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Approximate Reasoning in Expert Systems
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ON FOUNDATIONS OF APPROXIMATE REASONING

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This paper discusses basic elements required for developing a theory of approximate reasoning which is essential for designing knowledge-based systems which, in turn, can be directly utilized in expert systems. First, formal language will be used as a general framework of analysis. Secondly, semantic evaluation procedures will be developed. Thirdly, the concept of general logical systems will be introduced. Finally, some discussions on uncertainty measures and the problem of admissibility will be discussed.

Keywords: Approximate reasoning, dispersions, formal language, general logical systems, semantic evaluations, uncertainty measures.

1. INTRODUCTION

The analysis of knowledge-based systems consists mainly of developing appropriate measures of uncertainty and establishing theories of inference, i.e., systems consisting of axioms, rules and deductions based recursively on them. For example, in statistical systems, semantic evaluations are in the form of probability logic (PL) which supplies the basic operators and relations for the calculus of probability. In addition, to this basic structure is added various theories such as the theory of martingales, the theory of random sets and the theories of hypotheses testing and estimation. In order to establish a unified approach to uncertainty modeling, which includes PL and Zadeh's fuzzy logic (FL), we must first consider formal language and the symbolization of cognitive processes from a general viewpoint. Specifically, in section 2, aspects of formal language needed for our purpose will be established. In section 3, semantic evaluation procedures (realizations of formal language) will be carried out in a general algebraic structure known as deduction category. If sufficient logical and set-like properties are added to this structure, the resulting structure, called a topos,

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is very useful in both generalizing and analyzing of set theory for a multi-valued truth space. On the other hand, often additional structures are data driven, i.e., determined from empirical considerations of problems at hand and do not lead to a topos, but, to other systems involving logical connectors, i.e., logics, and set theory evaluations. For example, the class of operators (nt, &, or) may be modelled in terms of general negations, t-norms and t-conorms which, in general, are not compatible with topos structure (Indeed, a topos structure is most compatible with a particular type of logic namely intuitionistic logic). This leads to the umbrella-concept of general logical systems and dispersions which is given in section 4. General logical systems are semantic evaluations within a category-like structure. A common example of this is characterized by the four-tuple (nt, &, or, $\mathcal{F}$) where $\mathcal{F}$ is a mapping from the class of all sets into the class of all functions, called the dispersion map. Finally, we discuss uncertainty measures, including their admissibility, in section 5. Uncertainty measures, such as probability measures, Choquet's capacity-type measures, can be described in two equivalent ways: as set-functions (numerical-valued functions on sets) and as semantic evaluations of relations, special cases of dispersions.

2. FORMAL LANGUAGE AS A GENERAL FRAMEWORK OF ANALYSIS.

A (higher order) formal language $\mathcal{L}$ consists of a core, Core($\mathcal{L}$), syntax, $\mathrm{Th}_{\mathrm{Syn}}(\mathcal{L})$, basic variables part, $\overline{\mathrm{Var}}(\mathcal{L})$, well-formed expressions part, Wfex($\mathcal{L}$), and free individual variables and bound individual variables mappings, FV($\mathcal{L}$), BnV($\mathcal{L}$), and finally an additional theory $\mathrm{Th}_{K}(\mathcal{L})$ (optional).

Core($\mathcal{L}$) is the type-category which normally includes as its "objects" Ob(Core($\mathcal{L}$)) the sorts, Ob($\mathcal{L}$) refers to objects of $\mathcal{L}$, Ar($\mathcal{L}$) refers to basic relations of $\mathcal{L}$ and all compound (i.e., cartesian product, exponentiation, etc.) of sorts; and which has as its class of arrows, Ar(Core($\mathcal{L}$)), the function symbols (including Loc($\mathcal{L}$) = the logical connectors, and Quant($\mathcal{L}$) = the quantifiers) of $\mathcal{L}$, including the distinguished symbols representing signature (σ), substitution ([ ]), identity, projection, cartesian product, (set) membership, (set) equivalence, (set) abstraction, truth, negation, conjunction, disjunction, implication ($\Rightarrow$), universal quantifiers, etc.

$\mathrm{Th}_{\mathrm{Syn}}(\mathcal{L})$ (theory of syntax of $\mathcal{L}$) consists of, as any theory, a set of syntax axioms and rules, from which, recursively, the deducts of the syntax are obtained. The deducts of the theory are Wfex($\mathcal{L}$).

$\overline{\mathrm{Var}}(\mathcal{L})$ forms a deduction category (see e.g., Eytan [3]) (Var($\mathcal{L}$), $\mathcal{R}(\mathcal{L})$), where Var($\mathcal{L}$) is a category consisting of Var(Ob($\mathcal{L}$)), the set of basic objects and Var(Ar($\mathcal{L}$)), the set of arrows of $\mathcal{L}$, including Wfv($\mathcal{L}$), the class of projection-arrows or individual variables of $\mathcal{L}$ and Wfc($\mathcal{L}$), the individual constants, and where $\mathcal{R}(\mathcal{L})$: Var($\mathcal{L}$) $\rightarrow$ Rel($\mathcal{L}$), is a contravariant functor, where $\mathcal{R}(\mathcal{L})$ acting on Var(Ob($\mathcal{L}$)) yields Var(Rel($\mathcal{L}$)) $\subseteq$ Preord (the category of preordered sets, where order here is determined by the predicate symbol $\vdash$ representing formal deduction for $\mathcal{L}$, specified in $\mathrm{Th}_{K}(\mathcal{L})$ and $\mathcal{R}(\mathcal{L})$ acts compatibly on Var(Ar($\mathcal{L}$))).

Roughly speaking, $Wfex(\mathcal{L}) = Var(Ar(\mathcal{L})) \cup Var(Rel(\mathcal{L}))$ involving projections using the substitution symbol [ ] together with logical connectors such as negations, conjunctions, etc.

$FV(\mathcal{L}) : Wfex(\mathcal{L}) \to \mathcal{P}(Wfv(\mathcal{L}))$, where $FV(\mathcal{L})(\varphi)$, for any $\varphi \in Wfex(\mathcal{L})$, determines the set of all individual variables of φ that are not governed by quantifiers.

Similarly,

$$BnV(\mathcal{L}) : Wfex(\mathcal{L}) \to \mathcal{P}(Wfv(\mathcal{L}))$$

determines the set of all (quantifiably) bound individual variables of any given φ. In turn, the class of sentences of $\mathcal{L}$, $Sent(\mathcal{L})$, forms a subset of $Wfex(\mathcal{L})$ consisting of those φ for which $FV(\mathcal{L})(\varphi) = \phi$.

Finally, the theory $Th_K(\mathcal{L})$ is added theory in addition to $Th_{Syn}(\mathcal{L})$, operating upon the class of all sequents, $Seq(\mathcal{L}) = \{(\varphi \vdash \theta \mid \varphi, \theta \in Wfex(\mathcal{L})\}$ with the usual structure of a theory: axioms, rules, resulting in $De(Th_K(\mathcal{L}))$, the deducts or theorems for $Th_K(\mathcal{L})$. For further details, see Goodman and Nguyen [6], Chapter 2.

3. APPROXIMATE REASONING AS A SEMANTIC-BASED DEDUCTION PROCESS.

Based upon the formal language structure presented in the previous section, we develop now semantic evaluation procedures in order to investigate approximate reasoning.

3.1. Semantic Evaluations with Values in a Deduction Category.

For reasons which will appear later, we consider first semantic evaluations with values in a deduction category. Our approach is based on Coste [2], Eytan [3],[4] and Johnstone [7]. See also Goodman and Nguyen [6].

Let $\mathcal{L}$ be a given formal language, and $(\mathcal{C}, \mathcal{R})$ a deduction category. By a semantic evaluation map we mean:

$$\| \cdot \| : \mathcal{L} \to (\mathcal{C}, \mathcal{R}), \text{ where}$$

(i) $\|Sort(\mathcal{L})\| = \|Ob(\mathcal{L})\| \cup \|Var(Ar(\mathcal{L}))\| \cup \|Var(Rel(\mathcal{L}))\|$, $\|Var(\mathcal{L})\| = (\|Var(Ob(\mathcal{L}))\|, \|Var(Ar(\mathcal{L}))\|)$, a subcategory of $\mathcal{C}$, and where

$$Var(Rel(\mathcal{L})) = \bigcup_{j \in Ob(\mathcal{L})} Var(Rel_j(\mathcal{L})),$$

$$\|Var(Rel(\mathcal{L}))\| \subseteq Rel(\mathcal{C}, \mathcal{R}) \overset{d}{=} \bigcup_{a \in Ob(\mathcal{C})} Rel_a(\mathcal{C}, \mathcal{R}),$$

$$\mathcal{R}(\mathcal{L})(j) = Var(Rel_j(\mathcal{L})) \overset{d}{=} \{s \mid s \in Rel(\mathcal{L}) \text{ with } \sigma(s) = j\},$$

$$Rel_a(\mathcal{C}, \mathcal{R}) \overset{d}{=} \{r \mid r \in Rel(\mathcal{C}, \mathcal{R}) \text{ with } \|\sigma\|(r) = a\},$$

$\|\sigma\|$ being the semantic evaluation map for $(\mathcal{C},\mathcal{R})$, the evaluation of the original function symbol σ representing (formal) signature for $\mathcal{L}$,

$$\|\mathrm{Var}(\mathrm{Rel}_j(\mathcal{L}))\| \subseteq \mathrm{Rel}_{\|j\|}(\mathcal{C},\mathcal{R}) ,$$

and where, for any $j \in \mathrm{Var}(\mathrm{Ob}(\mathcal{L}))$, noting $\|j\| \in \mathrm{Ob}(\mathcal{C})$, and compatibly for any $(f : i \to j) \in \mathrm{Var}(\mathrm{Ar}(\mathcal{L}))$,

$$\|(f : i \to j)\| = \|f\| : \|i\| \to \|j\| , \text{ etc.}$$

Note that $\|\cdot\| : \mathrm{Var}(\mathcal{L}) \to \mathcal{C}$ is a functor. In turn, $(\|\cdot\|, \tau_{\|\cdot\|}) : \overline{\mathrm{Var}}(\mathcal{L}) \to (\mathcal{C},\mathcal{R})$ is an arrow between deduction categories, where $\tau_{\|\cdot\|} : \mathcal{R}(\mathcal{L}) \dot{\to} \mathcal{R}_o\|\cdot\|$ is a natural transform, where for any $i \in \mathrm{Var}(\mathrm{Ob}(\mathcal{L}))$, and $r \in \mathcal{R}(\mathcal{L})(i)$, $\tau_{\|\cdot\|}(r) = \|r\|$, and where we note that for any $s \in \mathrm{Var}(\mathrm{Rel}_j(\mathcal{L}))$, i.e., $\sigma(s) = j$, $\mathcal{R}(\mathcal{L})(f)(s) = s[f] \in \mathrm{Var}(\mathrm{Rel}_i(\mathcal{L}))$, where $\cdot[\cdot\cdot]$ is the distinguished substitution symbol in $\mathcal{L}$, and where

$$\|s[f]\| = \tau_{\|\cdot\|}(i)(s[f]) = \mathcal{R}(\|f\|)(\|s\|) \in \mathcal{R}(\|i\|) , \ (\|i\| \in \mathrm{Ob}(\mathcal{C})) .$$

We have already evaluated some distinguished function symbols σ, $\cdot[\cdot\cdot]$. More generally,

$$\|\mathrm{Cor}(\mathcal{L})\| = (\|\mathrm{Ob}(\mathrm{Core}(\mathcal{L}))\| , \|\mathrm{Ar}(\mathrm{Core}(\mathcal{L}))\|) ,$$

where $\|\mathrm{Ob}(\mathrm{Core}(\mathcal{L}))\|$ = compound of $\|\mathrm{Sort}(\mathcal{L})\|$ and where $\|\mathrm{Ar}(\mathrm{Core}(\mathcal{L}))\|$ is evaluated typically as follows:

If $(F : \mathrm{Ob}(\mathcal{L}) \times \mathrm{Ar}(\mathcal{L}) \to \mathrm{Rel}(\mathcal{L})) \in \mathrm{Ar}(\mathrm{Core}(\mathcal{L})) = \mathrm{FuncSymb}(\mathcal{L})$, then

$\|F : \mathrm{Ob}(\mathcal{L}) \times \mathrm{Ar}(\mathcal{L}) \to \mathrm{Rel}(\mathcal{L})\|$ is

$\|F\| : \|\mathrm{Var}(\mathrm{Ob}(\mathcal{L}))\| \times \|\mathrm{Var}(\mathrm{Ar}(\mathcal{L}))\| \to \|\mathrm{Var}(\mathrm{Rel}(\mathcal{L}))\|$, i.e.,

(abusing notation slightly)

$\|F\| : \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ar}(\mathcal{C}) \to \mathrm{Rel}(\mathcal{C},\mathcal{R})$

for some choice of $\|F\|$.

Remark: It is understood that the semantic evaluation here is compatible with the basic $\mathrm{Th}_{\mathrm{Syn}}(\mathcal{L})$, i.e., $\|\cdot\|$ is a model for $\mathrm{Th}_{\mathrm{Syn}}(\mathcal{L})$. Thus,

(ii) Using the above evaluations,

$$\|\mathrm{Wfex}(\mathcal{L})\| = \|\mathrm{Var}(\mathrm{Ar}(\mathcal{L}))\| \cup \|\mathrm{Var}(\mathrm{Rel}(\mathcal{L}))\|$$

where for any $x_m^{(i)}, y_m^{(i)} \in Wfv_i(\mathfrak{X})$, $z_m^{(k)} \in Wfv_k(\mathfrak{X})$, $x_m^{(i)}, y_m^{(i)}, z_m^{(i)}$ projections, $m \overset{d}{=} i \times i \times k$, $(f : i \to k)$, $(g : i \to \ell)$, noting product $\langle f,g\rangle : i \to j \times \ell$, and any $r \in Var(Rel_{k\times\ell}(\mathfrak{X}))$, $s \in Var_k(Rel(\mathfrak{X}))$ and using the distinguished logical connector symbol $(\& : Rel(\mathfrak{X}) \times Rel(\mathfrak{X}) \to Rel(\mathfrak{X}))$, $\varphi \overset{d}{=} ((r[\langle f\circ x_m^{(i)}, g\circ y_m^{(i)}\rangle]) \ \& \ (s[z_m^{(k)}]) \in Wff(\mathfrak{X})$, the class of all well-formed formulas, noting $Wff(\mathfrak{X}) \subseteq Wfex(\mathfrak{X})$ and that also we can consider $\varphi \in Rel_m(\mathfrak{X})$,

$$\|\varphi\| = (\mathfrak{R}\langle\|f\|\circ\|x_m^{(i)}\|, \|g\|\circ\|y_m^{(i)}\|\rangle(\|r\|))\cdot\|\&\|(\mathfrak{R}(\|z_m^{(k)}\|)(\|s\|)$$
$$\in Rel_{\|m\|}(\mathfrak{C},\mathfrak{R}) ,$$

noting $\|m\| = \|i\| \times \|i\| \times \|k\| \in Ob(\mathfrak{C})$.

(iii) One additional level of evaluations is required. Note that $\|x^{(i)}\|$, $\|y^{(i)}\|$, $\|z^{(k)}\|$ are compatibly projection arrows for $\mathfrak{C}$. However, this does not lead to a specific value for $\|\varphi\|$. In order to determine specific value of $\|\varphi\|$, we choose specific "values" for all the (free) individual variables. In effect, we replace typically $\|x^{(i)}\|$ by some arrow $(\tau : 1_{\|i\|} \to \|i\|) \in Ar(\mathfrak{C})$ (a subobject of $\|i\|$, i.e., τ is a monic), where $1_{\|i\|}$ is a terminal object (at $\|i\|$) , and denote any such functional relation by $\omega(x^{(i)}) = \tau$, so that a complete evaluation of φ is now $\|\varphi\|_\omega = \|\varphi\|$ with all individual variables $x^{(i)}$, $y^{(i)}$, $z^{(k)}$ replaced by $\omega(x^{(i)})$, $\omega(y^{(i)})$, $\omega(z^{(k)})$, respectively.

(iv) Often (but not always $\|\cdot\|$ will be such that $\|\cdot\|$ is compatible with $Th_K(\mathfrak{X})$ and the corresponding axioms, rules and deducts within $(\mathfrak{C},\mathfrak{R})$, then we call $\|\cdot\|$ a model for $Th_K(\mathfrak{X})$, in particular:

(a) All properties in $\mathfrak{L}$ involving $Th_K(\mathfrak{L})$ correspond to properties in $(\mathfrak{C},\mathfrak{R})$.

(b) $\|\cdot\|$ is order preserving, i.e., for any $(\varphi \vdash \psi) \in De(Th_K(\mathfrak{L}))$, $\|\varphi\| \le \|\psi\|$.

3.2 Some Particularizations.

(a) Let $(\mathfrak{C},\mathfrak{R}) = (SET,(0,1)^{\cdot})$, where $H^{\cdot}$, for any poset H , is the exponential (contravariant) functor from $\mathfrak{C}$ to Preord. The previous semantic evaluat.. will correspond to semantics of a classical set theory when $Th_K(\mathfrak{X})$ is appropriately chosen such as Zermalo-Fraenkel (Z-F) set theory, where we note relations are interpreted as membership functions.

(b) Let $(\mathfrak{C},\mathfrak{R}) = (SET,[0,1]^{\cdot})$, the corresponding semantic evaluation is a multi-valued logic. In particular, when $Th_K(\mathfrak{X})$ is a set theory, such as Z-F , then the semantic evaluation becomes a modified semantics of classical set theory where now relations are extended to include fuzzy relations, i.e., relations where membership functions have values in [0,1] .

(c) More generally, let $(\mathcal{C},\mathcal{H}) = (\text{SET},H^{\cdot})$, where H is a poset. Note that in (b), only a partial fuzzification is achieved through the semantics, we seek a richer structure which will allow semantic evaluations to be completely generalized. To relax the restriction in (b), we will use Benabou's construction (see, e.g., Coste [2]), where any deduction category $(\mathcal{C},\mathcal{H})$, with possible additional structures up to a formal topos structure, can be imbedded in the deduction category $(\mathcal{Y}(\mathcal{C},\mathcal{H}),\text{Sub})$ with corresponding possible additional structures up to a formal topos. We also note that a deduction category of that form always implies $\mathcal{Y}(\mathcal{C},\mathcal{H})$ is an ordinary category with possible additional structures up to a topos. Note that a deduction category is an appropriate vehicle for semantic evaluations, and when $Th_K(\mathcal{X})$ is a logical set theory, intuitionistic in form, in order for $\|\cdot\| : \mathcal{X} \to (\mathcal{C},\mathcal{H})$ to be a model, $(\mathcal{C},\mathcal{H})$ necessarily must be a formal topos, i.e., $(\mathcal{C},\mathcal{H})$ must have compatible structures with $Th_K(\mathcal{X})$. When $(\mathcal{C},\mathcal{H})$ is a formal topos, we may, without loss of generality, always consider a standarized form $(\mathcal{Y}(\mathcal{C},\mathcal{H}),\text{Sub})$, in which $(\mathcal{C},\mathcal{H})$ is imbedded in a natural sense. In addition, $\mathcal{Y}(\mathcal{C},\mathcal{H})$ is a topos, and therefore allows more structures in $\mathcal{X}$ to be evaluated. Indeed, when $Th_K(\mathcal{X})$ is an intuitionistic set theory, $(\mathcal{Y}(\mathcal{C},\mathcal{H}),\text{Sub})$ is a full semantic fuzzification of $\mathcal{X}$. Roughly speaking, $\mathcal{Y}(\mathcal{C},\mathcal{H})$ has for its objects (i,φ) , $\varphi \in \mathcal{H}(i)$, (or $\mathcal{H}(i \times i)$ depending on the particular construction); arrows of $\mathcal{Y}(\mathcal{C},\mathcal{H})$ generalize the Zadeh's functional transform on fuzzy sets. Thus, in a real sense, $(\mathcal{Y}(\mathcal{C},\mathcal{H}),\text{Sub})$ is a full generalization of Zadeh's fuzzy set theory. However, we must note that equality as interpreted here will be different, in general, from equality in Zadeh's sense. Indeed, equality is an intuitionistic equality. In fact, when Eytan [3] applies these constructions to $(\text{SET},H^{\cdot})$ and purposely constraints equality to be classical ordinary equality, $(\mathcal{Y}(\mathcal{C},\mathcal{H}),\text{Sub})$ is no longer a formal topos.

We have already mentioned that topoi and formal topoi may be considered as intuitionistic-like set theories (being the evaluations of such in $\mathcal{X}$) , but in addition, going in another direction, it can be shown that if $Th_K(\mathcal{X})$ is strenghtened to be a slightly weaken form of Z-F (called $Th_{Zer}(\mathcal{X})$) (see Johnstone [7]), and when $(\mathcal{C},\text{Sub})$, without loss of generality, is not only a formal topos, but in addition has other properties (well-pointedness, partial transitivity, etc.), then $\|\cdot\| : \mathcal{X} \to (\mathcal{C},\text{Sub})$ will be indeed a model for $Th_{Zer}(\mathcal{X})$. Thus another justification for considering topoi as ranges of semantic evaluations (and formal topoi) is their close identifications with various forms of set theories.

Moreover, $Th_K(\mathcal{X})$ is a <u>sound</u> and <u>complete</u> theory for semantic evaluations within formal topoi.

3.3 Approximate Reasoning.

In this section, we apply the previous developments of formal language and semantic evaluations, but with no $Th_K(\mathfrak{X})$ imposed, to approximate reasoning.

Note that (e.g., Zadeh [10]) approximate reasoning (e.g., inexact reasoning in clinical decision-support systems) is commonly used in expert systems. The data base in a knowledge-based system comprises:

(i) facts expressed by sentences,

(ii) rules expressed by conditional sentences (where conditioning is expressed as a binary logical connective).

All sentences in a knowledge-based system have an associated certainty factor, i.e., a numerical value in [0,1] expressing the degree of confirmation associated with the sentence in question. The basic problem in the analysis of uncertainty consists of assigning an associated degree of uncertainty to conclusions from various combinations of hypotheses from (i) using (ii). Following Zadeh, we note that when (i) and (ii) are imprecise, due, e.g., to natural language, the computation of certainty factors is not obtainable in classical form. But, if we consider a weaker replacement - linguistic certainty for numerical certainty - then fuzzy logic (FL) may be used as a basis to carry out approximate reasoning. By the latter, it is meant classical deduction reasoning modified by the use of imprecise sentences (and thus hypotheses and conclusions modified likewise).

To illustrate the use of FL in deriving inference rules for quantified propositions, we will examine below some example from Zadeh [10] and show that these deductions are semantically consistent.

Let X be a finite set, and A, B, C be attributes with domains in X. Let Q_1, Q_2 be linguistic quantifiers, e.g., most, with domains in $[0,1]$. We write ϕ_A, e.g., for the membership or possibility distribution of A, and $Q_1 \circledast Q_2$ is the product of fuzzy numbers.

Consider the following deduction:

$$\begin{array}{l} P_1 \overset{d}{=} Q_1 \text{ A's are B's} \\ P_2 \overset{d}{=} Q_2 \text{ (A and B)'s are C's} \\ \hline P_3 \overset{d}{=} (Q_1 \circledast Q_2) \text{ A's are (B and C)'s} \end{array}$$

Let $u = \sum_x (\phi_A(x) \wedge \phi_B(x) / \sum_x \phi_A(x)$

$$v = \sum_x (\phi_A(x) \wedge \phi_B(x) \wedge \phi_C(x)) / \sum_x (\phi_A(x) \wedge \phi_B(x))$$

$$w = \sum_x (\phi_A(x) \wedge \phi_B(x) \wedge \phi_C(x)) / \sum_x \phi_A(x)$$

Now

$$\|P_3\| = \Phi_{Q_1 \circledast Q_2}(w) \overset{d}{=} \sup_{\substack{s,t \\ s\cdot t=w}} (\Phi_{Q_1}(s) \wedge \Phi_{Q_2}(t))$$

$$= \sup_{s} (\Phi_{Q_1}(s) \wedge \Phi_{C_2}(w|s))$$

$$\geq \Phi_{Q_1}(u) \wedge \Phi_{Q_2}(v) = \|P_1 \text{ and } P_2\| \quad (*)$$

Next, if $z \overset{d}{=} \sum_x (\Phi_A(x) \wedge \Phi_C(x)) / \sum_x \Phi_A(x)$

then:

$$\Phi_{(\geq \circ Q_1 \circledast Q_2)}(z) \overset{d}{=} \sup_{\substack{s,t \\ s\cdot t\leq z}} (\Phi_{Q_1}(s) \wedge \Phi_{Q_2}(t))$$

$$\geq \sup_{\substack{s,t \\ s\cdot t=z}} (\Phi_{Q_1}(s) \wedge \Phi_{Q_2}(t))$$

$$\geq \Phi_{Q_1 \circledast Q_2}(w) \quad (**) \text{, since}$$

$$v \leq \sum_x (\Phi_A(x) \wedge \Phi_C(x)) / \sum_x (\Phi_A(x) \wedge \Phi_B(x)) = \frac{z}{u},$$
and $w = u \cdot v$.

The inequality (*) means that P_1 and $P_2 \vdash P_3$, and (**) means $P_3 \vdash$ Wff whose truth value is $\Phi_{\geq \circ Q_1 \circledast Q_2}(z)$

For a more general setting, see the deduction rules found in Eytan [3], see also Goodman and Nguyen [6].

4. GENERAL LOGICAL SYSTEMS.

The framework developed in sections 2 and 3 is now used to formulate rigorously the concepts of dispersions and general logical systems, in which associated calculus can be developed for uncertainty analysis in Expert Systems.

Let us first define a general logical system as a semantic evaluation $\|\cdot\| \; \mathcal{X} \to$ (GOG,Sub) , where

$$\text{GOG} \overset{d}{=} \bigcup_{\alpha\in I} (\text{Gog}(H), \text{Logic}_\alpha) \text{ , where}$$

I is an index set, $Gog(H)$ is a category of the form $\mathcal{Y}$(SET,H) previously mentioned and where Logic_α is some logic. By logic we mean a collection of semantic evaluations $\|\cdot\| : \mathcal{Y} \to (\mathcal{Y}, \mathcal{X})$ which have the same values over $\mathrm{Loc}(\mathfrak{X}) \cup \mathrm{Quant}(\mathfrak{X})$. Often, $GOG = (\mathrm{Gog}(H), \mathrm{Logic}_{\alpha_o})$ where $\mathrm{Logic}_{\alpha_o} = (nt, \&, or)$. For any $i \in \mathrm{Var}(\mathrm{Ob}(\mathfrak{X}))$, define

$$\|i\| \overset{d}{=} (\|i\|^{(1)}, \|i\|^{(2)}) = (X_i, \theta_i) \in \mathrm{Ob}(\mathrm{Gog}(H)),\ X_i \in \mathrm{Ob}(SET),\ \theta_i : X_i \to H .$$

Define generalized set A_i as $A_i \overset{d}{=} i$ with membership function $\phi_{A_i} \overset{d}{=} \|i\|^{(2)}$. For any $j \in \mathrm{Var}(\mathrm{Ob}(\mathfrak{X}))$ such that $\|j\|^{(2)} = 0_H : X \to 0_H$ (a least element of H) , identify $\|j\| = (X_j, 0_H)$ with ϕ . For j such that $\|j\|^{(2)} = 1_H : X_j \to 1_H$ (a greatest element of H) , identify $\|j\| = (X_j, 1_H)$ with X_j .

For $X \in \mathrm{Ob}(SET)$, consider

$$\mathfrak{F}(X) \overset{d}{=} \{A_i \mid i \in \mathrm{Var}(\mathrm{Ob}(\mathfrak{X})), \|i\|^{(1)} = X\}$$

and assume there exists $i_X \in \mathfrak{F}(X)$ such that $\|i_X\| = X$ by identification. We call $\mathfrak{F}(X)$ a class of generalized subsets A_i of X .

$$\mathfrak{T}(X) \overset{d}{=} \{\|i\|^{(2)} \mid i \in \mathrm{Var}(\mathrm{Ob}(\mathfrak{X})),\ \|i\|^{(2)} = X\} \subseteq H^X ,$$

we call $\mathfrak{T}(X)$ a class of dispersions or fuzzy set membership functions $\phi_{A_i} = \theta_i$ over X .

Depending on $\|\cdot\|$ of course, often we may have $\mathfrak{T}(X) = H^X$ for all $X \in \mathrm{Ob}(SET)$.

Assume $\|\cdot\|^{(2)} : \mathrm{Var}(\mathrm{Ob}(\mathfrak{X})) \to \mathrm{Ar}(SET)$ is injective, consider the related membership function map ϕ , where

$$\phi : \mathrm{Ob}(SET) \to \mathrm{Ar}(SET) ,$$

where for all

$$X \in \mathrm{Ob}(SET) ,\ \phi(X) \in \mathfrak{T}(X)^{\mathfrak{F}(X)} ,$$

where for all

$$A_i \in \mathfrak{F}(X) ,\ \phi(X)(A_i) = \phi_{A_i} = \|i\|^{(2)} = \theta_i .$$

Note that $\phi(X) : \mathfrak{F}(X) \to \mathfrak{T}(X)$ is also injective.

Examples.

α) Boolean system. Let $\mathcal{L}$ be the formal language of ordinary predicate calculus. For each set X , $\mathcal{F}(X)$ is the class of subsets of X , and $\mathcal{F}(X)$ is the class of all ordinary set membership functions over X .

β) Zadeh's system. Let $\mathcal{L}$ be the formal language with Lukasiewicz logic $\aleph_1$. For each set X , $\mathcal{F}(X)$ is the class of all generalized sets on X , and $\mathcal{F}(X) = [0,1]^X$, the class of all dispersions over X .

γ) Probabilistic system. Let $\mathcal{L}$ be the formal language of probability logic. Let $Y \in Ob(SET)$, and choose $X \subseteq \mathcal{P}(Y)$, where X is a σ-albegra of subsets of Y . Then $\mathcal{F}(X)$ corresponds bijectively to $\mathcal{F}(X)$, the class of all probability measures over (Y,X) .

5. UNCERTAINTY MEASURES AND ADMISSIBILITY.

We now proceed to specify a useful class of dispersions by considering additional structures at the (ω) evaluation level. These dispersions will be called uncertainty measures. See e.g., Bellman [1] for a discussion of uncertainty in Artificial Intelligence. These measures are useful for uncertainty modeling in Expert Systems. Finally, we will discuss the problem of admissibility of uncertainty measures arising from a paper by Lindley [8].

5.1 Uncertainty Measures.

Let $i \in Var(Ob(\mathcal{X}))$ such that $\|i\| = (\|i\|^{(1)}, \|i\|^{(2)}) = (X_i, \theta_i)$ with $X_i = \mathcal{A} \subseteq \mathcal{P}(Y)$, $Y \in Ob(SET)$, and take $H = [0,1]$. Then a dispersion θ_i is a set-function: $\mathcal{A} \to [0,1]$. This leads to the following class of uncertainty measures:

Definitions. By an uncertainty measure, we mean a mapping $\mu : \mathcal{A} \to [0,1]$.

We say that μ is monotone increasing iff $\forall C,D \in \mathcal{A}$, $C \subseteq D$, we have $\mu(C) \le \mu(D)$; similarly, μ is said to be monotone decreasing iff $\mu(C) \ge \mu(D)$. In addition, let $(\overset{\cdot}{\bullet}, \overset{\cdot\cdot}{\bullet}) = (\cap,\cup)$ or $= (\cup,\cap)$ (assuming that $\mathcal{A}$ is either closed under finite intersions or closed under finite unions) define, for any integer $n \ge 1$, the Choquet operators

$$\Delta_n, \nabla_n : \mathop{\times}_{j=1}^{n} \mathcal{A} \to [0,1] \text{ , where for any } \dot{A}_n = (A_1, A_2, \ldots, A_n) ,$$

$$A_j \in \mathcal{A}, \ j = 1,2,\ldots,n,$$

$$\Delta_n(\mu, \overset{\cdot\cdot}{\bullet}, \dot{A}_n) \overset{d}{=} \sum_{K \subseteq \{1,\ldots,n\}} (-1)^{\text{card } K} \mu(\overset{\cdot\cdot}{\bullet}_{j \in K} A_j \overset{\cdot\cdot}{\bullet} A_n)$$

$$\Delta_n(\mu, *', *'', \bar{A}_n) = \mu\Big(\mathop{*'}_{j\in\{1,\ldots,n\}} A_j\Big) - \sum_{\phi\neq K\subseteq\{1,\ldots,n\}} (-1)^{\text{card } K+1} \mu\Big(\mathop{*''}_{j\in K} A_j\Big)$$

we say that μ is a <u>plausibility</u> or <u>upper probability measure</u> iff $\forall n$ and $\bar{A}_n$, we have

$$\Delta_n(\mu, \cup, \bar{A}_n) \leq 0 \quad \text{and} \quad \mu(\phi) = 0 \ .$$

μ is a <u>belief</u> or <u>credibility</u> or <u>lower probability measure</u> iff $\forall n$ and $\bar{A}_n$, we have

$$\Delta_n(\mu, \cap, \bar{A}_n) \geq 0 \quad \text{and} \quad \mu(Y) = 1 \ .$$

μ is a <u>disbelief</u> or <u>incredibility measure</u> iff $\forall n$ and $\bar{A}_n$, we have

$$\Delta_n(\mu, \cap, \bar{A}_n) \leq 0 \quad \text{and} \quad \mu(Y) = 0 \ .$$

For the relationships between the above measures, we refer the reader to Goodman and Nguyen [6].

5.2 On Admissibility of Uncertainty Measures

Lindley [8] proposed a method for checking the admissibility of uncertainty measures as follows:

As Savage [9] has pointed out, the concept of score functions is useful when considering knowledge of experts. Therefore, a reasonable framework for admissibility can be formulated as follows: consider a score function consisting of two functions f and g from some closed interval I (of the real line $\mathbb{R}$) to $\mathbb{R}^+$ with the following interpretation:

Assuming that $\mu : \mathcal{A} \to I$, where $\mathcal{A}$ is an algebra of subsets of a space X. For $A \in \mathcal{A}$, if $\mu(A) = x$, then the score is $f(x)$ if A is false and the score is $g(x)$ if A is true. Note that one can also consider the case of multi-valued logic.

Next, assuming that the score is additive, i.e., if $A_1, \ldots, A_n \in \mathcal{A}$ and $\mu(A_i) = x_i$, $i = 1, 2, \ldots, n$, then the total score will be

$$(1 - A_1) f(x_1) + A_1 g(x_1) + \cdots + (1 - A_n) f(x_n) + A_n g(x_n)$$

with the abuse of notation: $A_i = 0$ or $A_i = 1$ according to A_i is false or true.

Consider the assumptions:

(a) f and g are of class C^1 over the interior of I.

(b) There exist α and β, with $\alpha < \beta$, in the interior of $\underline{I}$ such that:

$f'(x) < 0$ for $x < \alpha$, $f'(\alpha) = 0$, and $f'(x) > 0$ for $x > \alpha$,
$g'(x) < 0$ for $x < \beta$, $g'(\beta) = 0$, and $g'(x) > 0$ for $x > \beta$.

Define the transform $A \to P_\mu(\mu(A))$ as:

$$P_\mu(x) = f'(x)[f'(x) - g'(x)]^{-1}, \quad (\mu(A)=x).$$

Then it is easy to see that $\forall x \in [\alpha,\beta]$, we have $P_\mu(x) \in [0,1]$.

Next, $(x_1,\ldots,x_n)$ is said to be inadmissible if there exist $(y_1,y_2,\ldots,y_n)$ such that the total score based on the y's is strictly less than that based on the x's.

Lindley showed that all admissible values are in $[\alpha,\beta]$. One of the key result of Lindley is the following necessary condition of admissibility of μ: If $\mu(A) = x$ and $\mu(A') = y$, where A' denotes set-complement of A, then (x,y) is admissible implies $P_\mu(x) + P_\mu(y) = 1$, i.e., the transform P_μ is a finitely additive probability measure.

Note that the above necessary condition is equivalent to

(1) $f'(x)f'(y) = g'(x)g'(y)$, $\forall x,y \in [\alpha,\beta]$.

This result is then used to check whether or not a given uncertainty measure μ is admissible. For example, let μ be a posibility measure, and A,B disjoint, then $P_\mu(\mu(A \cup B)) = P_\mu(\mu(A) \vee \mu(B))$ is different from $P_\mu(\mu(A)) + P_\mu(\mu(B))$, therefore μ is inadmissible.

However, it is interesting to check whether finitely additive probability measures themselves are admissible!

(i) First, let μ be a finitely additive probability measure, and let $\mu(A) = x$, $\mu(A') = 1 - x$. Then if $(x, 1-x)$ is admissible, we must have

(2) $f'(x)f'(1-x) = g'(x)g'(1-x)$, $\forall x \in [0,1]$.

But, it is easy to give score functions (f,g) satisfying conditions (a) and (b) and yet, (2) does not hold. For example, $f(x) = 2x^2$, $g(x) = (1-x)^2$. This fact clearly shows that Lindley's argument depends heavily on the chosen score functions, a fact familiar in general decision theory.

(ii) In fact, it can be shown, see Goodman [5], that for any score functions satisfying (a), (b), except the trivial ones which lead to $P_\mu(x) = x$, any non-atomic probability measure is inadmissible in

Lindley's sense! The proof of this is based partly on Lindley's Theorem 2.

Note that Lindley's problem of admissibility of uncertainty measures can be shown to be imbeddable within the problem of admissibility of probability measures with respect to some distinguished class of probability measures.

By somewhat modifying the score functions, one can show that all uncertainty measures are admissible.

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