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Some New Results Concerning Random Sets and Fuzzy Sets

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This paper is a continuation of past work showing direct connections between fuzzy set theory and classical probability theory through the use of random sets. More specifically:

(1) The Negoita-Ralescu Representation Theorem connecting fuzzy and flou sets has been extended to characterize those probability spaces which admit arbitrary probability distributions (and arbitrary stochastic processes, more generally) as the class of all non-atomic probability spaces.

(2) A number of characterizations has been obtained for the one-point coverage equivalencies of random closed intervals with fuzzy sets. This includes the classes of translation type random intervals, statistically independent end-point and centering parameter random intervals, and nested random intervals.

(3) It is shown that all nested random sets must be of the same form as the canonical random set: $S_U(A) = \phi_A^{-1}$ $[U,1]$, where U is uniformly distributed over $[0,1]$, and as a consequence of this, there is only one nested random set one-point coverage equivalent to any given fuzzy set A, namely, $S_U(A)$.

(4) The entire solution class for the one-point coverage problem for any given fuzzy subset A of a finite space X is characterized by being in a one-to-one, onto correspondence with a particular convex, closed hyperplane-bounded subspace $\mathbf{R}(A)$ of $\mathbb{R}^{a_o}$, where $a_o \triangleq$ cardinality of the class of all ordinary subsets of X with two or more distinct elements of X. This result is extended to the finite multiple-point coverage problem.

(5) Entropy is introduced as a criterion for ordering random sets within the class S(A) of all one-point coverage equivalent random sets to a given fuzzy set A. It is shown that the maximal entropy random set in S(A) is T(A), the random set whose membership function process is a statistically independent zero-one process with $\Pr(\phi_{T(A)}(x) = 1) = \Pr(x \in T(A)) = \phi_A(x)$, for all $x \in X$. On the other hand, the minimal entropy random set within S(A) is much more difficult to obtain. It is demonstrated that the search for such may be restricted to the relatively small vertex set V(A) of R(A). In fact, $\tilde{S}_U(A)$ and S(A), the singleton-valued random set corresponding to the random variable over X having probability function ϕ_A, when $\Sigma(\phi_A(x))$ over all $x \in X$ is ≤ 1, always lie in V(A). Through a simple example, it is shown that at times $S_U(A)$ may achieve the minimum entropy, at other times, $\tilde{S}(A)$ may achieve the minimum entropy, and at other times, neither will yield the minimum value.

INFORMATION SCIENCES 34, 93–113 (1984)

Some New Results Concerning Random Sets and Fuzzy Sets

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ABSTRACT

This paper is a continuation of past work showing direct connections between fuzzy set theory and classical probability theory through the use of random sets. This includes characterizations of random intervals one-point-coverage-equivalent to fuzzy sets, determination of all nested random sets equivalent to fuzzy sets, the solution of the one- and multiple-point coverage problems for random sets in finite spaces, and the use of entropy for ordering random sets within the class of one-point-coverage-equivalent ones for any given fuzzy set. Finally, a connection between random variables, random sets, and fuzzy sets is pointed out.

INTRODUCTION

During the past few years a number of investigations has been carried out directly connecting fuzzy sets with random sets [1–4]. In addition, there has been work in interpreting fuzzy sets through subjective probability, where in effect, the membership function value $\phi_A(x)$ of the fuzzy set A at point x is considered the same as the probability of a zero-one random variable attaining the value one, which is approximated by an experiment [5, 6]. Also, it is appropriate to mention investigations into structures which generalize both fuzzy set and probabilistic concepts [7–9] and the extensive work concerning fuzzy probabilities and fuzzy random variables [10, 11]. However, in this paper only the first-mentioned connections between fuzzy sets and probability will be considered.

A SUMMARY OF THE BASIC RELATIONS BETWEEN FUZZY AND RANDOM SETS

1. Given any space X and any fuzzy subset A of X with membership function $\phi_A : X \to [0,1]$, there exist (in general, many) random subsets $S(A)$ of

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X such that $S(A)$ and A are one-point equivalent, i.e.,

$$\phi_A(x) = \Pr(x \in S(A)), \qquad \text{all } x \in X. \tag{1}$$

(See [1], [2].) By a random subset S of X is meant the measurable set-valued mapping $S: \Omega \to \mathscr{P}(X)$ for appropriate choices of probability spaces $(\Omega, \mathscr{D}, \Pr)$ and $(\mathscr{P}(X), \mathscr{C}, \Pr \circ S^{-1})$, where $\mathscr{C}$ is a σ-algebra over some subclass of the power class $\mathscr{P}(X)$ of X which contains at least the classes $\mathscr{C}_{\{x\}}$ of all sets containing x, for each $x \in X$. (See also [12].) Two simple examples of one-point-equivalent random subsets of X to a given fuzzy subset A are: $S_U(A) = \phi_A^{-1}([U, 1])$, where U is a random variable uniformly distributed over $[0,1]$; and $T(A)$, where the membership function $\phi_{T(A)}$, a random function which is zero-one valued, is such that $\phi_{T(A)}(x)$ for index $x \in X$ is a mutually statistically independent zero-one stochastic process with

$$\Pr\big(\phi_{T(A)}(x) = 1\big) = \phi_A(x), \qquad \text{all } x \in X. \tag{2}$$

In general, $S_U(A)$ and $T(A)$ are distinct [unless A is an ordinary subset of X, in which case necessarily $S(A) \equiv A$], the former being a nested random set while the latter is highly disconnected. (See [1], [2] for various properties.) Note that the mappings $S_U(\cdot), T(\cdot): \mathscr{F}(X) \to \mathscr{R}(X)$ are injective, where $\mathscr{F}(X)$ is the class of all fuzzy subsets of X and $\mathscr{R}(X)$ is the class of all random subsets of X.

2. Conversely, given any space X and random subset S of X, there is a unique fuzzy subset $\mathscr{A}(S)$ of X such that $\mathscr{A}(S)$ and S are one-point-coverage equivalent: namely, $\mathscr{A}(S)$ defined by

$$\phi_{\mathscr{A}(S)}(x) = \Pr(x \in S) \ \big(= \Pr \circ S^{-1}(\mathscr{C}_{\{x\}})\big), \qquad \text{all } x \in X. \tag{3}$$

3. Combining results 1 and 2, the one-point coverage mapping $\mathscr{A}: \mathscr{R}(X) \to \mathscr{F}(X)$ is many-to-one surjective. Recently, some of these results have been used independently in randomized test and confidence region theory [13].

4. S_U and T are special cases of mappings of the form

$$\mathscr{S}^J : \mathop{\times}_{j \in J} \mathscr{F}(X_j) \to \mathop{\times}_{j \in J} \mathscr{R}(X_j) \qquad (\text{well-defined joint random sets}), \tag{4}$$

where marginally $\mathscr{S}: \mathscr{F}(X_j) \to \mathscr{R}(X_j)$ is such that for any $A_j \in \mathscr{F}(X_j)$, $\mathscr{S}(A_j)$ is one-point-coverage equivalent to A_j, written from here on as

$$\mathscr{S}(A_j) \cong A_j, \tag{5}$$

for all $j \in J$. If $\bullet : \times_{j \in J} \mathscr{P}(X_j) \to \mathscr{P}(X)$ is an ordinary set operator,

$\odot : \times_{j \in J} \mathscr{F}(X_j) \to \mathscr{F}(X)$ is a fuzzy-set operator, and $\mathscr{S}^J$ is as above, a natural question is to determine if $\odot$ and $\bullet$ are in an isomorphic-like relation relative to $\mathscr{S}$ and $\cong$, i.e.,

$$\odot \mathscr{B} \cong \mathscr{S}(\odot \mathscr{B}) \cong \bullet \mathscr{S}^J(\mathscr{B}), \qquad \text{all } \mathscr{B} \in \underset{j \in J}{\times} \mathscr{F}(X_j). \tag{6}$$

Investigations into the conditions when such relations hold for various classes of ordinary-set operators, fuzzy-set operators, and mappings $\mathscr{S}^J$ have been carried out [2, 4, 14]. These results provide motivation for the choice of fuzzy-set operator or operators which extend ordinary set operators. For example, let $J = \{1,2\}$, and let U_1 and U_2 be two random variables each uniformly distributed over $[0,1]$ with joint distribution K to be specified. Then define $\mathscr{S}_K^J = (S_{U_1}(\cdot), S_{U_2}(\cdot))$ for $K = K_1$, corresponding to $U_1 = U_2$, and for $K = K_2$, corresponding to U_1 and U_2 being statistically independent. Let $\cap$ be ordinary intersection with $X_1 = X_2 = X$ arbitrary fixed. Let $\cap_j$ be fuzzy-set intersection defined as follows for $j = 1,2$:

$$\phi_{B \cap_1 C}(x) = \min(\phi_B(x), \phi_C(x)), \tag{7}$$

$$\phi_{B \cap_2 C}(x) = \phi_B(x) \cdot \phi_C(x), \tag{8}$$

for all $B, C \in \mathscr{F}(X)$ and all $x \in X$. It then follows that $\cap_j$ and $\cap$ are in an isomorphic-like relation with respect to $\mathscr{S}_{K_j}^J$ and $\cong$, for $j = 1,2$. (There are many other possible definitions for fuzzy-set intersection which lead to isomorphic-like relations with ordinary intersection. See [3].)

FURTHER RELATIONS BETWEEN RANDOM SETS AND FUZZY SETS

It is convenient first to establish the following necessary and sufficient conditions under which a uniformly distributed random variable may be used, in terms of nonatomic probability spaces and other criteria.

THEOREM 1. *Let $\mathscr{P} = (\Omega, \mathscr{D}, \Pr)$ be a probability space. Define a flou class $\mathscr{A} = (A_\alpha)_{\alpha \in [0,1]} \subseteq \mathscr{D}$ to be a nested class of sets which are nonincreasing inclusion-wise with respect to the index α and which are continuous with respect to union and intersection relative to the index α. Then the following statements are equivalent:*

1. *There is a random variable $U : \Omega \to [0,1]$ uniformly distributed.*
2. *For any probability distribution function F over $\mathbb{R}$ there is a random variable $V : \Omega \to \mathbb{R}$ such that V has probability distribution function F.*

3. *There is a flou subclass $\mathscr{A} = (A_\alpha)_{\alpha \in [0,1]}$ of $\mathscr{D}$ such that for all $\alpha \in [0,1]$,*

$$\Pr(A_\alpha) = 1 - \alpha. \tag{9}$$

4. *$\mathscr{P}$ is a nonatomic probability space.*

Proof. 1 implies 2: Let

$$V = F^{\dagger}(U), \tag{10}$$

where the pseudoinverse $F^{\dagger}$ is defined by

$$F^{\dagger}(t) = \inf F^{-1}(\inf\{x \mid t \leqslant x \in \text{range}(F)\}), \tag{11}$$

whence for all $t \in [0,1]$, $x \in R$,

$$F^{\dagger}(t) \leqslant x \quad \text{iff} \quad t \leqslant F(x). \tag{12}$$

1 implies 3: Let

$$A_\alpha = U^{-1}([\alpha,1]) \qquad \text{for all} \quad \alpha \in [0,1]. \tag{13}$$

3 implies 1: For all $\omega \in \Omega$, define

$$U(\omega) = \sup\{\alpha \mid \alpha \in [0,1] \ \&\ \omega \in A_\alpha\}, \tag{14}$$

which implies for all $\alpha \in [0,1]$

$$\Pr(U^{-1}([0,\alpha])) = \Pr(\Omega \setminus A_\alpha) = \alpha. \tag{15}$$

4 implies 3: Use, e.g., the standard result [15, pp. 168, 174].

3 implies 4: Suppose there is an atom $B \in \mathscr{D}$. Then use the continuity of probability to obtain desired contradiction. ■

Thus, from now on, $\mathscr{P}$ is assumed to be some fixed nonatomic probability space.

Let $A \in \mathscr{F}(X)$ be arbitrary. Consider now the mapping S_U without regard to the random variable U. That is, define the family of all level sets associated with A as

$$\text{lev}(A) = (\phi_A^{-1}([\alpha,1]))_{\alpha \in [0,1]} \tag{16}$$

and $\mathscr{L}ev(X)$ as the collection of all level families lev(A) for all $A \in \mathscr{F}(X)$. Operations between level families are defined componentwise in terms of ordinary set operations at each level α, including complements, intersections, unions, subset relations, functional transforms, projections, etc. Independent of the random set approach discussed earlier, Negoita and Ralescu [16] have developed isomorphic relations for many of the operations defined over $\mathscr{L}ev(X)$ and fuzzy-set operators, analogous to some of the isomorphic-like relations developed within a random set context as mentioned in point 4 above. These results may be combined into the following theorem:

THEOREM 2 [16, 1]. *Let X be any fixed space, and define $\mathscr{FL}(X)$ as the collection of all flou classes of X: $\mathscr{A} = (A_\alpha)_{\alpha \in [0,1]} \subseteq \mathscr{P}(X)$ with $A_0 = X$. Then*

1. *$\mathscr{FL}(X) = \mathscr{L}ev(X)$, where for any $\mathscr{A} \in \mathscr{FL}(X)$,*

$$\mathscr{A} = \mathrm{lev}(B_{\mathscr{A}}), \qquad B_{\mathscr{A}} \in \mathscr{F}(X), \tag{17}$$

where

$$\phi_{B_{\mathscr{A}}}(x) = \sup\{\alpha \mid \alpha \in [0,1] \;\&\; x \in A_\alpha\}, \qquad \text{all } x \in X. \tag{18}$$

Conversely, if $A \in \mathscr{F}(X)$, then lev$(A) \in \mathscr{FL}(X)$.

2. *Let the collection of all nested random subsets of X be denoted by $\mathscr{NR}(X)$, where it is assumed that for any $S \in \mathscr{FL}(X)$,* range$(S) \in \mathscr{FL}(X)$. *Then $\mathscr{F}(X)$, $\mathscr{FL}(X)$, and $\mathscr{NR}(X)$ are all bijectively related as in the following diagram:*

$$\begin{tikzcd}
 & \mathscr{FL}(X) \\
\mathscr{F}(X) \arrow[ur, "\mathrm{lev}"] \arrow[dr, "S_U"'] & \\
 & \mathscr{NR}(X) \arrow[uu, "\mathrm{range}"']
\end{tikzcd}$$

isomorphisms or isomorphic-like relations hold for all of the abovementioned operations defined over each of the three spaces.

Proof. The only thing new to show is that S_U is surjective. Let S be any nested random subset of X with range$(S) = (A_\alpha)_{\alpha \in [0,1]} \in \mathscr{FL}(X)$. Thus, there is a random variable $V: \Omega \to [0,1]$ such that for all $\omega \in \Omega$, $S(\omega) = A_{V(\omega)}$. By part 1 of this theorem,

$$S(\omega) = \phi^{-1}_{B_{\mathrm{range}(S)}}([V(\omega), 1]), \qquad \text{all } \omega \in \Omega. \tag{19}$$

Then letting F be the probability distribution function for V and using the proof of Theorem 1, for any $x \in X$,

$$x \in S \quad \text{iff} \quad V \leqslant \phi_{B_{(\text{range } S)}}(x) \quad \text{iff} \quad F^{\dagger}(U) \leqslant \phi_{B_{(\text{range } S)}}(x)$$

$$\text{iff} \quad U \leqslant F\left(\phi_{B_{(\text{range } S)}}(x)\right) \quad \text{iff} \quad x \in S_U(C), \tag{20}$$

where

$$\phi_C(x) = F\left(\phi_{B_{(\text{range } S)}}(x)\right), \qquad \text{all } x \in X \tag{21}$$

Hence, $S = S_U$ (in distribution). ■

An immediate consequence of Theorem 2 is that for any $A \in \mathscr{F}(X)$, there is a unique nested random subset of X, $S(A) \equiv A$, namely, $S(A) = S_U(A)$.

THE ONE-POINT RANDOM SET COVERAGE PROBLEM

Returning to points 1–3 in the section after the introduction, a basic question may be posed: Given $A \in \mathscr{F}(X)$, what other random subsets S of X exist besides $S_U(A)$ and $T(A)$ such that $S \equiv A$? Some results are given in [4], [2], where a family of random subsets of X is obtained, which includes $S_U(A)$ and $T(A)$ as members, all one-point-coverage-equivalent to a given A. However, this does not exhaust all possible such random sets. The problem of determining all possible random sets one-point-coverage equivalent to a given fuzzy set is called the one-point-coverage problem. (The problem of determining all random sets m-point-coverage-equivalent to a given fuzzy set is treated in the following section.) Let

$$\mathscr{S}_1(A) = \{ S \mid S \in \mathscr{R}(X) \ \&\ A \equiv S \}. \tag{22}$$

Suppose, from now on, that X is any space with $n = \operatorname{card}(X) \geqslant 3$, and order all 2^n sets $C \in \mathscr{P}(X)$ by the order $\leq$, where if $\operatorname{card}(C') < \operatorname{card}(C'')$ then $C' \leq C''$, and if $\operatorname{card}(C') = \operatorname{card}(C'')$ we determine an arbitrary but fixed and consistent order also. The notation

$$\binom{m}{j} = \frac{m!}{(m-j)!\,j!}, \qquad j = 0, 1, \ldots, m.$$

$$\mathbf{1}_m = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \mathbf{0}_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad a_n = 2^n - n - 1 \tag{23}$$

will be used, and denoting vector and matrix transposes by superscript Tr, we define

$$\kappa_{(n)}^{\mathrm{Tr}} = \left(1 \cdot \mathbf{1}_{\binom{\xi}{2}}^{\mathrm{T}}, 2 \cdot \mathbf{1}_{\binom{\xi}{3}}^{\mathrm{T}}, \ldots, (n-1) \cdot \mathbf{1}_{\binom{\xi}{n}}^{\mathrm{T}}\right) \quad (1 \text{ by } a_n); \tag{24}$$

$$\xi_x = (\phi_C(x))_{C \in \mathscr{P}(X), \mathrm{card}(C) \geqslant 2} \ (a_n \text{ by } 1), \quad \text{all } x \in X;$$

$$\xi_{(n)} = (\xi_x)_{x \in X} \ (a_n \text{ by } n); \tag{25}$$

$$\gamma^{(2)}(S) = (\Pr(S = C))_{C \in \mathscr{P}(X), \mathrm{card}(C) \geqslant 2} \quad (a_n \text{ by } 1), \qquad \text{all } S \in \mathscr{R}(X); \tag{26}$$

$$\nu(A) = (\phi_A(x))_{x \in X} \ (n \text{ by } 1), \quad \tau(A) = \max\left(\left(\sum_{x \in X} \phi_A(x)\right) - 1, 0\right), \quad \text{all } A \in \mathscr{F}(X) \tag{27}$$

THEOREM 3. *For any finite space X (with notation as above) and any $A \in \mathscr{F}(X)$:*

1. *We have*

$$\mathscr{S}_1(A) = \{ S \mid S \in \mathscr{R}(X) \ \& \ \gamma^{(2)}(S) \in \mathscr{R}_1(A) \}, \tag{28}$$

where

$$\mathscr{R}_1(A) = \{ W \mid W \in \mathbf{R}^{a_n} \ \textit{satisfying} \ (30)\text{–}(32) \}, \tag{29}$$

for

$$\tau(A) \leqslant \kappa_{(n)}^{\mathrm{Tr}} \cdot W \tag{30}$$

$$\nu(A) \geqslant \xi_{(n)}^{\mathrm{Tr}} \cdot W \tag{31}$$

$$0_{a_n} \leqslant W. \tag{32}$$

2. *$\mathscr{R}_1(A)$ is a closed convex region in $\mathbf{R}^{a_n}$ having in general $1 + n + a_n = 2^n$ hyperplane bounds with an uncountable infinity of possible elements $\gamma^{(2)}(S)$ in it, or equivalently, $S \in \mathscr{S}_1(A)$.*

3. *There is a bijective correspondence between* $\mathcal{S}_1(A)$ *and* $\mathcal{R}_1(A)$, *where for any given* $W \in \mathcal{R}_1(A)$, $S \in \mathcal{S}_1(A)$ *is uniquely determined (in distribution) by*

$$\gamma^{(2)}(S) = W, \tag{33}$$

$$\Pr(S = \varnothing) = 1 - \sum_{x \in X} \phi_A(x) + \kappa_{(n)}^{\mathrm{Tr}} \cdot W, \tag{34}$$

$$\Pr(S = \{x\}) = \phi_A(x) - 1_{a_n}^{\mathrm{Tr}} \cdot W, \qquad \textit{all } x \in X. \tag{35}$$

■

Define now the vertex set $\mathcal{V}_1(A)$ of $\mathcal{R}_1(A)$ as the set of all $W \in \mathcal{R}_1(A)$ such that equality holds in Equations (30)–(32) for a_n linearly independent columns of $(\kappa_{(n)}, \xi_{(n)}, I_{a_n})$. There can be at most

$$\binom{2^n}{a_n} = \binom{2^n}{n+1}$$

vertex elements. It also follows that for all $A \in \mathcal{F}(X)$:

(a) We have

$$\operatorname{range}(S_U(A)) = \left\{ \phi_A^{-1}([y_j(A), 1]) \middle| j = 1, \ldots, r \right\}, \tag{36}$$

where

$$\operatorname{range}(\phi_A) = \{ y_1(A), \ldots, y_r(A) \}, \qquad 0 \leqslant y_1(A) < \cdots < y_r(A) \leqslant 1. \tag{37}$$

Hence,

$$\Pr(S_U(A) = C) = \begin{cases} y_j(A) - y_{j-1}(A), & \text{if } C = \phi_A^{-1}([y_j(A), 1]) \quad \text{for } j = 1, \ldots, r, \\ & \qquad y_0(A) = 0, \\ 0 & , \text{ otherwise}, \end{cases} \tag{38}$$

and $\gamma^{(2)}(S_U(A)) \in \mathcal{V}_1(A)$, where

$$\kappa_{(n)}^{\mathrm{Tr}} \cdot \gamma^{(2)}(S_U(A)) = \sum_{x \in X} \phi_A(x) \geqslant \tau(A), \tag{39}$$

$$\xi_{(n)}^{\mathrm{Tr}} \cdot \gamma^{(2)}(S_U(A)) = \nu(A). \tag{40}$$

(b) $$\Pr(T(A) = C) = \left(\prod_{x \in C} \phi_A(x) \right) \left(\prod_{x \in X \setminus C} [1 - \phi_A(x)] \right), \qquad \text{all } C \in \mathcal{P}(X). \tag{41}$$

and $\gamma^{(2)}(T(A)) \in \text{interior}(\mathscr{R}_1(A))$ in general, where

$$\kappa_{(n)}^{Tr} \cdot \gamma^{(2)}(T(A)) = \Big(\sum_{x \in X} \phi_A(x) \Big) - 1 + \prod_{x \in X} [1 - \phi_A(x)] \geq \tau(A) \tag{42}$$

and

$$\xi_x^{Tr} \cdot \gamma^{(2)}(T(A)) = \phi_A(x) \cdot \Big(1 - \prod_{y \in X \setminus \{x\}} [1 - \phi_A(y)] \Big) \leq \phi_A(x), \qquad \text{all } x \in X \tag{43}$$

(c) It should be noted that the condition given in Equation (30) is superfluous iff

$$\sum_{x \in X} \phi_A(x) \leq 1, \tag{44}$$

i.e., ϕ_A is either an ordinary probability function [equality holding in (44)] or a deficient probability function [inequality holding in (44)] over X, in which case a random set $S'(A) \in \mathscr{S}_1(A)$, where $S'(A)$ may be identified with a random variable over X having ϕ_A as its probability function (possibly deficient). Thus

$$\gamma^{(2)}(S'(A)) = 0_{u_n}, \tag{45}$$

and hence $\gamma^{(2)}(S'(A)) \in \mathscr{V}_1(A)$. Also,

$$\Pr(S'(A) = \varnothing) = 1 - \sum_{x \in X} \phi_A(x), \tag{46}$$

$$\Pr(S'(A) = \{x\}) = \phi_A(x), \qquad \text{all } x \in X. \tag{47}$$

THE MULTIPLE-POINT RANDOM SET COVERAGE PROBLEM

For any integer $m \geq 1$, define $\mathscr{P}_{(m)}(X)$ as the collection of all nonvacuous subsets of X with cardinality $\leq m$. Then define

$$\begin{aligned}\mathscr{F}_{(m)}(X) = \{ f \mid f : \mathscr{P}_{(m)}(X) \to [0,1] \;\&\; & \text{there is an } S \in \mathscr{R}(X) \text{ such that} \\ & f(C) = \Pr(C \subseteq S), \text{ all } C \in \mathscr{P}_{(m)}(X) \} \\ \subseteq \{ f \mid f : \mathscr{P}_{(m)}(X) \to [0,1] \}. & \end{aligned} \tag{48}$$

with, in general, strict subset inclusion holding above. That is, there are functions $f : \mathscr{P}_{(m)} \to [0,1]$ which are not the m-point-coverage function of some random subset of X. A simple example of this is generated for the case $m = 2$ by first noting the basic constraints for any random subset S of X and any $x, y \in X$:

$$\max(\Pr(x \in S) + \Pr(y \in S) - 1, 0) \leqslant \Pr(\{x, y\} \subseteq S)$$
$$\leqslant \min(\Pr(x \in S), \Pr(y \in S)) \qquad (49)$$

and then choosing, e.g., any such f with at least some $x, y \in X$ such that

$$\min(f(\{x\}), f(\{y\})) < f(\{x, y\}). \qquad (50)$$

This situation contrasts sharply with the case $m = 1$, where indeed

$$\mathscr{F}(X) = \mathscr{F}_{(1)}(X) = \{f \mid f : \mathscr{P}_{(1)}(X) \to [0,1]\}, \qquad (51)$$

abusing notation somewhat in identifying A with ϕ_A for any $A \in \mathscr{F}(X)$.

In the following discussion, X need not be finite nor even discrete. Clearly,

$$\mathscr{F}_{(1)}(X) \supseteq \mathscr{F}_{(2)}(X) \supseteq \mathscr{F}_{(3)}(X) \supseteq \cdots \supseteq \mathscr{F}_{(\infty)}(X) = \bigcap_{j=1}^{\infty} \mathscr{F}_{(j)}(X) \qquad (52)$$

and if for any $f \in \mathscr{F}_{(m)}(X)$

$$\mathscr{S}_{(m)}(f) = \{S \mid S \in \mathscr{R}(X) \;\&\; f(C) = \Pr(C \subseteq S), \text{ all } C \in \mathscr{P}_{(m)}(X)\}, \qquad (53)$$

then

$$\mathscr{S}_1(f) = \mathscr{S}_{(1)}(f) \supseteq \mathscr{S}_{(2)}(f) \supseteq \mathscr{S}_{(3)}(f) \supseteq \cdots \supseteq \mathscr{S}_{(m)}(f). \qquad (54)$$

Letting $\mathscr{J}(X)$ be the class of all finite subsets of X,

$$\mathscr{F}_{(\infty)}(X) = \{f \mid f : \mathscr{J}(X) \to [0,1] \;\&\; \text{there is an } S \in \mathscr{R}(X) \text{ such that}$$
$$f(C) = \Pr(C \subseteq S), \text{ all } C \in \mathscr{J}(X)\} \qquad (55)$$

Define also, for any $f \in \mathscr{F}_{(\infty)}(X)$,

$$\mathscr{S}_{(\infty)}(f) = \{S \mid S \in \mathscr{R}(X) \;\&\; f(C) = \Pr(C \subseteq S) \text{ all } C \in \mathscr{J}(X)\} \qquad (56)$$

Then gathering all of these definitions together yields the following theorem:

THEOREM 4. *For any space X:*

1. $\mathcal{F}_{(\infty)}(X)$ *is the class of all doubt measures over* $\mathcal{J}(X)$. (See [1], [17].)
2. *For any integer $m \geqslant 1$, $\mathcal{R}(X)$ is disjointly partitioned as*

$$\mathcal{R}(X) = \dot{\bigcup_{f \in \mathcal{F}_{(m)}(X)}} \mathcal{S}_{(m)}(f). \tag{57}$$

3. *For any $f \in \mathcal{F}_{(\infty)}(X)$, there is a unique $S_f \in \mathcal{R}(X)$ such that*

$$\mathcal{S}_{(\infty)}(f) = \{S_f\}, \tag{58}$$

and the converse holds if for each $S \in \mathcal{R}(X)$ we determine the coverage function f_S by

$$f_S(C) = \Pr(C \subseteq S), \qquad \text{all } C \in \mathcal{J}(X). \tag{59}$$

4. $\mathcal{F}_{(\infty)}(X)$, $\mathcal{R}(X)$, *and $\mathcal{V}(X)$, the class of all zero-one stochastic processes over X, are all in a bijective relationship as follows:*

$$\begin{tikzcd} & \mathcal{R}(X) \arrow[d, "\phi"] \\ \mathcal{F}_{(\infty)}(X) \arrow[ur, "\mathcal{S}_{(\infty)}"] \arrow[r, "\rho"'] & \mathcal{V}(X) \end{tikzcd},$$

where for any $S \in \mathcal{R}(X)$ we have $\phi(S) = (\phi_S(x))_{x \in X}$, and ρ is determined by the Kolmogorov extension theorem applied to the relations below:

5. *For any $V = (V(x))_{x \in X} \in \mathcal{V}(X)$ and any $C, D \in \mathcal{J}(X)$ disjoint,*

$$\Pr\Big(\Big(\mathop{\&}_{x \in C}[V(x) = 1]\Big) \,\&\, \Big(\mathop{\&}_{x \in D}[V(x) = 0]\Big)\Big) = \sum_{K \subseteq D} (-1)^{\mathrm{card}(K)} f_V(C \cup K) \tag{60}$$

where $f_V \in \mathcal{F}_{(\infty)}(X)$ is given, for any $B \in \mathcal{J}(X)$, by

$$f_V(B) = \Pr\Big(\mathop{\&}_{x \in B}[V(x) = 1]\Big). \tag{61}$$

(See [14].) ∎

Thus in terms of degree of coverages, $\mathcal{F}(X)$ and $\mathcal{R}(X)$ represent the opposite extremes.

Returning to the case for X finite, let $m \geqslant 1$ be arbitrarily fixed with $\text{card}(X) = n \geqslant 3$. Define for any $S \in \mathscr{R}(X)$, recalling the total order defined over $\mathscr{P}(X)$,

$$\gamma^{(m+1)}(S) = (\Pr(S = C))_{C \in \mathscr{P}(X),\, \text{card}(C) \geqslant m+1} \qquad (a_{n,m} \text{ by } 1), \tag{62}$$

$$a_{n,m} = \sum_{k=m+1}^{n} \binom{n}{k}, \tag{63}$$

$$\mathscr{C}_{B,j}(X) = \{ C \mid B \subseteq C \in \mathscr{P}(X) \ \&\ \text{card}(C) = j \},$$

$$\text{all } C \in \mathscr{P}(X), \quad 0 \leqslant j \leqslant n. \tag{64}$$

Also, define for any $n \geqslant m \geqslant r, s \geqslant 0$ and all $f \in \mathscr{F}_{(m)}(X)$ and $C \in \mathscr{P}(X)$, $\text{card}(C) = r$,

$$\nu_{r,0}(f, C) = f(C) \qquad \text{for} \quad s = 0, \tag{65}$$

$$\nu_{r,s}(f, C) = f(C) + \sum_{t=1}^{s} \left((-1)^t \sum_{B \in \mathscr{C}_{C,r-t}} f(B) \right), \qquad \text{for} \quad s \geqslant 1, \tag{66}$$

$$(\gtrless)_j = \begin{cases} \geqslant & \text{iff} \quad j \text{ is even}, \\ \leqslant & \text{iff} \quad j \text{ is odd}. \end{cases} \tag{67}$$

THEOREM 5. *For any finite space X, with notation as above, for any $f \in \mathscr{F}_{(m)}(X)$:*

1. *We have*

$$\mathscr{S}_{(m)}(f) = \left\{ S \mid S \in \mathscr{R}(X) \ \&\ \gamma^{(m+1)}(S) \in \mathscr{R}_{(m)}(f) \right\}, \tag{68}$$

where

$$\mathscr{R}_{(m)}(f) = \{ W \mid W \in \mathbb{R}^{a_{n,m}} \text{ - } \textit{satisfying } (70), (71) \}, \tag{69}$$

with

$$(-1)^j \nu_{m-j,j}(f, C) (\gtrless)_j \sum_{i=m+1}^{n} \left[\binom{i-m+j-1}{j} \sum_{B \in \mathscr{C}_{C,i}(X)} W_B \right] \tag{70}$$

for all $C \in \mathscr{P}(X)$ *with* $\mathrm{card}(C) = m - j$, *for* $j = 0, 1, 2, \ldots, m$,

$$W_B \geqslant 0, \qquad \text{all } B \in \mathscr{P}(X) \text{ with } \mathrm{card}(B) \geqslant m+1, \quad \text{all } f \in \mathscr{F}_{(m)}(X); \tag{71}$$

and where

$$W = (W_B)_{B \in \mathscr{P}(X),\, \mathrm{card}(B) \geqslant m+1} \in \mathbf{R}^{a_{n,m}}. \tag{72}$$

2. $\mathscr{R}_{(m)}(f)$ *is a closed convex region in* $\mathbf{R}^{a_{n,m}}$ *having, in general,*

$$a_{n,m} + \sum_{i=0}^{m} \binom{n}{i} = 2^n$$

hyperplane bounds, and thus, in general, there is an uncountable infinity of possible $\gamma^{(m+1)}(S)$*'s in it, or equivalently* $S \in \mathscr{S}_{(m)}(f)$.

3. *There is a bijective correspondence between* $\mathscr{S}_{(m)}(f)$ *and* $\mathscr{R}_{(m)}(f)$, *where for any given* $W \in \mathscr{R}_{(m)}(f)$, $S \in \mathscr{S}_{(m)}(f)$ *is uniquely determined by*

$$\gamma^{(m+1)}(S) = W, \tag{73}$$

$$\Pr(S = C) = \nu_{m-j,j}(f, C) + (-1)^{j+1} \sum_{i=m+1}^{n} \left[\binom{i-m+j-1}{j} \sum_{B \in \mathscr{C}_{C,i}(X)} W_B \right] \tag{74}$$

for all $C \in \mathscr{P}(X)$ *with* $\mathrm{card}(C) = m - j$, $j = 0, 1, \ldots, m$.

Outline of proofs for Theorems 3 and 5. First obtain Theorem 3 by simply extending the basic identity in Equation (1) in terms of $\Pr(S = C)$'s together with the standard probability constraints, and solve in terms of $\gamma^{(2)}(S)$. Theorem 5 is obtained by a tedious induction. The key computational identity useful in the proofs is

$$\sum_{B \in \mathscr{C}_{C,k}(X)} \left(\sum_{D \in \mathscr{C}_{B,j}(X)} f(D) \right) = \binom{j-l}{k-l} \sum_{D \in \mathscr{C}_{C,j}(X)} f(D) \tag{75}$$

for any $f : \mathscr{P}(X) \to \mathbf{R}$, $j \geqslant k \geqslant l$, and any $C \in \mathscr{P}(X)$ with $\mathrm{card}(C) = l$. ∎

USE OF ENTROPY AND OTHER CRITERIA FOR COVERAGE PROBLEMS

Another basic problem associated with fuzzy sets, or more generally, multiple-point coverage functions $f \in \mathscr{F}_{(m)}(X)$ and random sets $S \in \mathscr{S}_{(m)}(f)$, is the determination of meaningful criteria for ordering in some sense the elements of $\mathscr{S}_{(m)}(f)$ in terms of "best" representing f. Among natural criteria, should be mentioned the expected n-volume if $X \subseteq \boldsymbol{R}^n$, the expected cardinality if X is finite, and the entropy. Consider first volume and cardinality. The following important result must be taken into account. This is based on an earlier result of Robbins [18] which was also later independently rediscovered by Pratt [19], and in turn used by Hooper [13] in demonstrating that most common figures of merit for randomized test procedures and randomized confidence sets involve the random sets determining these procedures only through their one-point coverage functions.

THEOREM 6 [18, 19]. *Let m be any positive integer, X any space, and* $S \in \mathscr{R}(X)$. *Then (assuming the usual measurability conditions), letting* f_S *be the coverage function of S [see Equation* (59)]:

1. *If* $X \subseteq R^n$ *for some fixed positive integer n and* vol_n *denotes n-dimensional Lebesgue measure, then for any* $m \geqslant 1$

$$E([\mathrm{vol}_n(S)]^m) = \int_{(x_1,\ldots,x_m \in R^n)} \cdots \int f_S(\{x_1,\ldots,x_m\})\,dx_1 \cdots dx_m. \tag{76}$$

2. *If X is a finite space,*

$$E([\mathrm{card}(S)]^m) = \sum_{(x_1,\ldots,x_m \in X)} \cdots \sum f_S(\{x_1,\ldots,x_m\}). \tag{77}$$

Proof. Let $g: \mathscr{B} \times \mathscr{C} \to \boldsymbol{R}$, where $\mathrm{range}(S) \subseteq \mathscr{B} \subseteq \mathscr{P}(\boldsymbol{R}^n)$, $\mathscr{C} \subseteq \boldsymbol{R}^{nm}$, and let $\mu: \mathscr{A} \to \boldsymbol{R}$ be a measure relative to the measure space $(\mathscr{C}, \mathscr{A}, \mu)$. Then apply Fubini's interchange theorem to the integral of g with respect to the joint measure $\Pr \circ S^{-1} \times \mu$, and specialize for $g(B, x) \equiv \phi_B(x_1) \cdots \phi_B(x_m)$, $B \in \mathscr{B}$, $x = (x_1, \ldots, x_m) \in \mathscr{C}$, with $\mathscr{C}$ and $\mathscr{A}$ chosen accordingly, when first $\mu = \mathrm{vol}_{nm}$ (Lebesgue measure) and then $\mu = \mathrm{card}$ (counting measure). ■

Thus, for any given $f \in \mathscr{F}_{(m)}(X)$ we have $E([\mathrm{vol}_n(S)]^t) = \mathrm{constant}_t$, $t = 1, \ldots, m$, regardless of the S chosen from $\mathscr{S}_{(m)}(X)$. Thus another criterion must be sought which is sensitive to variable $S \in \mathscr{S}_{(m)}(X)$.

Consider now entropy as a possible criterion. In this paper only the case of base space X finite will be treated. For any $S \in \mathscr{R}(X)$, define the entropy for S analogously to the usual definition for probability functions:

$$\mathrm{Ent}(S) = \sum_{C \in \mathscr{P}(X)} [-\Pr(S = C) \log \Pr(S = C)]. \tag{78}$$

The results of Theorem 4, part 4, specialized to this case, yield dually that S

may be identified with a random vector ϕ_S over the space $\{0,1\}^n$, assuming card$(X) = n$.

THEOREM 7. *Let* card$(X) = n$. *Then for any* $A \in \mathscr{F}(X)$

$$\sup_{S \in \mathscr{S}_1(A)} (\mathrm{Ent}(S)) = \sum_{x \in X} \{-\phi_A(x)\log\phi_A(x) - [1-\phi_A(x)]\log[1-\phi_A(x)]\}, \tag{79}$$

occurring uniquely for $S = T(A)$.

Proof. Either use the fundamental information-theory inequality applied to ϕ_S —which is equivalent to a specialization of the well-known result that given any marginal probability functions, the joint probability function maximizing the entropy corresponds to the marginal random variables being statistically independent—or specialize the exponential family characterization of maximal entropy to this case (see, e.g., Jaynes [20] for the general case). ■

On the other hand, the minimal-entropy problem here poses more difficulties. It can be shown that:

THEOREM 8. *Let* card$(X) = n$. *Then for any* $A \in \mathscr{F}(X)$

$$\inf_{S \in \mathscr{S}_1(A)} \mathrm{Ent}(S) = \min_{\gamma^{(2)}(S) \in \mathscr{V}_1(A)} \mathrm{Ent}(S). \tag{80}$$

Proof. By Theorem 3, part 3, the minimization problem reduces to the routine minimizing of the sum of a strictly convex function $(-x\log x)$ of linear combinations of components of $\gamma^{(2)}(S)$ over the region $\mathscr{R}_1(A)$. ■

Even though [see remarks (a) and (c) following Theorem 3] $\gamma^{(2)}(S_U(A)) \in \mathscr{V}_1(A)$, and if $\sum_{x \in X}\phi_A(x) \leq 1$, then $S'(A)$ is well defined with $\gamma^{(2)}(S'(A)) \in \mathscr{V}_1(A)$, it is not always true that the global minimal entropy occurs for either one. A simple example illustrates this. Let $n = 2$ with $X = \{x_1, x_2\}$, and $A \in \mathscr{F}(X)$ with $\phi_A(x_1) \leq \phi_A(x_2)$. Let $h(x) = -x\log x$, all x, and define $y_1 = \phi_A(x_1)$ and $y_2 = \phi_A(x_2)$. Finally, let $G(A) = \inf_{S \in \mathscr{S}_1(A)} \mathrm{Ent}(S)$. Then:

(i) If $\tau(A) = 0$ and $y_2 > \frac{1}{2}$, then

$$G(A) = h(y_1) + h(y_2) + h(1 - y_1 - y_2), \tag{81}$$

occurring uniquely for $S = S'(A)$. (See (45)-(47).)

(ii) If either $\tau(A) = 0$ and $y_2 < \frac{1}{2}$ or $\tau(A) > 0$ and $y_1 \geq \frac{1}{2}$, then

$$G(A) = h(y_1) + h(y_2 - y_1) + h(1 - y_2), \tag{82}$$

occurring uniquely for $S = S_U(A)$.

(iii) If $\tau(A) > 0$ and $y_1 < \frac{1}{2}$, then

$$G(A) = h(1-y_1) + h(1-y_2) + h(y_1+y_2-1), \tag{83}$$

occurring uniquely for $S = S''(A)$, where $S''(A)$ is determined by

$$\Pr(S''(A) = \{x_1\}) = 1 - y_2, \qquad \Pr(S''(A) = \{x_2\}) = 1 - y_1,$$
$$\Pr(S''(A) = X) = \tau(A), \qquad \Pr(S''(A) = \varnothing) = 0. \tag{84}$$

Extensions of the entropy problem to multiple-coverage functions and general spaces X have yet to be addressed.

ONE-POINT-COVERAGE FUNCTIONS AND RANDOM INTERVALS

An important class of random sets is the random intervals. In this section the one-point coverage problem is specialized to the case where the base space $X = \boldsymbol{R}$, $A \in \mathscr{F}(\boldsymbol{R})$, and $\mathscr{S}_1(A)$ is replaced by a more restrictive class $\mathscr{S}_1(A, \mathscr{Q})$, the class of all $S \in \mathscr{Q}$ such that $S \cong A$, for various classes $\mathscr{Q} \subseteq \mathscr{R}(\boldsymbol{R})$ of random closed intervals of $\boldsymbol{R}$.

Define first $\mathscr{Q}_1$ as the class of all random closed intervals and $\mathscr{Q}_2$ as the class of all random closed intervals S with $\Pr(S = \varnothing) = 0$. Then:

1. There is a natural identification between $\mathscr{Q}_1$ and the class of all bivariate random variables

$$Z = \begin{pmatrix} V \\ W \end{pmatrix}$$

over the upper half diagonal plane in $\boldsymbol{R}^2$, and between $\mathscr{Q}_2$ and the class of all random variables over $\boldsymbol{R}^2$. (See [21] for related results.) From now on, let $S = [V, W]$ denote a random interval with V and W r.v.'s over $\boldsymbol{R}$; denote the marginal probability distribution function for V by F_1, that for W by F_2, the joint probability distribution function by F, etc. The convention $[a, b] = \varnothing$ for $a > b$ will be used.

2. For any $A \in \mathscr{F}(\boldsymbol{R})$, we have $S = [V, W] \in \mathscr{S}_1(A, \mathscr{Q}_1)$ iff

$$\phi_A(x) = F_1(x) - F(x, x), \qquad \text{all } x \in \boldsymbol{R}, \tag{85}$$

the solution of which for F_1 and F in terms of ϕ_A may be complicated unless ϕ_A is further specified.

3. Letting $\mathscr{Q}_3$ be the class of all random intervals $S = [V, W]$ with V and W statistically independent, we have

$$\Pr(S \neq \varnothing) = \int_{x \in R} F_1(x)\, dF_2(x), \tag{86}$$

and for any $A \in \mathscr{F}(\boldsymbol{R})$, we have $S = [V, W] \in \mathscr{S}_1(A, \mathscr{Q}_3)$ iff

$$\phi_A(x) = F_1(x) \cdot [1 - F_2(x)], \qquad \text{all } x \in \boldsymbol{R}, \tag{87}$$

which implies that $\log \phi_A$ is of bounded variation with $\lim_{x \to \pm\infty} \phi_A(x) = 0$.

4. If $A \in \mathscr{F}(\boldsymbol{R})$ is such that ϕ_A is unimodal (which will be taken in the sense that possibly a neighborhood of modal points exists and continuity from the right holds) at some x_0 (say) at which $\phi_A(x_0) = 1$, then $S = [V, W] \in \mathscr{S}_1(A, \mathscr{Q}_3)$, where

$$F_1(x) = \begin{Bmatrix} \phi_A(x) & \text{if} & x \leqslant x_0 \\ 1 & \text{if} & x \geqslant x_0 \end{Bmatrix}, \qquad F_2(x) = \begin{Bmatrix} 0 & \text{if} & x \leqslant x_0 \\ 1 - \phi_A(x) & \text{if} & x \geqslant x_0 \end{Bmatrix}. \tag{88}$$

5. Let $\mathscr{Q}_4$ be the class of all random intervals $S = [V, W]$ with V and W statistically independent and identically distributed. If $A \in \mathscr{F}(\boldsymbol{R})$, then $\mathscr{S}_1(A, \mathscr{Q}_4) \neq \varnothing$ iff $\lim_{x \to \pm\infty} \phi_A(x) = 0$ and ϕ_A is unimodal at some x_0 with $\phi_A(x_0) \leqslant \frac{1}{4}$, in which case $\mathscr{S}_1(A, \mathscr{Q}_4) = \{S(A)\}$, where $S(A) = [V, W]$ is determined by solving Equation (87) for $F_1 = F_2$ in terms of ϕ_A:

$$F_1(x) = F_2(x) = \begin{cases} \frac{1}{2}\left\{1 - [1 - 4\phi_A(x)]^{1/2}\right\} & \text{if } x < x_0, \\ \frac{1}{2}\left\{1 + [1 - 4\phi_A(x)]^{1/2}\right\} & \text{if } x \geqslant x_0. \end{cases} \tag{89}$$

6. Let $\mathscr{Q}_5$ be the class of all nested random closed intervals. Then using Theorem 2,

$$\mathscr{Q}_5 = \{ S_U(A) \mid A \in \mathscr{F}(\boldsymbol{R})\ \&\ \phi_A \text{ is continuous and either unimodal, nonincreasing, or nondecreasing over } \boldsymbol{R} \}, \tag{90}$$

where as usual U is a fixed random variable uniformly distributed over $[0, 1]$.

Also, for any $A \in \mathscr{F}(\boldsymbol{R})$ with ϕ_A continuous and either unimodal, nonincreasing, or nondecreasing, $\mathscr{S}_1(A, \mathscr{Q}_5) = \{S_U(A)\}$, where it should be noted that

$$S_U(A) = \begin{cases} (-\infty, \sup \phi_A^{-1}(U)] & \text{if } \phi_A \text{ is nonincreasing,} \\ [\inf \phi_A^{-1}(U), +\infty] & \text{if } \phi_A \text{ is nondecreasing.} \end{cases} \tag{91}$$

In a related vein, note that if ϕ_A is any probability distribution function over $\boldsymbol{R}$ corresponding to random variable Z say, then Z can be identified with $\phi_A^{\dagger}(U)$ (see Theorem 1), $S_U(A) = [Z, +\infty)$, and

$$\phi_A(x) = \Pr(Z \leqslant x) = \Pr(x \in [Z, +\infty)), \qquad \text{all } x \in \boldsymbol{R}. \tag{92}$$

Two more interesting classes of random intervals remain to be discussed.

THEOREM 9. *Let $\mathscr{Q}_6$ denote the class of all fixed-length random closed intervals. Let $A \in \mathscr{F}(\boldsymbol{R})$. Then $\mathscr{S}_1(A, \mathscr{Q}_6) \neq \varnothing$ iff there exists a probability distribution function F_A and a positive real constant b_A such that for all $x \in X$,*

$$\phi_A(x) = F_A(x + b_A) - F_A(x - b_A) \tag{93}$$

iff ϕ_A is integrable over $\boldsymbol{R}$, where we define

$$2b_A = \int_{x \in R} \phi_A(x)\, dx \tag{94}$$

and F_A, where

$$F_A(x + (2k+1)b_A) = \sum_{j=-\infty}^{k} \phi_A(x + 2jb_A),$$

$$\text{all } x \in [0, 2b_A], \quad k = 0, \pm 1, \pm 2, \ldots \tag{95}$$

is a legitimate probability distribution function, in which case $\mathscr{S}_1(A, \mathscr{Q}_6) = \{S(A)\}$, where

$$S(A) = [V - b_A, V + b_A], \tag{96}$$

where V is a random variable having probability distribution function F_A. ∎

THEOREM 10. *Let $\mathscr{Q}_7$ denote the class of random closed intervals of the form $S = [V - W_1, V + W_2]$, where V is a random variable over $\boldsymbol{R}$ statistically indepen-*

dent of random variables W_1 and W_2 jointly defined over $\mathbf{R}^+ \times \mathbf{R}^+$. Then for any $A \in \mathscr{F}(\mathbf{R})$, $\mathscr{S}_1(A, \mathscr{Q}_7) \neq \emptyset$ iff

$$\phi_A = f_A \circledast G_A, \tag{97}$$

where $f_A : \mathbf{R} \to [0,1]$ is unimodal at 0 with $f_A(0) = 1$, $\lim_{x \to \pm\infty} f_A(x) = 0$, G_A is some probability distribution function over $\mathbf{R}$, and $\circledast$ denotes the convolution operator; in which case $S(A) = [V - W_1, V + W_2] \in \mathscr{S}_1(A, \mathscr{Q}_7)$, where V has probability distribution function G_A, W_1 has probability distribution function $1 - f_A(-\cdot)$ over $\mathbf{R}^+$, and W_2 has probability distribution function $1 - f_A(\cdot)$ over $\mathbf{R}^+$, with W_1, W_2 jointly arbitrary and statistically independent of V. ∎

It follows, e.g., from [22], that any $A \in \mathscr{F}(\mathbf{R})$ for which ϕ_A is uniformly continuous and integrable over $\mathbf{R}$ may be arbitrarily uniformly closely approximated, in the one-point-coverage sense, up to some scalar multiple, by some S in $\mathscr{Q}_7$. (Approximate ϕ_A by $c \cdot f \circledast G$, where $c = c_0/[(2\pi)^{1/2}\sigma]$, $c_0 = \int_{x \in \mathbf{R}} \phi_A(x)\,dx$, and $f = (2\pi)^{1/2}\sigma f_\sigma$, f_σ being the probability density function for Gaussian distribution $N(0, \sigma^2)$.)

A CONNECTION BETWEEN FUZZY SETS, RANDOM SETS, AND RANDOM VARIABLES

So far in this paper, connections have been established between the membership functions of fuzzy sets and the one-point coverage functions of random sets. Recently [23] it has been shown that random variable evaluation functions may also be directly related to membership functions and one-point coverage functions. This result is restated.

If Y is any space of elementary events and $\mathscr{B} \subseteq \mathscr{P}(Y)$ is any collection of compound or elementary events for a random variable V over Y (i.e., $\mathscr{B} \subseteq \mathscr{A}$, the σ-algebra over Y for V), then the function $g_V : \mathscr{B} \to [0,1]$, where for any $B \in \mathscr{B}$ we have $g_V(B) = \Pr(V \in B)$, is called the evaluation function for V over the event collection $\mathscr{B}$. Recall the notation f_S for the one-point coverage function for any $S \in \mathscr{R}(X)$ for any space X.

THEOREM 11 [23].

1. *Let X be any space and $S \in \mathscr{R}(X)$. Then X may be identified as a collection of events $\mathscr{B}$ for some random variable V over (say) Y such that*

$$S = \mathscr{C}_{(V)} \tag{98}$$

and whose evaluation function over $\mathscr{B}$ is the same as f_S.

2. *Let V be any random variable over (say) Y, and $\mathscr{D}$ any collection of events for V. Then, letting $X = \mathscr{D}$ and defining S by Equation* (98), *the evaluation g_V for V over $\mathscr{D}$ coincides with f_S.*

3. *Let X be any space, and let $A \in \mathscr{F}(X)$ be arbitrary. Let $S \in \mathscr{R}(X)$ be arbitrary with $S \in \mathscr{S}_1(A)$, such as $S_U(A)$, $T(A)$, for example. Then applying part 1 above, X may be identified as a collection of events $\mathscr{D}$ for some random variable V over (say) Y such that Equation* (98) *holds and ϕ_A, f_S, and g_V all coincide. That is, any fuzzy set is one-point-equivalent to the evaluation function of suitably chosen random variables over event collections.* ■

One consequence of Theorem 11 is a new interpretation for possibilities, or equivalently, values $\phi_A(x)$ for $x \in X$, where $A \in \mathscr{F}(X)$ is given, through the membership function $\phi_A : X \to [0,1]$: For any $x \in X$,

$$\text{Possibility of } x \in A = \text{Poss}(x \in A) = \Pr(x \in S(A)) = \Pr(V(A) \in x), \quad (99)$$

and since the events $x \in X$ can also be considered compound or elementary events for $V(A)$ over Y which may well be overlapping and perhaps exhaustive, possibilities need not sum to unity when X is discrete. However, when—and only when—ϕ_A is an ordinary or deficient probability function, possibilities will sum to unity or to less than unity, possibilities and probabilities coincide, and the events for $V(A)$ are all necessarily elementary and disjoint, with $V(A)$ and $S(A)$ also being identifiable.

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