rheinboldt on the a–posteriori estimation of the errors in the finite element method [5,6], this subject became an increasingly important aspect of the application of this method. During the last years several codes including different estimators have been developped [14,23,25,26,28] and nowadays there are many different estimators in use for a given problem (see, for instance, [12,13,21,24] and references there in).

A standard measure of the quality of an estimator is the so called *effectivity index*

$$\text{eff} = \frac{\text{estimated error}}{\text{true error}} \ .$$

For a given problem an estimator is said to be *equivalent* to the error if the effectivity index is bounded below and above by two strictly positive constants independently of the meshsize:

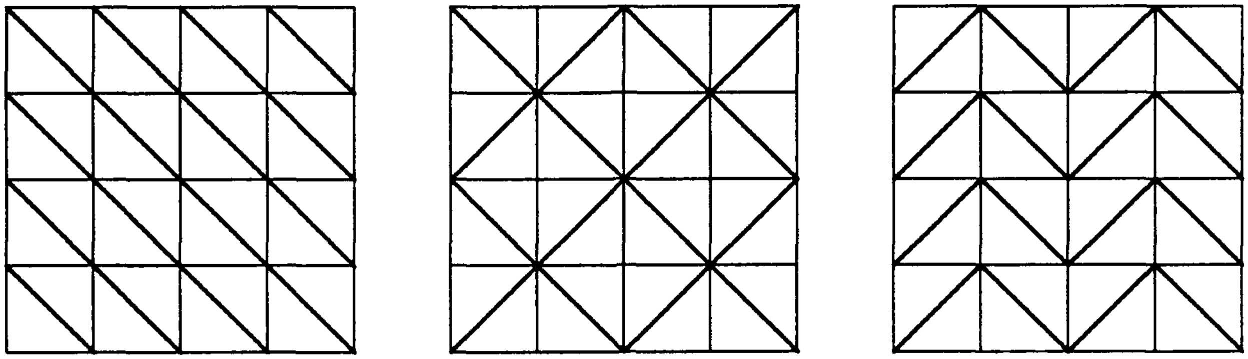
$$c \leq \text{eff} \leq C \ ;$$

these constants may depend on the class of functions under consideration. (Here and thereafter, c and C will denote constants not necessarily the same at each occurrence, but always independent of the meshsize).

A property that has been considered highly relevant to measure the potential quality of an estimator is the so called *asymptotic exactness*. Roughly speaking, an estimator is asymptotically exact for a particular problem if its effectivity index converges to one when the meshsize aproaches to zero.

In the one dimensional case Babuška and Rheinboldt [7,8] made a complete analysis of asymptotically exact error estimators. For two dimensional elliptic problems, several estimators have been proved to be asymptotically exact when used on almost uniform patches of rectangular or triangular meshes, provided the solution of the problem is smooth enough [2,11,17,18,19].

In particular, for linear triangular elements, some well known local estimators like Bank-Weiser's [15] and Zienkiewicz-Zhu's [31] are asymptotically exact on uniform meshes as that in Figure 1.1.a but not on other rather uniform meshes as those in Figures 1.1.b and 1.1.c. (See [19] for Bank-Weiser's estimator and [9] for Zienkiewicz-Zhu's).



(a) (b) (c)

Figure 1.1

A–posteriori error indicators (i.e.: estimators per element) are employed in adaptive processes to identify those portions of the mesh with bigger errors in order to generate a new refined mesh. Usually, the meshes generated by these adaptive processes are regular (in the sense of a minimal angle condition) but not uniform as in Figure 1.1.a. Very likely, all the used estimators are not asymptotically exact on the meshes that are adaptively constructed. However, the estimators actually in use are equivalent to the error for any regular family of meshes with bounds on the effectivity index depending only on the regularity of the mesh. Anyway, in no case these bounds are known explicitly. To increase the accuracy of the indicators and estimators, various correction factors derived by computational tests are used.

In this paper we shall analyze a particular estimator based on Babuška-Miller's [3]; (this type of estimator is used, for instance, in [25]). We shall prove again the equivalence of this estimator for the Laplace equation, but in such a way that it will be able to compute asymptotic bounds of its effectivity index in terms of the geometry of the mesh and on the smoothness of the solution. We shall show that these bounds are sharp and that their dependence on the geometry of the mesh is optimal. Finally we shall present similar results for the elasticity problem.

2. The error estimator. Let us consider as our first model problem the Laplace equation with mixed boundary conditions. Let Ω be a bounded polygon in $\mathbf{R}^2$ and let its boundary $\partial\Omega$ be split into two parts Γ_d and Γ_n (Γ_d of positive length). Let u be the solution of the problem

$$\begin{cases} -\Delta u = f \,, & \text{in } \Omega \,, \\ u = 0 \,, & \text{on } \Gamma_\mathrm{d} \,, \\ \frac{\partial u}{\partial n} = g \,, & \text{on } \Gamma_\mathrm{n} \,, \end{cases} \tag{2.1}$$

where $\mathbf{n}$ is the unit outer normal vector to $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_n)$.

We shall use the standard notation for Sobolev spaces $H^m(D)$, their norms $\|\cdot\|_{m,D}$ and seminorms $|\cdot|_{m,D}$. Let $H^1_{\Gamma_d}(\Omega) := \{v \in H^1(\Omega) : v|_{\Gamma_d} = 0\}$. $|\cdot|_{1,\Omega}$ is a norm on that space; it is the energy norm of this model problem.

Let $\{\mathcal{T}_h\}$ be a regular family of triangulations of Ω (i.e.: the minimal angle of all the triangles is bounded below by a positive constant, the same for all the meshes); as usual h stands for the maximal meshsize and we assume that, when the edge of a triangle intersects $\partial\Omega$, it is completely contained either in Γ_d or in Γ_n. The meshes are not assumed to be quasiuniform.

Let $u_h \in V_h := \{v \in H^1_{\Gamma_d}(\Omega) : v|_T \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}_h\}$ be the piecewise linear finite element approximate solution of problem (2.1). ($\mathcal{P}_m(T)$ denotes the set of polynomial functions defined on T of degree not greater than m). Let $e := u - u_h$ denote the error of this approximation.

Integrating by parts we obtain for any $v \in H^1_{\Gamma_d}(\Omega)$:

$$\int_\Omega \nabla e \cdot \nabla v = \int_\Omega \nabla u \cdot \nabla v - \sum_{T\in\mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v = \int_\Omega f v + \int_{\Gamma_n} g v - \sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial u_h}{\partial \mathbf{n}_T} v \ ,$$

where for each triangle T, $\mathbf{n}_T$ is its unit outer normal vector.

Let us call Γ_i the union of all the interior edges of the triangulation $\mathcal{T}_h$. For each edge $\ell \subset \Gamma_i$ let us choose an arbitrary normal direction $\mathbf{n}$ and denote the two triangles sharing this edge T_{in} and T_{out}, where the normal $\mathbf{n}$ is outwards T_{in}. Let

$$\left[\!\left[\frac{\partial u_h}{\partial \mathbf{n}}\right]\!\right]_\ell := \nabla\left(u_h|_{T_{\text{out}}}\right)\cdot \mathbf{n} - \nabla\left(u_h|_{T_{\text{in}}}\right)\cdot\mathbf{n}$$

denote the jump of $\frac{\partial u_h}{\partial \mathbf{n}}$ accross the edge ℓ; this value is independent of the choice of $\mathbf{n}$.

With this notation we may now write the so called *residual equation*:

$$\text{(2.2)} \qquad \int_\Omega \nabla e\cdot\nabla v = \int_\Omega f v + \int_{\Gamma_n}\left(g - \frac{\partial u_h}{\partial \mathbf{n}}\right) v + \sum_{\ell\subset\Gamma_i}\int_\ell \left[\!\left[\frac{\partial u_h}{\partial \mathbf{n}}\right]\!\right]_\ell v\ , \quad \forall v \in H^1_{\Gamma_d}(\Omega)\ .$$

This equation relates the error e with the interior residuals $f|_T = -\left[\Delta(u|_T) - \Delta(u_h|_T)\right]$, the boundary residual $\left(g - \frac{\partial u_h}{\partial \mathbf{n}}\right)$ and the jumps of the gradient of the finite element approximation $\left[\!\left[\frac{\partial u_h}{\partial \mathbf{n}}\right]\!\right]_\ell$. Several estimators have been obtained by approximating the error as the solution of this equation [3,15,16,29]. The estimator that we shall consider is a slight variation of Babuška-Miller's [3] that Verfürth describes for the Stokes problem [29].

For any triangle $T \in \mathcal{T}_h$, let E_T be the set of its three edges and let

$$\Pi_T f := \frac{1}{|T|}\int_T f$$

be the $L^2(T)$–projection of f onto the constants. For any edge $\ell \in \Gamma_n$, let

$$\Pi_\ell g := \frac{1}{|\ell|}\int_\ell g$$

be the $L^2(\ell)$–projection of g onto the constants. For each edge ℓ of the triangulation, let

$$J_\ell := \begin{cases} \left[\frac{\partial u_h}{\partial n}\right]_\ell, & \text{if } \ell \subset \Gamma_i\,, \\ 2\left(\Pi_\ell g - \frac{\partial u_h}{\partial n}\big|_\ell\right), & \text{if } \ell \subset \Gamma_n\,, \\ 0, & \text{if } \ell \subset \Gamma_d\,. \end{cases} \tag{2.3}$$

We define as an estimator of the local energy error $|e|_{1,T}$,

$$\eta_T := \left[|T|^2(\Pi_T f)^2 + \frac{1}{2}\sum_{\ell\in E_T}|\ell|^2 J_\ell^2\right]^{\frac{1}{2}}. \tag{2.4}$$

Although we deal with the Laplace model problem, this approach is valid for any divergence type operator with piecewise constant coefficients if the meshes are such that the interfaces of the coefficients coincide with boundaries of elements.

3. Equivalence between the error and the estimator. The ideas of Verfürth [30] can be directly applied to our simpler model problem to prove the following theorem without any further assumption on the mesh and for any problem (2.1) with solution $u \in H^1(\Omega)$.

THEOREM 3.1. *With the definitions and assumptions of Section 2, there exist two positive constants C and C' only depending on the regularity of the mesh such that*

$$|e|_{1,\Omega} \le C\left[\sum_{T\in\mathcal{T}_h}\left(\eta_T^2 + |T|\,\|f-\Pi_T f\|_{0,T}^2 + \sum_{\ell\subset(\partial T\cap\Gamma_n)}|\ell|\,\|g-\Pi_\ell g\|_{0,\ell}^2\right)\right]^{\frac{1}{2}} \tag{3.1}$$

and

(3.2)

$$\eta_T \le C'\left[|e|_{1,\widetilde{T}} + \left(\sum_{T'\subset\widetilde{T}}|T'|\,\|f-\Pi_{T'} f\|_{0,T'}^2\right)^{\frac{1}{2}} + \left(\sum_{\ell\subset(\partial T\cap\Gamma_n)}|\ell|\,\|g-\Pi_\ell g\|_{0,\ell}^2\right)^{\frac{1}{2}}\right],$$

where $\widetilde{T} := \bigcup\{T' \in \mathcal{T}_h \;:\; T \text{ and } T' \text{ have a common edge}\}$.

Proof. The proof will not be given here because it is essentially identical to that in [30]. □

These bounds show that whenever the data f and g are locally smooth, if the error is properly $\mathcal{O}(h^s)$ $(0 < s \leq 1)$, then the estimator

$$\eta_\Omega := \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{\frac{1}{2}} \tag{3.3}$$

is globally equivalent to the error. In fact, we have the following theorem.

THEOREM 3.2. *In addition to the assumptions of Section 2, let us assume that there exists a triangulation* $\mathcal{T}$ *such that*

$$f|_T \in H^1(T) \ , \ \forall T \in \mathcal{T} \tag{3.4}$$

and $\forall T \in \mathcal{T} \ : \ \partial T \cap \Gamma_n \neq \emptyset$

$$g|_\ell \in H^1(\ell) \ , \ \forall \ell \in \partial T \cap \Gamma_n \ ; \tag{3.5}$$

let us also assume that all the triangulations $\mathcal{T}_h$ *are refinements of* $\mathcal{T}$.

If there exist constants $C^* > 0$ *and* $s \in (0,1]$ *not depending on* h *such that*

$$|e|_{1,\Omega} \geq C^* h^s, \tag{3.6}$$

then there exist two positive constants c *and* C *such that*

$$c\eta_\Omega \leq |e|_{1,\Omega} \leq C\eta_\Omega \ . \tag{3.7}$$

Proof. By using (3.1–5), the regularity of the meshes and the standard aproximation properties of the projections $\Pi_T f$ and $\Pi_\ell g$, we may write

$$|e|_{1,\Omega}^2 \leq C \left(\eta_\Omega^2 + h^4 \sum_{T \in \mathcal{T}} |f|_{1,T}^2 + h^3 \sum_{\ell \in \mathcal{L}} |g|_{1.\ell}^2 \right)$$

where $\mathcal{L} := \{\ell \text{ edge of } T \in \mathcal{T} \ : \ \ell \subset \Gamma_n\}$, and

$$\eta_\Omega \leq C' \left[|e|_{1,\Omega} + h^2 \left(\sum_{T \in \mathcal{T}} |f|_{1,T}^2 \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_{\ell \in \mathcal{L}} |g|_{1.\ell}^2 \right)^{\frac{1}{2}} \right] ,$$

since each $\widetilde{T}$ is the union of at most 4 triangles of $\mathcal{T}_h$. Hence, by using (3.6) the theorem is proved. □

Remark 3.1. The assumptions about the existence of $\mathcal{T}$ is made only to cover those cases where f and g are piecewise smooth and the meshes are such that the interfaces of the data coincide with boundaries of the elements. On the other hand, these local smoothness assumptions (3.4) and (3.5) can be weakened; in fact, if $f|_T \in H^\epsilon(T)$ and $g|_\ell \in H^{\frac{1}{2}+\epsilon}(\ell)$ for some $\epsilon > 0$, then the conclusion of Theorem 3.2 and all what follows are valid. □

Remark 3.2. The error is always properly $\mathcal{O}(h^s)$ (i.e.: assumption (3.6) is valid) except for trivial cases (see [3]). □

We shall now describe a variation of Verfürth's proof of Theorem 3.1 that will give computable asymptotic approximations of the constants c and C in the equivalence (3.7) (assuming slightly stringent hypothesis for the upper bound). In the following sections, we shall show that the constants obtained in this way are almost achievable.

Let $V_p := \{v \in H^1_{\Gamma_d}(\Omega) : v|_T \in \mathcal{P}_p(T),\ \forall T \in \mathcal{T}_h\}$ (in particular $V_1 = V_h$) and, for $p \geq 2$, let $u_p \in V_p$ be the finite element approximate solution of problem (2.1) in this space. Let $e_p := u_p - u_h$, then

$$|e|_{1,\Omega} \leq |u - u_p|_{1,\Omega} + |e_p|_{1,\Omega} \,. \tag{3.8}$$

It is known [10] that, if $u \in H^{1+\epsilon}(\Omega)$ for some $\epsilon > 0$ and if the family of meshes is quasiuniform, then

$$|u - u_p|_{1,\Omega} \leq C h^{\min\{p,\epsilon\}} p^{-\epsilon} \|u\|_{1+\epsilon,\Omega} \,, \tag{3.9}$$

with a constant C independent of u, h and p. Therefore, if the solution is smooth enough, say $u \in H^{1+\epsilon}(\Omega)$ for some $\epsilon > 1$, $|u - u_p|_{1,\Omega}$ is asymptotically negligible with respect to the error $|e|_{1,\Omega}$ for any $p \geq 2$. Instead, if the solution $u \in H^{1+\epsilon}(\Omega)$ for some $\epsilon \in (0,1]$, the error $|e|_{1,\Omega}$ is expected to be $\mathcal{O}(h^\epsilon)$ and, in this case, $|u - u_p|_{1,\Omega}$ will be negligible with respect to $|e|_{1,\Omega}$ only for p big enough. In any case, even for h small or for p big enough (or both together), the term $|u - u_p|_{1,\Omega}$ can be neglected in (3.8). Therefore, it is enough to bound $|e_p|_{1,\Omega}$. Now,

$$|e_p|^2_{1,\Omega} = \int_\Omega \nabla e_p \cdot \nabla e_p = \int_\Omega \nabla(u_p - u) \cdot \nabla e_p + \int_\Omega \nabla e \cdot \nabla e_p = \int_\Omega \nabla e \cdot \nabla e_p \,, \tag{3.10}$$

where we have used that $e_p \in V_p$.

For any continuous function v defined on Ω, let v^I denote its Lagrange piecewise linear interpolant on the mesh $\mathcal{T}_h$. Since $e_p^I \in V_h$, then from (3.10) we have

$$|e_p|^2_{1,\Omega} = \int_\Omega \nabla e \cdot \nabla(e_p - e_p^I) \,,$$

and by using the residual equation (2.2) we may write

$$|e_p|^2_{1,\Omega} = \int_\Omega f(e_p - e_p^I) + \int_{\Gamma_n}\left(g - \frac{\partial u_h}{\partial \mathbf{n}}\right)(e_p - e_p^I) + \sum_{\ell\subset\Gamma_i}\int_\ell \left[\!\left[\frac{\partial u_h}{\partial \mathbf{n}}\right]\!\right]_\ell (e_p - e_p^I)$$

$$= \sum_{T\in\mathcal{T}_h}\left[\int_T f(e_p - e_p^I) + \int_{\partial T\cap\Gamma_n}\left(g - \frac{\partial u_h}{\partial \mathbf{n}}\right)(e_p - e_p^I) + \frac{1}{2}\sum_{\ell\subset(\partial T\cap\Gamma_i)}\int_\ell \left[\!\left[\frac{\partial u_h}{\partial \mathbf{n}}\right]\!\right]_\ell (e_p - e_p^I)\right]$$

$$= \sum_{T\in\mathcal{T}_h}\left[\Pi_T f\int_T(e_p - e_p^I) + \frac{1}{2}\sum_{\ell\in E_T} J_\ell \int_\ell (e_p - e_p^I)\right] + \delta(e_p - e_p^I)\ ,$$

where

$$\delta(v) := \sum_{T\in\mathcal{T}_h}\int_T (f - \Pi_T f)v + \sum_{\ell\subset\Gamma_n}\int_\ell (g - \Pi_\ell g)v\ .$$

Using the definition (2.4) of the local estimator η_T and Cauchy–Schwarz inequality we have:

$$\text{(3.11)}\quad |e_p|^2_{1,\Omega} \le \sum_{T\in\mathcal{T}_h}\eta_T\left[\left|\frac{1}{|T|}\int_T(e_p - e_p^I)\right|^2 + \frac{1}{2}\sum_{\ell\in E_T}\left|\frac{1}{|\ell|}\int_\ell (e_p - e_p^I)\right|^2\right]^{\frac{1}{2}} + \left|\delta(e_p - e_p^I)\right|\ .$$

This last expression allows us to prove the following theorem.

THEOREM 3.3. *Under the assumptions of Theorem 3.2,*

$$|e|_{1,\Omega} \le \left[\sum_{T\in\mathcal{T}_h}\left(\mathbf{C}_T^p\right)^2\eta_T^2\right]^{\frac{1}{2}} + |u - u_p|_{1,\Omega}$$

$$\text{(3.12)}\qquad + \ C\left[h^2\left(\sum_{T\in\mathcal{T}}|f|^2_{1,T}\right)^{\frac{1}{2}} + h^{\frac{3}{2}}\left(\sum_{\ell\in\mathcal{L}}|g|^2_{1,\ell}\right)^{\frac{1}{2}}\right]\ ,$$

where

$$\text{(3.13)}\qquad \left(\mathbf{C}_T^p\right)^2 := \sup_{v\in\mathcal{P}_p\setminus\mathcal{P}_0}\frac{\left|\frac{1}{|T|}\int_T(v - v^I)\right|^2 + \frac{1}{2}\sum_{\ell\in E_T}\left|\frac{1}{|\ell|}\int_\ell(v - v^I)\right|^2}{|v|^2_{1,T}}$$

Proof. According to (3.11) and the definition of $\mathbf{C}_T^p$:

$$|e_p|^2_{1,\Omega} \le \sum_{T\in\mathcal{T}_h}\mathbf{C}_T^p\eta_T|e_p|_{1,T} + \left|\delta(e_p - e_p^I)\right| \le \left[\sum_{T\in\mathcal{T}_h}\left(\mathbf{C}_T^p\right)^2\eta_T^2\right]^{\frac{1}{2}}|e_p|_{1,\Omega} + \left|\delta(e_p - e_p^I)\right|\ .$$

Proceeding as in Theorem 3.2 we prove that

$$|\delta(e_p - e_p^I)| \leq C \left[h^2 \left(\sum_{T \in \mathcal{T}} |f|_{1,T}^2 \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_{\ell \in \mathcal{L}} |g|_{1,\ell}^2 \right)^{\frac{1}{2}} \right] |e_p|_{1,\Omega} \ .$$

So, by using (3.8) we conclude the theorem. □

The constants $\mathbf{C}_T^p$ in this theorem depend on the degree p used to make $|u - u_p|_{1,\Omega}$ negligible in (3.12). However the next theorem shows that this dependence is very weak.

THEOREM 3.4. *Let $\mathbf{C}_T^p$ be defined by (3.13); there exists a constant C_T only depending on the shape of the triangle T such that $\forall p \geq 2$*

$$\mathbf{C}_T^p \leq C_T \log^{\frac{1}{2}} p$$

Proof. Let $\widehat{T} := \{(x,y) \ : \ x \geq 0, \ y \geq 0, \text{ and } x + y \leq 1\}$. For any polynomial $\widehat{v}$ of degree $p \geq 2$

$$\|\widehat{v}\|_{L^\infty(\widehat{T})} \leq C \log^{\frac{1}{2}} p \ \|\widehat{v}\|_{1,\widehat{T}}$$

with C independent of $\widehat{v}$ and p; (this is an inmediate consequence of Theorem 6.2 in [1]). Since $(\widehat{v} - \widehat{v}^I)$ vanishes for $\widehat{v} \in \mathcal{P}_0$, then

$$\|\widehat{v} - \widehat{v}^I\|_{L^\infty(\widehat{T})} \leq C \log^{\frac{1}{2}} p \ |\widehat{v}|_{1,\widehat{T}} \ ,$$

and so, for any $v \in \mathcal{P}_p(T)$, by changing coordinates to the triangle $\widehat{T}$ we obtain

$$\|v - v^I\|_{L^\infty(T)} \leq C_T \log^{\frac{1}{2}} p \ |v|_{1,\widehat{T}}$$

with a constant C_T only depending on the shape of the triangle. Using this inequality in the definition (3.13) of $\mathbf{C}_T^p$ we conclude the theorem. □

In the following section we shall compute the costants $\mathbf{C}_T^p$ for different values of p and we shall analyze their dependence on the shape of the triangle T. On the other hand, for the lower bound in (3.7) we have the following theorem.

THEOREM 3.5. *For each mesh $\mathcal{T}_h$, let $w \in H^1_{\Gamma_d}(\Omega)$ (eventually depending on the mesh) be such that for all the triangles $T \in \mathcal{T}_h$,*

$$\int_T (\Pi_T f) w = |T|^2 (\Pi_T f)^2 \ , \tag{3.14}$$

$$\sum_{\ell \in E_T} \int_\ell J_\ell w = \sum_{\ell \in E_T} |\ell|^2 J_\ell^2 \tag{3.15}$$

and

$$\exists \mathbf{C}'_T > 0 \ : \ |w|_{1,T} \leq \mathbf{C}'_T \eta_T \ , \tag{3.16}$$

where $\mathbf{C}'_T$ may depend on the shape of the triangle but not on its size h_T.

Then, under the assumptions of Theorem 3.2,

$$\eta_\Omega \leq \left(\sup_{T \in \mathcal{T}_h} \mathbf{C}'_T \right) \left\{ |e|_{1,\Omega} + C \left[h^2 \left(\sum_{T \in \mathcal{T}} |f|^2_{1,T} \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_{\ell \in \mathcal{L}} |g|^2_{1,\ell} \right)^{\frac{1}{2}} \right] \right\} . \tag{3.17}$$

Proof. By using (3.14) and (3.15) in the definition of η_T, the residual equation (2.2) and the definition of δ, we have

$$\sum_{T \in \mathcal{T}_h} \eta_T^2 = \sum_{T \in \mathcal{T}_h} \left[\int_T (\Pi_T f) w + \frac{1}{2} \sum_{\ell \in E_T} \int_\ell J_\ell w \right] = \int_\Omega \nabla e \cdot \nabla w - \delta(w)$$

and hence,

$$\sum_{T \in \mathcal{T}_h} \eta_T^2 \leq |e|_{1,\Omega} |w|_{1,\Omega} + |\delta(w)| \ .$$

Now

$$\left| \sum_{T \in \mathcal{T}_h} \int_T (f - \Pi_T f) w \right| = \left| \sum_{T \in \mathcal{T}_h} \int_T (f - \Pi_T f)(w - \Pi_T w) \right| \leq C \sum_{T \in \mathcal{T}_h} |T| \ |f|_{1,T} |w|_{1,T}$$

and

$$\left| \sum_{\ell \subset \Gamma_n} \int_\ell (g - \Pi_\ell g) w \right| = \left| \sum_{\ell \subset \Gamma_n} \int_\ell (g - \Pi_\ell g)(w - \Pi_\ell w) \right| \leq C \sum_{\ell \subset \Gamma_n} |\ell|^{\frac{3}{2}} |g|_{1,\ell} |w|_{1,T_\ell} \ ,$$

where T_ℓ is the triangle in $\mathcal{T}_h$ such that $\ell \subset \partial T_\ell$. Therefore, by using (3.16), we have

$$\sum_{T \in \mathcal{T}_h} \eta_T^2 \leq \left\{ |e|_{1,\Omega} + C \left[h^2 \left(\sum_{T \in \mathcal{T}} |f|^2_{1,T} \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_{\ell \in \mathcal{L}} |g|^2_{1,\ell} \right)^{\frac{1}{2}} \right] \right\} |w|_{1,\Omega}$$

$$\leq \left(\sup_{T \in \mathcal{T}_h} \mathbf{C}'_T \right) \left\{ |e|_{1,\Omega} + C \left[h^2 \left(\sum_{T \in \mathcal{T}} |f|^2_{1,T} \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_{\ell \in \mathcal{L}} |g|^2_{1,\ell} \right)^{\frac{1}{2}} \right] \right\} \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{\frac{1}{2}}$$

and hence we obtain (3.17). $\square$

In the next section we shall exhibit functions w satisfying the hypothesis of Theorem 3.5 and we shall show how to calculate the constant $\mathbf{C}'_T$.

4. Computation of the bounds. In order to compute the constants $\mathbf{C}_T^p$ of Theorem 3.3, let $z_0 \in \mathcal{P}_p(T)$ be the solution of the weak finite dimensional problem

$$\int_T \nabla z_0 \cdot \nabla v = \frac{1}{|T|}\int_T (v - v^I) \ , \quad \forall v \in \mathcal{P}_p(T) \tag{4.1}$$

and, for i=1,2,3, let $z_i \in \mathcal{P}_p(T)$ be the solution of

$$\int_T \nabla z_i \cdot \nabla v = \frac{1}{\sqrt{2}|\ell_i|}\int_{\ell_i} (v - v^I) \ , \quad \forall v \in \mathcal{P}_p(T) \ , \tag{4.2}$$

where ℓ_i, $i = 1,2,3$, are the three edges of T. We may write

$$\left(\mathbf{C}_T^p\right)^2 = \sup_{v\in\mathcal{P}_p\setminus\mathcal{P}_0} \frac{\sum_{i=0}^3 \left(\int_T \nabla z_i \cdot \nabla v\right)^2}{|v|_{1,T}^2} = \sup_{v\in\mathcal{Z}\setminus\mathcal{P}_0} \frac{\sum_{i=0}^3 \left(\int_T \nabla z_i \cdot \nabla v\right)^2}{|v|_{1,T}^2} \ ,$$

where $\mathcal{Z}$ is the subspace of $\mathcal{P}_p(T)$ spanned by $\{z_i\}_{i=0}^3$.

For $v = \sum_{i=0}^3 v_i z_i \in \mathcal{Z}$ we may write $|v|_{1,T}^2 = \mathbf{v}^t\mathbf{C}\mathbf{v}$, where $\mathbf{v} := (v_0, \ldots, v_3)$ and $\mathbf{C} \in \mathbf{R}^{4\times 4}$ is the symmetric matrix of entries $C_{i,j} := \int_T \nabla z_i \cdot \nabla z_j$, $i,j = 0,\ldots,3$. On the other hand $\sum_{i=0}^3 \left(\int_T \nabla z_i \cdot \nabla v\right)^2 = \mathbf{v}^t\mathbf{C}^2\mathbf{v}$. Therefore,

$$\left(\mathbf{C}_T^p\right)^2 = \sup_{\mathbf{v}\in\mathbf{R}^4 \ : \ \mathbf{v}^t\mathbf{C}\mathbf{v}\neq 0} \frac{\mathbf{v}^t\mathbf{C}^2\mathbf{v}}{\mathbf{v}^t\mathbf{C}\mathbf{v}} = \sup_{\mathbf{v}\neq 0} \frac{\mathbf{v}^t\mathbf{C}\mathbf{v}}{\mathbf{v}^t\mathbf{v}}$$

is the spectral ratio of $\mathbf{C}$.

So, to compute the constants for any degree $p \geq 2$ and any triangle T, we only need the solutions z_i of problems (4.1) and (4.2). These functions are the p–degree finite element solutions of elementary elliptic problems on the triangle T with a mesh consisting of this only triangle; they have been computed by using the code PROBE [27]. Our computations show that for any triangle T and for any degree $p = 2,3,\ldots,8$,

$$0.548 \ \log^{\frac{1}{2}} p \ \sin^{-\frac{1}{2}}\left(\frac{\alpha_T}{2}\right) \leq \mathbf{C}_T^p \leq 0.813 \ \log^{\frac{1}{2}} p \ \sin^{-\frac{1}{2}}\left(\frac{\alpha_T}{2}\right) \ , \tag{4.3}$$

where α_T is the minimum angle of T. These constants $\mathbf{C}_T^p$ also depend on the other angles of T; however this dependence is very weak. In fact, the estimate (4.3) is valid for all the triangles with minimum angle α_T, independently of the size of the other angles.

From Theorem 3.4 we know that for any fixed triangle the constants $\mathbf{C}_T^p$ are bounded above by $\log^{\frac{1}{2}} p$; our computations show that, actually, they are almost proportional to $\log^{\frac{1}{2}} p$. On the other hand, for a fixed degree $p \geq 2$, the constants depend on the geometry; they essentially depend on the minimum angle and in fact they deteriorate when this angle is very small, but the square roots in (4.3) makes this dependence to be weak.

Now, we shall describe how to compute the constants $\mathbf{C}'_T$ of Theorem 3.5. To this goal we need a function $w \in H^1_{\Gamma_d}(\Omega)$ satisfying (3.14), (3.15) and (3.16) with constants $\mathbf{C}'_T$ as small as possible. We define this function w in each triangle but in such a way that it satisfies the required global smoothness. For any edge ℓ of the triangulation we choose a continuous function ψ_ℓ vanishing at both ends of the edge and such that its average $c_\ell := \frac{1}{|\ell|}\int_\ell \psi_\ell \neq 0$. To guarantee that $w \in H^1_{\Gamma_d}(\Omega)$ we consider only those functions w whose restrictions to ℓ are a multiple of ψ_ℓ satisfying (3.15); therefore $w|_\ell = \frac{|\ell| J_\ell}{c_\ell}\psi_\ell$.

We shall introduce some notation in order to define w in the interior of each triangle T. Let $\mathbf{r} := (\Pi_T f, J_{\ell_1}, J_{\ell_2}, J_{\ell_3}) \in \mathbf{R}^4$ and $\mathbf{D} := \mathrm{diag}(|T|, \frac{\ell_1}{\sqrt{2}}, \frac{\ell_2}{\sqrt{2}}, \frac{\ell_3}{\sqrt{2}})$; then $\eta_T^2 = \mathbf{r}^t\mathbf{D}^2\mathbf{r}$. Let

$$W_{\mathbf{r}}^T := \left\{ w \in H^1(T) \ : \ \int_T w = |T|^2 (\Pi_T f) \quad \text{and} \quad w|_{\ell_i} = \frac{|\ell_i| J_{\ell_i}}{c_{\ell_i}} \psi_{\ell_i}, \ i = 1,2,3 \right\} ;$$

$W_{\mathbf{r}}^T$ is an affine subspace of $H^1(T)$ parallel to the subspace

$$K_{\mathbf{r}}^T := \left\{ w \in H^1(T) \ : \ \int_T w = 0 \quad \text{and} \quad w|_{\ell_i} = 0, \ i = 1,2,3 \right\} .$$

Let $w_{\mathbf{r}}^T \in W_{\mathbf{r}}^T$ be such that

$$\int_T |\nabla w_{\mathbf{r}}^T|^2 = \min_{w \in W_{\mathbf{r}}^T} \int_T |\nabla w|^2 ; \tag{4.4}$$

then $w_{\mathbf{r}}^T$ satisfies (3.14), (3.15) and (3.16) with a constant

$$\mathbf{C}'_T = \left(\sup_{\mathbf{r} \neq 0} \frac{\int_T |\nabla w_{\mathbf{r}}^T|^2}{\mathbf{r}^t \mathbf{D}^2 \mathbf{r}} \right)^{\frac{1}{2}} .$$

To compute this constant we need to calculate $\int_T |\nabla w_{\mathbf{r}}^T|^2$ for any $\mathbf{r} \in \mathbf{R}^4$. Let us remark that (4.4) holds if and only if $w_{\mathbf{r}}^T \in W_{\mathbf{r}}^T$ satisfies $\int_T \nabla w_{\mathbf{r}}^T \cdot \nabla w = 0$, $\forall w \in K_{\mathbf{r}}^T$.

Let w_0 be the solution of the Dirichlet problem

$$\begin{cases} -\Delta w_0 = 1 , & \text{in } T , \\ w_0|_{\partial T} = 0 , \end{cases} \tag{4.6}$$

then

$$\int_T \nabla w_0 \cdot \nabla w = \int_T w + \int_{\partial T} \frac{\partial w_0}{\partial \mathbf{n}} w = 0 , \quad \forall w \in K_{\mathbf{r}}^T .$$

For $i = 1,2,3$, let w_i be the solution of

$$\begin{cases} -\Delta w_i = 0 , & \text{in } T , \\ w_i|_{\ell_i} = \psi_{\ell_i} , & w_i|_{\partial T \setminus \ell_i} = 0 , \end{cases} \tag{4.7}$$

then also

$$\int_T \nabla w_i \cdot \nabla w = \int_{\partial T} \frac{\partial w_i}{\partial \mathbf{n}} w = 0 \ , \quad \forall w \in K_{\mathbf{r}}^T.$$

Hence,

$$w_{\mathbf{r}}^T = c_{\mathbf{r}} w_0 + \sum_{i=1}^{3} \frac{|\ell_i| J_{\ell_i}}{c_{\ell_i}} w_i$$

with a constant $c_{\mathbf{r}}$ such that $\int_T w_{\mathbf{r}}^T = |T|^2 (\Pi_T f)$ is satisfied; that is:

$$c_{\mathbf{r}} := \frac{1}{\int_T |\nabla w_0|^2} \left[|T|^2 (\Pi_T f) - \sum_{i=1}^{3} \frac{|\ell_i| J_{\ell_i}}{c_{\ell_i}} \int_T w_i \right] \ ;$$

(we have used that, because of (4.6), $\int_T |\nabla w_0|^2 = \int_T w_0$).

Finally, because of (4.7), $\int_T \nabla w_i \cdot \nabla w_0 = 0$, and so

$$\int_T |\nabla w_{\mathbf{r}}^T|^2 = \int_T \left| \nabla \left(\sum_{i=1}^{3} \frac{|\ell_i| J_{\ell_i}}{c_{\ell_i}} w_i \right) \right|^2 + \frac{1}{\int_T |\nabla w_0|^2} \left[|T|^2 (\Pi_T f) - \sum_{i=1}^{3} \frac{|\ell_i| J_{\ell_i}}{c_{\ell_i}} \int_T w_i \right]^2 ,$$

which is a quadratic form on $\mathbf{r}$. Therefore, the computation of the constant $\mathbf{C}'_T$ by means of (4.5) reduces to a simple eigenvalue problem which can be easily solved once the solutions w_i of the Dirichlet problems (4.6) and (4.7) are known. In our computations we have also used the code PROBE to solve numerically these problems.

The function $w \in H^1_{\Gamma_d}(\Omega)$ obtained by patching together all the $w_{\mathbf{r}}^T$ for $T \in \mathcal{T}_h$, gives the best possible constants for each triangle for a given choice of the edge functions ψ_ℓ. After some experimentation we choose ψ_ℓ as quadratic functions vanishing at both ends of the edge. This choice gives constants satisfying for any triangle T:

$$3.45 \sin^{-\frac{1}{2}} \left(\frac{\alpha_T}{2} \right) \le \mathbf{C}'_T \le 5.85 \sin^{-\frac{1}{2}} \left(\frac{\alpha_T}{2} \right) \ , \tag{4.8}$$

where α_T is the minimum angle of T. Once again, $\mathbf{C}'_T$ is almost proportional to $\sin^{-\frac{1}{2}}\left(\frac{\alpha_T}{2}\right)$ and practically independent of the size of the othe angles of the triangle.

Finally, by using (4.3) and (4.8) and Theorems 3.3 and 3.5, we obtain

$$\begin{aligned} 0.171 \sin^{\frac{1}{2}} \left(\frac{\alpha}{2} \right) \ \eta_\Omega - C \left[h^2 \left(\sum_{T \in \mathcal{T}} |f|^2_{1,T} \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_{\ell \in \mathcal{L}} |g|^2_{1,\ell} \right)^{\frac{1}{2}} \right] \le |e|_{1,\Omega} \\ \le 0.813 \ \log^{\frac{1}{2}} p \ \sin^{-\frac{1}{2}} \left(\frac{\alpha}{2} \right) \ \eta_\Omega + |u - u_p|_{1,\Omega} + \ C \left[h^2 \left(\sum_{T \in \mathcal{T}} |f|^2_{1,T} \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_{\ell \in \mathcal{L}} |g|^2_{1.\ell} \right)^{\frac{1}{2}} \right] , \end{aligned} \tag{4.9}$$

where α is the minimum angle of the mesh $\mathcal{T}_h$. The bounds (4.9) can be made more accurate for specific values of the minimal angle α and of the degree $p \geq 2$; Table 4.1 shows values of the constants $\mathbf{C}'_\alpha$ and $\mathbf{C}^p_\alpha$ for the estimate

$$\mathbf{C}'_\alpha\, \eta_\Omega - C\left[h^2\left(\sum_{T\in\mathcal{T}} |f|^2_{1,T}\right)^{\frac{1}{2}} + h^{\frac{3}{2}}\left(\sum_{\ell\in\mathcal{L}} |g|^2_{1,\ell}\right)^{\frac{1}{2}}\right] \leq |e|_{1,\Omega}$$

$$\leq \mathbf{C}^p_\alpha\, \eta_\Omega + |u - u_p|_{1,\Omega} + C\left[h^2\left(\sum_{T\in\mathcal{T}} |f|^2_{1,T}\right)^{\frac{1}{2}} + h^{\frac{3}{2}}\left(\sum_{\ell\in\mathcal{L}} |g|^2_{1.\ell}\right)^{\frac{1}{2}}\right],$$

for different values of α and p.

α	$\mathbf{C}'_\alpha$	$\mathbf{C}^p_\alpha$			
		$p=2$	$p=4$	$p=6$	$p=8$
7.5°	0.051	2.390	3.306	3.660	3.988
15.0°	0.072	1.682	2.341	2.609	2.839
22.5°	0.087	1.363	1.918	2.156	2.343
30.0°	0.099	1.169	1.670	1.895	2.058
37.5°	0.108	1.035	1.508	1.727	1.876
45.0°	0.115	0.939	1.400	1.615	1.757
52.5°	0.119	0.875	1.334	1.547	1.684
60.0°	0.121	0.850	1.309	1.522	1.657

Table 4.1. Constants of equivalence.

5. Sharpness of the bounds. We shall analyize the sharpness of the estimates obtained in the previous section by considering a simple example. In particular, we shall show that the dependence of these bounds on the geometry of the mesh is optimal.

Let us consider a particular case of problem (2.1) where Ω is a rectangle as in Figure 5.1, Γ_d consist of the two vertical edges of Ω and Γ_n of the horizontal ones; let f be a constant and $g = 0$. The solution is a quadratic polynomial in x (and it does not depend on y). Let $\mathcal{T}_h$ be a family of uniform meshes like that in Figure 5.1.

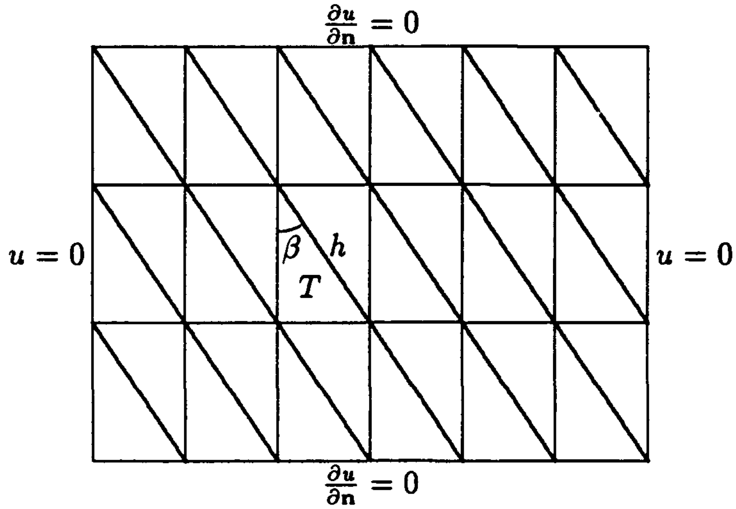


Figure 5.1

Since the solution is quadratic and the Neumann boundary conditions are zero, for any of these meshes the finite element approximation is exact at the nodes. Therefore, it is possible to compute explicitly the true error and the estimator. The error is the same for all the triangles; it only depends on the meshsize h and on the angle β which measures the regularity of the mesh (see Fig. 5.1). For all the elements disjoint with Γ_d the estimator is also the same; for those elements with an edge ℓ on the boundary Γ_d, the estimator will be smaller since, according to (2.3), the corresponding "jump" $J_\ell = 0$. However, since the proportion of the elements with an edge on Γ_d goes to zero when the mesh is refined, the global effectivity index is in this case, asymptotically equal to the local one $\mathrm{eff}_T := \frac{\eta_T}{|e|_{1,T}}$.

An explicit computation gives $\mathrm{eff}_T^2 = 18\cot\beta$. Let α denote, as before, the smallest angle of the mesh. If $\beta \leq \frac{\pi}{4}$ (as in Fig. 5.1), then $\alpha = \beta$ and it is simple to prove that for this problem

$$\mathrm{eff}_T > 2.62\sin^{-\frac{1}{2}}\left(\frac{\alpha}{2}\right) . \tag{5.1}$$

On the other hand, if $\beta > \frac{\pi}{4}$, the smallest angle is $\alpha = \frac{\pi}{2} - \beta$ and in this case

$$\mathrm{eff}_T < 6.86\sin^{\frac{1}{2}}\left(\frac{\alpha}{2}\right) . \tag{5.2}$$

Since f and g are constant and u_2 coincides with u, then (4.9) gives for this problem:

$$1.47\sin^{\frac{1}{2}}\left(\frac{\alpha}{2}\right) \leq \mathrm{eff} \leq 5.84\sin^{-\frac{1}{2}}\left(\frac{\alpha}{2}\right) . \tag{5.3}$$

The effectivity indexes (5.1) and (5.2) corresponding to different meshes show the sharpness of the bounds in (5.3) and the optimality of the terms $\sin^{\pm\frac{1}{2}}\left(\frac{\alpha}{2}\right)$ for their dependence on the regularity of the mesh.

6. The elasticity problem. We shall show how the techniques described above can be applied to a different problem. Let us consider the 2D linear elastic equations; let Ω, Γ_n, Γ_d, $\mathbf{n}$, $\mathcal{T}_h$, Γ_i, $\mathcal{T}$ and $\mathcal{L}$ be as in Sections 2 and 3; let $\mathbf{H}^1_{\Gamma_d}(\Omega) := \{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{v}|_{\Gamma_d} = 0\}$ be the space of admisssible displacements; let $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma} : H^1(\Omega)^2 \to \mathbf{R}^{2\times 2}$ be the strain and stress tensors defined by:

$$\varepsilon_{ij}(\mathbf{v}) := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) , \quad i,j = 1,2$$

and

$$\sigma_{ij}(\mathbf{v}) := \lambda \sum_{k=1}^{2} \varepsilon_{kk}(\mathbf{v})\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{v}) , \quad i,j = 1,2 ,$$

where λ and μ are the Lame coefficients that depend on the Young modulus E and the Poisson's ratio ν of the material:

$$\lambda := \frac{E\nu}{(1+\nu)(1-2\nu)} , \quad \mu := \frac{E}{2(1+\nu)} , \qquad E > 0 , \quad 0 < \nu < \frac{1}{2} .$$

Given a body force $\mathbf{f} \in L^2(\Omega)^2$ and a prescribed traction $\mathbf{g} \in L^2(\Gamma_n)^2$ with components locally smooth as described in Theorem 3.2, let $\mathbf{u}$ be the solution of the boundary value problem:

$$(6.1) \qquad \begin{cases} -(\lambda+\mu)\nabla(\mathrm{div}\mathbf{u}) - \mu\Delta\mathbf{u} = \mathbf{f} , & \text{in } \Omega , \\ \mathbf{u} = 0 , \quad \text{on } \Gamma_d , \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{g} , \quad \text{on } \Gamma_n ; \end{cases}$$

For $\mathbf{v}$ and $\mathbf{w} \in \mathbf{H}^1_{\Gamma_d}(\Omega)$ let

$$a(\mathbf{v},\mathbf{w}) := \int_\Omega \sum_{i,j=1}^{2} \sigma_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{w}) ;$$

a is a continuous symmetric bilinear form. By using Korn's inequality (for instance, see [20]), it is proved that a is coercive and so, the energy norm $\|\cdot\|_\Omega := a(\cdot,\cdot)^{\frac{1}{2}}$ is equivalent to the usual Sobolev norm $\|\cdot\|_{1,\Omega}$ on $\mathbf{H}^1_{\Gamma_d}(\Omega)$. Problem (6.1) has a unique solution $\mathbf{u} \in \mathbf{H}^1_{\Gamma_d}(\Omega)$ and it satisfies the weak formulation of this problem:

$$(6.2) \qquad a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\Gamma_n} \mathbf{g}\cdot\mathbf{v} , \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_d}(\Omega) .$$

Let $\mathbf{u}_h \in \mathbf{V}_h := \{\mathbf{v} \in \mathbf{H}^1_{\Gamma_d}(\Omega) : \mathbf{v}|_T \in \mathcal{P}_1(T)^2, \ \forall T \in \mathcal{T}_h\}$ be the piecewise linear finite element approximate solution of problem (6.2). Proceeding as in Section 2, it is proved that the error $\mathbf{e} := \mathbf{u} - \mathbf{u}_h$ satisfies the residual equation:

$$a(\mathbf{e},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\Gamma_n} [\mathbf{g} - \boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}]\cdot\mathbf{v} + \sum_{\ell\subset\Gamma_i} \int_\ell [\![\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}]\!]_\ell \cdot \mathbf{v} , \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_d}(\Omega) .$$

For any triangle $T \in \mathcal{T}_h$ and for any edge $\ell \in \Gamma_n$, let $\Pi_T \mathbf{f}$ and $\Pi_\ell \mathbf{g}$ be the local projections of the data defined as before; let

$$\mathbf{J}_\ell := \begin{cases} [\![\sigma(\mathbf{u})\mathbf{n}]\!]_\ell , & \text{if } \ell \subset \Gamma_i , \\ 2\left\{\Pi_\ell \mathbf{g} - [\sigma(\mathbf{u}_h)\mathbf{n}]\,|_\ell\right\} , & \text{if } \ell \subset \Gamma_n , \\ 0 , & \text{if } \ell \subset \Gamma_d , \end{cases}$$

and let

$$\eta_T := \left[|T|^2 |\Pi_T \mathbf{f}|^2 + \frac{1}{2} \sum_{\ell \in E_T} |\ell|^2 |\mathbf{J}_\ell|^2 \right]^{\frac{1}{2}} .$$

The proofs of the theorems in Section 3 can be immediately extended to this problem. Let $\mathbf{u}_p \in \left\{ \mathbf{v} \in \mathbf{H}^1_{\Gamma_d}(\Omega) : \mathbf{v}|_T \in \mathcal{P}_p(T)^2,\ \forall T \in \mathcal{T}_h \right\}$ be the approximate finite element solution of problem (6.2) in this space and, for any $U \subset \Omega$, let $\|\cdot\|_U^2 := \int_U \sum_{i,j=1}^2 \sigma_{ij}(\cdot)\varepsilon_{ij}(\cdot)$.

THEOREM 6.1. *With the definitions and assumptions introduced above*

$$\begin{aligned} \|\mathbf{e}\|_\Omega \le & \left[\sum_{T \in \mathcal{T}_h} \left(\mathbf{C}_T^p\right)^2 \eta_T^2 \right]^{\frac{1}{2}} + \|\mathbf{u} - \mathbf{u}_p\|_\Omega \\ & + C \left[h^2 \left(\sum_{T \in \mathcal{T}} |\mathbf{f}|_{1,T}^2 \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_{\ell \in \mathcal{L}} |\mathbf{g}|_{1,\ell}^2 \right)^{\frac{1}{2}} \right] , \end{aligned} \tag{6.3}$$

where

$$\left(\mathbf{C}_T^p\right)^2 := \sup_{\mathbf{v} \in \mathcal{P}_p^2 \,:\, \|\mathbf{v}\|_T \neq 0} \frac{\left| \frac{1}{|T|} \int_T (\mathbf{v} - \mathbf{v}^I) \right|^2 + \frac{1}{2} \sum_{\ell \in E_T} \left| \frac{1}{|\ell|} \int_\ell (\mathbf{v} - \mathbf{v}^I) \right|^2}{\|\mathbf{v}\|_T^2} . \tag{6.4}$$

□

THEOREM 6.2. *Let* $\mathbf{w} \in \mathbf{H}^1_{\Gamma_d}(\Omega)$ *be such that for all the triangles* $T \in \mathcal{T}_h$,

$$\int_T (\Pi_T \mathbf{f}) \cdot \mathbf{w} = |T|^2 |\Pi_T \mathbf{f}|^2 ,$$

$$\sum_{\ell \in E_T} \int_\ell \mathbf{J}_\ell \cdot \mathbf{w} = \sum_{\ell \in E_T} |\ell|^2 \mathbf{J}_\ell^2$$

and

$$\exists \mathbf{C}'_T > 0 \ :\ \|\mathbf{w}\|_T \le \mathbf{C}'_T \eta_T ,$$

where $\mathbf{C}'_T$ may depend on the shape of the triangle but not on its size h_T. Then

$$\eta_\Omega \leq \left(\sup_{T\in\mathcal{T}_h} \mathbf{C}'_T\right)\left\{\|\mathbf{e}\|_\Omega + C\left[h^2\left(\sum_{T\in\mathcal{T}} |\mathbf{f}|^2_{1,T}\right)^{\frac{1}{2}} + h^{\frac{3}{2}}\left(\sum_{\ell\in\mathcal{L}} |\mathbf{g}|^2_{1,\ell}\right)^{\frac{1}{2}}\right]\right\} .$$

□

The constants $\mathbf{C}^p_T$ and $\mathbf{C}'_T$ can be computed by techniques analogous to those in Section 4. They depend very weakly on the Poisson's ratio ν. The values of $\mathbf{C}'_T$ are almost proportional to $\sin^{-\frac{1}{2}}(\frac{\alpha_T}{2})$ (α_T the minimum angle of T) as for the Laplace equation. Instead, for any fixed degree $p \geq 2$, our computations show that $\mathbf{C}^p_T$ are almost proportional to $\sin^{-\frac{3}{2}}(\frac{\alpha_T}{2})$; the exponent $-\frac{3}{2}$ indicates a much stronger dependence on the regularity of the mesh.

Remark 6.1. The increase of the factor $\sin^{-\frac{1}{2}}(\frac{\alpha_T}{2})$ to $\sin^{-\frac{3}{2}}(\frac{\alpha_T}{2})$ is due to the constant in Korn's inequality. Let us show it in the case that T is a triangle as that in Figure 6.1.

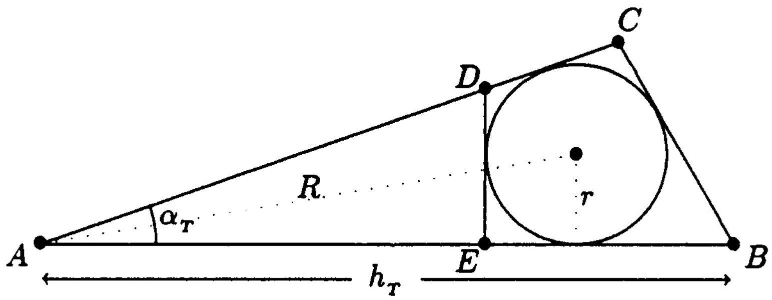


Figure 6.1

In [22] it is shown that, for any function $\mathbf{v} \in H^1(T)^2$,

$$|\mathbf{v}|^2_{1,T} \leq C_1\left(\frac{h_T}{r}\right)^2 \log\left(\frac{4h_T}{r}\right)\|\mathbf{v}\|^2_T + C_2\left(\frac{h_T}{r}\right)^2 |\mathbf{v}|^2_{1,Q} , \tag{6.5}$$

where h_T is the diameter of T, Q is the biggest circle contained in T and r is the length of its radius (see Fig 6.1). The estimate (6.5) is optimal.

If $|\mathbf{v}|^2_{1,Q}$ were used in the denominator of (6.4) instead of $\|\mathbf{v}\|^2_T$, the term $\sin^{-\frac{1}{2}}(\frac{\alpha_T}{2})$ would appear as in (4.3). On the other hand, for functions $\tilde{\mathbf{v}} \in \mathcal{P}_p(T)^2$ with three degrees of freedom fixed at the vertexes B, C (to avoid rigid motions),

$$\frac{|\tilde{\mathbf{v}}|_{1,Q}}{\|\tilde{\mathbf{v}}\|_T} \leq \frac{|\tilde{\mathbf{v}}|_{1,Q}}{\|\tilde{\mathbf{v}}\|_{\tilde{Q}}} \leq C_3 ,$$

where $\widetilde{Q}$ is the quadrilateral of vertexes B, C, D, E in Figure 6.1; the constant C_3 depends on the regularity of $\widetilde{Q}$ (i.e.: on the quotient $\frac{\mathrm{diam}(\widetilde{Q})}{r}$), but not on the small angle α_T. Since for α_T small,

$$\sin\left(\frac{\alpha_T}{2}\right) = \frac{r}{R} \approx \frac{r}{h_T} ,$$

then, for these functions we have that $\frac{|\tilde{\mathbf{v}}|_{1,T}}{\|\tilde{\mathbf{v}}\|_T}$ is bounded by $\sin^{-1}(\frac{\alpha_T}{2})$ (neglecting in (6.5) the logarithmic term). Therefore, since in (6.4) the supremum can be taken over these functions $\tilde{\mathbf{v}}$, we can expect $\mathbf{C}'_T$ to be proportional to $\sin^{-\frac{3}{2}}(\frac{\alpha_T}{2})$. □

The following table gives the values of the constants $\mathbf{C}'_\alpha$ and $\mathbf{C}^p_\alpha$ in the estimate

$$\mathbf{C}'_\alpha\ \eta_\Omega - C\left[h^2\left(\sum_{T\in\mathcal{T}} |\mathbf{f}|^2_{1,T}\right)^{\frac{1}{2}} + h^{\frac{3}{2}}\left(\sum_{\ell\in\mathcal{L}} |\mathbf{g}|^2_{1,\ell}\right)^{\frac{1}{2}}\right] \le \|\mathbf{e}\|_\Omega$$

$$\le \mathbf{C}^p_\alpha\ \eta_\Omega + \|u - u_p\|_\Omega + C\left[h^2\left(\sum_{T\in\mathcal{T}} |\mathbf{f}|^2_{1,T}\right)^{\frac{1}{2}} + h^{\frac{3}{2}}\left(\sum_{\ell\in\mathcal{L}} |\mathbf{g}|^2_{1.\ell}\right)^{\frac{1}{2}}\right] ,$$

in terms of the minimum angle α, for $p = 2$ and for different values of the Poisson's ratio.

	$\nu = .15$		$\nu = .30$		$\nu = .45$	
α	$\mathbf{C}'_\alpha$	$\mathbf{C}^p_\alpha$	$\mathbf{C}'_\alpha$	$\mathbf{C}^p_\alpha$	$\mathbf{C}'_\alpha$	$\mathbf{C}^p_\alpha$
7.5°	0.042	31.34	0.038	30.31	0.022	28.48
15.0°	0.060	11.10	0.054	10.81	0.032	10.26
22.5°	0.075	6.05	0.066	5.96	0.040	5.74
30.0°	0.088	3.93	0.078	3.92	0.047	3.85
37.5°	0.100	2.81	0.089	2.84	0.054	2.85
45.0°	0.113	2.12	0.101	2.19	0.061	2.23
52.5°	0.125	1.67	0.113	1.75	0.069	1.82
60.0°	0.136	1.36	0.124	1.44	0.077	1.52

Table 6.1. Constants of equivalence for different Poisson's ratios.

7. Conclusions and computational aspects..

1. The error estimator can either underestimate or overestimate the true error. If the solution is unsmooth the accuracy of the estimator could deteriorate (but not drastically— we have to consider a higher degree p in (3.12) and the deterioration is logarithmic)

2. The main factor in the accuracy of the estimator is the geometry of the elements. The geometry (angle α) has to be understood in conection with the differential equation. For example when an elliptic differential operator $\sum_{i,j=1,2} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$ (a_{ij} constants) is

considered, the equation can be transformed into the Laplace equation by an affine transformation which will modify the angles of the triangles. The constants arising in this case are those of the transformed mesh.

3. The accuracy of the estimator depends on the relation of the axes of anisotropy of the solution (i.e.: the eigenvectors of its Hessian matrix) and the orientation of the triangles. If the main axe and the orientation of the triangles are orthogonal, the error is overestimated; instead, it is underestimated if they are parallel.

4. The estimates we derived are theoretical and they allow us to define correction factors; for example, for the Laplace equation and a uniform mesh of equilateral triangles we can use (from table 4.1) $\sqrt{(0.121 \cdot 0.850)} \approx 0.32$. If we rather needed a safe estimator we should use a greater corrector factor (say 1.5).

5. For the elasticity equations, our estimates show a larger sensitivity with respect to the minimal angle. This effect grows for larger Poisson's ratio.

6. In practice, the bounds on the effectivity index are expected to be better than in our theoretical analysis. However, (5.1) and (5.2) show that they cannot be much better without additional restrictions. Of course, the examples yielding (5.1) and (5.2) are more or less extreme cases. For a detailed computational analysis we refer to [4].

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