

AD-A241 215



ON PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington DC 20503

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE August 1991	3. REPORT TYPE AND DATES COVERED professional paper	
4. TITLE AND SUBTITLE CHARACTERIZATIONS OF N-ARY FUZZY SET OPERATIONS WHICH INDUCE HOMOMORPHIC RANDOM SET OPERATIONS		5. FUNDING NUMBERS In-house funding	
6. AUTHOR(S) I. R. Goodman		8. PERFORMING ORGANIZATION REPORT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Ocean Systems Center San Diego, CA 92152-5000			
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Naval Ocean Systems Center San Diego, CA 92152-5000		10. SPONSORING/MONITORING AGENCY REPORT NUMBER	
11. SUPPLEMENTARY NOTES			
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.		12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) Let X be any given base space and let $F(X)$ denote the class of all fuzzy subsets of X . $P(X)$ the class of all ordinary subsets of X , and $S(X)$ the class of all random subsets of X . It is now known that a partitioning of $S(X)$ exists whose components $S_A(X)$ are indexed by arbitrary $A \in F(X)$, with equivalence for each component determined by the one point coverage relation $\phi_A(x) = \Pr(x \in S(A))$, for all $x \in X$, $S(A)$ arbitrary in $S_A(X)$, with $S : F(X) \rightarrow S(X)$ denoting any corresponding choice-mapping. Suppose $\star$ is an n -ary operation on $F(X) \times \dots \times F(X)$ and S (as above) is given. $\star$ is said to induce a weak n -homomorphism if there exists $\tau : P(X) \times \dots \times P(X) \rightarrow P(X)$ such that for any $A_1, \dots, A_n \in F(X)$ and thus $S(A_k) \in S_{A_k}(X)$, $k = 1, \dots, n$, a joint distribution of $(S(A_1), \dots, S(A_n))$ exists such that for all $x \in X$, $\phi_{\star(A_1, \dots, A_n)}(x) = \Pr(x \in S(\star(A_1, \dots, A_n))) = \Pr(x \in (\tau(S(A_1), \dots, S(A_n))))$. Characterizations are obtained for the class of n -ary fuzzy set operations $\star$ which induce weak n -homomorphisms with corresponding $\star$ satisfying nested cases of measurability, continuity, and general composability. Published in the <i>Fuzzy Information and Decision Processes</i> , North-Holland Publishing Co., 1982.			
14. SUBJECT TERMS fuzzy sets, level sets, homomorphisms, fuzzy set operations, random sets, <i>Report 5</i>		15. NUMBER OF PAGES	
16. PRICE CODE		17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	
18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED		19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	
20. LIMITATION OF ABSTRACT SAME AS REPORT			

91-12116



91 10 1 092

UNCLASSIFIED

21a. NAME OF RESPONSIBLE INDIVIDUAL I. R. Goodman	21b. TELEPHONE (include Area Code) (619) 553-4014	21c. OFFICE SYMBOL Code 421

CHARACTERIZATIONS OF N-ARY FUZZY SET OPERATIONS WHICH INDUCE HOMOMORPHIC RANDOM SET OPERATIONS

I.R. Goodman

Naval Ocean Systems Center
Surveillance Systems Department, Code 7223
San Diego, California 92152

ABSTRACT

Let X be any given base space and let $F(X)$ denote the class of all fuzzy subsets of X , $P(X)$ the class of all ordinary subsets of X , and $S(X)$ the class of all random subsets of X . It is now known that a partitioning of $S(X)$ exists whose components $S_A(X)$ are indexed by arbitrary $A \in F(X)$, with equivalence for each component determined by the one point coverage relation $\phi_A(x) = \Pr(x \in S(A))$, for all $x \in X$, $S(A)$ arbitrary in $S_A(X)$, with $S: F(X) \rightarrow S(X)$ denoting any corresponding choice-mapping.

Suppose $\odot$ is an n -ary operation on $F(X) \times \dots \times F(X)$ and S (as above) is given. $\odot$ is said to induce a weak n -homomorphism iff there exists $\star: P(X) \times \dots \times P(X) \rightarrow P(X)$ such that for any $A_1, \dots, A_n \in F(X)$, and thus $S(A_k) \in S_{A_k}(X)$, $k=1, \dots, n$, a joint distribution of $(S(A_1), \dots, S(A_n))$ exists such that for all $x \in X$, $\phi_{\odot(A_1, \dots, A_n)}(x) \equiv \Pr(x \in S(\odot(A_1, \dots, A_n))) = \Pr(x \in \star(S(A_1), \dots, S(A_n)))$.

Characterizations are obtained for the class of all n -ary fuzzy set operations $\odot$ which induce weak n -homomorphisms with corresponding $\star$ satisfying nested cases of measurability, continuity, and general compositibility.

Accession For	
NTIS ORIGIN	
DTIC TAB	
Unannounced	
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	20

KEYWORDS

Fuzzy sets, level sets, homomorphisms, fuzzy set operations, (ordinary) set operations, random sets.

1. INTRODUCTION

With the introduction of Zadeh's basic paper in 1965 [21], a new approach to the modeling of uncertainty began. The fledgling discipline of fuzzy set theory has now grown to encompass well over two thousand papers. (See the 1800 references in [7], as of 1979.) Applications of fuzzy set techniques have been made to a wide variety of subjects, too numerous to be mentioned here. (See, e.g., Dubois and Prade's recent text [2].) As early as 1971, Zadeh himself recognized the role that level sets¹ played with respect to fuzzy sets [22]. In 1975 Negoita and Ralescu [14] (see also [17]) obtained a representation theorem for fuzzy sets and certain of their operations. (See also [16] for another level set approach.) Independent of this development, Goodman [4] in an unpublished paper in 1976 derived a simple homomorphic relationship between fuzzy and random sets and their operations - in this case the levels of the set involved were randomized uniformly (See also Nguyen's reformulation of this result [15]. A listing of papers contrasting the classical probability and fuzzy set approaches is given in the introductory section of [6]. Additional comments on the controversy may be found in [1], [3], [13], [18] - [20].)

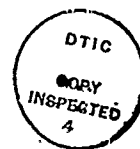
The two recent (independent) papers of Höhle [9] and Goodman [6] establish that, at least from a formal viewpoint, there exist systematic connections between fuzzy set theory and its operations, and probability theory and corresponding operations, via the concept of random sets. In the latter paper, one of the connections (SU-type) makes use of the basic level set mapping, used earlier in [4], while another (T₀-type) is related to the product probability measure

obtained from statistically independent 0-1 marginal random variables with parameters related to a given fuzzy set. Specifically, it was shown (summarized in Theorem 1 of this paper) that for any given fuzzy subset A of space X , there always exist random subsets $S(A)$ of X (in general, non-unique) such that A and $S(A)$ are equivalent under all one point coverages, i.e., $\phi_A(x) = \Pr(x \in S(A))$, for all constant $x \in X$. Furthermore, most of the common fuzzy set operations, such as union, intersection, complementation, subsetting, cartesian product, and functional transform correspond homomorphically under the mapping $A \rightarrow S(A)$, for all fuzzy A , to ordinary operations for random sets (summarized in Theorem 2 here). Consequently, these fuzzy set operations can be reinterpreted directly in terms of corresponding ordinary set operations among random sets. One application of this connection is to deductive reasoning, where both fuzzy set and probabilistic techniques can be compatibly utilized [5]. (See also, [6], Theorems 5-7.)

The thrust of this paper is to expand earlier unary and binary homomorphic relations between fuzzy and random set operations, by obtaining systematic characterizations for three classes of n -ary fuzzy set operations, which under the mapping $A \rightarrow S(A)$ for any fuzzy subset A of X , yield homomorphic images, i.e., ordinary set operations applicable to the random sets $S(A)$:

1. generalized composition Binary operations are presented in Theorem 3, unary operations, in Theorem 4, and n -ary operations in Theorem 6;
2. continuous (Theorem 5 demonstrates the general case, Corollary 1 exhibits characterizations for statistically independent image random sets, and Corollary 2 is concerned with a certain simplified subclass of continuous operations);
3. measurable (Theorem 7).

In the case of binary fuzzy set operations leading to homomorphic random set compositions (Theorem 3), the charac-



terizing structure for the membership function is a simple sum of at most four terms, each term being also an elementary combination of the individual component membership functions and/or a functional form that satisfies a constraint equation (Eq. (2)) — see also Examples 1 and 2 for further clarification. In turn, this leads to 16 subtypes of binary operations — all explicitly given in Table 1. These include as special cases: fuzzy union, intersection, bounded summation, product, etc. For unary operations, on the other hand, only identity, fuzzy complement, the hull set reducing operation, and the universal set expanding operation, as given in Table 2, lead to homomorphic random set compositions. Another relatively simple characterization is obtained in Corollary 1, for n -ary fuzzy set operations leading to continuous homomorphic random set operations, when the n random set factors are chosen to be statistically independent, and when the mapping $A \rightarrow S(A)$ is either of two types: $S(A) = S_U(A)$ or $S(A) = T_A$ (see Theorem 1). Part (iii) of Theorem 6 also exhibits a straight forward characterization — a sum of products of factor membership functions and/or their complements — for n -ary fuzzy set operations leading to homomorphic random set operations which are in the form of compositions and for which the individual random set factors are chosen to be statistically independent. Theorem 5 and a specialization of it, Corollary 2, present very general — and somewhat more complicated — characterizations for fuzzy set operations leading to arbitrary continuous random set image operations. In this case, a typical n -ary fuzzy set operation is shown to be determined by a (pointwise-dependent in structure) finite signed sum of probabilities corresponding to each random set factor cover of a given fixed finite subset of X which is also disjoint from another such subset of X . Theorem 6, which treats the general measurability situation can be interpreted as being a uniform limiting case of reapplications of Theorem 5.

2. PRELIMINARY DEFINITIONS AND RESULTS

Let X denote a given base space, $P(X)$ its power class; $G(X)$, often identified with the latter, the class of all membership functions of elements of $P(X)$; $F(X)$ the class of all fuzzy subsets A — identified with the class of fuzzy set (or subset) membership functions $\phi_A : X \rightarrow [0, 1]$; $S(X)$ the class of all random subsets — identified with the class of all probability measures on $P(X)$, with suitably chosen σ -algebras containing the sets $C_{\{x\}} = \{B \mid x \in B \in P(X)\} : x \in X$. (See [6]). Let $S', S'' \in S(X)$ and $A \in F(X)$. Define $A \approx S'$ (or $S' \approx A$) by, $\phi_A(x) = \Pr(x \in S')$, for all $x \in X$. Define $A \lesssim S'$ by, $\phi_A(x) \leq \Pr(x \in S')$, for all $x \in X$. Note "df" means "is defined to be equal to".

To provide necessary background, two previously established theorems are summarized and reformulated.

Theorem 1 — (Goodman [6], Theorem 3)

- (i) $S(X) = \cup_{A \in F(X)} S_A(X)$,

where $S_A(X) \stackrel{\text{df}}{=} \{S(A) \mid S(A) \in S(X) \text{ \& } A \approx S(A)\}$;

$\cup$ indicating disjoint union.

- (ii) Each $S_A(X)$ is nonvacuous, and in fact, $S_U(A), T_A \in S_A(X)$, where $S_U(A) \stackrel{\text{df}}{=} \phi_A^{-1}[U, 1]$, U a uniform random variable over $[0, 1]$ is determined by ϕ_{T_A} which corresponds to the product probability measure of the statistically independent 0-1 random variables $\phi_{T_A}(x)$ with $\Pr(\phi_{T_A}(x) = 1) = \phi_A(x)$; all $x \in X$.

- (iii) If $A \in P(X)$, then $S_U(A) = T_A = A$ and $\{A\} = S_A(X)$

If $W \in S(X)$, then $A(W) \in F(X)$ is defined by $\phi_{A(W)}(x) \stackrel{\text{df}}{=} \Pr(x \in W), x \in X$. Note, by definition, $A(W) \approx W$. Note also, $W, S_U(A(W)), T_{A(W)} \in S_{A(W)}$ are in general all distinct.

Fuzzy set operations $\oplus, \cdot, \otimes, \text{proj}_1, \lambda$ are defined as.

$$\phi_A \oplus \phi_B(x) \equiv \phi_A(x) + \phi_B(x) - \phi_A(x) \cdot \phi_B(x);$$

$$\phi_A \cdot \phi_B(x) \equiv \phi_A(x) \cdot \phi_B(x);$$

$$\phi_A \otimes \phi_B(x, y) \equiv \phi_A(x) \cdot \phi_B(y);$$

$$\phi_{\text{proj}_1(D)}(x) \equiv \sup_{y \in D} \phi(x, y);$$

$$\phi_{A \lambda B}(x) \equiv \lambda \phi_A(x) + (1 - \lambda) \phi_B(x);$$

$$\lambda \text{ is identified with } \phi_\lambda(x) \equiv \lambda; 0 \leq \lambda \leq 1.$$

(The fuzzy set operations $\cup, \cap, X, \neg, f(\cdot)$, etc., are the usual ones.)

Theorem 2 — (Modification of Goodman [6], Theorem 5)

For any $A, B \in F(X), C \in F(Y), f: X \rightarrow Y, D \in F(X \times Y)$:

$$S_U(A \cup B) = S_U(A) \cup S_U(B)$$

$$S_U(A \cap B) = S_U(A) \cap S_U(B)$$

$$S_U(AXC) = S_U(A) \times S_U(C)$$

$$S_U(X \neg A) = X \neg S_{1-U}(A)$$

$$(\text{excluding the boundary } \phi_A^{-1}(1 - U))$$

$$S_U(f(A)) \subseteq f(S_U(A))$$

$$\tilde{S}_U(f^{-1}(C)) = f^{-1}(S_U(C))$$

$$S_U(\text{proj}_1(D)) \approx \text{proj}_1(D) \lesssim \text{proj}_1(S_U(D))$$

$$T_A \oplus T_B \approx T_A \cup T_B$$

$$T_A \cdot T_B \approx T_A \cap T_B$$

$$T_A \otimes T_C \approx T_A \times T_C$$

$$T_{X \neg A} \approx X \neg T_A$$

$$T_{f(A)} \lesssim f(T_A)$$

$$T_{f^{-1}(C)} \approx f^{-1}(T_C)$$

$$T_{\text{proj}_1(D)} \approx \text{proj}_1(D) \lesssim \text{proj}_1(T_D)$$

$$T_{A \lambda B} \approx (\lambda \cap T_A) \cup ((1 - \lambda) \cap T_B),$$

where T_A, T_B, T_C are to be chosen to be mutually statistically independent.

Let $S : F(X) \rightarrow S(X)$ be such that for each $A \in F(X)$, $S(A) \in S_A(X)$ for some fixed choice of $S(A)$. Then S is called a choice function.

We will be concerned with the specification of joint distributions of $(S(A_1), \dots, S(A_n))$ for all finite n and all $A_1, \dots, A_n$

$$\tau: X/\mathbb{R} \cong \mathbb{R}^{F(X)} \rightarrow X/\{0,1\} \cong \{0,1\}^{F(X)} \cong FF(X),$$

$$\text{where for any}$$

$$f = (x_A)_{A \in F(X)} \in \mathbb{R}^{F(X)}, \tau(f) \stackrel{\text{def}}{=} (\psi(x_A))_{A \in F(X)}.$$

Let $\gamma: F(X) \times F(X) \rightarrow M$, be any given marginally consistent correlation function where $M \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} : -1 \leq r \leq 1 \right\}$ and let $m: F(X) \rightarrow \{0\}$ be the zero mean function. Then a unique gaussian stochastic process $V = (V_A)_{A \in F(X)}$ can always be

constructed over $\mathbb{R}^{F(X)}$ corresponding to probability measure μ , etc., which has the specified γ and m . (See, e.g. [10], for more details.) Then, since it readily follows that τ is a measurable mapping, $\tau(V) = (\psi(V_A))_{A \in F(X)}$ is a stochastic process over $[0, 1]^{F(X)}$ with the following properties:

- (i) For any $A \in F(X)$, the marginal random variable $\nu_A \stackrel{\text{def}}{=} \psi(V_A)$ is uniformly distributed over $[0, 1]$.
- (ii) Subject to consistency for γ , for $n \geq 1$ and any $A_1, \dots, A_n \in F(X)$, $x \in X$, $\delta_n(A_1, \dots, A_n; x)$ can be made arbitrary within natural constraints (again see [6], Lemmas 1, 2 and Theorem 3). In particular, for $n = 2$, $\delta_2(A_1, A_2; x)$ may be chosen arbitrarily to satisfy Eq. (2) (for the usual consistency checks). (This follows directly from a continuity argument applied to the two boundary values in Eq. (2), which correspond to $r = 1$ (yielding $U_{A_1} = U_{A_2}$) and $r = -1$ (yielding $U_{A_1} = 1 - U_{A_2}$), for $\gamma(A_1, A_2)$.)
- (iii) The following two special cases of (ii) are of particular importance:

- (a) Choose all U_A , $A \in F(X)$ to be statistically independent and identically distributed – with common distribution being a uniform one over $[0, 1]$. Thus all $S_{U_A}(A)$, $A \in F(X)$, are statistically independent. (This corresponds to correlations $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = I_2$.)
- (b) Choose all $U_A \equiv U$, $A \in F(X)$. Thus all $S_{U_A}(A) \equiv S_U(A)$, $A \in F(X)$ are highly statistically dependent. (This corresponds to correlation matrices $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.)

Finally, note that no matter what choice function S is used, for $n \geq 1$ fixed, $S(A_1), \dots, S(A_n)$ can always be chosen to be statistically independent marginal random sets of the product probability measure $\mu = \times \mu_A$, each μ_A corresponding to $S(A)$, $A \in F(X)$.

3. THE BASIC PROBLEM

With the preliminary results established, we may now pose the basic problem. Let S be a general choice function and assume $(S(A))_{A \in F(X)}$ is some well-defined stochastic process over $P(X)$. Hence, $(S(A_1), \dots, S(A_n))$ for all $A_1, \dots, A_n \in F(X)$ can be assigned consistent joint distributions. Then for certain classes of ordinary set operations $\star: P(X) \times \dots \times P(X) \rightarrow P(X)$ we seek corresponding classes of fuzzy set operations $\oplus: F(X) \times \dots \times F(X) \rightarrow F(X)$ such that $\oplus$ induces a weak n -homomorphism relative to $\star$, i.e.,

$$S(\oplus(A_1, \dots, A_n)) \approx \star(S(A_1), \dots, S(A_n))$$

$$\approx \star(S(A_1), \dots, S(A_n)), \quad (9)$$

for all $A_1, \dots, A_n \in F(X)$.

Conversely, given any fuzzy set operation $\oplus$, it is of interest to determine if some ordinary set operation $\star$ exists satisfying (9).

We will consider three general (nested) cases for $\star$: composition, continuous, measurable.

Consider first the composition case. We say ordinary set operation $\star$ is a generalized composition iff $h_k: X \rightarrow X$,

$k = 1, \dots, n$ and for each $x \in X$, $\exists g_x: \{0, 1\} \times \dots \times \{0, 1\} \rightarrow \{0, 1\}$ such that for all $B_1, \dots, B_n \in P(X)$,

$$\phi_{\star(B_1, \dots, B_n)}(x) = g_x(\phi_{B_1}(h_1(x)), \dots, \phi_{B_n}(h_n(x))). \quad (10)$$

Note that for any given $B_1, \dots, B_n, h_1, \dots, h_n, x$ as above

$\star(B_1, \dots, B_n)$ can be any of 2^{2^n} possible combinations for g_x , involving ordinary unions, intersections and complements of the sets $h_1^{-1}(B_1), \dots, h_n^{-1}(B_n)$.

4. BINARY AND UNARY COMPOSITIONS

Theorem 3

Let $n = 2$ and S be a given choice function with all bivariate $(S(A_1), S(A_2))$ assumed to have specified consistent joint distributions.

- (i) Then binary fuzzy set operation $\oplus$ induces a weak 2-homomorphism relative to some $\star$ being a generalized composition (as in Eq. (10)) iff $\oplus$ has the following form:

$$\begin{aligned} \phi_{\oplus(A_1, A_2)}(x) = & g_x(0, 0) \cdot (1 - \phi_{A_1}(h_1(x)) - \phi_{A_2}(h_2(x)) \\ & + \delta_2(A_1, A_2; x)) \\ & + g_x(1, 0) \cdot (\phi_{A_1}(h_1(x)) - \delta_2(A_1, A_2; x)) \\ & + g_x(0, 1) \cdot (\phi_{A_2}(h_2(x)) - \delta_2(A_1, A_2; x)) \\ & + g_x(1, 1) \cdot \delta_2(A_1, A_2; x), \end{aligned} \quad (11)$$

for all $A_1, A_2 \in F(X)$ and $x \in X$: $g_x(i, j) \in \{0, 1\}$

- (ii) By varying the distributions of $S(A_1), S(A_2)$ and the function $g_x: \{0, 1\} \times \{0, 1\} \rightarrow \{0, 1\}$, the $\delta_2(A_1, A_2; x)$ can be chosen arbitrary fixed to satisfy Eq. (2), up to consistency. Note, in addition, that the statistical independence case $\delta_2(A_1, A_2; x) = \phi_{A_1}(x) \cdot \phi_{A_2}(x)$ always satisfies Eq. 2. In general, the coefficients $g_x(i, j)$ may be arbitrary $\in \{0, 1\}$, for $i, j \in \{0, 1\}$: for all $A_1, A_2 \in F(X), x \in X$.

$$\begin{aligned} \text{(Proof)} \quad \phi_{\oplus(A_1, A_2)}(x) &= \Pr(x \in \star(S(A_1), S(A_2))) \\ &= \Pr(\star(S(A_1), S(A_2))(x) = 1) \\ &= \Pr(\{g_x(\phi_{S(A_1)}(h_1(x)), \phi_{S(A_2)}(h_2(x))) = 1\}) \\ &= \sum_{(i, j) \in \{0, 1\} \times \{0, 1\}} \Pr(\phi_{S(A_1)}(h_1(x)) = i \\ &\quad \& \phi_{S(A_2)}(h_2(x)) = j) \end{aligned}$$

which yields the form in Eq. (11). By again, use of Eqs. (5)–(8).

Note the application of Theorem 3 (via Examples 1 and 2 previously discussed) to the special cases: $S(A) = S_{U_A}(A)$,

for either $U_A \equiv U$, or U_A mutually statistically independent, for all $A \in F(X)$, where U and each U_A are identically distributed uniform over $[0,1]$; and T_A .

Table 1 exhibits, for a given $S; x \in X; A_1, A_2 \in F(X); h_1, h_2$: the 16 possible types of binary fuzzy set operations $\otimes$ inducing weak homomorphic ordinary set operations $*$ in compositional form. By following the comments at the end of Example 1, a similar table (now using δ_3 in addition to δ_2) of $2^{2^3} = 256$ entries may be constructed, etc. (See Theorem 6 for the general composition case.)

Thus we see that several different binary fuzzy set operations can induce the same homomorphic image — binary random set operation — for the same fixed choice function, by varying the statistical dependence (or stochastic process) between the random sets.

The analogue to Theorem 3 for unary operations is an important special case:

Theorem 4

Let $n = 1$ and S be a given choice function. Then unary fuzzy set operation $\otimes$ induces a weak 1-homomorphism relative to some $*$ being a generalized composition iff $\otimes$ has the following form:

$$\phi_{\star(A)}(x) = g_x(0) \cdot (1 - \phi_A(h(x))) + g_x(1) \cdot \phi_A(h(x)), \quad (12)$$

where as $g_x: \{0,1\} \rightarrow \{0,1\}$ varies, $g_x(0), g_x(1)$ may be arbitrary $\in \{0,1\}$; for all $A \in F(X), x \in X$. In this situation, for any $x \in X, B \in P(X)$,

$$x \in \star(B) \text{ iff } x \in g(B)(x), \quad (13)$$

where for any $y \in X$,

$$g(B)(y) \stackrel{\text{df}}{=} \bigcup_{i \in g_y^{-1}(1)} (\cap_{i=1} h^{-1}(B) \cup h^{-1}(B)) \quad (14)$$

again using the convention $\cap(\cdot) = X, \cup(\cdot) = \emptyset$.

Table 1. Tabulation of Possible Binary Fuzzy Set Operations Inducing Weak Homomorphic Ordinary Set Operations in Compositional Form.

Case	$g_x^{-1}(1)$	$\star(S(A_1), S(A_2))$	$\phi_{\otimes(A_1, A_2)}(x)$
1	$\emptyset$	$\emptyset$	0
2	$\{(0,0)\}$	$X \rightarrow (h_1^{-1}(S(A_1)) \cup h_2^{-1}(S(A_2)))$	$1 - \phi_{A_1}(h_1(x)) - \phi_{A_2}(h_2(x)) + \delta_2$
3	$\{(0,1)\}$	$h_2^{-1}(S(A_2)) \rightarrow h_1^{-1}(S(A_1))$	$\phi_{A_2}(h_2(x)) - \delta_2$
4	$\{(1,0)\}$	$h_1^{-1}(S(A_1)) \rightarrow h_2^{-1}(S(A_2))$	$\phi_{A_1}(h_1(x)) - \delta_2$
5	$\{(1,1)\}$	$h_1^{-1}(S(A_1)) \cap h_2^{-1}(S(A_2))$	δ_2
6	$\{(0,0), (0,1)\}$	$X \rightarrow h_1^{-1}(S(A_1))$	$1 - \phi_{A_1}(h_1(x))$
7	$\{(0,0), (1,0)\}$	$X \rightarrow h_2^{-1}(S(A_2))$	$1 - \phi_{A_2}(h_2(x))$
8	$\{(0,0), (1,1)\}$	$X \rightarrow (h_1^{-1}(S(A_1)) \Delta h_2^{-1}(S(A_2)))$	$1 - \phi_{A_1}(h_1(x)) - \phi_{A_2}(h_2(x)) + 2\delta_2$
9	$\{(0,1), (1,0)\}$	$h_1^{-1}(S(A_1)) \Delta h_2^{-1}(S(A_2))$	$\phi_{A_1}(h_1(x)) + \phi_{A_2}(h_2(x)) - 2\delta_2$
10	$\{(0,1), (1,1)\}$	$h_2^{-1}(S(A_2))$	$\phi_{A_2}(h_2(x))$
11	$\{(1,0), (1,1)\}$	$h_1^{-1}(S(A_1))$	$\phi_{A_1}(h_1(x))$
12	$\{(0,0), (0,1), (1,0)\}$	$X \rightarrow (h_1^{-1}(S(A_1)) \cap h_2^{-1}(S(A_2)))$	$1 - \delta_2$
13	$\{(0,0), (0,1), (1,1)\}$	$h_2^{-1}(S(A_2)) \cup (X \rightarrow h_1^{-1}(S(A_1)))$	$1 - \phi_{A_1}(h_1(x)) + \delta_2$
14	$\{(0,0), (1,0), (1,1)\}$	$h_1^{-1}(S(A_1)) \cup (X \rightarrow h_2^{-1}(S(A_2)))$	$1 - \phi_{A_2}(h_2(x)) + \delta_2$
15	$\{(0,1), (1,0), (1,1)\}$	$h_1^{-1}(S(A_1)) \cup h_2^{-1}(S(A_2))$	$\phi_{A_1}(h_1(x)) + \phi_{A_2}(h_2(x)) - \delta_2$
16	$\{(0,0), (0,1), (1,0), (1,1)\}$	X	1

$\delta_2 = \delta_2(A_1, A_2, x)$ may be arbitrary satisfying Eq. (2). g_x is assumed to be the same for all $x \in X$.

Table 2 lists the four possible unary fuzzy set operations leading to weak homomorphic ordinary set operations in composition form.

Table 2. Tabulation of Possible Unary Fuzzy Set Operations Inducing Weak Homomorphic Ordinary Operations in Compositional Form.

Case	$g_x^{-1}(1)$	$\star(S(A))$	$\phi_{\otimes}(A)(x)$
1	$\emptyset$	$\emptyset$	0
2	$\{0\}$	$X \rightarrow h^{-1}(S(A))$	$1 - \phi_A(h(x))$
3	$\{1\}$	$h^{-1}(S(A))$	$\phi_A(h(x))$
4	$\{0,1\}$	X	1

5. GENERAL RESULTS

The following more general results require some topological background. (See, e.g., [11] for all relevant definitions and general relations.)

Let $\mathcal{D}$ be the discrete topology on $\{0,1\}$. Thus the open sets for $\mathcal{D}$ consist of $P(\{0,1\})$. Then the natural topology $\mathcal{F}$ for $G(X)$, noting $G(X) = X \times \{0,1\} = \{0,1\}^X$, is the cartesian

product (or, equivalently, pointwise convergence) topology. Open sets for $\mathcal{F}$ consist typically of all unions of sets of the form

$$\mathcal{O}(z,a) \stackrel{\text{def}}{=} \{ \phi_B \mid \phi_B \in G(X) \text{ \& } \phi_B(z_i) = a_i, \text{ for } i = 1, \dots, q \}, \quad (15)$$

where

$$z \stackrel{\text{def}}{=} \begin{pmatrix} z_1 \\ \vdots \\ z_q \end{pmatrix}, \quad a \stackrel{\text{def}}{=} \begin{pmatrix} a_1 \\ \vdots \\ a_q \end{pmatrix}; \quad z_i \in X, a_i \in \{0,1\}; i = 1, \dots, q.$$

$z \in X \times \dots \times X$, $a \in \{0,1\} \times \dots \times \{0,1\}$ (q factors); $q \geq 1$, can all be chosen arbitrary. In turn, let $\mathcal{V}$ be the natural cartesian product topology for $G(X) \times \dots \times G(X)$ (n factors). In this case, the open sets for $\mathcal{V}$ consist of all unions of sets of the form

$$\mathcal{O}_1 \times \dots \times \mathcal{O}_n, \text{ where each } \mathcal{O}_i \text{ is arbitrary open for } \mathcal{F}.$$

Note that for any ordinary set operation $\star$, there is a unique corresponding $\phi_{\star} : G(X) \times \dots \times G(X) \rightarrow G(X)$ (n factors).

Then for any $x \in X$, $B_1, \dots, B_n \in P(X)$,

$$\phi_{\star}(B_1, \dots, B_n) = \phi_{\star}(\phi_{B_1}, \dots, \phi_{B_n}), \quad (16)$$

and

$$\begin{aligned} x \in \star(B_1, \dots, B_n) &\text{ iff } \phi_{\star}(B_1, \dots, B_n)(x) = 1 \\ &\text{ iff } (\phi_{\star}(\phi_{B_1}, \dots, \phi_{B_n}))(x) = 1 \\ &\text{ iff } (\phi_{B_1}, \dots, \phi_{B_n}) \in \phi_{\star}^{-1}(\mathcal{O}(x,1)). \end{aligned} \quad (17)$$

In particular we can let $B_j = S(A_j) : A_j \in F(X); j = 1, \dots, n$. We say fuzzy set operation $\otimes$ induces a weak n -homomorphism relative to ordinary set operation $\star$ being continuous iff $\phi_{\otimes}$ is continuous with respect to topologies $(\mathcal{V}, \mathcal{F})$ and Eq. (9) holds.

$\mathcal{D}$ is compact, Hausdorff, and second countable and separable. Consequently, the product topologies (by Tychonoff's Theorem) $\mathcal{F}$ and $\mathcal{V}$ are also compact and Hausdorff. If X

is at most countably infinite space, then $\mathcal{F}$ and $\mathcal{V}$ will also be second countable and separable.

Theorem 5

Let $n \geq 1$ be arbitrary fixed and let S be a given choice function with all $(S(A_1), \dots, S(A_n))$ assumed to have a specified consistent distribution.

(i) Then n -ary fuzzy set operation $\otimes$ induces a weak n -homomorphism relative to some ordinary set operation $\star$ being continuous iff $\otimes$ has the following form:

$$\begin{aligned} \phi_{\otimes}(A_1, \dots, A_n)(x) &= \Pr \left(\bigcap_{j=1}^n (M_{\nu,j} \subseteq S(A_j) \subseteq X \rightarrow N_{\nu,j}) \right) \\ &= \sum_{\text{over all } B_j} \left\{ (-1)^{\text{card}(B)} + 1 \right. \\ &\quad \left. \cdot \Pr \left(\bigcap_{j=1}^n (M(B_j) \subseteq S(A_j) \subseteq X \rightarrow N(B_j)) \right) \right\}. \end{aligned} \quad (18)$$

for all $A_1, \dots, A_n \in F(X)$ and $x \in X$, where not depending on the A_i 's, $\exists$ finite (or vacuous) sets $M_{\nu,j}, N_{\nu,j} \in P(X)$ with $M(B,j) \stackrel{\text{def}}{=} \bigcup_{\nu \in B} M_{\nu,j}, N(B,j) \stackrel{\text{def}}{=} \bigcup_{\nu \in B} N_{\nu,j}$; for each $x \in X, K_x$ is

some finite index set.

(ii) Not only may the $(S(A_1), \dots, S(A_n))$ be varied in joint distribution, but also the $M_{\nu,j}$'s and $N_{\nu,j}$'s may be chosen arbitrary finite (or vacuous) $\in P(X)$; K_x may be chosen also as an arbitrary finite index set, for each $x \in X$.

(Proof:

Using Eq. (17), for any ordinary continuous n -set operation $\star$, for any $B_1, \dots, B_n \in P(X)$ and $x \in X$,

$$\begin{aligned} x \in \star(B_1, \dots, B_n) &\text{ iff } (\phi_{B_1}, \dots, \phi_{B_n}) \in \bigcup_{\nu \in K_x} \bigcap_{j=1}^n \mathcal{O}(z_{\nu,j}, a_{\nu,j}) \end{aligned}$$

for some finite index set K_x , etc.

$$\text{iff } \bigcap_{\nu \in K_x} \left(\bigcap_{j=1}^n (M_{\nu,j} \subseteq B_j \subseteq X \rightarrow N_{\nu,j}) \right). \quad (19)$$

where

$$M_{\nu,j} \stackrel{\text{def}}{=} \{ z_{\nu,j} \mid 1 \leq i \leq q_{\nu,j} \text{ such that } a_{\nu,j,i} = 1 \} \quad (20a)$$

$$N_{\nu,j} \stackrel{\text{def}}{=} \{ z_{\nu,j} \mid 1 \leq i \leq q_{\nu,j} \text{ such that } a_{\nu,j,i} = 0 \}. \quad (20b)$$

Corollary 1

Suppose for given choice function S , that, for any $A_1, \dots, A_n \in F(X)$, $S(A_1), \dots, S(A_n)$ are always mutually statistically independent. Then the characterization for $\otimes$ in Theorem 5 can be made to depend on $A_1, \dots, A_n$ through $\phi_{A_1}, \dots, \phi_{A_n}$ directly and not on probability statements involving $S(A_j), j = 1, \dots, n$, for any $A_1, \dots, A_n \in F(X)$, when either case 1 or

case 2 holds for choice of S:

Case 1: For any $A \in F(X)$, $S(A) \equiv S_{U_A}(A)$, where as marginal random variables, each U_A is uniformly distributed over $\{0,1\}$, but jointly all U_A 's for different A's are mutually statistically independent. In this case, the characterizing equation, Eq. (18), becomes

$$\phi_{\oplus}(A_1, \dots, A_n)(x) = \sum_{\substack{\text{over all } B, \\ B \neq \emptyset \in P(K_X), \\ \text{for } M(B,j) \cap N(B,j) \\ = \emptyset, \text{ for all } j, 1 \leq j \leq n}} (-1)^{\text{card } B+1} \cdot \prod_{j=1}^n \omega'(B, A_j), \quad (21)$$

$$\omega'(B, A_j) \stackrel{\text{df}}{=} \inf_{y \in M(B,j)} (\phi_{A_j}(y)) - \inf_{y \in N(B,j)} (\phi_{A_j}^{-1}(y, 1)) \quad (22)$$

Case 2: For any $A \in F(X)$, $S(A) \equiv T_A$, where jointly all of the T_A 's are mutually statistically independent. In this case, the characterizing equation, Eq. (18), becomes

$$\phi_{\oplus}(A_1, \dots, A_n)(x) = \sum_{\substack{\text{over all } B, \\ B \neq \emptyset \in P(K_X), \\ \text{for } M(B,j) \cap N(B,j) \\ = \emptyset, \text{ for all } j, 1 \leq j \leq n}} (-1)^{\text{card } B+1} \cdot \prod_{j=1}^n \omega''(B, A_j), \quad (23)$$

$$\omega''(B, A_j) \stackrel{\text{df}}{=} \prod_{(y \in M(B,j))} \phi_{A_j}(y) \cdot \prod_{(y \in N(B,j))} (1 - \phi_{A_j}(y)). \quad (24)$$

Corollary 2

Let $n \geq 1$ be arbitrary and S a given choice function with all $(S(A_1), \dots, S(A_n))$ assumed to have a specified distribution.

- (i) Then n-ary fuzzy set operation $\oplus$ induces a weak n-homomorphism relative to some ordinary set operation $\star$ being continuous such that the sum in Eq. (18) becomes a non-negative one, i.e., all terms for non-singleton B are vacuous, i.e.

$$\phi_{\oplus}(A_1, \dots, A_n)(x) = \sum_{v \in K_X} \Pr \left(\bigcap_{j=1}^n M_{v_j} \subseteq S(A_j) \subseteq X \rightarrow N_{v_j} \right), \quad (25)$$

and thus in Eq. (19), defining $\star$, Or is replaced by

Or (disjoint), iff for all $x \in X$ and each pair $v_1, v_2 \in K_X$,

with $v_1 \neq v_2$, there is an integer $j, 1 \leq j \leq n$, such that either $M_{v_1, j} \cap N_{v_2, j} \neq \emptyset$ or $M_{v_2, j} \cap N_{v_1, j} \neq \emptyset$, such that Eq. (25) holds.

- (ii) If also S is chosen so that $S(A_1), \dots, S(A_n)$ are all mutually statistically independent, for all $A_1, \dots, A_n \in F(X)$, then the characterizing equations Eq. (21)

for case (1) and Eq. (23) for case 2 — in (i) become here, respectively

$$\phi_{\oplus}(A_1, \dots, A_n)(x) = \sum_{v \in K_X} \prod_{j=1}^n \omega'(\{v\}, A_j), \quad (21')$$

$$\phi_{\oplus}(A_1, \dots, A_n)(x) = \sum_{v \in K_X} \prod_{j=1}^n \omega''(\{v\}, A_j), \quad (23')$$

where ω' is given in Eq. (22) and ω'' in Eq. (24), for $B = \{v\}$.

Note that for Corollary 1 and 2, the construction of induced homomorphic $\star$, given $\oplus$, remains formally the same as in the general case as given in Eq. (19).

Consider now the case of generalized composition.

Theorem 6

Let $n \geq 1$ be arbitrary fixed and let S be a given choice function with all $(S(A_1), \dots, S(A_n))$ assumed to have a specified consistent distribution.

- (i) When $\star$ is a generalized composition, $\star$ is continuous, since, for any $x \in X$,

$$\oplus^{-1}(\mathcal{O}(x, 1)) = \bigcup_{(a_1, \dots, a_n) \in \mathcal{E}_X^{-1}(1)} \bigcap_{j=1}^n \mathcal{O}(h_j(x), a_j). \quad (26)$$

- (ii) n-ary fuzzy set operation $\oplus$ induces a weak n-homomorphism relative to some ordinary set operation $\star$ being a generalized composition iff $\oplus$ has the following form:

$$\phi_{\oplus}(A_1, \dots, A_n)(x) = \sum_{\substack{(a_1, \dots, a_n) \in \\ \{0,1\} \times \dots \times \{0,1\}}} \{g_x(a_1, \dots, a_n) \cdot \Pr(x \in (\bigcap_{j=1}^n h_j^{-1}(S(A_j)) \cup \bigcap_{j=1}^n h_j^{-1}(S(A_j))))\}, \quad (27)$$

$$\left(\begin{matrix} 1 \leq j \leq n, \\ \text{such that } a_j = 1 \end{matrix} \right) \quad \left(\begin{matrix} 1 \leq j \leq n, \\ \text{such that } a_j = 0 \end{matrix} \right)$$

where

$$g_x : \{0,1\} \times \dots \times \{0,1\} \rightarrow \{0,1\} \text{ (n factors)}$$

- (iii) By varying the distributions of $(S(A_1), \dots, S(A_n))$ and the function g_x , the coefficients $g_x(a_1, \dots, a_n)$ can be arbitrarily chosen $\in \{0,1\}$, $x \in X$. There are 2^{2^n} such g_x 's, for each $x \in X$.

- (iv) If, S is such that $S(A_1), \dots, S(A_n)$ are always statistically independent, then Eq. (27) reduces to

$$\phi_{\star}(A_1, \dots, A_n)(x) = \sum_{(a_1, \dots, a_n) \in \{0,1\} \times \dots \times \{0,1\}} \left\{ \prod_{\substack{1 \leq j \leq n \\ \text{such that } a_j = 1}} \phi_{A_j}(h_j(x)) \cdot \prod_{\substack{1 \leq j \leq n \\ \text{such that } a_j = 0}} (1 - \phi_{A_j}(h_j(x))) \right\} \quad (28)$$

Finally, we treat the general situation of an n -homomorphic induced ordinary set operation being only measurable with respect to suitably chosen σ -algebras relative to topologies $\mathcal{T}$ and $\mathcal{F}$.

Consider again the fixed compact Hausdorff topological spaces $(G(X), \mathcal{T})$ and $(G(X) \times \cdots \times G(X), \mathcal{V})$ (n -factors), and choice function S , where for any $A_1, \dots, A_n \in F(X)$, $(S(A_1), \dots, S(A_n))$ has a specified joint distribution. Thus, each marginal random subset $S(A_j)$ of X corresponds to probability space $(P(X), \mathcal{A}_j, q_j)$, as well as to probability space $(G(X), \mathcal{B}_j, p_j)$. Recall from section 2 the requirement that the σ -algebra $\mathcal{A}_j$ must contain all $C_{\{x\}}$, $x \in X$. Thus, $\mathcal{B}_j$ must contain all $\mathcal{O}(x, 1)$, $x \in X$, and hence all countable unions and intersections of $\mathcal{O}(x, 1)$, $x \in X$ and $\mathcal{O}(y, 0)$, $y \in X$. Recall also that each $\mathcal{O}(x, 1)$ and $\mathcal{O}(y, 0)$ are closed-open-compact subsets of $G(X)$ with respect to $\mathcal{T}$; it follows that each $\mathcal{B}_j \supseteq \mathcal{B}_0$, the Baire σ -algebra generated by all compact - G_δ (i.e., compact and represented by an at most countable intersection of open) sets in $\mathcal{T}$. (See [8], sections 51, 52 for general background.) Note that any typical compact - G_δ set here is

of the form $\mathcal{O}(Z, a)$, where $Z = \{z_j\}_{j=1}^m \in \prod_{j=1}^m G(X)$

and $a = \{a_j\}_{j=1}^m \in \prod_{j=1}^m \{0, 1\}$, where m may be a finite

positive integer or $+\infty$. (See Halmos [8], Example 4, p. 219, for a related result.) If X is at most countably infinite, then since $\mathcal{T}$ will be separable, every compact set in $\mathcal{T}$ will also be a G_δ set and thus $\mathcal{B}$ the Borel σ -algebra, generated by all compact-open subsets of $\mathcal{T}$, will be the same as $\mathcal{B}_0$ ([8], Theorem E, p. 218). For X not countable, then generally $\mathcal{B}_0 \subset \mathcal{B}$ (proper subclass). While $\mathcal{B}$ has certain desirable properties, it does not possess the regularity properties that $\mathcal{B}_0$ does. Analogous results hold for $(S(A_1), \dots, S(A_n))$ and its corresponding probability space $(G(X) \times \cdots \times G(X), \mathcal{E}, p)$, where σ -algebras $\mathcal{E} \supseteq \mathcal{E}' \supseteq \mathcal{B}_0$, with, in general $\mathcal{B}_0 \subset \mathcal{E}'$ is the σ -algebra generated by $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$; $\mathcal{B}_0$ is the Baire σ -algebra, relative to $\mathcal{V}$, generated by $\mathcal{B}_0 \times \cdots \times \mathcal{B}_0$, and $\mathcal{E}$ is the Borel σ -algebra generated by $\mathcal{B} \times \cdots \times \mathcal{B}$. Thus, in order to utilize regularity properties, we will say a mapping $\psi: G(X) \times \cdots \times G(X) \rightarrow G(X)$ is (Baire) measurable iff $\psi^{-1}(\mathcal{B}_0) \subseteq \mathcal{B}_0$. (Clearly, if ψ is continuous, it is measurable.)

Theorem 7

Let $n \geq 1$ be arbitrarily fixed and S a given choice function with $(S(A_1), \dots, S(A_n))$ having a consistent specified distribution for each $A_1, \dots, A_n \in F(X)$, where, motivated by the previous discussion, it is assumed that the σ -algebra corresponding to each $S(A_j)$ is $\mathcal{B}_0$, and the σ -algebra corresponding to $(S(A_1), \dots, S(A_n))$ is $\mathcal{B}_0$. Then n -ary fuzzy set operation $\otimes$ induces a weak n -homomorphism relative to some ordinary n -ary set operation $*$ being measurable iff $\otimes$ has the following form, for all $x \in X$, $A_1, \dots, A_n \in F(X)$:

$$\begin{aligned} \phi_{\otimes}(A_1, \dots, A_n)(x) &= \lim_{\ell \rightarrow +\infty} \phi_{\otimes_\ell}(A_1, \dots, A_n)(x) \\ &= \lim_{\ell \rightarrow +\infty} \phi_{\otimes'_\ell}(A_1, \dots, A_n)(x), \end{aligned} \quad (29)$$

where for each positive integer ℓ , $\otimes_\ell(A_1, \dots, A_n)$ is defined from Eq. (18) with $\otimes$ replaced by $\otimes_\ell$ and each K_x by (finite) $K_{x,\ell}$, where the limit in Eq. (29) is uniform with respect to all $x \in X$, and where $\otimes'_\ell$ is defined from Eq. (18) with $\otimes$ replaced by $\otimes'_\ell$ and each K_x by $K'_{x,\ell} \stackrel{\text{def}}{=} \bigcup_{t=\ell}^{+\infty} K_{x,t}$ (at most, countably infinite, and consequently the last part of Eq. (18) must be replaced by a limit).

6. SUMMARY AND CONCLUSIONS

Those n -ary fuzzy set operations have been characterized which induce weak homomorphic n -ary ordinary set operations between random sets, given: (1) any given choice function, i.e., mapping between $F(X)$ and $S(X)$ which picks for each given fuzzy subset of X an equivalent random subset of X up to one point coverages, and (2) any compatible stochastic process. In particular, most ordinary fuzzy set operations and their generalizations were shown to induce naturally corresponding operations among random sets. (For example, see Tables 1, 2, the remarks in Theorem 6, and Eq. (19).) When the choice function is either $S(A) = \bigcup_{A \in \mathcal{A}} A$ or $S(A) = T_A$,

A an arbitrary fuzzy subset of X , a large class of operations exists (including essentially all of the usual ones and their generalizations) such that fuzzy sets formed from finite combination of such operations applied to more primitive fuzzy sets can be identified with corresponding random sets formed from similar combinations of ordinary set operations applied to their primitive random set components. Combining this result with a previous one ([16], as Theorems 6 and 7), allows for complete flexibility in using fuzzy and/or random sets as inputs in describing an unknown quantity V based on all of this information.

Future work will be directed towards applications to specific systems of fuzzy set operations and further investigation of the relations among choice functions, compatible stochastic processes over $P(X)$, and weak homomorphisms. Relative to the latter, it is of some interest to determine for a given n -ary ordinary set operation $*$, what are the possible corresponding n -ary fuzzy set operations $\otimes$ which induce $*$ homomorphically (weakly, for example), as the choice function and/or joint distributions of random sets are made to vary over all combinations. Conversely, for a fixed fuzzy set operation, what possible random set operations does it induce as the choice function and/or joint distribution of random sets vary?

¹ By a level set (or α -cut), we mean the ordinary set $\phi_A^{-1}[\alpha, 1] \subseteq X$, given fuzzy set membership function $\phi_A: X \rightarrow [0, 1]$. $0 \leq \alpha \leq 1$

7. REFERENCES

- [1] Arbib, M.A., Review of three books on fuzzy sets, *Bull. Am. Math Soc.* 83, no. 5 (Sept. 1977), 946-951.
- [2] Dubois, D. and H. Prade, *Fuzzy Sets and Systems*, Academic Press (1980).
- [3] Gaines, B.R., "B.R.", "Fuzzy and probability uncertainty logics", *Info. & Control*, 38 (1978), 154-169.
- [4] Goodman, I.R., Some relations between fuzzy sets and random sets. (April, 1976) (unpublished).
- [5] Goodman, I.R., "Identification of fuzzy sets with a class of canonically induced random sets", *Proc. 19th IEEE Conf. on Decis. & Control*: Albuquerque, NM (Dec., 1980) 352-357. Longer version in *Naval Research Laboratory Report 8415*, (Sept. 15, 1980) Washington, D.C.
- [6] Goodman, I.R., "Fuzzy sets as equivalence classes of random sets", to appear in *Recent Developments in Fuzzy Set and Possibility Theory*, R.R. Yager (ed.), Pergamon Press (1981).
- [7] Gupta, M.M., R.K. Ragade, R.R. Yager (eds.), *Advances in Fuzzy Set Theory and Applications*, North Holland Publishing Co. (1979).
- [8] Halmos, P., *Measure Theory*, D. Van Nostrand Company (1961).
- [9] Höhle, U., "A mathematical theory of uncertainty (fuzzy experiments and their realizations)", *Proc. Int'l. Congress on Appl. Res. & Cyber.* (Dec., 1980), Acapulco, Mex.
- [10] Ibragimov, I.A. and Y.A. Rozanov, *Gaussian Random Processes*, Springer-Verlag Company (1978), Chapter I.
- [11] Kelley, J.L., *General Topology*, Springer-Verlag (1975).
- [12] Loeve, M., *Probability Theory*, D. Van Nostrand Company (1963).
- [13] Manes, E.G., Review of 'Fuzzy Switching and Automata: Theory and Applications', *SIAM Review*, 23, no. 2 (April, 1981), 271-273.
- [14] Negoita, C.V. and D.A. Ralescu, "Representation theorems for fuzzy concepts", *Kybernetes*, 4 (1975), 169-174.
- [15] Nguyen, H.T., "Some mathematical tools for linguistic probabilities", *Fuzzy Sets and Systems*, 2 (1979), 53-65.
- [16] Radecki, T., "Level fuzzy sets", *Jour. of Cybernetics*, 7 (1977), 189-198.
- [17] Ralescu, D., "A survey of the representation of fuzzy concepts and its applications", in *Advances in Fuzzy Set Theory and Applications*, Gupta, Ragade, Yager (eds.), North Holland (1979).
- [18] Schefe, P., "On foundations of reasoning with uncertain facts and vague concepts", *Internatl. Journ. Man-Machine Stud.*, 12 (Jan., 1980), 35-62.
- [19] Tong, R.M. and J. Efsthathiou, "A critical assessment of truth functional modification and its use in approximate reasoning", *Dept. of Elec. Engin. & Comp. Sci., Univ. of Calif., Berkeley* (1980). Submitted to *Fuzzy Sets and Systems*.
- [20] Tribus, M., "Fuzzy sets and bayesian methods applied to the problem of literature search", *IEEE Trans. Sys., Man and Cyber.*, SMC-10, no. 8 (Aug., 1980), 501-502.
- [21] Zadeh, L.A., "Fuzzy sets", *Inform. Control*, 8, 338-353.
- [22] Zadeh, L.A., "Similarity relations and fuzzy orderings", *Info. Sciences*, 3 (1971), 177-200.