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Exact Performance of Filtered and Weighted Energy Detector with Mismatched Frequency and Time Locations and Characteristics

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PREFACE

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The characteristic function of the output of a filtered and weighted energy detector, subjected to a nonstationary Gaussian input signal in the presence of colored Gaussian noise, is derived in closed form. This result allows and accounts for mismatch in the receiving filter frequency characteristics relative to the received signal and noise spectra. It also allows for uncertainty in the time location and/or duration of the received signal, by way of arbitrary time weighting of the squared filter output.

Programs for evaluation of the receiver operating characteristics, for white noise inputs as well as colored noise inputs, are furnished in BASIC and exercised for a number of examples. This approach allows for precise quantitative investigation of the degradation associated with filtered energy detection in the presence of uncertainties.

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14. SUBJECT TERMS (continued)

colored input noise
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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

Δ	time sampling increment, figure 1
k,m	integers denoting discrete time, figure 1 and (1)
$x(k\Delta)$	digital input sequence, figure 1 and (1)
$h(m\Delta)$	digital filter, figure 1 and (5)
$y(k\Delta)$	filter output sequence, figure 1 and (5)
$w(k\Delta)$	summer weights, figure 1, (6) and (7)
z	system output, figure 1 and (6)
$n(k\Delta)$	input noise sequence, (1)
$s(k\Delta)$	input signal sequence, (1)
$g(k\Delta)$	signal gating sequence, (1) and (4)
overbar	ensemble average, (2)
$R_n(m\Delta)$	input noise covariance, (2)
$R_s(m\Delta)$	input signal covariance, (3)
k_a	input signal start time integer, (4)
k_b	input signal stop time integer, (4)
$y_s(k\Delta)$	filter output signal, (5) and (13)
$y_n(k\Delta)$	filter output noise, (5) and (8)
k_c	observation start time integer, (7)
k_d	observation stop time integer, (7)
$k_1\Delta, k_2\Delta$	general covariance times, (9)
C_n	filter output noise covariance, (9)
$\phi_h(j\Delta)$	filter correlation function, (10)
σ_n^2	input noise power, (11) and (45)
C_s	filter output signal covariance, (15)
K_1, K_2	auxiliary integers, (18)

K_o	auxiliary integer, (19)
C_y	covariance of total filter output, (23)
a_k	auxiliary random variables, (24)
C_a	covariance of $\{a_k\}$, (26)
C	matrix with elements (26)
Q	normalized modal matrix of eigenvectors of C, (27)
Λ	diagonal matrix of eigenvalues $\{\lambda_k\}$ of C, (27)
T	transpose of matrix, (27)
A	column vector of $\{a_k\}$, (28)
B	linear transformation of A, (29)
$\{b_k\}$	elements of B, (30)
$f_z(\xi)$	characteristic function of output z, (32)
τ	argument (delay) of correlation functions, (33)
J	number of input signal covariance components, (33)
α_j	input signal scaling, (33)
τ_{sj}	input signal time constant, (33)
σ_s^2	input signal power, (34)
f	frequency, (35)
$G_s(f)$	input signal spectrum, (35)
ω	$2\pi f$, (35)
a_j	input signal exponential parameter, (36)
M	number of input noise covariance components, (44)
β_m	input noise scaling, (44)
τ_{nm}	input noise time constant, (44)
b_m	input noise exponential parameter, (46)

N	number of filter components, (47)
ψ_n	filter scaling, (47)
τ_{hn}	filter time constant, (47)
c_n	filter exponential parameter, (49)
ν_n	auxiliary filter constant, (51)
$\underline{H}(z)$	filter z transform, (52)
$\oplus$	convolution, (55) and (A-2)
$C(k\Delta)$	auxiliary function, (56)
μ_m, γ_n	auxiliary constants, (60)
S	auxiliary function, (65) and (69)
g	auxiliary function, (68)
$f_o(\xi)$	characteristic function for noise only, (73)
$p_o(u)$	probability density function for noise only, (73)
Δ_o	sampling increment in ξ applied to $f_o(\xi)$, (73)
$\tilde{p}_o(u)$	approximation to $p_o(u)$, (73) and (74)
N_o	size of FFT for $\tilde{p}_o(u)$, (74)
δ_o	increment in u for $\tilde{p}_o(u)$, (74) and (75)
u_o	maximum useful u for $\tilde{p}_o(u)$, (75)
$f_1(\xi)$	characteristic function for signal present, (76)
$p_1(u)$	probability density function for signal present, (76)
Δ_1	sampling increment in ξ applied to $f_1(\xi)$, (76)
$\tilde{p}_1(u)$	approximation to $p_1(u)$, (76) and (77)
N_1	size of FFT for $\tilde{p}_1(u)$, (77)
δ_1	increment in u for $\tilde{p}_1(u)$, (77) and (78)
u_1	maximum useful u for $\tilde{p}_1(u)$, (78)
$E_k(u)$	exceedance distribution corresponding to $f_k(\xi)$, (80)
Im	imaginary part, (80)

P_D	detection probability
P_F	false alarm probability
CDF	cumulative distribution function
EDF	exceedance distribution function
R	input signal to noise ratio σ_s^2/σ_n^2, (82)
$\delta_b(\tau)$	unit impulse train with spacing b, (A-2)

EXACT PERFORMANCE OF FILTERED AND WEIGHTED ENERGY DETECTOR WITH MISMATCHED FREQUENCY AND TIME LOCATIONS AND CHARACTERISTICS

INTRODUCTION

The detection of weak random signals of unknown duration, bandwidth, time location, and frequency shift in the presence of colored noise is often accomplished in practice by filtering the received waveform to the frequency band of interest, squaring the filter output, and time-weighting this quantity prior to accumulation and threshold comparison. The performance of this processor obviously depends on the spectra of the received signal and noise processes as well as on the transfer characteristics of the filter employed. Lack of detailed knowledge of the signal spectral characteristics or frequency shift will often dictate that a fairly broad (mismatched) filter passband be utilized in order that the signal be passed when present. Similarly, uncertainty about the signal location or duration will mandate that the receiver observation time be lengthened in order to ensure capture of any impingent signal energy.

Additionally, the received signal process is often characterized as a finite-duration burst of a stationary process. Since the signal time location and duration are often unknown, the receiving filter must be kept open at all times, thereby allowing the filter output noise to reach its full-strength steady state value. By contrast, a gated input signal leads to a filter output component which gradually builds up in time and then decays slowly after the signal is turned off. These nonstationary effects, coupled with the particular time weighting employed during the observation interval, influence the performance of the detector in a complicated nonobvious fashion.

There is a need to be able to calculate exactly the performance of this type of signal processor in the presence of such deleterious factors, so that the degradations associated with lack of knowledge can be ascertained quantitatively. Furthermore, it will not suffice to resort to Gaussian approximations for the processor output, because the number of samples employed are not large enough to utilize the central limit theorem, especially on the tails of the distribution (small false alarm probabilities) where we are interested. This need is addressed in this report for the case of Gaussian input signals in Gaussian noise, with the result that programs are furnished for exactly assessing the performance of the mismatched energy detector for a very wide variety of characteristics and selections of parameters. In particular, no presumptions are made about the input signal time-bandwidth product or about the size of the product of observation time and filter bandwidth.

SYSTEM DESCRIPTION AND ASSUMPTIONS

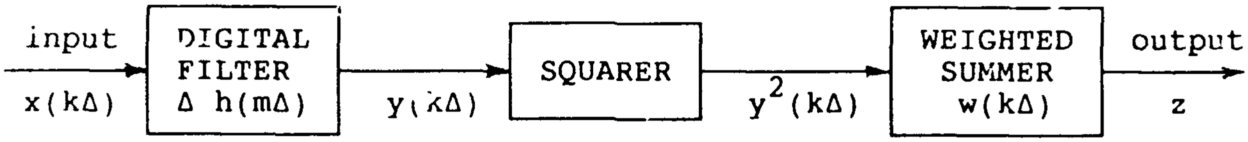

Figure 1. Block Diagram of Detector

The detector of interest is the digital processor depicted in figure 1; time sampling increment Δ is arbitrary but should be approximately matched to the coherence times of the input signal and filter. The digital input sequence $x(k\Delta)$, filter impulse response $\Delta\, h(m\Delta)$, and summer weights $w(k\Delta)$ are all real. Input $x(k\Delta)$ consists of either noise-alone or gated signal plus noise, according to model

$$x(k\Delta) = \left\{ \begin{matrix} n(k\Delta) \\ \text{or} \\ s(k\Delta)\; g(k\Delta) + n(k\Delta) \end{matrix} \right\} \quad \text{for all } k \; . \tag{1}$$

Input noise $n(k\Delta)$ is present for all time and is a stationary zero-mean discrete Gaussian sequence with covariance

$$R_n(m\Delta) = \overline{n(k\Delta)\; n(k\Delta - m\Delta)} \quad \text{for all } m,k \; , \tag{2}$$

where an overbar denotes an ensemble average. The numerical evaluation of the values of the noise covariance, directly from a specified noise spectrum $G_n(f)$, is considered in appendix A.

Underlying input signal $s(k\Delta)$ in (1) (if present) is also a stationary zero-mean discrete Gaussian sequence with covariance

$$R_s(m\Delta) = \overline{s(k\Delta)\, s(k\Delta-m\Delta)} \quad \text{for all } m,k \, . \tag{3}$$

However, input signal samples $s(k\Delta)$ are gated by function $g(k\Delta)$ which is nonzero only in a limited time region:

$$g(k\Delta) = 1 \quad \text{for } k_a \le k \le k_b \text{ , zero otherwise .} \tag{4}$$

This results in a gated burst of stationary signal sequence $s(k\Delta)$ being inputted to digital filter $h(m\Delta)$; the input signal starting and ending times $k_a\Delta$ and $k_b\Delta$, respectively, are generally unknown, except in an approximate way. This generality allows for consideration of input signals of unknown arrival time and duration at the detector input.

The filter output $y(k\Delta)$ can be conveniently broken into signal and noise components, in accordance with (1), and is available by means of discrete convolution

$$y(k\Delta) = \sum_m \Delta\, h(m\Delta)\, x(k\Delta - m\Delta) = y_s(k\Delta) + y_n(k\Delta) \, , \tag{5}$$

when an input signal is present. (Summations without limits are over $(-\infty,+\infty)$.) However, due to the gating in (1), filter output signal $y_s(k\Delta)$ will have transient phases, including a buildup just after time $k_a\Delta$ and a decay just after time $k_b\Delta$. This nonstationary behavior is included and exactly accounted for in the following analysis.

The filter output (5) is then squared, scaled by weights $w(k\Delta)$, and summed to give system output

$$z = \sum_k w(k\Delta)\ y^2(k\Delta)\ , \tag{6}$$

where the weights are nonzero only in a limited observation time specified by

$$w(k\Delta) \neq 0 \quad \text{for } k_c \le k \le k_d\ . \tag{7}$$

These weights need not be uniform; for example, they could be exponential. Output z is compared with a threshold for a declaration of signal absent versus signal present.

Observation time limits $k_c\Delta$ and $k_d\Delta$ are under the control of the receiver processor, and would hopefully match fairly well the time interval over which the gated input signal is received; see (4). But, in any event, the degree of generality in (7) allows for investigation of observation times that might be too short, thereby losing some signal energy, or too long, thereby picking up additional unwanted noise components; both situations lead to deterioration of the detectability of weak signals and should be avoided if possible. An illustration of the parameters is given in figure 2, for signal-only present.

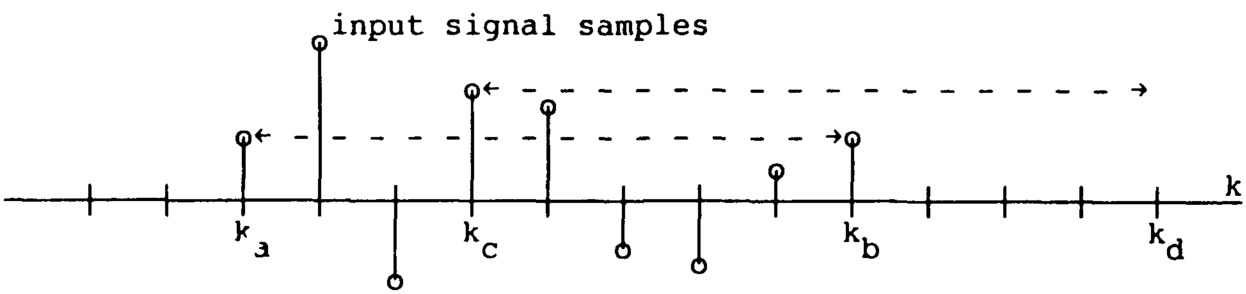

Figure 2. Gated and Observation Intervals; No Noise

FILTER OUTPUT NOISE CHARACTERIZATION

The filter output noise sequence is obtained from discrete convolution (5) and (1) as

$$y_n(k\Delta) = \Delta \sum_m h(m\Delta)\, n(k\Delta - m\Delta) \quad \text{for all } k \;, \tag{8}$$

where it is presumed that the filter $h(m\Delta)$ has been open to noise input $n(k\Delta)$ for all time; this results in noise output $y_n(k\Delta)$ being a zero-mean stationary Gaussian sequence. The filter output noise covariance at general times $k_1\Delta$ and $k_2\Delta$ is given by

$$C_n(k_1\Delta, k_2\Delta) = \overline{y_n(k_1\Delta)\; y_n(k_2\Delta)} =$$

$$= \Delta^2 \sum_{mp} h(m\Delta)\; h(p\Delta)\; \overline{n(k_1\Delta - m\Delta)\; n(k_2\Delta - p\Delta)} =$$

$$= \Delta^2 \sum_{mp} h(m\Delta)\; h(p\Delta)\; R_n(k_1\Delta - k_2\Delta + p\Delta - m\Delta) =$$

$$= \sum_j \phi_h(j\Delta)\; R_n(k_1\Delta - k_2\Delta - j\Delta) \;, \tag{9}$$

where the filter correlation function is defined as

$$\phi_h(j\Delta) = \Delta^2 \sum_m h(m\Delta)\; h(m\Delta - j\Delta) \quad \text{for all } j \;. \tag{10}$$

The right-hand side of (9) is a function only of the time difference $k_1\Delta - k_2\Delta$, in keeping with the stationarity of filter output noise $y_n(k\Delta)$.

If the input noise in figure 1 is white, its covariance in (2) becomes

$$C_n(m\Delta) = \sigma_n^2 \, \delta_{mo} \; , \tag{11}$$

where σ_n^2 is the input noise power. In this case, the filter output noise covariance in (9) simplifies to

$$C_n(k_1\Delta, k_2\Delta) = \sigma_n^2 \, \phi_h(k_1\Delta - k_2\Delta) \quad \text{for white input noise} \; . \tag{12}$$

The numerical evaluation of filter correlation function $\phi_h(j\Delta)$ in (10), directly from a specified transfer function H(f), is considered in appendix B. A possible problem with discontinuous functions is treated in appendix C.

FILTER OUTPUT SIGNAL CHARACTERIZATION

The filter output signal sequence (when present) follows from (5) and (1) as

$$y_s(k\Delta) = \Delta \sum_m h(k\Delta - m\Delta)\ s(m\Delta)\ g(m\Delta) \quad \text{for all } k\ . \qquad (13)$$

The finite duration of gating sequence $g(k\Delta)$ in (4), as well as the realizability of impulse response $\Delta\ h(m\Delta)$, will serve to terminate the summation in (13) at finite limits.

We presume that filter $h(m\Delta)$ in figure 1 is realizable; that is,

$$h(m\Delta) = 0 \quad \text{for } m < 0\ . \qquad (14)$$

This makes filter output signal $y_s(k\Delta)$ in (13) equal to zero for $k < k_a$. On the other hand, if the filter has an infinite-duration impulse response $\Delta\ h(m\Delta)$, then $y_s(k\Delta)$ is nonzero for $k \geq k_a$; see figure 3. However, the filter output noise $y_n(k\Delta)$ will dominate the signal output for $k \gg k_b$, meaning that performance of the detector will be poor in the case when $k_d \gg k_b$. That is, too long an observation time in (7), relative to the signal duration, is detrimental to signal detectability.

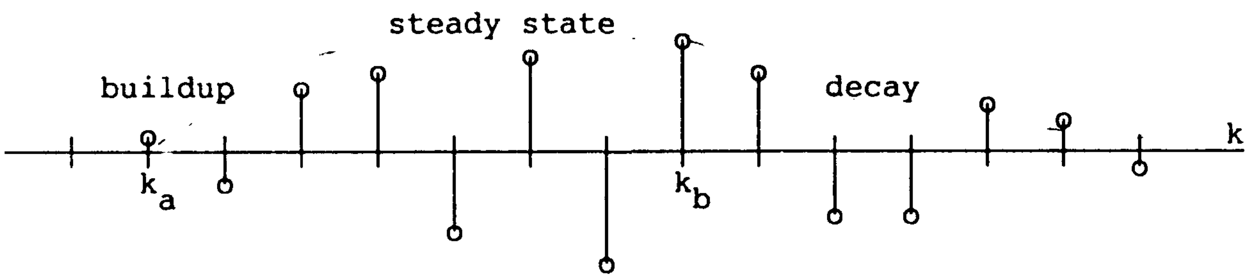

Figure 3. Filter Output Signal Buildup and Decay

The signal output, $y_s(k\Delta)$ in (13), from filter $h(m\Delta)$ is <u>not</u> stationary. The filter output signal covariance is defined as

$$C_s(k_1\Delta,k_2\Delta) = \overline{y_s(k_1\Delta)\; y_s(k_2\Delta)} = C_s(k_2\Delta,k_1\Delta) \quad \text{for all } k_1,k_2. \qquad (15)$$

This function is zero for $k_1 < k_a$ or $k_2 < k_a$; therefore, in the following, we can confine calculation of (15) to $k_1 \geq k_a$ and $k_2 \geq k_a$. The particular filter output signal sample $y_s(k_a\Delta)$ is nonzero only if $h(0) \neq 0$.

Substitution of (13) in (15) and the use of (3) yields

$$C_s(k_1\Delta,k_2\Delta) = \Delta^2 \sum_{mp} h(k_1\Delta-m\Delta)\; h(k_2\Delta-p\Delta)\; \overline{s(m\Delta)\; s(p\Delta)}\; g(m\Delta)\; g(p\Delta) =$$

$$= \Delta^2 \sum_{mp} h(k_1\Delta-m\Delta)\; h(k_2\Delta-p\Delta)\; R_s(m\Delta-p\Delta)\; g(m\Delta)\; g(p\Delta) \; . \qquad (16)$$

When we take explicit account of realizability condition (14) and the finite duration of unity gating function $g(k\Delta)$ in (4), the filter output signal covariance in (16) can be efficiently computed according to

$$C_s(k_1\Delta,k_2\Delta) = \Delta^2 \sum_{m=k_a}^{K_1} h(k_1\Delta-m\Delta) \sum_{p=k_a}^{K_2} h(k_2\Delta-p\Delta)\; R_s(m\Delta-p\Delta) \; , \qquad (17)$$

where

$$K_1 = \min(k_1,k_b) \; , \quad K_2 = \min(k_2,k_b) \; . \qquad (18)$$

Since $k_b \geq k_a$, $k_1 \geq k_a$, $k_2 \geq k_a$, it follows that $K_1 \geq k_a$ and $K_2 \geq k_a$, thereby making the summations in (17) treat only nonzero entries.

SUMMER CONSIDERATIONS

The weighted summer in figure 1 and (6) - (7) takes its samples at times $k\Delta$, where $k_c \le k \le k_d$. If $k_d < k_a$, then <u>no</u> signal gets into the summer, and the signal at the input cannot be detected at all, whether present or not. Since this is not a useful application of the detector in figure 1, the only situation we will address is that where $k_d \ge k_a$, so that <u>some</u> signal (when present) contributes to the summer output.

There are then two possibilities for observation start time $k_c\Delta$, as indicated in figure 4 below. If $k_c < k_a$ (case 1), the summer is taking in noise-only samples for $k_c \le k < k_a$; this will degrade performance of the detector but is sometimes unavoidable when the signal onset time, $k_a\Delta$, is unknown. In this case, we only need to compute signal output covariance $C_s(k_1\Delta, k_2\Delta)$ for $k_a \le k_1, k_2 \le k_d$, since $C_s = 0$ for $k_1 < k_a$ or $k_2 < k_a$.

On the other hand, if $k_c \ge k_a$ (case 2), signal is taken into the summer on all samples available to it. However, if k_c is considerably larger than k_a, a significant portion of the signal contribution can be lost; this degradation of performance is

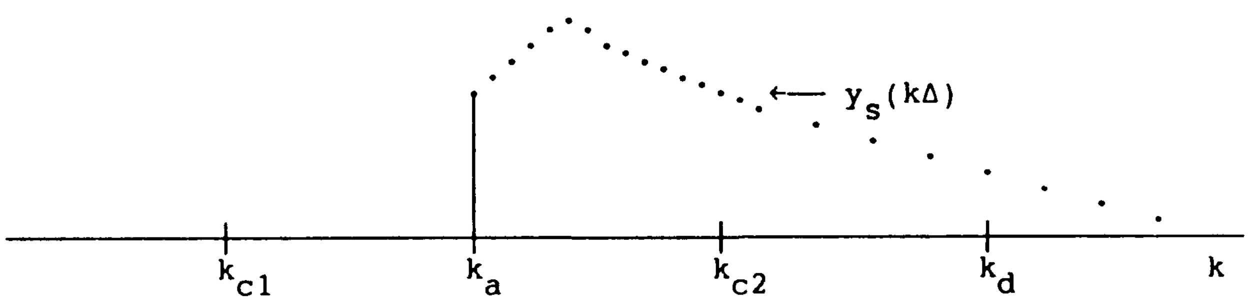

Figure 4. Filter Output Signal $y_s(k\Delta)$

sometimes unavoidable when k_a is unknown. In this latter case, we only need to compute $C_s(k_1\Delta,k_2\Delta)$ for $k_c \le k_1,k_2 \le k_d$.

The general rule is that we need to compute, from (17), the filter output signal covariance $C_s(k_1\Delta,k_2\Delta)$ for

$$K_o \equiv \max(k_a,k_c) \le k_1,k_2 \le k_d \; . \qquad (19)$$

Then, if $k_c < k_a$, use 0 for $C_s(k_1\Delta,k_2\Delta)$ for those values where $k_1 < k_a$ or $k_2 < k_a$. Of course, noise covariance $C_n(k_1\Delta,k_2\Delta)$ in (9) or (12) must be computed for $k_c \le k_1,k_2 \le k_d$ in all cases.

Since, from (19),

$$k_1 \ge \max(k_a,k_c) \ge k_a, \quad \text{then} \quad K_1 = \min(k_1,k_b) \ge k_a \; ; \qquad (20)$$

also, since

$$k_2 \ge \max(k_a,k_c) \ge k_a, \quad \text{then} \quad K_2 = \min(k_2,k_b) \ge k_a \; . \qquad (21)$$

Therefore, the sums in (17) always have some entries; that is, the upper limit is never less than the lower limit. The only restriction is

$$k_a \le k_d \; . \qquad (22)$$

Of course, we must always have $k_a \le k_b$ and $k_c \le k_d$.

In summary, the filter output signal covariance follows from (17) and (18), while the filter output noise covariance is available in (9) or (12). The region where the filter output signal covariance $C_s(k_1\Delta,k_2\Delta)$ calculation is nonzero is crosshatched in figure 5 for the case where $k_c < k_a$. If $k_c \ge k_a$, then $C_s(k_1\Delta,k_2\Delta)$ is nonzero in the larger square indicated in figure 5.

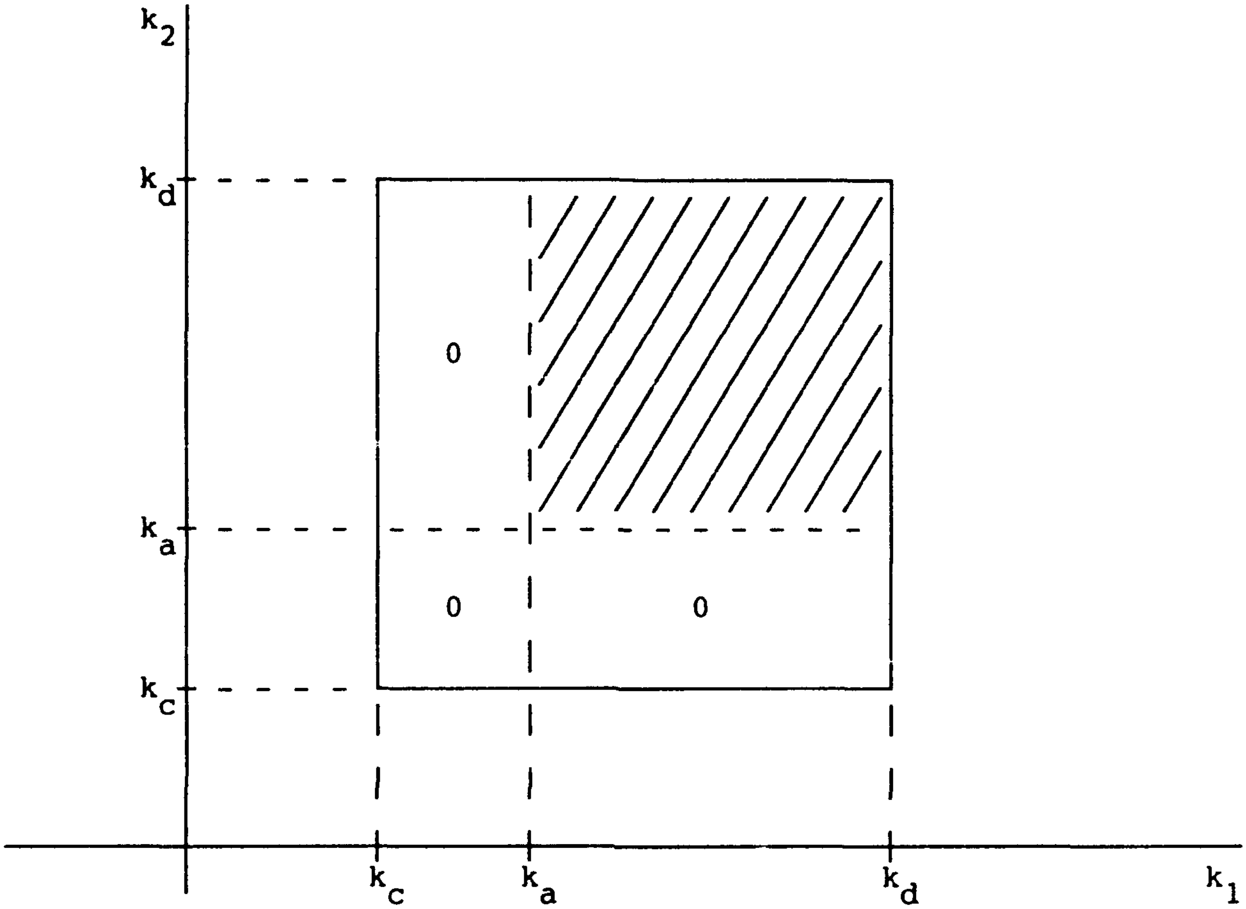

Figure 5. Calculation Region for $C_s(k_1\Delta, k_2\Delta)$

CHARACTERISTIC FUNCTION OF SYSTEM OUTPUT z

The system output z is given in (6) as a sum of weighted and squared correlated zero-mean Gaussian random variables $y(k\Delta)$. Also, the covariance of the noise component $y_n(k\Delta)$ is given by (9) or (12), while the covariance of the signal component $y_s(k\Delta)$ (if present) is given by (17) - (18) and figure 5. Therefore, the covariance of $y(k\Delta)$ is given by

$$C_y(k_1\Delta, k_2\Delta) = C_s(k_1\Delta, k_2\Delta) + C_n(k_1\Delta, k_2\Delta) \; . \tag{23}$$

The system input signal-to-noise ratio is $R_s(0)/R_n(0)$ in terms of input covariances (2) and (3).

For nonnegative weights $\{w(k\Delta)\}$, we define random variables

$$a_k = \sqrt{w(k\Delta)}\; y(k\Delta) \quad \text{for } k_c \le k \le k_d \; . \tag{24}$$

Then, (6) - (7) yields the output in the form

$$z = \sum_{k=k_c}^{k_d} a_k^2 \; , \tag{25}$$

where zero-mean Gaussian random variables $\{a_k\}$ have covariance

$$C_a(k_1, k_2) = \overline{a_{k_1} a_{k_2}} = \sqrt{w(k_1\Delta)\; w(k_2\Delta)}\; C_y(k_1\Delta, k_2\Delta) \; . \tag{26}$$

In order to find the characteristic function of random variable z in (25), we consider the square symmetric covariance matrix C with elements (26) for $k_c \le k_1, k_2 \le k_d$. Let Q be the normalized modal matrix of the orthonormal eigenvectors of C and let Λ be the diagonal matrix of the eigenvalues $\{\lambda_k\}$ of C; see [1; pages 36 - 39]. Then we have

$$Q^T Q = Q Q^T = I , \quad Q^T C Q = \Lambda . \tag{27}$$

Now let column vector A be made up of elements $\{a_k\}$ for $k_c \le k \le k_d$, as defined in (24), in which case (25) and (26) can be expressed as

$$z = A^T A , \quad C = \overline{A A^T}. \tag{28}$$

Also, let column vector B be defined by linear transformation

$$B = Q^T A ; \quad \text{then} \quad A = Q B . \tag{29}$$

Substitution of (29) into (28) yields

$$z = A^T A = B^T Q^T Q B = B^T B = \sum_{k=k_c}^{k_d} b_k^2 , \tag{30}$$

where $\{b_k\}$ are the elements of B. At the same time, the covariance matrix of vector B is

$$\overline{B B^T} = Q^T \overline{A A^T} Q = Q^T C Q = \Lambda , \tag{31}$$

where we used (29), (28), and (27). Thus, the sum for z in (30) is composed of uncorrelated (and therefore independent) zero-mean Gaussian random variables $\{b_k\}$, where the variance of element b_k is eigenvalue λ_k.

The characteristic function of random variable z in (30) is, using the independence and Gaussian character of $\{b_k\}$,

$$f_z(\xi) = \overline{\exp(i\xi z)} = \prod_{k=k_c}^{k_d} \overline{\exp\left(i\xi b_k^2\right)} =$$

$$= \prod_{k=k_c}^{k_d} \int du \ \exp(i\xi u^2) \ (2\pi\lambda_k)^{-\frac{1}{2}} \exp\left(\frac{-u^2}{2\lambda_k}\right) =$$

$$= \prod_{k=k_c}^{k_d} \left(1 - i2\xi\lambda_k\right)^{-\frac{1}{2}} . \qquad (32)$$

Although this final result is given as a finite product of k_d-k_c+1 principal-value square roots, it can be computed as a single principal-value square root of a finite product of complex first-order polynomials, provided that the location of the product in the complex plane is tracked from $\xi = 0$; for example, see [2; pages B-1 and B-2].

COVARIANCE AND FILTER EXAMPLES

INPUT SIGNAL CHARACTERIZATION

The input signal covariance will be taken to be a sum of damped exponentials:

$$R_s(\tau) = \sum_{j=1}^{J} \alpha_j \exp(-|\tau|/\tau_{sj}) \quad \text{for all } \tau \ , \tag{33}$$

where the J exponential parameters $\{\tau_{sj}\}$ are all distinct. The scalings $\{\alpha_j\}$ and time constants $\{\tau_{sj}\}$ can be complex, provided that they occur in complex conjugate pairs; that is, input signal covariance $R_s(\tau)$ must be real for all τ. The origin value,

$$R_s(0) = \sum_{j=1}^{J} \alpha_j = \sigma_s^2 \ , \tag{34}$$

is the input power of stationary signal sequence $s(k\Delta)$ in (1) and (3), prior to gating. The input signal spectrum corresponding to (33) is, with $\omega = 2\pi f$,

$$G_s(f) = \int d\tau \, \exp(-i2\pi f\tau) \, R_s(\tau) = \sum_{j=1}^{J} \frac{2\, \alpha_j \, \tau_{sj}}{1 + (\omega\, \tau_{sj})^2} \ , \tag{35}$$

which is seen to contain 2J distinct poles in the complex f-plane. $G_s(f)$ can contain up to 2J-2 zeros.

For time sampling increment Δ, the sampled input signal covariance is, from (33),

$$R_s(k\Delta) = \sum_{j=1}^{J} \alpha_j \exp(-a_j|k|) \ ; \quad a_j = \frac{\Delta}{\tau_{sj}} \ . \tag{36}$$

The J exponential parameters $\{a_j\}$ are distinct, but they can be complex; in fact, $\{\alpha_j\}$ and $\{a_j\}$ can both be complex provided that they occur in complex conjugate pairs such that $R_s(k\Delta)$ is real for all k. In fact, some of the scalings $\{\alpha_j\}$ can be negative, provided that total spectrum $G_s(f)$ in (35) is nonnegative for all f. The exponential parameters $\{a_j\}$ in (36) must also satisfy Re $a_j > 0$ in order that the covariance tend to zero at $k = \pm\infty$.

For a particular pair of complex conjugate terms, say

$$\alpha_1 = \alpha_o \ , \quad \alpha_2 = \alpha_o^* \ , \quad \tau_{s1} = \tau_o \ , \quad \tau_{s2} = \tau_o^* \ , \quad a_1 = a_o \ , \quad a_2 = a_o^* , \tag{37}$$

the corresponding part of the (continuous) input signal covariance (33) can be expressed in the alternative form

$$R_s^o(\tau) = \alpha_o \exp(-|\tau|/\tau_o) + \alpha_o^* \exp(-|\tau|/\tau_o^*) =$$

$$= 2 \exp(-\omega_r|\tau|) \ [\alpha_r \cos(\omega_i|\tau|) + \alpha_i \sin(\omega_i|\tau|)] \ , \tag{38}$$

where we have defined the real and imaginary parts

$$\alpha_o = \alpha_r + i\alpha_i \ , \qquad \frac{1}{\tau_o} = \omega_r + i\omega_i \ . \tag{39}$$

The sampled signal covariance for component (38) is

$$R_s^o(k\Delta) = 2 \exp(-a_r|k|) \ [\alpha_r \cos(a_i|k|) + \alpha_i \sin(a_i|k|)] \ , \tag{40}$$

where $a_r = \omega_r \Delta$ and $a_i = \omega_i \Delta$.

The spectral component associated with component covariance (38) is, by reference to (35) and (39),

$$G_s^o(f) = \frac{2\ \alpha_o\ \tau_o}{1 + (\omega\ \tau_o)^2} + \frac{2\ \alpha_o^*\ \tau_o^*}{1 + (\omega\ \tau_o^*)^2} =$$

$$= 2\ \mathrm{Re}\ \frac{2\ \alpha_o/\tau_o}{\omega^2 + 1/\tau_o^2} = 4\ \mathrm{Re}\ \frac{(\alpha_r + i\alpha_i)(\omega_r + i\omega_i)}{\omega^2 + (\omega_r + i\omega_i)^2} =$$

$$= 4\ \frac{\alpha_r\ \omega_r\ \left(\omega_r^2 + \omega_i^2 + \omega^2\right) + \alpha_i\ \omega_i\ \left(\omega_r^2 + \omega_i^2 - \omega^2\right)}{\left(\omega_r^2 + (\omega - \omega_i)^2\right)\left(\omega_r^2 + (\omega + \omega_i)^2\right)} =$$

$$= 2\ \frac{\alpha_r\ \omega_r - \alpha_i(\omega - \omega_i)}{\omega_r^2 + (\omega - \omega_i)^2} + 2\ \frac{\alpha_r\ \omega_r + \alpha_i(\omega + \omega_i)}{\omega_r^2 + (\omega + \omega_i)^2} = \tag{41}$$

$$= \frac{\alpha_i + i\alpha_r}{\omega + \omega_i + i\omega_r} + \frac{\alpha_i - i\alpha_r}{\omega + \omega_i - i\omega_r} + \frac{-\alpha_i + i\alpha_r}{\omega - \omega_i + i\omega_r} + \frac{-\alpha_i - i\alpha_r}{\omega - \omega_i - i\omega_r}\ .$$

If this spectral component $G_s^o(f)$ goes negative for some ranges of f, it must be accompanied by additional terms in summation (35) to keep the total spectrum $G_s(f)$ nonnegative for all frequencies f. Spectral component (41) has four poles in the complex ω-plane at locations $\omega = \omega_i \pm i\omega_r$ and $\omega = -\omega_i \pm i\omega_r$.

Even if coefficients $\{\alpha_j\}$ and $\{\tau_{sj}\}$ in (35) are purely real and the poles are distinct, oscillatory behavior of spectrum $G_s(f)$ is still attainable. For example, with

$$J = 3, \qquad \{a_j\} = 1,\ 8,\ 15, \qquad \{\alpha_j\} = 11,\ -48,\ 75, \tag{42}$$

the spectrum (35) is displayed versus $\omega\Delta$ in figure 6. Actually plotted is the scaled spectrum

$$\frac{G_s(f)}{2\Delta} = \sum_{j=1}^{J} \frac{\alpha_j \ a_j}{(\omega\Delta)^2 + a_j^2} , \tag{43}$$

where we have used (36) to eliminate $\{\tau_{sj}\}$. The negative coefficient for α_2 causes the dip near $\omega\Delta = 4$.

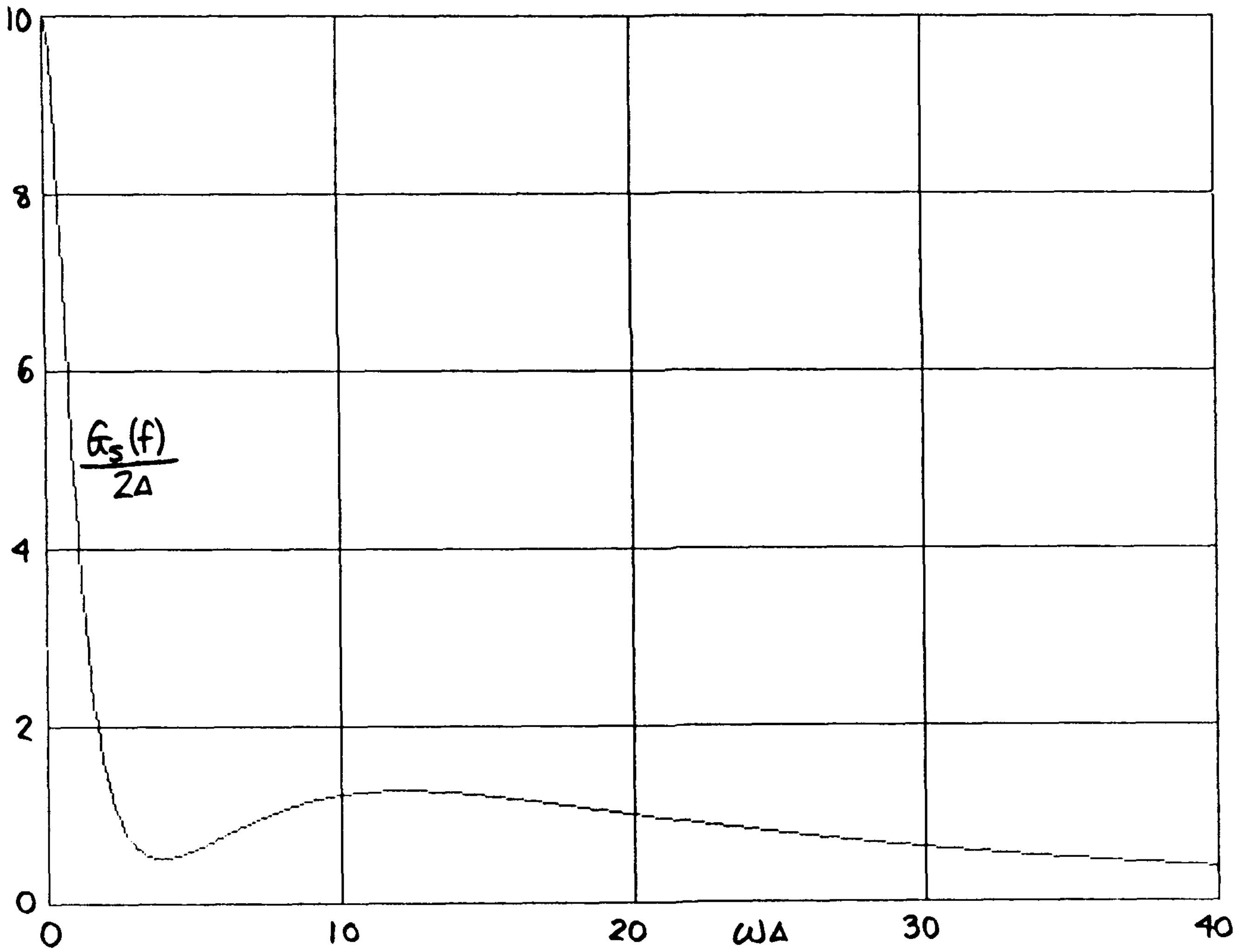

Figure 6. Spectrum (43) for Parameters (42)

INPUT NOISE CHARACTERIZATION

The presentation in this subsection exactly parallels that above for the input signal. The input noise covariance is again taken to be a sum of damped exponentials of the form

$$R_n(\tau) = \sum_{m=1}^{M} \beta_m \exp(-|\tau|/\tau_{nm}) \quad \text{for all } \tau \ , \tag{44}$$

where the M (complex) noise time constants $\{\tau_{nm}\}$ are all distinct from one another. (Some of them may equal some of the signal time constants $\{\tau_{sj}\}$ in (33), if desired.) The origin value,

$$R_n(0) = \sum_{m=1}^{M} \beta_m = \sigma_n^2 \ , \tag{45}$$

is the input power of noise sequence $n(k\Delta)$ in (1).

For time sampling increment Δ, the sampled input noise covariance is, from (44),

$$R_n(k\Delta) = \sum_{m=1}^{M} \beta_m \exp(-b_m|k|) \ ; \quad b_m = \frac{\Delta}{\tau_{nm}} \ . \tag{46}$$

The noise exponential parameters $\{b_m\}$ are distinct, but they can be complex; however, $\text{Re}\ b_m > 0$. (Some of the $\{b_m\}$ can equal some of the $\{a_j\}$ in (36).) The special case of white noise, $R_n(k\Delta) = \sigma_n^2\ \delta_{ko}$, is handled by choosing $M = 1$, $\beta_1 = \sigma_n^2$, $\tau_{n1} \to 0$, $b_1 \to +\infty$.

FILTER CHARACTERIZATION

We first consider an analog filter with a rational transfer function with N distinct poles, namely

$$H(f) = \sum_{n=1}^{N} \frac{\psi_n}{1 + i2\pi f \tau_{hn}} , \qquad (47)$$

where $\{\tau_{hn}\}$ are distinct (complex) time constants. H(f) can have up to N-1 zeros. The corresponding impulse response is

$$h(\tau) = \begin{cases} \sum_{n=1}^{N} \frac{\psi_n}{\tau_{hn}} \exp\left(\frac{-\tau}{\tau_{hn}}\right) & \text{for } \tau \geq 0 \\ 0 & \text{for } \tau < 0 \end{cases} , \qquad (48)$$

which is required to be real for all τ. The sampled impulse response is then of the form

$$\Delta\, h(m\Delta) = \sum_{n=1}^{N} \psi_n\, c_n \exp(-c_n m) \quad \text{for } m \geq 0 , \quad c_n = \frac{\Delta}{\tau_{hn}} , \qquad (49)$$

where the N filter exponential parameters $\{c_n\}$ are distinct from one another; also, Re $c_n > 0$.

The filter correlation function is available by substitution of (49) in (10), with the result

$$\phi_h(j\Delta) = \sum_{n=1}^{N} \psi_n\, c_n\, \nu_n \exp(-c_n |j|) \quad \text{for all } j , \qquad (50)$$

where

$$\nu_n = \sum_{k=1}^{N} \frac{\psi_k\, c_k}{1 - \exp(-c_n - c_k)} \quad \text{for } 1 \leq n \leq N . \qquad (51)$$

The form of filter correlation $\phi_h(j\Delta)$ in (50) is identical to the sampled input signal covariance (36), in that (50) is also composed of N distinct exponentials; hence, the comments and examples given there are relevant here also.

A useful alternative exists to the calculation of constants $\{\nu_n\}$ via (51). Namely, define z transform filter

$$\underline{H}(z) = \sum_{m=0}^{\infty} z^{-m} \, \Delta \, h(m\Delta) \; . \tag{52}$$

Replace n by k in (49) and substitute the result into (52); then an interchange of summations yields

$$\underline{H}(z) = \sum_{k=1}^{N} \frac{\psi_k \; c_k}{1 - \frac{1}{z} \exp(-c_k)} \; . \tag{53}$$

It then follows immediately by comparison with (51) that

$$\nu_n = \underline{H}(\exp(c_n)) \quad \text{for } 1 \le n \le N \; . \tag{54}$$

When $\underline{H}(z)$ is available as a rational function, (54) can be used directly, instead of summation (51).

The exponential parameters $\{c_n\}$, in impulse response (49) and corresponding N-pole filter (53), can be complex, provided that they occur in complex conjugate pairs and that their corresponding coefficients $\{\psi_n\}$ are also conjugate pairs. That is, the sum of two conjugate terms of the form of (49) must be real for all $m \ge 0$. This generality allows for oscillatory impulse responses such as yielded by narrowband filters.

FILTER OUTPUT NOISE COVARIANCE

In the sequel, we will find use for the following closed form for the discrete convolution of two exponentials:

$$\exp[-c|k|] \oplus \exp[-b|k|] \equiv \sum_j \exp[-c|j|-b|k-j|] =$$

$$= \begin{Bmatrix} \dfrac{\sinh(c)\ \exp(-b|k|) - \sinh(b)\ \exp(-c|k|)}{\cosh(c) - \cosh(b)} & \text{for } b \neq c \\ \left[|k| + \dfrac{\cosh(c)}{\sinh(c)}\right] \exp(-c|k|) & \text{for } b = c \end{Bmatrix} . \quad (55)$$

The filter output noise covariance is given by (9). If we define

$$C(k\Delta) = \sum_j \phi_h(j\Delta)\ R_n(k\Delta - j\Delta) \ , \quad (56)$$

then the stationarity of filter output noise $y_n(k\Delta)$ in (8) allows us to express (9) as

$$C_n(k_1\Delta, k_2\Delta) = C(k_1\Delta - k_2\Delta) \ . \quad (57)$$

Therefore, we can concentrate on evaluation of $C(k\Delta)$ in (56) for representative filter correlations ϕ_h and input noise covariances R_n.

In particular, we will use the general filter correlation (50) and input noise covariance (46), where we presume that none of the filter parameters $\{c_n\}$ are equal to any of the input noise parameters $\{b_m\}$. That is,

$$c_n \neq b_m \quad \text{for all } n,m \, . \tag{58}$$

Substitution of (50) and (46) in (56) and use of (55) results in

$$C(k\Delta) = \sum_j \sum_{n=1}^{N} \psi_n \, c_n \, \nu_n \, \exp(-c_n|j|) \sum_{m=1}^{M} \beta_m \, \exp(-b_m|k-j|) =$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{M} \psi_n \, c_n \, \nu_n \, \beta_m \sum_j \exp[-c_n|j|-b_m|k-j|] =$$

$$= \sum_{m=1}^{M} \mu_m \, \exp(-b_m|k|) + \sum_{n=1}^{N} \gamma_n \, \exp(-c_n|k|) \, , \tag{59}$$

where we defined auxiliary constants

$$\mu_m = \beta_m \sum_{n=1}^{N} \frac{\psi_n \, c_n \, \nu_n \, \sinh(c_n)}{\cosh(c_n) - \cosh(b_m)} \quad \text{for } 1 \le m \le M \, ,$$

$$\gamma_n = \psi_n \, c_n \, \nu_n \sum_{m=1}^{M} \frac{\beta_m \, \sinh(b_m)}{\cosh(b_m) - \cosh(c_n)} \quad \text{for } 1 \le n \le N \, . \tag{60}$$

Condition (58) keeps all the denominators in (60) from becoming zero. On the other hand, if one of the filter parameters $\{c_n\}$ is equal to one of the noise parameters $\{b_m\}$, then it is merely necessary to utilize instead the second line of (55) for that particular n,m pair in the j sum in the middle line of (59).

End result (59) for the filter output noise covariance is a compact expression that is capable of being quickly computed, once constants $\{\mu_m\}$ and $\{\gamma_n\}$ have been evaluated by means of (60) and stored. Again, we encounter a sum of decaying exponentials, albeit of M + N terms now.

FILTER OUTPUT NOISE COVARIANCE FOR WHITE NOISE INPUT

When the input noise is white, then as noted under (46), we have for the input noise covariance,

$$R_n(k\Delta) = \sigma_n^2\,\delta_{ko}\ , \quad \text{that is,}\ \ M = 1\ ,\ \beta_1 = \sigma_n^2\ ,\ b_1 \rightarrow +\infty\ . \qquad (61)$$

Use of this result in (60) yields $\mu_1 \rightarrow 0$, $\gamma_n \rightarrow \sigma_n^2\,\psi_n\,c_n\,\nu_n$, in which case the filter output noise covariance (59) becomes

$$C(k\Delta) = \sigma_n^2 \sum_{n=1}^{N} \psi_n\,c_n\,\nu_n\,\exp(-c_n|k|)\ , \qquad (62)$$

where $\{\nu_n\}$ are given by (51).

Actually, (62) is a special case of the general white noise result obtained by substitution of (61) into (56); namely, for arbitrary filter $\Delta\,h(m\Delta)$, the filter output noise covariance is

$$C(k\Delta) = \sigma_n^2\,\phi_h(k\Delta) \quad \text{for white noise input}\ , \qquad (63)$$

where filter correlation ϕ_h is given by (10).

FILTER OUTPUT SIGNAL COVARIANCE

The nonstationary filter output signal covariance is given by double summation (17) in conjunction with (18). If we substitute input signal covariance $R_s(k\Delta)$ given by (36), along with filter impulse response $\Delta\ h(m\Delta)$ given by (49), into (17), there follows

$$C_s(k_1\Delta,k_2\Delta) = \sum_{m=k_a}^{K_1} \sum_{n=1}^{N} \psi_n\ c_n\ \exp\left(-c_n(k_1-m)\right) \sum_{p=k_a}^{K_2} \sum_{q=1}^{N} \psi_q\ c_q\ \times$$

$$\times\ \exp\left(-c_q(k_2-p)\right) \sum_{j=1}^{J} \alpha_j\ \exp\left(-a_j|m-p|\right) =$$

$$= \sum_{j=1}^{J} \alpha_j \sum_{n=1}^{N} \psi_n\ c_n\ \exp\left(-c_n k_1\right) \sum_{q=1}^{N} \psi_q\ c_q\ \exp\left(-c_q k_2\right)\ \times$$

$$\times \sum_{m=k_a}^{K_1} \exp\left(c_n m\right) \sum_{p=k_a}^{K_2} \exp\left(c_q p\ -\ a_j|m-p|\right)\ . \qquad (64)$$

In order to evaluate the innermost double summation in (64), we let $A = \exp(a_j)$, $B = \exp(c_q)$, $C = \exp(c_n)$ and we presume that $k_1 \le k_2$. It then follows from (18) that $K_1 \le K_2$, and the double sum in the last line of (64) can be developed thusly:

$$S(A,B,C,k_a,K_1,K_2) \equiv \sum_{m=k_a}^{K_1} C^m \sum_{p=k_a}^{K_2} B^p\ A^{-|m-p|} = \qquad (65)$$

$$= \sum_{m=k_a}^{K_1} C^m \left[\sum_{p=k_a}^{m} B^p\ A^{p-m} + \sum_{p=m}^{K_2} B^p\ A^{m-p}\ -\ B^m\right] =$$

$$= \sum_{m=k_a}^{K_1} (C/A)^m \sum_{p=k_a}^{m} (BA)^p + \sum_{m=k_a}^{K_1} (CA)^m \sum_{p=m}^{K_2} (B/A)^p - \sum_{m=k_a}^{K_1} (CB)^m \ . \qquad (66)$$

Now we use the fact that

$$\sum_{m=L}^{M} z^m = \frac{z^L - z^{M+1}}{1 - z} \quad \text{for } z \neq 1 \ , \quad L \leq M \ , \qquad (67)$$

and we define auxiliary function

$$g(z,k) \equiv \frac{z^k}{1 - z} \quad \text{for } z \neq 1 \ . \qquad (68)$$

Then, after some amount of manipulation, the sums in (66) can be expressed in the closed form

$$S(A,B,C,k_a,K_1,K_2) = g(CB,k_a) \frac{A^2 - CB}{(A-B)(A-C)} + g(CB,K_1+1) \frac{B(A^2 - 1)}{(A-B)(1-BA)}$$
$$+ g(B/A,K_2+1) \left[g(CA,K_1+1) - g(CA,k_a)\right] - g(BA,k_a)\ g(C/A,K_1+1) \ , \qquad (69)$$

provided that $B \neq A$ and $C \neq A$. (B cannot equal 1/A, nor can C equal 1/A or 1/B, because the real parts of c_n and a_j are always positive.)

We now employ definition (65) in order to express the filter output signal covariance in (64) in the final form

$$C_s(k_1\Delta,k_2\Delta) = \sum_{j=1}^{J} \alpha_j \sum_{n=1}^{N} \psi_n\ c_n \exp\left(-c_n k_1\right) \sum_{q=1}^{N} \psi_q\ c_q \exp\left(-c_q k_2\right) \times$$
$$\times\ S\left(\exp(a_j),\exp(c_q),\exp(c_n),k_a,K_1,K_2\right) \ , \qquad (70)$$

provided that <u>none</u> of the filter parameters $\{c_n\}$ are equal to any of the input signal parameters $\{a_j\}$. That is,

$$c_n \neq a_j \quad \text{for all } n,j \ . \tag{71}$$

This result, (70), cannot be further reduced without the various parameters being specified. The triple summation will not be overly time consuming unless the input signal or filter are characterized by a large number of poles, that is, large J or N.

The result in (70) holds for $k_1 \leq k_2$; the range for $k_1 > k_2$ is most easily handled by use of symmetry relation (15). Also, a significant computational aid is available by using the recurrence relation for the $g(z,k)$ function in (68), namely $g(z,k) = z\, g(z,k-1)$.

In summary, k_a, k_b, k_c, k_d are given integers satisfying $k_a \leq k_b$, $k_c \leq k_d$, $k_a \leq k_d$. Also, $K_o = \max(k_a,k_c)$, $K_1 = \min(k_1,k_b)$, $K_2 = \min(k_2,k_b)$. Since we keep $k_1 \leq k_2$, then $K_1 \leq K_2$. Integers k_1 and k_2 must vary over the range

$$K_o \leq k_1 \leq k_2 \leq k_d \ , \tag{72}$$

meaning that K_1 varies in the range $\min(K_o,k_b)$ to $\min(k_d,k_b)$. From (69), we see that we have to compute function values $g(CB,k_a)$, $g(CA,k_a)$, $g(BA,k_a)$, as well as the arrays of function values of $g(CB,k)$, $g(CA,k)$, $g(B/A,k)$, $g(C/A,k)$ for k in the range $\min(K_o,k_b)+1$ to $\min(k_d,k_b)+1$.

PROGRAMS

A program listing for the white noise input case is presented in appendix D; it consists of two separate parts. The first part computes the filter output signal and noise covariance matrices for unity input signal and noise powers, the total filter output covariance matrix of elements C_y in (23) for the various signal-to-noise ratios of interest, the weighted covariance matrix C of elements C_a in (26) and its corresponding eigenvalue matrix Λ in (27). The output of these eigenvalues to storage completes the first part.

The second part lists the program that takes in these eigenvalues and computes the exceedance distribution functions, for noise-alone as well as for signal present, from characteristic function (32); these are the false alarm and detection probabilities, respectively. The precise method of handling all the widely different signal-to-noise ratios of interest is presented in the next section. At the end of appendix D, there is also a listing of the simulation program utilized to verify these theoretical derivations and results.

The corresponding program for the colored noise case is listed in appendix E. A similar breakdown into two parts has been adopted; however, once the eigenvalues have been computed at the end of the first part, the identical program in part 2 of appendix D is used for distribution calculations and is therefore not listed again. Also, the simulation program use to check the colored noise case is similar to that in part 3 of appendix D.

ON CALCULATION OF RECEIVER OPERATING CHARACTERISTICS

The evaluation of the receiver operating characteristics for a number of different input signal-to-noise ratios typically involves the Fourier transforms of a set of characteristic functions of the decision variable, which frequently have widely different ranges and rates of variation. In order to make these calculations tractable and easily plotted, it is necessary to resort to FFTs [2] with carefully chosen sizes and sampling increments. We now present the reasoning behind our choices of these parameters and their interrelationships.

For ease of presentation, we will consider the numerical evaluation of a probability density function $p_o(u)$ from a given (noise-only) characteristic function $f_o(\xi)$; the extension to an exceedance distribution function [2] is immediate. We have

$$p_o(u) = \frac{1}{2\pi} \int d\xi \, \exp(-i\xi u) \, f_o(\xi) =$$

$$\cong \frac{1}{2\pi} \Delta_o \sum_k \exp(-ik\Delta_o u) \, f_o(k\Delta_o) \equiv \tilde{p}_o(u) \, , \qquad (73)$$

where the trapezoidal approximation with sampling increment Δ_o in ξ has been employed. The approximation $\tilde{p}_o(u)$ defined by (73) is an aliased version of $p_o(u)$ and has period $u_o = 2\pi/\Delta_o$ in u. In order to keep aliasing effects negligible in $\tilde{p}_o(u)$, it will be necessary to choose ξ increment Δ_o small enough that the aliasing lobes at separation u_o don't overlap. Therefore, we can restrict the evaluation of $\tilde{p}_o(u)$ to N_o equally spaced points over that interval u_o, according to

$$\tilde{p}_o\left(\frac{2\pi n}{N_o\Delta_o}\right) = \frac{1}{2\pi}\,\Delta_o \sum_k \exp(-i2\pi kn/N_o)\; f_o(k\Delta_o) \quad \text{for } 0 \le n \le N_o-1 \;. \tag{74}$$

Furthermore, by collapsing the sequence $\{f_o(k\Delta_o)\}$ modulo N_o, (74) can be accomplished exactly and efficiently as an N_o-point FFT when N_o is a power of 2; see [2]. The increment in argument u of $\tilde{p}_o(u)$ in (74) and the maximum useful value of u are

$$\delta_o = \frac{2\pi}{N_o\Delta_o}\,, \quad u_o = \frac{2\pi}{\Delta_o}\;. \tag{75}$$

Now suppose that we also want to evaluate the probability density function $p_1(u)$ corresponding to a different (signal present) characteristic function $f_1(\xi)$ according to

$$p_1(u) = \frac{1}{2\pi}\int d\xi\, \exp(-i\xi u)\; f_1(\xi) =$$

$$\cong \frac{1}{2\pi}\,\Delta_1 \sum_k \exp(-ik\Delta_1 u)\; f_1(k\Delta_1) \equiv \tilde{p}_1(u)\;, \tag{76}$$

where the sampling increment in ξ is now Δ_1. By identical reasoning to that above, we can get samples of periodic (aliased) approximation $\tilde{p}_1(u)$ according to

$$\tilde{p}_1\left(\frac{2\pi n}{N_1\Delta_1}\right) = \frac{1}{2\pi}\,\Delta_1 \sum_k \exp(-i2\pi kn/N_1)\; f_1(k\Delta_1) \quad \text{for } 0 \le n \le N_1-1 \;, \tag{77}$$

which can be realized as an N_1-point FFT. The increment in u and the maximum useful value of u for this latter case are

$$\delta_1 = \frac{2\pi}{N_1\Delta_1}\,, \quad u_1 = \frac{2\pi}{\Delta_1}\;. \tag{78}$$

If we now want to eliminate the arguments of functions $\tilde{p}_o(u)$ and $\tilde{p}_1(u)$ and be able to directly plot $\tilde{p}_1(u)$ versus $\tilde{p}_o(u)$ (that is, y versus x), this can be easily accomplished if we force the u increments in (74) and (77) to be equal, namely, $\delta_1 = \delta_o$. But, then, according to (75) and (78), this means choosing

$$N_1\Delta_1 = N_o\Delta_o \ . \tag{79}$$

Since integers N_o and N_1 will be limited to powers of 2, this will mean repeated halving of the ξ increments Δ_o and Δ_1 in (73) and (76), respectively. The values of $\tilde{p}_1$ yielded by (77) can then be directly plotted versus those values of $\tilde{p}_o$ yielded by (74), up to $n = \min(N_o,N_1) = N_o$.

There are several alternative procedures that could have been adopted. For example, one option is to halve the ξ increment, that is, take $\Delta_1 = \Delta_o/2$, but keep $N_1 = N_o$. This will require discarding every other value put out by FFT (74), in order that the remaining values occur at the same u arguments yielded by (77). This is wasteful of FFT (74) and has not been adopted here. Nor have we employed the possibility of interpolation of (77) to determine approximately what the values of $\tilde{p}_1$ would be at the arguments of $\tilde{p}_o$ in (74).

The need to decrease the size of the ξ increment from Δ_o in (74) to Δ_1 in (77) is fundamental. It arises from the fact that as the input signal-to-noise ratio is increased, the contribution of the corresponding probability density function $p_k(u)$, $k = 0,1,2,\ldots$, moves to higher values on the u scale (thereby leading to the desired higher detection probabilities). In order that approximation $\tilde{p}_k(u)$ not be severely aliased, the ξ increment Δ_k must therefore be decreased; for example, see upper limits u_o and u_1 in (75) and (78), respectively.

With these points in mind, the following procedure has been adopted and utilized in the programs in this report. First, for the noise-only probability density function $p_o(u)$, a satisfactory value for ξ-increment Δ_o in (73) is found such that aliasing is insignificant in $\tilde{p}_o(u)$. This selection of Δ_o is arrived at by looking at a plot of (74) and modifying Δ_o appropriately by trial and error. An FFT size of $N_o = 128$ is utilized to evaluate (74), which is then stored; this size for N_o has generally proven to be sufficient to keep track of the variations of $p_o(u)$.

When we encounter the characteristic function $f_1(\xi)$ and probability density function $p_1(u)$ for the first (lowest) nonzero signal-to-noise ratio of interest, it is usually necessary to decrease the ξ increment to $\Delta_1 = \Delta_o/2$ or $\Delta_o/4$ in order to control the aliasing inherent in approximation $\tilde{p}_1(u)$. At the same time, N_1 is scaled up by 2 or 4, according to (79), thereby maintaining the same u arguments for (74) and (77), as desired. Again, a plot of aliased version $\tilde{p}_1(u)$, obtained by means of (77), serves as the decision point for making an acceptable choice of value for Δ_1.

For the successively larger signal-to-noise ratios and their corresponding probability density functions $p_k(u)$, $k=1,2,\ldots$, occasional repeated halvings of the ξ increment according to

$\Delta_k = \Delta_{k-1}/2$ are made, but only when needed to keep aliasing insignificant in $\tilde{p}_k(u)$. At those times that such a halving is necessary and made, the FFT size is doubled according to $N_k = 2\ N_{k-1}$. Otherwise, Δ_k and N_k are kept at their same values as used for the k-1 run. By the time the largest signal-to-noise ratio case of interest is reached, the FFT size N_k can get rather large. We are limited in our application to a maximum FFT size of 16384; if aliasing is significant at this size, we are unable to accurately numerically handle these larger signal-to-noise ratios by the adopted procedure. Interpolation would then be the recommended alternative, as alluded to above.

The procedure above has been explained in terms of probability density functions rather than the exceedance distribution functions which are of actual interest. That is, the relevant exceedance distribution functions are plots of the detection probabilities for various signal-to-noise ratios versus the false alarm probability. The only basic change is to replace the top lines of (73) and (76) by the exceedance distributions, which are also obtained by means of Fourier transforms, namely [2; (5) - (6)]

$$E_k(u) = \frac{1}{2} + \int_{0+}^{+\infty} d\xi \ \mathrm{Im}\left\{\exp(-i\xi u)\ \frac{f_k(\xi)}{\pi\xi}\right\} , \qquad (80)$$

for k = 0,1,2,...

RESULTS

COMPARISON WITH SIMULATION

In figure 7, the cumulative (CDF) and exceedance distribution functions (EDF) for a white noise input excitation are presented. The input signal and filter are both characterized by one pole, that is, J = 1 in (33) - (36) and N = 1 in (47) - (50). The simulation employed 1E6 trials and verifies the theoretical calculations down to the probability level of 1E-5 plotted. A listing of the simulation program, including all the parameter values that were utilized here, is given at the end of appendix D. The gating and observation parameters are $k_a = 4$, $k_b = 11$ and $k_c = 2$, $k_d = 16$, respectively; thus, it can be seen that the transient buildup and decay are a prominent part of the filter output signal for this particular example.

The results in figure 8 pertain to a colored noise input with one pole, that is, M = 1 in (44) - (46). Again, 1E6 trials were used in the simulation and they confirm the theoretical result down to the 1E-5 level of probabilities. A listing of the simulation program is given at the end of appendix D.

Absolute time is unimportant to the performance of the energy detector in figure 1. That is, any common constant can be added to or subtracted from k_a, k_b, k_c, k_d, without changing the receiver operating characteristics. Thus, k_a could always be selected as 0 without loss of generality.

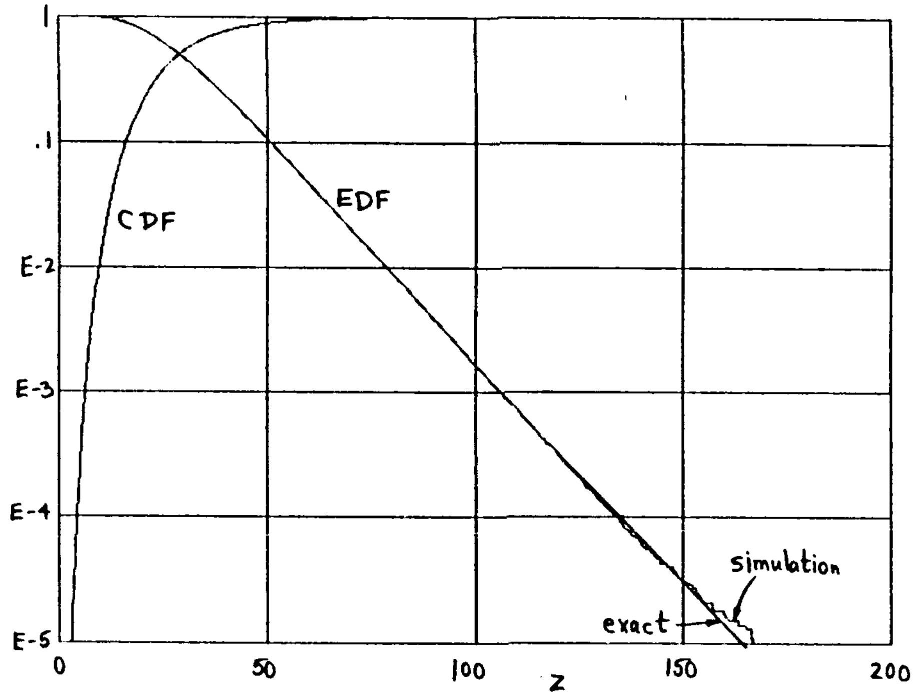

Figure 7. Cumulative and Exceedance Distributions for White Noise

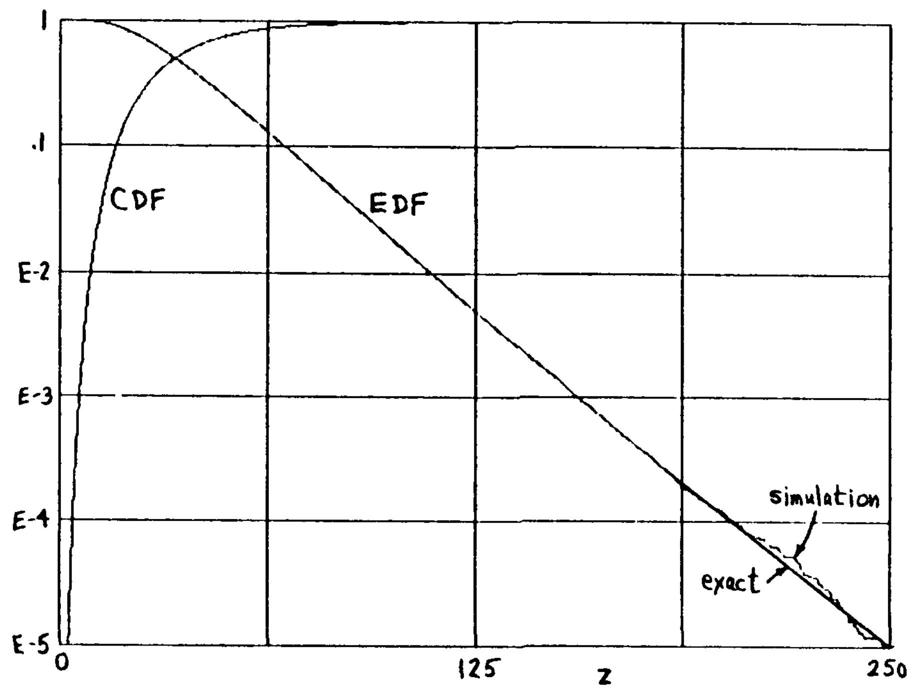

Figure 8. Cumulative & Exceedance Distributions for Colored Noise

RECEIVER OPERATING CHARACTERISTICS

The receiver operating characteristics, that is, detection probability P_D versus false alarm probability P_F for a set of signal-to-noise ratios, for the system in figure 1 will be considered in this section. The first example is evaluated for the following set of parameters, where all times are in seconds. Also, all parameters are real.

$$\begin{aligned}
&k_a = 4,\ k_b = 11,\ k_c = 2,\ k_d = 16,\ \Delta = .2, \\
&J = 3,\ \{\alpha_j\} = \{11 \quad -48 \quad 75\},\ \{\tau_{sj}\} = \{1 \quad 1/8 \quad 1/15\}, \\
&M = 2,\ \{\beta_m\} = \{39 \quad 60\},\ \{\tau_{nm}\} = \{.2 \quad .4\}, \\
&N = 4,\ \{\psi_n\} = \{1 \quad 2 \quad 3 \quad 4\},\ \{\tau_{hn}\} = \{.3 \quad .5 \quad .7 \quad .9\}.
\end{aligned} \tag{81}$$

Twenty nonzero values of the system input signal-to-noise ratio

$$R = \frac{\sigma_s^2}{\sigma_n^2} \tag{82}$$

were utilized, namely R = 5(1.2)27.8 dB. For computation of the exceedance distribution function P_F directly from the characteristic function, the initial ξ increment Δ_o in (73) and (74) was taken as .00025; this yielded round-off errors less than 1E-15 in the false alarm probability calculation. Repeated occasional halving of the ξ increment, as explained in the previous section, was utilized, eventually requiring an FFT size of 16384 for the largest signal-to-noise ratios cases for P_D.

The receiver operating characteristics are plotted on normal probability paper in figure 9 and cover a wide range of values, ranging from 1E-10 for P_F to .999 for P_D. Since Gaussian random variables would plot on such paper as a set of parallel straight lines, the curvatures of these results illustrate that the Gaussian assumption for decision variable z in figure 1 is not warranted, at least for this example. The major reason for this behavior is the relatively small number of samples used in the detector; see the top line of (81). This is also a partial explanation for the rather large values of the per-sample signal-to-noise ratio R required at the high quality region at the top left of figure 9. Another factor to notice is that the input signal duration is only $k_b\Delta - k_a\Delta = 1.4$ seconds, meaning that the filter output signal never reaches steady state before the input signal is turned off; all these transient signal effects and their limitations on performance have been included exactly in the analysis and numerical results presented here.

The effect of halving the sampling increment Δ in figure 1 is considered in the next example but done in such a manner as to keep the total input signal duration the same. That is, the parameter values in (81) are kept the same except for the following changes:

$$k_a = 8, \; k_b = 22, \; k_c = 4, \; k_d = 32, \; \Delta = .1. \tag{83}$$

Notice that all absolute times, such as $k_a\Delta$, are kept fixed. The corresponding receiver operating characteristics are plotted in figure 10. Performance has been degraded by a couple of dB. For example, to realize P_F = 1E-3 and P_D = .5, the per-sample input signal-to-noise ratio R must now be 16 dB, whereas it was 13.8 dB in the previous figure. This is a degradation of 2.2 dB. The main reason for this behavior is again the inability of the filter output signal to reach steady state and thereby contribute significantly to the summer output. By contrast, the filter output noise is in steady state (by assumption) for both examples.

We now return to the original sampling increment $\Delta = .2$ seconds employed in (81). When the observation interval coincides with the input signal nonzero-excitation duration, that is, $k_a = k_c = 4$, $k_b = k_d = 11$, the receiver operating characteristics in figure 11 illustrate additional improvement. For example, probabilities $P_F = 1E-3$ and $P_D = .5$ can now be realized with R = 13.1 dB, which is an improvement of .7 dB relative to the signal-to-noise ratio required for the broader observation interval, $k_c = 2$, $k_d = 16$, used in (81).

Finally, an example with a wide observation interval, namely $k_c = 0$, $k_d = 25$, is displayed in figure 12. The desired probabilities can now be achieved only if the input signal-to-noise ratio is increased to 14.7 dB, a degradation of 1.6 dB relative to the best case above.

These examples illustrate the utility of being able to investigate quantitatively and accurately the effects of nonstationarity in a system, without having to make questionable assumptions about, for example, how closely steady state was or was not realized. They also afford a dependable verification or rejection of the Gaussian approximation for the decision variable; in a related work [3], the latter approximation was found to be too optimistic for most working ranges of this detection system.

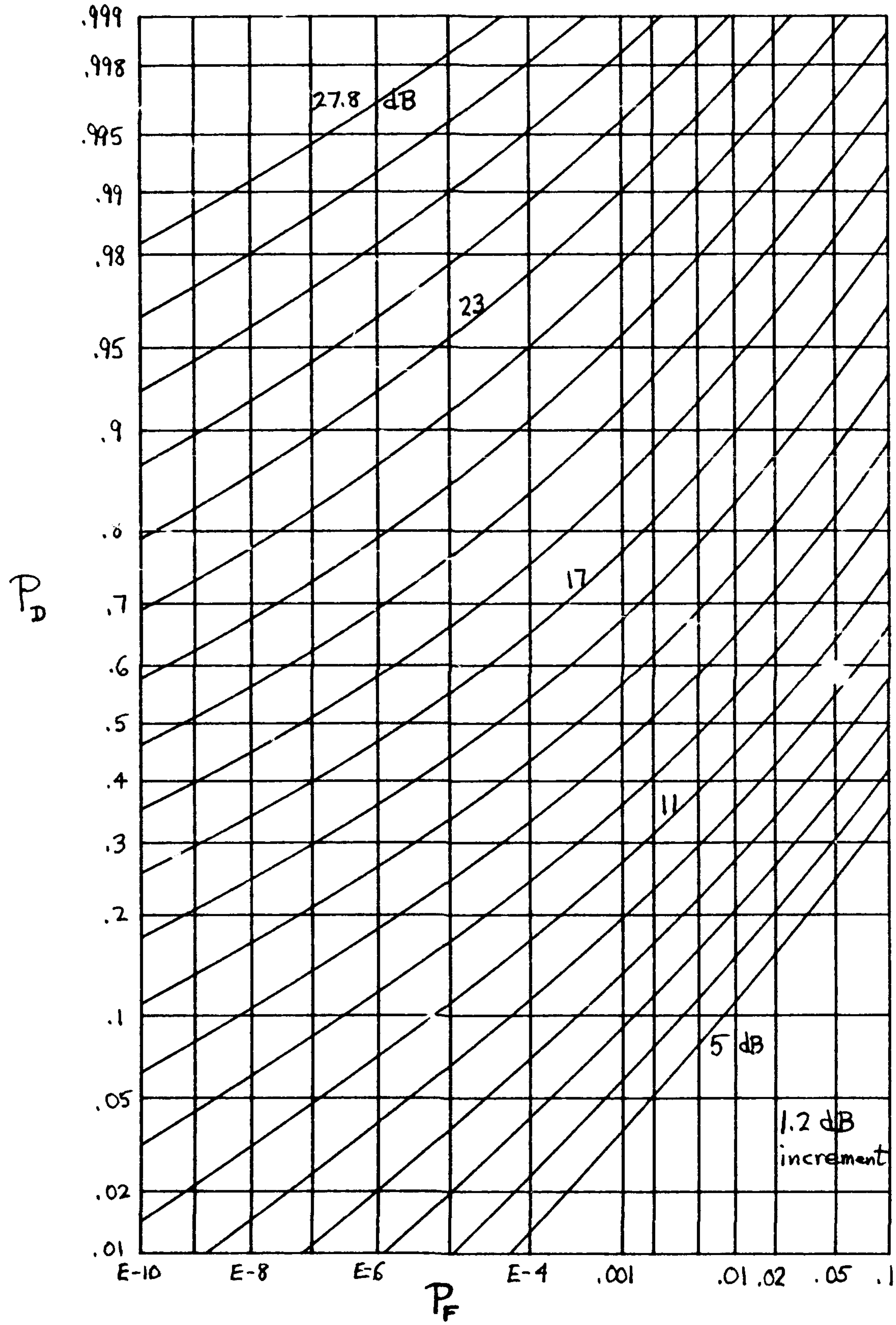

Figure 9. Receiver Operating Characteristics; Example A

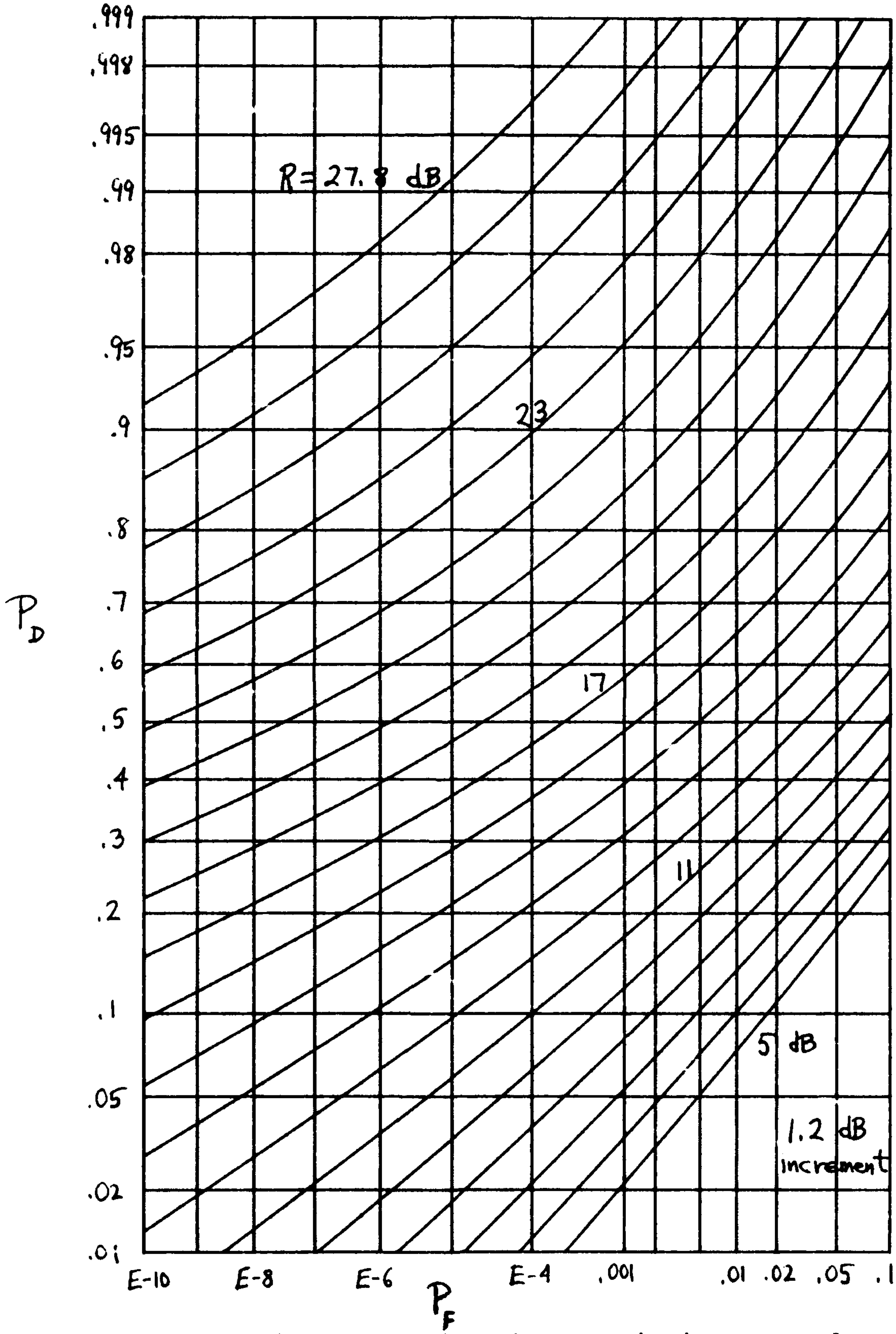

Figure 10. Receiver Operating Characteristics; Example B

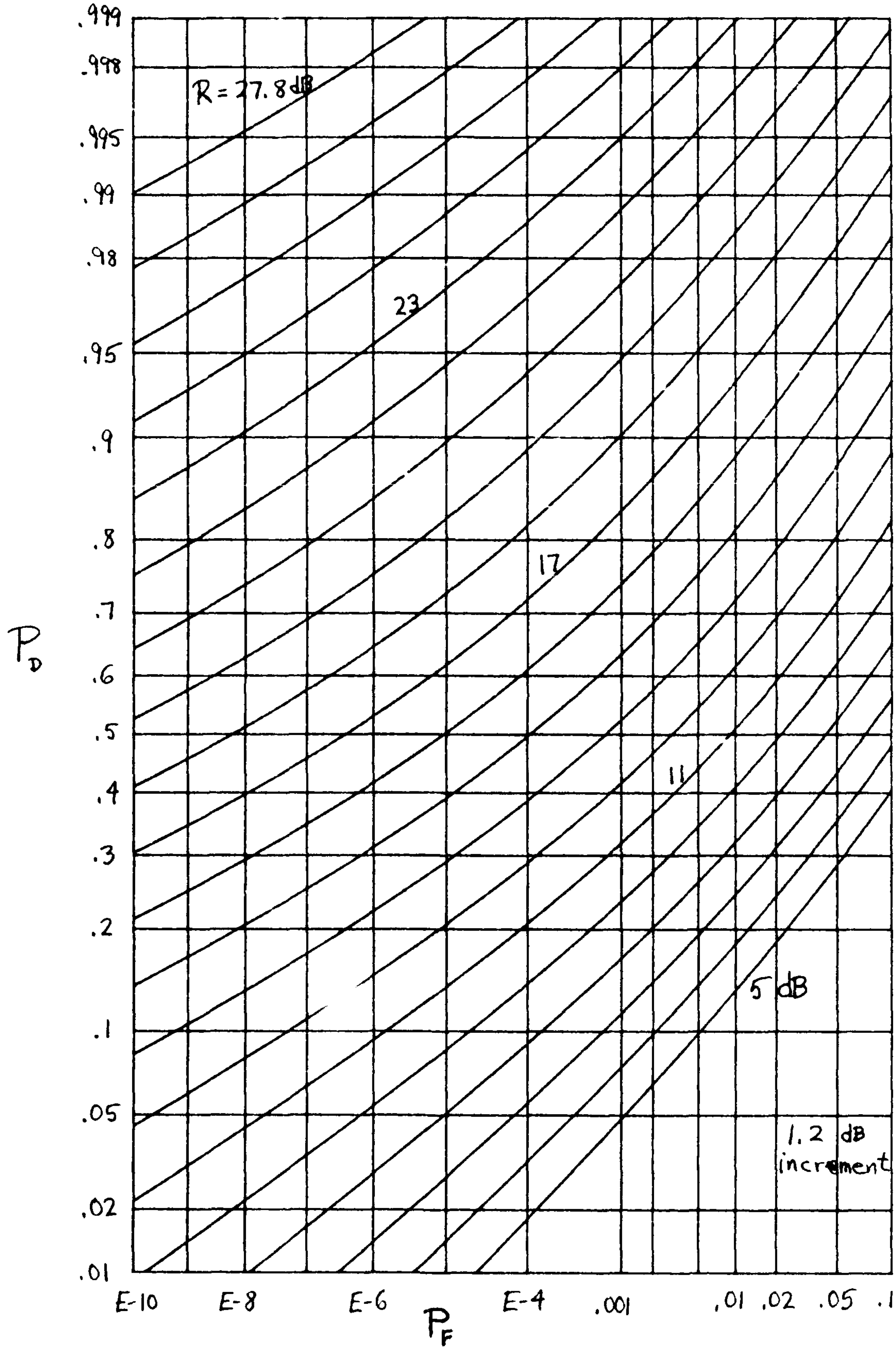

Figure 11. Receiver Operating Characteristics; Example C

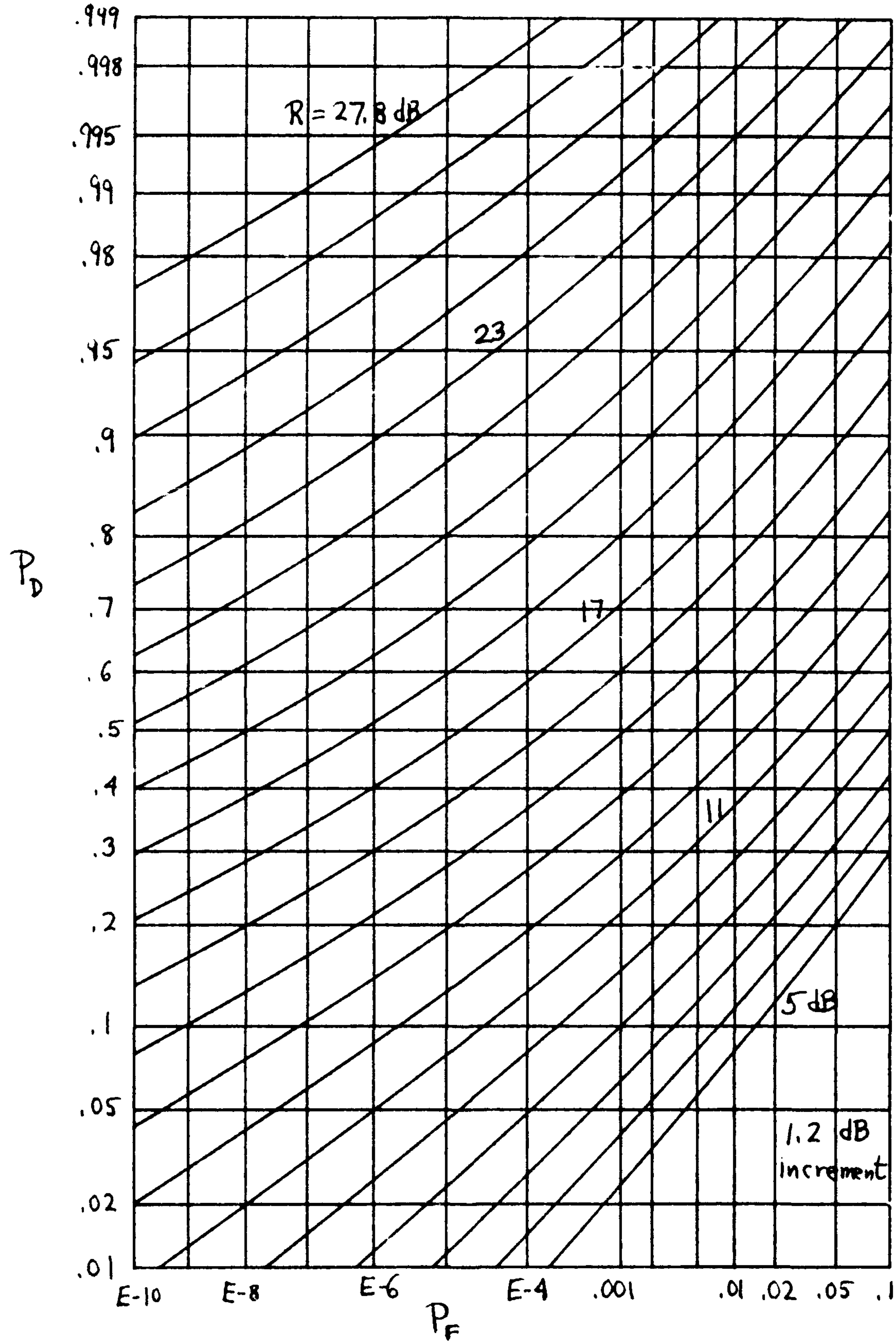

Figure 12. Receiver Operating Characteristics; Example D

SUMMARY

Programs have been written which enable exact analysis of performance of the mismatched bandpass energy detector of figure 1, both for the white noise input case as well as the colored noise input case. These results allow for arbitrary sampling time increment Δ and for arbitrary input signal spectra, input noise spectra, and filter transfer functions. By these means, exact quantitative evaluations and degradations can be determined for various combinations of uncertainty regarding the input signal time location and duration as well as its center frequency, bandwidth, and spectrum. Included in the analysis and programs are the buildup and decay (or any portion thereof) of the nonstationary output signal from the filter, when excited by a burst-like input signal, regardless of the lengths or locations of the excitation and observation intervals. Both programs have been compared with simulation results and confirmed down to as low a probability level as possible consistent with the number of trials used.

APPENDIX A. NUMERICAL EVALUATION OF NOISE COVARIANCE R_n

If the noise covariance $R_n(\tau)$ is not available in analytic form, but the noise spectrum $G_n(f)$ is specified, the following numerical procedure can be employed to evaluate the required samples $R_n(k\Delta)$. We begin with

$$R_n(\tau) = \int df \exp(i2\pi f\tau)\, G_n(f) =$$

$$\cong \Delta_f \sum_m \exp(i2\pi m\Delta_f\tau)\, G_n(m\Delta_f) \equiv \tilde{R}_n(\tau) \; , \tag{A-1}$$

where the trapezoidal rule with frequency increment Δ_f was used. But $\tilde{R}_n(\tau)$ in (A-1) can be developed as

$$\tilde{R}_n(\tau) = \int df \exp(i2\pi f\tau)\, G_n(f)\, \Delta_f\, \delta_{\Delta_f}(f) =$$

$$= R_n(\tau) \circledast \delta_{1/\Delta_f}(\tau) = \sum_m R_n\left(\tau - \frac{m}{\Delta_f}\right) . \tag{A-2}$$

Now suppose that the noise covariance

$$R_n(\tau) \cong 0 \quad \text{for } |\tau| > \tau_o . \tag{A-3}$$

Then, in order to avoid significant aliasing in (A-2), we must take Δ_f small enough that

$$\frac{1}{2\Delta_f} > \tau_o \; , \tag{A-4}$$

in which case (A-2) yields

$$\tilde{R}_n(\tau) \cong R_n(\tau) \quad \text{for } |\tau| < \tau_o < \frac{1}{2\Delta_f} . \tag{A-5}$$

The required samples of $\tilde{R}_n(\tau)$ follow from (A-1) according to

$$\tilde{R}_n(k\Delta) = \Delta_f \sum_m \exp(i2\pi m\Delta_f k\Delta)\; G_n(m\Delta_f) \; . \tag{A-6}$$

Now take frequency increment

$$\Delta_f = \frac{1}{N\Delta} \, , \tag{A-7}$$

where, according to (A-4), integer N must be taken large enough that $N\Delta/2 > \tau_o$, that is,

$$N > \frac{2\tau_o}{\Delta} \; . \tag{A-8}$$

Then, from (A-5) - (A-7),

$$R_n(k\Delta) \cong \tilde{R}_n(k\Delta) = \frac{1}{N\Delta} \sum_m \exp(i2\pi mk/N)\; G_n\!\left(\frac{m}{N\Delta}\right) \quad \text{for } |k| < \frac{N}{2} \; . \tag{A-9}$$

Alternatively, this can be expressed as

$$R_n(k\Delta) \cong \tilde{R}_n(k\Delta) = \frac{1}{N\Delta} \sum_{m=0}^{N-1} \exp(i2\pi mk/N)\; \tilde{G}_n\!\left(\frac{m}{N\Delta}\right) \quad \text{for } |k| < \frac{N}{2} \, , \tag{A-10}$$

where collapsed (prealiased) noise spectrum and sequence

$$\tilde{G}_n(f) \equiv \sum_j G_n\!\left(f - \frac{j}{\Delta}\right) , \quad \tilde{G}_n\!\left(\frac{m}{N\Delta}\right) = \sum_j G_n\!\left(\frac{m}{N\Delta} - \frac{j}{\Delta}\right) \quad \text{for } 0 \le m \le N-1 \; . \tag{A-11}$$

The procedure in (A-10) can be efficiently realized as an N-point FFT of N nonzero numbers. The major conditions that must be met are (A-8) in conjunction with (A-3).

APPENDIX B. NUMERICAL EVALUATION OF FILTER CORRELATION ϕ_h

Suppose that we want digital filter correlation

$$\phi_h(j\Delta) = \Delta^2 \sum_m h(m\Delta)\, h(m\Delta - j\Delta) \ , \qquad \text{(B-1)}$$

defined in (10), but all that we have is the filter transfer function $H(f)$, where

$$h(\tau) = \int df \exp(i2\pi f\tau)\, H(f) \quad \text{for all } \tau \ . \qquad \text{(B-2)}$$

Impulse response $h(\tau)$ is presumed real. The direct relationship between $\phi_h(j\Delta)$ and $H(f)$ is the subject of this appendix.

First, define periodic function

$$\tilde{H}(f) \equiv \sum_m H\left(f - \frac{m}{\Delta}\right) \quad \text{for all } f \ . \qquad \text{(B-3)}$$

Then, there follows

$$\tilde{H}(f) = H(f) \circledast \delta_{1/\Delta}(f) =$$

$$= \int d\tau \exp(-i2\pi f\tau)\, h(\tau)\, \Delta\, \delta_\Delta(\tau) = \sum_m \exp(-i2\pi f\Delta m)\, \Delta\, h(m\Delta) \ , \qquad \text{(B-4)}$$

where we used the fact that convolution in the frequency domain is equivalent to Fourier transformation of a product in the time domain. That is, $\tilde{H}(f)$ is the Fourier transform of the samples of impulse response $h(\tau)$ taken at time increment Δ. However, care must be taken at points of discontinuity of $h(\tau)$; for example, if $h(\tau)$ is discontinuous at $k\Delta$, the contribution to the right-hand side of (B-4), at m = k, is $\frac{1}{2}[h(k\Delta+) + h(k\Delta-)]$. This point and an example are discussed more fully in appendix C.

Now, there follows from (B-1) and (B-2),

$$\phi_h(j\Delta) = \Delta^2 \sum_m \int df \exp(i2\pi f\Delta m)\ H(f) \int du \exp[-i2\pi u\Delta(m-j)]\ H^*(u) =$$

$$= \Delta \iint df\ du\ H(f)\ H^*(u) \exp(i2\pi u\Delta j)\ \Delta \sum_m \exp[i2\pi(f - u)\Delta m] =$$

$$= \Delta \iint df\ du\ H(f)\ H^*(u) \exp(i2\pi u\Delta j) \sum_m \delta\left(f - u - \frac{m}{\Delta}\right) =$$

$$= \Delta \sum_m \int df\ H(f)\ H^*\left(f - \frac{m}{\Delta}\right) \exp\left[i2\pi\left(f - \frac{m}{\Delta}\right)\Delta j\right] =$$

$$= \Delta \int df \exp(i2\pi f\Delta j)\ H(f) \sum_m H^*\left(f - \frac{m}{\Delta}\right) =$$

$$= \Delta \int df \exp(i2\pi f\Delta j)\ H(f)\ \tilde{H}^*(f) \ , \qquad \text{(B-5)}$$

where we used (B-3).

Now, let I_n denote the following interval of length $1/\Delta$ on the f axis:

$$I_n \equiv \left(\frac{n-\frac{1}{2}}{\Delta}, \frac{n+\frac{1}{2}}{\Delta}\right) . \qquad \text{(B-6)}$$

Then, the filter correlation ϕ_h follows from (B-5) according to

$$\phi_h(j\Delta) = \Delta \sum_n \int_{I_n} df \exp(i2\pi f\Delta j)\ H(f)\ \tilde{H}^*(f) =$$

$$= \Delta \sum_n \int_{I_o} du \exp\left[i2\pi\left(u + \frac{n}{\Delta}\right)\Delta j\right] H\left(u + \frac{n}{\Delta}\right) \tilde{H}^*(u) =$$

$$= \Delta \int_{I_o} du \exp(i2\pi u\Delta j) \tilde{H}^*(u) \sum_n H\left(u + \frac{n}{\Delta}\right) =$$

$$= \Delta \int_{-.5/\Delta}^{.5/\Delta} du \exp(i2\pi u\Delta j) |\tilde{H}(u)|^2 , \qquad \text{(B-7)}$$

where we let $u = f - n/\Delta$ and used (B-3) and the periodicity of $\tilde{H}(f)$. This result indicates that we must first alias the given transfer function $H(f)$ according to (B-3) and then Fourier transform its magnitude-square over an interval of length $1/\Delta$.

An alternative approach that leads to the same result (B-7) is to use (B-2) and (B-3) in the form

$$h(m\Delta) = \int df \exp(i2\pi f\Delta m) H(f) = \int_{I_o} df \exp(i2\pi f\Delta m) \tilde{H}(f) . \qquad \text{(B-8)}$$

Use of this latter result in (B-1) leads to filter correlation

$$\phi_h(j\Delta) = \Delta^2 \sum_m \int_{I_o} df \exp(i2\pi f\Delta m) \tilde{H}(f) \int_{I_o} du \exp[-i2\pi u\Delta(m-j)] \tilde{H}^*(u) =$$

$$= \Delta \iint_{I_o} df\, du \exp(i2\pi u\Delta j) \tilde{H}(f) \tilde{H}^*(u) \Delta \sum_m \exp[i2\pi(f - u)\Delta m] =$$

$$= \Delta \iint_{I_o} df\, du \exp(i2\pi u\Delta j) \tilde{H}(f) \tilde{H}^*(u) \sum_m \delta\left(f - u - \frac{m}{\Delta}\right) =$$

$$= \Delta \int_{I_o} df \exp(i2\pi f\Delta j) \; |\tilde{H}(f)|^2 , \qquad \text{(B-9)}$$

since only one impulse, at u = f, lies in interval I_o. Result (B-9) is identical to (B-7). Notice that we do <u>not</u> get

$$\Delta \int df \exp(i2\pi f\Delta j) \; |H(f)|^2 = \Delta \int_{I_o} df \exp(i2\pi f\Delta j) \sum_n \left| H\left(f + \frac{n}{\Delta}\right)\right|^2 . \qquad \text{(B-10)}$$

As for the actual numerical evaluation of (B-9), suppose we sample the integral on f with increment $\Delta_f = 1/(K\Delta)$ and use the trapezoidal approximation; then there follows

$$\phi_h(j\Delta) \cong \frac{1}{K} \sum_{k=-K/2}^{K/2} w_k \exp\left(i2\pi \frac{k}{K} j\right) \left|\tilde{H}\left(\frac{k}{K\Delta}\right)\right|^2 , \qquad \text{(B-11)}$$

where weights $\{w_k\}$ are given by

$$w_k = \begin{Bmatrix} \frac{1}{2} & \text{for } |k| = K/2 \\ 1 & \text{for } |k| < K/2 \end{Bmatrix} . \qquad \text{(B-12)}$$

By using the periodicity K of the exp and $\tilde{H}$ terms in (B-11), that sum may be written exactly in the standard FFT form

$$\phi_h(j\Delta) \cong \frac{1}{K} \sum_{k=0}^{K-1} \exp(i2\pi jk/K) \left|\tilde{H}\left(\frac{k}{K\Delta}\right)\right|^2 \quad \text{for } |j| < \frac{K}{2} . \qquad \text{(B-13)}$$

This is a good approximation, provided that the FFT size K satisfies $K > 2\tau_h/\Delta$, where τ_h is the effective duration of filter correlation $\phi_h(\tau)$.

APPENDIX C. DISPLACED SAMPLING AND FOURIER TRANSFORM OF DISCONTINUOUS FUNCTION

GENERAL RELATIONS

Let a(t) and A(f) be a Fourier transform pair:

$$A(f) = \int dt \, \exp(-i2\pi ft) \; a(t) \; . \qquad \text{(C-1)}$$

Also, let b(t) and B(f) be a Fourier transform pair. Then, Parseval's theorem states that

$$\int dt \; a(t) \; b^*(t) = \int df \; A(f) \; B^*(f) \; . \qquad \text{(C-2)}$$

We now apply this relation to the case where

$$b(t) = \Delta \; \delta_\Delta(t - \alpha) \; , \quad B(f) = \delta_{1/\Delta}(f) \; \exp(-i2\pi f\alpha) \; . \qquad \text{(C-3)}$$

There follows

$$L \equiv \Delta \sum_n a(n\Delta + \alpha) = \int dt \; a(t) \; \Delta \; \delta_\Delta(t - \alpha) = \qquad \text{(C-4)}$$

$$= \int df \; A(f) \; \delta_{1/\Delta}(f) \; \exp(i2\pi f\alpha) = \sum_n A\left(\frac{n}{\Delta}\right) \exp(i2\pi n\alpha/\Delta) \equiv R \; . \qquad \text{(C-5)}$$

If the samples required in the sums in (C-4) and (C-5) encounter a discontinuity of a(t) or A(f), the value used in (C-4) or (C-5) must be taken as the average value approached from both sides of the discontinuity. Also, the sums in (C-4) and (C-5) may have to be interpreted as principal value, if necessary for convergence.

EXAMPLE

$$a(t) = \begin{Bmatrix} \exp[-(b+ic)t] & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{Bmatrix} , \quad b > 0 . \tag{C-6}$$

$$A(f) = \frac{1}{b + i(c + 2\pi f)} \quad \text{for all } f . \tag{C-7}$$

Function a(t) is discontinuous at t = 0, while A(f) is continuous for all f.

Let $0 < \alpha < \Delta$; then the sum in (C-4) is

$$L = \Delta \sum_{n=0}^{\infty} \exp[-(b+ic)(n\Delta+\alpha)] = \frac{\Delta \exp[-\alpha(b+ic)]}{1 - \exp[-\Delta(b+ic)]} \quad \text{for } 0 < \alpha < \Delta . \tag{C-8}$$

At the same time, the sum in (C-5) is

$$R = \sum_n \frac{\exp(i2\pi n\alpha/\Delta)}{b + i(c + 2\pi n/\Delta)} \quad \text{for all } \alpha . \tag{C-9}$$

When $\alpha \neq 0$ (or an integer multiple of Δ), the phase factor in the numerator of (C-9) yields a convergent sum for the real and imaginary parts, without the need for a principal value interpretation. It has also been verified numerically that (C-8) and (C-9) are equal, as predicted by (C-4) and (C-5); that is,

$$\sum_n \frac{\exp(i2\pi n\alpha/\Delta)}{b + i(c + 2\pi n/\Delta)} = \frac{\Delta \exp[-\alpha(b+ic)]}{1 - \exp[-\Delta(b+ic)]} \quad \text{for } 0 < \alpha < \Delta . \tag{C-10}$$

On the other hand, if $\alpha = 0$, then the sum in (C-4) is, accounting for the discontinuity of a(t),

$$L = \frac{\Delta}{2} + \Delta \sum_{n=1}^{\infty} \exp[-(b+ic)n\Delta] = \frac{\Delta}{2} \frac{1 + \exp[-\Delta(b+ic)]}{1 - \exp[-\Delta(b+ic)]} . \qquad \text{(C-11)}$$

Also, then, the sum in (C-5) is

$$R = \sum_n \frac{1}{b + i(c + 2\pi n/\Delta)} = \sum_n \frac{b - i(c + 2\pi n/\Delta)}{b^2 + (c + 2\pi n/\Delta)^2} , \qquad \text{(C-12)}$$

the imaginary part of which must be interpreted as a principal value sum. Again, it has been numerically verified (next page) that (C-11) and (C-12) are equal; that is,

$$\sum_n \frac{1}{b + i(c + 2\pi n/\Delta)} = \frac{\Delta}{2} \frac{1 + \exp[-\Delta(b+ic)]}{1 - \exp[-\Delta(b+ic)]} . \qquad \text{(C-13)}$$

Notice that the limit of (C-10) as $\alpha \to 0+$ does <u>not</u> yield (C-13).

If we apply result (C-11) to $\tilde{H}(f)$ in the last entry in (B-4), we have, for example (304),

$$\tilde{H}(f) = \frac{a}{2} \frac{1 + \exp(-a-i2\pi f\Delta)}{1 - \exp(-a-i2\pi f\Delta)} . \qquad \text{(C-14)}$$

By contrast, the limit of (C-10) as $\alpha \to 0+$ yields

$$\frac{a}{1 - \exp(-a-i2\pi f\Delta)} . \qquad \text{(C-15)}$$

BASIC PROGRAM FOR NUMERICAL VERIFICATION

```
10     Bs=.57                        !  b
20     Cs=.71                        !  c
30     De=.93                        !  delta
40     Al=.1                         !  alpha
50     PRINT Bs,Cs,De,Al
60     CALL Exp(-Al*Bs,-Al*Cs,Nr,Ni)
70     CALL Exp(-De*Bs,-De*Cs,Er,Ei)
80     CALL Div(De*Nr,De*Ni,1-Er,-Ei,Lr,Li)
90     IF Al=0. THEN Lr=Lr-.5*De
100    PRINT Lr,Li                   !  LEFT-HAND SIDE L
110    A=2*PI/De
120    B=A*Al
130    B2=Bs*Bs
140    Rr=Ri=0.
150    DOUBLE Ns                     !  INTEGER
160    FOR Ns=-1E5 TO 1E5            !  PRINCIPAL VALUE SUM
170    T=B*Ns
180    C=COS(T)
190    S=SIN(T)
200    T=Cs+A*Ns
210    D=B2+T*T
220    Dr=(C*Bs+S*T)/D
230    Di=(S*Bs-C*T)/D
240    Rr=Rr+Dr
250    Ri=Ri+Di
260    NEXT Ns
270    PRINT Rr,Ri                   !  RIGHT-HAND SIDE R
280    END
290    !
300    SUB Div(X1,Y1,X2,Y2,U,V)      !  Z1/Z2
310    T=X2*X2+Y2*Y2
320    U=(X1*X2+Y1*Y2)/T
330    V=(Y1*X2-X1*Y2)/T
340    SUBEND
350    !
360    SUB Exp(X,Y,U,V)              !  EXP(Z)
370    E=EXP(X)
380    U=E*COS(Y)
390    V=E*SIN(Y)
400    SUBEND
```

```
.57                  .71                  .93             0
.729357897734        -.80564044434
.729357647983        -.805640755434

.57                  .71                  .93             .001
1.19310532814        -.806028668009
1.19353711305        -.806028668087

.57                  .71                  .93             .1
1.07135515262        -.839119622089
1.07135537888        -.839119622093
```

APPENDIX D. PROGRAMS FOR WHITE INPUT NOISE

Three BASIC programs are listed in this appendix. All the signal, noise, and filter parameters are restricted to be real here. The first program computes the filter output signal and noise covariance matrices for a white noise input, the sums of these matrices scaled by the various signal-to-noise ratios and then weighted, and the eigenvalues which are then stored. The particular subroutine listed here, SUB Svd, actually computes the eigenvectors in addition; a faster alternative would be to replace this subroutine by one which calculates only the eigenvalues. Twenty different nonzero values of the input signal-to-noise ratio are allowed in the program and must be chosen and input by the user.

These eigenvalues serve as the input to the second program which computes the cumulative and exceedance distribution functions according to the method given in [2] and then plots the receiver operating characteristics by elimination of the threshold variable according to the method described in (73) - (80) in the main text.

The third program is the simulation program used to check the two programs above. It is written for single-pole processes and allows for a colored input noise process but could be easily modified to handle more general processes. A run of 1E6 trials took 7 hours on the Hewlett-Packard 9000 computer.

```
 10  ! COMPUTE COVARIANCE MATRICES AND EIGENVALUES FOR WHITE
 20  ! NOISE INPUT; STORE EIGENVALUES IN "EIG" IN LINE 1720
 30    Ka=-10                   !  INPUT SIGNAL START
 40    Kb=5                     !  INPUT SIGNAL END; Kb >= Ka
 50    Kc=0                     !  ACCUMULATOR  START
 60    Kd=5                     !  ACCUMULATOR  END; Kd >= Kc,Ka
 70    Delta=.2                 !  TIME SAMPLING INCREMENT (SECONDS)
 80    J=3                      !  NUMBER OF SIGNAL COMPONENTS
 90    DATA 11.,-48.,75.        !  SIGNAL SCALINGS (WE SET SUM = 1)
100    DATA 1.,.125,.066667 !  SIGNAL TIME CONSTANTS (SECONDS)
110    N=4                      !  NUMBER OF FILTER COMPONENTS
120    DATA 1.,2.,3.,4.         !  FILTER SCALINGS (ARBITRARY)
130    DATA .3,.5,.7,.9         !  FILTER TIME CONSTANTS (SECONDS)
140    Nr=20                    !  NUMBER OF SIGNAL-TO-NOISE RATIOS
150    DATA 0,1,2,3,4,5,6,7,8,9,10          !  INPUT SNRS IN DB
160    DATA 11,12,13,14,15,16,17,18,19
170    IF (Ka<=Kb) AND (Kc<=Kd) AND (Ka<=Kd) THEN 200
180    PRINT "PROBLEM WITH PARAMETERS"
190    STOP
200    DIM Alpha(10),A(10),Psi(10),C(10),R(20),W(100)
210    DIM Pc(10),Ec(10),Pcv(10),Cn(200),Ea(10),E(0,1000)
220    DIM Gca(100),Gcda(100),Gcb(100),Gbda(100),Cs(5000)
230    DIM Cw(0,10000),Ca(0,10000),D(100),Eig(0,2000)
240    DOUBLE Ka,Kb,Kc,Kd,J,N,Nr,Ns,Ks,Kdc,Ko  !  INTEGERS
250    DOUBLE L1,L2,L,L11,Js,Qs,Ki,Ks1,K1,Ks2,K2,I
260    REDIM Alpha(1:J),A(1:J)
270    READ Alpha(*),A(*)
280    MAT Alpha=Alpha/(SUM(Alpha))
290    MAT A=(Delta)/A
300    REDIM Psi(1:N),C(1:N)
310    READ Psi(*),C(*)
320    MAT C=(Delta)/C
330    REDIM R(1:Nr)
340    READ R(*)
350    REDIM W(Kc:Kd)
360    CALL Weights(Kc,Kd,W(*))
370    MAT W=SQR(W)
380    REDIM Pc(1:N),Ec(1:N) ! FILTER OUTPUT NOISE COVARIANCE
390    FOR Ns=1 TO N
400    C=C(Ns)
410    Pc(Ns)=Psi(Ns)*C
420    Ec(Ns)=EXP(-C)
430    NEXT Ns
440    REDIM Pcv(1:N)
450    FOR Ns=1 TO N
460    E=Ec(Ns)
470    S=0.
480    FOR Ks=1 TO N
490    S=S+Pc(Ks)/(1.-E*Ec(Ks))
500    NEXT Ks
510    Pcv(Ns)=Pc(Ns)*S
520    NEXT Ns
530    Kdc=Kd-Kc
540    REDIM Cn(0:Kdc)
550    MAT Cn=(0.)
560    FOR Ns=1 TO N
570    Pcv=Pcv(Ns)
580    E=Ec(Ns)
```

```
590   P=1.
600   Cn(0)=Cn(0)+Pcv
610   FOR Ks=1 TO Kdc
620   P=P*E
630   Cn(Ks)=Cn(Ks)+Pcv*P
640   NEXT Ks
650   NEXT Ns
660   FOR Js=1 TO J          !  FILTER OUTPUT SIGNAL COVARIANCE
670   A(Js)=EXP(A(Js))
680   NEXT Js
690   Ko=MAX(Ka,Kc)
700   REDIM Ea(1:N),E(1:N,Ko:Kd)
710   FOR Ns=1 TO N
720   E=Ec(Ns)
730   Ea(Ns)=1./E
740   E(Ns,Ko)=EXP(-C(Ns)*Ko)
750   FOR Ks=Ko+1 TO Kd
760   E(Ns,Ks)=E(Ns,Ks-1)*E
770   NEXT Ks
780   NEXT Ns
790   L1=MIN(Ko,Kb)+1
800   L2=MIN(Kd,Kb)+1
810   REDIM Gca(L1:L2),Gcda(L1:L2),Gcb(L1:L2),Gbda(L1:L2)
820   L=Kd-Ko+1
830   REDIM Cs(1:L*(L+1)/2)
840   L11=L1+1
850   FOR Js=1 TO J
860   Al=Alpha(Js)
870   A=A(Js)
880   A1=A*A-1.
890   FOR Ns=1 TO N
900   C=Ea(Ns)
910   Ca=C*A
920   Cda=C/A
930   Ca1=1.-Ca
940   Ac=A-C
950   Ae=A1*Pc(Ns)
960   Gca=Ca^Ka/Ca1
970   Gca(L1)=Ca^L1/Ca1
980   Gcda(L1)=Cda^L1/(1.-Cda)
990   FOR Ks=L11 TO L2
1000  Gca(Ks)=Ca*Gca(Ks-1)
1010  Gcda(Ks)=Cda*Gcda(Ks-1)
1020  NEXT Ks
1030  FOR Qs=1 TO N
1040  B=Ea(Qs)
1050  Aee=Ae*Pc(Qs)
1060  Ba=B*A
1070  Bda=B/A
1080  Cb=C*B
1090  Ba1=1.-Ba
1100  Cb1=1.-Cb
1110  Gba=Ba^Ka/Ba1
1120  Ab=A-B
1130  F=B*A1/(Ab*Ba1)
1140  S1=Cb^Ka/Cb1*(A1+Cb1)/(Ab*Ac)
1150  Gcb(L1)=Cb^L1/Cb1
1160  Gbda(L1)=Bda^L1/(1.-Bda)
```

```
1170    FOR Ks=L11 TO L2
1180    Gcb(Ks)=Cb*Gcb(Ks-1)
1190    Gbda(Ks)=Bda*Gbda(Ks-1)
1200    NEXT Ks
1210    Ki=0
1220    FOR Ks1=Kc TO Kd
1230    K1=MIN(Ks1,Kb)+1
1240    S2=S1+Gcb(K1)*F-Gba*Gcda(K1)
1250    S3=Gca(K1)-Gca
1260    E=Aee*E(Ns,Ks1)
1270    FOR Ks2=Ks1 TO Kd
1280    Ki=Ki+1
1290    K2=MIN(Ks2,Kb)+1
1300    S=S2+Gbda(K2)*S3       !  SUM S
1310    Cs(Ki)=Cs(Ki)+E*E(Qs,Ks2)*S
1320    NEXT Ks2
1330    NEXT Ks1
1340    NEXT Qs
1350    NEXT Ns
1360    NEXT Js
1370    L=Kd-Kc+1              !  TOTAL WEIGHTED COVARIANCE MATRIX
1380    REDIM Cw(1:L,1:L),Ca(1:L,1:L),D(1:L),Eig(0:Nr,1:L)
1390    L11=Kc-1
1400    R=0.
1410    FOR I=0 TO Nr
1420    Ki=0
1430    IF I=0 THEN 1450
1440    R=10.^(R(I)*.1)        !  INPUT POWER SIGNAL-TO-NOISE RATIO
1450    FOR Ks1=Kc TO Kd
1460    W=W(Ks1)               !  WEIGHTS (w(k))
1470    FOR Ks2=Ks1 TO Kd
1480    IF Ks1>=Ko THEN 1510
1490    Cs=0.
1500    GOTO 1530
1510    Ki=Ki+1
1520    Cs=Cs(Ki)
1530    Pr=W*W(Ks2)*(Cn(Ks2-Ks1)+R*Cs)
1540    L1=Ks1-L11
1550    L2=Ks2-L11
1560    Cw(L1,L2)=Cw(L2,L1)=Pr
1570    NEXT Ks2
1580    NEXT Ks1
1590    CALL Svd(L,L,Cw(*),Ca(*),D(*))  !  EIGENVALUES
1600    MAT SORT D(*) DES
1610    IF D(L)>0. THEN 1640
1620    PRINT "PROBLEM: SOME NON-POSITIVE EIGENVALUES"
1630    PAUSE
1640    PRINT "I =";I;"   CONDITION NUMBER =";D(1)/D(L)
1650    PRINT D(*)
1660    PRINT
1670    FOR Ks=1 TO L
1680    Eig(I,Ks)=D(Ks)        !  STORE EIGENVALUES
1690    NEXT Ks
1700    NEXT I
1710    MASS STORAGE IS ":CS80,7"
1720    ASSIGN #1 TO "EIG"
1730    PRINT #1;Nr,Kc,Kd,Eig(*)
1740    ASSIGN #1 TO *
1750    END
1760    !
```

```
1770    SUB Weights(DOUBLE Kc,Kd,REAL W(*))
1780    DOUBLE Ks               !  INTEGER
1790    F=1.                    !  DECAY FACTOR (DIMENSIONLESS)
1800    W(Kd)=1.
1810    FOR Ks=Kd-1 TO Kc STEP -1
1820    W(Ks)=W(Ks+1)*F         !  EXPONENTIAL WEIGHTS
1830    NEXT Ks
1840    SUBEND
1850    !
1860    SUB Svd(DOUBLE M,N,REAL A(*),V(*),W(*))
1870    ALLOCATE Rv1(1:N)       !  NUMERICAL RECIPES, PAGES 60-64
1880    IF M>=N THEN 1910       !  A(*) IS OVER-WRITTEN
1890    PRINT "M<N IS DISALLOWED"
1900    PAUSE
1910    DOUBLE I,J,K,L,Its,Nm,Jj ! INTEGERS (NOT DOUBLE PRECISION)
1920    G=Scale=Anorm=0.
1930    FOR I=1 TO N
1940    L=I+1
1950    Rv1(I)=Scale*G
1960    G=S=Scale=0.
1970    IF I>M THEN 2250
1980    FOR K=I TO M
1990    Scale=Scale+ABS(A(K,I))
2000    NEXT K
2010    IF Scale=0. THEN 2250
2020    FOR K=I TO M
2030    Aa=A(K,I)=A(K,I)/Scale
2040    S=S+Aa*Aa
2050    NEXT K
2060    F=A(I,I)
2070    G=-SQR(S)
2080    IF F<0. THEN G=-G
2090    H=F*G-S
2100    A(I,I)=F-G
2110    IF I=N THEN 2220
2120    FOR J=L TO N
2130    S=0.
2140    FOR K=I TO M
2150    S=S+A(K,I)*A(K,J)
2160    NEXT K
2170    F=S/H
2180    FOR K=I TO M
2190    A(K,J)=A(K,J)+F*A(K,I)
2200    NEXT K
2210    NEXT J
2220    FOR K=I TO M
2230    A(K,I)=A(K,I)*Scale
2240    NEXT K
2250    W(I)=Scale*G
2260    G=S=Scale=0.
2270    IF (I>M) OR (I=N) THEN 2570
2280    FOR K=L TO N
2290    Scale=Scale+ABS(A(I,K))
2300    NEXT K
2310    IF Scale=0. THEN 2570
```

```
2320    FOR K=L TO N
2330    Aa=A(I,K)=A(I,K)/Scale
2340    S=S+Aa*Aa
2350    NEXT K
2360    F=A(I,L)
2370    G=-SQR(S)
2380    IF F<0. THEN G=-G
2390    H=F*G-S
2400    A(I,L)=F-G
2410    FOR K=L TO N
2420    Rv1(K)=A(I,K)/H
2430    NEXT K
2440    IF I=M THEN 2540
2450    FOR J=L TO M
2460    S=0.
2470    FOR K=L TO N
2480    S=S+A(J,K)*A(I,K)
2490    NEXT K
2500    FOR K=L TO N
2510    A(J,K)=A(J,K)+S*Rv1(K)
2520    NEXT K
2530    NEXT J
2540    FOR K=L TO N
2550    A(I,K)=A(I,K)*Scale
2560    NEXT K
2570    Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
2580    NEXT I
2590    FOR I=N TO 1 STEP -1
2600    IF I>=N THEN 2770
2610    IF G=0. THEN 2740
2620    FOR J=L TO N
2630    V(J,I)=A(I,J)/A(I,L)/G
2640    NEXT J
2650    FOR J=L TO N
2660    S=0.
2670    FOR K=L TO N
2680    S=S+A(I,K)*V(K,J)
2690    NEXT K
2700    FOR K=L TO N
2710    V(K,J)=V(K,J)+S*V(K,I)
2720    NEXT K
2730    NEXT J
2740    FOR J=L TO N
2750    V(I,J)=V(J,I)=0.
2760    NEXT J
2770    V(I,I)=1.
2780    G=Rv1(I)
2790    L=I
2800    NEXT I
2810    FOR I=N TO 1 STEP -1
2820    L=I+1
2830    G=W(I)
2840    IF I>=N THEN 2880
2850    FOR J=L TO N
2860    A(I,J)=0.
2870    NEXT J
```

```
2880    IF G=0. THEN 3050
2890    G=1./G
2900    IF I=N THEN 3010
2910    FOR J=L TO N
2920    S=0.
2930    FOR K=L TO M
2940    S=S+A(K,I)*A(K,J)
2950    NEXT K
2960    F=S/A(I,I)*G
2970    FOR K=I TO M
2980    A(K,J)=A(K,J)+F*A(K,I)
2990    NEXT K
3000    NEXT J
3010    FOR J=I TO M
3020    A(J,I)=A(J,I)*G
3030    NEXT J
3040    GOTO 3080
3050    FOR J=I TO M
3060    A(J,I)=0.
3070    NEXT J
3080    A(I,I)=A(I,I)+1.
3090    NEXT I
3100    FOR K=N TO 1 STEP -1
3110    FOR Its=1 TO 30
3120    FOR L=K TO 1 STEP -1
3130    Nm=L-1
3140    IF (ABS(Rv1(L))+Anorm)=Anorm THEN 3360
3150    IF (ABS(W(Nm))+Anorm)=Anorm THEN 3170
3160    NEXT L
3170    C=0.
3180    S=1.
3190    FOR I=L TO K
3200    F=S*Rv1(I)
3210    Rv1(I)=C*Rv1(I)
3220    IF (ABS(F)+Anorm)=Anorm THEN 3360
3230    G=W(I)
3240    H=SQR(F*F+G*G)
3250    W(I)=H
3260    H=1./H
3270    C=G*H
3280    S=-F*H
3290    FOR J=1 TO M
3300    Y=A(J,Nm)
3310    Z=A(J,I)
3320    A(J,Nm)=Y*C+Z*S
3330    A(J,I)=-Y*S+Z*C
3340    NEXT J
3350    NEXT I
3360    Z=W(K)
3370    IF L<>K THEN 3440
3380    IF Z>=0. THEN 3430
3390    W(K)=-Z
3400    FOR J=1 TO N
3410    V(J,K)=-V(J,K)
3420    NEXT J
3430    GOTO 3970
```

```
3440    IF Its<30 THEN 3470
3450    PRINT "NO CONVERGENCE IN 30 ITERATIONS"
3460    PAUSE
3470    X=W(L)
3480    Nm=K-1
3490    Y=W(Nm)
3500    G=Rv1(Nm)
3510    H=Rv1(K)
3520    F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
3530    G=SQR(F*F+1.)
3540    Aa=ABS(G)
3550    IF F<0. THEN Aa=-Aa
3560    F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
3570    C=S=1.
3580    FOR J=L TO Nm
3590    I=J+1
3600    G=Rv1(I)
3610    Y=W(I)
3620    H=S*G
3630    G=C*G
3640    Z=SQR(F*F+H*H)
3650    Rv1(J)=Z
3660    C=F/Z
3670    S=H/Z
3680    F=X*C+G*S
3690    G=-X*S+G*C
3700    H=Y*S
3710    Y=Y*C
3720    FOR Jj=1 TO N
3730    X=V(Jj,J)
3740    Z=V(Jj,I)
3750    V(Jj,J)=X*C+Z*S
3760    V(Jj,I)=-X*S+Z*C
3770    NEXT Jj
3780    Z=SQR(F*F+H*H)
3790    W(J)=Z
3800    IF Z=0. THEN 3840
3810    Z=1./Z
3820    C=F*Z
3830    S=H*Z
3840    F=C*G+S*Y
3850    X=-S*G+C*Y
3860    FOR Jj=1 TO M
3870    Y=A(Jj,J)
3880    Z=A(Jj,I)
3890    A(Jj,J)=Y*C+Z*S
3900    A(Jj,I)=-Y*S+Z*C
3910    NEXT Jj
3920    NEXT J
3930    Rv1(L)=0.
3940    Rv1(K)=F
3950    W(K)=X
3960    NEXT Its
3970    NEXT K
3980    SUBEND
```

```
10  ! PLOT ROC FROM EIGENVALUES "EIG", LINES 120 & 1060
20    Delxi=.08                !  INITIAL INCREMENT ON CHAR. FUNCTION
30    M=128                    !  INITIAL SIZE OF FFT
40    Bs=0.                    !  SHIFT b
50    Tol0=1.E-36              !  TOLERANCE FOR FALSE ALARM PROBABILITY
60    Tol1=1.E-32              !  TOLERANCE FOR DETECTION PROBABILITIES
70    DOUBLE M,Nr,Kc,Kd,Kdc,Ms,I,Ks,Ns,Im,Mm  !  INTEGERS
80    DIM E(0:20,1:100),D(1:100),Cos(0:4096)  !  20 NONZERO SNRs
90    DIM X(0:16383),Y(0:16383),Pr(0:20,0:127)
100   PRINT "Delxi =";Delxi,"M =";M,"b =";Bs
110   MASS STORAGE IS ":CS80,7"
120     ASSIGN #1 TO "EIG"
130   READ #1;Nr,Kc,Kd         !  NO. OF SNRs, START & END OF SUMMER
140   Kdc=Kd-Kc+1
150   REDIM E(0:Nr,1:Kdc),D(1:Kdc)
160   READ #1;E(*)             !  EIGENVALUES
170   GINIT
180   PLOTTER IS "GRAPHICS"
190   GRAPHICS ON
200   REDIM Cos(0:M/4),X(0:M-1),Y(0:M-1)
210   A=2.*PI/M
220   FOR Ms=0 TO M/4
230   Cos(Ms)=COS(A*Ms)        !  INITIAL QUARTER-COSINE TABLE
240   NEXT Ms
250   Tol=Tol0
260   FOR I=0 TO Nr            !  Nr NONZERO SIGNAL-TO-NOISE RATIOS
270   IF I>0 THEN Tol=Tol1
280   FOR Ks=1 TO Kdc
290   D(Ks)=E(I,Ks)            !  COPY EIGENVALUES
300   NEXT Ks
310   Mux=SUM(D)               !  MEAN OF RANDOM VARIABLE x
320   R=0.                     !  ARGUMENT OF SQUARE ROOT
330   P=1.                     !  POLARITY INDICATOR
340   Muy=Mux+Bs               !  MEAN OF y = x + b
350   MAT X=(0.)
360   MAT Y=(0.)
370   Y(0)=.5*Delxi*Muy
380   Ns=0
390   Ns=Ns+1                  !  LOOP ON xi
400   Xi=Delxi*Ns              !  ARGUMENT xi OF CHAR. FN.
410   Txi=Xi*2.                !  CALCULATION OF
420   Pr=1.                    !  CHARACTERISTIC
430   Pi=0.                    !  FUNCTION fy(xi)
440   FOR Ks=1 TO Kdc
450   Te=Txi*D(Ks)
460   Pe=Pr+Te*Pi
470   Pi=Pi-Te*Pr
480   Pr=Pe
490   NEXT Ks
500   CALL Sqr(Pr,Pi,Sr,Si)
510   De=Sr*Sr+Si*Si
520   Fyr=Sr/De
530   Fyi=-Si/De
540   Ro=R
550   R=ATN(Fyi/Fyr)
560   IF ABS(R-Ro)>1.6 THEN P=-P
570   IF P>0. THEN 600
```

```
 580    Fyr=-Fyr
 590    Fyi=-Fyi
 600    Ms=Ns MODULO M          !  COLLAPSING
 610    A=Fyr/Ns
 620    B=Fyi/Ns
 630    X(Ms)=X(Ms)+A
 640    Y(Ms)=Y(Ms)+B
 650    IF A*A+B*B<Tol THEN 670
 660    GOTO 390
 670    CALL Fft14(M,Cos(*),X(*),Y(*))
 680    GCLEAR
 690    WINDOW 0,M,-18,0
 700    LINE TYPE 3
 710    GRID M/8,1
 720    LINE TYPE 1
 730    FOR Ks=0 TO M-1
 740    T=Y(Ks)/PI-Ks/M
 750    Y(Ks)=Pr=.5+T                    ! EXCEEDANCE PROBABILITY IN Y(*)
 760    IF Pr>=1.E-16 THEN Y=LGT(Pr)!Y(Ks)=PROB(x>2 PI Ks/(M Delxi)-Bs)
 770    IF Pr<=-1.E-16 THEN Y=-32.-LGT(-Pr)
 780    IF ABS(Pr)<1.E-16 THEN Y=-16.
 790    PLOT Ks,Y
 800    NEXT Ks
 810    PENUP
 820    IF I=Nr THEN 940
 830    PRINT I;
 840    INPUT "SCALE FFT SIZE BY 1,2,4,8,...:",Im
 850    Mm=MIN(M*Im,16384)
 860    IF Mm=M THEN 940
 870    Delxi=Delxi*M/Mm
 880    M=Mm
 890    REDIM Cos(0:M/4),X(0:M-1),Y(0:M-1)
 900    A=2.*PI/M
 910    FOR Ms=0 TO M/4
 920    Cos(Ms)=COS(A*Ms)       !  QUARTER-COSINE TABLE; <= 4096
 930    NEXT Ms
 940    FOR Ks=0 TO 127
 950    Pr(I,Ks)=Y(Ks)          !  STORE FOR ROC PLOT
 960    NEXT Ks
 970    NEXT I
 980    FOR Ks=0 TO 127
 990    IF Pr(0,Ks)<.1 THEN 1010
1000    NEXT Ks
1010    Ms=Ks-1
1020    FOR Ks=Ms TO 127
1030    IF Pr(0,Ks)<1E-10 THEN 1050
1040    NEXT Ks
1050    Ns=Ks
1060      ASSIGN #1 TO "ROC"
1070    PRINT #1;Ms,Ns,Nr
1080    PRINT #1;Pr(*)
1090    ASSIGN #1 TO *
1100    DIM Pf(1:14),Pd(1:18)
1110    DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4
1120    DATA .001,.002,.005,.01,.02,.05,.1
1130    READ Pf(*)
```

```
1140 DATA .01,.02,.05,.1,.2,.3,.4,.5,.6,.7
1150 DATA .8,.9,.95,.98,.99,.995,.998,.999
1160 READ Pd(*)
1170 FOR I=1 TO 14
1180 Pf(I)=FNInvphi(Pf(I))
1190 NEXT I
1200 FOR I=1 TO 18
1210 Pd(I)=FNInvphi(Pd(I))
1220 NEXT I
1230 GCLEAR
1240 X1=Pf(1)
1250 X2=Pf(14)
1260 Y1=Pd(1)
1270 Y2=Pd(18)
1280 WINDOW X1,X2,Y1,Y2
1290 LINE TYPE 3
1300 FOR I=1 TO 14
1310 MOVE Pf(I),Y1
1320 DRAW Pf(I),Y2
1330 NEXT I
1340 FOR I=1 TO 18
1350 MOVE X1,Pd(I)
1360 DRAW X2,Pd(I)
1370 NEXT I
1380 PENUP
1390 LINE TYPE 1
1400 FOR Ks=Ms TO Ns
1410 Pr(0,Ks)=FNInvphi(Pr(0,Ks)) ! FALSE ALARM PROBABILITY
1420 NEXT Ks
1430 FOR I=1 TO Nr
1440 FOR Ks=Ms TO Ns
1450 X=Pr(0,Ks)
1460 Pr=Pr(I,Ks)
1470 IF Pr>.9999 THEN 1510
1480 Pr(I,Ks)=Y=FNInvphi(Pr)
1490 PLOT X,Y
1500 IF Pr<.01 THEN 1520
1510 NEXT Ks
1520 PENUP
1530 NEXT I
1540 PAUSE
1550 END
1560 !
1570 DEF FNArg(X,Y)          !  PRINCIPAL ARG(Z)
1580 IF X=0. THEN RETURN .5*PI*SGN(Y)
1590 A=ATN(Y/X)
1600 IF X>0. THEN RETURN A
1610 IF Y<0. THEN RETURN A-PI
1620 RETURN A+PI
1630 FNEND
1640 !
1650 SUB Sqr(X,Y,U,V)        !  PRINCIPAL SQR(Z)
1660 F=SQR(SQR(X*X+Y*Y))
1670 T=.5*FNArg(X,Y)
1680 U=F*COS(T)
1690 V=F*SIN(T)
1700 SUBEND
1710 !
```

```
1720    DEF FNInvphi(X)        !  AMS 55, 26.2.23
1730    IF X=.5 THEN RETURN 0.
1740    P=MIN(X,1.-X)
1750    P=MAX(P,1.E-20)
1760    T=-LOG(P)
1770    T=SQR(T+T)
1780    P=1.+T*(1.432788+T*(.189269+T*.001308))
1790    P=T-(2.515517+T*(.802853+T*.010328))/P
1800    IF X<.5 THEN P=-P
1810    RETURN P
1820    FNEND
1830    !
1840    SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
1850    DOUBLE Log2n,N1,N2,N3,N4,J,K    !  INTEGERS < 2^31 = 2,147,483,648
1860    DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1870    IF N=1 THEN SUBEXIT
1880    IF N>2 THEN 1960
1890    A=X(0)+X(1)
1900    X(1)=X(0)-X(1)
1910    X(0)=A
1920    A=Y(0)+Y(1)
1930    Y(1)=Y(0)-Y(1)
1940    Y(0)=A
1950    SUBEXIT
1960    A=LOG(N)/LOG(2.)
1970    Log2n=A
1980    IF ABS(A-Log2n)<1.E-8 THEN 2010
1990    PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
2000    PAUSE
2010    N1=N/4
2020    N2=N1+1
2030    N3=N2+1
2040    N4=N3+N1
2050    FOR I1=1 TO Log2n
2060    I2=2^(Log2n-I1)
2070    I3=2*I2
2080    I4=N/I3
2090    FOR I5=1 TO I2
2100    I6=(I5-1)*I4+1
2110    IF I6<=N2 THEN 2150
2120    A1=-Cos(N4-I6-1)
2130    A2=-Cos(I6-N1-1)
2140    GOTO 2170
2150    A1=Cos(I6-1)
2160    A2=-Cos(N3-I6-1)
2170    FOR I7=0 TO N-I3 STEP I3
2180    I8=I7+I5-1
2190    I9=I8+I2
2200    T1=X(I8)
2210    T2=X(I9)
2220    T3=Y(I8)
2230    T4=Y(I9)
```

```
2240    A3=T1-T2
2250    A4=T3-T4
2260    X(I8)=T1+T2
2270    Y(I8)=T3+T4
2280    X(I9)=A1*A3-A2*A4
2290    Y(I9)=A1*A4+A2*A3
2300    NEXT I7
2310    NEXT I5
2320    NEXT I1
2330    I1=Log2n+1
2340    FOR I2=1 TO 14
2350    L(I2-1)=1
2360    IF I2>Log2n THEN 2380
2370    L(I2-1)=2^(I1-I2)
2380    NEXT I2
2390    K=0
2400    FOR I1=1 TO L(13)
2410    FOR I2=I1 TO L(12) STEP L(13)
2420    FOR I3=I2 TO L(11) STEP L(12)
2430    FOR I4=I3 TO L(10) STEP L(11)
2440    FOR I5=I4 TO L(9) STEP L(10)
2450    FOR I6=I5 TO L(8) STEP L(9)
2460    FOR I7=I6 TO L(7) STEP L(8)
2470    FOR I8=I7 TO L(6) STEP L(7)
2480    FOR I9=I8 TO L(5) STEP L(6)
2490    FOR I10=I9 TO L(4) STEP L(5)
2500    FOR I11=I10 TO L(3) STEP L(4)
2510    FOR I12=I11 TO L(2) STEP L(3)
2520    FOR I13=I12 TO L(1) STEP L(2)
2530    FOR I14=I13 TO L(0) STEP L(1)
2540    J=I14-1
2550    IF K>J THEN 2620
2560    A=X(K)
2570    X(K)=X(J)
2580    X(J)=A
2590    A=Y(K)
2600    Y(K)=Y(J)
2610    Y(J)=A
2620    K=K+1
2630    NEXT I14
2640    NEXT I13
2650    NEXT I12
2660    NEXT I11
2670    NEXT I10
2680    NEXT I9
2690    NEXT I8
2700    NEXT I7
2710    NEXT I6
2720    NEXT I5
2730    NEXT I4
2740    NEXT I3
2750    NEXT I2
2760    NEXT I1
2770    SUBEND
```

```
10  ! SIMULATION FOR ONE-POLE PROCESSES
20    As=.85                   !  INPUT SIGNAL PARAMETER
30    Bs=.77                   !  INPUT NOISE  PARAMETER; LINE 160
40    Cs=1.13                  !  FILTER PARAMETER
50    R=.7                     !  INPUT POWER SNR
60    Ka=4
70    Kb=11
80    Kc=2
90    Kd=16
100   Bins=1000
110   Num=1E6                  !  NUMBER OF TRIALS
120   Deltaz=.25               !  BIN SIZE FOR OUTPUT Z
130   DOUBLE Ka,Kb,Kc,Kd,Bins,Num,It,K,M
140   DIM E(100),S(100),Yn(100),Ys(100),T(1000)
150   Ea=EXP(-As)
160   Eb=EXP(-Bs)              !  WHITE NOISE: FOR Bs=inf, SET Eb=0
170   Ec=EXP(-Cs)
180   Fa=SQR(R*(1.-Ea*Ea))
190   Fb=SQR(1.-Eb*Eb)
200   REDIM E(0:Kd-Ka)
210   E(0)=Cs
220   FOR K=1 TO Kd-Ka
230   E(K)=E(K-1)*Ec
240   NEXT K
250   REDIM S(-30:Kd),Yn(Kc-1:Kd),Ys(Ka:Kd),T(1:Bins)
260   FOR It=1 TO Num
270   S(-30)=Nk=Yk=0.
280   FOR K=-29 TO Kc-1        !  ALLOW STEADY STATE
290   R1=2.*RND-1.             !  BOX-MULLER
300   R2=2.*RND-1.
310   R3=R1*R1+R2*R2
320   IF R3>1. THEN 290
330   R3=SQR(-2.*LOG(R3)/R3)
340   R1=R1*R3
350   R2=R2*R3
360   S(K)=Fa*R1+Ea*S(K-1) !  FILTER INPUT  SIGNAL
370   Nk=Fb*R2+Eb*Nk           !  FILTER INPUT  NOISE
380   Yk=Cs*Nk+Ec*Yk           !  FILTER OUTPUT NOISE
390   NEXT K
400   Yn(Kc-1)=Yk
410   FOR K=Kc TO Kd
420   R1=2.*RND-1.
430   R2=2.*RND-1.
440   R3=R1*R1+R2*R2
450   IF R3>1. THEN 420
460   R3=SQR(-2.*LOG(R3)/R3)
470   R1=R1*R3
480   R2=R2*R3
490   M=K-1
```

```
500     S(K)=Fa*R1+Ea*S(M)    !  FILTER INPUT  SIGNAL
510     Nk=Fb*R2+Eb*Nk        !  FILTER INPUT  NOISE
520     Yn(K)=Cs*Nk+Ec*Yn(M) !  FILTER OUTPUT NOISE
530     NEXT K
540     FOR K=Ka TO Kd
550     S=0.
560     FOR M=Ka TO MIN(K,Kb)
570     S=S+E(K-M)*S(M)       !  FILTER OUTPUT SIGNAL
580     NEXT M
590     Ys(K)=S
600     NEXT K
610     Z=0.
620     FOR K=Kc TO Ka-1
630     T=Yn(K)
640     Z=Z+T*T
650     NEXT K
660     FOR K=Ka TO Kd
670     T=Ys(K)+Yn(K)
680     Z=Z+T*T
690     NEXT K
700     K=INT(Z/Deltaz)+1
710     K=MIN(K,Bins)
720     T(K)=T(K)+1.
730     NEXT It
740     FOR K=Bins-1 TO 1 STEP -1
750     T(K)=T(K)+T(K+1)
760     NEXT K
770     MAT T=T/(Num)
780     GINIT
790     PLOTTER IS "GRAPHICS"
800     GRAPHICS ON
810     WINDOW 0,Bins,-5,0
820     GRID Bins/4,1
830     FOR K=1 TO Bins
840     T=T(K)
850     IF T>0. THEN 870
860     GOTO 890
870     PLOT K,LGT(T)
880     NEXT K
890     PENUP
900     FOR K=Bins TO 1 STEP -1
910     T=1.-T(K)
920     IF T>0. THEN 940
930     GOTO 960
940     PLOT K,LGT(T)
950     NEXT K
960     PENUP
970     PAUSE
980     DUMP GRAPHICS
990     END
```

APPENDIX E. PROGRAM FOR COLORED INPUT NOISE

One program is listed in this appendix. It computes the covariance matrices and eigenvalues as described in appendix D, but now for a colored noise input. The program for computing the receiver operating characteristics from the eigenvalues is identical to that listed above and therefore has not been repeated here.

```
10    ! COMPUTE COVARIANCE MATRICES AND EIGENVALUES
20    ! STORE EIGENVALUES IN "EIG" IN LINE 2260
30      Ka=-10                  !  INPUT SIGNAL START
40      Kb=5                    !  INPUT SIGNAL END; Kb >= Ka
50      Kc=0                    !  ACCUMULATOR  START
60      Kd=5                    !  ACCUMULATOR  END; Kd >= Kc,Ka
70      Delta=.2                !  TIME SAMPLING INCREMENT (SECONDS)
80      J=3                     !  NUMBER OF SIGNAL COMPONENTS
90      DATA 11.,-48.,75.       !  SIGNAL SCALINGS (WE SET SUM = 1)
100     DATA 1.,.125,.066667    !  SIGNAL TIME CONSTANTS (SECONDS)
110     M=2                     !  NUMBER OF NOISE COMPONENTS
120     DATA 39.,60.            !  NOISE SCALINGS (WE SET SUM = 1)
130     DATA .2,.4              !  NOISE TIME CONSTANTS (SECONDS)
140     N=4                     !  NUMBER OF FILTER COMPONENTS
150     DATA 1.,2.,3.,4.        !  FILTER SCALINGS (ARBITRARY)
160     DATA .3,.5,.7,.9        !  FILTER TIME CONSTANTS (SECONDS)
170     Nr=20                   !  NUMBER OF SIGNAL-TO-NOISE RATIOS
180     DATA 0,1,2,3,4,5,6,7,8,9,10             !  INPUT SNRS IN DB
190     DATA 11,12,13,14,15,16,17,18,19
200     IF (Ka<=Kb) AND (Kc<=Kd) AND (Ka<=Kd) THEN 230
210     PRINT "PROBLEM WITH PARAMETERS"
220     STOP
230     DIM Alpha(10),A(10),Beta(10),B(10),Psi(10),C(10),R(20)
240     DIM W(100),Pc(10),Ec(10),Pcv(10),Eb(10),Chb(10),Shb(10)
250     DIM Chc(10),Shc(10),Mu(10),Gamma(10),Cn(200),Ea(10)
260     DIM E(0,1000),Gca(100),Gcda(100),Gcb(100),Gbda(100)
270     DIM Cs(5000),Cw(0,10000),Ca(0,10000),D(100),Eig(0,2000)
280     DOUBLE Ka,Kb,Kc,Kd,J,M,N,Nr,Ns,Ks,Ms,Kdc,Ko
290     DOUBLE L1,L2,L,L11,Js,Qs,Ki,Ks1,K1,Ks2,K2,I
300     REDIM Alpha(1:J),A(1:J)
310     READ Alpha(*),A(*)
320     MAT Alpha=Alpha/(SUM(Alpha))
330     MAT A=(Delta)/A
340     REDIM Beta(1:M),B(1:M)
350     READ Beta(*),B(*)
360     MAT Beta=Beta/(SUM(Beta))
370     MAT B=(Delta)/B
380     REDIM Psi(1:N),C(1:N)
390     READ Psi(*),C(*)
400     MAT C=(Delta)/C
410     REDIM R(1:Nr)
420     READ R(*)
430     REDIM W(Kc:Kd)
440     CALL Weights(Kc,Kd,W(*))
450     MAT W=SQR(W)
```

```
460   REDIM Pc(1:N),Ec(1:N) ! FILTER OUTPUT NOISE COVARIANCE
470   FOR Ns=1 TO N
480   C=C(Ns)
490   Pc(Ns)=Psi(Ns)*C
500   Ec(Ns)=EXP(-C)
510   NEXT Ns
520   REDIM Pcv(1:N)
530   FOR Ns=1 TO N
540   E=Ec(Ns)
550   S=0.
560   FOR Ks=1 TO N
570   S=S+Pc(Ks)/(1.-E*Ec(Ks))
580   NEXT Ks
590   Pcv(Ns)=Pc(Ns)*S
600   NEXT Ns
610   REDIM Eb(1:M)
620   FOR Ms=1 TO M
630   Eb(Ms)=EXP(-B(Ms))
640   NEXT Ms
650   REDIM Chb(1:M),Shb(1:M)
660   FOR Ms=1 TO M
670   E=Eb(Ms)
680   R=1./E
690   Chb(Ms)=.5*(R+E)
700   Shb(Ms)=.5*(R-E)
710   NEXT Ms
720   REDIM Chc(1:N),Shc(1:N)
730   FOR Ns=1 TO N
740   E=Ec(Ns)
750   R=1./E
760   Chc(Ns)=.5*(R+E)
770   Shc(Ns)=.5*(R-E)
780   NEXT Ns
790   REDIM Mu(1:M)
800   FOR Ms=1 TO M
810   Chb=Chb(Ms)
820   S=0.
830   FOR Ns=1 TO N
840   S=S+Pcv(Ns)*Shc(Ns)/(Chc(Ns)-Chb)
850   NEXT Ns
860   Mu(Ms)=Beta(Ms)*S
870   NEXT Ms
880   REDIM Gamma(1:N)
890   FOR Ns=1 TO N
900   Chc=Chc(Ns)
910   S=0.
920   FOR Ms=1 TO M
930   S=S+Beta(Ms)*Shb(Ms)/(Chb(Ms)-Chc)
940   NEXT Ms
950   Gamma(Ns)=Pcv(Ns)*S
960   NEXT Ns
970   Kdc=Kd-Kc
980   REDIM Cn(0:Kdc)
990   MAT Cn=(0.)
1000  FOR Ms=1 TO M
1010  Mu=Mu(Ms)
1020  E=Eb(Ms)
1030  P=1.
1040  Cn(0)=Cn(0)+Mu
1050  FOR Ks=1 TO Kdc
1060  P=P*E
```

```
1070    Cn(Ks)=Cn(Ks)+Mu*P
1080    NEXT Ks
1090    NEXT Ms
1100    FOR Ns=1 TO N
1110    Ga=Gamma(Ns)
1120    E=Ec(Ns)
1130    P=1.
1140    Cn(0)=Cn(0)+Ga
1150    FOR Ks=1 TO Kdc
1160    P=P*E
1170    Cn(Ks)=Cn(Ks)+Ga*P
1180    NEXT Ks
1190    NEXT Ns
1200    FOR Js=1 TO J          !  FILTER OUTPUT SIGNAL COVARIANCE
1210    A(Js)=EXP(A(Js))
1220    NEXT Js
1230    Ko=MAX(Ka,Kc)
1240    REDIM Ea(1:N),E(1:N,Ko:Kd)
1250    FOR Ns=1 TO N
1260    E=Ec(Ns)
1270    Ea(Ns)=1./E
1280    E(Ns,Ko)=EXP(-C(Ns)*Ko)
1290    FOR Ks=Ko+1 TO Kd
1300    E(Ns,Ks)=E(Ns,Ks-1)*E
1310    NEXT Ks
1320    NEXT Ns
1330    L1=MIN(Ko,Kb)+1
1340    L2=MIN(Kd,Kb)+1
1350    REDIM Gca(L1:L2),Gcda(L1:L2),Gcb(L1:L2),Gbda(L1:L2)
1360    L=Kd-Ko+1
1370    REDIM Cs(1:L*(L+1)/2)
1380    L11=L1+1
1390    FOR Js=1 TO J
1400    Al=Alpha(Js)
1410    A=A(Js)
1420    A1=A*A-1.
1430    FOR Ns=1 TO N
1440    C=Ea(Ns)
1450    Ca=C*A
1460    Cda=C/A
1470    Ca1=1.-Ca
1480    Ac=A-C
1490    Ae=A1*Pc(Ns)
1500    Gca=Ca^Ka/Ca1
1510    Gca(L1)=Ca^L1/Ca1
1520    Gcda(L1)=Cda^L1/(1.-Cda)
1530    FOR Ks=L11 TO L2
1540    Gca(Ks)=Ca*Gca(Ks-1)
1550    Gcda(Ks)=Cda*Gcda(Ks-1)
1560    NEXT Ks
1570    FOR Os=1 TO N
1580    B=Ea(Os)
1590    Aee=Ae*Pc(Os)
1600    Ba=B*A
1610    Bda=B/A
1620    [illegible]*B
1630    Ba1=1.-Ba
1640    Cb1=1.-Cb
1650    Gba=Ba^Ka/Ba1
1660    Ab=A-[illegible]
1670    [illegible]=B*A1/(Ab*Ba1)
```

```
1680    S1=Cb^Ka/Cb1*(A1+Cb1)/(Ab*Ac)
1690    Gcb(L1)=Cb^L1/Cb1
1700    Gbda(L1)=Bda^L1/(1.-Bda)
1710    FOR Ks=L11 TO L2
1720    Gcb(Ks)=Cb*Gcb(Ks-1)
1730    Gbda(Ks)=Bda*Gbda(Ks-1)
1740    NEXT Ks
1750    Ki=0
1760    FOR Ks1=Ko TO Kd
1770    K1=MIN(Ks1,Kb)+1
1780    S2=S1+Gcb(K1)*F-Gba*Gcda(K1)
1790    S3=Gca(K1)-Gca
1800    E=Aee*E(Ns,Ks1)
1810    FOR Ks2=Ks1 TO Kd
1820    Ki=Ki+1
1830    K2=MIN(Ks2,Kb)+1
1840    S=S2+Gbda(K2)*S3       !  SUM S
1850    Cs(Ki)=Cs(Ki)+E*E(Qs,Ks2)*S
1860    NEXT Ks2
1870    NEXT Ks1
1880    NEXT Qs
1890    NEXT Ns
1900    NEXT Js
1910    L=Kd-Kc+1              !  TOTAL WEIGHTED COVARIANCE MATRIX
1920    REDIM Cw(1:L,1:L),Ca(1:L,1:L),D(1:L),Eig(0:Nr,1:L)
1930    L11=Kc-1
1940    R=0.
1950    FOR I=0 TO Nr
1960    Ki=0
1970    IF I=0 THEN 1990
1980    R=10.^(R(I)*.1)        !  INPUT POWER SIGNAL-TO-NOISE RATIO
1990    FOR Ks1=Kc TO Kd
2000    W=W(Ks1)               !  WEIGHTS (w(k))
2010    FOR Ks2=Ks1 TO Kd
2020    IF Ks1>=Ko THEN 2050
2030    Cs=0.
2040    GOTO 2070
2050    Ki=Ki+1
2060    Cs=Cs(Ki)
2070    Pr=W*W(Ks2)*(Cn(Ks2-Ks1)+R*Cs)
2080    L1=Ks1-L11
2090    L2=Ks2-L11
2100    Cw(L1,L2)=Cw(L2,L1)=Pr
2110    NEXT Ks2
2120    NEXT Ks1
2130    CALL Svd(L,L,Cw(*),Ca(*),D(*))   !  EIGENVALUES
2140    MAT SORT D(*) DES
2150    IF D(L)>0. THEN 2180
2160    PRINT "PROBLEM: SOME NON-POSITIVE EIGENVALUES"
2170    PAUSE
2180    PRINT "I =";I;"   CONDITION NUMBER =";D(1)/D(L)
2190    PRINT D(*)
2200    PRINT
2210    FOR Ks=1 TO L
2220    Eig(I,Ks)=D(Ks)        !  STORE EIGENVALUES
2230    NEXT Ks
2240    NEXT I
2250    MASS STORAGE IS ":CS80,7"
2260    ASSIGN #1 TO "EIG"
2270    PRINT #1;Nr,Kc,Kd,Eig(*)
2280    ASSIGN #1 TO *
2290    END
```

REFERENCES

[1] F. B. Hildebrand, **Methods of Applied Mathematics**, Prentice-Hall, Inc., New York, NY, 1954.

[2] A. H. Nuttall, **Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions**, NUSC Technical Report 7023, Naval Underwater Systems Center, New London, CT, 1 October 1983.

[3] A. H. Nuttall, **Operating Characteristics for Weighted Energy Detector with Gaussian Signals**, NUSC Technical Report 8753, Naval Underwater Systems Center, New London, CT, 16 July 1990.