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MULTICHANNEL DETECTION USING THE DISCRETE-TIME MODEL-BASED INNOVATIONS APPROACH

Dr. James H. Michels

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CHAPTER 1 INTRODUCTION

1.1. Background

The detection problem for a known deterministic signal in additive Gaussian noise has been solved using a correlator or matched filter [46,55,43,19 Part I]. For the case of a random signal in white noise, a two stage process is utilized which first estimates the signal and then correlates this estimate with the received waveform. This detection scheme is referred to as the estimator-correlator [48-51,53,19 Part III]. The advantage of this latter approach is its practical implementation structure as well as its intuitive justification in that the best obtainable estimate of the signal is used in the correlation stage. As noted by Kailath [54], however, the estimator-correlator method described in [51] fails when the signal and noise are correlated. These limitations were overcome for continuous-time processes by Kailath [53,56] using the innovations representation. In addition, the approach based on innovations representations can be applied to non-Gaussian signal processes. This detection problem is expressed as

$$H_0: x(t) = w(t) \quad 0 \leq t \leq T \quad (1.1.1a)$$

$$H_1: x(t) = s(t) + w(t) \quad 0 \leq t \leq T \quad (1.1.1b)$$

where T is the observation interval and H_i $i=0,1$ are the hypotheses corresponding to signal absent and present, respectively; $s(t)$ is the (not necessarily Gaussian) random signal and $w(t)$ is Gaussian white noise. Under H_1 , the optimal minimum mean-squared error (MMSE) causal estimate of $s(t)$ is

$$\hat{s}(t) = E [s(t) | x(t'), t' < t]. \quad (1.1.2)$$

Using the definition of the innovations [57]; ie.

$$v(t) = x(t) - \hat{s}(t), \quad (1.1.3)$$

the innovations theorem for continuous-time processes (ie. the covariance of $v(t)$ and $w(t)$ are identical), and the property that $v(t)$ is Gaussian if $w(t)$ is Gaussian

[53], Kailath transformed the detection problem of eq (1.1.1) to the equivalent detection problem:

$$H_0: x(t) = v(t) \quad 0 \leq t \leq T. \quad (1.1.4a)$$

$$H_1: x(t) = \hat{s}(t) + v(t) \quad 0 \leq t \leq T \quad (1.1.4b)$$

In this form of the detection problem, $\hat{s}(t)$ is viewed as a conditionally-known signal. In [53], it was shown that the likelihood ratio for the problem defined in eqs(1.1.4) has the same form as the estimator-correlator for the Gaussian signal in the white Gaussian noise case. Although this likelihood ratio was derived assuming $s(t)$ and $w(t)$ as statistically independent, this restriction was later [56,58] relaxed to include the situation where the signal could be correlated with past noise. It is interesting to note that the innovations approach does not use a Karhunen-Loeve expansion which requires $s(t)$ and $w(t)$ to be statistically independent. This latter generalization by Kailath [56,58] indicates that the innovations approach may offer a robustness in detection problems involving feedback, multipath, etc.

Unlike the continuous-time case, discrete innovations processes are not necessarily Gaussian unless both $w(t)$ and $s(t)$ are Gaussian. In addition, the covariance of the innovations is not equivalent to that of the white noise [53,57]. Thus, the likelihood ratio for the discrete time case can not be computed using the estimator-correlator structure. For this case, a likelihood ratio using discrete-time innovations processes was developed [14, 24]. In [14], it is shown for non-Gaussian processes that this likelihood ratio, called the Innovations-Based Detection Algorithm (IBDA), asymptotically approaches the likelihood ratio for the Gaussian problem. The approximation is due to the use of the Central Limit Theorem, so that an asymptotic approximation is obtained using a large number of data samples. Furthermore, the structure of the likelihood ratio is obtained by assuming a specific parametric model of the observation processes for each hypothesis and by designing a prediction error filtering stage for each. Adaptive algorithms are then used to estimate the unknown model parameters. Since the order of the model under H_1 is usually assumed to be greater than that under H_0 (and with distinct coefficients) the filter F_1 designed for the hypothesis H_1 data is of larger order than the filter F_0 designed for the hypothesis H_0 data. Therefore,

the error output from the F_0 filter given that H_1 is true will be greater than the error output from the filter designed for H_1 . These error differences are used in the computation of the likelihood ratio to increase or decrease its value relative to the threshold level. In [40], it is shown that this likelihood ratio is a generalization of several important special cases. These include (1) the detection of a deterministic signal in additive white Gaussian noise, (2) the detection of a non-white Gaussian signal process in additive white Gaussian noise, (3) the moving target detection (MTD) algorithm for the detection of a deterministic signal in non-white Gaussian noise [41,42] and (4) the detection of a non-white Gaussian signal process in additive non-white Gaussian noise [43]. It is also shown [40] that the IBDA reduces to the algorithm developed [44] for the detection of a deterministic signal in additive non-white Gaussian noise of unknown correlation statistics. An experimental investigation using two implementations of the IBDA was conducted in [16] for an airport surveillance radar application and performance comparisons made with three MTD algorithms.

Several early analyses involving model-based parametric detection approaches are presented in [81,45,43,62] where the original pioneering work appears to be presented by Schweppe in [1]. Most of the interest, however, appears to have taken place within the past few years [14,16,20,23,24,44,46,61,80,82]. The principal advantage of characterizing the observation processes for each hypothesis via a parametric model is that well known algorithms can be utilized to estimate the parameters. In [46], a likelihood ratio test was considered for two known autoregressive (AR) models. In [82], a more general formulation considers AR and autoregressive-moving average (ARMA) models to detect a Gaussian signal in white Gaussian noise, both with unknown statistics. Thus, the IBDA [14,24] mentioned above considers the more general detection problem of a non-Gaussian signal in non-white plus white noise. As noted in [83], however, different modeling approaches will generally yield differences in receiver performance. In addition, the problem of modeling observation processes in this detection scheme is more difficult than model-fitting a time series process. This is due to the fact that one set of observation data is given in the detection problem, and the problem is to determine which of the two filters is estimating the parameters properly. In [20,83], Zhang investigated the detection performance improvement over the IBDA [24] in a radar application,

when 'a priori' information was used to predetermine the process parameters; specifically, a reference channel which provided data from range cells adjacent to the "test cell" was used to determine the filter coefficients under H_0 . A significant detection performance improvement over the IBDA was reported. It thus appears that performance improvements can be made in the model-based detection schemes through investigations of alternative algorithms and implementation schemes for the detection problem with unknown statistics.

In development of the model-based detection approaches, the emphasis is placed on characterizing the observation data received under each hypothesis with approximate models. If the models fail to fit the physical processes, performance degradations will result. Model fitting of observation data via time series analysis has received considerable attention [26,27,30,31,36,63-67]. Multichannel time series analyses have also been investigated [28,30,31,35,64(PartII),65,67] with emphasis in the areas of geophysics, biophysics and economics. Furthermore, multichannel algorithms for parameter estimation have received some attention [6,7,8,10,17,59,60]. However, the multichannel model fitting problem remains an open area for future research. In particular, a requirement for computationally efficient and numerically stable multichannel algorithms remains as does a requirement for a multichannel model order selection criterion.

1.2 Research Objectives and Approach

This dissertation considers the binary multichannel detection problem for a random signal vector in additive non-white interference plus white Gaussian noise using a model-based approach. The observation processes are allowed to have partial correlation in time and space (ie. across the channels). The principal research objective is the investigation of multichannel model-based detection methods to determine their performance relative to the single channel case. The model-based approach considers the underlying processes to be characterized by parametric models [such as the autoregressive(AR) description] and implements a likelihood ratio which is sensitive to the differences between the estimated model parameters under each hypothesis. Specifically, we give prime consideration to multichannel autoregressive processes [7-10,17,28,29] as the underlying process model descriptions in the proposed investigation. Thus, the model-based approach is based on the contention that the coefficients of the received processes are distinct for each of the two hypotheses (i.e., signal present or absent). Decision, therefore, is based upon measures which are sensitive to the differences in the process coefficients for each hypothesis.

The detection procedure which is developed and analyzed here establishes a method capable of processing observation data from 'distinct' multiple sources each containing information regarding the presence or absence of the signal. The term 'distinct' is used here to indicate that the channel processes may be dissimilar in terms of the methods used to observe them. However, the basic assumption is that these processes can be related by a multichannel time series model; ie., the underlying physical mechanisms giving rise to the observed random processes obtained under each hypothesis can be represented by a mathematical model which approximates its statistical characteristics. Thus, the applicability of multichannel time series models and the performance of multichannel parameter estimation algorithms to the detection problem considered here is an important area for further investigation. The emphasis in the analysis presented here is the development of multichannel model-based detection approaches and their assessment for improved detection performance over that of the single channel case. The extraction of this additional information is achieved by removing the redundant (i.e. correlated) information from the observation data. This is achieved via estimation methods which "whiten" the data in time and space (i.e., channels). The resulting uncorrelated error residual

processes contain all the useful or 'new' information about the processes in a compact form and are utilized to determine a sufficient statistic for hypothesis selection. As shown in chapter 6, these processes enable one to efficiently perform a calculation of the likelihood ratio. In addition, the likelihood ratio considered here has an implementation framework amenable to adaptive processing methods. Furthermore, the detection procedure accounts for the temporal and cross-channel correlation of the signal and clutter processes and thus provides coherent detection over multiple observations both in time and across channels. Also, no restrictive assumptions are made regarding correlation between the signal and clutter processes. As a result, the detection method should retain considerable robustness in detection problems involving feedback or radar multipath conditions [21,40]. In this dissertation, multichannel likelihood ratios are first developed for jointly stationary, complex, Gaussian baseband processes which also have stationary bandpass processes. However, a likelihood ratio for a class of processes with more generally correlated Gaussian quadrature components is also developed. These latter processes are not restricted to have stationary bandpass processes.

In addition to the likelihood ratio implementation, an approach has been developed for the synthesis of multichannel autoregressive (AR) random processes. The procedure allows for variable temporal and cross-channel correlation properties subject to specific constraints for correlation functions. The resulting synthesized processes provide a 'fit' in a minimum mean squared error (MMSE) sense to the correlation functions which are specified in terms of temporal and cross-channel correlation parameters of the processes. Computer simulation results are presented showing the case of a two channel AR process with various temporal and cross-channel correlation. A method is also suggested to generalize the synthesized outputs to obtain complex processes with jointly Gaussian quadrature components where the usual assumptions associated with a complex Gaussian process are relaxed. These processes provide an input to the likelihood ratio architectures to obtain detection performance as a function of channel signal-to-noise ratio as well as the process correlation parameters.

As a third contribution, ergodicity considerations are used to develop analytic expressions for the variance of the time-averaged auto- and cross-channel correlation function estimators for discrete, complex processes. These expressions provide a performance measure which can be used to specify the

window size of the observation interval required to achieve a specific value of this variance. A unique aspect of this development is the determination of the functional dependence of these expressions in terms of the process characteristics such as their variances as well as their temporal and cross-channel correlations. In addition, the analytic expressions are developed for the general case of complex processes with jointly Gaussian quadrature components where the usual assumptions associated with a complex Gaussian process are relaxed. Validity of the analytic expressions is presented using the multichannel process synthesis method noted above. This analysis is of particular significance for the case of 'unknown' process parameters where the generalized likelihood ratio is evaluated using parameter estimation algorithms based on time-averaging methods. In this case, detection performance limits result due to the errors associated with the time-averaged estimates.

Finally, the multichannel approach addressed here is applicable in a wide range of detection problems. Examples of application areas can be found in radar technology, biomedical applications, geophysical research, image processing, data compression, speech analysis and channel equalization. A significant contribution of the detection algorithm considered here is the capability to utilize signal processing procedures to deal with partially correlated observation data. For the active radar case, we view the multichannel observation data as processes which arise through a simultaneous excitation of the surveillance volume with a multiplicity of waveforms. For passive detection applications, we view the processes as arising from internal physical mechanisms which give rise to the emission of radiation which may, in general, contain partial correlation when observed over specific bands. In biomedical applications, we might consider the processing of EEG waves where we may seek to detect a weak brain potential among other brain signals. For the purpose of validating the theoretical results developed here, performance evaluations are carried out in terms of receiver operating characteristics. Both analytic as well as Monte Carlo approaches are utilized in the analysis.

In chapter 2.0, we review multichannel linear filtering and estimation methods which are used in the implementation of the detection methods used in this dissertation. The detection problem is defined in chapter 3.0. In chapter 4.0, we present a novel method for the synthesis of multichannel random processes composed of additive signal, non-white (clutter) and white noise processes. Both

the signal and non-white clutter interference are characterized as random vector processes using multichannel autoregressive (AR) processes. In addition, the method enables one to control the correlation and variances of these processes both in time and across channels. Thus, the method provides a means for the generation of input observation processes for the detection algorithm in which parameters such as the temporal and cross-channel correlation as well as the signal-to-noise (S/N) and clutter-to-noise (C/N) ratios can be varied. In chapter 5.0, expressions for the variance of time-averaged estimators are presented. These results are derived to demonstrate the detection performance limits which result due to the errors associated with these estimates. Likelihood ratios for the detection problem with wide-sense stationary, Gaussian signal and interference processes are developed in chapter 6.0. In addition, implementation architectures for these likelihood ratios are presented. In chapter 7.0, we describe the procedure used to determine detection performance. Specifically, the Monte-Carlo method is described under the conditions where the statistics of the underlying processes are assumed to be 'known' and 'unknown'. In one of the appendices, we present a review of detection using multiple observations with an extension to the multichannel case. We also consider the two cases where the signal processes are totally correlated and uncorrelated, respectively, in time and across channels. These two cases are shown to bound the partially correlated signal in white noise detection results presented and thus provide an analytic assessment to validate the results presented here. Finally, in chapter 8, we present a summary of the conclusions and recommendations for future considerations.

CHAPTER 2 REVIEW OF LINEAR FILTERING AND ESTIMATION ALGORITHMS

In this chapter, we derive the mathematical description for several important linear prediction error filters (PEF). These include the multichannel forward and backward prediction error filters which utilize a transversal (tapped delay line) structure as well as the multichannel lattice filter. In chapter 6, these filtering operations are used in the implementation architectures for the multichannel likelihood ratios. Initially, however, we present a preliminary discussion of the complex processes which we consider in this dissertation. These include the autoregressive (AR) processes which provide a parametric description of the processes. Finally, we present a derivation of the Levinson-Wiggins-Robinson and the Strand-Nuttall algorithms. These algorithms can be used to obtain estimates of the AR process parameters using the observation data. In this dissertation, however, the Strand-Nuttall algorithm is used to provide the estimates of these AR parameters which in turn are used to set the weights in the prediction error filters described in chapter 6. Whereas, the Levinson-Wiggins-Robinson algorithm is used in a process synthesis procedure described in chapter 4.

2.1 DEFINITION OF THE OBSERVATION PROCESS

Consider the complex Gaussian vector process $\underline{x}(n)$ such that

$$\underline{x}(n) = \underline{s}(n) + \underline{c}(n) + \underline{w}(n) \quad n = 1, 2, \dots, N \quad (2.1.1)$$

where $\underline{x}(n)$ is a discrete, zero mean, wide-sense jointly stationary $J \times 1$ received baseband observation vector consisting of observations from J channels and $\underline{s}(n)$, $\underline{c}(n)$ and $\underline{w}(n)$ are zero mean, complex Gaussian random $J \times 1$ vector processes describing the signal, non-white interference (clutter) and white noise, respectively. The vector processes $\underline{s}(n)$ and $\underline{c}(n)$ individually contain an arbitrary correlation in time and between channels. In addition, correlation can be considered between $\underline{s}(n)$ and past values of $\underline{c}(n)$ and $\underline{w}(n)$. We will consider this point further in chapter 3. The white noise vector $\underline{w}(n)$ is uncorrelated with itself in time, but not necessarily across channels, so that

$$E[\underline{w}(n)\underline{w}^H(k)] = \begin{cases} [0] & n \neq k \\ R_{ww}(0) & n = k \end{cases} \quad (2.1.2)$$

where $R_{ww}(0)$ is the $J \times J$ correlation matrix of $\underline{w}(n)$. The notation H denotes the Hermitian operation; ie., the complex conjugate transpose. The $N \times N$ correlation matrix for the observation data is expressed as

$$[R_{xx}]_N = E[\underline{x}_{1,N}(\underline{x}_{1,N})^H] \quad (2.1.3)$$

where

$$\underline{x}_{1,N}^T = [\underline{x}^T(1) \ \underline{x}^T(2) \dots \underline{x}^T(N)] \quad (2.1.4a)$$

and

$$\underline{x}^T(k) = [x_1(k) \ x_2(k) \dots x_J(k)]. \quad (2.1.4b)$$

Under the condition of wide-sense stationarity, $[R_{xx}]_N$ is a Hermitian, positive semi-definite matrix [3]. Furthermore, this matrix can be written in block form as

$$[R_{xx}]_N = \begin{bmatrix} R_{xx}(0) & R_{xx}(-1) & \dots & R_{xx}(-N+1) \\ R_{xx}(1) & R_{xx}(0) & \dots & R_{xx}(-N+2) \\ \dots & \dots & \dots & \dots \\ R_{xx}(N-1) & R_{xx}(N-2) & \dots & R_{xx}(0) \end{bmatrix} = \begin{bmatrix} R_{xx}^H(0) & R_{xx}^H(1) & \dots & R_{xx}^H(N-1) \\ R_{xx}^H(-1) & R_{xx}^H(0) & \dots & R_{xx}^H(N-2) \\ \dots & \dots & \dots & \dots \\ R_{xx}^H(-N+1) & R_{xx}^H(-N+2) & \dots & R_{xx}^H(0) \end{bmatrix} \quad (2.1.5)$$

where

$$R_{xx}(l) = E[\underline{x}(k)\underline{x}^H(k-l)] \quad k = 1, 2, \dots, N \\ l = 0, \pm 1, \dots, \pm (N-1) \quad (2.1.6)$$

and the last expression in eq(2.1.5) results because $R_{xx}(l) = R_{xx}^H(-l)$. It is noted, however, that for $l \neq 0$, the block matrices of $[R_{xx}]_N$ are not, in general, Hermitian; i.e., $R_{xx}(l) \neq R_{xx}^H(l)$ for $l \neq 0$. Note that $[R_{xx}]_N$ is written in block form where each block, as defined in eq(2.1.6), is a $J \times J$ correlation matrix over the J channels. We also note that $[R_{xx}]_N$ is block Toeplitz. This property is utilized in recursive algorithms to solve equations involving $[R_{xx}]_N$.

2.2. DEFINITION OF THE AUTOREGRESSIVE (AR) PROCESS

In this analysis, the multichannel signal and non-white clutter vector processes denoted as $\underline{s}(n)$ and $\underline{c}(n)$, respectively, are modeled as multichannel forward autoregressive (AR) processes; ie., the multichannel $J \times 1$ vector $\underline{g}(n)$ with $\underline{g} = s, c$ is expressed as

$$\underline{g}(n) = - \sum_{k=1}^{M_g} A_g^H(k) \underline{g}(n-k) + \underline{u}_g(n) \quad \underline{g} = s, c \quad (2.2.1)$$

where $A_g^H(k)$ is the k th $J \times J$ matrix coefficient for AR process $\underline{g}(n)$ of model order M_g and $\underline{u}_g(n)$ is the $J \times 1$ forward white noise driving vector with covariance matrix $[\Sigma_f]_{u_g}$. We note that $A_g^H(k)$ is expressed in terms of the Hermitian operation for notational convenience, but is not treated here as a Hermitian matrix. It will be shown later that this notation enables us to express the Yule-Walker equation [eq(2.5.5)] in terms of row vectors containing the Hermitian transpose operation. The vectors $\underline{u}_g(n)$ from eq(2.2.1) and $\underline{w}(n)$ from eq(2.1.1) can be expressed as

$$\underline{u}_g(n) = C_{v_g} \underline{v}_g(n) \quad \underline{g} = s, c \quad (2.2.2a)$$

and

$$\underline{w}(n) = C_w \underline{v}_w(n) \quad (2.2.2b)$$

where $\underline{v}_g(n)$ and $\underline{v}_w(n)$ are white noise vectors with $J \times J$ diagonal covariance matrices D_{v_g} and the identity matrix I , respectively. Eqs(2.2.2) enable us to obtain

$$[\Sigma_f]_{u_g} \equiv E[\underline{u}_g(n) \underline{u}_g^H(n)] = E[C_{v_g} \underline{v}_g(n) \underline{v}_g^H(n) C_{v_g}^H] \quad \underline{g} = s, c \quad (2.2.3a)$$

$$= C_{v_g} E[\underline{v}_g(n) \underline{v}_g^H(n)] C_{v_g}^H \quad \underline{g} = s, c \quad (2.2.3b)$$

$$= C_{v_g} D_{v_g} C_{v_g}^H \quad \underline{g} = s, c \quad (2.2.3c)$$

and

$$R_{ww}(0) \equiv E[\underline{w}(n)\underline{w}^H(n)] = E[C_w \underline{w}(n) \underline{w}^H(n) C_w^H] \quad (2.2.4a)$$

$$= C_w E[\underline{w}(n) \underline{w}^H(n)] C_w^H \quad (2.2.4b)$$

$$= C_w C_w^H. \quad (2.2.4c)$$

For the special case

$$D_{v_g} = E[\underline{v}_g(n) \underline{v}_g^H(n)] = I \quad g = s, c \quad (2.2.5)$$

eq(2.2.3c) reduces to

$$[\Sigma_f]_{u_g} = C_{v_g} C_{v_g}^H. \quad (2.2.6)$$

We will discuss the implications of eq(2.2.5) in section 4.6. Eqs(2.2.4c) and (2.2.6) indicate that the $J \times J$ matrices C_w and C_{v_g} can be obtained by the Cholesky decomposition of $R_{ww}(0)$ and $[\Sigma_f]_{u_g}$, respectively. These relationships will be utilized in the process synthesis procedure described in chapter 4. Alternatively, eq(2.2.2a) could have been expressed as

$$\underline{u}_g(n) = L_{z_g} \underline{z}_g(n) \quad g = s, c \quad (2.2.7)$$

where L_{z_g} is a unit diagonal lower triangular matrix and the vector process $\underline{z}_g(n)$ is a white noise vector uncorrelated in time, whose components are also uncorrelated across channels. The covariance matrix of $\underline{u}_g(n)$ is expressed as

$$[\Sigma_f]_{u_g} \equiv E[\underline{u}_g(n) \underline{u}_g^H(n)] = E[L_{z_g} \underline{z}_g(n) \underline{z}_g^H(n) L_{z_g}^H] \quad g = s, c \quad (2.2.8a)$$

$$= L_{z_g} E[\underline{z}_g(n) \underline{z}_g^H(n)] L_{z_g}^H \quad g = s, c \quad (2.2.8b)$$

$$= L_{z_g} D_{z_g} L_{z_g}^H \quad g = s, c \quad (2.2.8c)$$

where

$$D_{z_g} = E[z_g(n)z_g^H(n)]. \quad (2.2.9)$$

Eq(2.2.8c) denotes the LDL^H decomposition of $[\Sigma_f]_{u_g}$ provided D_{z_g} is a real, diagonal matrix. Furthermore, the channel variances are contained along the diagonal elements of D_{z_g} .

2.3. THE YULE-WALKER EQUATIONS

In this section, we present the relationship between the matrix coefficients $A_g^H(k)$ of eq(2.2.1), the covariance matrix $[\Sigma_f]_{u_g}$ of the forward AR white noise driving vector defined in eqs(2.2.3), and the known correlation matrix $[R_{gg}]$ of the vector $g(n)$. The latter is defined as

$$[R_{gg}] = \dots_{n-M_g} (g_{n,n-M_g})^H]. \quad (2.3.1)$$

where

$$g_{n-M_g,n}^T = [g^T(n-M_g) g^T(n-M_g+1) \dots g^T(n)] \quad g = s,c. \quad (2.3.2)$$

We first introduce the reversed order correlation matrix $[\tilde{R}_{gg}]_{M_g+1}$ defined as

$$[\tilde{R}_{gg}] = E[g_{n,n-M_g} (g_{n,n-M_g})^H] \quad g = s,c \quad (2.3.3)$$

where

$$g_{n,n-M_g}^T = [g^T(n) g^T(n-1) \dots g^T(n-M_g)] \quad g = s,c \quad (2.3.4)$$

and $\sim$ denotes time order reversal. We note that the matrix $[\tilde{R}_{gg}]$ is the correlation matrix of $\tilde{g}_{n-M_g, n} = g_{n, n-M_g}$ in contrast to $[R_{gg}]$ which is the correlation matrix of the vector $g_{n-M_g, n}$ defined in eq(2.3.1).

The relationship between $A_g^H(k)$, $[\Sigma_f]_{u_g}$ and $[\tilde{R}_{gg}]$ is expressed by the augmented Yule-Walker equations [68,69]; ie., dropping the subscript on the correlation matrix

$$A_g^H [\tilde{R}_{gg}] = \{[\Sigma_f]_{u_g}^H [0] \dots [0]\} \quad g = s, c \quad (2.3.5)$$

where

$$A_g^H = [I \ A_g^H(1) \ A_g^H(2) \dots A_g^H(M_g)] \quad g = s, c \quad (2.3.6a)$$

$$[\Sigma_f]_{u_g}^H = E[u_g(n) u_g^H(n)] = [\Sigma_f]_{u_g} \quad g = s, c \quad (2.3.6b)$$

$$u_g^T(n) = [u_1^g(n) \ u_2^g(n) \dots u_J^g(n)] \quad g = s, c. \quad (2.3.6c)$$

The corresponding equation for the wide-sense stationary, backward AR process is expressed as

$$g(n) = - \sum_{k=1}^{M_g} B_g^H(k) g(n+k) + u_{gb}(n) \quad g = s, c \quad (2.3.7)$$

where we specifically denote the backward white noise driving vector $u_{gb}(n)$ with the subscript b. The covariance matrix of $u_{gb}(n)$ is expressed as

$$[\Sigma_b]_{u_g} = E[u_{gb}(n) u_{gb}^H(n-l)] \quad (2.3.8)$$

The corresponding equation for the backward Yule-Walker equations are expressed as

$$B_g^H [\tilde{R}_{gg}] = \{[0] \dots [0] [\Sigma_b] u_g^H\} \quad (2.3.9)$$

where

$$B_g^H = [B_g^H(M_g) \dots B_g^H(1) \mathbf{I}]. \quad (2.3.10)$$

2.4 MULTICHANNEL LINEAR FILTERING AND INNOVATIONS PROCESSES

2.4.1. Forward Prediction Error Filter

The $J \times 1$ error vector output of a multichannel linear prediction error filter of order p is expressed as

$$\varepsilon_p(n) = \mathbf{x}(n) - \hat{\mathbf{x}}_p(n|n-1, \dots, n-p) \quad (2.4.1a)$$

$$= \mathbf{x}(n) + \sum_{k=1}^p A_p^H(k) \mathbf{x}(n-k) \quad (2.4.1b)$$

$$= \sum_{k=0}^p A_p^H(k) \mathbf{x}(n-k) \quad (2.4.1c)$$

where

$$\hat{\mathbf{x}}_p(n|n-1, \dots, n-p) = - \sum_{k=1}^p A_p^H(k) \mathbf{x}(n-k), \quad (2.4.1d)$$

$A_p^H(k)$ $k=1, 2, \dots, p$ are the $J \times J$ matrix coefficients of the linear predictor, $A_p^H(0) = \mathbf{I}$

where $\mathbf{I}$ is the $J \times J$ identity matrix and the subscript p distinguishes the matrix coefficients as belonging to a filter of order p . We again note that the Hermitian notation is used on the matrix coefficients for notational convenience and that these coefficients are not necessarily Hermitian. The filter structure for this implementation is shown in Fig. 2.4.1. For the single channel case, when the scalar $\varepsilon_p(n)$ is minimized in a minimum mean-squared error (MMSE) sense, the resulting process is called a discrete-time innovations process [1,24,53,54,57]. In [59], $\varepsilon_p(n)$ is referred to as a multichannel innovations process. We note,

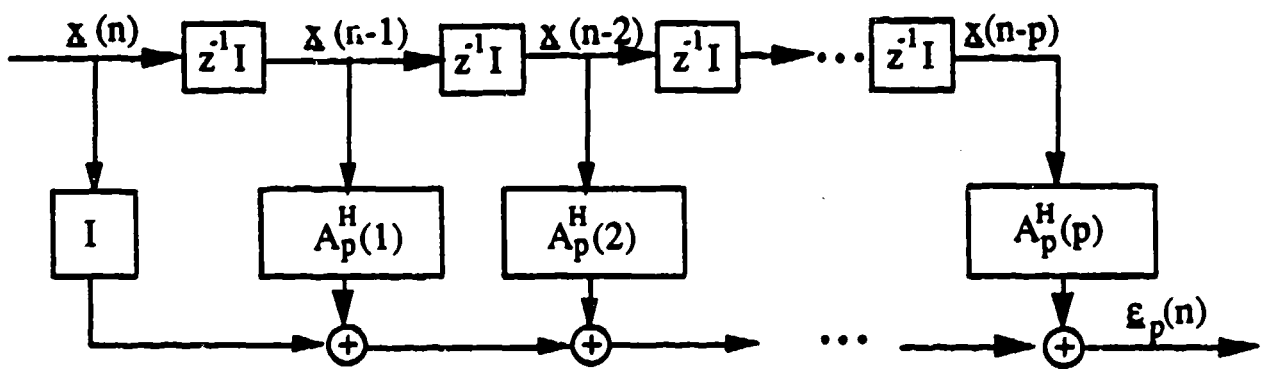


Figure 2.4.1 Multichannel Forward Prediction Error Filter.

however, that although $\underline{x}_p(n)$ is 'whitened' in time, it is not necessarily 'white' across channels. We discuss this further in chapter 6. In matrix form, we can rewrite eq(2.4.1c) as

$$\underline{x}_p(n) = \Delta_p^H \underline{x}_{n,n-p} \quad (2.4.2)$$

where

$$\Delta_p^H = [I \ A_p^H(1) \ A_p^H(2) \ \dots \ A_p^H(p)] \quad (2.4.3a)$$

and

$$\underline{x}_{n,n-p}^T = [\underline{x}^T(n) \ \underline{x}^T(n-1) \ \underline{x}^T(n-2) \ \dots \ \underline{x}^T(n-p)] . \quad (2.4.3b)$$

Following a presentation similar to that found in [17], we post-multiply both sides of eq(2.4.2) by $\underline{x}_{n,n-p}^H$ and take the expectation operation so that

$$E[\underline{x}_p(n) \underline{x}_{n,n-p}^H] = E[\Delta_p^H \underline{x}_{n,n-p} \underline{x}_{n,n-p}^H] \quad (2.4.4a)$$

$$= \Delta_p^H E[\underline{x}_{n,n-p} \underline{x}_{n,n-p}^H] \quad (2.4.4b)$$

$$= \Delta_p^H [\tilde{R}_{\underline{x}\underline{x}}]_{p+1} \quad (2.4.4c)$$

where the subscript $p+1$ denotes the dimension of the correlation matrix in terms of the $J \times J$ blocks [see eq(2.5.6)], $[\tilde{R}_{\underline{x}\underline{x}}]_{p+1}$ is defined as

$$[\tilde{R}_{\underline{x}\underline{x}}]_{p+1} = E[\underline{x}_{n,n-p} \underline{x}_{n,n-p}^H] \quad (2.4.5)$$

and

$$[R_{\underline{x}\underline{x}}]_{p+1} = E[\underline{x}_{n-p,n} \underline{x}_{n-p,n}^H]. \quad (2.4.6)$$

Again, $\sim$ denotes time order reversal of the vector $\underline{x}_{n-p,n}$. Using eq(2.4.3b), we now expand the LHS of eq(2.4.4a) as

$$E[\underline{x}_p(n) \underline{x}_{n,n-p}^H] = E\{\underline{x}_p(n) [\underline{x}^H(n) \ \underline{x}^H(n-1) \ \dots \ \underline{x}^H(n-p)]\} \quad (2.4.7a)$$

$$= \left\{ E[\epsilon_p(n)\mathbf{x}^H(n)] E[\epsilon_p(n)\mathbf{x}^H(n-1)] \dots E[\epsilon_p(n)\mathbf{x}^H(n-p)] \right\} \quad (2.4.7b)$$

and determine the coefficients of Δ_p^H subject to the condition that $\epsilon_p(n)$ is a minimum mean-squared error process. Under this condition, the orthogonality principle holds (see Appendix A); ie.,

$$E[\epsilon_p(n)\mathbf{x}^H(n-k)] = [0] \quad 1 \leq k \leq p \quad (2.4.8)$$

so that eq(2.4.7b) becomes

$$E[\epsilon_p(n) \mathbf{x}_{n,n-p}^H] = \left\{ E[\epsilon_p(n)\mathbf{x}^H(n)] [0] [0] \dots [0] \right\} \quad (2.4.9)$$

where $[0]$ is a $J \times J$ null matrix. Solving eq(2.4.1b) for $\mathbf{x}(n)$, we obtain

$$\mathbf{x}(n) = \epsilon_p(n) - \sum_{k=1}^p A_p^H(k) \mathbf{x}(n-k) \quad (2.4.10a)$$

so that

$$E[\epsilon_p(n)\mathbf{x}^H(n)] = E[\epsilon_p(n)\epsilon_p^H(n)] - \sum_{k=1}^p E[\epsilon_p(n)\mathbf{x}^H(n-k)] A_p(k). \quad (2.4.10b)$$

Again, using eq(2.4.8), the summation term in eq(2.4.10b) is a null matrix. Substituting the remaining term into eq(2.4.9) yields

$$E[\epsilon_p(n) \mathbf{x}_{n,n-p}^H] = \left\{ E[\epsilon_p(n)\epsilon_p^H(n)] [0] [0] \dots [0] \right\} \quad (2.4.11a)$$

$$= \left\{ [\Sigma_f]_p^H [0] [0] \dots [0] \right\} \quad (2.4.11b)$$

where

$$[\Sigma_f]_p = E[\epsilon_p(n)\epsilon_p^H(n)] = [\Sigma_f]_p^H \quad (2.4.12)$$

is the $J \times J$ Hermitian forward prediction error covariance matrix for the p th order filter. Since eqs(2.4.4c) and (2.4.11b), are identical expressions, equality of the RHS of these equations provides

$$\Delta_p^H [\tilde{R}_{xx}]_{p+1} = \{ [\Sigma_f]_p^H [0] [0] \dots [0] \}. \quad (2.4.13)$$

Taking the Hermitian transpose of eq(2.4.13) and noting that $[\tilde{R}_{xx}]$ is Hermitian, we obtain

$$[\tilde{R}_{xx}]_{p+1} \Delta_p = [\Sigma_f]_p \underline{1} \quad (2.4.14)$$

where

$$\underline{1}^T = \{ I [0] [0] \dots [0] \}. \quad (2.4.15)$$

Eqs(2.4.13) and (2.4.14) are the augmented normal equations for a forward multichannel prediction error filter. Thus, Δ_p contains the minimum mean-squared error (MMSE) matrix coefficients for a linear prediction error filter of order p . We also note that eqs (2.4.13) and (2.3.1) are identical in form. This implies that the MMSE estimate of the AR observation process is obtained provided p is equal to the order of the observation process and the prediction error filter coefficients are identically equal to the AR process coefficients. Figure 2.4.2 shows in block form what we define as process generation (synthesis) and the filtering (analysis) procedure. Finally, we note that the use of the Hermitian notation on the matrix coefficients of eq(2.4.1b) resulted in a consistent notation in eqs(2.4.13) and (2.4.14); ie., the row vector of matrices expressed by Δ_p^H contains matrices subjected to the Hermitian operator while the column vector Δ_p does not. This consistent notation was found not to be used in some of the literature.

With $\underline{x}(n)$ and $\underline{u}(n)$ initially considered as non-random[†] processes, further insight can be obtained by considering the z-transform of

$$\underline{x}(n) = - \sum_{k=1}^M A_M^H(k) \underline{x}(n-k) + \underline{u}(n) \quad (2.4.16)$$

[†] We initially consider non-random processes since the z-transform of a random process is not defined.

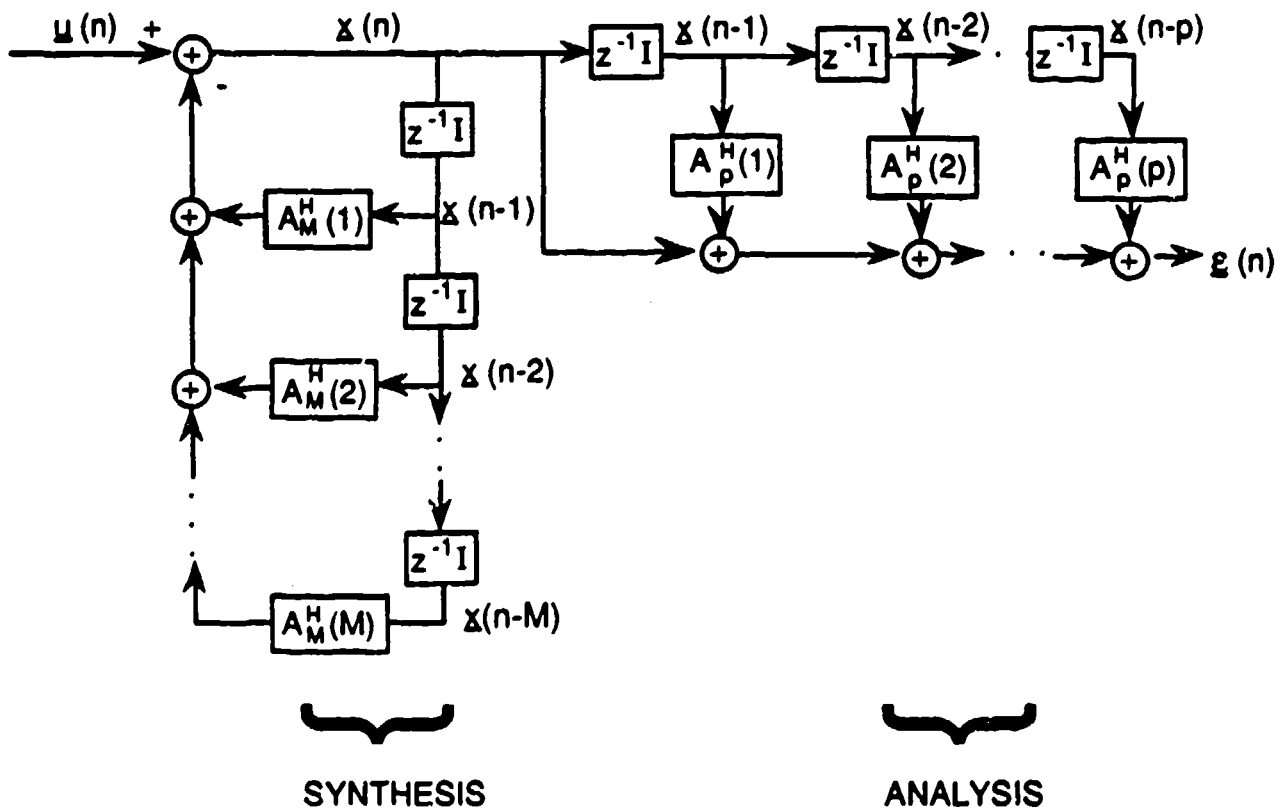


Figure 2.4.2 Synthesis and Analysis Procedure.

so that

$$\underline{X}(z) = - \sum_{k=1}^M A_M^H(k) \underline{X}(z) z^{-k} + \underline{U}(z). \quad (2.4.17)$$

Bringing the summation to the LHS, we can write

$$\left[\sum_{k=0}^M A_M^H(k) z^{-k} \right] \underline{X}(z) = \underline{U}(z) \quad (2.4.18)$$

where $A_M^H(0) = I$. We now define a filter representation for the model process as

$$H_M(z) = \left[\sum_{k=0}^M A_M^H(k) z^{-k} \right]^{-1} \quad (2.4.19)$$

so that

$$\underline{X}(z) = H_M(z) \underline{U}(z). \quad (2.4.20)$$

Eq (2.4.20) indicates that $\underline{X}(z)$ is the output of the filter $H_M(z)$ with input $\underline{U}(z)$. Similarly, the z-transform of eq (2.4.1c) where $\underline{x}(n)$ and $\underline{e}_p(n)$ are initially considered non-random is expressed as

$$\underline{E}(z) = \left[\sum_{k=0}^p A_p^H(k) z^{-k} \right] \underline{X}(z) \quad (2.4.21a)$$

$$= H_f(z) \underline{X}(z) \quad (2.4.21b)$$

where the filter transfer function is expressed as

$$H_f(z) = \left[\sum_{k=0}^p A_p^H(k) z^{-k} \right] \quad (2.4.22)$$

and

$$A_p^H(0) = I. \quad (2.4.23)$$

When $p \geq M$, consider the case

$$A_p^H(k) = \begin{cases} A_M^H(k) & k \leq M \\ [0] & k > M. \end{cases} \quad (2.4.24)$$

We then have

$$H_f(z) = H_M^{-1}(z). \quad (2.4.25)$$

At this point, we can now consider the input and outputs of these filters to be random. Using eqs (2.4.20) and (2.4.25) in eq(2.4.21b)

$$E(z) = H_M^{-1}(z) H_M(z) U(z) = U(z) \quad (2.4.26)$$

In the time domain, eq (2.4.26) is equivalent to

$$e_p(n) = u(n). \quad (2.4.27)$$

And so, under the condition that $p=M$ and the prediction error coefficients are identical to the coefficients of the AR model process and under the assumption that the AR process is the exact model of the observed process, the prediction error filter output $e_p(n)$ is a white noise vector equivalent to the AR model white noise driving vector. However, it must be emphasized that the use of an AR process with a white noise driving function is usually an approximate representation; i.e. it is not used to describe the underlying physical mechanisms which give rise to the random processes. Rather, it is a representation which has a system transfer function given by eq(2.4.19). We therefore make a distinction between the model of the processes (synthesis) and the estimation process (analysis) [12]. In general, the output $e_p(n)$ of the linear predictor of finite order p is not a white noise vector output due to the approximate representation of physical processes by an AR model. In this dissertation, the observation processes often consist of an AR model in additive noise. In general, these are

autoregressive moving-average (ARMA) processes. The error vector is also non-white due to the fact that we often do not have 'a priori' knowledge regarding the exact values of the coefficients for the model of the processes. As a result, we must estimate these coefficients from the observation data as we obtain it. With a limited amount of data, the filter coefficients are only estimates of the process coefficients.

For stationary processes, these coefficients could be determined through estimates of the multichannel correlation matrix lag values and the Levinson-Wiggins-Robinson algorithm [6]. Other methods, however, have been proposed by Strand-Nuttall and Morf-Vieira [7,8,9,10] with improved performance using limited data. For non-stationary processes, adaptive schemes are considered. We will address this topic in future investigations. However, in section 6.3 we propose an estimator and develop a likelihood ratio which is applicable to the case where the associated bandpass processes could be non-stationary.

At this point, we note that eq(2.4.27) resulted from the analysis procedure [via a linear prediction error filter with coefficients given by eq(2.4.24)] of the processes synthesized by eq(2.4.16). If the components of $\underline{u}(n)$ are assumed to be uncorrelated across channels, the resulting components of $\underline{\epsilon}_p(n)$ are also uncorrelated in time and space (i.e., channels). In general, however, the components of $\underline{u}(n)$ may possess arbitrary correlation between the J channel elements. Therefore, the components of $\underline{\epsilon}_p(n)$ will retain a residual correlation over the channels due to the spatial correlation of the components of $\underline{u}(n)$.

Since the matrix $R_{\epsilon\epsilon}(0) \equiv [\Sigma_f]_p = E [\underline{\epsilon}_p(n)\underline{\epsilon}_p^H(n)]$ is Hermitian[†], and positive semi-definite, we can perform an LDL^H decomposition,* where L is a unit diagonal lower triangular matrix and D is a diagonal matrix, such that

$$R_{\epsilon\epsilon}(0) = [\Sigma_f]_p = L_\gamma D_\gamma L_\gamma^H. \quad (2.4.28)$$

Solving for D_γ

[†] It is noted that in general the correlation matrix $R_{\epsilon\epsilon}(l)$ is not Hermitian for $l \neq 0$.

* Other decompositions such as Cholesky (see Chapter 6.0) or unitary[1] can be considered. The motivation for the LDL^H decomposition is noted in [5].

$$D_{\gamma} = L_{\gamma}^{-1} [\Sigma_f]_p [L_{\gamma}^{-1}]^H \quad (2.4.29a)$$

$$= L_{\gamma}^{-1} E[\xi_p(n) \xi_p^H(n)] [L_{\gamma}^{-1}]^H \quad (2.4.29b)$$

$$= E[L_{\gamma}^{-1} \xi_p(n) \xi_p^H(n) [L_{\gamma}^{-1}]^H] \quad (2.4.29c)$$

$$= E[\gamma_p(n) \gamma_p^H(n)] \quad (2.4.29d)$$

where

$$\gamma_p(n) = L_{\gamma}^{-1} \xi_p(n). \quad (2.4.30)$$

Because D_{γ} is a diagonal matrix, the vector $\gamma_p(n)$ contains uncorrelated elements. Also,

$$E[\xi_p(n) \xi_p^H(n')] = [0] \quad n \neq n'. \quad (2.4.31)$$

Then, using eq (2.4.30) to solve for $\xi_p(n)$ and substituting this result in (2.4.31), we obtain

$$E[L_{\gamma} \gamma_p(n) \gamma_p^H(n') L_{\gamma}^H] = [0] \quad n \neq n' \quad (2.4.32)$$

so that

$$L_{\gamma} E[\gamma_p(n) \gamma_p^H(n')] L_{\gamma}^H = [0] \quad n \neq n'. \quad (2.4.33)$$

Premultiplying and postmultiplying eq(2.4.33) by the inverses of L_{γ} and L_{γ}^H , respectively, it follows that

$$E[\gamma_p(n) \gamma_p^H(n')] = [0] \quad n \neq n'. \quad (2.4.34)$$

Eq (2.4.34) proves the temporal decorrelation of $\gamma_p(n)$ while eq(2.4.29d) states its spatial decorrelation. When

$$A_p^H \approx \begin{cases} A_M^H(k) & k \leq M \\ [0] & k > M, \end{cases} \quad (2.4.35)$$

the output of the first filter stage $\epsilon_p(n)$ converges toward $u(n)$ as noted above, so that $[\Sigma_f]_p \approx [\Sigma_f]_u$. Referring to eqs(2.2.8a) and (2.4.28), $L_z D_z L_z^H \approx L_\gamma D_\gamma L_\gamma^H$. From the uniqueness of the LDL^H decomposition, it follows that $L_z \approx L_\gamma$. Using eq(2.2.7) in (2.4.30),

$$\gamma_p(n) = L_\gamma^{-1} \epsilon_p(n) \approx L_z^{-1} u(n) \quad (2.4.36a)$$

$$\approx L_z^{-1} L_z z(n) \quad (2.4.36b)$$

$$\approx z(n). \quad (2.4.36c)$$

Thus, as the filter coefficients converge to the coefficients of the AR process, $\gamma_p(n)$ approximates the spatially and temporally whitened vector $z(n)$. In addition, $\epsilon_p(n)$ has been obtained through a causal and causally invertible transformation of the original observation process $x(n)$ [59]; ie. the input vector $x(n)$ could be recovered through an inverse filter operation on $\epsilon_p(n)$. Furthermore, since L_γ^{-1} is a unit diagonal, lower triangular matrix, it enables us to obtain a 'spatially causal' and invertible transformation of $\epsilon_p(n)$ to $\gamma_p(n)$.

2.4.2. Backward Prediction Error Filter

Let the backward linear prediction estimate of the $J \times 1$ observation vector $x(n-p)$ using the previous p data samples be defined as

$$\hat{x}_p(n-p|n-p+1, \dots, n) = - \sum_{k=1}^p B_p^H(k) x(n-p+k) \quad (2.4.37)$$

so that the $J \times 1$ backward linear prediction error vector from a filter of order p becomes

$$\beta_p(n) = \underline{x}(n-p) - \hat{\underline{x}}_p(n-p|n-p+1, \dots, n) \quad (2.4.38a)$$

$$= \underline{x}(n-p) + \sum_{k=1}^p B_p^H(k) \underline{x}(n-p+k) \quad (2.4.38b)$$

$$= \sum_{k=0}^p B_p^H(k) \underline{x}(n-p+k) \quad (2.4.38c)$$

where $B_p^H(k)$ $k=1,2,\dots,p$ are the $J \times J$ matrix coefficients of backward linear prediction and $B_p(0) = I$ where I is the $J \times J$ identity matrix. The filter structure for this implementation is shown in Fig 2.4.3.

Using the matrix form and following a similar discussion as in the previous section, we can write

$$\beta_p(n) = \underline{B}_p^H(k) \underline{x}_{n,n-p} \quad (2.4.39)$$

where

$$\underline{B}_p^H = [B_p^H(p) B_p^H(p-1) \dots B_p^H(1) I]. \quad (2.4.40)$$

It follows that

$$E[\beta_p(n) \underline{x}_{n,n-p}^H] = E[\underline{B}_p^H \underline{x}_{n,n-p} \underline{x}_{n,n-p}^H] \quad (2.4.41a)$$

$$= \underline{B}_p^H E[\underline{x}_{n,n-p} \underline{x}_{n,n-p}^H] \quad (2.4.41b)$$

$$= \underline{B}_p^H [\tilde{R}_{\underline{x}\underline{x}}]_{p+1}. \quad (2.4.41c)$$

Likewise, we can obtain

$$E[\beta_p(n) \underline{x}_{n,n-p}^H] = E\{\beta_p(n) [\underline{x}^H(n) \underline{x}^H(n-1) \dots \underline{x}^H(n-p)]\} \quad (2.4.42a)$$

$$= \{E[\beta_p(n) \underline{x}^H(n)] E[\beta_p(n) \underline{x}^H(n-1)] \dots E[\beta_p(n) \underline{x}^H(n-p)]\}. \quad (2.4.42b)$$

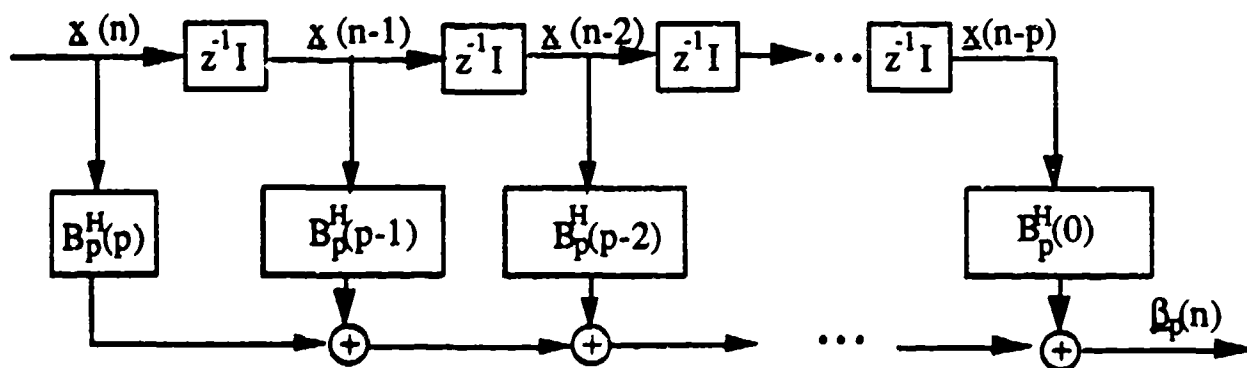


Fig 2.4.3 Multichannel Backward Prediction Error Filter

Since we are estimating $\underline{x}(n-p)$ using the data from $\underline{x}(n)$ to $\underline{x}(n-p+1)$, the orthogonality principle becomes

$$E[\underline{\beta}_p(n)\underline{x}^H(n-k)] = [0] \quad 0 \leq k \leq (p-1) \quad (2.4.43)$$

so that eq(2.4.42b) becomes

$$E[\underline{\beta}_p(n)\underline{x}_{n,n-p}^H] = \{ [0] \dots [0] E[\underline{\beta}_p(n)\underline{x}^H(n-p)] \}. \quad (2.4.44)$$

Taking the Hermitian transpose of both sides of eq(2.4.38b) and solving for $\underline{x}^H(n-p)$

$$\underline{x}^H(n-p) = \underline{\beta}_p^H(n) - \sum_{k=1}^p \underline{x}^H(n-p+k)B_p(k). \quad (2.4.45)$$

Premultiplying both sides of eq(2.4.45) by $\underline{\beta}_p(n)$ and taking the expectation, we have

$$E[\underline{\beta}_p(n)\underline{x}^H(n-p)] = E[\underline{\beta}_p(n)\underline{\beta}_p^H(n)] - \sum_{k=1}^p E[\underline{\beta}_p(n)\underline{x}^H(n-p+k)]B_p(k). \quad (2.4.46)$$

Note that the summation term in eq(2.4.46) is a null matrix from the orthogonality condition of eq(2.4.43). Substituting the remaining term into eq(2.4.44), yields

$$E[\underline{\beta}_p(n)\underline{x}_{n,n-p}^H] = \{ [0] \dots [0] [\Sigma_b]_p^H \} \quad (2.4.47)$$

where

$$[\Sigma_b]_p = E[\underline{\beta}_p(n)\underline{\beta}_p^H(n)] = [\Sigma_b]_p^H \quad (2.4.48)$$

is the Hermitian, $J \times J$ multichannel backward prediction error variance matrix for the p th order filter. Combining eq(2.4.40c) and (2.4.47), we have

$$\underline{B}_p^H [\tilde{\mathbf{R}}_{\underline{xx}}]_{p+1} = \{ [0] [0] \dots [0] [\Sigma_b]_p^H \}. \quad (2.4.49)$$

Taking the Hermitian transpose of eq(2.4.49) and noting that $[\tilde{R}_{xx}]_{p+1}$ is Hermitian, we obtain

$$[\tilde{R}_{xx}]_{p+1} B_p = [\Sigma_b]_p^H \tilde{I} \quad (2.4.50)$$

where $\tilde{I}$ is the reversed order vector of matrices defined in eq(2.4.15); ie.,

$$\tilde{I}^T = \{ [0] \dots [0] I \}. \quad (2.4.51)$$

2.4.3 The Multichannel Lattice Filter

The forward and backward prediction error filters introduced in the previous sections are now combined into a single structure. From eq(2.4.1c), the forward error vector from a pth order prediction error filter (PEF) is expressed as

$$\epsilon_p(n) = \sum_{k=0}^p A_p^H(k) x(n-k). \quad (2.4.52)$$

In the next section, we will define an update equation for the pth order multichannel forward and backward PEF coefficients $A_p(k)$ and $B_p(p)$ using eqs(2.5.2) and (2.5.4) which are expressed as

$$A_p(k) = A_{p-1}(k) + B_{p-1}(p-k)[\Gamma_f]_p \quad k=1,2,\dots,p-1. \quad (2.4.53a)$$

and

$$B_p(k) = B_{p-1}(k) + A_{p-1}(p-k)[\Gamma_b]_p \quad k=1,2,\dots,p-1 \quad (2.4.53b)$$

Substituting eq(2.4.53a) into (2.4.52), we have

$$\epsilon_p(n) = \sum_{k=0}^p A_{p-1}^H(k) x(n-k) + [\Gamma_f]_p^H \sum_{k=0}^p B_{p-1}^H(p-k) x(n-k). \quad (2.4.54)$$

Noting that $A_{p-1}^H(P) = B_{p-1}^H(P) = [0]$ at $k=P$ and interchanging the notation on the second summation in eq(2.4.54), we have

$$\varepsilon_p(n) = \sum_{k=0}^{p-1} A_{p-1}^H(k) \Delta(n-k) + [\Gamma_f]_p^H \sum_{k=0}^{p-1} B_{p-1}^H(k) \Delta(n-P+k). \quad (2.4.55)$$

Similarly, the corresponding backward PEF output for the p th order filter defined in eq(2.4.38c) is expressed as

$$\beta_p(n) = \sum_{k=0}^p B_p^H(k) \Delta(n-p+k). \quad (2.4.56)$$

For a backward PEF of order $p-1$ at time $n-1$, we have

$$\beta_{p-1}(n-1) = \sum_{k=0}^p B_{p-1}^H(k) \Delta(n-p+k) \quad (2.4.57)$$

while for a forward PEF of order $p-1$ at time n , eq(2.4.52) yields

$$\varepsilon_{p-1}(n) = \sum_{k=0}^{p-1} A_{p-1}^H(k) \Delta(n-k). \quad (2.4.58)$$

These last two equations can be substituted into eq(2.4.55) so that

$$\varepsilon_p(n) = \varepsilon_{p-1}(n) + [\Gamma_f]_p^H \beta_{p-1}(n-1) \quad (2.4.59a)$$

$$= \varepsilon_{p-1}(n) + A_p^H(P) \beta_{p-1}(n-1). \quad (2.4.59b)$$

Similarly, we can substitute eq(2.4.53b) into (2.4.56) to obtain

$$\beta_p(n) = \sum_{k=0}^p B_{p-1}^H(k) x(n-P+k) + [\Gamma_b]_p^H \sum_{k=0}^p A_{p-1}^H(P-k) x(n-P+k) \quad (2.4.60a)$$

$$= \sum_{k=0}^p B_{p-1}^H(k) x(n-P+k) + [\Gamma_b]_p^H \sum_{k=0}^p A_{p-1}^H(P-k) x(n-k) \quad (2.4.60b)$$

where we again used the fact that $A_{p-1}^H(P) = B_{p-1}^H(P) = [0]$ at $k=P$ and an interchange of notation on the second summation in eq(2.4.54). Using eqs(2.4.57) and (2.4.52), we obtain

$$\beta_p(n) = \beta_{p-1}(n-1) + [\Gamma_b]_p^H \epsilon_{p-1}(n) \quad (2.4.61a)$$

$$= \beta_{p-1}(n-1) + B_p^H(P) \epsilon_{p-1}(n). \quad (2.4.61b)$$

Eqs(2.4.59) and (2.4.61) are the order update recursions for stage p multichannel lattice predictor. The initial conditions are

$$\epsilon_0(n) = \beta_0(n) = x(n). \quad (2.4.62)$$

The structure of the filter which satisfies eqs(2.4.59) and (2.4.61) is shown in Fig. 2.4.4.

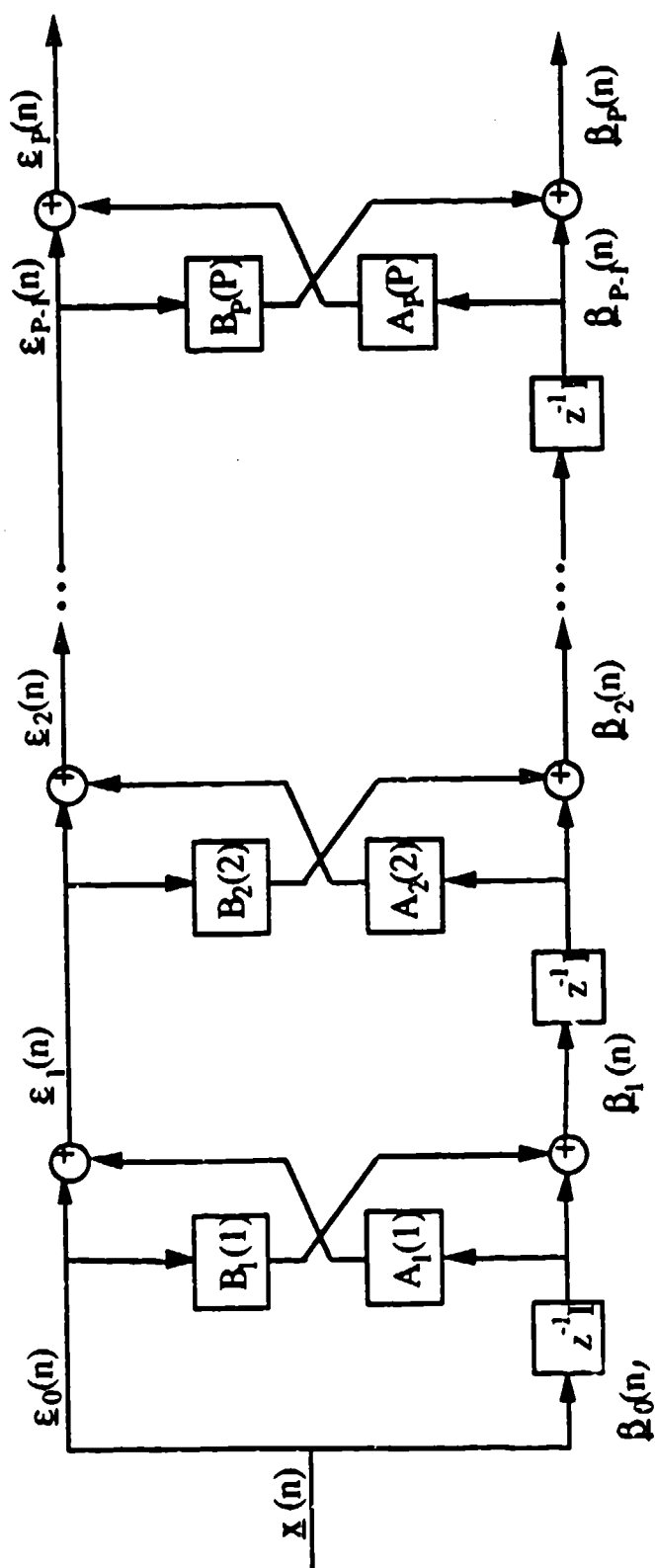


Figure 2.4.4 Multichannel Lattice Filter

2.5 PARAMETER ESTIMATION ALGORITHMS

In this section, we derive the Levinson-Wiggins-Robinson [6] and the Strand-Nuttall [8,10] algorithms which are used to estimate the coefficients of the autoregressive processes.

2.5.1. Levinson-Wiggins-Robinson Recursion

We now develop a recursive procedure to determine the multichannel forward and backward matrix coefficients. The procedure was initially presented by Wiggins and Robinson [6] based on the scalar version of Levinson [163]. The approach presented in this subsection, however, is similar to that used by Strand [8]. Consider the p th order update of the multichannel forward PEF coefficient expressed in the form

$$A_p = \begin{bmatrix} A_p(0) \\ A_p(1) \\ A_p(2) \\ \vdots \\ A_p(p-1) \\ A_p(p) \end{bmatrix} = \begin{bmatrix} A_{p-1}(0) \\ A_{p-1}(1) \\ A_{p-1}(2) \\ \vdots \\ A_{p-1}(p-1) \\ [0] \end{bmatrix} + \begin{bmatrix} [0] \\ B_{p-1}(p-1) \\ \vdots \\ B_{p-1}(2) \\ B_{p-1}(1) \\ B_{p-1}(0) \end{bmatrix} [\Gamma_f]_p \quad (2.5.1)$$

where $[\Gamma_f]_p$ is a constant matrix called the forward reflection coefficient matrix and $A_p(0)=A_{p-1}(0)=B_{p-1}(0) = I$, the $J \times J$ identity matrix. The update equation is therefore expressed as

$$A_p(k) = A_{p-1}(k) + B_{p-1}(p-k)[\Gamma_f]_p \quad k=1,2,\dots,p-1 \quad (2.5.2a)$$

$$= [\Gamma_f]_p \quad k=p \quad (2.5.2b)$$

$$= [0] \quad k>p \quad (2.5.2c)$$

$$= A_p(0) = I \quad k=0. \quad (2.5.2d)$$

Consider the order update of the backward PEF coefficient as

$$\mathbf{B}_p = \begin{bmatrix} \mathbf{B}_p(p) \\ \mathbf{B}_p(p-1) \\ \vdots \\ \mathbf{B}_p(1) \\ \mathbf{B}_p(0) \end{bmatrix} = \begin{bmatrix} [0] \\ \mathbf{B}_{p-1}(p-1) \\ \vdots \\ \mathbf{B}_{p-1}(1) \\ \mathbf{B}_{p-1}(0) \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{p-1}(0) \\ \mathbf{A}_{p-1}(1) \\ \vdots \\ \mathbf{A}_{p-1}(p-1) \\ [0] \end{bmatrix} [\Gamma_b]_p \quad (2.5.3)$$

where $[\Gamma_b]_p$ is the backward reflection coefficient matrix. The update equation is expressed as

$$\mathbf{B}_p(k) = \mathbf{B}_{p-1}(k) + \mathbf{A}_{p-1}(p-k)[\Gamma_b]_p \quad k=1,2,\dots,p-1 \quad (2.5.4a)$$

$$= [\Gamma_b]_p \quad k=p \quad (2.5.4b)$$

$$= [0] \quad k > p \quad (2.5.4c)$$

$$= \mathbf{B}_p(0) = \mathbf{I} \quad k=0. \quad (2.5.4d)$$

We now determine the conditions under which the above equations hold. First, we take the Hermitian transpose of eq(2.5.1) and postmultiply both sides of the equation by $[\tilde{\mathbf{R}}_{\mathbf{X}\mathbf{X}}]_{p+1}$. For the LHS, eq(2.4.13) provides

$$\Delta_p^H [\tilde{\mathbf{R}}_{\mathbf{X}\mathbf{X}}]_{p+1} \approx \{ [\Sigma_f]_p^H [0] \dots [0] \} \quad (2.5.5a)$$

where $[\Sigma_f]_p^H$ is the $J \times J$ forward error covariance matrix for a p th order filter.

Each side of eq(2.5.5a) is a vector of matrices containing $p+1$ matrix elements. Taking the Hermitian transpose of Δ_p in eq(2.5.1) and using it in (2.5.5a), we also have

$$\Delta_p^H [\tilde{\mathbf{R}}_{\mathbf{X}\mathbf{X}}]_{p+1} = \{ \Delta_{p-1}^H [0] \} [\tilde{\mathbf{R}}_{\mathbf{X}\mathbf{X}}]_{p+1} + [\Gamma_f]_p^H \{ [0] \mathbf{B}_{p-1}^H \} [\tilde{\mathbf{R}}_{\mathbf{X}\mathbf{X}}]_{p+1} \quad (2.5.5b)$$

so that

$$\{ [\Sigma_f]_p^H [0] \dots [0] \} = \{ \Delta_{p-1}^H [0] \} [\tilde{\mathbf{R}}_{\mathbf{X}\mathbf{X}}]_{p+1} + [\Gamma_f]_p^H \{ [0] \mathbf{B}_{p-1}^H \} [\tilde{\mathbf{R}}_{\mathbf{X}\mathbf{X}}]_{p+1} \quad (2.5.5c)$$

Let us now introduce a partitioned form of $[\tilde{R}_{xx}]_{p+1}$ such that

$$[\tilde{R}_{xx}]_{p+1} = \begin{bmatrix} R_{xx}(0) & R_{xx}(1) & R_{xx}(2) & \dots & R_{xx}(p) \\ R_{xx}(-1) & R_{xx}(0) & R_{xx}(1) & \dots & R_{xx}(p-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{xx}(1-p) & R_{xx}(2-p) & R_{xx}(3-p) & \dots & R_{xx}(1) \\ R_{xx}(-p) & R_{xx}(1-p) & R_{xx}(2-p) & \dots & R_{xx}(0) \end{bmatrix} = \begin{bmatrix} [\tilde{R}_{xx}]_p & (\underline{S}_{xx})_p \\ (\underline{S}_{xx})_p^H & [R_{xx}(0)] \end{bmatrix} \quad (2.5.6)$$

where $(\underline{S}_{xx})_p$ is a vector containing p matrix elements and is defined as

$$(\underline{S}_{xx})_p^H = [R_{xx}(-p), R_{xx}(1-p), \dots, R_{xx}(-1)]. \quad (2.5.7)$$

We now consider

$$\{\Delta_{p-1}^H [0]\} [\tilde{R}_{xx}]_{p+1} = \{\Delta_{p-1}^H [0]\} \begin{bmatrix} [\tilde{R}_{xx}]_p & (\underline{S}_{xx})_p \\ (\underline{S}_{xx})_p^H & [R_{xx}(0)] \end{bmatrix} \quad (2.5.8a)$$

$$= \{\Delta_{p-1}^H [\tilde{R}_{xx}]_p \quad \Delta_{p-1}^H (\underline{S}_{xx})_p\}. \quad (2.5.8b)$$

However, we can reduce the order of eq(2.5.5a) by one and obtain the Hermitian transposed vector which will now contain p matrix elements, so that

$$\Delta_{p-1}^H [\tilde{R}_{xx}]_p = \{[\Sigma_f]_{p-1}^H [0] \dots [0]\}. \quad (2.5.9)$$

Let us now define

$$[\Delta]_p^H \equiv \Delta_{p-1}^H (\underline{S}_{xx})_p = \{I \quad A_{p-1}^H(1) \quad A_{p-1}^H(2) \quad \dots \quad A_{p-1}^H(p-1)\} \begin{bmatrix} R_{xx}(p) \\ R_{xx}(p-1) \\ \vdots \\ R_{xx}(1) \end{bmatrix} \quad (2.5.10a)$$

$$= R_{xx}(p) + \sum_{k=1}^{p-1} A_{p-1}^H(k) R_{xx}(p-k). \quad (2.5.10b)$$

Substituting eqs(2.5.9) and (2.5.10a) in (2.5.8b), we have

$$\{\Delta_{p-1}^H [0]\} [\tilde{R}_{xx}]_{p+1} = \{[\Sigma_f]_{p-1}^H [0] \dots [0] [\Delta]_p^H\}. \quad (2.5.11)$$

We will utilize this expression later. We now repartitioned $[\tilde{R}_{xx}]_{p+1}$ into the form given by

$$[\tilde{R}_{xx}]_{p+1} = \begin{bmatrix} R_{xx}(0) & R_{xx}(1) & R_{xx}(2) & \dots & R_{xx}(p) \\ R_{xx}(-1) & R_{xx}(0) & R_{xx}(1) & \dots & R_{xx}(p-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{xx}(1-p) & R_{xx}(2-p) & R_{xx}(3-p) & \dots & R_{xx}(1) \\ R_{xx}(-p) & R_{xx}(1-p) & R_{xx}(2-p) & \dots & R_{xx}(0) \end{bmatrix} = \begin{bmatrix} R_{xx}(0) (R_{xx})_p^H \\ (R_{xx})_p [\tilde{R}_{xx}]_p \end{bmatrix} \quad (2.5.12)$$

where

$$(R_{xx})_p^H = [R_{xx}(1) R_{xx}(2) \dots R_{xx}(p)]. \quad (2.5.13)$$

We now consider

$$\{[0] \ B_{p-1}^H\} [\tilde{R}_{xx}]_{p+1} = \{[0] \ B_{p-1}^H\} \begin{bmatrix} R_{xx}(0) (R_{xx})_p^H \\ (R_{xx})_p [\tilde{R}_{xx}]_p \end{bmatrix} \quad (2.5.14a)$$

$$= \{B_{p-1}^H (R_{xx})_p \ B_{p-1}^H [\tilde{R}_{xx}]_p\}. \quad (2.5.14b)$$

However, reducing eq(2.4.49) by one order, we have

$$B_{p-1}^H [\tilde{R}_{xx}]_p = \{[0] \dots [0] [\Sigma_b]_{p-1}^H\}. \quad (2.5.15)$$

Let us now define

$$[\Delta']_p^H = B_{p-1}^H (R_{xx})_p \quad (2.5.16a)$$

$$= \{ B_{p-1}^H(p-1) B_{p-1}^H(p-2) \dots B_{p-1}^H(1) I \} \begin{bmatrix} R_{xx}(-1) \\ R_{xx}(-2) \\ \vdots \\ R_{xx}(-p) \end{bmatrix} \quad (2.5.16b)$$

$$= R_{xx}(-p) + \sum_{k=1}^{p-1} B_{p-1}^H(k) R_{xx}(k-p). \quad (2.5.16c)$$

Substituting eqs(2.5.15) and (2.5.16a) into (2.5.14b), yields

$$\{ [0] B_{p-1}^H \} [\tilde{R}_{xx}]_{p+1} = \{ [\Delta']_p^H [0] \dots [0] [\Sigma_b]_{p-1}^H \}. \quad (2.5.17)$$

Using eqs(2.5.11) and (2.5.17) in (2.5.5c), we obtain

$$\begin{aligned} \{ [\Sigma_f]_p^H [0] \dots [0] \} &= \{ [\Sigma_f]_{p-1}^H [0] \dots [0] [\Delta]_p^H \} \\ &\quad + [\Gamma_f]_p^H \{ [\Delta']_p^H [0] \dots [0] [\Sigma_b]_{p-1}^H \}. \end{aligned} \quad (2.5.18)$$

And so, in order for eq(2.5.1) to hold, we must satisfy eq(2.5.18). From eq(2.5.18), we have

$$[\Sigma_f]_p^H = [\Sigma_f]_{p-1}^H + [\Gamma_f]_p^H [\Delta']_p^H \quad (2.5.19)$$

and

$$[0] = [\Delta]_p^H + [\Gamma_f]_p^H [\Sigma_b]_{p-1}^H. \quad (2.5.20)$$

From eqs(2.5.20) and (2.5.2b), we have

$$[\Gamma_f]_p^H = A_p^H(p) = - [\Delta]_p^H \{ [\Sigma_b]_{p-1}^H \}^{-1}. \quad (2.5.21)$$

We must now return to eq(2.5.3) and carry out a similar procedure that was carried out for eq(2.5.1); ie., we post-multiply B_p^H by $[\tilde{R}_{xx}]_{p+1}$ so that

$$\underline{B}_p^H [\tilde{R}_{xx}]_{p+1} = \{ [0] \dots [0] [\Sigma_b]_p^H \}. \quad (2.5.22)$$

For the first term on the RHS of eq(2.5.3), we postmultiply by the partitioned form of eq(2.5.12). In this case, we obtain the same results as expressed in eqs(2.5.14a) through (2.5.17). For the second term on the RHS, we postmultiply by the partitioned form of eq(2.5.6) so that we obtain the same results as expressed in eqs(2.5.8) through (2.5.11). Substituting these results into eq(2.5.22), we have

$$\begin{aligned} \{ [0] \dots [0] [\Sigma_b]_p^H \} &= \{ [\Delta']_p^H [0] \dots [0] [\Sigma_b]_{p-1}^H \} \\ &+ [\Gamma_b]_p^H \{ [\Sigma_f]_{p-1}^H [0] \dots [0] [\Delta]_p^H \}. \end{aligned} \quad (2.5.23)$$

Eq(2.5.23) must therefore be maintained in order for the recursion in eq(2.5.3) to hold. From the above equation, we have

$$[\Sigma_b]_p^H = [\Sigma_b]_{p-1}^H + [\Gamma_b]_p^H [\Delta]_p^H \quad (2.5.24)$$

$$[0] = [\Delta']_p^H + [\Gamma_b]_p^H [\Sigma_f]_{p-1}^H. \quad (2.5.25)$$

From eqs(2.5.25) and (2.5.4b), we have

$$[\Gamma_b]_p^H = B_p^H(p) = - [\Delta']_p^H \{ [\Sigma_f]_{p-1}^H \}^{-1}. \quad (2.5.26)$$

Before summarizing the pertinent recursion equations used to estimate the forward and backward coefficient matrices, we can at this point develop several important relationships. Using equations (2.5.21) and (2.5.25) in (2.5.19), we obtain

$$[\Sigma_f]_p^H = [\Sigma_f]_{p-1}^H + [\Gamma_f]_p^H [\Delta']_p^H \quad (2.5.27a)$$

$$= [\Sigma_f]_{p-1}^H - A_p^H(p) [\Gamma_b]_p^H [\Sigma_f]_{p-1}^H. \quad (2.5.27b)$$

Using eq(2.5.26) in (2.5.27b), we have

$$[\Sigma_f]_p^H = [\Sigma_f]_{p-1}^H - A_p^H(p) B_p^H(p) [\Sigma_f]_{p-1}^H \quad (2.5.28a)$$

so that

$$[\Sigma_f]_p^H = \{I - A_p^H(p) B_p^H(p)\} [\Sigma_f]_{p-1}^H \quad (2.5.28b)$$

Similarly,

$$[\Sigma_b]_p^H = \{I - B_p^H(p) A_p^H(p)\} [\Sigma_b]_{p-1}^H \quad (2.5.29)$$

Let us also consider the matrix identity [17]*

$$\{\Delta_{p-1}^H \ [0]\} [\tilde{R}_{xx}]_{p+1} \begin{bmatrix} [0] \\ B_{p-1} \end{bmatrix} = \left[\{[0] \ B_{p-1}^H\} [\tilde{R}_{xx}]_{p+1} \begin{bmatrix} \Delta_{p-1} \\ [0] \end{bmatrix} \right]^H \quad (2.5.30)$$

which follows since $[\tilde{R}_{xx}]_{p+1} = [\tilde{R}_{xx}]_{p+1}^H$. Using eqs(2.5.11) and (2.5.17) in (2.5.30), we can write

$$\begin{aligned} & ([\Sigma_f]_{p-1}^H \ [0] \ \dots \ [0] \ [\Delta]_p^H) \begin{bmatrix} [0] \\ B_{p-1}(p-1) \\ \vdots \\ B_{p-1}(1) \\ I \end{bmatrix} = \\ & \left[\{[\Delta]_p^H \ [0] \ \dots \ [0] \ [\Sigma_b]_{p-1}^H\} \begin{bmatrix} I \\ A_{p-1}(1) \\ A_{p-1}(2) \\ \vdots \\ A_{p-1}(p-1) \\ [0] \end{bmatrix} \right]^H \end{aligned} \quad (2.5.31)$$

* We note the typographical error in eq(15.77) in ref [17].

so that after carrying out the matrix multiplications in eq(2.5.31), we have

$$[\Delta]_p^H = [\Delta']_p. \quad (2.5.32)$$

Eq(2.5.32) is useful since it reduces the computations involved in the determination of the reflection coefficient matrices; ie., eqs(2.5.21) and (2.5.26) can both be expressed in terms of $[\Delta]_p$, so that

$$[\Gamma_f]_p^H = A_p^H(p) = - [\Delta]_p^H \{ [\Sigma_b]_{p-1}^H \}^{-1} \quad (2.5.33a)$$

$$[\Gamma_b]_p^H = B_p^H(p) = - [\Delta]_p \{ [\Sigma_f]_{p-1}^H \}^{-1}. \quad (2.5.33b)$$

In the single channel case, the scalar forward and backward prediction error variances are equal. Since $[\Delta]_p$ is a scalar, it follows from eqs(2.5.33) that $a_p^*(p) = b_p(p)$.

We again note that the expressions developed here are in terms of the Hermitian operator as opposed to those often shown in the literature. This stems from the notation used to express the autoregressive equation in eq(2.2.1) and provides a more consistent notation as stated previously. We now summarize the pertinent equations for the recursion to update the forward and backward PEF coefficients. First, we have the definitions

$$[\Delta]_p^H \equiv R_{xx}(p) + \sum_{k=1}^{p-1} A_{p-1}^H(k) R_{xx}(p-k) \quad (2.5.34a)$$

$$[\Delta']_p^H \equiv R_{xx}(-p) + \sum_{k=1}^{p-1} B_{p-1}^H(k) R_{xx}(k-p). \quad (2.5.34b)$$

However, as shown previously

$$[\Delta]_p^H = [\Delta']_p. \quad (2.5.35)$$

Hence, we need use only one of the equations in eqs(2.5.34). Next, from eqs(2.5.33)

$$[\Gamma_f]_p^H = A_p^H(p) = -[\Delta]_p^H \{[\Sigma_b]_{p-1}^H\}^{-1} \quad (2.5.36a)$$

$$[\Gamma_b]_p^H = B_p^H(p) = -[\Delta]_p^H \{[\Sigma_f]_{p-1}^H\}^{-1} \quad (2.5.36b)$$

where

$$[\Sigma_f]_p^H = \{I - A_p^H(p) B_p^H(p)\} [\Sigma_f]_{p-1}^H. \quad (2.5.37a)$$

$$[\Sigma_b]_p^H = \{I - B_p^H(p) A_p^H(p)\} [\Sigma_b]_{p-1}^H. \quad (2.5.37b)$$

Then,

$$A_p^H(k) = \begin{cases} A_{p-1}^H(k) + [\Gamma_f]_p^H B_{p-1}^H(p-k) & k=1, 2, \dots, p-1 \\ [\Gamma_f]_p^H & k=p \end{cases} \quad (2.5.38a)$$

$$B_p^H(k) = \begin{cases} B_{p-1}^H(k) + [\Gamma_b]_p^H A_{p-1}^H(p-k) & k=1, 2, \dots, p-1 \\ [\Gamma_b]_p^H & k=p. \end{cases} \quad (2.5.38b)$$

We initiate the procedure using the initial conditions for eqs(2.5.37) as $[\Sigma_f]_0^H = [\Sigma_b]_0^H = R_{xx}(0)$ so that eqs(2.5.38) provide

$$A_1^H(1) = [\Gamma_f]_1^H = -[\Delta]_1^H \{[\Sigma_b]_0^H\}^{-1} \quad (2.5.39a)$$

$$= -R_{xx}(1)R_{xx}^{-1}(0) \quad (2.5.39b)$$

and

$$B_1^H(1) = [\Gamma_b]_1^H = -[\Delta]_1^H \{[\Sigma_f]_0^H\}^{-1} \quad (2.5.39c)$$

$$= -R_{xx}(-1)R_{xx}^{-1}(0). \quad (2.5.39d)$$

The procedure continues recursively and continues until the desired order is reached.

2.5.2. The Strand-Nuttall Method

The multichannel harmonic mean algorithm which is a generalization of the Burg method was proposed independently by Strand [8] and Nuttall [10]. In the single channel case, Burg [35] estimated the reflection coefficients directly from the data subject to the constraint of minimizing a performance index consisting of the mean-square values of the estimates of the forward and backward prediction errors. In this case, the scalar forward coefficients of linear prediction are the complex conjugates of the backward coefficients [87,126,38]. In the multichannel case, however, the forward and backward reflection coefficient matrices are related by eqs(2.5.33a) and (2.5.33b), respectively. These equations indicate that the multichannel forward and backward reflection coefficient matrices have a more complicated relationship than in the scalar case. Specifically, they are related through the forward and backward error covariance matrices. Solving eqs(2.5.33) for $-\Delta_p^H$ and noting that the error covariance matrices are Hermitian, we have

$$-\Delta_p^H = A_p^H(p) [\Sigma_b]_{p-1}^H = [\Sigma_f]_{p-1}^H B_p(p). \quad (2.5.40)$$

From eq(2.5.38) at $k=p$, we have

$$-\Delta_p^H = [\Gamma_f]_p^H [\Sigma_b]_{p-1}^H = [\Sigma_f]_{p-1}^H [\Gamma_b]_p. \quad (2.5.41)$$

These equations show the relationship between the forward and backward reflection coefficients and error covariance matrices as well as the matrix $[\Delta]_p^H$.

In the Strand-Nuttall algorithm, it is this latter matrix that we solve for. In eq(2.5.41), note that the error covariances can be obtained from stage $p-1$. Two additional well known relationships between the forward and backward error residuals can be expressed as [8,10,38]

$$\epsilon_p(n) = \epsilon_{p-1}(n) + A_p^H(p) \beta_{p-1}(n-1) \quad (2.5.42)$$

and

$$\beta_p(n) = \beta_{p-1}(n-1) + B_p^H(p) \varepsilon_{p-1}(n). \quad (2.5.43)$$

In this section, we will establish a performance function which depends upon estimates of the forward and backward error covariance matrices. This performance function will be minimized with respect to the matrix $[\Delta]_p^H$. In the multichannel case, the reflection coefficients are chosen such that wqs(2.5.41) through (2.5.43) are satisfied. These are the constraint equations under which $[\Delta]_p$ will be selected to minimize the performance function. Using the notation of Nuttall [10], we consider the unbiased error residual matrices over the available data for a filter of order p as

$$E_p \equiv \frac{1}{N-p} \sum_{k=p+1}^N \varepsilon_p(k) \varepsilon_p^H(k) = E_p^H \quad (2.5.44a)$$

$$F_p \equiv \frac{1}{N-p} \sum_{k=p+1}^N \beta_p(k) \beta_p^H(k) = F_p^H. \quad (2.5.44b)$$

Using eqs(2.5.42) and (2.5.43) in the above equations and expanding yields

$$E_p \equiv \frac{1}{N-p} \sum_{k=p+1}^N [\varepsilon_{p-1}(k) \varepsilon_{p-1}^H(k) + A_p^H(p) \beta_{p-1}(k-1) \varepsilon_{p-1}^H(k) + \varepsilon_{p-1}(k) \beta_{p-1}^H(k-1) A_p(p) + A_p^H(p) \beta_{p-1}(k-1) \beta_{p-1}^H(k-1) A_p(p)] \quad (2.5.45a)$$

and

$$F_p \equiv \frac{1}{N-p} \sum_{k=p+1}^N [\beta_{p-1}(k-1) \beta_{p-1}^H(k-1) + B_p^H(p) \varepsilon_{p-1}(k) \beta_{p-1}^H(k-1) + \beta_{p-1}(k-1) \varepsilon_{p-1}^H(k) B_p(p) + B_p^H(p) \varepsilon_{p-1}(k) \varepsilon_{p-1}^H(k) B_p(p)] \quad (2.5.45b)$$

so that

$$E_p = [\hat{R}_{ff}]_{p-1} + A_p^H(p) [\hat{R}_{fb}]_{p-1}^H + [\hat{R}_{fb}]_{p-1} A_p(p) + A_p^H(p) [\hat{R}_{bb}]_{p-1} A_p(p) \quad (2.5.46a)$$

$$F_p = [\hat{R}_{bb}]_{p-1} + B_p^H(p) [\hat{R}_{fb}]_{p-1} + [\hat{R}_{fb}]_{p-1}^H B_p(p) + B_p^H(p) [\hat{R}_{ff}]_{p-1} B_p(p) \quad (2.5.46b)$$

where the estimated time-averaged error covariance matrices are expressed as

$$[\hat{R}_{ff}]_{p-1} = \frac{1}{N-p} \sum_{k=p+1}^N \epsilon_{p-1}(k) \epsilon_{p-1}^H(k) \quad (2.5.47a)$$

$$[\hat{R}_{fb}]_{p-1} = \frac{1}{N-p} \sum_{k=p+1}^N \epsilon_{p-1}(k) \beta_{p-1}^H(k-1) \quad (2.5.47b)$$

$$[\hat{R}_{fb}]_{p-1}^H = \frac{1}{N-p} \sum_{k=p+1}^N \beta_{p-1}(k-1) \epsilon_{p-1}^H(k) \quad (2.5.47c)$$

$$[\hat{R}_{bb}]_{p-1} = \frac{1}{N-p} \sum_{k=p+1}^N \beta_{p-1}(k-1) \beta_{p-1}^H(k-1). \quad (2.5.47d)$$

In the Strand-Nuttall algorithm, we minimize the sum of the traces of the weighted error matrices

$$I_{sn}([\Delta]_p) = \text{tr}(\Lambda_{p-1} E_p) + \text{tr}(\Omega_{p-1} F_p) \quad (2.5.48a)$$

where Λ_{p-1} and Ω_{p-1} are Hermitian, positive definite matrices which weight the matrices E_p and F_p , respectively. These matrices provide arbitrary weights and their selection will be determined later. Eq(2.5.48a) is the performance function mentioned previously. In the single channel case, we can consider minimizing an index of performance $I_p(\Gamma_p)$ expressed here as the weighted sum of the mean-squared values of the forward and backward prediction errors $\epsilon_p(n)$ and $\beta_p(n)$, respectively; ie.,

$$I_p(\Gamma_p) = aE[|\epsilon_p(n)|^2] + (1-a)E[|\beta_p(n)|^2]. \quad (2.5.48b)$$

Burg considered the minimization of eq(2.5.48b) for the case where $a=1/2$. We will show that the minimization of eq(2.5.48a) as developed here, leads to a generalization of the results obtained by the minimization of eq(2.5.48b) with $a=1/2$. Using eq(2.5.40), let us define G_p^H such that

$$G_p^H = -[\Delta]_p^H = A_p^H(p) [\Sigma_b]_{p-1}^H = [\Sigma_f]_{p-1}^H B_p(p). \quad (2.5.49)$$

Changing notation for the Hermitian error covariance matrices to a form similar to Nuttall's [10], let

$$U_{p-1} \equiv [\Sigma_f]_{p-1} \quad (2.5.50a)$$

$$V_{p-1} \equiv [\Sigma_b]_{p-1} \quad (2.5.50b)$$

so that

$$A_p^H(p) = G_p^H [V_{p-1}^{-1}]^H \quad (2.5.51a)$$

$$B_p^H(p) = G_p [U_{p-1}^{-1}]^H. \quad (2.5.51b)$$

Using the last two equations together with eqs(2.5.46), we can write

$$\begin{aligned} \Lambda_{p-1} E_p + \Omega_{p-1} F_p = \Lambda_{p-1} \{ & [\hat{R}_{ff}]_{p-1} + G_p^H [V_{p-1}^{-1}]^H [\hat{R}_{fb}]_{p-1}^H + [\hat{R}_{fb}]_{p-1} [V_{p-1}^{-1}] G_p \\ & + G_p^H [V_{p-1}^{-1}]^H [\hat{R}_{bb}]_{p-1} [V_{p-1}^{-1}] G_p \} \\ \Omega_{p-1} \{ & [\hat{R}_{bb}]_{p-1} + G_p [U_{p-1}^{-1}]^H [\hat{R}_{fb}]_{p-1}^H + [\hat{R}_{fb}]_{p-1}^H [U_{p-1}^{-1}] G_p^H \\ & + G_p [U_{p-1}^{-1}]^H [\hat{R}_{ff}]_{p-1} [U_{p-1}^{-1}] G_p^H \}. \end{aligned} \quad (2.5.52)$$

Taking the trace of eq(2.5.52) and rearranging terms, we obtain

$$\begin{aligned}
\text{tr}[\Lambda_{p-1}E_p + \Omega_{p-1}F_p] &= \\
&= \text{tr} \left\{ \Lambda_{p-1}[\hat{R}_{ff}]_{p-1} + \Omega_{p-1}[\hat{R}_{bb}]_{p-1} + \Lambda_{p-1}G_p^H[V_{p-1}^{-1}]^H[\hat{R}_{fb}]_{p-1}^H \right. \\
&\quad \left. + \Omega_{p-1}[\hat{R}_{fb}]_{p-1}^H[U_{p-1}^{-1}]G_p^H \right\} \\
&+ \text{tr} \left\{ [\Lambda_{p-1}[\hat{R}_{fb}]_{p-1}[V_{p-1}^{-1}] + \Lambda_{p-1}G_p^H[V_{p-1}^{-1}]^H[\hat{R}_{bb}]_{p-1}[V_{p-1}^{-1}]]G_p \right. \\
&\quad \left. + \Omega_{p-1}G_p[U_{p-1}^{-1}]^H[\hat{R}_{fb}]_{p-1} + \Omega_{p-1}G_p[U_{p-1}^{-1}]^H[\hat{R}_{ff}]_{p-1}[U_{p-1}^{-1}]G_p^H \right\}.
\end{aligned} \tag{2.5.53}$$

We now consider the matrix relation

$$\text{tr}[KL] = \text{tr}[LK] \tag{2.5.54}$$

where K and L are conformable matrices. Noting that all the above matrices are $J \times J$, the last two terms in the second trace of eq(2.5.53) can be reexpressed (ie., with $K = \Omega_{p-1}G_p$ and L defined accordingly for each of these last two terms). G_p can then be factored out in the second trace term so that

$$\begin{aligned}
\text{tr}[\Lambda_{p-1}E_p + \Omega_{p-1}F_p] &= \\
&= \text{tr} \left\{ \Lambda_{p-1}[\hat{R}_{ff}]_{p-1} + \Omega_{p-1}[\hat{R}_{bb}]_{p-1} + \Lambda_{p-1}G_p^H[V_{p-1}^{-1}]^H[\hat{R}_{fb}]_{p-1}^H \right. \\
&\quad \left. + \Omega_{p-1}[\hat{R}_{fb}]_{p-1}^H[U_{p-1}^{-1}]G_p^H \right\} \\
&+ \text{tr} \left\{ [\Lambda_{p-1}[\hat{R}_{fb}]_{p-1}[V_{p-1}^{-1}] + \Lambda_{p-1}G_p^H[V_{p-1}^{-1}]^H[\hat{R}_{bb}]_{p-1}[V_{p-1}^{-1}]] \right. \\
&\quad \left. + [U_{p-1}^{-1}]^H[\hat{R}_{fb}]_{p-1}\Omega_{p-1} + [U_{p-1}^{-1}]^H[\hat{R}_{ff}]_{p-1}[U_{p-1}^{-1}]G_p^H\Omega_{p-1}]G_p \right\}.
\end{aligned} \tag{2.5.55}$$

Nuttall now minimizes eq(2.5.55) by choice of G_p^H (in the Hermitian notation used here), subject to eqs(2.5.41) through (2.5.43). Eq(2.5.49) can then be used to solve for the coefficients $A_p^H(p)$ and $B_p^H(p)$ which achieve this

minimization. Nuttall [10] points out that the above equation is minimized by setting the coefficient of G_p [contained in the large square brackets of the second trace term of eq(2.5.55)] equal to a null matrix [85]. Premultiplying this coefficient by $[\Lambda_{p-1}^{-1}]$ and post multiplying by $[\Omega_{p-1}^{-1}]$, we obtain

$$[\hat{R}_{fb}]_{p-1} [V_{p-1}^{-1}] [\Omega_{p-1}^{-1}] + G_p^H [V_{p-1}^{-1}]^H [\hat{R}_{bb}]_{p-1} [V_{p-1}^{-1}] \Omega_{p-1}^{-1} \\ + [\Lambda_{p-1}^{-1}] [U_{p-1}^{-1}]^H [\hat{R}_{fb}]_{p-1} + \Lambda_{p-1}^{-1} [U_{p-1}^{-1}]^H [\hat{R}_{ff}]_{p-1} [U_{p-1}^{-1}] G_p^H = [0] \quad (2.5.56a)$$

so that after rearranging terms

$$- G_p^H [V_{p-1}^{-1}]^H [\hat{R}_{bb}]_{p-1} [V_{p-1}^{-1}] \Omega_{p-1}^{-1} - \Lambda_{p-1}^{-1} [U_{p-1}^{-1}]^H [\hat{R}_{ff}]_{p-1} [U_{p-1}^{-1}] G_p^H = \\ = [\hat{R}_{fb}]_{p-1} [V_{p-1}^{-1}] [\Omega_{p-1}^{-1}] + [\Lambda_{p-1}^{-1}] [U_{p-1}^{-1}]^H [\hat{R}_{fb}]_{p-1} \cdot \quad (2.5.56b)$$

However, from eq(2.5.49)

$$G_p^H \equiv - [\Delta]_p^H \quad (2.5.57)$$

so that

$$[\hat{\Delta}]_p^H \mathcal{A}^H + \mathcal{B}^H [\hat{\Delta}]_p^H = \mathcal{C}^H \quad (2.5.58)$$

where $[\hat{\Delta}]_p^H$ is recognized as an estimate since limited data is being used in this equation and

$$\mathcal{A}^H = [V_{p-1}^{-1}]^H [\hat{R}_{bb}]_{p-1} [V_{p-1}^{-1}] \Omega_{p-1}^{-1} \quad (2.5.59a)$$

$$\mathcal{B}^H = \Lambda_{p-1}^{-1} [U_{p-1}^{-1}]^H [\hat{R}_{ff}]_{p-1} [U_{p-1}^{-1}] \quad (2.5.59b)$$

$$\mathcal{C}^H = [\hat{R}_{fb}]_{p-1} [V_{p-1}^{-1}] [\Omega_{p-1}^{-1}] + [\Lambda_{p-1}^{-1}] [U_{p-1}^{-1}]^H [\hat{R}_{fb}]_{p-1} \cdot \quad (2.5.59c)$$

In reference [10], Nuttall examined several forms of the weighting matrices Λ_{p-1} and Ω_{p-1} and showed that stable conditions are achieved if

$$\Lambda_{p-1} = U_{p-1}^{-1} \equiv [\Sigma_f]_{p-1}^{-1} \quad (2.5.60)$$

and

$$\Omega_{p-1} = V_{p-1}^{-1} \equiv [\Sigma_b]_{p-1}^{-1} . \quad (2.5.61)$$

Thus, the constant matrices $\mathcal{A}^H$, $\mathcal{B}^H$ and $\mathcal{C}^H$ simplify to

$$\mathcal{A}^H = [V_{p-1}^{-1}]^H [\hat{R}_{bb}]_{p-1} = [V_{p-1}^{-1}]^H [\hat{R}_{bb}]_{p-1}^H \quad (2.5.62a)$$

$$\mathcal{B}^H = [\hat{R}_{ff}]_{p-1} [U_{p-1}^{-1}] = [\hat{R}_{ff}]_{p-1}^H [U_{p-1}^{-1}]^H \quad (2.5.62b)$$

$$\mathcal{C}^H = 2[\hat{R}_{fb}]_{p-1} \quad (2.5.62c)$$

where $[\hat{R}_{bb}]_{p-1}$, $[\hat{R}_{ff}]_{p-1}$ and $[U_{p-1}^{-1}]$ are Hermitian. We note that $\mathcal{A}^H$ and $\mathcal{B}^H$ each contain the product of two matrices. The matrices $[U_{p-1}^{-1}]$ and $[V_{p-1}^{-1}]$ are the forward and backward error variances for a filter of order $p-1$, respectively. They are obtained from the Levinson-Wiggins-Robinson recursion. The matrices $[\hat{R}_{ff}]_{p-1}$ and $[\hat{R}_{bb}]_{p-1}$, however, are time-averaged estimates of these variances using limited data [see eqs(2.5.47)]. Using eqs(2.5.62), eq(2.5.58) is now written as

$$[\hat{\Delta}]_p^H [V_{p-1}^{-1}]^H [\hat{R}_{bb}]_{p-1}^H + [\hat{R}_{ff}]_{p-1}^H [U_{p-1}^{-1}]^H [\hat{\Delta}]_p^H = 2[\hat{R}_{fb}]_{p-1} . \quad (2.5.63)$$

In the single channel case, each element is a scalar where

$$U_{p-1} = V_{p-1} \equiv \sigma_{p-1}^2 \quad (2.5.64)$$

and $A_p^H(p) = [\Gamma_f]_p^H = -[\hat{\Delta}]_p^H [V_{p-1}^{-1}]^H$ becomes

$$a_p^*(p) = \hat{\Gamma}_p^* = -\frac{\hat{\Delta}_p^*}{\sigma_{p-1}}. \quad (2.5.65)$$

From eqs(2.5.47), the terms $[\hat{R}_{bb}]_{p-1}$, $[\hat{R}_{ff}]_{p-1}$ and $[\hat{R}_{fb}]_{p-1}$ for the scalar case can be expressed as

$$[\hat{R}_{ff}]_{p-1} = [\hat{R}_{ff}]_{p-1}^* = \frac{1}{N-p} \sum_{k=p+1}^N |\epsilon_{p-1}(k)|^2 \quad (2.5.66a)$$

$$[\hat{R}_{fb}]_{p-1} = \frac{1}{N-p} \sum_{k=p+1}^N \epsilon_{p-1}(k) \beta_{p-1}^*(k-1) \quad (2.5.66b)$$

$$[\hat{R}_{bb}]_{p-1} = [\hat{R}_{bb}]_{p-1}^H = \frac{1}{N-p} \sum_{k=p+1}^N |\beta_{p-1}(k-1)|^2. \quad (2.5.66c)$$

Using eqs(2.5.64) through (2.5.66), the scalar version of eq(2.5.63) can be written as

$$-a_p^*(p) \left[\sum_{k=p+1}^N |\beta_{p-1}(k-1)|^2 \right] - \left[\sum_{k=p+1}^N |\epsilon_{p-1}(k)|^2 \right] a_p^*(p) = 2 \left[\sum_{k=p+1}^N \epsilon_{p-1}(k) \beta_{p-1}^*(k-1) \right]. \quad (2.5.67)$$

Solving eq(2.5.67) for $a_p^*(p)$, we have

$$a_p^*(p) = \hat{\Gamma}_p^* = -\frac{2 \sum_{k=p+1}^N [\epsilon_{p-1}(k) \beta_{p-1}^*(k-1)]}{\sum_{k=p+1}^N [|\epsilon_{p-1}(k)|^2 + |\beta_{p-1}(k-1)|^2]}. \quad (2.5.68)$$

The result expressed in eq(2.5.68) is the estimate of the single channel Burg reflection coefficient. This result is equivalent to that obtained by minimizing eq(2.5.48b) with respect to Γ_p [35]. Thus the results presented here are a generalization of the single channel Burg algorithm.

Before summarizing the pertinent equations in this algorithm, we will first consider a solution to eq(2.5.58) to determine $[\hat{\Delta}]$. Taking the Hermitian of both sides of eq(2.5.58), we have

$$\mathcal{A} [\hat{\Delta}]_p + [\hat{\Delta}]_p \mathcal{B} = \mathcal{C}. \quad (2.5.69)$$

Nuttall [10] considered a solution of this equation in the form

$$[\hat{\Delta}]_p = P Q^{-1} \quad (2.5.70)$$

where

$$P = \sum_{k=0}^{J-1} (-1)^k \mathcal{A}_k \mathcal{C} \mathcal{B}^{J-1-k} \quad (2.5.71a)$$

and

$$Q = \sum_{k=0}^J (-1)^k \alpha_k \mathcal{B}^{J-k} \quad (2.5.71b)$$

where P and Q are JxJ matrices, the scalar constants $\{\alpha_k\}$ $k=0,1,\dots,J$ are given by

$$\alpha_k = -\frac{1}{k} \text{tr}(\mathcal{A} \mathcal{A}_{k-1}) \quad 1 \leq k \leq J \quad (2.5.72a)$$

$$= 1 \quad k = 0 \quad (2.5.72b)$$

and the matrices $\{\mathcal{A}_k\}$ satisfy the recurrence relationship

$$\mathcal{A}_k = \mathcal{A} \mathcal{A}_{k-1} + \alpha_k \mathcal{I} \quad 1 \leq k \leq J \quad (2.5.73a)$$

$$= \mathcal{I} \quad k = 0. \quad (2.5.73b)$$

For the two channel case (ie., $J=2$), we have (see Appendix B)

$$P = \mathcal{C} \mathcal{B} - [\mathcal{A} - \text{tr}(\mathcal{A}) \mathcal{I}] \mathcal{C} \quad (2.5.74a)$$

and

$$Q = \mathcal{B} \mathcal{B} + \text{tr}(\mathcal{A}) \mathcal{B} - \frac{1}{2} \text{tr}\{\mathcal{A} [\mathcal{A} - \text{tr}(\mathcal{A}) \mathcal{I}]\} \mathcal{I}. \quad (2.5.74b)$$

Eq(2.5.74b) can be further simplified by considering the matrix characteristic equation

$$|sI - \mathbf{B}| = 0 \quad (2.5.75)$$

so that

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right| = \begin{vmatrix} s-b_{11} & -b_{12} \\ -b_{21} & s-b_{22} \end{vmatrix} = 0 \quad (2.5.76a)$$

or

$$s^2 - (b_{11} + b_{22})s + (b_{11}b_{22} - b_{21}b_{12}) = 0. \quad (2.5.76b)$$

From the Caley-Hamilton theorem, we can substitute $s=\mathbf{B}$, so that

$$\mathbf{B}^2 - (\text{tr}\mathbf{B})\mathbf{B} + (\det\mathbf{B})\mathbf{I} = [0] \quad (2.5.77a)$$

or

$$\mathbf{B}^2 = (\text{tr}\mathbf{B})\mathbf{B} - (\det\mathbf{B})\mathbf{I}. \quad (2.5.77b)$$

Using this relation in eq(2.5.74b), we have

$$\mathbf{Q} = (\text{tr}\mathbf{B})\mathbf{B} - (\det\mathbf{B})\mathbf{I} + \text{tr}(\mathbf{A})\mathbf{B} - \frac{1}{2} \text{tr}(\mathbf{A}^2)\mathbf{I} + \frac{1}{2} \text{tr}\{\mathbf{A} \text{tr}(\mathbf{A})\}\mathbf{I}. \quad (2.5.78)$$

However, using eq(2.5.77b) with $\mathbf{B} = \mathbf{A}$, we also have

$$\mathbf{A}^2 = \text{tr}(\mathbf{A})\mathbf{A} - (\det\mathbf{A})\mathbf{I} \quad (2.5.79a)$$

so that

$$\text{tr}(\mathbf{A}^2) = \text{tr}[\text{tr}(\mathbf{A})\mathbf{A} - (\det\mathbf{A})\mathbf{I}]\mathbf{I} \quad (2.5.79b)$$

$$= [(\text{tr}\mathbf{A})^2 - \text{tr}(\det\mathbf{A})]\mathbf{I} \quad (2.5.79c)$$

And so,

$$\mathbf{Q} = [\text{tr}\mathbf{A} + \text{tr}\mathbf{B}]\mathbf{B} - (\det\mathbf{B})\mathbf{I} - \frac{1}{2} \text{tr}[\text{tr}(\mathbf{A})\mathbf{A} - (\det\mathbf{A})\mathbf{I}]\mathbf{I} + \frac{1}{2} \text{tr}\{\mathbf{A} \text{tr}(\mathbf{A})\}\mathbf{I}. \quad (2.5.80)$$

But, using the relations

$$\text{tr}[(\det\mathbf{A})\mathbf{I}] = J(\det\mathbf{A})\mathbf{I} = (2\det\mathbf{A}) \quad (2.5.81a)$$

and

$$\text{tr}\{\text{tr}(\mathcal{A})\mathcal{A}\} = \text{tr}\{\mathcal{A}\text{tr}(\mathcal{A})\} \quad (2.5.81b)$$

in eq(2.5.80), we obtain

$$Q = [\text{tr}\mathcal{A} + \text{tr}\mathcal{B}]\mathcal{B} - (\det\mathcal{B})I + (\det\mathcal{A})I. \quad (2.5.82)$$

And so, using eqs(2.5.74a) and (2.5.82), eq(2.5.70) becomes

$$[\hat{\Delta}]_p = PQ^{-1} \quad (2.5.83a)$$

$$= \{\mathcal{C}\mathcal{B} - [\mathcal{A} - \text{tr}\mathcal{A}I]\mathcal{C}\} \{[\text{tr}\mathcal{A} + \text{tr}\mathcal{B}]\mathcal{B} + [\det\mathcal{A} - \det\mathcal{B}]I\}^{-1} \quad (2.5.83b)$$

which is the result expressed by eq(162) of ref [10].

We are not aware of a reference for the solution of the J=3 case. In Appendix B, however, we also show that for J=3

$$P = [\text{tr}\mathcal{A} + \text{tr}\mathcal{B}]\mathcal{C}\mathcal{B} + \left[\frac{1}{2}(\det\mathcal{A}) - (\det\mathcal{B})\right]\mathcal{C} - \mathcal{A}\mathcal{C}\mathcal{B} \quad (2.5.84a)$$

and

$$Q = [\text{tr}\mathcal{A} + \text{tr}\mathcal{B}][(\text{tr}\mathcal{B})\mathcal{B} - (\det\mathcal{B})I] - (\det\mathcal{B})\mathcal{B} + \frac{3}{2}(\det\mathcal{A})\mathcal{B} + \frac{1}{2}(\text{tr}\mathcal{A})(\det\mathcal{A})I \\ - (\det\mathcal{A})I - \frac{1}{2}(\text{tr}\mathcal{A})^2I + \frac{1}{2}[\text{tr}\mathcal{A}]^3I \quad (2.5.84b)$$

Using eqs(2.5.84) in (2.5.70)

$$[\hat{\Delta}]_p = PQ^{-1} \quad (2.5.85a)$$

$$= \left\{ [\text{tr}\mathcal{A} + \text{tr}\mathcal{B}]\mathcal{C}\mathcal{B} + \left[\frac{1}{2}(\det\mathcal{A}) - (\det\mathcal{B})\right]\mathcal{C} - \mathcal{A}\mathcal{C}\mathcal{B} \right\} \\ \cdot \left\{ [\text{tr}\mathcal{A} + \text{tr}\mathcal{B}][(\text{tr}\mathcal{B})\mathcal{B} - (\det\mathcal{B})I] - (\det\mathcal{B})\mathcal{B} + \frac{3}{2}(\det\mathcal{A})\mathcal{B} + \frac{1}{2}(\text{tr}\mathcal{A})(\det\mathcal{A})I - (\det\mathcal{A})I - \frac{1}{2}(\text{tr}\mathcal{A})^2I + \frac{1}{2}[\text{tr}\mathcal{A}]^3I \right\}^{-1}. \quad (2.5.85b)$$

We now summarize the pertinent equations in the Strand-Nuttall algorithm. The initialization begins with eqs(2.5.44) expressed as

$$E_0 = F_0 = \frac{1}{N} \sum_{k=1}^N \mathbf{x}_0(k) \mathbf{x}_0^H(k) = \frac{1}{N} \sum_{k=1}^N \mathbf{b}_0(k) \mathbf{b}_0^H(k) \quad (2.5.86a)$$

$$= \frac{1}{N} \sum_{k=1}^N \mathbf{x}(k) \mathbf{x}^H(k). \quad (2.5.86b)$$

The estimated time-averaged error covariance matrices of eqs(2.5.47) are now updated using

$$[\hat{\mathbf{R}}_{ff}]_{p-1} = \frac{1}{N-p} \sum_{k=p+1}^N \mathbf{x}_{p-1}(k) \mathbf{x}_{p-1}^H(k) \quad (2.5.87a)$$

$$[\hat{\mathbf{R}}_{fb}]_{p-1} = \frac{1}{N-p} \sum_{k=p+1}^N \mathbf{x}_{p-1}(k) \mathbf{b}_{p-1}^H(k-1) \quad (2.5.87b)$$

$$[\hat{\mathbf{R}}_{bb}]_{p-1} = \frac{1}{N-p} \sum_{k=p+1}^N \mathbf{b}_{p-1}(k-1) \mathbf{b}_{p-1}^H(k-1). \quad (2.5.87c)$$

We now determine $[\Sigma_f]_{p-1}$ and $[\Sigma_b]_{p-1}$ using eqs(2.5.37) from the Levinson-Wiggins-Robinson recursion. Their inverses provide the weights Λ_{p-1} and Ω_{p-1} as noted by eqs(2.5.60) and (2.5.61); ie.,

$$\Lambda_{p-1} = \mathbf{U}_{p-1}^{-1} \equiv [\Sigma_f]_{p-1}^{-1} \quad (2.5.88a)$$

and

$$\Omega_{p-1} = \mathbf{V}_{p-1}^{-1} \equiv [\Sigma_b]_{p-1}^{-1}. \quad (2.5.88b)$$

The partial correlation matrix denoted as $[\hat{\Delta}]_p^H$ can now be determined using eqs(2.5.58) and (2.5.59); ie.,

$$[\hat{\Delta}]_p^H \mathbf{A}^H + \mathbf{B}^H [\hat{\Delta}]_p^H = \mathbf{C}^H \quad (2.5.89)$$

where again $[\hat{\Delta}]_p^H$ is recognized as an estimate since limited data is being used and

$$\mathbf{A}^H = [\mathbf{V}_{p-1}^{-1}]^H [\hat{\mathbf{R}}_{bb}]_{p-1} [\mathbf{V}_{p-1}^{-1}] \boldsymbol{\Omega}_{p-1}^{-1} \quad (2.5.90a)$$

$$\mathbf{B}^H = \boldsymbol{\Lambda}_{p-1}^{-1} [\mathbf{U}_{p-1}^{-1}]^H [\hat{\mathbf{R}}_{ff}]_{p-1} [\mathbf{U}_{p-1}^{-1}] \quad (2.5.90b)$$

$$\mathbf{C}^H = [\hat{\mathbf{R}}_{fb}]_{p-1} [\mathbf{V}_{p-1}^{-1}] [\boldsymbol{\Omega}_{p-1}^{-1}] + [\boldsymbol{\Lambda}_{p-1}^{-1}] [\mathbf{U}_{p-1}^{-1}]^H [\hat{\mathbf{R}}_{fb}]_{p-1} . \quad (2.5.90c)$$

From eqs(2.5.35) and (2.5.36), we have

$$[\Delta]_p^H = [\Delta']_p \quad (2.5.91a)$$

$$[\Gamma_f]_p^H = \mathbf{A}_p^H(p) = - [\Delta]_p^H \{ [\boldsymbol{\Sigma}_b]_{p-1}^H \}^{-1} \quad (2.5.91b)$$

$$[\Gamma_b]_p^H = \mathbf{B}_p^H(p) = - [\Delta]_p \{ [\boldsymbol{\Sigma}_f]_{p-1}^H \}^{-1} . \quad (2.5.91c)$$

We can now update the forward and backward error covariance matrices using eqs(2.5.37) where

$$[\boldsymbol{\Sigma}_f]_p^H = \{ \mathbf{I} - \mathbf{A}_p^H(p) \mathbf{B}_p^H(p) \} [\boldsymbol{\Sigma}_f]_{p-1}^H \quad (2.5.92a)$$

$$[\boldsymbol{\Sigma}_b]_p^H = \{ \mathbf{I} - \mathbf{B}_p^H(p) \mathbf{A}_p^H(p) \} [\boldsymbol{\Sigma}_b]_{p-1}^H \quad (2.5.92b)$$

Eqs(2.5.42) and (2.5.43) are now used to obtain the update for $\mathbf{g}_p(n)$ and $\mathbf{\beta}_p(n)$; ie.,

$$\mathbf{g}_p(n) = \mathbf{g}_{p-1}(n) + \mathbf{A}_p^H(p) \mathbf{\beta}_{p-1}(n-1) \quad (2.5.93a)$$

and

$$\mathbf{\beta}_p(n) = \mathbf{\beta}_{p-1}(n-1) + \mathbf{B}_p^H(p) \mathbf{g}_{p-1}(n). \quad (2.5.93b)$$

In this manner, the recursion is continued until the desired order is reached.

In this chapter, we have provided a brief mathematical description for several important linear prediction error filters (PEF). These include the multichannel forward and backward prediction error filters which utilize a

transversal (tapped delay line) structure as well as the multichannel lattice filter. As noted previously, these filtering operations are used in the implementation architectures for the multichannel likelihood ratios discussed in chapter 6. We also presented a preliminary discussion of the complex processes which we consider in this dissertation. These include the autoregressive (AR) processes which provide a parametric description of the processes. Finally, we presented a derivation of the Levinson-Wiggins-Robinson and the Strand-Nuttall algorithms including a summary of the specific equations used in the recursion procedure. In this dissertation, the Strand-Nuttall algorithm is used to provide the estimates of the AR process parameters which in turn are used to set the weights in the prediction error filters described in chapter 6. Whereas, the Levinson-Wiggins-Robinson algorithm is used in a process synthesis procedure described in chapter 4.

CHAPTER 3 THE MULTICHANNEL BINARY DETECTION PROBLEM

In this chapter, we first define the multichannel detection problem using complex Gaussian baseband processes. In chapter 6, we will derive likelihood ratios for this problem. Second, we provide a description of the associated bandpass processes and discuss the conditions under which these processes are stationary. The motivation for this description is due to the fact that in chapter 6, we develop likelihood ratios which are applicable for a more general class of Gaussian processes for which the bandpass processes are non-stationary.

3.1 DISCUSSION OF THE DETECTION PROBLEM FOR BASEBAND PROCESSES

In this section, we define the multichannel binary detection problem in which the discrete-time received baseband observations are

$$H_0: \quad \underline{x}(n) = \underline{c}(n) \quad \underline{z}(n) \equiv \underline{d}(n) \quad n = 1, 2, \dots, N, \quad (3.1.1a)$$

$$H_1: \quad \underline{x}(n) = \underline{s}(n) + \underline{c}(n) + \underline{w}(n) \quad n = 1, 2, \dots, N \quad (3.1.1b)$$

where $\underline{x}(n)$ is a complex Gaussian, zero mean, wide-sense jointly stationary $J \times 1$ received observation vector consisting of observations from J channels and is defined as

$$\underline{x}^T(n) = [x_1(n) \ x_2(n) \ \dots \ x_J(n)] \quad (3.1.2)$$

where $x_j(n)$ $j = 1, 2, \dots, J$ is a scalar observation on channel j at time integer n . The detection problem, therefore, is to select between hypotheses H_0 and H_1 using the available observation processes $\underline{x}(n)$.

The vectors $\underline{s}(n)$, $\underline{c}(n)$ and $\underline{w}(n)$ are zero mean, complex Gaussian random $J \times 1$ vector processes describing the signal, non-white noise (clutter) and white noise, respectively. H_1 and H_0 denote the hypotheses under which the signal is present or absent, respectively. The vector $\underline{d}(n)$ denotes the total disturbance process consisting of the clutter and additive white noise. The vector processes

$\underline{s}(n)$ and $\underline{c}(n)$ each contain an arbitrary correlation in time and between channels. In addition, correlation can be considered between $\underline{s}(n)$ and past values of $\underline{c}(n)$ and $\underline{w}(n)$. In feedback communication and control problems, the signal may be affected by past signal as well as current and past noise, so that [21]

$$E[\underline{s}(n)\underline{w}^H(k)] = \begin{cases} [0] & n < k \\ R_{sw}(n, k) & n \geq k. \end{cases} \quad (3.1.3)$$

In the radar problem (or active sonar), the signal may be affected by the presence of multiple reflections from the background environment termed multipath (or reverberation) [40]. Thus, in general, the signal contains correlation with current and past values of the clutter such that

$$E[\underline{s}(n)\underline{c}^H(k)] = \begin{cases} [0] & n < k \\ R_{sc}(n, k) & n \geq k. \end{cases} \quad (3.1.4)$$

Finally, the vector $\underline{w}(n)$ is uncorrelated with itself in time, but not necessarily across channels, so that

$$E[\underline{w}(n)\underline{w}^H(k)] = \begin{cases} [0] & n \neq k \\ R_{ww}(0) & n = k \end{cases} \quad (3.1.5)$$

where $R_{ww}(0)$ is the $J \times J$ correlation matrix of $\underline{w}(n)$ specifying the correlation across channels.

Next, we introduce the quadrature form of $\underline{x}(n)$ and expressed it as

$$\underline{x}(n) = \underline{x}_I(n) + j \underline{x}_Q(n) \quad (3.1.6)$$

where $\underline{x}_I(n)$ and $\underline{x}_Q(n)$ are the in-phase and quadrature components of $\underline{x}(n)$, respectively, and

$$\underline{x}_I^T(n) = [x_{1I}(n) \ x_{2I}(n) \ \dots \ x_{JI}(n)] \quad (3.1.7a)$$

$$\underline{x}_Q^T(n) = [x_{1Q}(n) \ x_{2Q}(n) \ \dots \ x_{JQ}(n)]. \quad (3.1.7b)$$

As noted above, the detection problem is the selection between hypotheses H_0 and H_1 using the available observation processes $\underline{x}(n)$. As noted in chapter 2, we assume that the processes $\underline{x}(n)$ have a parametric model description such as the autoregressive (AR) or autoregressive moving-average (ARMA) models. In our detection procedures, we utilize the filtering methods and estimation algorithms discussed in chapter 2 to obtain prediction error vectors $\underline{\epsilon}_p(n)$, $n=1,2,\dots,N$. These error vectors have a causally invertible relationship with and are therefore statistically equivalent to the vectors $\underline{x}(n)$. In chapter 6, we develop likelihood ratios which are computed using these error vectors and compared to a threshold level Y_b in order to determine the hypothesis. A block diagram of the detection and estimation analysis procedure used in this dissertation is shown in Fig. 3.1. As noted above, we assume that the baseband processes are wide-sense jointly-stationary. In Appendices N and O, we show that the condition of wide-sense joint stationarity on the narrowband bandpass processes provides specific relationships between the auto- and cross-correlation sequences of the in-phase and quadrature components $\underline{x}_I(n)$ and $\underline{x}_Q(n)$, respectively. Thus, a more general class of processes results (for which the corresponding bandpass processes are non-stationary) when we have a more general correlation between the quadrature components of the complex process $\underline{x}(n)$. In section 6.3, we develop likelihood ratios for the more general class of Gaussian processes where the corresponding bandpass processes are non-stationary.

3.2 DESCRIPTION OF BASEBAND AND BANDPASS PROCESSES

The baseband processes are obtained by processing a bandpass process of spectral width B positioned about some carrier frequency f_c . In this dissertation, we will primarily consider B to be small with respect to f_c . However, a generalization to the case where this assumption is relaxed will also be made. We denote the i th channel continuous-time bandpass process at time t as

$$x_{i_{BP}}(t) = \text{Re}[x_i(t) \exp(j\omega_c t)] \quad (3.2.1)$$

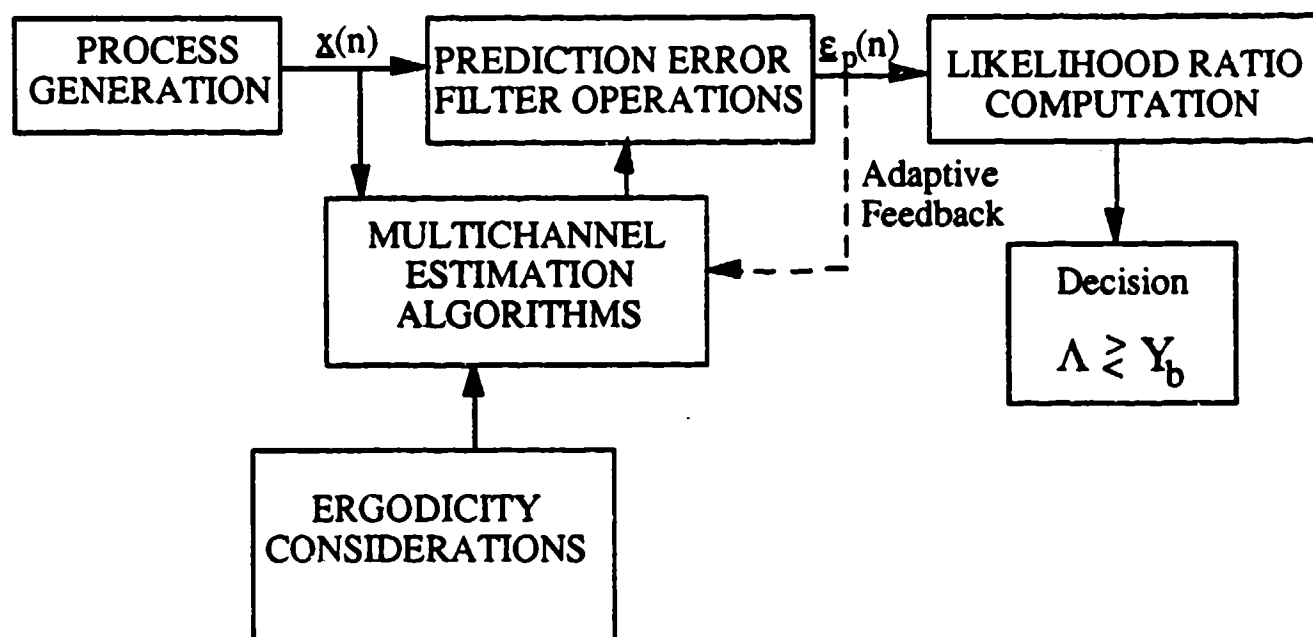


Figure 3.1 Block Diagram of the Detection and Estimation Analysis Approach

where the angular frequency $\omega_c = 2\pi f_c$. Since $x_i(t)$ is, in general, complex, we can express it as

$$x_i(t) = x_{iI}(t) + j x_{iQ}(t) \quad (3.2.2)$$

where $x_{iI}(t)$ and $x_{iQ}(t)$ are the in-phase and quadrature components of $x_i(t)$, respectively, both with respect to the carrier frequency f_c . We can now express eq(3.2.1) as

$$x_{iBP}(t) = \text{Re} \{ [x_{iI}(t) + j x_{iQ}(t)] \exp(j\omega_c t) \} \quad (3.2.3a)$$

$$= x_{iI}(t) \cos(\omega_c t) - x_{iQ}(t) \sin(\omega_c t). \quad (3.2.3b)$$

Alternatively, we can also express $x_i(t)$ in polar form such that

$$x_i(t) = A_i(t) \exp[j\theta_i(t)] \quad (3.2.4)$$

where the amplitude $A_i(t)$ is expressed as

$$A_i(t) = \sqrt{x_{iI}^2(t) + x_{iQ}^2(t)} \quad (3.2.5a)$$

and

$$\theta_i(t) = \tan^{-1}[x_{iQ}(t)/x_{iI}(t)]. \quad (3.2.5b)$$

The baseband process $x_i(t)$ contains all the information about the process. The discrete-time form of the quadrature components expressed in eq(3.2.2) can be obtained using the quadrature carrier multiplexing network depicted in Fig. 3.2. The low-pass filters are implemented to remove double frequency terms and each filter has a bandwidth $B/2$ for noise reduction. Finally, a sampler is used to obtain the discrete-time quadrature components $x_{iI}(n)$ and $x_{iQ}(n)$.

It is also useful to consider the real concatenated $2J \times 1$ observation vector process $\mathbf{r}(n)$ consisting of the quadrature components of $\mathbf{x}(n)$; ie.,

$$\mathbf{r}(n) = \begin{bmatrix} \mathbf{x}_I(n) \\ \mathbf{x}_Q(n) \end{bmatrix} \quad (3.2.6)$$

where the quadrature components are defined in eqs(3.1.7). The correlation matrix for this vector is

$$\mathbf{R}_x(l) = E[\mathbf{r}(n)\mathbf{r}^T(n-l)] \quad (3.2.7a)$$

$$= E \left\{ \begin{bmatrix} \mathbf{x}_I(n) \\ \mathbf{x}_Q(n) \end{bmatrix} \begin{bmatrix} \mathbf{x}_I^T(n-l) & \mathbf{x}_Q^T(n-l) \end{bmatrix} \right\} \quad (3.2.7b)$$

$$= \begin{bmatrix} R_x^{II}(l) & R_x^{IQ}(l) \\ R_x^{QI}(l) & R_x^{QQ}(l) \end{bmatrix}. \quad (3.2.7c)$$

The $J \times J$ block correlation matrices in eq(3.2.7c) are expressed as

$$R_x^{AB}(l) = E[\mathbf{x}_A(n)\mathbf{x}_B^T(n-l)] \quad A=I,Q \quad B=I,Q. \quad (3.2.8)$$

We define the 1×1 autocorrelation ($i=k$ and $A=B$) and cross-correlation ($i \neq k$ and/or $A \neq B$) sequences of the quadrature components as

$$R_{x_{ik}}^{AB}(l) = E[x_{iA}(n)x_{kB}^T(n-l)] \quad A=I,Q \quad B=I,Q \quad \text{all } i,k=1,2,\dots,J \quad (3.2.9)$$

where the subscripts i and k denote the channels.

In Appendices N and O, it is shown that the narrowband multichannel bandpass processes corresponding to these baseband processes are jointly wide-sense stationary processes when the quadrature components satisfy the specific constraints

$$R_{x_{ik}}^{II}(l) = R_{x_{ik}}^{QQ}(l) \quad \text{all } i, k=1,2,\dots,J \quad (3.2.10a)$$

and

$$R_{x_{ik}}^{IQ}(l) = -R_{x_{ik}}^{QI}(l) \quad \text{all } i, k=1,2,\dots,J. \quad (3.2.10b)$$

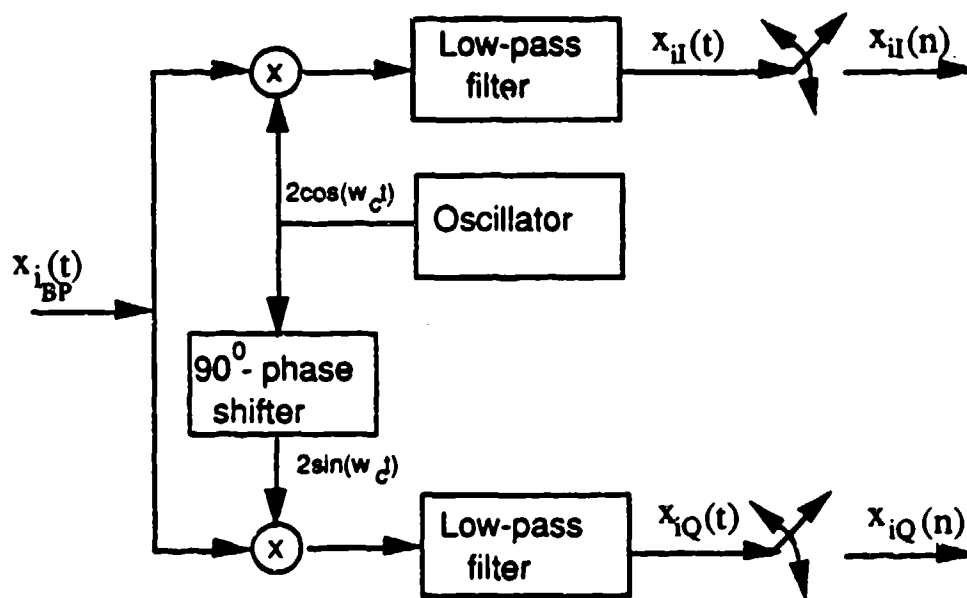


Figure 3.2 Quadrature Carrier Multiplexing Network

The condition of wide-sense stationarity on the baseband processes is assumed throughout this dissertation. In general, however, these specific correlation relationships among the quadrature components do not necessarily hold. Thus, the bandpass processes may not be jointly wide-sense stationary. In chapter 6, we develop a likelihood ratio not only for the special case described by eqs(3.2.10), but also for the case of the more generally correlated observation processes.

We now consider the complex auto- and cross-correlation functions and develop some useful properties of these functions for the special case given by eqs(3.2.10). For jointly wide-sense stationary processes, the complex cross-correlation ($i \neq k$) and autocorrelation ($i = k$) sequences are expressed (using the complex conjugate operation $*$) as

$$R_{x_{ik}}(l) = E[x_i(n)x_k^*(n-l)] \quad (3.2.11a)$$

$$= E\{[x_{iI}(n) + j x_{iQ}(n)][x_{kI}(n-l) - j x_{kQ}(n-l)]\} \quad (3.2.11b)$$

$$= \{E[x_{iI}(n)x_{kI}(n-l)] + E[x_{iQ}(n)x_{kQ}(n-l)]\} \\ + j\{E[x_{iQ}(n)x_{kI}(n-l)] - E[x_{iI}(n)x_{kQ}(n-l)]\} \quad (3.2.11c)$$

$$= \{R_{x_{ik}}^{II}(l) + R_{x_{ik}}^{QQ}(l)\} + j\{R_{x_{ik}}^{QI}(l) - R_{x_{ik}}^{IQ}(l)\} \quad (3.2.11d)$$

$$= R_{xC_{ik}}(l) + j R_{xD_{ik}}(l) \quad (3.2.11e)$$

where

$$R_{xC_{ik}}(l) = R_{x_{ik}}^{II}(l) + R_{x_{ik}}^{QQ}(l) \quad (3.2.12a)$$

$$R_{xD_{ik}}(l) = R_{x_{ik}}^{QI}(l) - R_{x_{ik}}^{IQ}(l). \quad (3.2.12b)$$

For wide-sense stationary baseband processes, we have the property that

$$R_{x_{ik}}(l) = R_{x_{ki}}^*(-l). \quad (3.2.13)$$

We will now consider the case of the observation processes on a single channel. In this case, we set $k=i$. Substituting eq(3.2.11e) with $k=i$ into eq(3.2.13), we obtain

$$R_{xC_{ii}}(l) + j R_{xD_{ii}}(l) = R_{xC_{ii}}(-l) - j R_{xD_{ii}}(-l) \quad (3.2.14)$$

so that

$$R_{xC_{ii}}(l) = R_{xC_{ii}}(-l) \quad (3.2.15a)$$

and

$$R_{xD_{ii}}(l) = -R_{xD_{ii}}(-l). \quad (3.2.15b)$$

Thus, $R_{xC_{ii}}(l)$ and $R_{xD_{ii}}(l)$ are even and odd, respectively. The property stated in eq(3.2.13) also holds for the autocorrelation sequences of the in-phase and quadrature components, so that

$$R_{x_{ii}}^{II}(l) = [R_{x_{ii}}^{II}(-l)]^* = R_{x_{ii}}^{II}(-l) \quad (3.2.16a)$$

and

$$R_{x_{ii}}^{QQ}(l) = [R_{x_{ii}}^{QQ}(-l)]^* = R_{x_{ii}}^{QQ}(-l) \quad (3.2.16b)$$

which indicates that these sequences are even. At this point, no conclusion can be made regarding the even or odd nature of the cross-correlation sequences of the in-phase and quadrature components. However, expressing these sequences in terms of their even and odd components, we obtain

$$R_{x_{ii}}^{QI}(l) = [R_{x_{ii}}^{QI}(l)]_{\text{even}} + [R_{x_{ii}}^{QI}(l)]_{\text{odd}}. \quad (3.2.17)$$

Noting that $R_{x_{ii}}^{IQ}(l) = R_{x_{ii}}^{QI}(-l)$ and using eq(3.2.3), it follows that

$$R_{x_{ii}}^{IQ}(l) = [R_{x_{ii}}^{QI}(-l)] = [R_{x_{ii}}^{QI}(-l)]_{\text{even}} + [R_{x_{ii}}^{QI}(-l)]_{\text{odd}}. \quad (3.2.18)$$

Applying the property of even and odd functions to the RHS of eq(3.2.18), however, we obtain

$$R_{x_{ii}}^{IQ}(l) = [R_{x_{ii}}^{QI}(l)]_{\text{even}} - [R_{x_{ii}}^{QI}(l)]_{\text{odd}} . \quad (3.2.19)$$

Solving for $[R_{x_{ii}}^{QI}(l)]_{\text{odd}}$ in eq(3.2.17) and substituting into eq(3.2.19), we have

$$R_{x_{ii}}^{IQ}(l) = [R_{x_{ii}}^{QI}(l)]_{\text{even}} - \{ R_{x_{ii}}^{QI}(l) - [R_{x_{ii}}^{QI}(l)]_{\text{even}} \} \quad (3.2.20a)$$

$$= - R_{x_{ii}}^{QI}(l) + 2[R_{x_{ii}}^{QI}(l)]_{\text{even}} . \quad (3.2.20b)$$

A similar equation can be obtained by solving for $[R_{x_{ii}}^{QI}(l)]_{\text{even}}$ in eq(3.2.17) and using it in (3.2.19), so that

$$R_{x_{ii}}^{IQ}(l) = R_{x_{ii}}^{QI}(l) - 2[R_{x_{ii}}^{QI}(l)]_{\text{odd}} . \quad (3.2.21)$$

These last two equations show the explicit dependence of $R_{x_{ii}}^{IQ}(l)$ on the even and odd nature of $R_{x_{ii}}^{QI}(l)$. When eq(3.2.10b) holds, so that $R_{x_{ii}}^{IQ}(l) = -R_{x_{ii}}^{QI}(l)$, it follows that $[R_{x_{ii}}^{QI}(l)]_{\text{odd}}$ must equal $R_{x_{ii}}^{QI}(l)$. Thus, $R_{x_{ii}}^{QI}(l)$ is odd for this special case.

In this chapter, we have defined the detection problem addresses in this dissertation. In addition, we have introduced and discussed the complex observation processes as well as their quadrature components. A general overview of the detection analysis approach was then presented. Finally, both the baseband and bandpass processes were described. Of particular note, were the constraints involving the correlation of the quadrature components leading to the condition of wide-sense joint stationarity of the bandpass processes. This condition is described in detail in Appendices N and O.

CHAPTER 4 SYNTHESIS OF MULTICHANNEL AUTOREGRESSIVE PROCESSES

4.1 INTRODUCTION

The modeling of multichannel time series has been given considerable attention since the classic work of Wiener [30]. However, as noted in [31], the utility of such models is dependent upon their suitability for the intended investigation. The objective here is the development of a multichannel time series methodology for the synthesis of multichannel autoregressive (AR) processes. The synthesized processes will be employed as observation data processes for the simulation and performance evaluation of multichannel detection and estimation problems. These problems involve vector observation processes containing random signals in additive noise disturbances such as clutter, reverberation and thermal white noise. It is, therefore, essential that we tailor our modeling procedure to control process parameters such as temporal and cross-channel correlation, the process variance, the channel signal-to-noise ratio (SNR) and clutter-to-noise ratio (CNR). In the evaluation of multichannel detection and estimation methods [2,127], the procedures described in this chapter provide a means to assess performance capability in terms of a receiver operating characteristic (ROC) by allowing for variations in these parameters.

In the method presented here, complex auto- and cross-channel correlation functions are described in terms of a function of process characteristics such as temporal and cross-channel correlation coefficients as well as the process variances. Necessary constraints for realizability of the correlation functions are provided. These functional forms are used in the multichannel Yule-Walker equation to solve for the matrix coefficients of the AR process via the Levinson-Wiggins-Robinson recursion algorithm [6]. Utilizing the coefficients in the multichannel AR linear difference equation, the vector processes are generated. Thus, the resulting processes provide a 'fit' to the parameterized correlation functions in a minimum mean squared error (MMSE) sense. A procedure is also described which enables the generation of processes consisting of sums of AR processes in additive white noise. Finally, a method is proposed to synthesize complex processes with jointly Gaussian quadrature components with more general auto- and cross-correlation. The resulting processes represent more general complex process where the associated bandpass processes are non-

stationary. Results are presented for two-channel, vector processes using several values of temporal/cross-channel correlation parameters and process variances as well as their estimated, time-averaged auto- and cross-channel correlation functions.

4.2 DEFINITION OF THE VECTOR PROCESS

We again consider the discrete complex Gaussian vector process $\underline{x}(n)$ presented in section 2.1 such that

$$\underline{x}(n) = \underline{s}(n) + \underline{c}(n) + \underline{w}(n) \quad n = 1, 2, \dots, N \quad (4.2.1a)$$

$$= \underline{s}(n) + \underline{d}(n) \quad n = 1, 2, \dots, N \quad (4.2.1b)$$

where $\underline{x}(n)$ is a discrete, zero-mean, wide-sense jointly stationary $J \times 1$ received baseband observation vector consisting of observations from J channels and $\underline{s}(n)$, $\underline{c}(n)$ and $\underline{w}(n)$ are zero mean, complex Gaussian $J \times 1$ vector processes describing the signal, non-white interference (clutter) and white noise, respectively. The vector $\underline{d}(n)$ is the summation of the clutter $\underline{c}(n)$ and white noise $\underline{w}(n)$ vectors and is referred to as the disturbance vector. In the synthesis procedure presented here, these vectors are assumed to be uncorrelated with each other. However, in section 8.2.6, a method is suggested to correlate $\underline{s}(n)$ with past values of $\underline{c}(n)$ and $\underline{w}(n)$ to model physical conditions such as feedback, radar multi-path or sonar reverberation. The vector $\underline{w}(n)$ is uncorrelated in time, but not necessarily across channels. The vector processes $\underline{s}(n)$ and $\underline{c}(n)$, however, contain an arbitrary correlation both in time and between channels.

As noted previously, the vector processes $\underline{s}(n)$ and $\underline{c}(n)$ are generated as multichannel autoregressive processes; ie., the multichannel $J \times 1$ vector process $\underline{g}(n)$ with $g=s, c$ is expressed as

$$\underline{g}(n) = - \sum_{k=1}^{M_g} \underline{A}_g^H(k) \underline{g}(n-k) + \underline{u}_g(n) \quad g = s, c \quad (4.2.2)$$

where $\underline{A}_g^H(k)$ is the k th $J \times J$ matrix coefficient for AR process $\underline{g}(n)$ of model order M_g and $\underline{u}_g(n)$ is a $J \times 1$ forward white noise driving vector with covariance matrix

$[\Sigma_f]_{u_g}$. We express $A_g^H(k)$ in terms of the Hermitian operation for notational convenience, but here it is not constrained to be a Hermitian matrix. Using a Cholesky or LDL^H decomposition of $[\Sigma_f]_{u_g}$, the vector $u_g(n)$ can be expressed, respectively, as

$$u_g(n) = C_{v_g} y_g(n) \quad g = s, c \quad (4.2.3a)$$

or

$$u_g(n) = L_{v_g} z_g(n) \quad g = s, c, \quad (4.2.3b)$$

where $y_g(n)$ and $z_g(n)$ are $J \times 1$ white noise vectors with the $J \times J$ identity covariance matrix I and diagonal $J \times J$ covariance matrix D_{v_g} , respectively. Finally, we can also obtain cross-channel correlation on the additive white noise vector process $w(n)$ using

$$w(n) = C_w y_w(n) \quad (4.2.4)$$

where $y_w(n)$ is a $J \times 1$ white noise vector and the $J \times J$ matrix C_w specifies the cross channel correlation.

4.3. CORRELATION FUNCTION SHAPING APPROACH

4.3.1 General Development

Consider the autocorrelation function $R_{g_{ii}}(l)$ and cross-correlation function $R_{g_{ij}}(l)$ for the i th and j th channel processes. We now consider modeling $R_{g_{ii}}(l)$ and $R_{g_{ij}}(l)$ with functional forms that will enable us to obtain a variety of spectral distributions. We express both of these functions in a unified form as

$$R_{g_{ij}}(l) = E[g_i(n)g_j^*(n-l)] \quad (4.3.1a)$$

$$= K_{g_{ij}} f_g(\lambda_{g_{ij}}, l - l_{g_{ij}}) \exp\{j\theta_{g_{ij}}(l)\} \quad g=s, c. \quad (4.3.1b)$$

The quantity $\lambda_{g_{ij}}$ is defined as the temporal cross-correlation parameter for $i \neq j$ and the temporal autocorrelation parameter for $i=j$. It provides a measure of the correlation between discrete samples of the process on a given channel ($i=j$) or between channels ($i \neq j$); $l_{g_{ij}}$ is the lag value at which the corresponding real function $f_g(\cdot)$ has a peak value of unity. We note that the cross-correlation function does not necessarily peak at lag zero as the autocorrelation function does; i.e. for $i=j$, $l_{g_{ii}} = 0$. The function $f_g(\cdot)$ is selected to specify the shape of the correlation function magnitude and will be considered below. The constant $K_{g_{ij}}$ is a real, normalizing coefficient. Normalizing eq(4.3.1b) so that $R_{g_{ij}}(0) = (\rho_{g_{ij}})\sigma_{g_{ii}}\sigma_{g_{jj}}$ enables us to obtain (see App A, Q or ref [3])

$$R_{g_{ij}}(l) = \frac{(\rho_{g_{ij}})\sigma_{g_{ii}}\sigma_{g_{jj}}f_g(\lambda_{g_{ij}}, l - l_{g_{ij}})}{f_g(\lambda_{g_{ij}}, l - l_{g_{ij}})|_{l=0}} \exp\{j[\theta_{g_{ij}}(l) - \theta_{g_{ij}}(0)]\} \quad g=s,c \quad (4.3.2a)$$

$$= \frac{|\rho_{g_{ij}}|\sigma_{g_{ii}}\sigma_{g_{jj}}f_g(\lambda_{g_{ij}}, l - l_{g_{ij}})}{f_g(\lambda_{g_{ij}}, l - l_{g_{ij}})|_{l=0}} \exp\{j[\theta_{g_{ij}}(l)]\} \quad g = s,c \quad (4.3.2b)$$

where, by definition [3],

$$\rho_{g_{ij}} = |\rho_{g_{ij}}| \exp[j\theta_{g_{ij}}(0)] \quad g = s,c. \quad (4.3.2c)$$

Expressions for $R_{g_{ji}}(l)$ are obtained from eq(4.3.2b) using the property

$$R_{g_{ji}}(l) = R_{g_{ij}}^*(-l). \quad (4.3.3)$$

These equations provide us with a useful description of the cross-correlation function in terms of the cross-correlation coefficient $\rho_{g_{ij}}$, the standard deviations $\sigma_{g_{ii}}$ and $\sigma_{g_{jj}}$ of the channel i and j processes, respectively, and the temporal cross-correlation parameter, $\lambda_{g_{ij}}$. For the autocorrelation function ($i=j$), we have $|\rho_{g_{ii}}|=1$ since any given channel process is totally correlated with

itself at zero lag. Also, $\theta_{g_{ii}}(0) = 0$ since $\theta_{g_{ij}}(l)$ is an odd function of l . Since the function $f_g(\cdot)$ for the autocorrelation function has a peak value of unity at $l=0$ and again noting that $l_{g_{ii}} = 0$, eq(4.3.2b) for $i=j$ reduces to

$$R_{g_{ii}}(l) = \sigma_{g_{ii}}^2 f_g(\lambda_{g_{ii}}, l) \exp\{j\theta_{g_{ii}}(l)\} \quad g=s,c. \quad (4.3.4)$$

At $l = 0$, eq.(4.3.4) becomes $R_{g_{ii}}(0) = \sigma_{g_{ii}}^2$ which is, as expected, the variance of the zero mean, channel i process.

In the above discussion, we have proposed the use of functional forms to characterize the magnitude and phase of the correlation functions. The motivation for this approach is that it will allow for flexibility in modeling random processes with various correlation and spectral shape. We note, however that at this point we have not constrained these functions to meet all the criteria that are necessary and sufficient to characterize correlation functions. In fact, determining all of these conditions in a general formulation is a difficult task. In section 4.4.1, we note several constraints for the correlation functions including the important condition of positive semi-definiteness of the correlation matrix. A more complete discussion is presented in [3].

EXAMPLE Gaussian Shaped Correlation Functions

In [3], the special cases of Gaussian, exponential and sinc shaped correlation functions are considered using the forms given in eqs(4.3.2b) and (4.3.4). For cross-correlation functions with Gaussian shaped magnitudes, for example, we use

$$f_g(\lambda_{g_{ij}}, l - l_{g_{ij}}) = (\lambda_{g_{ij}})^{(l-l_{g_{ij}})^2} \quad g = s,c \quad (4.3.5)$$

so that eq(4.3.2b) becomes

$$R_{g_{ij}}(l) = \frac{|\rho_{g_{ij}}| \sigma_{g_{ii}} \sigma_{g_{jj}} (\lambda_{g_{ij}})^{(l-l_{g_{ij}})^2}}{(\lambda_{g_{ij}})^{l_{g_{ij}}^2}} \exp\{j\theta_{g_{ij}}(l)\} \quad g = s,c. \quad (4.3.6)$$

For the autocorrelation function ($i=j$) with Gaussian shaped magnitudes, we have (dropping the subscript i for notational convenience)

$$R_g(l) = \sigma_g^2 f_g(\lambda_g, l) \exp[j\theta_g(l)] = \sigma_g^2 (\lambda_g)^{l^2} \exp[j\theta_g(l)] \quad g=s,c \quad (4.3.7)$$

where

$$f_g(\lambda_g, l) = (\lambda_g)^{l^2} = \exp[-2\pi^2 \mu_g^2 T^2 l^2] \quad g = s, c \quad (4.3.8a)$$

and

$$\lambda_g = \exp[-2\pi^2 \mu_g^2 T^2] \quad g = s, c \quad (4.3.8b)$$

is a real constant such that $0 \leq \lambda_g \leq 1$ and T is the sample period. In Appendix R and ref [3], we show that μ_g^2 with $g=s, c$ is the variance (which determines the width) of the corresponding Gaussian shaped spectra. If these expressions are normalized by the white noise variance σ_w^2 , we can obtain

$$r_s(l) = R_s(l) / \sigma_w^2 = (S/R) f_s(\lambda_s, l) \exp[j\theta_s(l)] \quad (4.3.9a)$$

and, assuming $c(n)$ and $w(n)$ to be uncorrelated,

$$r_d(l) = r_c(l) + r_w(l) \quad (4.3.9b)$$

$$= R_c(l) / \sigma_w^2 + R_w(l) / \sigma_w^2 \quad (4.3.9c)$$

$$= (C/R) f_c(\lambda_c, l) \exp[j\theta_c(l)] + \delta(l) \quad (4.3.9d)$$

where $S/R = \sigma_s^2 / \sigma_w^2$ and $C/R = \sigma_c^2 / \sigma_w^2$. When $c(n)$ and $w(n)$ are uncorrelated, we also have $\sigma_d^2 = \sigma_c^2 + \sigma_w^2$.

4.3.2 Correlation Functions for Doppler Shifted Processes

For certain investigations such as radar and sonar applications, the expression for the correlation functions can be modified using a linear phase shift term to explicitly account for a Doppler center frequency. In this case, eqs.(4.3.9a) and (4.3.9d) can be expressed as

$$r_s(l) = (S/R) f_s(\lambda_s, l) \exp[j\theta_s(l)] \exp[j2\pi v_s l T] \quad (4.3.10a)$$

and

$$r_d(l) = (C/R) f_c(\lambda_c, l) \exp[j\theta_c(l)] \exp[j2\pi v_c l T] + \delta(l) \quad (4.3.10b)$$

where v_s and v_c are the signal and clutter Doppler center frequencies, respectively. Eqs(4.3.10a) and (4.3.10b) are generalizations of the forms suggested in [23] for real autocorrelation functions to the complex form.

For a baseband signal $g_i(n)$ on channel i with a Doppler shifted frequency v_{g_i} , we can express this process as

$$g_i(n) = \{g_{iI}(n) + jg_{iQ}(n)\} \exp[j2\pi v_{g_i} nT] \quad (4.3.11)$$

where T is the pulse repetition period in radar applications. The cross-correlation function between $g_i(n)$ and $g_j(n)$ is therefore

$$R_{g_{ij}}(n, n-l) = E[g_i(n) g_j^*(n-l)] \quad (4.3.12a)$$

$$= E \{ [g_{iI}(n) + jg_{iQ}(n)] [g_{jI}(n-l) - jg_{jQ}(n-l)] \} \\ \cdot \exp[j2\pi v_{g_i} nT] \exp[-j2\pi v_{g_j} (n-l)T] \quad (4.3.12b)$$

$$= E \{ [g_{iI}(n)g_{jI}(n-l)] + [g_{iQ}(n)g_{jQ}(n-l)] \\ + j[g_{iQ}(n)g_{jI}(n-l)] - j[g_{iI}(n)g_{jQ}(n-l)] \} \\ \cdot \exp[j2\pi v_{g_i} nT] \exp[-j2\pi v_{g_j} (n-l)T]. \quad (4.3.12c)$$

Due to the wide-sense joint-stationary properties of the quadrature components, $R_{g_{ij}}(n, n-l)$ can be expressed as

$$R_{g_{ij}}(n, n-l) = [R_{gC_{ij}}(l) + jR_{gD_{ij}}(l)] \exp \{ j2\pi [(v_{g_i} - v_{g_j})nT + v_{g_j} lT] \} \quad (4.3.13)$$

where $R_{gC_{ij}}(l)$ and $R_{gD_{ij}}(l)$ are defined in eqs (3.2.12). We note that this cross-correlation function is a function of l and nT for $v_{g_i} \neq v_{g_j}$ and therefore, the processes $g_i(n)$ and $g_j(n)$ are not jointly wide-sense stationary. This situation would result, for example, when processing data from two or more radar systems operating at different center frequencies. However, since the time dependent term is deterministic, it can be removed in the pre-processing. This is achieved by selection of a reference channel and frequency multiplying the other channel signals so that the resulting Dopplers are all equal to that of the reference. The proper frequency multiplying terms are obtained as follows. First, consider the Doppler frequency on channel i expressed as

$$v_{g_i} = \frac{2v v_{oi}}{c} \quad (4.3.14)$$

where v is the radial velocity of the radar target, v_{oi} is the channel i carrier frequency and c is the velocity of light. We therefore have the relation between channels i and j , such that

$$\frac{v_{g_i}}{v_{oi}} = \frac{v_{g_j}}{v_{oj}} = \frac{2v}{c}. \quad (4.3.15)$$

If we select channel i as the reference channel so that $v_{g_i} = v_{g_R}$ and $v_{oi} = v_{oR}$ where v_{g_R} is the Doppler on the reference channel and v_{oR} is the reference channel carrier frequency, we have

$$v_{g_R} = \left[\frac{v_{oR}}{v_{oj}} \right] v_{g_j} \quad j = 1, 2, \dots, J. \quad (4.3.16)$$

And so, if the carrier frequency in the oscillator of Fig. 3.2 for each channel j is frequency multiplied by its appropriate factor (v_{oR}/v_{oj}) prior to multiplexing, then, all of the Dopplers will be equal to v_{g_R} and eq(4.3.13) reduces to

$$R_{g_{ij}}(l) = [R_{A_{ij}}^g(l) + jR_{B_{ij}}^g(l)]\exp[j2\pi v_{g_R} l T]. \quad (4.3.17)$$

This result depends only on lag l and therefore satisfies the wide-sense stationarity condition. The preprocessing proposed here, of course, would be performed subsequent to processing stages which might utilize the raw Doppler information contained on each channel.

In the process synthesis procedure, we will make use of eqs(4.3.2). After translating the dopplers to the reference frequency, the functional expression corresponding to eq(4.3.2b) is

$$R_{g_{ij}}(l) = \frac{|\rho_{g_{ij}}| \sigma_{g_{ij}} \sigma_{g_{ij}} f_g(\lambda_{g_{ij}}, l - l_{g_{ij}})}{f_g(\lambda_{g_{ij}}, l - l_{g_{ij}})|_{l=0}} \exp\{j[\theta_{g_{ij}}(l)]\} \exp(j2\pi v_{g_R} l T). \quad (4.3.18)$$

4.4. CONSTRAINTS FOR CORRELATION FUNCTIONS

In this section, we discuss constraints imposed on the parameters of the functional forms presented in the previous sections to ensure that these functions yield valid correlation functions.

4.4.1. General Constraints

Several important constraints can be developed for stationary complex processes in a straightforward manner by generalizing a discussion noted in [18]. Specifically, with the real constant α , we consider

$$E[|x_i(n+l) + \alpha x_j(n)|^2] = E[(x_i(n+l) + \alpha x_j(n))(x_i^*(n+l) + \alpha x_j^*(n))] \quad (4.4.1a)$$

$$= R_{ii}(0) + \alpha[R_{ij}(l) + R_{ji}(-l)] + \alpha^2 R_{jj}(0) \quad (4.4.1b)$$

$$= R_{ii}(0) + \alpha[R_{ij}(l) + R_{ij}^*(l)] + \alpha^2 R_{jj}(0) \quad (4.4.1c)$$

$$= R_{ii}(0) + 2\alpha \text{Re}(R_{ij}(l)) + \alpha^2 R_{jj}(0). \quad (4.4.1d)$$

The above quadratic is nonnegative for any α ; therefore, its discriminant is nonpositive so that from eq(4.4.1d)

$$[\operatorname{Re}\{R_{ij}(l)\}]^2 \leq R_{ii}(0)R_{jj}(0). \quad (4.4.2a)$$

Likewise, interchanging i and j in eq. (4.4.1a), we obtain

$$[\operatorname{Re}\{R_{ji}(l)\}]^2 \leq R_{ii}(0)R_{jj}(0). \quad (4.4.2b)$$

Alternative expressions can be obtained from eq.(4.4.1b) such that

$$[R_{ij}(l) + R_{ji}(-l)]^2 \leq 4R_{ii}(0)R_{jj}(0). \quad (4.4.3)$$

Using eq(4.3.6) for the Gaussian shaped cross-correlation function in eq(4.4.3), we can derive [see Appendix C] the constraint

$$|\rho_{g_{ij}}| \leq \frac{(\lambda_{g_{ij}})^2 |g_{ij}|^2}{|\cos[\theta_{g_{ij}}(l_{g_{ij}})]|}. \quad (4.4.4)$$

The result expressed by eq(4.4.4) provides a constraint for $|\rho_{g_{ij}}|$ in terms of an upper bound. It represents one of several conditions that must be satisfied by complex correlation functions.

4.4.2. Positive Semi-Definiteness Condition

The correlation sequences developed in eqs(4.3.2) and (4.3.4) are used to determine the elements of the multichannel correlation matrix, $[R_{\mathbf{x}}]_N$. This matrix must satisfy the condition of positive semi-definiteness; i.e. a matrix for which all eigenvalues are non-negative. Alternatively, all subminor matrices of the $JN \times JN$ matrix $[R_{\mathbf{x}}]_N$ have non-negative determinants [70,71] leading to $2^{JN}-1$ constraints for the terms $\rho_{g_{ij}}$, $\lambda_{g_{ij}}$, $\lambda_{g_{ii}}$, $\sigma_{g_{ii}}^2$, and $l_{g_{ij}}$. These are discussed further in [3].

It should be noted that the constraints on the model parameters discussed in this section provide necessary conditions but are not in themselves sufficient. The procedure for the synthesis of multichannel A/D processors adopted in this

dissertation is to utilize eq(4.4.4) and then verify that the resulting matrix is positive semi-definite. Investigation of additional constraints is suggested for future consideration.

4.5. SYNTHESIS PROCEDURE

In the single channel radar problem, efforts have been made to model radar clutter with AR processes of relatively low order [16,79,80]. The extension of these analyses to the multichannel case remains an open area of research. It has also been noted [36] that the sum of two AR processes or the sum of an AR process with additive white noise y_i is an autoregressive moving-average (ARMA) process. In this dissertation, we assume the applicability of the AR model in the representation of both signal and non-white noise (clutter) processes. Since we will also be considering these processes and the white noise processes in an additive manner, we are therefore also concerned with the ARMA process.

In this section, we discuss the synthesis procedure used to generate wide-sense stationary, multichannel autoregressive (AR) processes. Since the observation process under the two hypotheses H_i , $i=0,1$ will have different model order, the processes under the hypotheses H_i , $i=0,1$ will be considered separately. In chapter 7, these synthesized processes will be used as observation processes for a Monte-Carlo analysis in the performance evaluation of our multichannel detection algorithm.

In the process synthesis procedure, we could elect to characterize the observation processes $\mathbf{x}(n|H_i)$ as multichannel AR processes under hypotheses H_0 and H_1 such that

$$\mathbf{x}(n|H_i) = \sum_{k=1}^{M_i} \mathbf{A}_{M_i}^H(k|H_i) \mathbf{x}(n-k) + \mathbf{u}(n|H_i) \quad i = 0,1 \quad (4.5.1)$$

where M_i , $\mathbf{A}_{M_i}^H(k|H_i)$ and $\mathbf{u}(n|H_i)$ denote the model order, the matrix coefficients of the AR process and the white Gaussian noise driving term under each hypothesis, respectively. Eq (4.5.1) could be utilized to generate the processes under each hypothesis using coefficients which would have to be pre-determined in some ad-hoc manner in order to vary the correlation of the synthesized processes. Furthermore, this approach does not allow for individual control of

the signal and clutter processes; ie., we have no direct control of the signal and clutter temporal and cross-channel correlation as well as variations of the signal-to-noise (S/N) and/or signal-to-clutter (S/C) ratios. Thus, this approach is not considered further in this dissertation.

In the approach used here, the $J \times 1$ vectors $\underline{g}(n)$ and $\underline{c}(n)$ defined in eqs(3.1.1) are generated as distinct, multichannel AR processes for the signal and non-white noise, respectively. It is interesting to note, however, that each element of these vectors will be an ARMA process [10, also section 8.2.1]. The two processes $\underline{g}(n)$ and $\underline{c}(n)$ are controlled individually using the scheme shown in Fig. 4.5.1. This scheme is somewhat similar to that suggested in [23] for scalar process synthesis. In this work, however, the synthesis procedure is generalized to consider multichannel vector processes in which we are able to control the variance and temporal correlation on each channel, the cross-channel correlation, the time lag $l_{g_{ij}}$ between the processes as well as the individual channel signal-to-noise (SNR) and clutter-to-noise (CNR) ratios. In subsection 8.2.6, a method is suggested to synthesize a multichannel AR process containing delayed feedback or radar multi-path. The functions $H_g(z)$ $g=s,c$ are the linear model filter transfer functions for the synthesis of the signal and clutter processes, respectively. They serve to temporally correlate $\underline{g}(n)$ and $\underline{c}(n)$ while the matrices C_s , C_c and C_w provide the cross-channel correlation. Since $\underline{v}_g(n)$, $g=s,s,w$ is Gaussian, and recognizing that $\underline{x}(n|H_i)$ $i=0,1$ are obtained using linear operations, they also are Gaussian.

In section 4.3, we discussed a method to express the magnitude and phase of auto- and cross-correlation functions with functional forms. A specific example of a Gaussian shaped magnitude function was considered. Once these correlation functions have been specified to obtain a desired spectral shape, the Yule-Walker equation can be used to determine the AR coefficients for the process with a model order chosen to fit the desired spectrum within certain tolerance specifications.* In the procedure used here, correlation functions for the signal and non-white noise are specified separately in terms of $R_{g_{ij}}(l)$ and $R_{c_{ij}}(l)$, respectively, for $i,j=1,2,\dots,J$. The autoregressive power spectral density

* Determination of the model order is beyond the scope of this work. Here, it is assumed that the model order is known or pre-specified

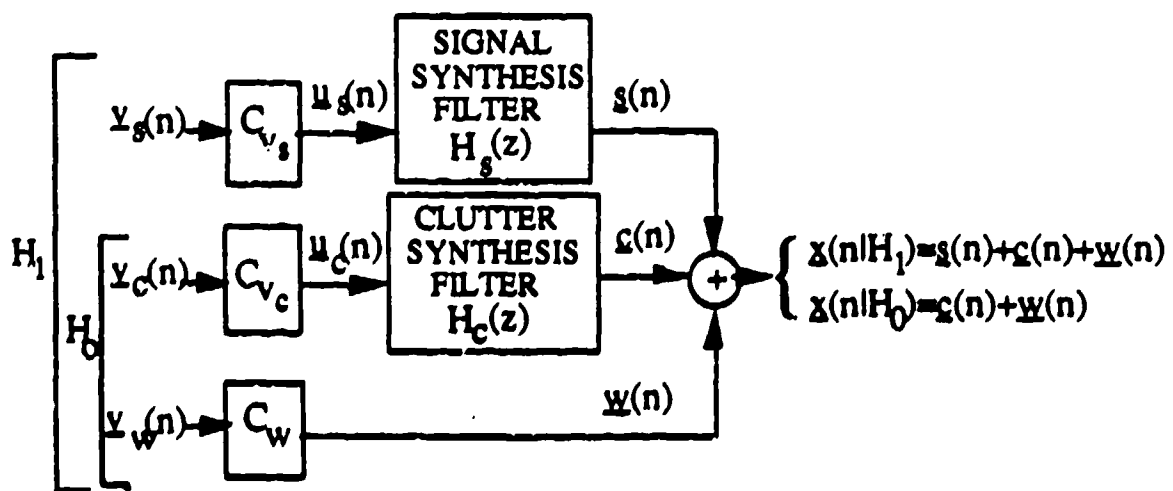


Figure 4.5.1 Block diagram for the process synthesis procedure

(PSD) which would result from the determined AR coefficients provides a fit to the desired spectrum. Alternatively, when these coefficients are used in eq(4.2.2), we are able to generate processes which provide a fit to the desired spectrum in a MMSE sense. In this dissertation, however, the process synthesis procedure is based upon the specification of the correlation sequences. In addition, we emphasize that in this work, we are concerned with parameter estimation rather than spectrum estimation.

Specifically, the following procedure is used to generate the time samples for both the AR signal and clutter processes:

- (1) The desired shapes of the autocorrelation and cross-correlation values are obtained using the $f(\cdot)$ functions by the methods of section 4.3.
- (2) The order of the AR process (for synthesis) is selected based upon a specified tolerance for fitting the desired spectrum.
- (3) The values of $R_{s_{ij}}(l)$ and $R_{c_{ij}}(l)$ are used to form the matrix elements of the signal and non-white noise correlation matrices designated as R_{ss} and R_{cc} , respectively.
- (4) Using the block Toeplitz form of $[\tilde{R}_{gg}]_{M_g+1}$, the multichannel Yule-Walker equations expressed by eq(2.3.1) are solved using the Levinson-Wiggins-Robinson recursion [6] to determine the matrix coefficients $A_g^H(k)$ and the Hermitian driving white noise covariance matrix $[\Sigma_f]_{u_g}^H$ for $k=1,2, \dots, M_g$. The matrix C_{v_g} is obtained from eq(2.2.6) using a Cholesky decomposition[†] of $[\Sigma_f]_{u_g}$; i.e.,

$$[\Sigma_f]_{u_g} = C_{v_g} C_{v_g}^H. \quad (4.5.2)$$

- (5) The values of $A_g^H(k)$ and C_{v_g} are now used in the generation of $\mathbf{g}(n)$ and $\mathbf{c}(n)$ via eqs(2.2.1) and (2.2.2a) expressed here for convenience; i.e.,

[†] We again note that other decompositions such as the L.D.L.^H could also be used.

$$\underline{g}(n) = - \sum_{k=1}^{M_g} A_g^H(k) \underline{g}(n-k) + \underline{u}_g(n) \quad g = s, c \quad (4.5.3)$$

where

$$\underline{u}_g(n) = C_v \underline{v}_g(n). \quad (4.5.4)$$

- (6) The vector $\underline{w}(n)$ is generated using eq(2.2.2b) where C_w can be obtained by specifying $R_{ww}(0)$ in eq(2.2.4c) and again using the Cholesky decomposition.
- (7) The processes $\underline{s}(n)$, $\underline{c}(n)$ and $\underline{w}(n)$ are added to yield $\underline{x}(n|H_1)$ or $\underline{x}(n|H_0)$ as shown in Fig. 4.5.1.

Thus, the vectors $\underline{u}_s(n)$, $\underline{u}_c(n)$ and $\underline{w}(n)$ in Fig 4.5.1 can be controlled to be zero-mean, white Gaussian noise vectors uncorrelated in time but with variable correlation across channels; ie.,

$$E[\underline{u}_g(n) \underline{u}_g^H(n-l)] = \begin{cases} [0] & l \neq 0 \\ [\Sigma_f]_{u_g} & l = 0 \end{cases} \quad g = s, c \quad (4.5.5a)$$

$$E[\underline{w}(n) \underline{w}^H(n-l)] = \begin{cases} [0] & l \neq 0 \\ R_{ww}(0) & l = 0 \end{cases} \quad (4.5.5b)$$

where, in general, the matrices $[\Sigma_f]_{u_g}$ and $R_{ww}(0)$ have non-zero off-diagonal components. Again, the functions $H_g(z)$ $g=s,c$ are the model filter transfer functions for the synthesis of the signal and clutter processes, respectively. They serve to temporally correlate $\underline{s}(n)$ and $\underline{c}(n)$ while the matrices C_{v_s} , C_{v_c} and C_w provide the cross-channel correlation.

For illustrational purposes, we now consider an example involving a single channel AR process in further detail. This example will provide us with further insight regarding the relationship between the correlation of the synthesized processes and the AR parameters.

EXAMPLE

In this example, we consider the synthesis of a single channel AR process of order two using real, Gaussian shaped correlation functions. The correlation function is, therefore, expressed as (dropping the subscript notation)

$$R(l) = \sigma^2(\lambda)^{|l|}. \quad (4.5.6)$$

The Yule-Walker equation for this case is

$$\begin{bmatrix} R(0) & R(1) & R(2) \\ R(-1) & R(0) & R(1) \\ R(-2) & R(-1) & R(0) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \\ a(2) \end{bmatrix} = \begin{bmatrix} \sigma_u^2 \\ 0 \\ 0 \end{bmatrix} \quad (4.5.7)$$

where σ^2 is the variance of the white noise driving term of the associated AR process. Using the functional form of eq(4.5.6) in (4.5.7) and solving for the coefficients, we have

$$a(1) = \frac{R(2)R(-1) - R(1)R(0)}{R(0)R(0) - R(1)R(-1)} = -\lambda(\lambda^2 + 1) \quad (4.5.8)$$

$$a(2) = \frac{R(1)R(1) - R(2)R(0)}{R(0)R(0) - R(1)R(-1)} = \lambda^2 \quad (4.5.9)$$

$$\sigma^2 = R(0) + a(1)R(+1) + a(2)R(+2) \quad (4.5.10a)$$

$$= \sigma^2 - \sigma^2\lambda^2(\lambda^2 + 1) + \sigma^2\lambda^6 \quad (4.5.10b)$$

$$= \sigma^2[1 - \lambda^2 - \lambda^4 + \lambda^6] \quad (4.5.10c)$$

We now consider the effect of these results when used in the AR equation (4.2.2) for the scalar case; i.e.,

$$g(n) = -a(1)g(n-1) - a(2)g(n-2) + u(n). \quad (4.5.11)$$

Using eqs(4.5.8) and (4.5.9) in (4.5.11), we have

$$g(n) = \lambda(\lambda^2 + 1)g(n-1) - \lambda^2g(n-2) + u(n) \quad (4.5.12)$$

where the white noise driving term $u(n)$ has the variance σ^2 . As λ approaches zero, we note from eq(4.5.6) that the correlation function approaches a discrete delta function with variance σ^2 . Therefore, $g(n)$ is expected to be an uncorrelated white noise process with variance σ^2 . Examination of eq(4.5.12) indicates that $g(n)$ is approaching the white noise process $u(n)$ for small λ since the AR coefficients are becoming vanishingly small. In addition, eq(4.5.10c) indicates that the variance of $u(n)$ approaches that of $g(n)$ as required. We, therefore, have a method for process synthesis that allows one to control the variance and temporal correlation over a wide range of values. Furthermore, for a specific σ^2 , the variance of the synthesized processes will remain fixed independent of the choice of λ .

4.6. MULTICHANNEL GAUSSIAN PROCESSES WITH UNCONSTRAINED QUADRATURE COMPONENTS

In [3], a procedure for the synthesis of multichannel AR processes was described where we were able to control not only the temporal and cross-channel correlation, but also the correlation of the quadrature components. However, the correlation of the quadrature components had to be achieved by an ad-hoc method. In this section, we present a direct method to obtain this correlation between the quadrature components in a manner that is more consistent with that presented in the previous section. Specifically, this method enables us to specify the temporal, cross-channel and quadrature component correlation directly within the correlation matrix used in the Yule-Walker equation.

We first consider the real, concatenated $2J \times 1$ observation vector $g(n)$ such that

$$g(n) = \begin{bmatrix} x_I(n) \\ x_Q(n) \end{bmatrix} \quad (4.6.1)$$

where

$$\mathbf{x}_I^T(n) = [x_{1I}(n) \ x_{2I}(n) \ \dots \ x_{JI}(n)] \quad (4.6.2a)$$

$$\mathbf{x}_Q^T(n) = [x_{1Q}(n) \ x_{2Q}(n) \ \dots \ x_{JQ}(n)] \quad (4.6.2b)$$

and J denotes the number of channels. The correlation matrix for this vector is

$$\mathbf{R}_{\mathbf{x}}(\parallel H_i) = E[\mathbf{x}(n\parallel H_i)\mathbf{x}^T(n\parallel H_i)] \quad (4.6.3a)$$

$$= \begin{bmatrix} \mathbf{R}_x^{II}(\parallel H_i) & \mathbf{R}_x^{IQ}(\parallel H_i) \\ \mathbf{R}_x^{QI}(\parallel H_i) & \mathbf{R}_x^{QQ}(\parallel H_i) \end{bmatrix} \quad i = 0,1 \quad (4.6.3b)$$

where the $J \times J$ block correlation matrices in eq(4.6.3b) are expressed as

$$\mathbf{R}_x^{AB}(\parallel H_i) = E[\mathbf{x}_A(n\parallel H_i)\mathbf{x}_B^T(n\parallel H_i)] \quad A=I,Q \ B=I,Q \quad i = 0,1. \quad (4.6.4)$$

We define the 1×1 autocorrelation ($j=k$ and $A=B$) and cross-correlation ($j \neq k$ and/or $A \neq B$) sequences of the quadrature components as

$$\begin{aligned} \mathbf{R}_{x_{jk}}^{AB}(\parallel H_i) &= E[x_{jA}(n\parallel H_i)x_{kB}^T(n\parallel H_i)] \quad A=I,Q \ B=I,Q \quad i = 0,1 \\ j,k &= 1,2,\dots,J \end{aligned} \quad (4.6.5)$$

where the subscripts j and k denote the channels. Considering the Yule-Walker equations for an order M process expressed in terms of the concatenated quadrature form (see Appendix M.4 where the identical form of these equations for prediction error filters is derived), we have

$$[\tilde{\mathbf{R}}_{\mathbf{x}}]_{M+1} \mathbf{A} = [\Sigma_f]_{M+1} \mathbf{I} \quad (4.6.6)$$

where the $2J(M+1)$ square matrix $[\tilde{\mathbf{R}}_{\mathbf{x}}]_{M+1}$ is expressed as

$$[\tilde{\mathbf{R}}_{\mathbf{x}}]_{M+1} = E[\mathbf{x}_{n,n-M}(\mathbf{x}_{n,n-M})^H] \quad (4.6.7a)$$

and

$$\underline{\mathbf{r}}_{n,n-M}^T = [\mathbf{r}^T(n) \mathbf{r}^T(n-1) \dots \mathbf{r}^T(n-M)] \quad (4.6.7b)$$

$$\mathbf{r}^T(k) = [r_1(k) \ r_2(k) \dots r_J(k)]. \quad (4.6.7c)$$

$$\underline{\mathbf{A}}^T = [\mathbf{I} \ \mathbf{A}^T(1) \ \mathbf{A}^T(2) \ \dots \ \mathbf{A}^T(M)] \quad (4.6.7d)$$

$$\underline{\mathbf{I}}^T = [\mathbf{I} \ [0] \ [0] \ \dots \ [0]] \quad (4.6.7e)$$

$$\mathbf{A}^T(k) = \begin{bmatrix} \mathbf{A}_{11}^T(k) & \mathbf{A}_{21}^T(k) \\ \mathbf{A}_{12}^T(k) & \mathbf{A}_{22}^T(k) \end{bmatrix} \quad (4.6.7f)$$

$$[\Sigma_f]_{u_r} = E[\underline{\mathbf{u}}_r(n) \underline{\mathbf{u}}_r^T(n)] \quad (4.6.7g)$$

$$\underline{\mathbf{u}}_r(n) = \begin{bmatrix} \underline{\mathbf{u}}_I(n) \\ \underline{\mathbf{u}}_Q(n) \end{bmatrix} \quad (4.6.7h)$$

where the elements of $\mathbf{A}^T(k)$ are $J \times J$ matrices and the quadrature vectors $\underline{\mathbf{u}}_I(n)$ and $\underline{\mathbf{u}}_Q(n)$ are each $J \times 1$. As an example, we consider eq(4.6.6) expressed in expanded form for an order one process as

$$\begin{bmatrix} [\mathbf{R}_x^{II}(0)] & [\mathbf{R}_x^{IQ}(0)] & [\mathbf{R}_x^{II}(1)] & [\mathbf{R}_x^{IQ}(1)] \\ [\mathbf{R}_x^{QI}(0)] & [\mathbf{R}_x^{QQ}(0)] & [\mathbf{R}_x^{QI}(1)] & [\mathbf{R}_x^{QQ}(1)] \\ [\mathbf{R}_x^{II}(-1)] & [\mathbf{R}_x^{IQ}(-1)] & [\mathbf{R}_x^{II}(0)] & [\mathbf{R}_x^{IQ}(0)] \\ [\mathbf{R}_x^{QI}(-1)] & [\mathbf{R}_x^{QQ}(-1)] & [\mathbf{R}_x^{QI}(0)] & [\mathbf{R}_x^{QQ}(0)] \end{bmatrix} \begin{bmatrix} \mathbf{I} & [0] \\ [0] & \mathbf{I} \\ \mathbf{A}_{11}(1) & \mathbf{A}_{12}(1) \\ \mathbf{A}_{21}(1) & \mathbf{A}_{22}(1) \end{bmatrix} = \begin{bmatrix} [\mathbf{R}_{u_r}^{II}(0)] & [\mathbf{R}_{u_r}^{IQ}(0)] \\ [\mathbf{R}_{u_r}^{QI}(0)] & [\mathbf{R}_{u_r}^{QQ}(0)] \\ [0] & [0] \\ [0] & [0] \end{bmatrix} \quad (4.6.8)$$

where

$$\mathbf{A}(k) = \begin{bmatrix} \mathbf{A}_{11}(1) & \mathbf{A}_{12}(1) \\ \mathbf{A}_{21}(1) & \mathbf{A}_{22}(1) \end{bmatrix} \quad (4.6.9a)$$

and

$$[\Sigma_f]_{u_r} = \begin{bmatrix} [R_{u_r}^{II}(0)] & [R_{u_r}^{IQ}(0)] \\ [R_{u_r}^{QI}(0)] & [R_{u_r}^{QQ}(0)] \end{bmatrix}. \quad (4.6.9b)$$

We now apply the same procedure using eq(4.6.6) as in the previous section; ie., correlation shaping functions are used for each of the scalar elements of $[\tilde{R}_{\Pi}]_{M+1}$. The Yule-Walker equations are then solved for the $2J \times 2J$ matrix coefficients $\mathcal{A}(k)$, $k=1,2,\dots,M$ and $[\Sigma_f]_{u_r}$. A Cholesky or LDL^T decomposition of $[\Sigma_f]_{u_r}$ provides the real, $2J \times 2J$ matrices C_t and L_p which serve the same role as C_v and L_z in the previous section. In this case, however, C_t and L_p are able to control not only the cross-channel correlation, but also the quadrature correlation that was specified in the matrix $[\tilde{R}_{\Pi}]_{M+1}$. This can be seen by considering

$$[\Sigma_f]_{u_r} = \frac{1}{2} C_t C_t^T \quad (4.6.10)$$

where the matrix C_t contains all the cross-channel and quadrature correlation information contained in $[\Sigma_f]_{u_r}$ and the factor of $(1/2)$ is used on the quadrature terms in order to provide a unit variance on the complex vector $\mathbf{t}(n)$ defined below. From eq(4.6.10), we obtain

$$C_t^{-1} [\Sigma_f]_{u_r} C_t^{-T} = \frac{1}{2} I \quad (4.6.11)$$

where I in this case is a $2J \times 2J$ identity matrix. Eq(4.6.11) can also be written as

$$C_t^{-1} [\Sigma_f]_{u_r} C_t^{-T} = E[C_t^{-1} \mathbf{u}_r(n) \mathbf{u}_r^T(n) C_t^{-T}] = \frac{1}{2} I. \quad (4.6.12)$$

Eq(4.6.12) implies the existence of the concatenated real vector $\mathbf{t}_r(r)$ such that

$$\mathbf{t}_r(n) = \mathbf{C}_t^{-1} \mathbf{u}_r(n) \quad (4.6.13)$$

and

$$E[\mathbf{t}_r(n) \mathbf{t}_r^T(n)] = \frac{1}{2} \mathbf{I} \quad (4.6.14)$$

where, again, $\mathbf{I}$ is a $2J \times 2J$ identity matrix. From eq(4.6.13), we have

$$\mathbf{u}_r(n) = \mathbf{C}_t \mathbf{t}_r(n) = \mathbf{C}_t \begin{bmatrix} \mathbf{t}_I(n) \\ \mathbf{t}_Q(n) \end{bmatrix}. \quad (4.6.15)$$

Thus, $\mathbf{u}_r(n)$ can now be generated using the matrix $\mathbf{C}_t$ [which is available from the solution of eq(4.6.10)] and the white noise vector $\mathbf{t}_r(n)$. From eq(4.6.14), we obtain the $J \times J$ matrices

$$E[\mathbf{t}_I(n) \mathbf{t}_I^T(n)] = \frac{1}{2} \mathbf{I} \quad (4.6.16a)$$

$$E[\mathbf{t}_Q(n) \mathbf{t}_Q^T(n)] = \frac{1}{2} \mathbf{I} \quad (4.6.16b)$$

and

$$E[\mathbf{t}_I(n) \mathbf{t}_Q^T(n)] = [0]. \quad (4.6.16c)$$

From eqs(4.6.16), the correlation matrix of the complex vector $\mathbf{t}(n) = \mathbf{t}_I(n) + j\mathbf{t}_Q(n)$ is found to be

$$E[\mathbf{t}(n) \mathbf{t}^H(n)] = E[\mathbf{t}_I(n) \mathbf{t}_I^T(n)] + E[\mathbf{t}_Q(n) \mathbf{t}_Q^T(n)] = \mathbf{I} \quad (4.6.17)$$

where, in this case, $\mathbf{I}$ is a $J \times J$ identity matrix. Thus, the complex white noise driving vector $\mathbf{t}(n)$ has unit variance. Likewise, the LDL^T of $[\Sigma_f]_{u_r}$ implies the existence of $\mathbf{p}_r(n)$ such that

$$\mathbf{u}_r(n) = \mathbf{L}_p \mathbf{p}_r(n) = \mathbf{L}_p \begin{bmatrix} \mathbf{p}_I(n) \\ \mathbf{p}_Q(n) \end{bmatrix} \quad (4.6.18)$$

where

$$E[p_I(n)p_I^T(n)] = \frac{1}{2}D_p^I \quad (4.6.19a)$$

$$E[p_Q(n)p_Q^T(n)] = \frac{1}{2}D_p^{QQ}. \quad (4.6.19b)$$

$$E[p_I(n)p_Q^T(n)] = [0] \quad (4.6.19c)$$

and the correlation matrix of the complex white noise driving vector $p(n)=p_I(n)+jp_Q(n)$ is

$$E[p(n)p^H(n)] = D_p. \quad (4.6.20)$$

This procedure provides a practical method for the synthesis of $u_r(n)$. As noted above, the solution of eq(4.6.6) also provides the matrices $\mathcal{A}(k)$, $k=1,2,\dots,M$. Thus, we can now synthesize the concatenated vector process $\mathbf{r}(n)$ using the concatenated form of the AR equation

$$\mathbf{r}(n) = - \sum_{k=1}^M \mathcal{A}^T(k) \mathbf{r}(n-k) + \mathbf{u}_r(n). \quad (4.6.21a)$$

$$= - \sum_{k=1}^M \mathcal{A}^T(k) \mathbf{r}(n-k) + \mathbf{C}_t \mathbf{t}_r(n) \quad (4.6.21b)$$

where eq(4.6.21b) follows from (4.6.15).

In this section, we have presented a method for generating AR processes in which we are able to control not only the temporal and cross-channel correlation, but also the correlation between quadrature components. The method allows one to specify these correlation properties directly in the correlation matrix of the concatenated quadrature form of the Yule-Walker equations presented in eq(4.6.6). The Yule-Walker equations are then solved for $\mathcal{A}^T(k)$ and $\mathbf{C}_t$ in order to synthesize $\mathbf{r}(n)$ via eqs(4.6.21). This procedure allows us to synthesize the more general processes discussed in Appendix N.

4.7. SYNTHESIS RESULTS

In this section, we illustrate examples of the synthesized AR processes in order to gain further insight into the synthesis method. In addition, we also consider the time-averaged and ensemble averaged auto- and cross-correlation sequences. Both single channel and multichannel processes are shown in subsections 4.7.1 and 4.7.2, respectively. In these sections, we note that the terms 'trials' and 'realizations' are used interchangeably. They refer to a statistically independent generation of a sequence of an AR process.

4.7.1. Single Channel Case

In this subsection, we utilize the AR process synthesis procedure described in section 4.5 to generate single channel autoregressive processes of order two. For the processes considered here, the elements of the correlation matrix are specified using the Gaussian autocorrelation shaping function. Their parameters are specified in Table 4.7.1. These elements are then used in the Yule-Walker equation. The two AR coefficients and driving white noise variance are then obtained and used in the scalar form of eq(4.5.3). This equation is then used to generate the required signal processes. For the processes shown in Figs. 4.7.1 through 4.7.8, the real, Gaussian autocorrelation function was used. We note, however, that an AR(2) process has an exponential autocorrelation function. Therefore, in these cases, we are generating processes which provide a 'fit' to the specified Gaussian autocorrelation function in a MMSE sense. Although the AR coefficients can be shown to be real, the processes $s(n)$ generated by eq(4.5.3) are complex since $u(n)$ is complex. Also, the resulting spectra will be even.

In Fig 4.7.1, we show the amplitude of the real part of the AR(2) process using a temporal correlation coefficient $\lambda_s=0.99$ and variances σ_s^2 ranging from 2.0 to 8.0. We note the variation in the amplitude of each of the processes as the variance changes. In Fig 4.7.2, we show the amplitude of the real part of the signal using a variance $\sigma_s^2=4.0$. In this case, however, we demonstrate the effect of varying the temporal correlation coefficient λ_s from 0.1 to 0.9999. We note that Figs 4.7.2a through 4.7.2c are plotted for 100 samples, while Fig.4.7.2d uses 200 samples. These figures illustrate the effect of λ_s in controlling the sample-to-

Figure	Order	σ_s^2	σ_w^2	λ_s	l_s	$R_s(0)$
4.7.1	AR(2)	2,4,6,8	-	0.99	0	2,4,6,8
4.7.2	AR(2)	4	-	0.1,0.9,0.99 0.9999	0	4
4.7.3	AR(2)	4	-	0.9999	0	4
4.7.4	AR(2)	4	-	0.95,0.8,0.1	0	4
4.7.5	AR(2)	4	-	0.95,0.8,0.1	0	4
4.7.6	AR(2)	4	-	0.95,0.8,0.1	0	4
4.7.7	AR(2)	4	1	0.95,0.8,0.1	0	5
4.7.8	AR(2)	4	1	0.95,0.8,0.1	0	5

Table 4.7.1 Parameters for single channel synthesized processes

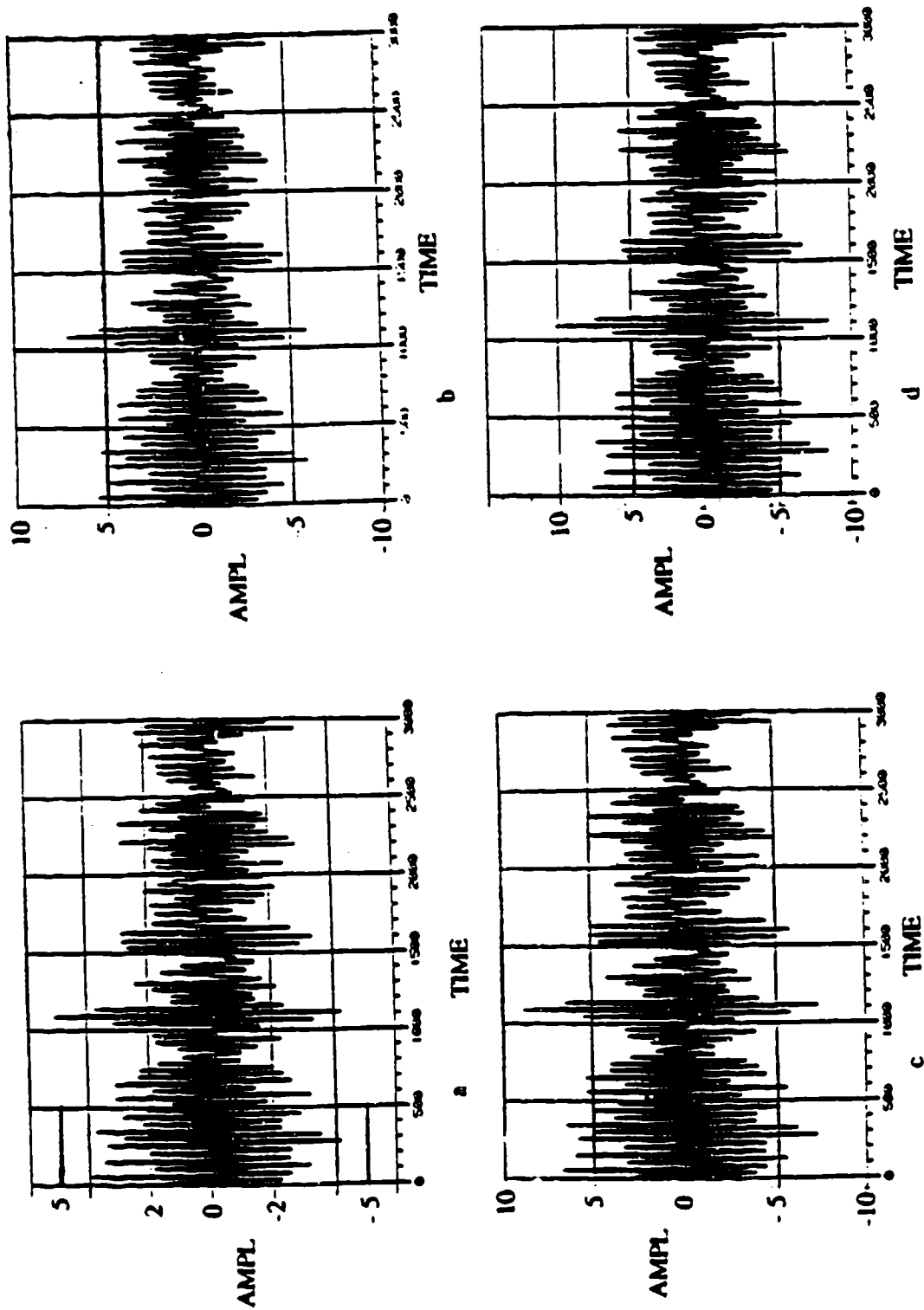


Fig. 4.7.1 Amplitude of the real part of an AR(2) signal process for $\lambda_s=0.99$ and 3000 sample observations a.) $\sigma_s^2=2.0$ b.) $\sigma_s^2=4.0$ c.) $\sigma_s^2=6.0$ d.) $\sigma_s^2=8.0$.

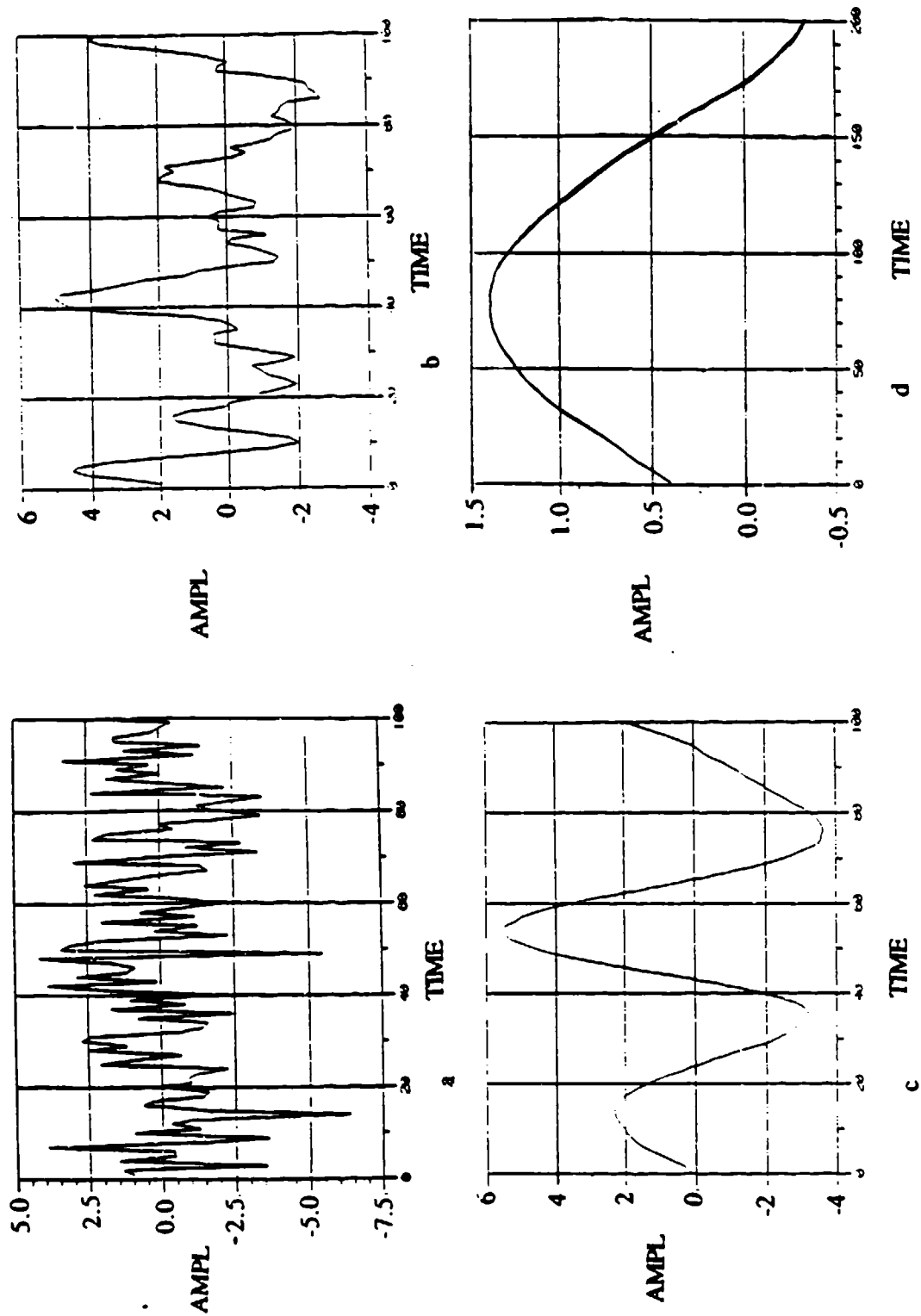


Fig. 4.7.2 Amplitude of the real part of the AR(2) signal process with a.) $\lambda_s=0.1$ b.) $\lambda_s=0.9$ c.) $\lambda_s=0.99$ d.) $\lambda_s=0.9999$.

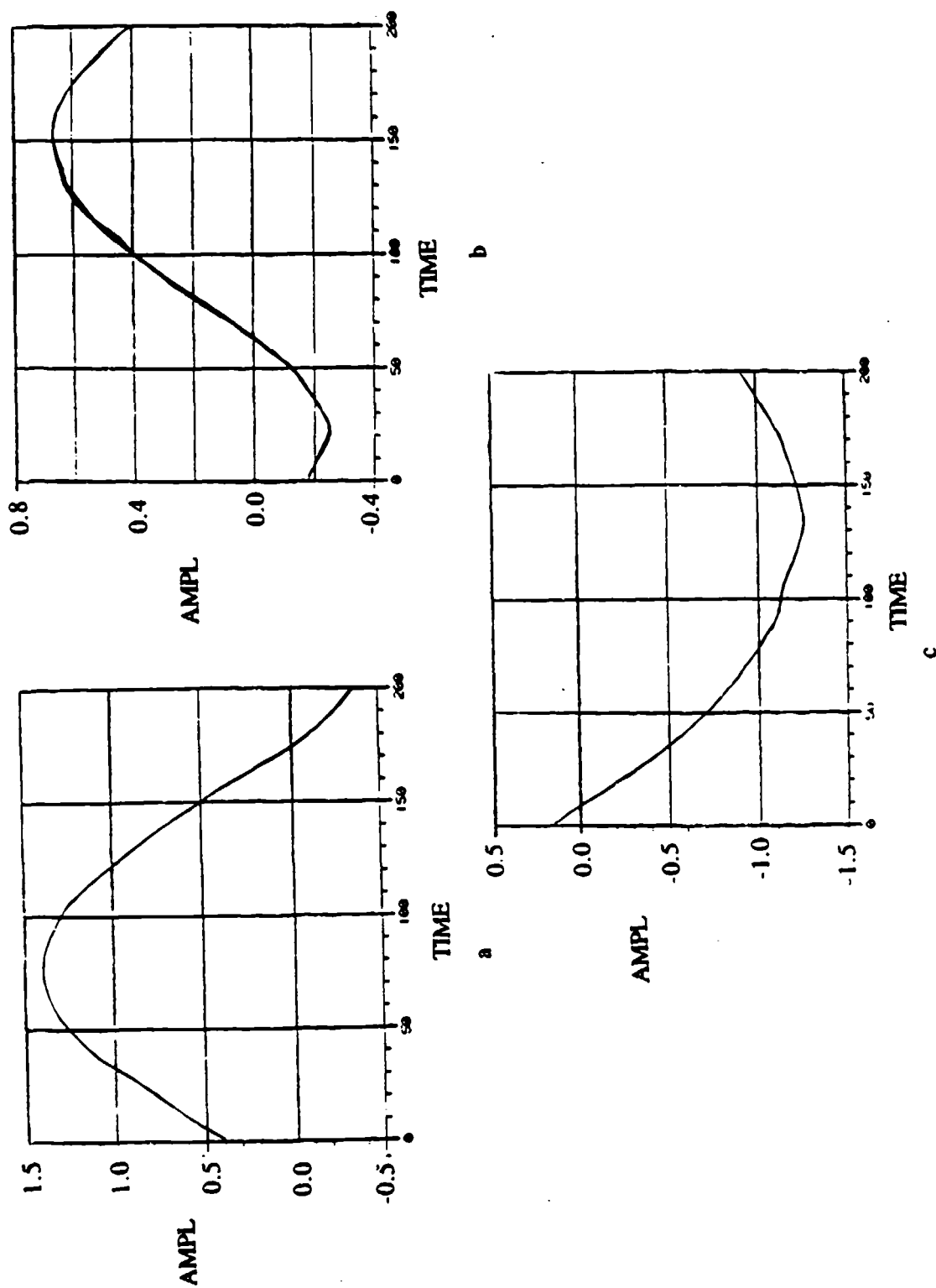


Fig. 4.7.3 Amplitude of the real part vs time for three independent trials of the AR(2) signal using $\lambda_s=0.9999$.

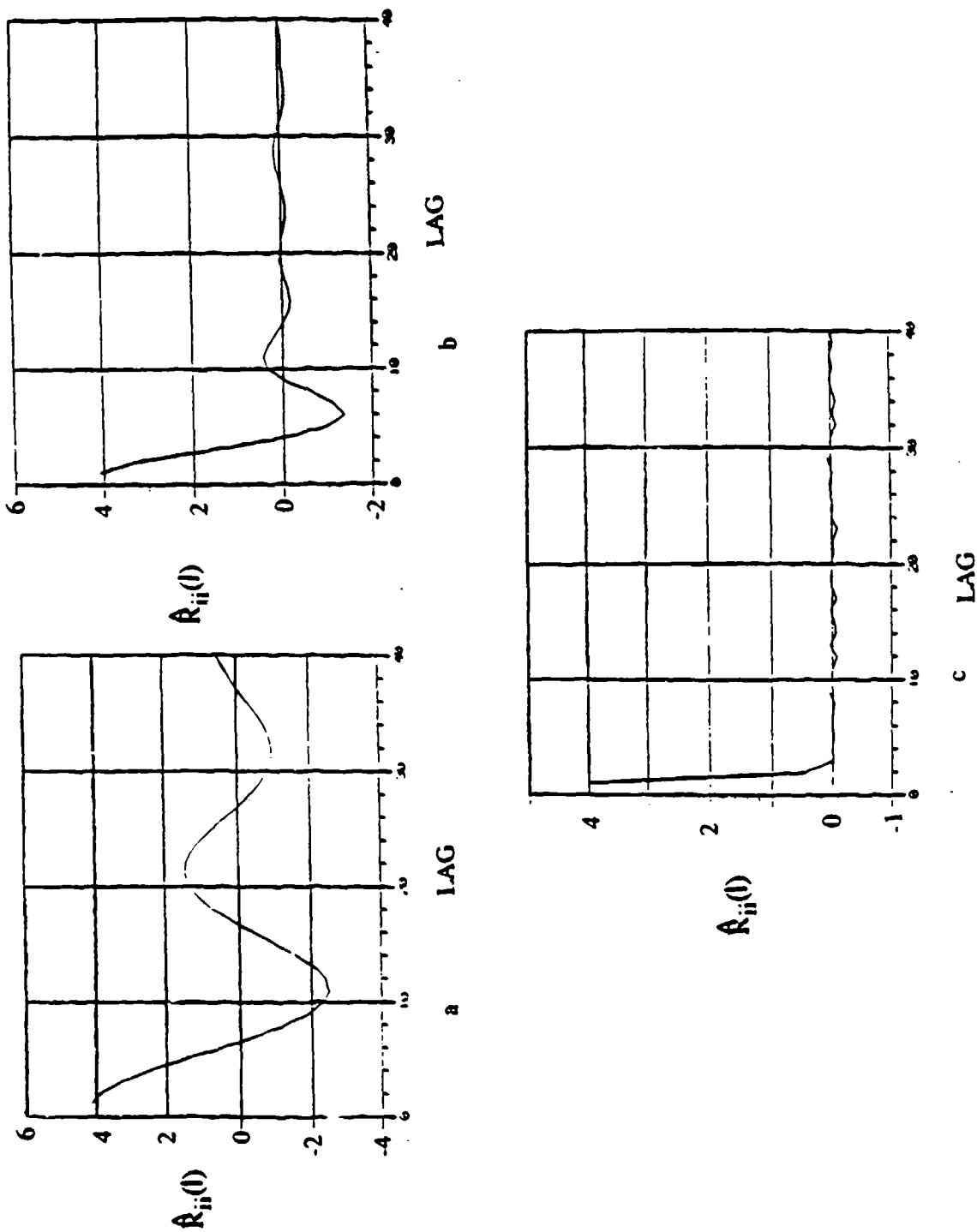


Fig. 4.7.4 Plots of the specified(---) and the estimated(—) time-averaged autocorrelation function versus lag l for an AR(2) process using 10,000 sample observations with a.) $\lambda_s=0.95$ b.) $\lambda_s=0.80$ c.) $\lambda_s=0.1$.

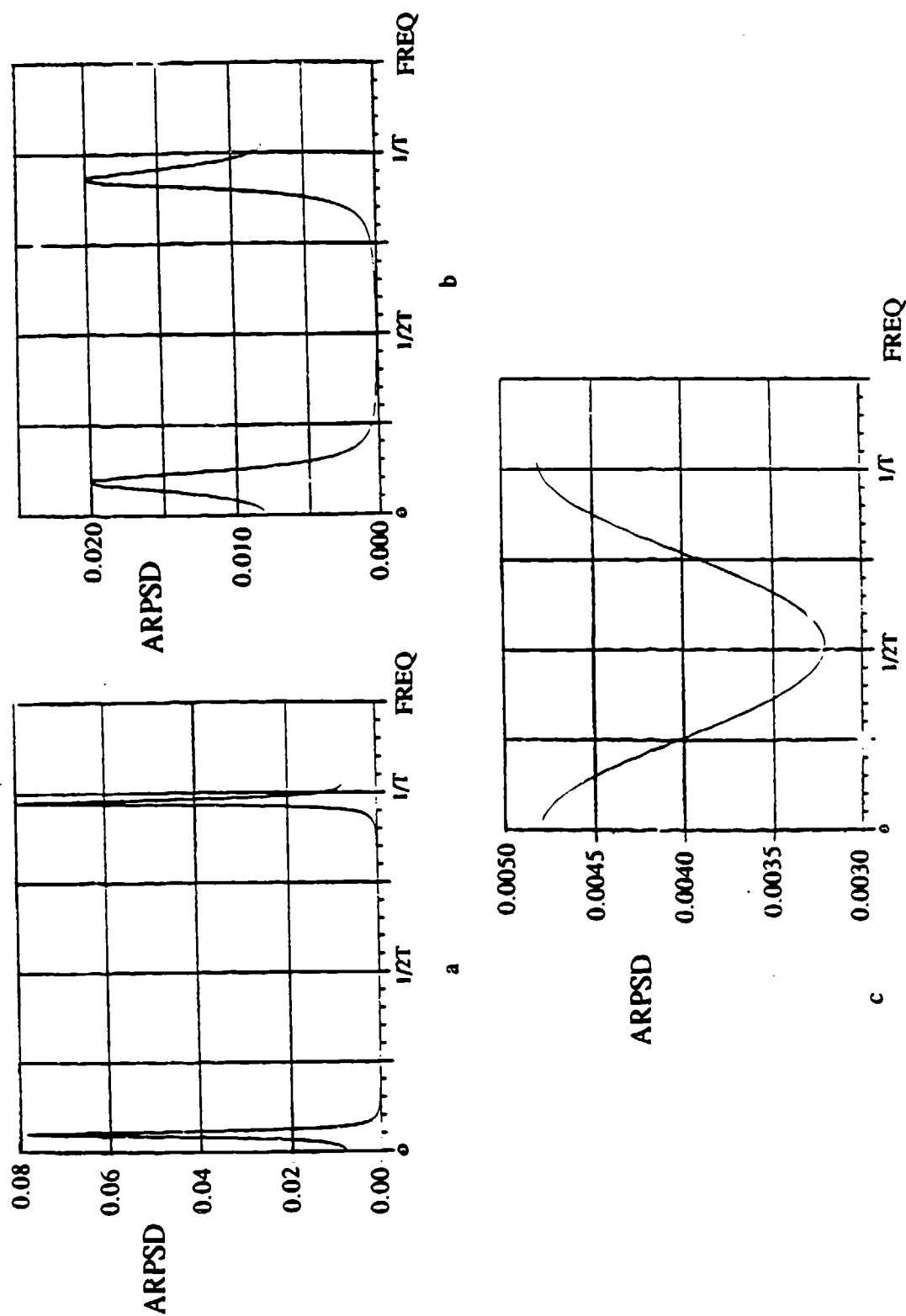


Fig. 4.7.5 Plots of the ARPSP computed using the AR(2) signal coefficients obtained from the Yule-Walker equation a.) $\lambda_s=0.95$ b.) $\lambda_s=0.80$ c.) $\lambda_s=0.1$.

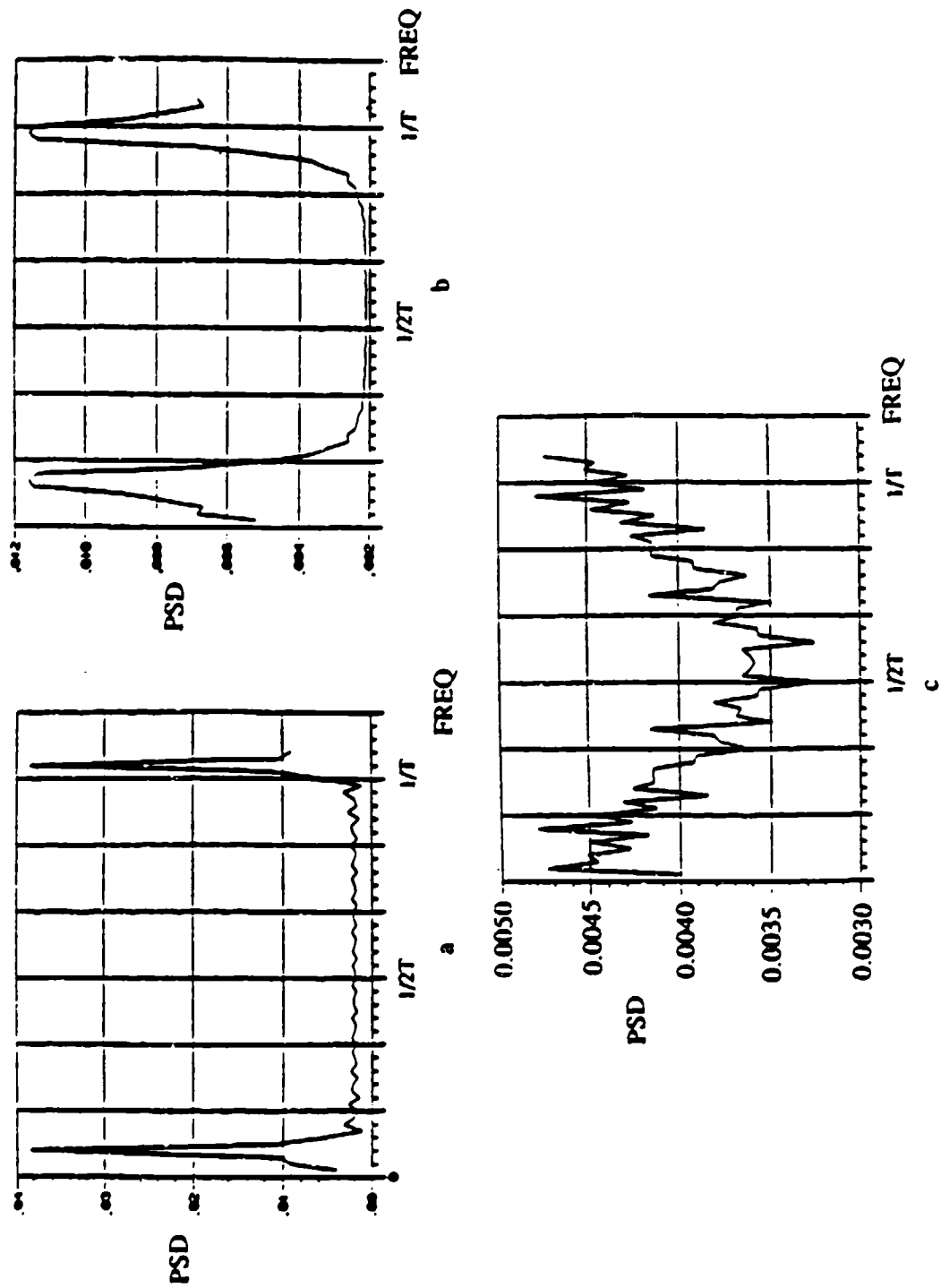


Fig. 4.7.6 PSD of the AR(2) signal using a zero-filled 64-point FFT of the estimated co.relation functions in Fig 4.7.4 with a.) $\lambda_s=0.95$ b.) $\lambda_s=0.80$ c.) $\lambda_s=0.1$.

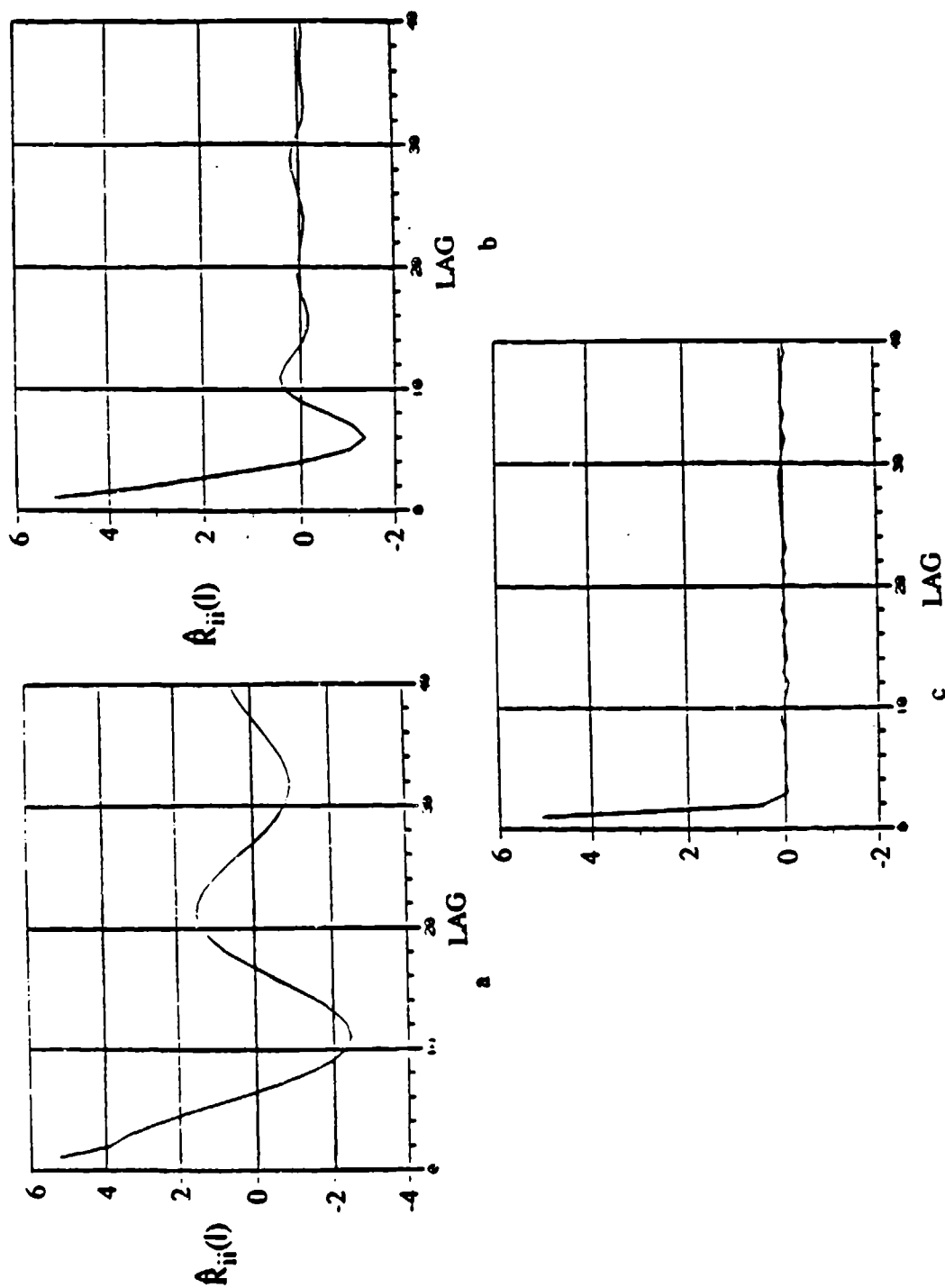


Fig. 4.7.7 Plots of the estimated correlation functions for the AR(2) signal $\sigma_s^2=4$ plus white noise $\sigma_w^2=1$ using 10,000 sample observations with a.) $\lambda_s=0.95$ b.) $\lambda_s=0.80$ c.) $\lambda_s=0.1$.

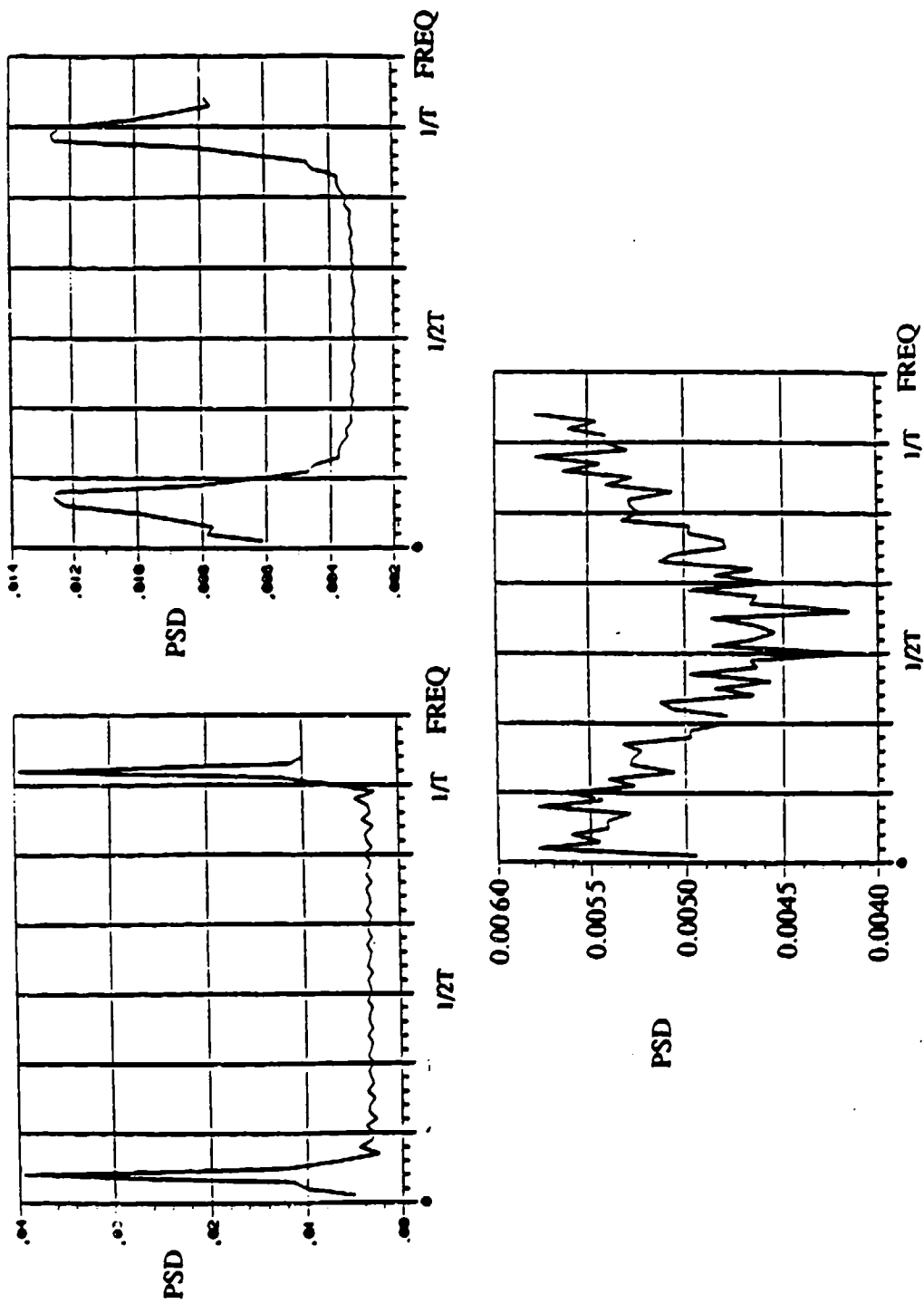


Fig. 4.7.8 PSD of the AR(2) signal $\sigma_s^2=4$ plus white noise $\sigma_w^2=1$ using a zero-filled 64-point FFT of the estimated correlation functions in Fig. 4.7.7 with a.) $\lambda_s=0.95$ b.) $\lambda_s=0.80$ c.) $\lambda_s=0.1$.

sample correlation; ie., for λ_s approaching zero as in Fig. 4.7.2a, the process becomes white. However, as λ_s approaches unity as in Fig. 4.7.2, the more slowly varying function of time indicates that these processes are approaching the case of total correlation sample-to-sample. It is worth noting, at this time, that the imaginary component of these complex processes follows the same behavior. This can be shown by first recognizing that the signal phase is expressed as

$$\theta_s(n) = \tan^{-1} \left[\frac{s_Q(n)}{s_I(n)} \right]. \quad (4.7.1)$$

As noted above, for λ_s approaches unity, the in-phase and quadrature signal components, $s_I(n)$ and $s_Q(n)$ respectively, are each becoming more slowly varying functions of time. Therefore, they are each approaching constant values within a single trial. Consequently, $\theta_s(n)$ approaches a constant. On the other hand, as λ_s approaches zero, $s_I(n)$ and $s_Q(n)$ each become more uncorrelated in time so that each exhibits random behavior along the sequence of a given realization. And so, $\theta_s(n)$ is random in time. The important point to be made here is that the parameter λ_s controls the amplitude as well as the phase correlation of the process. This capability will be utilized in chapter 7 to synthesize phase correlated signal processes. Therefore, we are able to investigate the coherent integration gain in the detection algorithms that are derived in chapter 6.

In Fig 4.7.3, we show results for three separate trials of the real part of the signal process for $\lambda_s = 0.9999$ and $\sigma_s^2 = 4.0$ over a window of 200 time samples.

The point to be noted here is the randomness associated with the initial amplitude (and phase via the previous discussion). As noted, however, after the initial amplitude is selected at random, the remaining samples are highly correlated within each trial as governed by the high value of the temporal correlation coefficient. This feature will enable us to model processes such as Swerling fluctuating signals [72] in radar applications, for example.

Fig 4.7.4 shows the results of the model fitting procedure. The dotted curves show the correlation functions prespecified using the functional shaping method. The solid curves are plots of the correlation function estimated using

10,000 samples of the synthesized processes over 40 lag values.[†] Figs 4.7.4a, 4.7.4b and 4.7.4c are plots for $\lambda_s = 0.95, 0.8$, and 0.1 , respectively. We first note the effect of λ_s on the correlation function; ie. as λ_s transitions from unity to zero, the correlation function ranges from a slowly varying function of the lag value to that approaching a delta function. Next, we note from Figs 4.7.4a and 4.7.4b that the correlation sequences of the AR(2) process provide an approximate fit to the predetermined correlation function. This is a result of the fact that it would require an AR process of infinite order to model the Gaussian shaped correlation exactly. These plots illustrate that the processes generated here are an approximation to those corresponding to the predetermined correlation function. In fact, they are the processes that 'fit' the known model in a MMSE sense. In Fig 4.7.5, we show the autoregressive process power spectral density (ARPSD) which is obtained using the AR coefficients obtained from the Yule-Walker equation. Fig 4.7.6 is the corresponding power spectral density (PSD) determined using a zero-filled 64 point FFT of the calculated correlation functions plotted in Fig 4.7.4. The plots for Figs 4.7.6a and 4.7.6b, show even spectra as we would expect since the correlation functions are real. However, we also note the dual peaks associated with the poles of the AR(2) model. The point to be noted here is the behavior of the spectra for various values of λ_s ; ie. as λ_s ranges from unity to zero, the spectra transitions from its peaked behavior to the broad distribution associated with white noise processes. It is also interesting to note that as λ_s decreases, the pronounced pole positions are diminished as shown in Fig 4.7.6c. This is a result of the fact that the AR coefficients are becoming vanishingly small as noted by eqs(4.5.8) and (4.5.9).

In Figs 4.7.7 and 4.7.8, we show plots of the autocorrelation function and power spectral density, respectively, estimated using 10,000 samples of the same signal process, but in the presence of unit variance additive white noise. We note that the correlation functions in Fig 4.7.7 overlay the corresponding functions shown in Fig 4.7.4 except that at lag zero, they have increased by a unit value due to the uncorrelated noise process; ie., at $l=0$, the autocorrelation sequences peak at 5 rather than 4. A comparison of Figs 4.7.6 and 4.7.8 shows the increased levels due to the additive noise.

[†] We note the offset in the display graphics for the correlation sequence plots.

4.7.2. Multichannel Case

In this subsection, we consider the generation of a two-channel vector process using the AR synthesis approach described in chapter 6. In this case, the multichannel Yule-Walker equation is solved for the forward AR matrix coefficients and the forward driving white noise covariance matrix. These matrices are used for process synthesis following the method described in section 4.5. We limit the results shown here to the case of real, Gaussian shaped correlation functions. In this section, we will qualitatively discuss the dependence of ergodicity on the correlation parameters. In sections 5.3 and 5.4, we will investigate quantitatively the ergodicity of multichannel AR processes.

Table 4.7.2 contains the parameters used to specify the Gaussian auto- and cross-correlation shaping functions described in eqs(4.3.7) and (4.3.6), respectively. These in turn are used in the correlation matrix of the multichannel Yule-Walker equation to solve for the matrix $[\Sigma_f]_u$ and $A(k)$ $k = 1, 2$. The matrix C is then obtained by taking the Cholesky decomposition of $[\Sigma_f]_u$. We note that these parameters satisfy the constraints discussed in section 4.4. Later, we will list the values of these matrices and discuss their significance later.

Figs 4.7.9 through 4.7.12 contain the results for processes with high temporal correlation on each channel. In Fig 4.7.9, $\lambda_{ii} = 0.9999$ on both channels with $|\rho_{12}| = 0.95$. In Figs 4.7.10 through 4.7.12, $\lambda_{ii} = 0.95$ on both channels, while the cross-correlation coefficient $|\rho_{12}|$ ranges from 0.99 to 0.0.

In Fig 4.7.9, plots a and b show $N_T = 200$ time sample observations of the synthesized processes for channels 1 and 2, respectively, for $N_R = 1$ realization (trial). A second realization is shown in plots c and d. A visual inspection of these two trials reveals the relatively high degree of cross-correlation between the two channels controlled by $|\rho_{12}| = 0.95$. Plots e and f show the corresponding ensemble averaged autocorrelation functions calculated using 1,000 trials. In plot f, we note that the experimentally obtained autocorrelation function evaluated for lag zero fell somewhat below the process variance of 4. The reason for this will be explained below. An overlay of six realizations of the corresponding time-averaged, biased, autocorrelation function for each channel is shown in plots g and h, respectively. In these plots, 10,000 time sample observations were used to estimate the functions over 64 lag values. Plots i and j show the temporal- and ensemble-averaged cross-correlation functions, respectively. Inspection of plots

Fig.	Order	σ_{11}^2	σ_{22}^2	$ \rho_{12} $	λ_{11}	λ_{22}	λ_{12}	l_{12}	$R_{12}(0)$
4.7.9	AR(2)	4	4	0.95	0.9999	0.9999	0.9999	0	3.80
4.7.10	AR(2)	4	4	0.99	0.95	0.95	0.95	0	3.96
4.7.11	AR(2)	4	4	0.5	0.95	0.95	0.95	0	2.0
4.7.12	AR(2)	4	4	0.0	0.95	0.95	arbitrary*	0	0.0
4.7.13	AR(2)	4	4	0.0	0.80	0.80	arbitrary*	0	0.0
4.7.14	AR(2)	4	4	0.0	0.40	0.40	arbitrary*	0	0.0
4.7.15	AR(2)	4	4	0.99	0.10	0.10	0.10	0	3.96
4.7.16	AR(2)	4	4	0.30	0.10	0.10	0.97	0	1.2
4.7.17	AR(2)	4	4	0.10	0.10	0.10	0.20	0	0.4
4.7.18	AR(2)	4	4	0.50	0.95	0.10	0.97	0	2.0
4.7.19	AR(2)	4	1	0.40	0.95	0.10	0.97	4	0.8
4.7.20	AR(4)	4	1	0.30	0.95	0.10	0.97	4	0.6

Table 4.7.2 Parameters for the two-channel synthesized processes

* Examination of eq(4.3.6) indicates that the value of λ_{12} can be chosen arbitrarily when $|\rho_{12}|=0$. A value of 0.95 was used in this case.

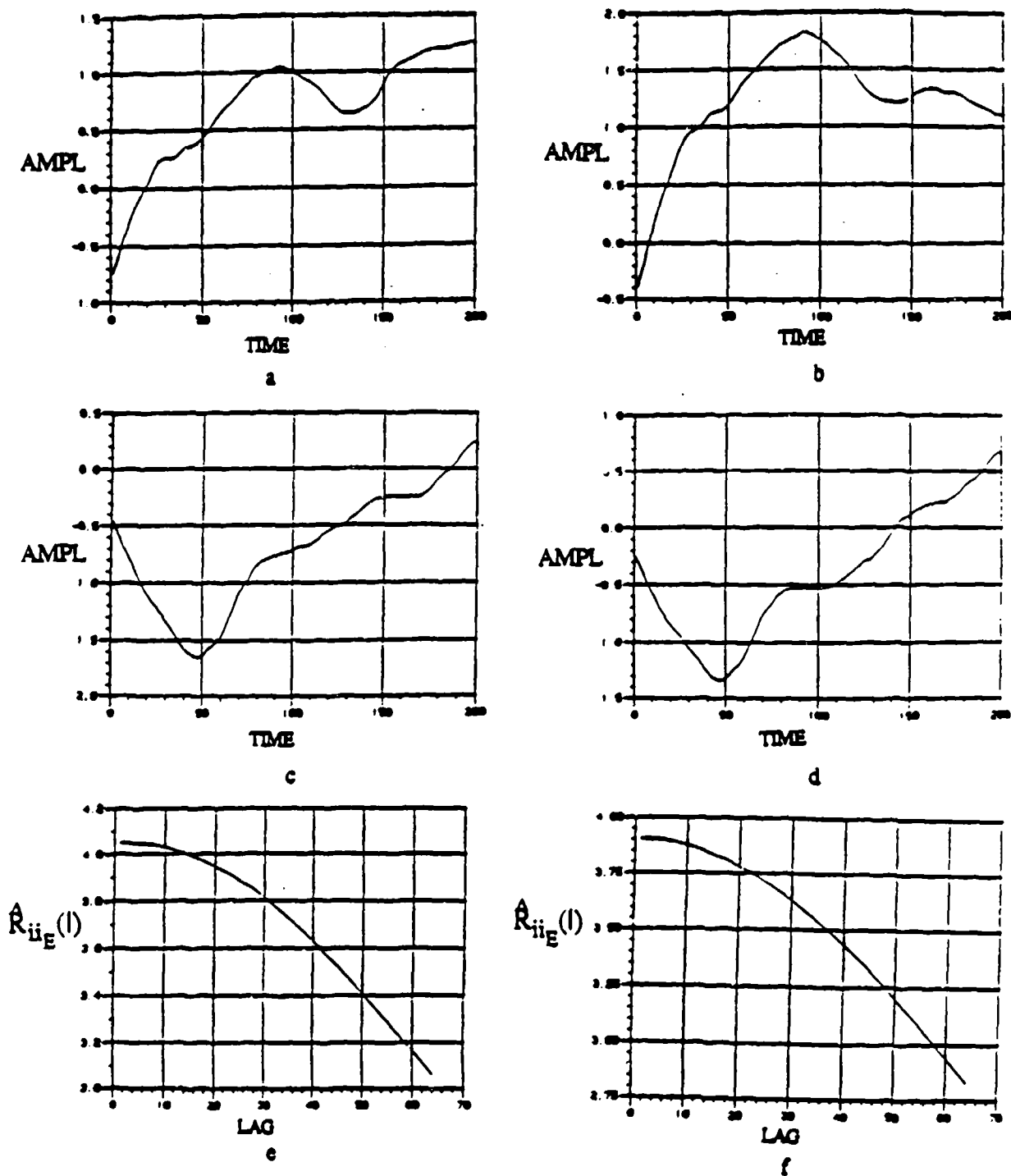


Fig. 4.7.9 Two channel AR(2) processes a.) channel 1 data (trial 1) b.) channel 2 data (trial 1) c.) channel 1 data (trial 2) d.) channel 2 data (trial 2) e.) ensemble averaged channel 1 autocorrelation (1,000 realizations) f.) ensemble averaged channel 2 autocorrelation (1,000 realizations); $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.9999$, $|p_{12}|=0.95$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.

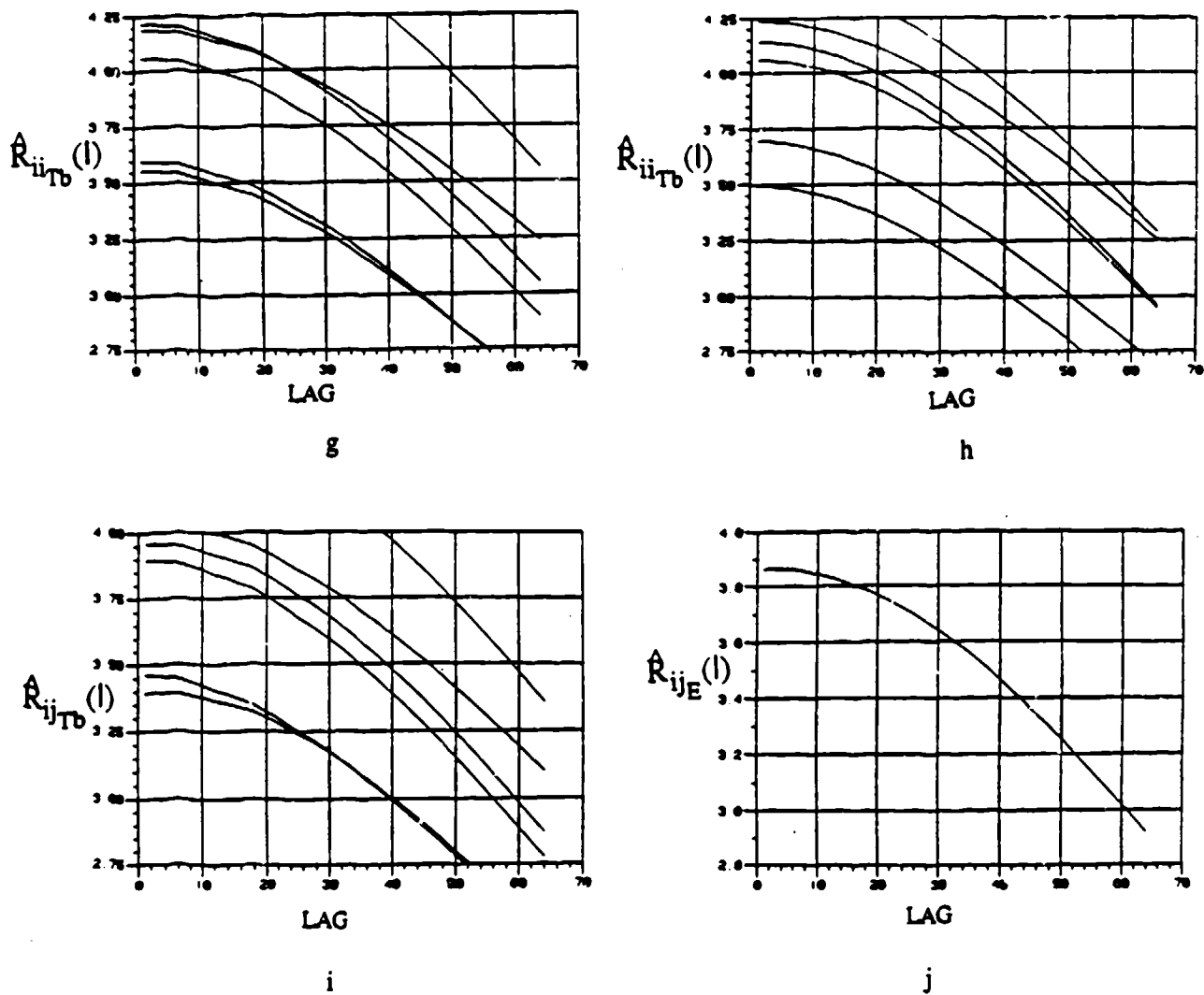
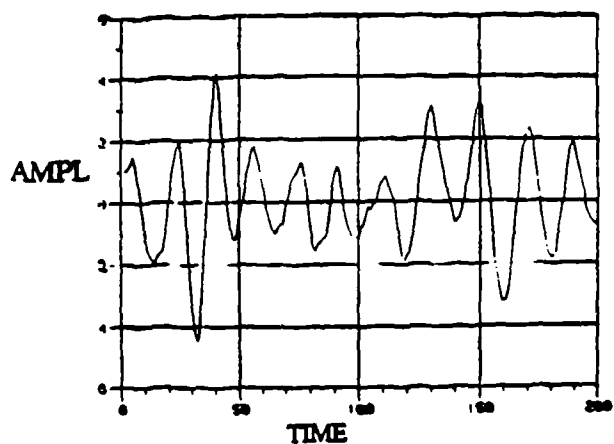
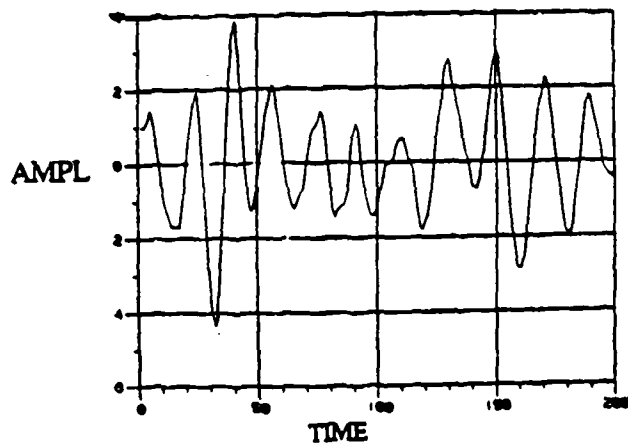


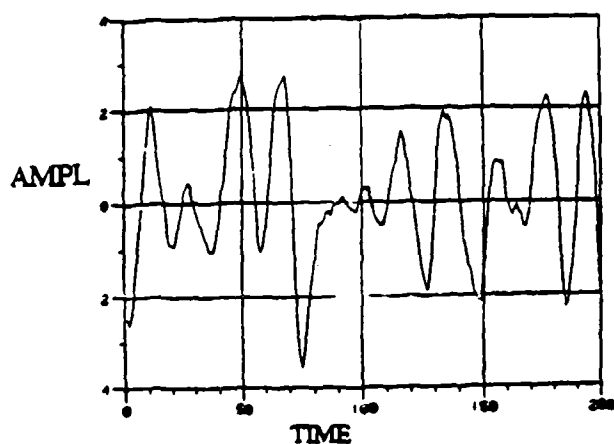
Fig. 4.7.9 (contin.) g.) time-averaged channel 1 autocorrelation (6 trials) h.)time averaged channel 2 autocorrelation (6 trials) i.) time-averaged cross- correlation (6 trials) j.) ensemble averaged cross-correlation (1,000 realizations); $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.9999$, $|\rho_{12}|=0.95$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.



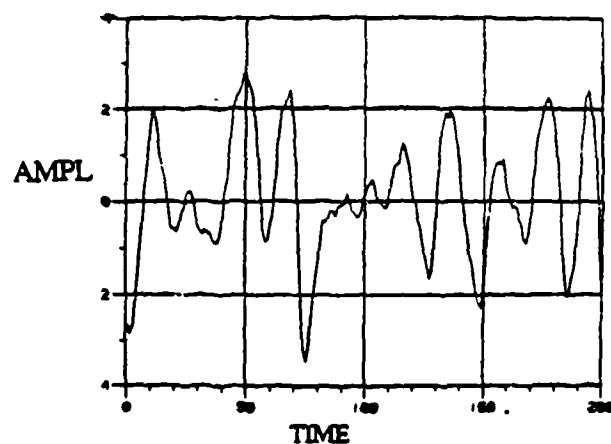
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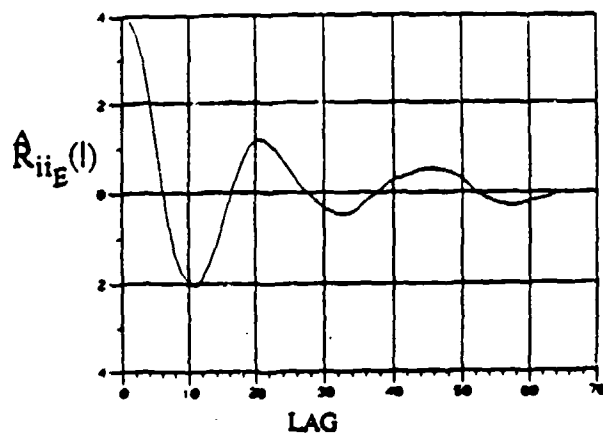
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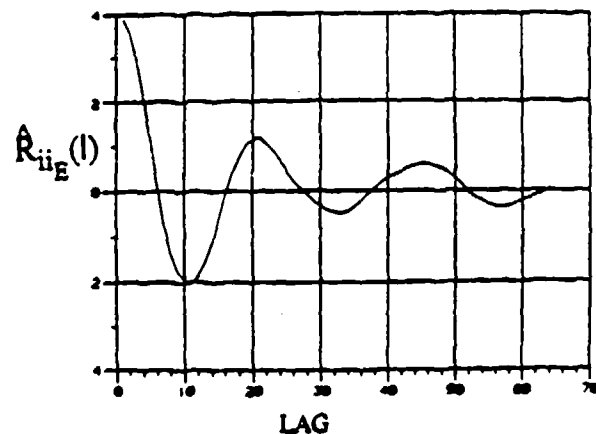
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d

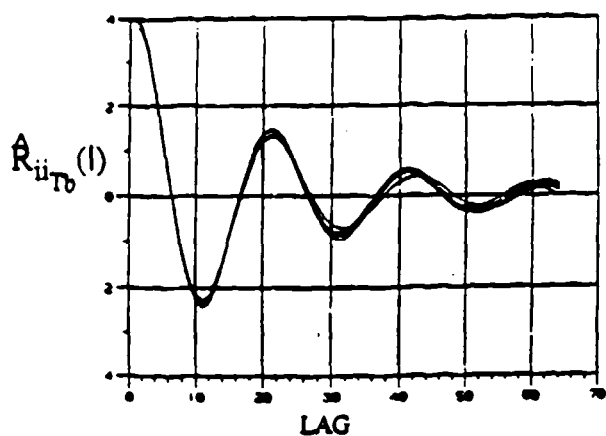


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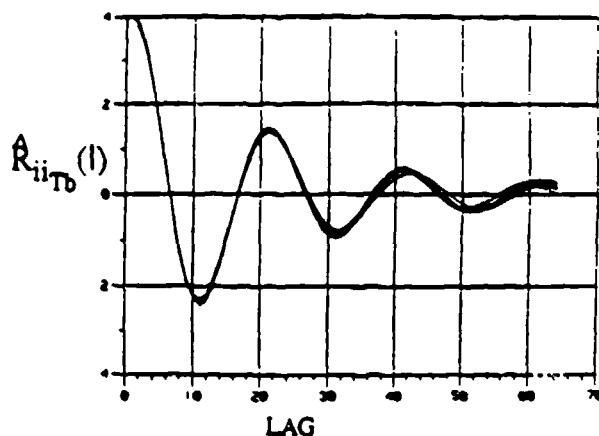


f

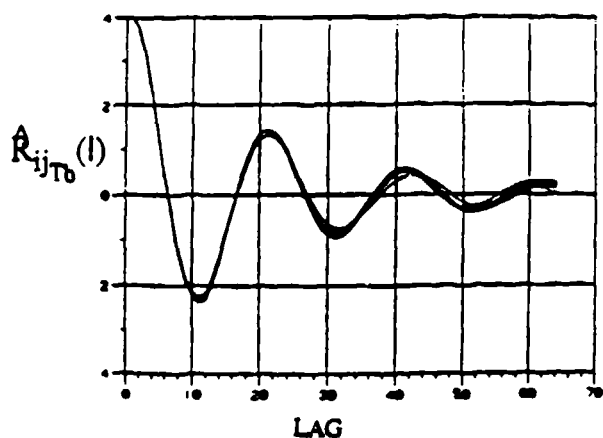
Fig. 4.7.10 Two channel AR(2) processes a.) channel 1 data (trial 1) b.) channel 2 data (trial 1) c.) channel 1 data (trial 2) d.) channel 2 data (trial 2) e.) ensemble averaged channel 1 autocorrelation (1,000 realizations) f.) ensemble averaged channel 2 autocorrelation (1,000 realizations); $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.95$, $|p_{12}|=0.99$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.



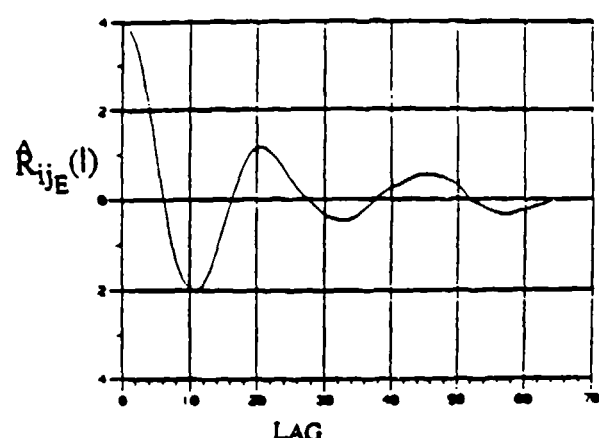
g



h



i



j

Fig. 4.7.10 (contin.) g.) time-averaged channel 1 autocorrelation (6 trials) h.) time averaged channel 2 autocorrelation (6 trials) i.) time-averaged cross-correlation (6 trials) j.) ensemble averaged cross-correlation (10,000 realizations); $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.95$, $|\rho_{12}|=0.99$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.

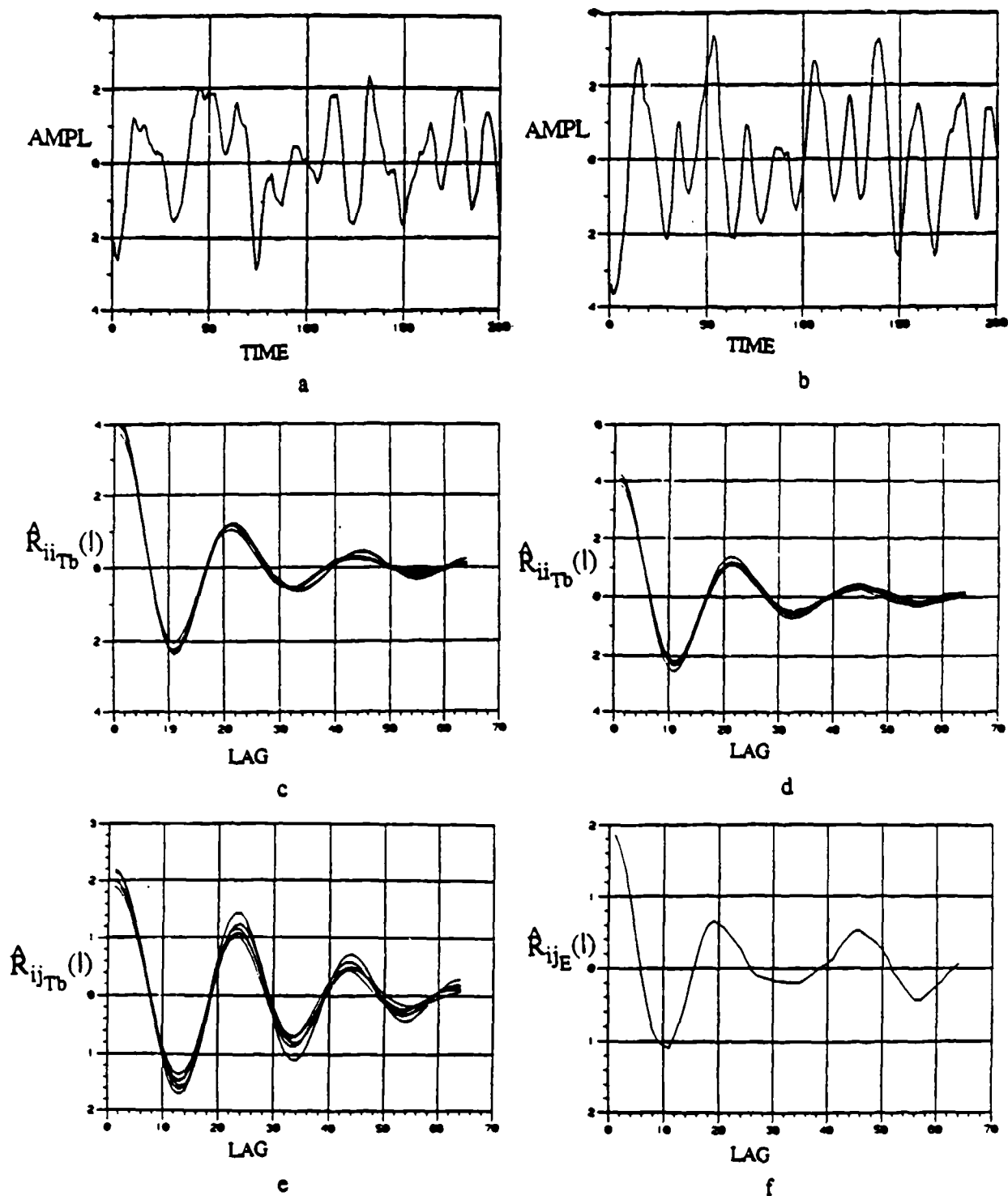
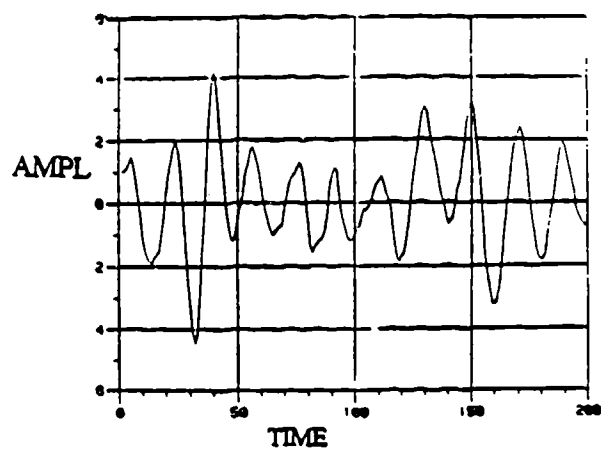
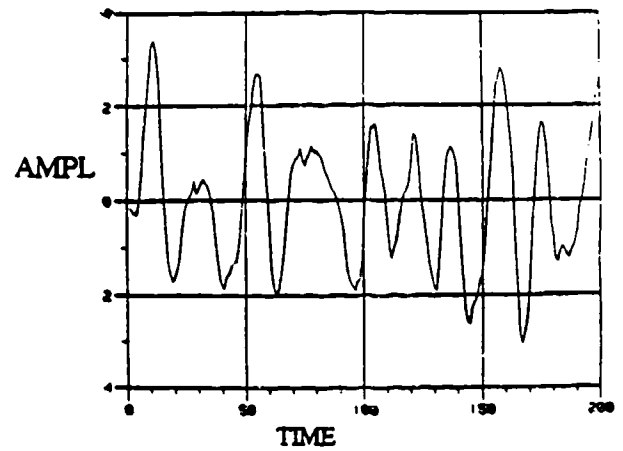


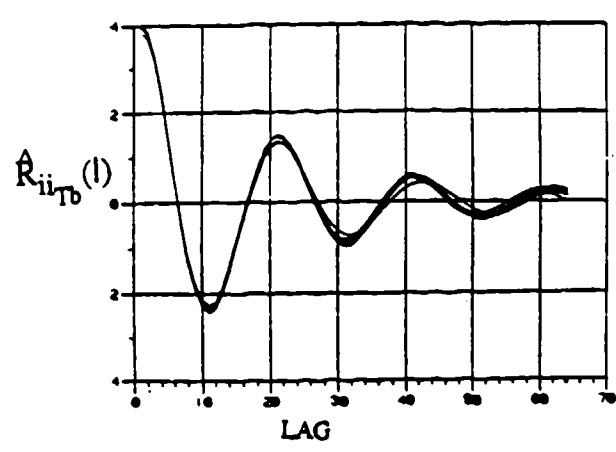
Fig. 4.7.11 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) time-averaged channel 1 autocorrelation (6 trials) d.) time-averaged channel 2 autocorrelation (6 trials) e.) time-averaged cross-correlation (6 trials) f.) ensemble averaged cross-correlation (10,000 realizations); $\sigma_{11}^2 = \sigma_{22}^2 = 4$, $|p_{12}| = 0.5$, $\lambda_{11} = \lambda_{22} = \lambda_{12} = 0.95$, $l_{12} = 0$.



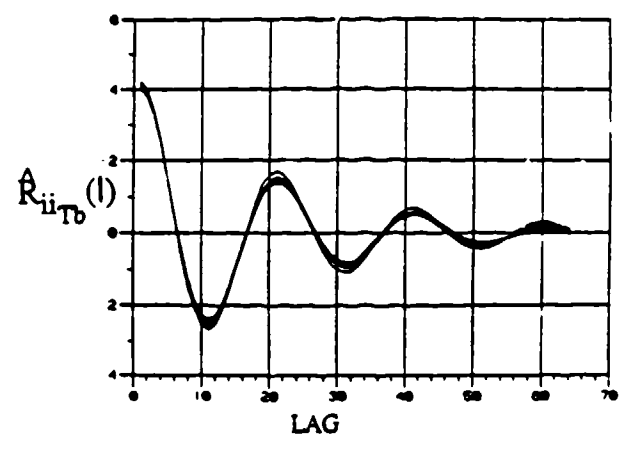
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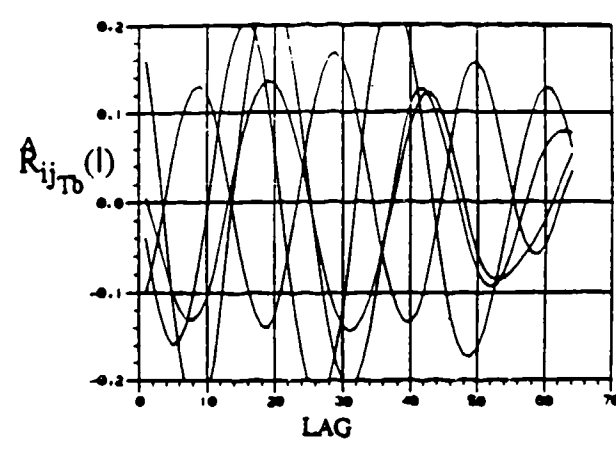
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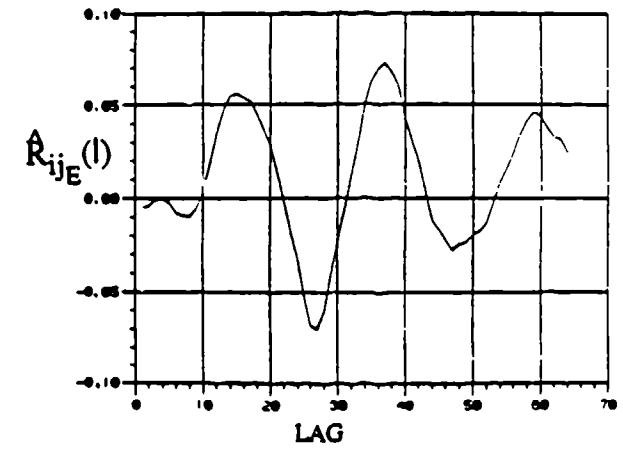
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d



e



f

Fig. 4.7.12 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) time-averaged channel 1 autocorrelation (6 trials) d.) time-averaged channel 2 autocorrelation (6 trials) e.) time-averaged cross-correlation (4 trials) f.) ensemble averaged cross-correlation (10,000 realizations); $\lambda_{11}=\lambda_{22}=0.95$, $|p_{12}|=0$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.

g and h indicates that the six realizations of the estimated time-averaged autocorrelation sequences, each based on 10,000 time observations, vary considerably from trial-to-trial. This implies that the error variances of the time-averaged autocorrelation sequences are large. Likewise, this same behavior is noted for the time averaged cross-correlation sequence in plot i. These results provide an indication that, although N_T is as high as 10,000, ergodicity cannot be assumed to hold for this window sample size; ie., the estimates of the auto- and cross-correlation sequences obtained by time averaging over a single realization will, in general, differ considerably from the ensemble averaged value when the temporal correlation coefficients λ_{ii} are high.

In Fig 4.7.10, the cross-correlation coefficient is increased slightly to 0.99 while the temporal correlation coefficients are reduced to 0.95 on both channels. In this figure, we show the same plots as displayed in Fig 4.7.9. We note in plots a and b, that both channel processes have become more uncorrelated in time as evidenced by their more rapid temporal fluctuation. However, noting the scale change on the plots a and b, we see that the cross-correlation between the channel processes has increased; ie., both waveforms are nearly identical. This high correlation results from the high value of $|\rho_{12}|=0.99$. Again, plots e and f show the ensemble averaged autocorrelation functions for each channel. Six realizations of the corresponding time-averaged autocorrelation sequences shown in plots g and h show a significant decrease in the error variance of these functions as compared to the previous figure. Likewise the six realizations of the time-averaged cross-correlation sequence in plot i also show a reduction in their error variance.

In Fig 4.7.11 and 4.7.12, the temporal correlation coefficients are held at 0.95 as in 4.7.10; however, the cross-correlation coefficient is reduced to 0.5 and 0.0, respectively. In each of these figures, plots a and b show a single realization of 200 sample observations. An overlay of six corresponding temporally-averaged autocorrelation sequences, based on 10,000 time sample observations each, is shown in plots c and d. Six trials of the temporal-averaged cross-correlation sequence are shown in plot e, while the corresponding ensemble averaged result, based on 10,000 realizations, is shown in plot f. A visual comparison of the temporally-averaged autocorrelation sequences in Figs 4.7.10 g and h, 4.7.11 c and d and 4.7.12 c and d indicates that the error variance

associated with these plots appears to remain constant. Again, we note that λ_{11} and λ_{22} have remained the same for each plot, although $|\rho_{12}|$ has changed significantly from 0.99 to zero. Close examination of the scale levels for the time-averaged cross-correlation functions among these figures also indicates that the error variances of these estimated functions has not changed; ie., the curves range in value approximately ± 0.2 . Thus, it appears that the error variance of the autocorrelation function does not vary as a function of $|\rho_{12}|$. In section 5.2.2, we show that this is true for a special case of correlation among the quadrature components of the processes.

The cross-correlation coefficient is held at zero in Figs 4.7.13 and 4.7.14, while the temporal correlation coefficients are decreased to 0.8 and 0.4, respectively. Plots a and b for Figs 4.7.12, 4.7.13 and 4.7.14 show the effect of the decreasing temporal correlation; ie., the processes are becoming more whitened as λ_{ii} decreases. The resulting autocorrelation functions shown in plots c and d for these cases also reflect this trend by approaching a delta function as λ_{ii} decreases. Close examination of each of these figures provides some indication that the variance of the cross-correlation function decreases as λ_{11} and λ_{22} decrease.

In Figs 4.7.15 through 4.7.20, similar plots are shown. Fig 4.7.15 shows the case of two temporally uncorrelated processes which are highly correlated across channels. In Fig 4.7.16, the temporal cross-correlation coefficient λ_{12} is raised to 0.97. The value of $|\rho_{12}|$ was lowered to 0.3 in order to satisfy the positive semi-definiteness constraint condition. In the expanded view of the cross-correlation function displayed in Fig 4.7.16f, we note that the resulting estimated values at these lags remain high out to the lag value of 2. Beyond $l=2$, however, the cross-correlation function drops significantly. This result will be discussed later. Fig 4.7.18 shows the interesting result obtained when the temporally uncorrelated signal on channel 2 is required to be correlated with the high temporally correlated channel 1 process. Fig 4.7.2.18c shows an overlay of the two sample functions shown in plots a and b. As this figure indicates, the noisy channel 2 process appears as a modulation on the channel 1 process. It is also noted that the autocorrelation functions for each channel have maxima and

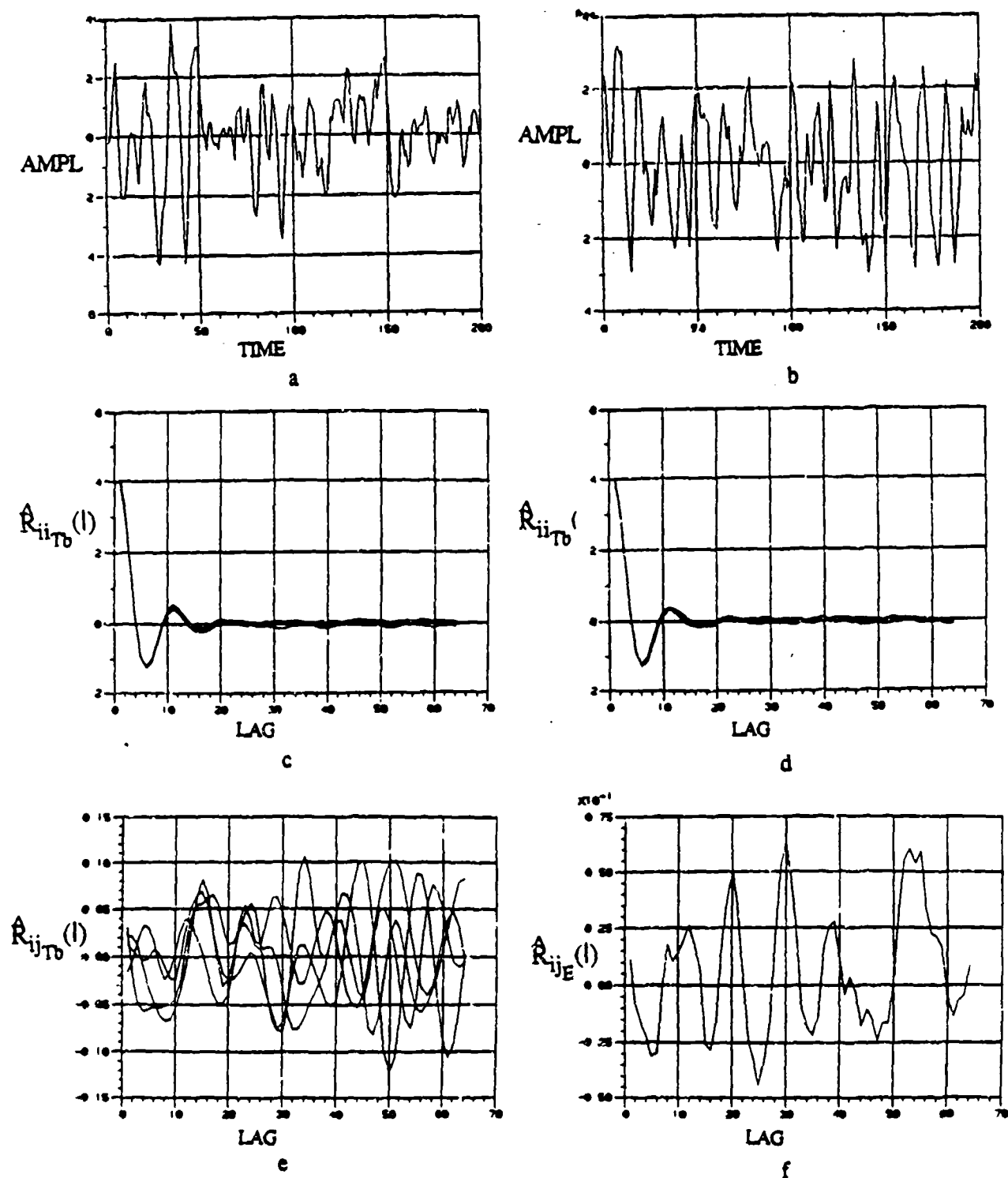


Fig. 4.7.13 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) time-averaged channel 1 autocorrelation (6 trials) d.) time-averaged channel 2 autocorrelation (6 trials) e.) time-averaged cross-correlation (6 trials) f.) ensemble averaged cross-correlation (10,000 realizations); $\lambda_{11}=\lambda_{22}=0.8$, $|\rho_{12}|=0$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.

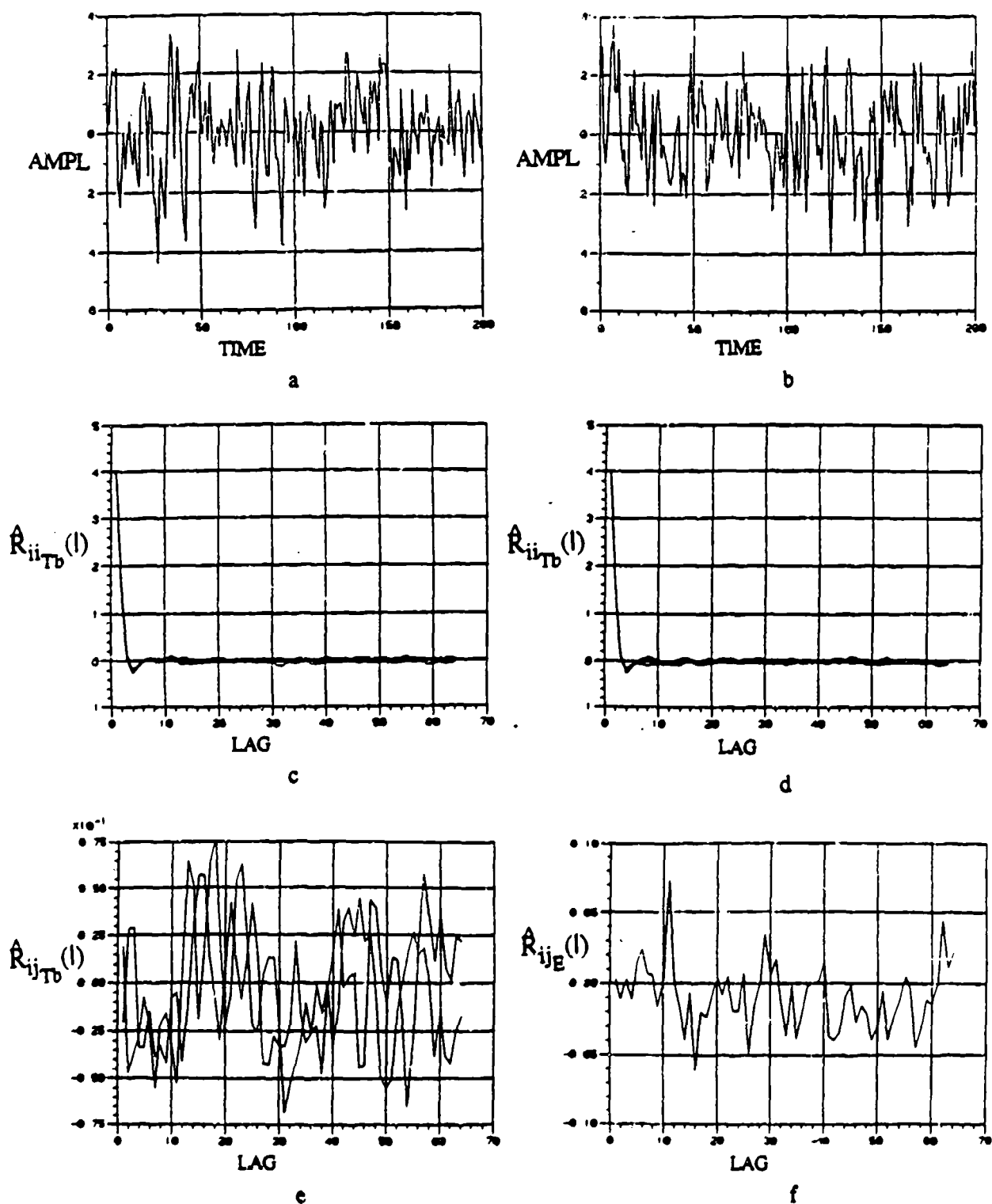


Fig. 4.7.14 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) time-averaged channel 1 autocorrelation (6 trials) d.) time-averaged channel 2 autocorrelation (6 trials) e.) time-averaged cross-correlation (6 trials) f.) ensemble averaged cross-correlation (10,000 realizations); $\lambda_{11}=\lambda_{22}=0.4$, $|p_{12}|=0$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.

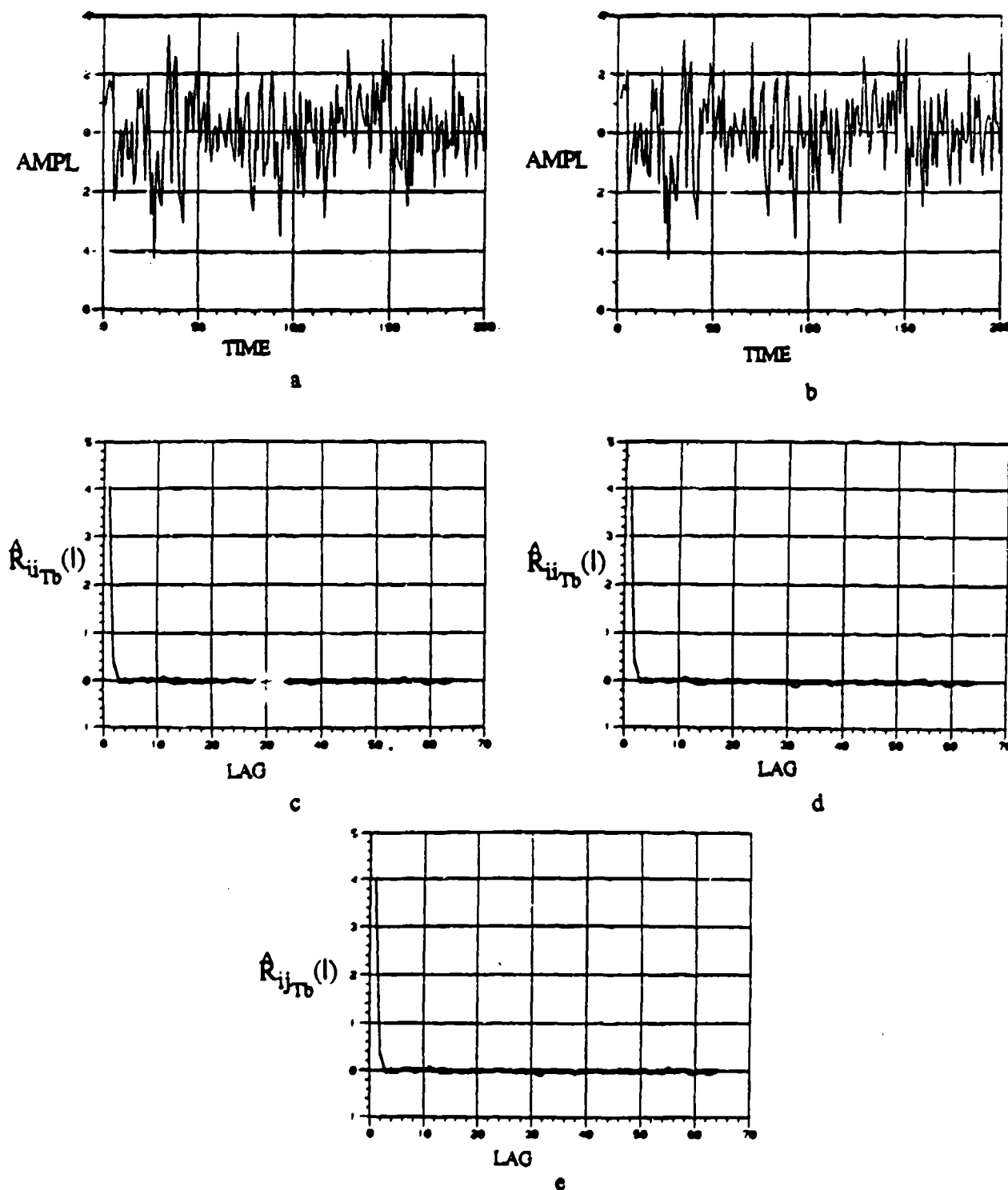


Fig.4.7.15 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) time-averaged channel 1 autocorrelation (6 trials) d.) time-averaged channel 2 autocorrelation (6 trials) e.) time-averaged cross-correlation (6 trials); $|\rho_{12}|=0.99$, $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.1$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.

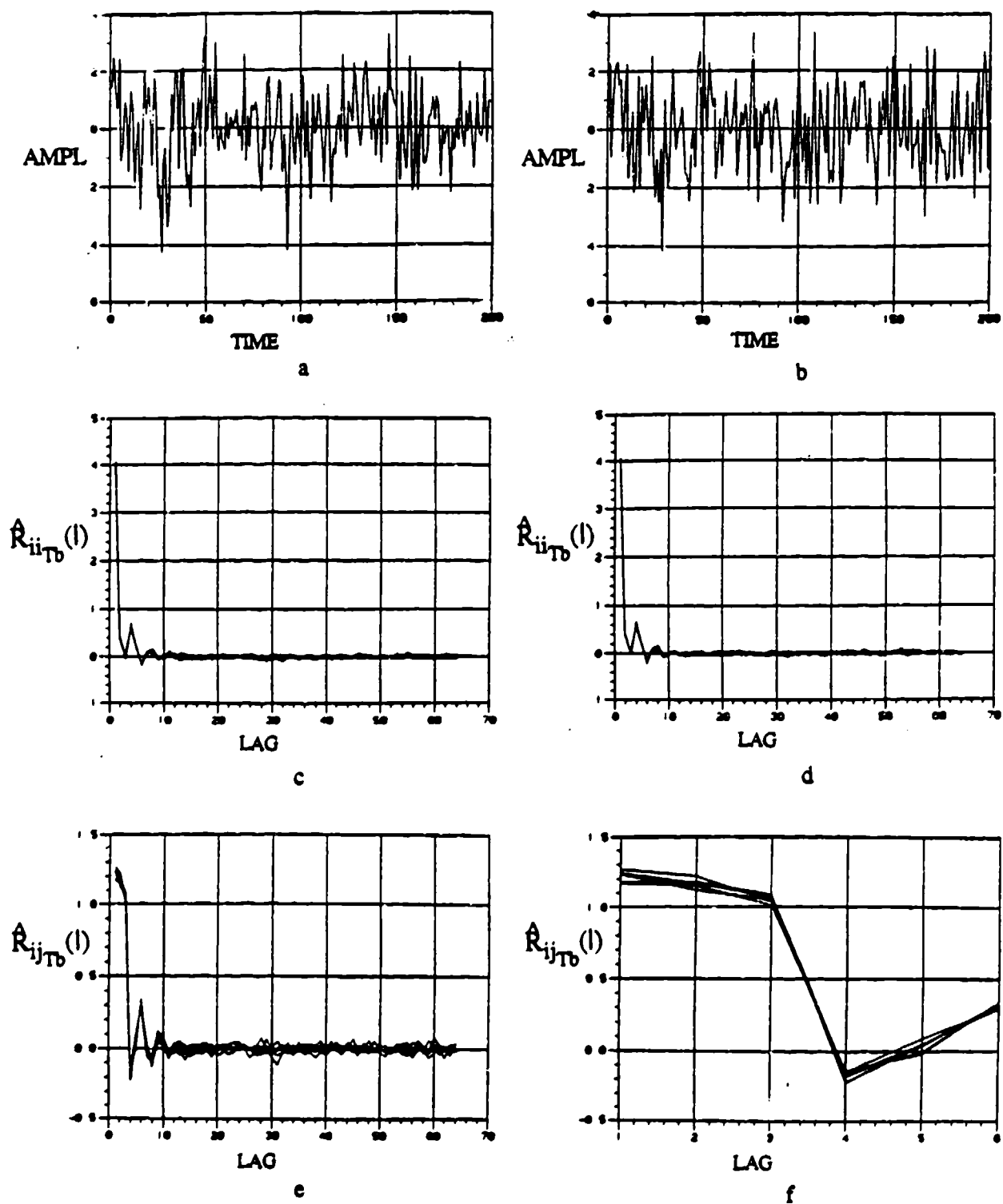


Fig. 4.7.16 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) time-averaged channel 1 autocorrelation (6 trials) d.) time-averaged channel 2 autocorrelation (6 trials) e.) time-averaged cross-correlation (64 lags-6 trials) f.) time-averaged cross-correlation (6 lags-6 trials); $\lambda_{11}=\lambda_{22}=0.1$, $\lambda_{12}=0.97$, $\sigma_{11}^2=\sigma_{22}^2=4$, $|\rho_{12}|=0.3$, $l_{12}=0$.

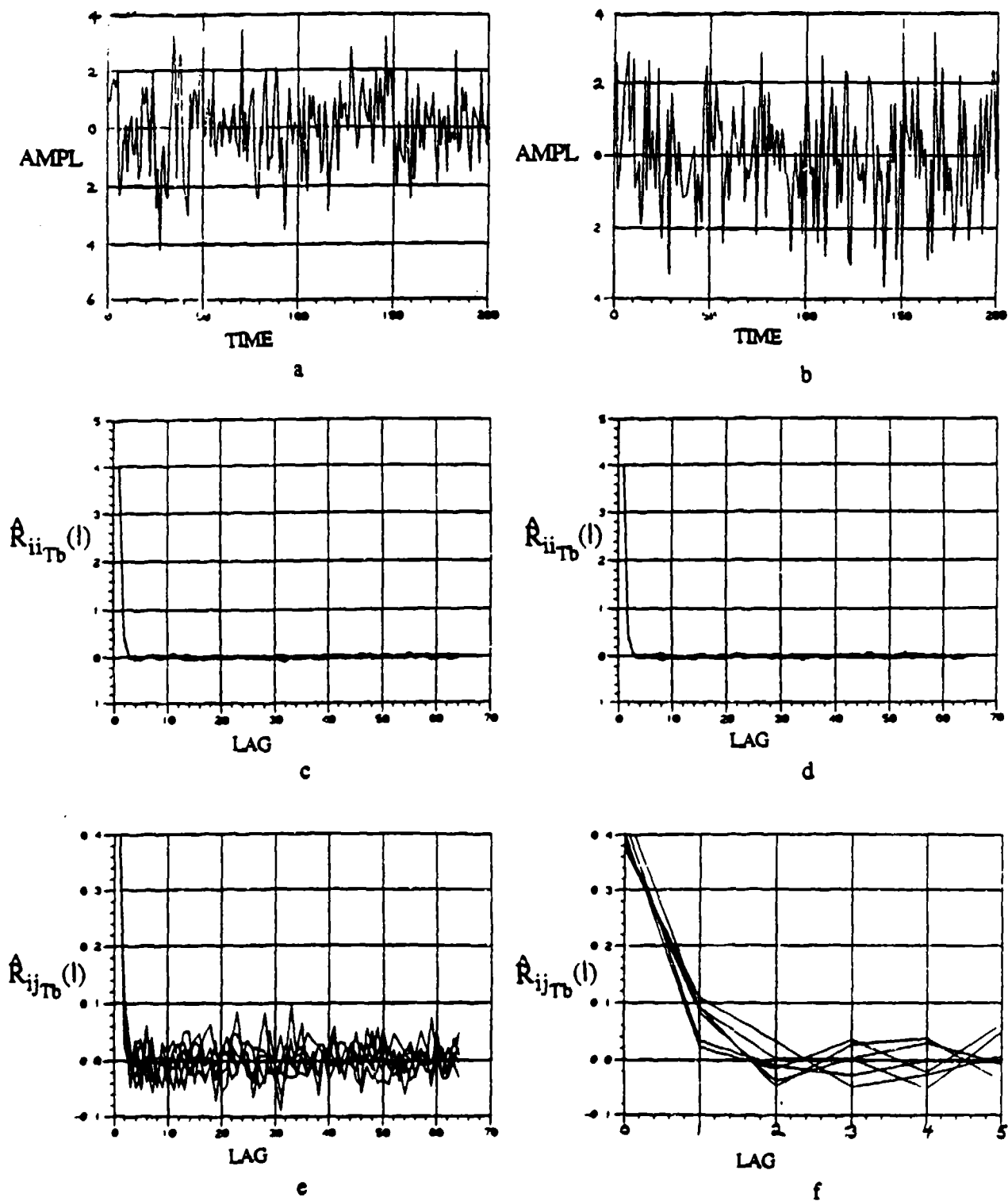


Fig. 4.7.17 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) time-averaged channel 1 autocorrelation (6 trials) d.) time-averaged channel 2 autocorrelation (6 trials) e.) time-averaged cross-correlation (64 lags-6 trials) f.) time-averaged cross-correlation (6 lags-6 trials); $\lambda_{11}=\lambda_{22}=0.1$, $\lambda_{12}=0.2$, $|\rho_{12}|=0.1$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.

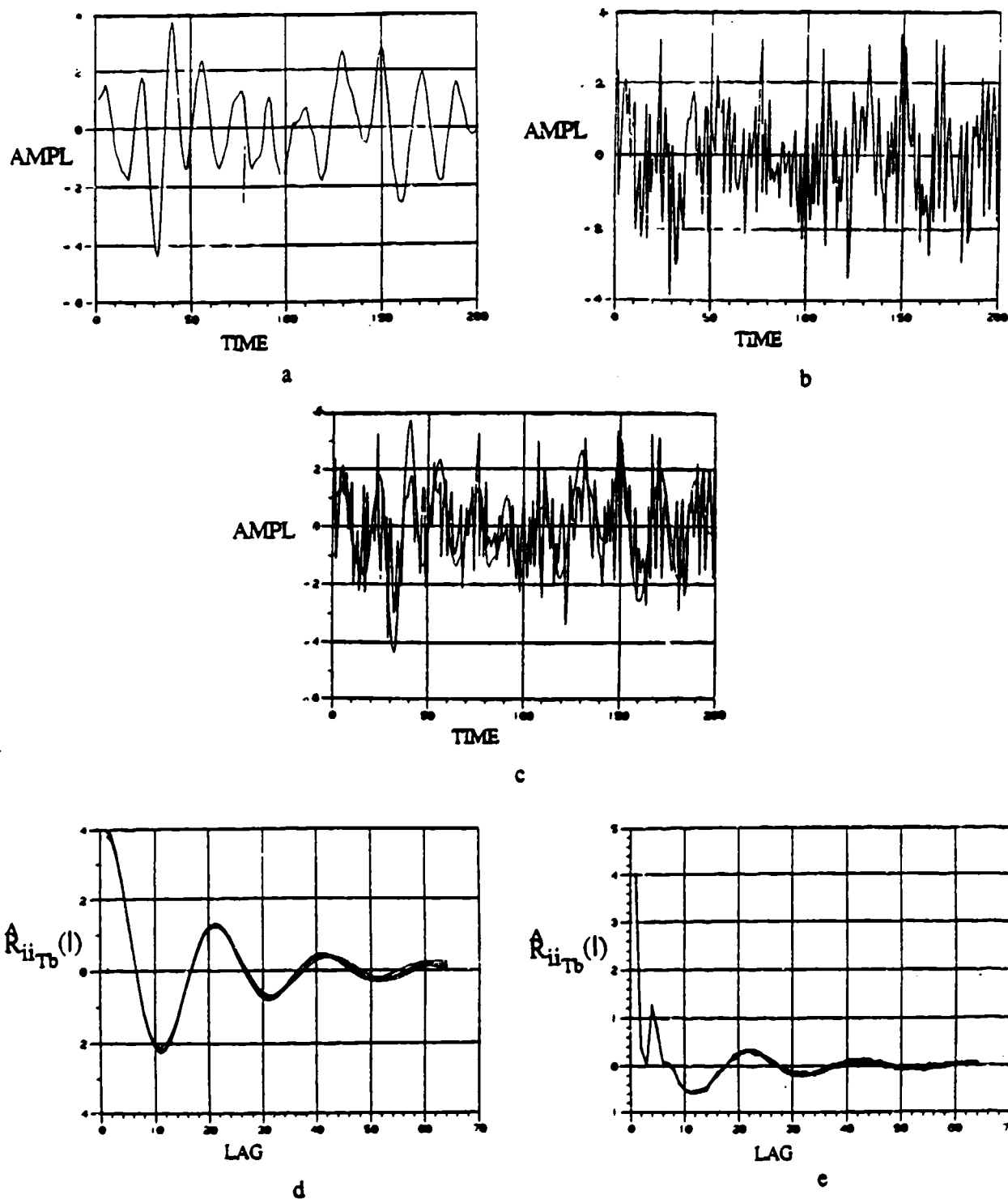


Fig. 4.7.18 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) overlay of channels 1 and 2 d.) time-averaged channel 1 auto- correlation (6 trials) e.) time-averaged channel 2 autocorrelation (6 trials); $\lambda_{11}=0.95$, $\lambda_{22}=0.1$, $\lambda_{12}=0.97$, $\lambda_{21}=0.97$, $|\rho_{12}|=0.5$, $\sigma_{11}^2=\sigma_{22}^2=4$, $I_{12}=0$.

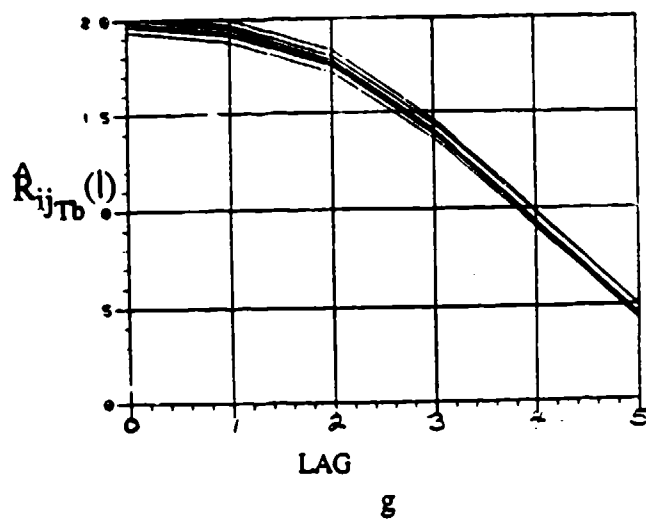
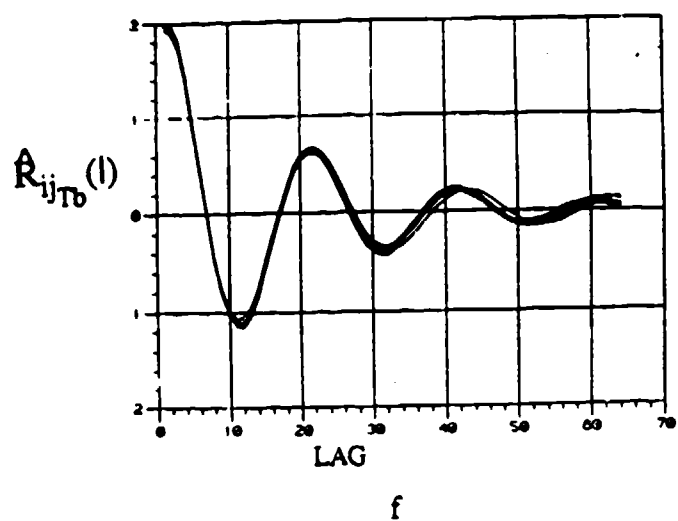


Fig 4.7.18 (contin.) f.) time-averaged cross-correlation (64 lags-6 trials) g.) time-averaged cross-correlation (6 lags-6 trials); $\lambda_{11}=0.95$, $\lambda_{22}=0.1$, $\lambda_{12}=0.97$, $|\rho_{12}|=0.5$, $\sigma_{11}^2=\sigma_{22}^2=4$, $l_{12}=0$.

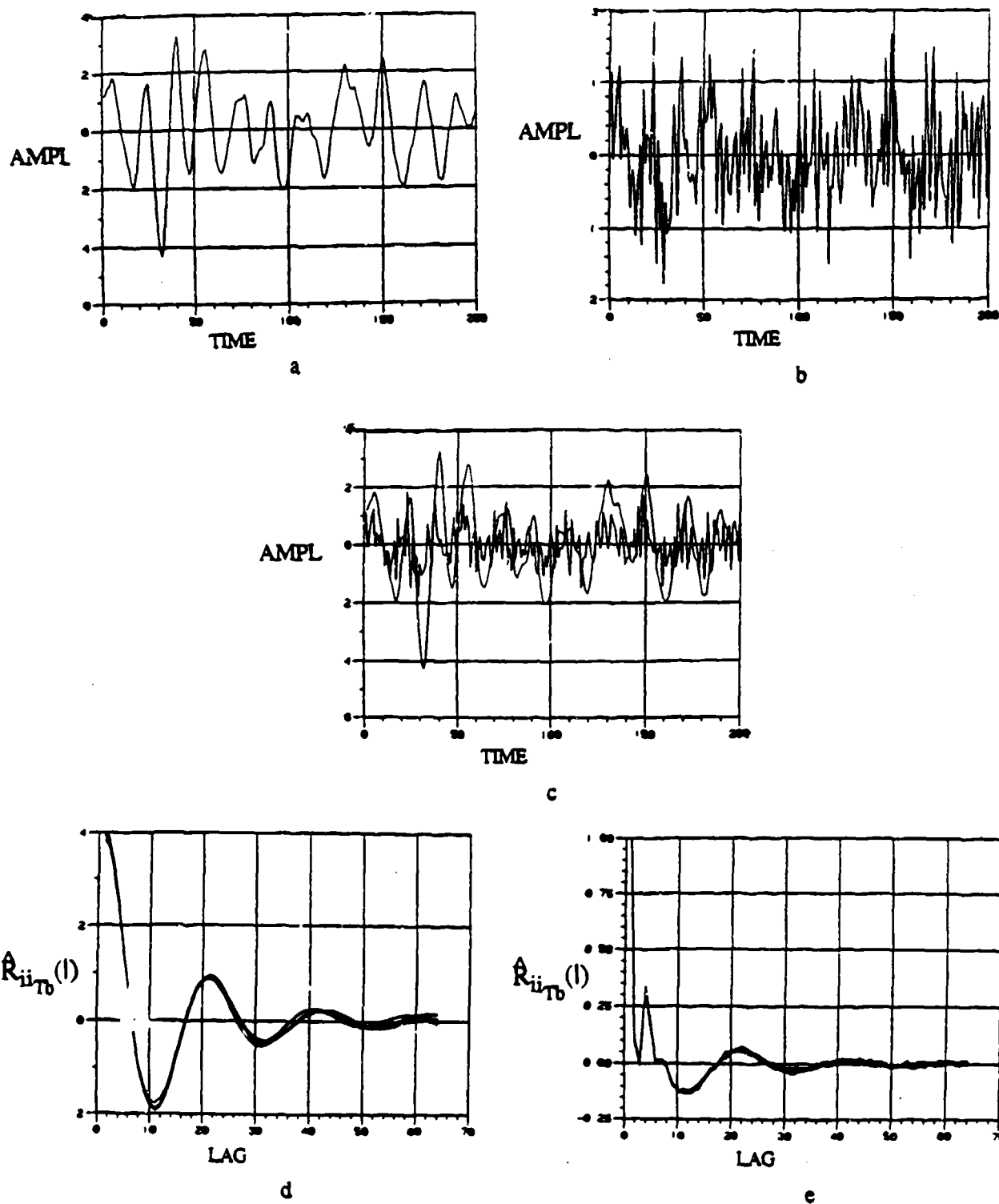
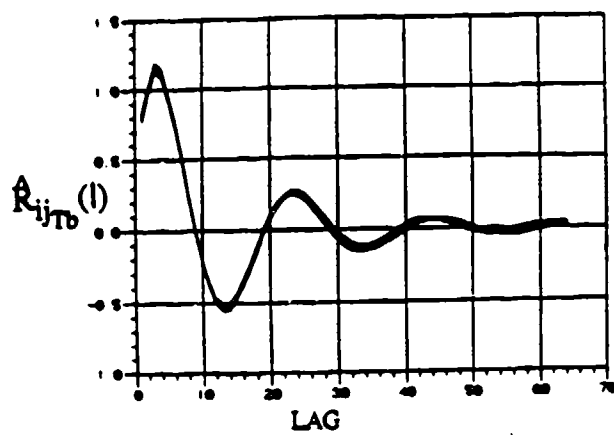
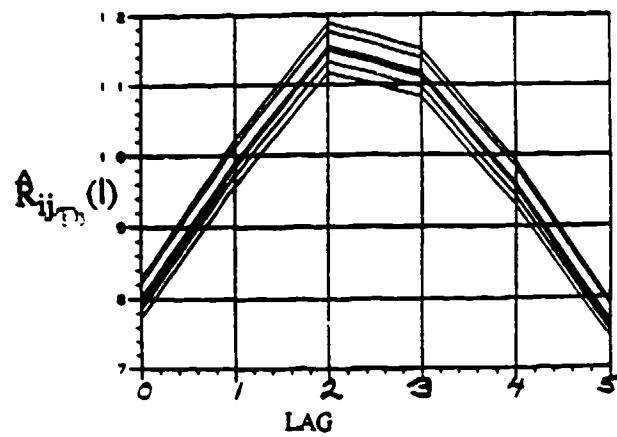


Fig. 4.7.19 Two channel AR(2) processes a.) channel 1 data b.) channel 2 data c.) overlay of channels 1 and 2 d.) time-averaged channel 1 auto- correlation (6 trials) e.) time-averaged channel 2 autocorrelation (6 trials); $\lambda_{11}=0.95$, $\lambda_{22}=0.1$, $\lambda_{12}=0.97$, $|p_{12}|=0.4$, $\sigma_{11}^2=4$, $\sigma_{22}^2=1$, $l_{12}=4$.



f



g

Fig 4.7.19 (contin.) f.) time-averaged cross-correlation (64 lags-6 trials) g.) time-averaged cross-correlation (6 lags-6 trials); $\lambda_{11}=0.95$, $\lambda_{22}=0.1$, $\lambda_{12}=0.97$, $|\rho_{12}|=0.4$, $\sigma_{11}^2=4$, $\sigma_{22}^2=1$, $l_{12}=4$.

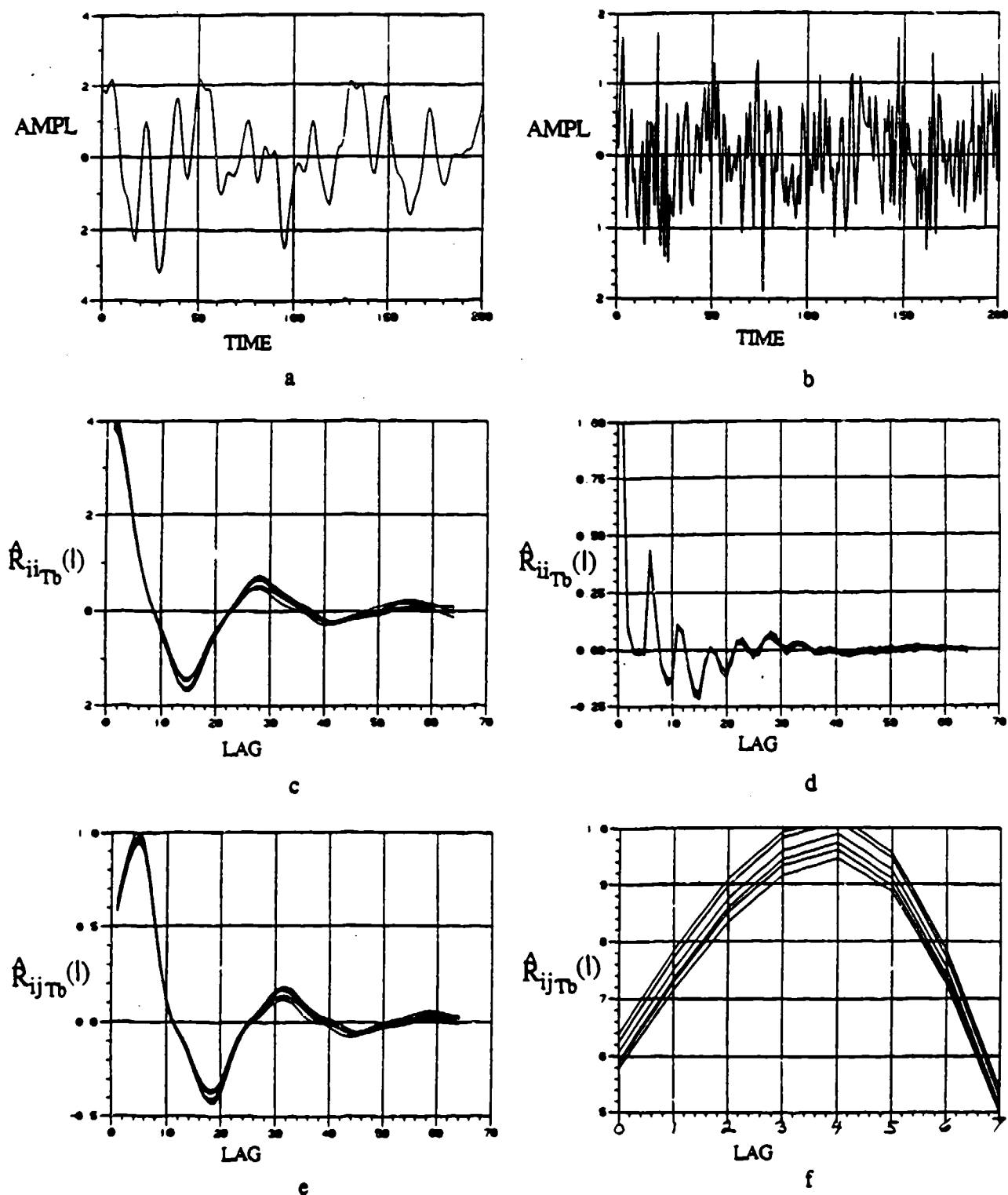


Fig. 4.7.20 Two channel AR(4) processes a.) channel 1 data b.) channel 2 data c.) time-averaged channel 1 autocorrelation (6 trials) d.) time-averaged channel 2 autocorrelation (6 trials) e.) time-averaged cross-correlation (64 lags-6 trials) f.) time-averaged cross-correlation (8 lags-6 trials); $\lambda_{11}=0.95$, $\lambda_{22}=0.1$, $\lambda_{12}=0.97$, $|\rho_{12}|=0.3$, $\sigma_{11}^2=4$, $\sigma_{22}^2=1$, $l_{12}=4$.

minima which occur at the same lag values. In addition, an interesting peak occurs at the third lag value in plot e. We will consider this later.

Finally, we consider the results shown in Figs 4.7.19 and 4.7.20, where l_{12} is specified to be 4. In these figures, we synthesize AR(2) and AR(4) processes, respectively. Plots a and b in each figure show 200 samples of the processes. As in the previous case, the difference in temporal correlation on each channel is noted. The overlay in Fig 4.7.19c shows the effect of the distinct variances on each channel as well as the moderate amount of cross-correlation. Six realizations of the corresponding autocorrelation functions are shown in plots d and e for Fig. 4.7.19 and plots c and d for Fig 4.7.20. The cross-correlation functions (six realizations) are displayed in the last two plots in each figure for 64 and 8 lag values, respectively. We note that in Fig 4.7.19g, the cross-correlation function peaks at a lag value of $l_{12}=2$ rather than $l_{12}=4$. Thus, the two channels of the AR(2) process, which is a function of its previous two samples, cannot be delayed with respect to each other by more than two lag values. In Fig 4.7.20f, however, we note that for the AR(4) process, the cross-correlation function does peak at $l_{12}=4$. We also mention that in the case of the AR(4) process described in Fig 4.7.20, the parameter $|p_{12}|$ had to be lowered to the value of 0.3. This was necessary to maintain the positive semi-definiteness requirement. Finally, in plots c and d for Fig 4.7.20, we again note that the maxima and minima occur at the same lag values. In addition, we note that for this AR(4) process, the peak values of the channel 2 autocorrelation function follow the shape of the channel 1 autocorrelation function. The rapid decrease in this function between the peak values is a measure of the 'noisy' channel 2 process. The periodic peak values, however, are a measure of the correlation on this channel resulting from its moderate degree of cross-correlation (ie., $|p_{12}|=0.3$) with the high temporally correlated channel 1 process.

Considerable insight into the process generation scheme can be obtained by examining the coefficients in Table 4.7.3 which were determined from a solution of the Yule-Walker equation. For processes corresponding to Fig 4.7.9, we first note that the white noise driving covariance matrix C has very low values compared to those in the A(1) and A(2) matrices. This will cause the resulting processes to be highly dependent upon the past data samples with minimal dependence upon the driving white noise vector. This increases the temporal

Fig.	A(1)	A(2)	C
4.7.9	$\begin{bmatrix} -1.98 & 4.54 \times 10^{-4} \\ 4.54 \times 10^{-4} & -1.98 \end{bmatrix}$	$\begin{bmatrix} 0.984 & -4.54 \times 10^{-4} \\ -4.54 \times 10^{-4} & -0.984 \end{bmatrix}$	$\begin{bmatrix} 0.0051 & 0.0 \\ 0.0049 & 0.0015 \end{bmatrix}$
4.7.10	$\begin{bmatrix} -1.81 & -3.6 \times 10^{-6} \\ 0.0 & -1.81 \end{bmatrix}$	$\begin{bmatrix} 0.903 & -3.8 \times 10^{-6} \\ 0.0 & 0.903 \end{bmatrix}$	$\begin{bmatrix} 0.269 & 0.0 \\ 0.266 & 0.038 \end{bmatrix}$
4.7.11	$\begin{bmatrix} -1.795 & -0.0313 \\ -0.0313 & -1.795 \end{bmatrix}$	$\begin{bmatrix} 0.903 & -6.23 \times 10^{-3} \\ -6.23 \times 10^{-3} & 0.903 \end{bmatrix}$	$\begin{bmatrix} 0.265 & 0.0 \\ 0.066 & 0.257 \end{bmatrix}$
4.7.12	$\begin{bmatrix} -1.81 & 0.0 \\ 0.0 & -1.81 \end{bmatrix}$	$\begin{bmatrix} 0.903 & 0.0 \\ 0.0 & 0.903 \end{bmatrix}$	$\begin{bmatrix} 0.269 & 0.0 \\ 0.0 & 0.269 \end{bmatrix}$
4.7.13	$\begin{bmatrix} -1.31 & 0.0 \\ 0.0 & -1.31 \end{bmatrix}$	$\begin{bmatrix} 0.64 & 0.0 \\ 0.0 & 0.64 \end{bmatrix}$	$\begin{bmatrix} 0.922 & 0.0 \\ 0.0 & 0.922 \end{bmatrix}$
4.7.14	$\begin{bmatrix} -0.464 & 0.0 \\ 0.0 & -0.464 \end{bmatrix}$	$\begin{bmatrix} 0.16 & 0.0 \\ 0.0 & 0.16 \end{bmatrix}$	$\begin{bmatrix} 1.81 & 0.0 \\ 0.0 & 1.81 \end{bmatrix}$
4.7.15	$\begin{bmatrix} -0.101 & 4.84 \times 10^{-7} \\ 4.78 \times 10^{-7} & -0.101 \end{bmatrix}$	$\begin{bmatrix} -0.01 & -2.9 \times 10^{-8} \\ -2.9 \times 10^{-8} & -0.01 \end{bmatrix}$	$\begin{bmatrix} 1.99 & 0.0 \\ 1.97 & 0.281 \end{bmatrix}$
4.7.16	$\begin{bmatrix} 0.072 & -0.334 \\ -0.334 & 0.072 \end{bmatrix}$	$\begin{bmatrix} 0.183 & -0.31 \\ -0.31 & 0.184 \end{bmatrix}$	$\begin{bmatrix} 1.82 & 0.0 \\ 0.74 & 0.166 \end{bmatrix}$
4.7.17	$\begin{bmatrix} -0.099 & -0.010 \\ -0.010 & -0.099 \end{bmatrix}$	$\begin{bmatrix} 0.099 & -0.019 \\ -0.019 & 0.099 \end{bmatrix}$	$\begin{bmatrix} 1.989 & 0.0 \\ 0.195 & 1.980 \end{bmatrix}$
4.7.18	$\begin{bmatrix} -1.792 & -0.0311 \\ -0.801 & 0.257 \end{bmatrix}$	$\begin{bmatrix} 0.9174 & -0.0292 \\ 0.0166 & 0.3545 \end{bmatrix}$	$\begin{bmatrix} 0.261 & 0.0 \\ 0.521 & 1.519 \end{bmatrix}$
4.7.19	$\begin{bmatrix} -1.632 & -0.149 \\ -0.782 & 0.162 \end{bmatrix}$	$\begin{bmatrix} 0.789 & -0.154 \\ 0.545 & 0.321 \end{bmatrix}$	$\begin{bmatrix} 0.213 & 0.0 \\ 0.639 & 0.608 \end{bmatrix}$
4.7.20	$\begin{bmatrix} -3.093 & -0.0418 \\ -4.053 & 0.283 \end{bmatrix}$	$\begin{bmatrix} 3.974 & -0.0386 \\ -8.2604 & 0.4039 \end{bmatrix}$	$\begin{bmatrix} 0.0665 & 0.0 \\ 0.5798 & 0.5834 \end{bmatrix}$
	A(3)	A(4)	
	$\begin{bmatrix} -2.573 & -0.0375 \\ -6.797 & 0.4123 \end{bmatrix}$	$\begin{bmatrix} 0.698 & -0.0384 \\ 2.085 & 0.452 \end{bmatrix}$	

Table 4.7.3 Coefficients used in the AR Equation for the Multichannel Cases of Section 4.7.2.

correlation on the channel processes. In addition, the c_{11} and c_{21} elements of this matrix are approximately equal, while c_{22} is much smaller. This will cause the elements of the vector $\underline{u}(n)$ in eq(4.5.4) to be nearly identical, thus contributing to the high cross-correlation. We again point out that in this case of high temporal correlation, the C matrix has small values. This will cause the variance of the white noise driving term to be small. If the overall variance of the process is required to be large, the synthesis process must be operated over a long initial transient period to allow the processes to reach the specified variance level. For this case, 2,000 samples were generated in the transient period and discarded before collecting data. This quantity was not sufficient to achieve the specified variance level and thus, slightly low values of the ensemble-averaged autocorrelation sequence resulted as shown in Fig. 4.7.9f.

For processes corresponding to those shown in Fig. 4.7.10, the c_{11} and c_{21} elements are again nearly equal and much greater than c_{22} so that the very high cross-correlation is obtained. We note, however, that all the C matrix elements have increased as compared to the previous case. This will cause the additive white noise driving variance to have a more significant effect on the resulting process. As a result, temporal correlation on each channel will decrease as expected since the λ_{11} and λ_{22} values have been decreased.

For processes corresponding to Fig. 4.7.12, we note that only c_{11} and c_{22} are non-zero. This will cause the vector $\underline{u}(n)$ to have totally uncorrelated elements, thus providing no cross-correlation as required since $|p_{12}|=0$. Examination of these elements for processes corresponding to Figs. 4.7.12 through 4.7.16 shows the increasing significance of the white noise driving term relative to the past values; ie., the C matrix eventually begins to weight the white noise vector $\underline{u}(n)$ higher than the $A(k)$ $k=1,2$ matrices weight the past samples. Thus, the temporal correlation decreases.

For the matrix elements corresponding to the Fig. 4.7.16 results, we make the following observations. First, the relatively high values of the C matrix elements compared to the $A(k)$ matrices again provides emphasis on the white noise driving term. Second, the relatively high and equal off-diagonal elements of the two $A(k)$ matrices provides high cross-correlation with respect to the two past sample values. This result is controlled by the high value of $\lambda_{12}=0.97$. Third, the value $|p_{12}|=0.3$ has a significant effect on the c_{21} element. It is close to

the maximum value for the cross-correlation coefficient that can be obtained under the constraint condition of positive semi-definiteness. The result is that the white noise vector is provided a moderately high cross-correlation. This case can be contrasted with the coefficients for Fig 4.7.17. In this case, the low value of $\lambda_{12}=0.2$ and $|p_{12}|=0.1$ causes the off-diagonal elements in the $A(k)$ matrices and the c_{21} to decrease significantly.

In this chapter, we have developed a time series methodology for the synthesis of multichannel autoregressive (AR) processes. The multichannel synthesis procedure provides a capability for the generation of processes with variable temporal and cross-channel correlation properties as well as signal-to-noise ratios. In addition, this method was generalized to provide a capability for generating AR processes in which we are able to control not only the temporal and cross-channel correlation, but also the correlation between quadrature components. The method allows one to specify these correlation properties directly in the correlation matrix of the concatenated quadrature form of the Yule-Walker equations presented in eq(4.6.6) and to synthesize the more general processes discussed in Appendix N. Finally, specific examples of synthesized scalar and vector processes are shown to illustrate the capability.

CHAPTER 5 ERGODICITY CONSIDERATIONS

5.1 INTRODUCTION

Ergodicity is the condition which enables time-averaged statistics of random processes to approximate those obtained by ensemble averages. This condition is often assumed in estimation and other signal processing applications. The ergodicity condition for auto- and cross-channel correlation functions expressed in terms of fundamental process parameters is derived in this dissertation. Specifically, analytic expressions are developed for the variance of the biased, time-averaged correlation functions for stationary discrete complex processes. These expressions relate the variance to the correlation functions of the underlying processes as well as the size of the observation window. When the variance is suitably small, the ergodic condition is said to hold; i.e., a time-average closely approximates the corresponding ensemble average. The analytic expressions for the variance provide a performance measure which can be used to specify the window size of the observation interval required to achieve a specific value of this variance. In this section, the functional forms of the correlation shaping functions described in section 4.3 are used to relate these variances to the process temporal and ensemble correlation parameters. A unique aspect of this development is the determination of the functional dependence of these expressions in terms of the process temporal and ensemble correlations. As a result, the analytic expressions provide a measure for the variance of the time-averaged correlation functions in terms of both the window of the sample integration size and these fundamental process parameters. As an example, the increase in sample size required to achieve a specific value of the variance of the time-averaged correlation functions is quantitatively related to the increase in temporal correlation as well as the variance of the underlying process. In addition, the analytic expressions are simplified for the general case of Gaussian processes with unconstrained quadrature components where the usual constraints associated with a complex Gaussian process are relaxed. Validity of the analytic expressions is presented using the multichannel process synthesis method described in chapter 4.0.

5.2. VARIANCE OF THE TIME-AVERAGED CORRELATION FUNCTIONS

5.2.1 Variance of the Time-Averaged Autocorrelation Function

The ensemble autocorrelation function is defined as the expectation of lagged products of a given stationary process when averaged over an ensemble of realizations. If this function is equal to the time-averaged autocorrelation function obtained from a single realization, the process is called autocorrelation ergodic. Consider the time-averaged estimate of the autocorrelation function using N_T observation samples

$$\hat{R}_{iiT}(l, N_T) = \begin{cases} \frac{1}{M} \sum_{n=0}^{N_T-l-1} x_i(n) x_i^*(n-l) & 0 \leq l \leq N_T-1 \\ \frac{1}{M} \sum_{n=0}^{N_T-|l|-1} x_i^*(n) x_i(n-|l|) & -(N_T-1) \leq l \leq 0. \end{cases} \quad (5.2.1)$$

For $M = N_T$, we obtain the biased[†] estimator $\hat{R}_{iiT_b}(l, N_T)$ while for $M = N_T - l$, we have the unbiased estimator $\hat{R}_{iiT_{ub}}(l, N_T)$ [3]. We now define

$$\phi(n, l) = x_i(n) x_i^*(n - l) \quad (5.2.2a)$$

and

$$R_{\phi\phi}(k, l) = E[\phi(n, l) \phi^*(n - k, l)] \quad (5.2.2b)$$

so that

$$C_{\phi\phi}(k, l) = E[\{ \phi(n, l) - E[\phi(n, l)] \} \{ \phi^*(n-k, l) - E[\phi^*(n-k, l)] \}] \quad (5.2.2c)$$

[†] In this dissertation, we primarily consider the biased estimator for the correlation functions with limited data since it ensures positive semi-definiteness. In [3], an estimator with unlimited data is also considered.

$$= R_{\phi\phi}(k,l) - E[\phi(n,l)]E[\phi^*(n-k,l)]. \quad (5.2.2d)$$

Using eq(5.2.1), the variance of $\hat{R}_{iiT_b}(l, N_T)$ can now be expressed as (see Appendix D)

$$V_{B_{ii}}(l, N_T) = E \left\{ [\hat{R}_{iiT_b}(l, N_T) - E[\hat{R}_{iiT_b}(l, N_T)]] [\hat{R}_{iiT_b}^*(l, N_T) - E[\hat{R}_{iiT_b}^*(l, N_T)]] \right\} \quad (5.2.3a)$$

$$= E[\hat{R}_{iiT_b}(l, N_T) \hat{R}_{iiT_b}^*(l, N_T)] - E[\hat{R}_{iiT_b}(l, N_T)] E[\hat{R}_{iiT_b}^*(l, N_T)] \quad (5.2.3b)$$

$$= \frac{1}{N_T} \sum_{k=-(N_T-||l|-1)}^{N_T-||l|-1} \left[1 - \frac{||l|+|k|}{N_T} \right] C_{\phi\phi}(k,l). \quad (5.2.3c)$$

In Appendices E through G, it is shown for the general case of Gaussian processes with zero-mean and unconstrained quadrature components that

$$V_{B_{ii}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-||l|-1)}^{N_T-||l|-1} \left[1 - \frac{||l|+|k|}{N_T} \right] \left[|R_{ii}(k)|^2 + \text{Re}\{F_{ii}(l,k)\} \right] \quad (5.2.4)$$

where

$$R_{ii}(k) = E[x_i(n)x_i^*(n-k)] \quad (5.2.5a)$$

and

$$F_{ii}(l,k) = E[x_i(n)x_i(n-l-k)]E[x_i^*(n-l)x_i^*(n-k)]. \quad (5.2.5b)$$

Using eq(E.6) in Appendix E, it can be shown in the special case where

$$R_{ii}^{\Pi}(l) = R_{ii}^{QQ}(l) \quad (5.2.6a)$$

and

$$R_{ii}^{IQ}(l) = -R_{ii}^{QI}(l) \quad (5.2.6b)$$

that

$$\operatorname{Re}\{F_{ij}(l,k)\} = 0 \quad \text{for all } l,k. \quad (5.2.7)$$

In this case, eq(5.2.4) becomes

$$V_{B_{ii}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T} \right] |R_{ii}(k)|^2. \quad (5.2.8)$$

5.2.2. Variance of the Time-Averaged Cross-Correlation Function

For the ensemble cross-correlation function, lagged products between two processes are averaged over an ensemble of realizations. Let the lagged product between the wide-sense jointly stationary processes $x_i(n)$ and $x_j(n)$ be denoted by

$$\beta(n,l) = x_i(n) x_j^*(n-l). \quad (5.2.9)$$

The autocorrelation function for $\beta(n,l)$ is expressed as

$$R_{\beta\beta}(k,l) = E[\beta(n,l)\beta^*(n-k,l)]. \quad (5.2.10)$$

From eq(5.2.9)

$$E[\beta(n,l)] = R_{ij}(l) \quad (5.2.11a)$$

$$E[\beta^*(n-k,l)] = R_{ij}^*(l). \quad (5.2.11b)$$

Using the same form as eq(5.2.2c)

$$C_{\beta\beta}(k,l) = R_{\beta\beta}(k,l) - E[\beta(n,l)] E[\beta^*(n-k,l)] \quad (5.2.12a)$$

$$= R_{\beta\beta}(k,l) - |R_{ij}(l)|^2. \quad (5.2.12b)$$

The variance of the time-averaged cross-correlation function is therefore expressed as

$$V_{Bij}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-||l|-1)}^{N_T-||l|-1} \left[1 - \frac{||l|+|k|}{N_T} \right] C_{\beta\beta}(k, l). \quad (5.2.13)$$

Using eq(5.2.12b) in (5.2.13), we have

$$V_{Bij}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-||l|-1)}^{N_T-||l|-1} \left[1 - \frac{||l|+|k|}{N_T} \right] [R_{\beta\beta}(k, l) - |R_{ij}(l)|^2]. \quad (5.2.14)$$

We now consider

$$R_{\beta\beta}(k, l) = E[\beta(n, l)\beta^*(n - k, l)] \quad (5.2.15a)$$

$$= E[x_i(n)x_j^*(n - l)x_i^*(n - k)x_j(n - l - k)] \quad (5.2.15b)$$

so that eq(5.2.14) becomes

$$V_{Bij}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-||l|-1)}^{N_T-||l|-1} \left[1 - \frac{||l|+|k|}{N_T} \right] \cdot \{ E[x_i(n)x_j^*(n - l)x_i^*(n - k)x_j(n - l - k)] - |R_{ij}(l)|^2 \}. \quad (5.2.16)$$

For Gaussian processes with zero-mean and unconstrained quadrature components, eq(5.2.15b) can be expressed as (see Appendix H)

$$\begin{aligned} R_{\beta\beta}(k, l) &= E[x_i(n)x_j^*(n - l)]E[x_i^*(n - k)x_j(n - l - k)] \\ &\quad + E[x_i(n)x_i^*(n - k)]E[x_j^*(n - l)x_j(n - l - k)] \\ &\quad + E[x_i(n)x_j(n - l - k)]E[x_j^*(n - l)x_i^*(n - k)] \end{aligned} \quad (5.2.17a)$$

$$= R_{ij}(l)R_{ij}^*(l) + R_{ii}(k)R_{jj}^*(k) + F_{ij}(l, k) \quad (5.2.17b)$$

$$= |R_{ij}(l)|^2 + R_{ii}(k)R_{jj}^*(k) + F_{ij}(l,k) \quad (5.2.17c)$$

where

$$F_{ij}(l,k) = E[x_i(n)x_j(n-l-k)]E[x_j^*(n-l)x_i^*(n-k)]. \quad (5.2.18)$$

Using eq(5.2.17c) in (5.2.14), we obtain

$$V_{Bij}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T} \right] [R_{ii}(k)R_{jj}^*(k) + F_{ij}(l,k)]. \quad (5.2.19)$$

In Appendix H, we show that the summation in eq(5.2.19) over positive and negative values cancels the imaginary terms in $R_{ii}(k)R_{jj}^*(k)$ and $F_{ij}(l,k)$, so

$$V_{Bij}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T} \right] \text{Re}[R_{ii}(k)R_{jj}^*(k) + F_{ij}(l,k)]. \quad (5.2.20)$$

We note that the term $\text{Re}[F_{ij}(l,k)]$ will contribute a dependence of $V_{Bij}(l, N_T)$ upon cross-correlation parameters such as the cross-correlation coefficient ρ_{ij} defined in section 4.3. From eqs(5.2.6), where the subscript ii is replaced by ij , $R_{ij}^{II}(l) = R_{ij}^{QQ}(l)$ and $R_{ij}^{IQ}(l) = -R_{ij}^{QI}(l)$. In Appendix O, we show that

these conditions hold when the bandpass processes are jointly stationary and narrowband. In Appendix H, we also show that in this case $F_{ij}(l,k)=0$ and, therefore, eq(5.2.9) reduces to

$$V_{Bij}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T} \right] \text{Re}[R_{ii}(k)R_{jj}^*(k)]. \quad (5.2.21)$$

5.2.3. Sample Variance of the Correlation Functions

Consider N_R realizations of the random process $x_i(n)$. Let each realization be indexed by the integer α ; $\alpha=1,2,\dots,N_R$. Corresponding to the realization with index α , let $\hat{R}_{ijT_b}(l, N_T|\alpha)$ be the biased, time-averaged cross-correlation function estimate using N_T observation samples. The sample variance of the time-averaged cross-correlation function estimate is computed from N_R realizations using the expression

$$\text{Var}[\hat{R}_{ijT_b}(l, N_T):N_R] = \frac{1}{N_R-1} \sum_{\alpha=1}^{N_R} | \hat{R}_{ijT_b}(l, N_T|\alpha) - \bar{\hat{R}}_{ijT_b}(l, N_R|\alpha) |^2 \quad (5.2.22)$$

where

$$\bar{\hat{R}}_{ijT_b}(l, N_R|\alpha) = \frac{1}{N_R} \sum_{\alpha=1}^{N_R} \hat{R}_{ijT_b}(l, N_T|\alpha). \quad (5.2.23)$$

The above results reduce to the sample variance of the time-averaged autocorrelation function when $i=j$.

5.2.4. Discussion

In section 4.3, we presented functional forms for the auto- and cross-correlation functions in terms of the process parameters. For example, an exponentially shaped autocorrelation function with positive magnitude can be expressed as

$$R_{ii}(k) = \sigma_{ii}^2 (\lambda_{ii})^{|k|} \exp[j\theta_{ii}(l)]. \quad (5.2.24)$$

where σ_{ii}^2 is the variance of the channel i process, λ_{ii} is a temporal correlation coefficient which is a measure of the correlation magnitude between consecutive samples such that $0 \leq \lambda_{ii} \leq 1$, and $\theta_{ii}(l)$ is the phase of the correlation function. Using eq(5.2.24) in eqs(5.2.8) and (5.2.21), we obtain

$$V_{B_{ii}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T} \right] \sigma_{ii}^4 (\lambda_{ii})^{2|k|} \quad (5.2.25)$$

and

$$V_{B_{ij}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T} \right] \sigma_{ii}^2 (\lambda_{i1})^{|k|} \sigma_{jj}^2 (\lambda_{22})^{|k|} \cdot \cos[\theta_{ii}(l) - \theta_{jj}(l)]. \quad (5.2.26)$$

We note in the special case described by the constraints $R_{ij}^{II}(l) = R_{ij}^{QQ}(l)$ and $R_{ij}^{IQ}(l) = -R_{ij}^{QI}(l)$, that $V_{B_{ij}}(l, N_T)$ is independent of $|p_{12}|$. When invoking the ergodic assumption, one would like the variances in eqs(5.2.25) and (5.2.26) to be suitably small so that the time-averaged correlation function is a satisfactory approximation to the ensemble correlation function. For a given λ_{ii} and σ_{ii}^2 , these analytic expressions provide a means for determining the value of N_T required to reduce the variance to a specified level. A unique aspect of this development is the determination of these expressions in terms of both the observation window size and the process parameters σ_{ii}^2 and λ_{ii} . These expressions indicate for $\lambda_{ii} < 1$, that their limit approaches zero as N_T approaches infinity. Thus, for stationary processes, ergodicity holds in all cases except for total temporal correlation (ie., $\lambda_{ii}=1$). However, as σ_{ii}^2 and λ_{ii} change, these equations provide a quantitative measure expressing the requirements on the observation window to obtain time-averaged correlation function estimates which yield a close approximation to the ensemble averaged correlation functions.

5.3. RESULTS

5.3.1. Computed Variance of the Time-Averaged Autocorrelation Function

In this section, we validate eq(5.25) for the exponential autocorrelation function of eq(5.2.24). Consider a real AR(1) process for which the autocorrelation function can be shown to be

$$R_{AR}(k) = R_{AR}(0) [-a(1)]^{|k|} \quad (5.3.1a)$$

$$= \sigma_{AR}^2 [-a(1)]^{|k|} \quad (5.3.1b)$$

where

$$R_{AR}(0) = \frac{\sigma_u^2}{1 - a^2(1)} = \sigma_{AR}^2 \quad (5.3.2)$$

and σ_u^2 , $a(1)$ and σ_{AR}^2 are the white noise driving variance, the AR(1) parameter,

and the variance of the AR(1) process, respectively. To be consistent with the AR(1) process used in this example, the constants in eq(5.2.24) were chosen to be $\lambda_{ii} = -a(1)$, $\sigma_{ii}^2 = \sigma_{AR}^2$ and $\theta_{ii}(l) = 0$ for all l . Table 5.3.1 contains the parameters used

in the process synthesis procedure. The variance σ_{ii}^2 was held fixed at 4 while λ_{ii} was varied using 0.1, 0.7 and 0.99.

Fig.	σ_{ii}^2	λ_{ii}	N_T	N_R
5.3.1	4	0.1	100	10,000
5.3.2	4	0.7	100	10,000
5.3.3	4	0.99	100	10,000

Table 5.3.1 Parameters used in the single channel process synthesis procedure.

Figs 5.3.1 through 5.3.3 show the autocorrelation functions and their variances. In plot a, we show six realizations of the biased, time-averaged autocorrelation function plotted over 64 lag values using $N_T=100$ time samples. The corresponding ensemble averaged result is shown in plot b using $N_R=10,000$. The sample variance $\text{Var}[\hat{R}_{iiT_b}(l, N_T):N_R]$ of the biased time-averaged autocorrelation function computed using eq(5.2.22) with $N_R=10,000$ is displayed in c. The corresponding analytic calculation of $V_{B_{ii}}(l, N_T)$ using eq(5.2.25) is shown in plot d. The decrease in this quantity as a function of l is a result of the weighting associated with the biased estimator. In plot e of Fig. 5.3.1, we show the corresponding sample variance using the unbiased estimate of the autocorrelation function [ie., eq(5.2.1) with $M=N_T-l$] and note the increase in this quantity as a function of l . A similar behavior is noted in Fig 5.12 of Ref [37] for continuous-time processes. The computed results are in excellent agreement with the analytic expressions (note the scale change between plots c and d in Fig 5.3.1).

In Fig 5.3.4, the maximum value of $V_{B_{ii}}(l, N_T)$ which occurs at $l=0$ is plotted (solid curve) as a function of λ_{ii} for $N_T=100$ and $N_T=1000$ using the analytic expression of eq(5.2.25). The corresponding sample variances of the time-averaged autocorrelation function estimates computed at lag zero using eq(5.2.22) with $i=j$ for the synthesized data processes are also plotted ($\bullet$) on this curve. These values were computed using N_R realizations of the functions. For $N_T=100$, $N_R=10,000$ was used while for $N_T=1000$, the number of realizations was reduced to $N_R=1,000$. Reducing N_R decreases the confidence level associated with the computation of the sample variance $\text{Var}[\hat{R}_{ijT_b}(l, N_T):N_R]$. However, the error bars representing one standard deviation from the mean are less than the size of the printed ($\bullet$). The results shown in this figure indicate the significant increase in the variance $V_{B_{ii}}(l, N_T)$ as a function of λ_{ii} . In addition, it provides a measure of the required observation window size necessary to achieve a specific level of the variance. For example, the curve for $N_T=1000$ has a very distinct knee for $\lambda_{ii} \approx 0.9$. For processes with a temporal correlation above this value, a larger observation data window would be required to reduce $V_{B_{ii}}(l, N_T)$ to values less than 0.2. We also note that for $N_T=100$, large variances can be obtained even for moderately low values of λ_{ii} .

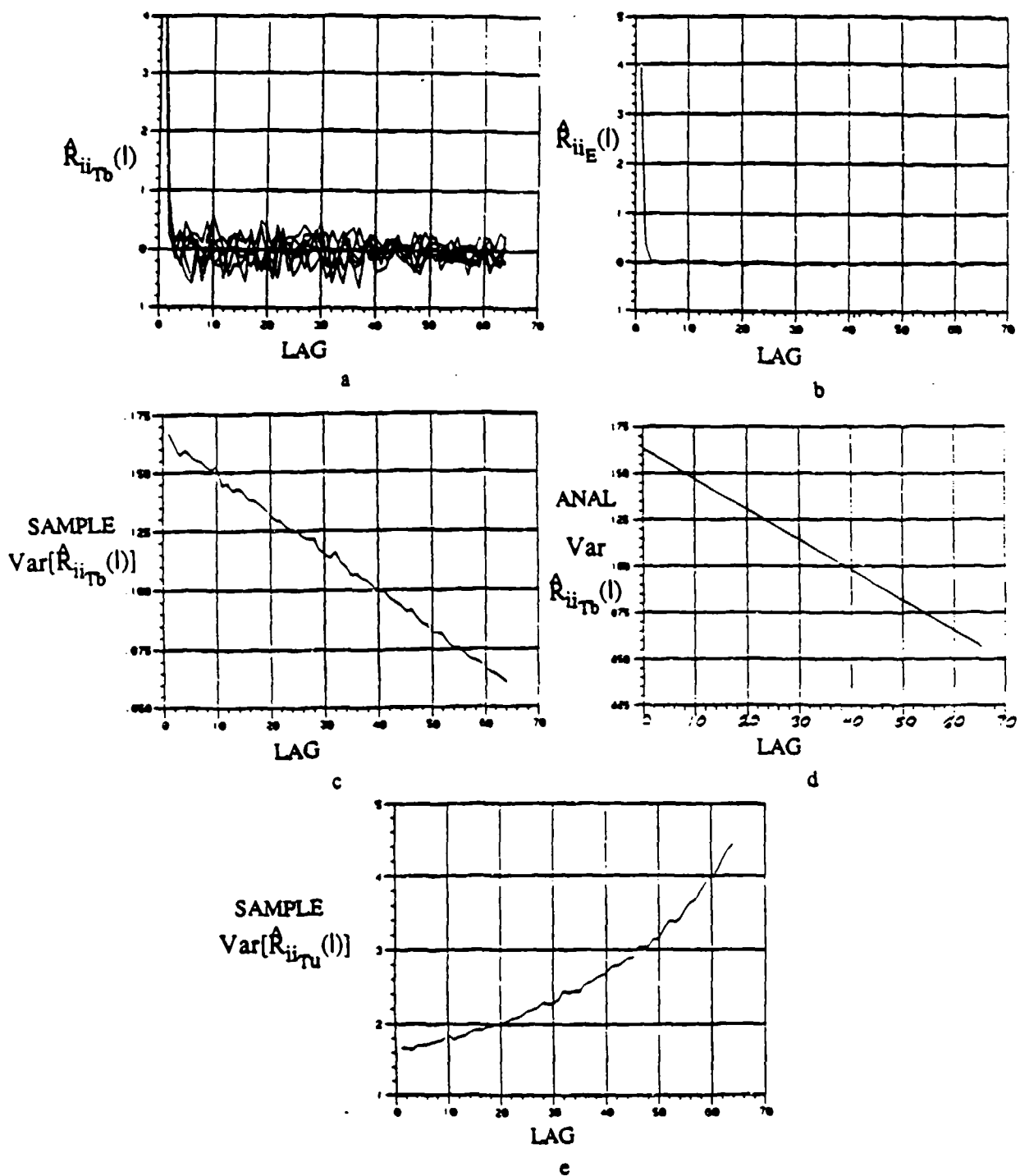


Fig. 5.3.1 Estimated autocorrelation function and its variance for an AR(1) process; $\lambda=0.1$, $\sigma_s^2=4$, a.) $\hat{R}_{ii_{Tb}}(l)$ (6-trials) using $N_T=100$ b.) ensemble-averaged $\hat{R}_{ii_E}(l)$ using 10,000 realizations c.) sample variance of $\hat{R}_{ii_{Tb}}(l)$ d.) analytical variance of $\hat{R}_{ii_{Tb}}(l)$ e.) sample variance of the unbiased $\hat{R}_{ii_{Tu}}(l)$.

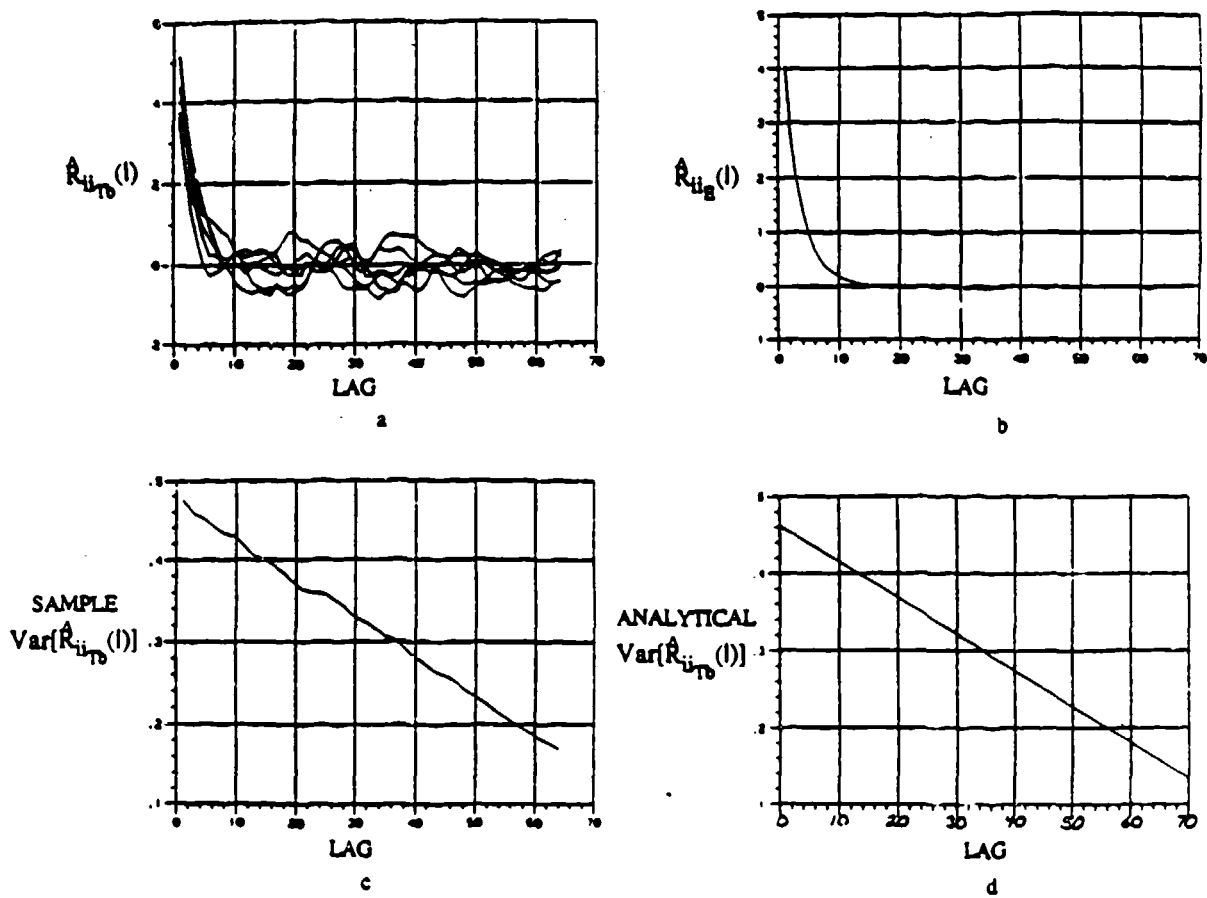


Fig. 5.3.2 Estimated autocorrelation function and its variance for an AR(1) process; $\lambda=0.7$, $\sigma_s^2=4$, a.) $\hat{R}_{ii_{Tb}}(l)$ (6-trials) using $N_T=100$ b.) ensemble-averaged $\hat{R}_{ii_E}(l)$ using 10,000 realizations c.) sample variance of $\hat{R}_{ii_{Tb}}(l)$ d.) analytical variance of $\hat{R}_{ii_{Tb}}(l)$.

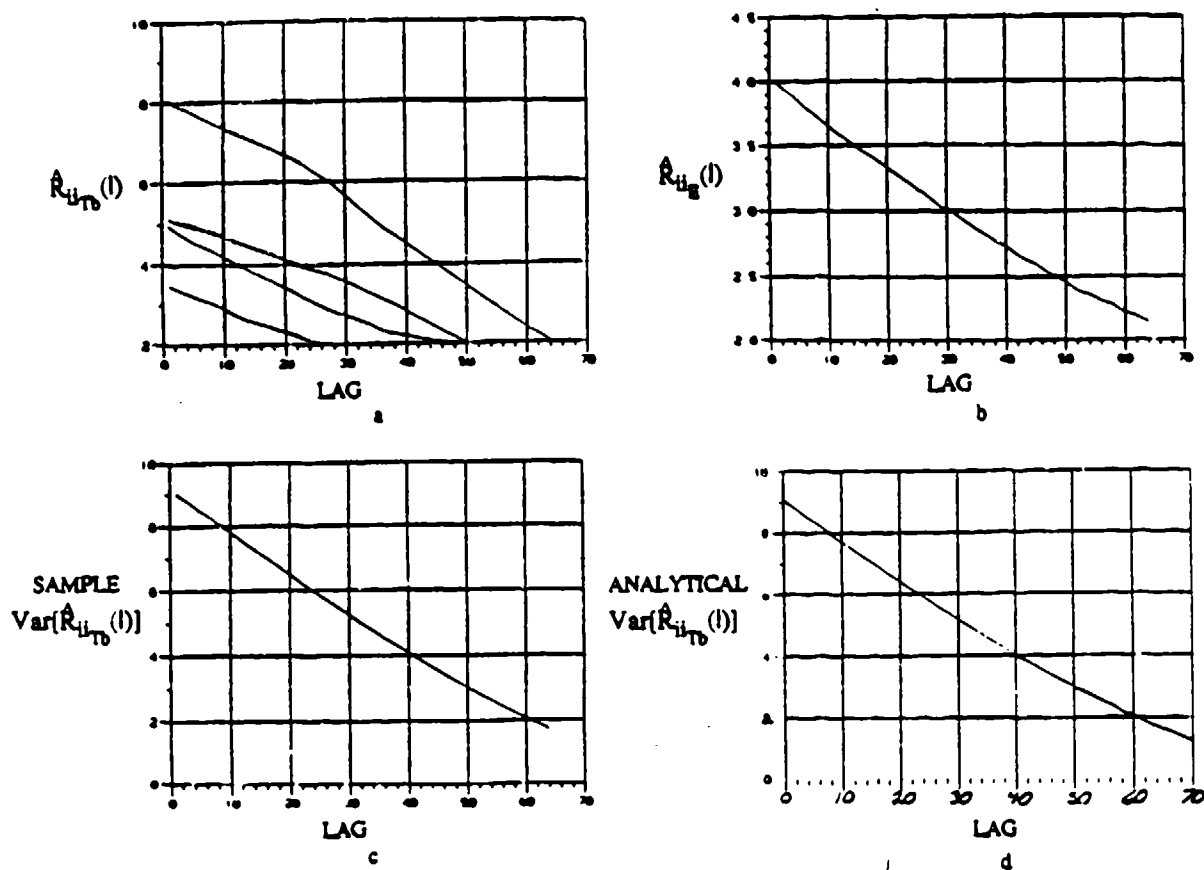


Fig. 5.3.3 Estimated autocorrelation function and its variance for an AR(1) process; $\lambda=0.99$, $\sigma_s^2=4$, a.) $\hat{R}_{ii_{Tb}}(l)$ (6-trials) using $N_T=100$ b.) ensemble-averaged $\hat{R}_{ii_E}(l)$ using 10,000 realizations c.) sample variance of $\hat{R}_{ii_{Tb}}(l)$ d.) analytical variance of $\hat{R}_{ii_{Tb}}(l)$.

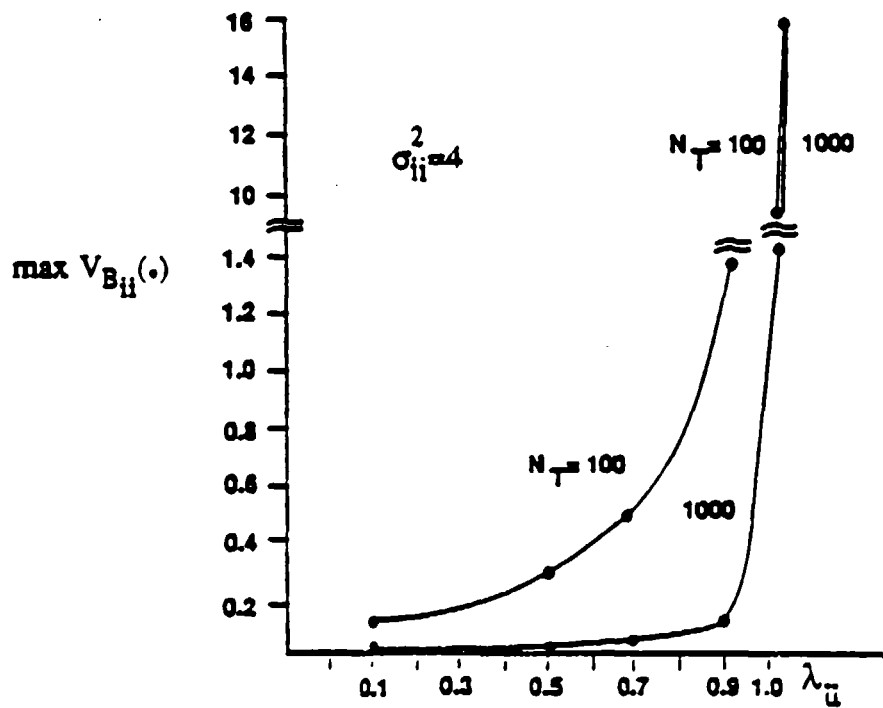


Fig 5.3.4 Maximum variance of the time-averaged autocorrelation function versus λ_{ii} for $\sigma_{ii}^2=4$; analytical (-) and computed (•).

5.3.2. Computed Variance of the Time-Averaged Cross-Correlation Function

We now consider the case of a two channel AR(1) process with real correlation functions. Table 5.3.2 contains the parameters used in the process synthesis procedure described in section 4.5 to specify the correlation functions of the Yule-Walker equation. This equation is then solved for the matrices $A(1)$ and $[\Sigma_f]_u$. Table 5.3.3 lists the AR coefficient matrix $A(1)$ and the Cholesky matrix C obtained by the Cholesky decomposition of $[\Sigma_f]_u$. We note in each of the $A(1)$ matrices that we maintain the conditions

$$-a_{11} = \lambda_{11} \quad (5.3.4a)$$

$$-a_{22} = \lambda_{22} \quad (5.3.4b)$$

and

$$a_{12} = a_{21} = 0. \quad (5.3.4c)$$

We also note that the cross-correlation between the process is controlled by the elements of the C matrices; ie., for high correlation, the c_{11} and c_{21} elements become nearly equal while c_{22} diminishes. For low correlation, the c_{21} element diminishes. In these cases, the variance of the processes σ_{11}^2 and σ_{22}^2 was held fixed at 4 while λ_{11} and λ_{22} had values of 0.1, 0.5 and 0.9. The cross-correlation coefficient $|\rho_{12}|$ had values of 0.99, 0.5 and 0.0. For the special case where $F_{ij}(l,k)=0$ [see discussion leading to eq(5.2.21)], the results are independent of the correlation coefficient $|\rho_{12}|$.

Figs 5.3.5 through 5.3.9 show the results for these variances based on the computed values of eq(5.2.22) and the analytic expression of eq(5.2.26). The computed results are in excellent agreement with the analytic expressions. In Fig 5.3.10, the maximum value of $V_{B_{ij}}(l, N_T)$ which occurs at $l=0$ is plotted (solid curves) as a function of $\lambda=\lambda_{11}=\lambda_{22}$ for $N_T = 100$ and $N_T = 1000$ using the analytic expression of eq(5.2.26) and $\sigma_{11}^2 = \sigma_{22}^2 = 4$. The corresponding maximum values of the sample variances of the time-averaged cross-correlation function estimates computed using the synthesized data processes are also plotted ($\bullet$) on this curve. These values were computed using $N_R = 1000$ realizations of the functions. In

Fig.	σ_{11}^2	σ_{22}^2	λ_{11}	λ_{22}	λ_{12}	$ \rho_{12} $	N_T	N_R	l_{12}
5.3.5	4	4	0.1	0.1	0.1	0.99	100	1000	0
5.3.6	4	4	0.1	0.1	0.1	0.5	100	1000	0
5.3.7	4	4	0.1	0.1	0.1	0.0	100	1000	0
5.3.8	4	4	0.5	0.5	0.5	0.5	100	1000	0
5.3.9	4	4	0.9	0.9	0.9	0.5	100	1000	0

Table 5.3.2 Parameters used in the two-channel AR(1) synthesis procedure

Figure	A(1)	C
5.3.5	$\begin{bmatrix} -0.1 & 4.7 \times 10^{-7} \\ 4.7 \times 10^{-7} & -0.1 \end{bmatrix}$	$\begin{bmatrix} 1.99 & 0.0 \\ 1.97 & 0.28 \end{bmatrix}$
5.3.6	$\begin{bmatrix} -0.1 & 0.0 \\ 0.0 & -0.1 \end{bmatrix}$	$\begin{bmatrix} 1.99 & 0.0 \\ 0.995 & 1.723 \end{bmatrix}$
5.3.7	$\begin{bmatrix} -0.1 & 0.0 \\ 0.0 & -0.1 \end{bmatrix}$	$\begin{bmatrix} 1.99 & 0.0 \\ 0.0 & 1.99 \end{bmatrix}$
5.3.8	$\begin{bmatrix} -0.5 & 0.0 \\ 0.0 & -0.5 \end{bmatrix}$	$\begin{bmatrix} 1.732 & 0.0 \\ 0.866 & 1.50 \end{bmatrix}$
5.3.9	$\begin{bmatrix} -0.9 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}$	$\begin{bmatrix} 0.872 & 0.0 \\ 0.436 & 0.755 \end{bmatrix}$

Table 5.3.3 Matrices A(1) and C used in the two-channel synthesis procedure.

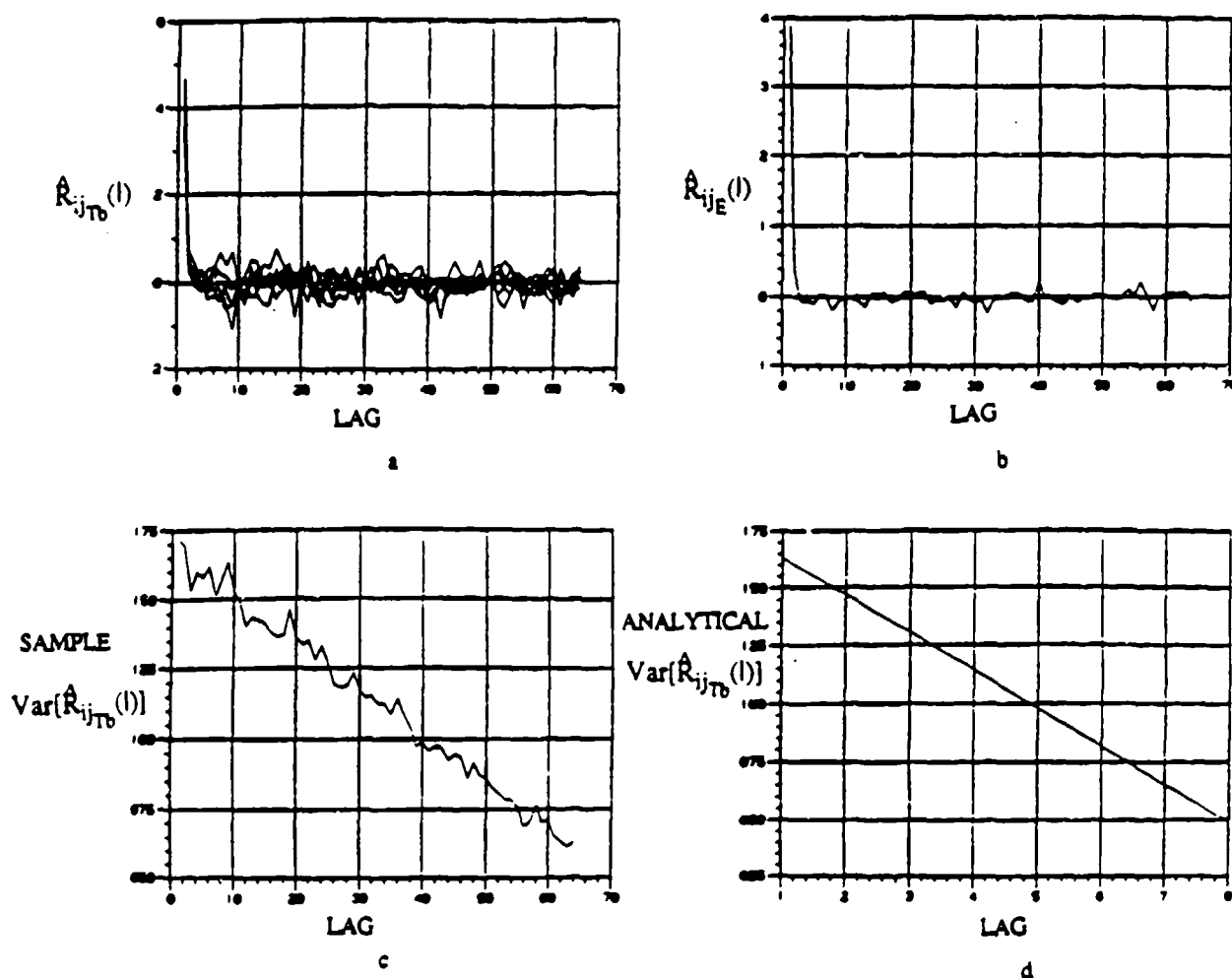


Fig. 5.3.5 Estimated cross-correlation function and its variance for an AR(1) process; $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.1$, $\sigma_s^2=4$, $|\rho_{12}|=0.99$, $l_{12}=0$ a.) $\hat{R}_{ijTb}(l)$ (6-trials) using $N_T=100$ b.) ensemble-averaged $\hat{R}_{ijE}(l)$ (1,000 realizations) c.) sample variance of $\hat{R}_{ijTb}(l)$ d.) analytical variance of $\hat{R}_{ijTb}(l)$.

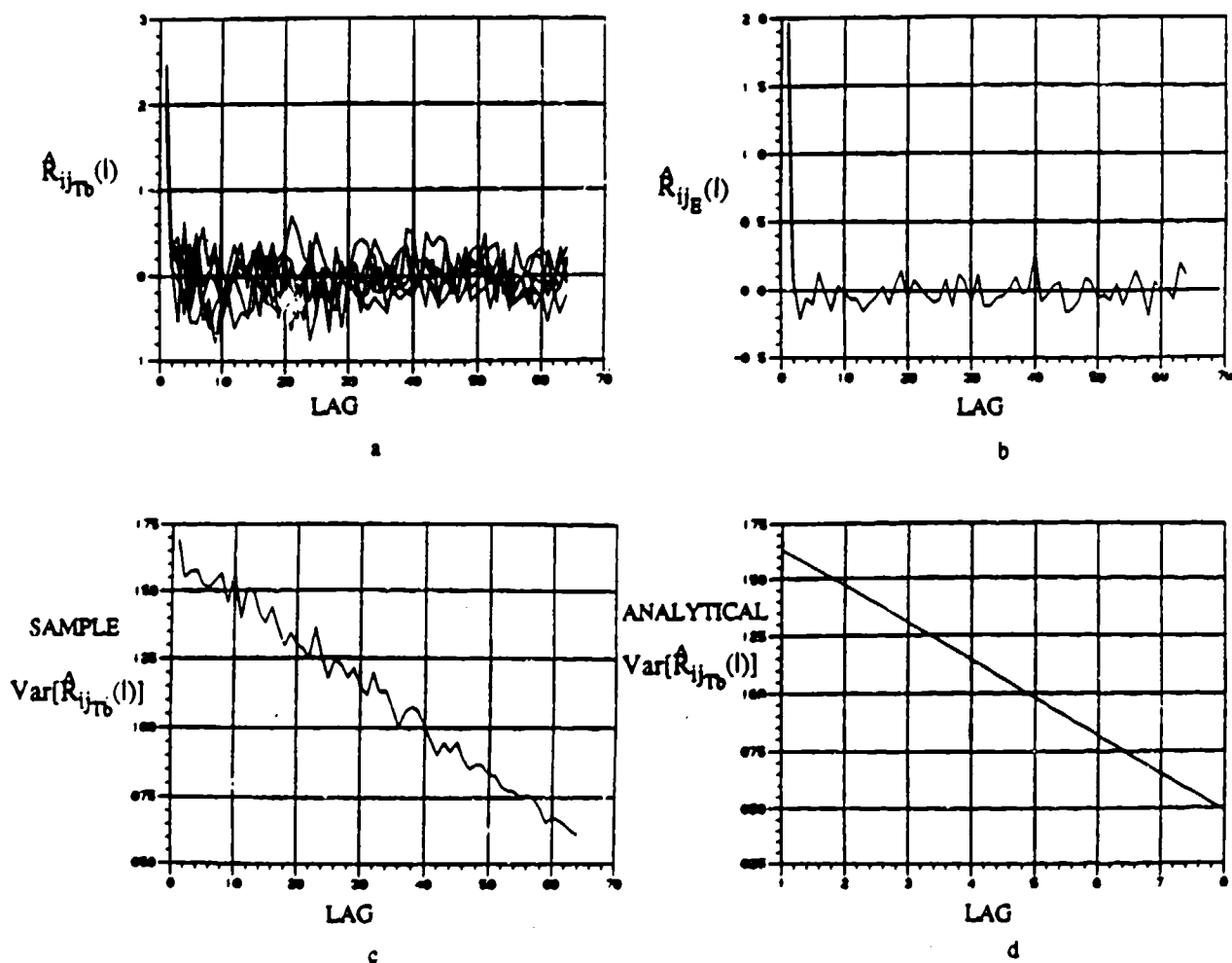


Fig. 5.3.6 Estimated cross-correlation function and its variance for an AR(1) process; $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.1$, $\sigma_s^2=4$, $|\rho_{12}|=0.5$, $l_{12}=0$ a.) $\hat{R}_{ijTb}(l)$ (6-trials) using $N_T=100$ b.) ensemble-averaged $\hat{R}_{ijE}(l)$ (1,000 realizations) c.) sample variance of $\hat{R}_{ijTb}(l)$ d.) analytical variance of $\hat{R}_{ijTb}(l)$.

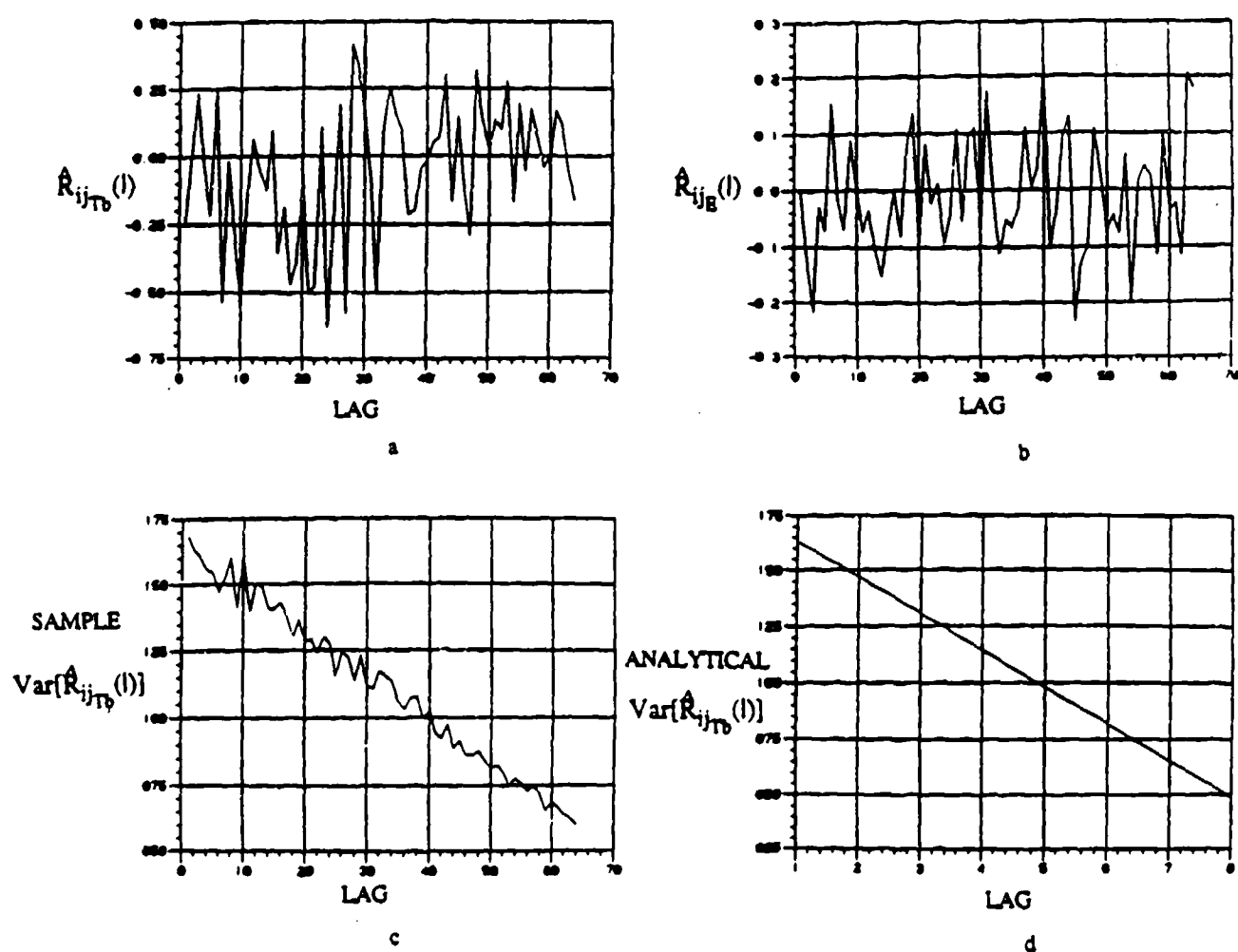


Fig. 5.3.7 Estimated cross-correlation function and its variance for an AR(1) process; $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.1$, $\sigma_s^2=4$, $|\rho_{12}|=0.0$, $l_{12}=0$ a.) $\hat{R}_{ijTb}(l)$ (6-trials) using $N_T=100$ b.) ensemble-averaged $\hat{R}_{ijE}(l)$ (1,000 realizations) c.) sample variance of $\hat{R}_{ijTb}(l)$ d.) analytical variance of $\hat{R}_{ijTb}(l)$.

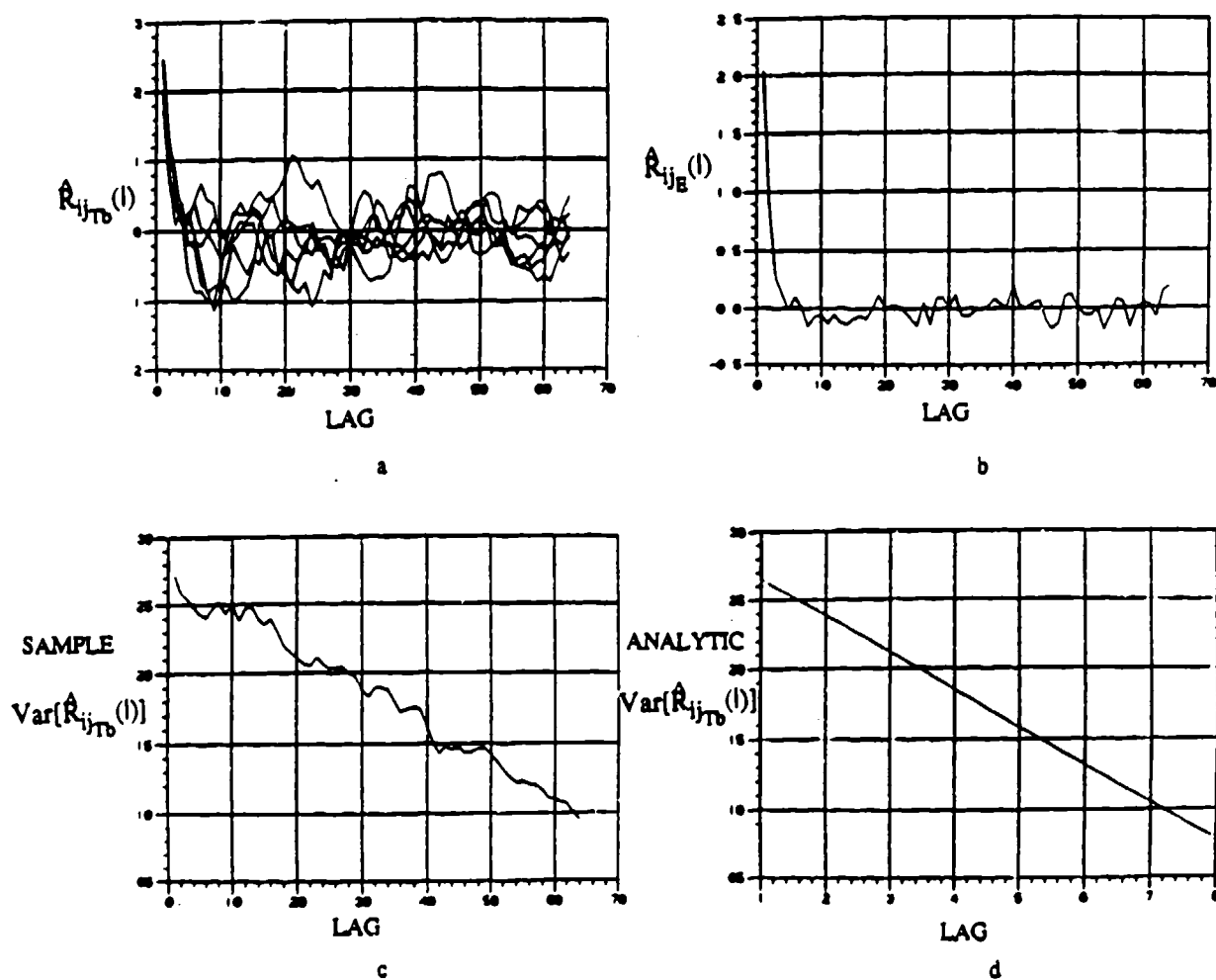


Fig. 5.3.8 Estimated cross-correlation function and its variance for an AR(1) process; $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.5$, $\sigma_s^2=4$, $|\rho_{12}|=0.5$, $l_{12}=0$ a.) $\hat{R}_{ijTb}(l)$ (6-trials) using $N_T=100$ b.) ensemble-averaged $\hat{R}_{ijE}(l)$ (1,000 realizations) c.) sample variance of $\hat{R}_{ijTb}(l)$ d.) analytical variance of $\hat{R}_{ijTb}(l)$.

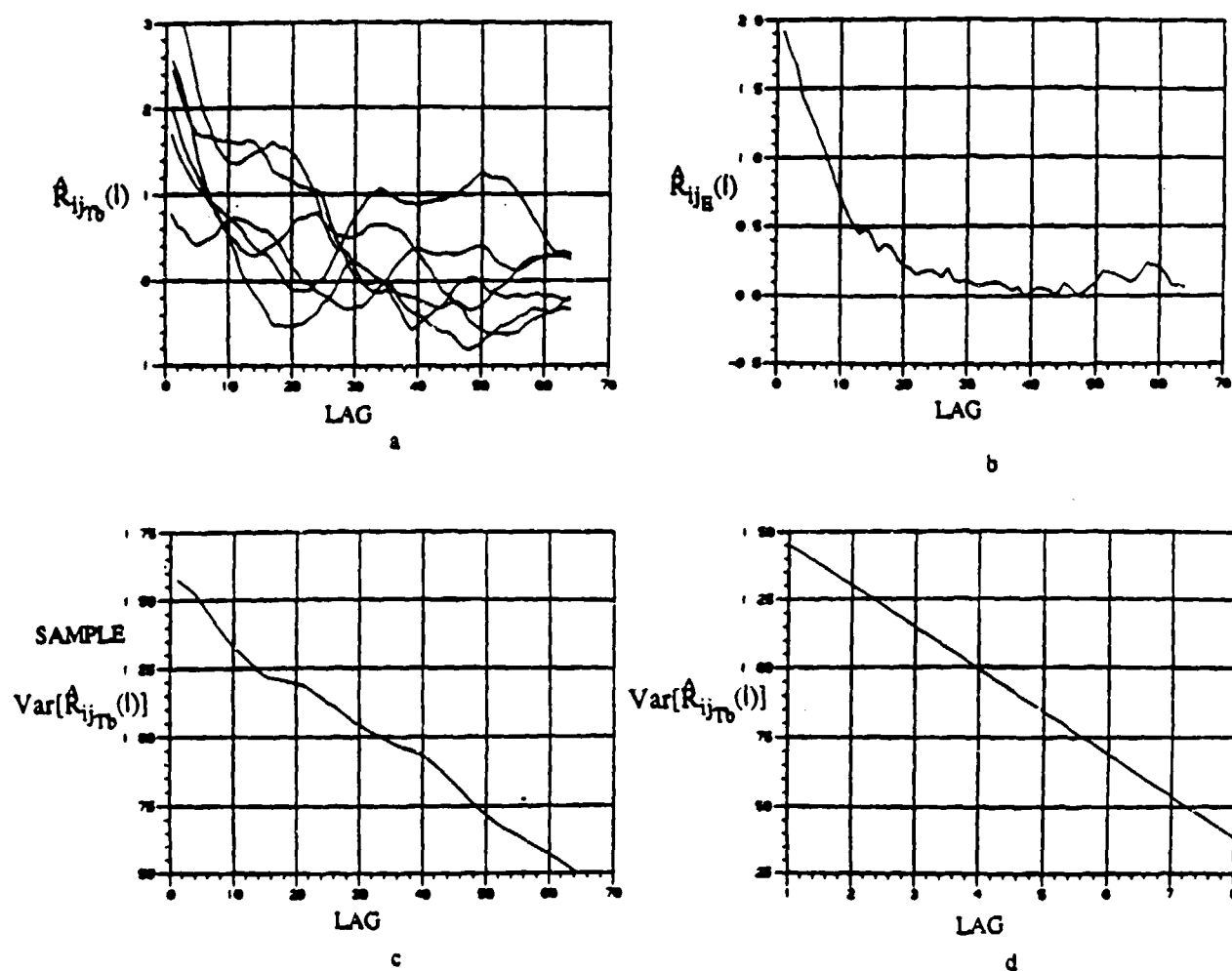


Fig. 5.3.9 Estimated cross-correlation function and its variance for an AR(1) process; $\lambda_{11}=\lambda_{22}=\lambda_{12}=0.9$, $\sigma_s^2=4$, $|p_{12}|=0.5$, $l_{12}=0$ a.) $\hat{R}_{ijTb}(l)$ (6-trials) using $N_T=100$ b.) ensemble-averaged $\hat{R}_{ijE}(l)$ (1,000 realizations) c.) sample variance of $\hat{R}_{ijTb}(l)$ d.) analytical variance of $\hat{R}_{ijTb}(l)$.

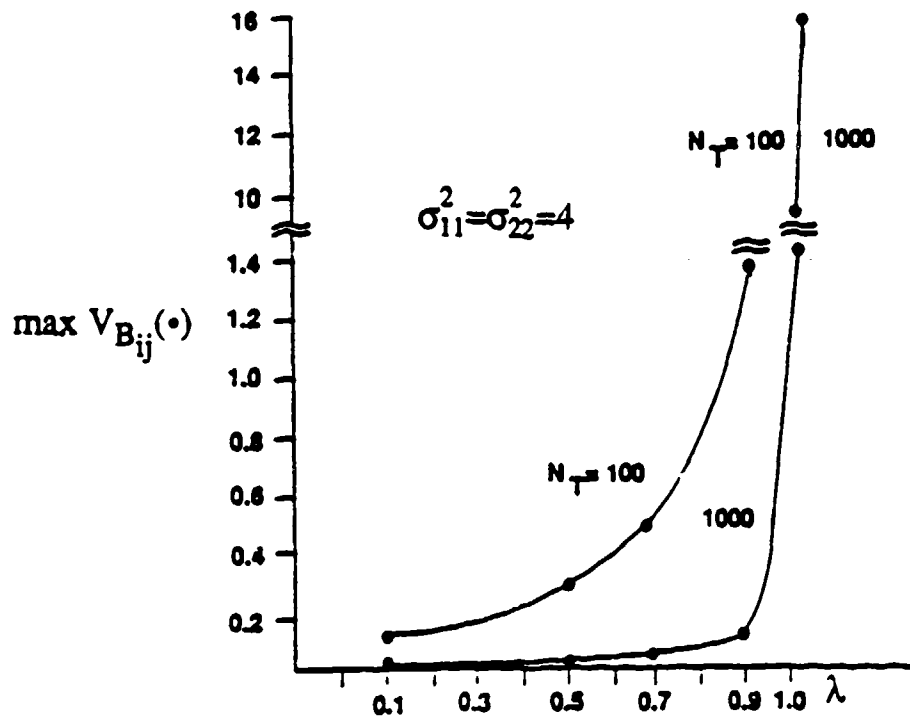


Fig. 5.3.10 Maximum variance of the time-averaged cross-correlation function versus $\lambda = \lambda_{11} = \lambda_{22}$ for $\sigma_{11}^2 = \sigma_{22}^2 = 4$; analytical(-) and computed ($\bullet$).

several of the figures, we note the increase in the sample variance of the cross-correlation functions $\text{Var}[\hat{R}_{ijT_b}(l, N_T): N_R]$ over that of the autocorrelation functions $\text{Var}[\hat{R}_{iiT_b}(l, N_T): N_R]$ presented in the previous section. This is due to the decrease in N_R from 10,000 to 1,000.

CHAPTER 6. MULTICHANNEL LIKELIHOOD RATIO

6.1. DERIVATION

In this section, we develop a multichannel likelihood ratio for the detection problem of eqs(3.1.1); ie.,

$$H_1: \underline{x}(n) = \underline{s}(n) + \underline{c}(n) + \underline{w}(n) \quad n = 1, 2, \dots, N \quad (6.1.1a)$$

$$H_0: \underline{x}(n) = \underline{c}(n) + \underline{w}(n) \quad n = 1, 2, \dots, N \quad (6.1.1b)$$

where each of the complex vectors are Gaussian, $J \times 1$ vectors and $\underline{x}(n)$ is the baseband observation vector. The methodology derived here stems from the considerations presented in [1] for real processes. Under hypothesis H_i , the multivariate joint Gaussian density can be written as the product of conditional densities so that

$$p_{\underline{x}}(\underline{x}_{1:N} | H_i) = p[\underline{x}(1) | H_i] \prod_{n=2}^N p[\underline{x}(n) | \underline{x}_{1:n-1}, H_i] \quad i=0,1 \quad (6.1.2)$$

where

$$\underline{x}_{1,n}^T = [\underline{x}^T(1) \underline{x}^T(2) \dots \underline{x}^T(n)] \quad (6.1.3)$$

$$\underline{x}^T(k) = [x_1(k) \ x_2(k) \dots x_J(k)] \quad (6.1.4)$$

and

$$\underline{x}_{1,1} = \underline{x}(1) \quad (6.1.5)$$

and all the conditional densities are Gaussian. The mean of the multivariate conditional density $p[\underline{x}(n) | \underline{x}_{1:n-1}, H_i]$ is $\hat{\underline{x}}(n|n-1, H_i)$; ie. the linear MMSE predictor of $\underline{x}(n)$ using past data $\underline{x}_{1:n-1}$ when H_i is true [44]. The $J \times J$ covariance matrix of this density function is the conditional covariance matrix $K_{\underline{x}}(n|n-1, H_i)$ such that

$$K_{\underline{x}}(n|n-1, H_i) = E\{[\underline{x}(n) - \hat{\underline{x}}(n|n-1, H_i)][\underline{x}(n) - \hat{\underline{x}}(n|n-1, H_i)]^H\} \quad (6.1.6a)$$

$$= E[\underline{\varepsilon}(n|H_i)\underline{\varepsilon}^H(n|H_i)] \quad n = 1, 2, \dots, N \quad i = 0, 1 \quad (6.1.6b)$$

where

$$\mathbf{e}(n|H_i) = \mathbf{x}(n) - \hat{\mathbf{x}}(n|n-1, H_i) \quad i = 0, 1 \quad (6.1.7)$$

is a zero-mean MMSE vector. In eq(2.4.1d), we defined the linear estimator $\hat{\mathbf{x}}(n|n-1, H_i)$ based on p past data samples. Assuming wide-sense joint stationarity on the narrowband bandpass processes, the conditional density functions can be expressed in terms of a quadratic form* such that

$$\begin{aligned} p[\mathbf{x}(n)|\mathbf{x}_{1,n-1}, H_i] &= \\ &= \frac{1}{(\pi)^J |K_x(n|n-1, H_i)|} \exp\{-[\mathbf{x}(n) - \hat{\mathbf{x}}(n|n-1, H_i)]^H [K_x(n|n-1, H_i)]^{-1} [\mathbf{x}(n) - \hat{\mathbf{x}}(n|n-1, H_i)]\} \\ & \quad i = 0, 1. \end{aligned} \quad (6.1.8)$$

where the notation $|A|$ denotes the determinant of the matrix A and

$$\hat{\mathbf{x}}(1|0, H_i) = \mathbf{0} \quad i = 0, 1. \quad (6.1.9)$$

Using eq(6.1.7) in (6.1.8), we obtain

$$\begin{aligned} p[\mathbf{x}(n)|\mathbf{x}_{1,n-1}, H_i] &= \\ &= \frac{1}{(\pi)^J |K_x(n|n-1, H_i)|} \exp\{-\mathbf{e}^H(n|H_i) [K_x(n|n-1, H_i)]^{-1} \mathbf{e}(n|H_i)\} \quad i = 0, 1 \end{aligned} \quad (6.1.10)$$

where

$$\begin{aligned} p[\mathbf{x}(1)|H_i] &= p[\mathbf{e}(1)|H_i] \\ &= \frac{1}{(\pi)^J |K_x(1|0, H_i)|} \exp\{-\mathbf{e}^H(1|H_i) [K_x(1|0, H_i)]^{-1} \mathbf{e}(1|H_i)\} \quad i = 0, 1. \end{aligned} \quad (6.1.11)$$

We can now express the likelihood ratio for the multivariate joint Gaussian density functions as

* The condition of wide-sense joint stationarity on the narrowband bandpass processes provides specific relationships between the auto- and cross-correlation sequences of the in-phase and quadrature components. These relationships enable the multivariate Gaussian density function to be expressed directly in terms of the complex random vectors as given by eq(6.1.8). A complete discussion of this topic is presented in Appendices N and O.

$$\Lambda_{H_1, H_0} = \frac{p_X(\underline{x}_{1,N}|H_1)}{p_X(\underline{x}_{1,N}|H_0)} \quad (6.1.12a)$$

$$= \frac{(p[\underline{x}(1)|H_1] \prod_{n=2}^N p[\underline{x}(n)|\underline{x}_{1,n-1}, H_1])}{(p[\underline{x}(1)|H_0] \prod_{n=2}^N p[\underline{x}(n)|\underline{x}_{1,n-1}, H_0])} \quad (6.1.12b)$$

where the last equality results from eq(6.1.2). Substituting eqs(6.1.10) and (6.1.11) into (6.1.12b) and taking the natural logarithm, we have

$$\ln \Lambda_{H_1, H_0} = \ln \left[\frac{\prod_{n=1}^N |K_x(n|n-1, H_0)| \exp \{ -\underline{\epsilon}^H(n|H_1) [K_x(n|n-1, H_1)]^{-1} \underline{\epsilon}(n|H_1) \}}{\prod_{n=1}^N |K_x(n|n-1, H_1)| \exp \{ -\underline{\epsilon}^H(n|H_0) [K_x(n|n-1, H_0)]^{-1} \underline{\epsilon}(n|H_0) \}} \right] \quad (6.1.13a)$$

$$= \sum_{n=1}^N \left[\ln \frac{|K_x(n|n-1, H_0)|}{|K_x(n|n-1, H_1)|} + \underline{\epsilon}^H(n|H_0) [K_x(n|n-1, H_0)]^{-1} \underline{\epsilon}(n|H_0) - \underline{\epsilon}^H(n|H_1) [K_x(n|n-1, H_1)]^{-1} \underline{\epsilon}(n|H_1) \right]. \quad (6.1.13b)$$

Eq (6.1.13b) can be simplified further by a diagonalization of the conditional covariance matrices and an orthogonal transformation of $\underline{\epsilon}(n|H_i)$ $i=0,1$ across channels. Since these matrices are Hermitian and positive semi-definite, several decompositions could be performed. In this dissertation, two decompositions are considered.

6.1.1. Whitening using an LDL^H Decomposition

We first consider an LDL^H decomposition of $K_x(n|n-1, H_i)$ such that

$$K_x(n|n-1, H_i) = L_\gamma(n|H_i) D_\gamma(n|H_i) L_\gamma^H(n|H_i) \quad i = 0,1 \quad (6.1.14)$$

where $L_{\gamma}(n|H_i)$ is a $J \times J$ lower triangular unit diagonal matrix and $D_{\gamma}(n|H_i)$ $i=0,1$ is a $J \times J$ diagonal matrix with real elements $d_{\gamma_{ij}}(n|H_i)$. In the discussion below, we note that $K_X(n|n-1, H_i)$ is a deterministic function [1] of the past observation data. Consequently, $L_{\gamma}(n|H_i)$ and $D_{\gamma}(n|H_i)$ are determined by the past data. Noting that $L_{\gamma}(n|H_i)$ is non-singular and solving for $D_{\gamma}(n|H_i)$, we have from eqs(6.1.14) and (6.1.6b)

$$D_{\gamma}(n|H_i) = L_{\gamma}^{-1}(n|H_i) K_X(n|n-1, H_i) [L_{\gamma}^H(n|H_i)]^{-1} \quad (6.1.15a)$$

$$= L_{\gamma}^{-1}(n|H_i) E[\underline{\epsilon}(n|H_i) \underline{\epsilon}^H(n|H_i)] [L_{\gamma}^{-1}(n|H_i)]^H \quad (6.1.15b)$$

$$= E \left\{ L_{\gamma}^{-1}(n|H_i) \underline{\epsilon}(n|H_i) \underline{\epsilon}^H(n|H_i) [L_{\gamma}^{-1}(n|H_i)]^H \right\} \quad (6.1.15c)$$

$$= E[\gamma(n|H_i) \gamma^H(n|H_i)] \quad i = 0,1 \quad (6.1.15d)$$

where

$$\gamma(n|H_i) = L_{\gamma}^{-1}(n|H_i) \underline{\epsilon}(n|H_i) \quad i = 0,1. \quad (6.1.16)$$

Since $D_{\gamma}(n|H_i)$ is a diagonal matrix, eq(6.1.15d) implies that $\gamma(n|H_i)$ contains uncorrelated elements across the channels. In Appendix A, it is shown that $\underline{\epsilon}(n|H_i)$ is a white noise process. Hence,

$$E[\underline{\epsilon}(n|H_i) \underline{\epsilon}^H(n'|H_i)] = [0] \quad n \neq n' \quad i = 0,1. \quad (6.1.17)$$

Solving for $\underline{\epsilon}(n|H_i)$ in eq(6.1.16), and substituting in eq(6.1.17), we have

$$E \{ L_{\gamma}(n|H_i) \gamma(n|H_i) \gamma^H(n'|H_i) [L_{\gamma}(n'|H_i)]^H \} = [0] \quad (6.1.18a)$$

so that

$$L_{\gamma}(n|H_i) E[\gamma(n|H_i) \gamma^H(n'|H_i)] [L_{\gamma}(n'|H_i)]^H = [0] \quad n \neq n' \quad i = 0,1. \quad (6.1.18b)$$

Since the matrices $L_{\gamma}(n|H_i)$ are non-singular, it follows that

$$E[\gamma(n|H_i)\gamma^H(n'|H_i)] = [0] \quad n \neq n' \quad i = 0,1. \quad (6.1.19)$$

In summary,

$$E[\gamma(n|H_i)\gamma^H(n'|H_i)] = \begin{cases} D_{\gamma}(n|H_i) & n=n' \\ [0] & n \neq n' \end{cases} \quad i = 0,1. \quad (6.1.20)$$

Eq(6.1.20) indicates that $\gamma(n|H_i)$ is uncorrelated in time and across channels.

From eq(6.1.14), we now have

$$K_X^{-1}(n|n-1, H_i) = [L_{\gamma}^H(n|H_i)]^{-1} D_{\gamma}^{-1}(n|H_i) L_{\gamma}^{-1}(n|H_i) \quad i=0,1. \quad (6.1.21)$$

and

$$|K_X(n|n-1, H_i)| = |L_{\gamma}(n|H_i)| |D_{\gamma}(n|H_i)| |L_{\gamma}^H(n|H_i)| \quad i=0,1 \quad (6.1.22a)$$

$$= |D_{\gamma}(n|H_i)| \quad (6.1.22b)$$

$$= \prod_{j=1}^J d_{\gamma_{jj}}(n|H_i) = \prod_{j=1}^J \sigma_{\gamma_j}^2(n|H_i) \quad i=0,1 \quad (6.1.22c)$$

where

$$|L_{\gamma}(n|H_i)| = 1 \quad i = 0,1 \quad (6.1.23)$$

and

$$d_{\gamma_{jj}}(n|H_i) = E[|\gamma_j(n|H_i)|^2] \equiv \sigma_{\gamma_j}^2(n|H_i) \quad i = 0,1. \quad (6.1.24)$$

The quantity $\sigma_{\gamma_j}^2(n|H_i)$ denotes the variance of $\gamma_j(n|H_i)$. Using eq(6.1.21), the quadratic terms in eq.(6.1.13b) become

$$\begin{aligned} \mathbf{E}^H(n|H_i) [K_X(n|n-1, H_i)]^{-1} \mathbf{E}(n|H_i) &= \\ &= \mathbf{E}^H(n|H_i) [L_{\gamma}^H(n|H_i)]^{-1} D_{\gamma}^{-1}(n|H_i) L_{\gamma}^{-1}(n|H_i) \mathbf{E}(n|H_i) \quad i=0,1 \end{aligned} \quad (6.1.25a)$$

$$= [L_{\gamma}^{-1}(n|H_i) \mathbf{x}(n|H_i)]^H D_{\gamma}^{-1}(n|H_i) [L_{\gamma}^{-1}(n|H_i) \mathbf{x}(n|H_i)] \quad i=0,1 \quad (6.1.25b)$$

$$= \mathbf{x}^H(n|H_i) D_{\gamma}^{-1}(n|H_i) \mathbf{x}(n|H_i) \quad i = 0,1 \quad (6.1.25c)$$

$$= \sum_{j=1}^J \frac{|\gamma_j(n|H_i)|^2}{2 \sigma_{\gamma_j}^2(n|H_i)} \quad i = 0,1 \quad (6.1.25d)$$

where the last two equations result from eqs(6.1.16) and (6.1.24), respectively. Using eqs(6.1.22c) and (6.1.25d) in eq(6.1.13b), we obtain

$$\ln \Lambda_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ \ln \frac{\sigma_{\gamma_j}^2(n|H_0)}{2 \sigma_{\gamma_j}^2(n|H_1)} + \frac{|\gamma_j(n|H_0)|^2}{2 \sigma_{\gamma_j}^2(n|H_0)} - \frac{|\gamma_j(n|H_1)|^2}{2 \sigma_{\gamma_j}^2(n|H_1)} \right\}. \quad (6.1.26)$$

Eq(6.1.26) may appear as the likelihood ratio for incoherent integration due to the appearance of the magnitude squared operation on each error residual $\gamma_j(n|H_i)$. We note, however, that in the transformation of the observation processes $\mathbf{x}(n|H_i)$ to the whitened $\mathbf{x}(n|H_i)$ error vector, we utilized both magnitude and phase information. Thus, eq(6.1.26) provides coherent integration not only over the N time samples, but also over the J channel processes. It is the multichannel likelihood ratio for wide-sense, joint stationary, Gaussian processes, and represents a generalization of the single channel likelihood ratio reported in [14,15,23,39] for Gaussian processes. In sections 2.4 and 6.4, we discuss a two stage filtering method using multichannel linear prediction filtering which generates $\mathbf{x}(n|H_i)$ when the correlation sequence under each of the two hypotheses is known. Then, two sets of filter coefficients and error variances can be obtained exactly, through the multichannel Yule-Walker equations. In this case, the error vector $\mathbf{x}(n|H_i)$ will be a MMSE output of the filter for which the hypothesis H_i is true. The other filter output, however, will increase in terms of its average output magnitude. Eq(6.1.26) indicates that when H_0 is true, the last term will contribute a larger value in a negative sense, causing the likelihood ratio to decrease. Alternatively, when H_1 is true, the second term increases in a

positive sense so that the likelihood ratio increases. For the unknown correlation case, the filter coefficients and error variances must be estimated. In this case, eq(6.1.26) becomes a generalized likelihood ratio and is therefore suboptimal. From one set of observation data, we must estimate the parameters (ie. the filter coefficients and error variances) for each filter assuming the appropriate hypothesis is true. In the practical implementation for this case, we must assume that the MMSE filters under each hypothesis have distinct orders so that the likelihood ratio will, in general, have a value other than zero. This assumption is justified, for example, when characterizing the observation data as autoregressive processes where the order of the process under hypothesis H_1 (signal present) is larger than that under H_0 (signal absent). For single channel processes, these considerations have been treated in [14,16].

6.1.2. Whitening using a Cholesky Decomposition

We now consider the decomposition of the positive semi-definite matrix $K_X(n|n-1, H_i)$ via the Cholesky decomposition such that

$$K_X(n|n-1, H_i) = C_\alpha(n|H_i) C_\alpha^H(n|H_i) \quad i = 0, 1 \quad (6.1.27)$$

where $C_\alpha(n|H_i)$ $i=0,1$ is a lower triangular $J \times J$ matrix with elements $c_{\alpha_{jk}}(n|H_i)$ $j \leq k$. We again note that $K_X(n|n-1, H_i)$ is considered as a deterministic function of the past data. Also, since this matrix is positive semi-definite, $C_\alpha(n|H_i)$ is non-singular. Therefore, from eq(6.1.27), we can obtain

$$I = C_\alpha^{-1}(n|H_i) K_X(n|n-1, H_i) [C_\alpha^H(n|H_i)]^{-1} \quad i = 0, 1 \quad (6.1.28a)$$

$$= C_\alpha^{-1}(n|H_i) E[\underline{\epsilon}(n|H_i) \underline{\epsilon}^H(n|H_i)] [C_\alpha^H(n|H_i)]^{-1} \quad (6.1.28b)$$

$$= E \{ C_\alpha^{-1}(n|H_i) \underline{\epsilon}(n|H_i) \underline{\epsilon}^H(n|H_i) [C_\alpha^H(n|H_i)]^{-1} \} \quad (6.1.28c)$$

$$= E[\underline{\alpha}(n|H_i) \underline{\alpha}^H(n|H_i)] \quad i = 0, 1 \quad (6.1.28d)$$

where

$$\underline{\alpha}(n|H_i) = C_{\alpha}^{-1}(n|H_i) \underline{\xi}(n|H_i) \quad i = 0,1. \quad (6.1.29)$$

The diagonal form of eq(6.1.28d) implies that $\underline{\alpha}(n|H_i)$ contains uncorrelated elements across the channels. Similar reasoning is again used as presented in eqs(6.1.17) through (6.1.20). We have from eq(6.1.29)

$$\underline{\xi}(n|H_i) = C_{\alpha}(n|H_i) \underline{\alpha}(n|H_i) \quad i = 0,1. \quad (6.1.30)$$

Using eq(6.1.30) in the orthogonality condition of eq(6.1.17)

$$E \{ C_{\alpha}(n|H_i) \underline{\alpha}(n|H_i) \underline{\alpha}^H(n'|H_i) C_{\alpha}^H(n'|H_i) \} = [0] \quad n \neq n' \quad i=0,1 \quad (6.1.31)$$

so that

$$C_{\alpha}(n|H_i) E[\underline{\alpha}(n|H_i) \underline{\alpha}^H(n'|H_i)] C_{\alpha}^H(n'|H_i) = [0] \quad n \neq n' \quad i=0,1. \quad (6.1.32)$$

Since the matrix $C_{\alpha}(n|H_i)$ is non-singular, it follows that

$$E[\underline{\alpha}(n|H_i) \underline{\alpha}^H(n'|H_i)] = [0] \quad n \neq n' \quad i=0,1 \quad (6.1.33)$$

so that using eqs(6.1.28d) and (6.1.33)

$$E[\underline{\alpha}(n|H_i) \underline{\alpha}^H(n'|H_i)] = \begin{cases} I & n=n' \\ [0] & n \neq n' \end{cases} \quad i = 0,1. \quad (6.1.34)$$

Eq(6.1.34) indicates that the elements of $\underline{\alpha}(n|H_i)$ have unit variances and are uncorrelated both in time and across channels. From eq(6.1.27), we obtain

$$K_x^{-1}(n|n-1, H_i) = [C_{\alpha}^H(n|H_i)]^{-1} C_{\alpha}^{-1}(n|H_i) \quad i = 0,1 \quad (6.1.35)$$

and

$$|K_x(n|n-1, H_i)| = |C_\alpha(n|H_i)| |C_\alpha^H(n|H_i)| \quad i = 0,1 \quad (6.1.36a)$$

$$= \prod_{j=1}^J c_{\alpha jj}(n|H_i) \prod_{j=1}^J c_{\alpha jj}^*(n|H_i) \quad i = 0,1 \quad (6.1.36b)$$

$$= \prod_{j=1}^J |c_{\alpha jj}(n|H_i)|^2 \quad i = 0,1. \quad (6.1.36c)$$

Using eq(6.1.35), the quadratic terms in eq(6.1.13b) become

$$\begin{aligned} \underline{\epsilon}^H(n|H_i) [K_x(n|n-1, H_i)]^{-1} \underline{\epsilon}(n|H_i) &= \\ &= \underline{\epsilon}^H(n|H_i) [C_\alpha^H(n|H_i)]^{-1} C_\alpha^{-1}(n|H_i) \underline{\epsilon}(n|H_i) \quad i = 0,1 \end{aligned} \quad (6.1.37a)$$

$$= [C_\alpha^{-1}(n|H_i) \underline{\epsilon}(n|H_i)]^H [C_\alpha^{-1}(n|H_i) \underline{\epsilon}(n|H_i)] \quad i = 0,1 \quad (6.1.37b)$$

$$= \underline{\alpha}^H(n|H_i) \underline{\alpha}(n|H_i) \quad i = 0,1 \quad (6.1.37c)$$

$$= \sum_{j=1}^J |\alpha_j(n|H_i)|^2 \quad i = 0,1 \quad (6.1.37d)$$

where we have used eq(6.1.29) to obtain (6.1.37c). Using eqs(6.1.36c) and (6.1.37d) in eq(6.1.13b), we obtain

$$\ln \Lambda_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ \ln \frac{|c_{\alpha jj}(n|H_0)|^2}{|c_{\alpha jj}(n|H_1)|^2} + |\alpha_j(n|H_0)|^2 - |\alpha_j(n|H_1)|^2 \right\}. \quad (6.1.38)$$

A comparison of eqs(6.1.38) and (6.1.26) reveals that when using the Cholesky matrix to decorrelate the error vector processes $\underline{\epsilon}(n|H_i)$, the resulting uncorrelated error vector output $\underline{\alpha}(n|H_i)$ is such that the elements have unit variance. From the equality of the LHS of eqs(6.1.25d) and (6.1.37d), we have

$$\sum_{j=1}^J \frac{|y_j(n|H_i)|^2}{\sigma_{\gamma_j}(n|H_i)} = \sum_{j=1}^J |\alpha_{ij}(n|H_i)|^2 \quad i = 0,1. \quad (6.1.39)$$

Similarly, from the equality of eqs(6.1.22c) and (6.1.36c)

$$\prod_{j=1}^J \sigma_{\gamma_j}^2(n|H_i) = \prod_{j=1}^J |c_{\alpha_{ij}}(n|H_i)|^2 \quad i = 0,1. \quad (6.1.40)$$

Thus,

$$\ln \frac{\prod_{j=1}^J \sigma_{\gamma_j}^2(n|H_0)}{\prod_{j=1}^J \sigma_{\gamma_j}^2(n|H_1)} = \ln \frac{\prod_{j=1}^J |c_{\alpha_{ij}}(n|H_0)|^2}{\prod_{j=1}^J |c_{\alpha_{ij}}(n|H_1)|^2} \quad i = 0,1 \quad (6.1.41a)$$

or

$$\sum_{j=1}^J \ln \frac{\sigma_{\gamma_j}^2(n|H_0)}{\sigma_{\gamma_j}^2(n|H_1)} = \sum_{j=1}^J \ln \frac{|c_{\alpha_{ij}}(n|H_0)|^2}{|c_{\alpha_{ij}}(n|H_1)|^2} \quad i = 0,1. \quad (6.1.41b)$$

Thus, each term in eqs(6.1.26) and (6.1.38) are identical. However, in general, we note that each j th term in these equations are not equal for $J > 1$.

6.2. THE SPECIAL CASE OF AN UNCORRELATED SIGNAL PROCESS

In [40], several limiting forms of the single channel ($J=1$) likelihood ratio are discussed. They include (1) the detection of a deterministic signal in additive white Gaussian noise, (2) the detection of a non-white Gaussian signal process in additive white Gaussian noise, (3) the moving target detection (MTD) algorithm for the detection of a deterministic signal in non-white Gaussian noise [41,42] and (4) the detection of a non-white Gaussian signal process in additive non-white Gaussian noise [43]. It is also shown in [40] that this likelihood ratio contains the

algorithm developed [44] for the detection of a deterministic signal in additive non-white Gaussian noise of unknown correlation statistics.

One limiting form of this likelihood ratio which has not been noted is the case of uncorrelated (both in time and across channels) Gaussian signal processes in additive white noise. In this case, the signal is a white noise process. Since past values of such a process are uncorrelated with present and future values, the coefficients in the linear prediction error filters are zeroes. Thus, the error signals in this case are identical to the observation processes. For known variance terms, the error variances under each hypothesis for the scalar error $\gamma_j(n|H_i)$ on each channel j

$$\sigma_{\gamma_j}^2(H_0) = \sigma_{w_j}^2 \quad j = 1, 2, \dots, J \quad (6.2.1a)$$

and

$$\sigma_{\gamma_j}^2(H_1) = \sigma_{s_j}^2 + \sigma_{w_j}^2 \quad j = 1, 2, \dots, J \quad (6.2.1b)$$

where $\sigma_{s_j}^2$ and $\sigma_{w_j}^2$ are the known variances associated with the j th channel signal and white noise, respectively. Absorbing the constant term into the threshold, eq(6.1.26) becomes

$$\ln \Lambda_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ \frac{|x_j(n)|^2}{\sigma_{w_j}^2} - \frac{|x_j(n)|^2}{\sigma_{s_j}^2 + \sigma_{w_j}^2} \right\} \quad (6.2.2a)$$

$$= \sum_{j=1}^J \sum_{n=1}^N \left[\frac{\sigma_{s_j}^2}{\sigma_{w_j}^2 (\sigma_{s_j}^2 + \sigma_{w_j}^2)} |x_j(n)|^2 \right]. \quad (6.2.2b)$$

Eq(6.2.2b) is the likelihood ratio for complex Gaussian uncorrelated signal processes in additive white noise. For $J=1$, its form is noted in [19 Part I, Chap. 2] for the case of real processes. This is a case where an incoherent integrator is optimal; ie., if pulse-to-pulse phase information were utilized in the coherent

integration procedure, it would not result in an improvement in detection performance. Thus eq(6.2.2b) is also informative from another point of view. We recognize that it has the form of an incoherent integrator [13]. In section 6.4, we discuss the implementation of eq(6.1.26) with a dual filter strategy, where each filter is designed for the hypothesis H_i $i=0,1$, respectively. The detection results obtained using this equation will be compared to analytic expressions for coherent pulse train detection presented in Appendix K. In addition, we will plot the detection results obtained when the coefficients of these filters are set to zero; i.e., the results obtained when the likelihood ratio is equivalent to eq(6.2.2b). These results are compared to the analytic expressions for incoherent pulse train detection also presented in Appendix K.

6.3. GAUSSIAN PROCESSES WITH UNCONSTRAINED QUADRATURE COMPONENTS

In this section, we generalize the derivation of the likelihood ratio to Gaussian processes with unconstrained quadrature components. We will therefore consider a zero-mean, complex baseband process $x_i(n)$ for channel i such that

$$x_i(n) = x_{iI}(n) + j x_{iQ}(n) \quad n = 1, 2, \dots, N \quad (6.3.1)$$

where $x_{iI}(n)$ and $x_{iQ}(n)$ are the real and imaginary components of the process $x_i(n)$, respectively. We now define the vectors

$$\underline{x}_{I1,N}^T = [x_I^T(1) \ x_I^T(2) \ \dots \ x_I^T(N)] \quad (6.3.2a)$$

and

$$\underline{x}_{Q1,N}^T = [x_Q^T(1) \ x_Q^T(2) \ \dots \ x_Q^T(N)] \quad (6.3.2b)$$

where

$$\underline{x}_I^T(k) = [x_{1I}(k) \ x_{2I}(k) \ \dots \ x_{JI}(k)] \quad (6.3.2c)$$

$$\underline{x}_Q^T(k) = [x_{1Q}(k) \ x_{2Q}(k) \ \dots \ x_{JQ}(k)]. \quad (6.3.2d)$$

Note that

$$\underline{x}_{I,1} = \underline{x}_I(1) \quad (6.3.3a)$$

$$\underline{x}_{Q,1} = \underline{x}_Q(1). \quad (6.3.3b)$$

We now consider the concatenated real vector $\underline{I}_{1,N}$ defined as

$$\underline{I}_{1,N}^T = [\underline{x}_{I,1,N}^T \quad \underline{x}_{Q,1,N}^T]. \quad (6.3.4)$$

Under hypothesis H_i $i=0,1$ the real vector $\underline{I}_{1,N}$ is defined to be Gaussian if and only if its density function is of the form

$$p_R(\underline{I}_{1,N}|H_i) = \frac{1}{[(2\pi)^{2JN} |K_{\underline{I}}|]^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} [\underline{I}_{1,N} - \bar{\underline{I}}_{1,N}]^T K_{\underline{I}}^{-1} [\underline{I}_{1,N} - \bar{\underline{I}}_{1,N}] \right\} \quad (6.3.5)$$

$$K_{\underline{I}} = E \left\{ [\underline{I}_{1,N} - \bar{\underline{I}}_{1,N}] [\underline{I}_{1,N} - \bar{\underline{I}}_{1,N}]^T \right\} \quad (6.3.6a)$$

and

$$\bar{\underline{I}}_{1,N} = E[\underline{I}_{1,N}]. \quad (6.3.6b)$$

Since the means are assumed to be known 'a priori', they can be subtracted from the data and there is no loss in generality by assuming zero-mean processes. By definition, $p_R(\underline{I}_{1,N}|H_i)$ is a $2JN$ -dimensional multivariate density function for the real random variables contained in eqs(6.3.4). We now express $p_R(\underline{I}_{1,N}|H_i)$ as the product of conditional densities so that

$$p_R(\underline{I}_{1,N}|H_i) = p_R[\underline{I}(1)|H_i] \prod_{n=2}^N p_R[\underline{I}(n)|\underline{I}_{1,n-1}, H_i] \quad i = 0,1 \quad (6.3.7)$$

where

$$\underline{I}^T(k) = [\underline{x}_I^T(k) \quad \underline{x}_Q^T(k)] \quad (6.3.8a)$$

$$= [x_{1I}(k) \quad x_{2I}(k) \quad \dots \quad x_{JI}(k) \quad x_{1Q}(k) \quad x_{2Q}(k) \quad \dots \quad x_{JQ}(k)] \quad (6.3.8b)$$

$$\underline{\mathbf{L}}_{1,1}^T = [\underline{\mathbf{x}}_I^T(1) \underline{\mathbf{x}}_Q^T(1)] \quad (6.3.8c)$$

and all the conditional densities are multivariate jointly Gaussian. The mean of the multivariate conditional density $p_R[\underline{\mathbf{r}}(n)|\underline{\mathbf{r}}_{1,n-1}, H_i]$ is $\hat{\underline{\mathbf{r}}}(n|n-1, H_i)$; ie., the linear MMSE predictor of $\underline{\mathbf{r}}(n)$ using past data $\underline{\mathbf{r}}_{1,n-1}$ and assuming H_i is true. The $2J \times 2J$ covariance matrix of this density function is the conditional covariance matrix $K_r(n|n-1, H_i)$ such that

$$K_r(n|n-1, H_i) = E \{ [\underline{\mathbf{r}}(n) - \hat{\underline{\mathbf{r}}}(n|n-1, H_i)][\underline{\mathbf{r}}(n) - \hat{\underline{\mathbf{r}}}(n|n-1, H_i)]^T \} \quad (6.3.9a)$$

$$= E [\underline{\mathbf{e}}(n|H_i)\underline{\mathbf{e}}^T(n|H_i)] \quad n = 1, 2, \dots, N \quad i = 0, 1 \quad (6.3.9b)$$

where

$$\underline{\mathbf{e}}(n|H_i) = \underline{\mathbf{r}}(n) - \hat{\underline{\mathbf{r}}}(n|n-1, H_i) \quad i = 0, 1 \quad (6.3.10a)$$

is the zero-mean $2J \times 1$ error vector. When $\hat{\underline{\mathbf{r}}}(n|n-1, H_i)$ is obtained using a linear combination of p past data samples, we define

$$\hat{\underline{\mathbf{r}}}_p(n|n-1, \dots, n-p, H_i) = - \sum_{k=1}^p \mathcal{A}_p^T(k|H_i) \begin{bmatrix} \underline{\mathbf{x}}_I(n-k) \\ \underline{\mathbf{x}}_Q(n-k) \end{bmatrix} \quad (6.3.10b)$$

where the $2J \times 2J$ block matrix $\mathcal{A}_p^T(k|H_i)$ is expressed in terms of its $J \times J$ block elements such that (dropping the notation H_i)

$$\mathcal{A}_p^T(k) = \begin{bmatrix} A_{p11}^T(k) & A_{p21}^T(k) \\ A_{p12}^T(k) & A_{p22}^T(k) \end{bmatrix}. \quad (6.3.10c)$$

We note that this real estimator is of higher dimension than the complex estimator defined in eq(2.4.1d). The increased dimension provides this estimator with the necessary degrees of freedom to whiten quadrature components of processes with more general correlation between the quadrature components than is allowed by the conventional complex Gaussian process as was discussed in section 4.6. In Appendix M, we show that the complex estimator of eq(2.4.1d) is

limited to the special case where $A_{p11}^T(k) = A_{p22}^T(k)$ and $A_{p12}^T(k) = -A_{p21}^T(k)$ and whitens the quadrature components only for the special conditions $R_{xx}^{II}(l) = R_{xx}^{QQ}(l)$ and $R_{xx}^{QI}(l) = -R_{xx}^{IQ}(l)$. The conditional density functions can now be expressed in terms of the quadratic form such that

$$p_R[\underline{r}(n)|\underline{r}_{1,n}, H_i] = \frac{1}{[(2\pi)^{2JN} |K_r(n|n-1, H_i)|]^{\frac{1}{2}}} \cdot \exp \left\{ -\frac{1}{2} [\underline{r}(n) - \hat{\underline{r}}(n|n-1, H_i)]^T [K_r(n|n-1, H_i)]^{-1} [\underline{r}(n) - \hat{\underline{r}}(n|n-1, H_i)] \right\} \quad i = 0, 1 \quad (6.3.11)$$

and

$$\hat{\underline{r}}(1|0, H_i) = \underline{0} \quad i = 0, 1. \quad (6.3.12)$$

Using eq(6.3.10a) in (6.3.11), we obtain

$$p_R[\underline{r}(n)|\underline{r}_{1,n}, H_i] = \frac{1}{[(2\pi)^{2JN} |K_r(n|n-1, H_i)|]^{\frac{1}{2}}} \cdot \exp \left\{ -\frac{1}{2} \underline{e}^T(n|H_i) [K_r(n|n-1, H_i)]^{-1} \underline{e}(n|H_i) \right\} \quad i = 0, 1 \quad (6.3.13)$$

where

$$p_R[\underline{r}(1)|H_i] = p_R[\underline{e}(1)|H_i] \quad i = 0, 1 \quad (6.3.14a)$$

$$= \frac{1}{[(2\pi)^{2JN} |K_r(1|0, H_i)|]^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} \underline{e}^T(1|H_i) [K_r(1|0, H_i)]^{-1} \underline{e}(1|H_i) \right\} \quad i = 0, 1. \quad (6.3.14b)$$

We can now express the likelihood ratio for the multivariate joint Gaussian density function of the quadrature components as

$$\Lambda_{H_1, H_0}^Q = \frac{p_R(\underline{L}_{1,N}|H_1)}{p_R(\underline{L}_{1,N}|H_0)} \quad (6.3.15a)$$

$$= \frac{(p_R[l(1)|H_1] \prod_{n=2}^N p_R[l(n)|\underline{L}_{1,n-1}, H_1])}{(p_R[l(1)|H_0] \prod_{n=2}^N p_R[l(n)|\underline{L}_{1,n-1}, H_0])} \quad (6.3.15b)$$

where the last equality results from eq(6.3.7) and superscript Q denotes the quadrature form. Substituting eqs(6.3.13) and (6.3.14) into (6.3.15b), we have

$$\Lambda_{H_1, H_0}^Q = \frac{\prod_{n=1}^N |K_r(n|n-1, H_0)|^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underline{e}^T(n|H_1) [K_r(n|n-1, H_1)]^{-1} \underline{e}(n|H_1) \right\}}{\prod_{n=1}^N |K_r(n|n-1, H_1)|^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underline{e}^T(n|H_0) [K_r(n|n-1, H_0)]^{-1} \underline{e}(n|H_0) \right\}} \quad (6.3.16a)$$

Taking the natural logarithm, we obtain

$$\ln \Lambda_{H_1, H_0}^Q = \frac{1}{2} \sum_{n=1}^N \left[\ln \frac{|K_r(n|n-1, H_0)|}{|K_r(n|n-1, H_1)|} + \underline{e}^T(n|H_0) [K_r(n|n-1, H_0)]^{-1} \underline{e}(n|H_0) - \underline{e}^T(n|H_1) [K_r(n|n-1, H_1)]^{-1} \underline{e}(n|H_1) \right] \quad (6.3.16b)$$

The error vector $\underline{e}(n|H_i)$ $i=0,1$ in eq(6.3.16b) has been 'whitened' temporally. However, it still retains correlation across channels. Furthermore, all the quadrature components retain cross-channel correlation. We will now consider a diagonalization of $K_r(n|n-1, H_0)$ which will decorrelate the vector $\underline{e}(n|H_i)$ across channels and between quadrature components. Although several decompositions could be used, we will again focus on the LDL^T and Cholesky CC^T decompositions.

6.3.1. Whitening Across Channels and Quadrature Components via LDL^T Decomposition

In this section, we perform an LDL^T decomposition of the positive semi-definite matrix $K_r(n|n-1, H_i)$ to remove the cross-channel and quadrature

correlation. We again note that the matrix $K_r(n|n-1, H_i)$ is a deterministic function [1] of the past observation data. Therefore,

$$K_r(n|n-1, H_i) = L_\psi(n|H_i) D_\psi(n|H_i) L_\psi^T(n|H_i) \quad i = 0, 1 \quad (6.3.17)$$

where $L_\psi(n|H_i)$ is a unit diagonal, lower triangular $2J \times 2J$ matrix. Solving eq(6.3.17) for $D_\psi(n|H_i)$, we obtain

$$D_\psi(n|H_i) = L_\psi^{-1}(n|H_i) K_r(n|n-1, H_i) [L_\psi^T(n|H_i)]^{-1} \quad i = 0, 1 \quad (6.3.18a)$$

$$= E \{ L_\psi^{-1}(n|H_i) \underline{e}(n|H_i) \underline{e}^T(n|H_i) [L_\psi^T(n|H_i)]^{-1} \} \quad i = 0, 1 \quad (6.3.18b)$$

$$= E[\underline{\psi}(n|H_i) \underline{\psi}^T(n|H_i)] \quad i = 0, 1 \quad (6.3.18c)$$

where

$$\underline{\psi}(n|H_i) = L_\psi^{-1}(n|H_i) \underline{e}(n|H_i) \quad i = 0, 1 \quad (6.3.19a)$$

$$= L_\psi^{-1}(n|H_i) \begin{bmatrix} \underline{e}_I(n|H_i) \\ \underline{e}_Q(n|H_i) \end{bmatrix} \quad i = 0, 1. \quad (6.3.19b)$$

The vector $\underline{\psi}(n|H_i)$ $i = 0, 1$ can be expressed as

$$\underline{\psi}^T(n|H_i) = [\underline{\psi}_A^T(n|H_i) \quad \underline{\psi}_B^T(n|H_i)] \quad i = 0, 1. \quad (6.3.20)$$

The diagonal form of eq(6.3.18c) implies that $\underline{\psi}(n|H_i)$ is uncorrelated across channels and quadrature components; thus,

$$E[\underline{\psi}_A(n|H_i) \underline{\psi}_B^T(n|H_i)] = [0] \quad i = 0, 1. \quad (6.3.21)$$

From eq(6.3.17), we now have

$$[K_r(n|n-1, H_i)]^{-1} = [L_\psi^T(n|H_i)]^{-1} D_\psi^{-1}(n|H_i) L_\psi^{-1}(n|H_i) \quad i = 0, 1 \quad (6.3.22)$$

and

$$|K_r(n|n-1, H_i)| = |L_\psi(n|H_i)| |D_\psi(n|H_i)| |L_\psi^T(n|H_i)| \quad i = 0,1 \quad (6.3.23a)$$

$$= |D_\psi(n|H_i)| \quad i = 0,1 \quad (6.3.23b)$$

$$= \prod_{j=1}^J d_{jj_A}^\psi(n|H_i) d_{jj_B}^\psi(n|H_i) \quad i = 0,1 \quad (6.3.23c)$$

$$= \prod_{j=1}^J \sigma_{\psi_{j_A}}^2(n|H_i) \sigma_{\psi_{j_B}}^2(n|H_i) \quad i = 0,1 \quad (6.3.23d)$$

where

$$|L_\psi(n|H_i)| = 1 \quad i = 0,1 \quad (6.3.24)$$

and

$$d_{jj_A}^\psi(n|H_i) = E[(\psi_{j_A}(n|H_i))^2] \equiv \sigma_{\psi_{j_A}}^2(n|H_i) \quad i = 0,1 \quad (6.3.25a)$$

and

$$d_{jj_B}^\psi(n|H_i) = E[(\psi_{j_B}(n|H_i))^2] \equiv \sigma_{\psi_{j_B}}^2(n|H_i) \quad i = 0,1. \quad (6.3.25b)$$

The quantities $\sigma_{\psi_{j_A}}^2(n|H_i)$ and $\sigma_{\psi_{j_B}}^2(n|H_i)$ denote the variances of the A and

B components, respectively. Using eq(6.3.22), the A and B component terms in eq(6.3.16b) become

$$\begin{aligned} \underline{e}^T(n|H_i) [K_r(n|n-1, H_i)]^{-1} \underline{e}(n|H_i) &= \\ &= \underline{e}^T(n|H_i) [L_\psi^T(n|H_i)]^{-1} D_\psi^{-1}(n|H_i) L_\psi^{-1}(n|H_i) \underline{e}(n|H_i) \quad i=0,1 \end{aligned} \quad (6.3.26a)$$

$$= [L_\psi^{-1}(n|H_i) \underline{e}(n|H_i)]^T D_\psi^{-1}(n|H_i) L_\psi^{-1}(n|H_i) \underline{e}(n|H_i) \quad i=0,1 \quad (6.3.26b)$$

$$= \underline{\psi}^T(n|H_i) D_\psi^{-1}(n|H_i) \underline{\psi}(n|H_i) \quad i = 0,1 \quad (6.3.26c)$$

$$= \sum_{j=1}^J \frac{[\psi_{j_A}(n|H_i)]^2}{\sigma_{\psi_{j_A}}^2} + \frac{[\psi_{j_B}(n|H_i)]^2}{\sigma_{\psi_{j_B}}^2} \quad i = 0,1 \quad (6.3.26d)$$

where the last two equations result from eqs(6.3.19a) and (6.3.25), respectively. Using eqs(6.3.23c) and (6.3.26d) in eq(6.3.16b), we obtain

$$\ln \Lambda_{H_1, H_0}^Q = \frac{1}{2} \sum_{j=1}^J \sum_{n=1}^N \left[\ln \frac{\sigma_{\psi_{jA}}^2(n|H_0)}{\sigma_{\psi_{jA}}^2(n|H_1)} + \ln \frac{\sigma_{\psi_{jB}}^2(n|H_0)}{\sigma_{\psi_{jB}}^2(n|H_1)} + \frac{[\psi_{jA}(n|H_0)]^2}{\sigma_{\psi_{jA}}^2(n|H_0)} + \frac{[\psi_{jB}(n|H_0)]^2}{\sigma_{\psi_{jB}}^2(n|H_0)} - \frac{[\psi_{jA}(n|H_1)]^2}{\sigma_{\psi_{jA}}^2(n|H_1)} - \frac{[\psi_{jB}(n|H_1)]^2}{\sigma_{\psi_{jB}}^2(n|H_1)} \right] \quad (6.3.27)$$

Eq(6.3.27) is the generalization of the multichannel likelihood ratio expressed in eq(6.1.26) for processes with correlated Gaussian quadrature components. For the special case where $\underline{x}(n)$ has the correlation properties

$$R_{ij}^{II}(l) = R_{ij}^{QQ}(l) \quad \text{all } i, j \quad (6.3.28a)$$

and

$$R_{ij}^{IQ}(l) = -R_{ij}^{QI}(l) \quad \text{all } i, j \quad (6.3.28b)$$

it can be shown (see Appendix I) that the complex process $\gamma(n|H_i)$ discussed in section 6.1 and the quadrature components presented in this section can be related as

$$\sum_{j=1}^J \left[\ln \frac{\sigma_{\psi_{jA}}^2(n|H_0)}{\sigma_{\psi_{jA}}^2(n|H_1)} + \ln \frac{\sigma_{\psi_{jB}}^2(n|H_0)}{\sigma_{\psi_{jB}}^2(n|H_1)} \right] = 2 \sum_{j=1}^J \left[\ln \frac{\sigma_{\gamma_j}^2(n|H_0)}{\sigma_{\gamma_j}^2(n|H_1)} \right] \quad (6.3.29a)$$

and

$$\sum_{j=1}^J \left[\frac{[\psi_{jA}(n|H_i)]^2}{\sigma_{\psi_{jA}}^2} + \frac{[\psi_{jB}(n|H_i)]^2}{\sigma_{\psi_{jB}}^2} \right] = 2 \sum_{j=1}^J \frac{|\gamma_j(n|H_i)|^2}{\sigma_{\gamma_j(n|H_i)}^2} \quad (6.3.29b)$$

Using these expressions, eq(6.3.27) reduces to that of eq(6.1.26).

6.3.2. Whitening across Channels and Quadrature Components via a Cholesky Decomposition

We now consider the decomposition of $K_r(n|n-1, H_i)$ using the Cholesky decomposition such that

$$K_r(n|n-1, H_i) = C_\tau(n|H_i) C_\tau^T(n|H_i) \quad i = 0, 1 \quad (6.3.30)$$

where $C_\tau(n|H_i)$ $i=0,1$ is a lower triangular $2J \times 2J$ matrix. From eq(6.3.30), we can obtain

$$I_{2J} = C_\tau^{-1}(n|H_i) K_r(n|n-1, H_i) [C_\tau^T(n|H_i)]^{-1} \quad i = 0, 1 \quad (6.3.31a)$$

$$= C_\tau^{-1}(n|H_i) E [\underline{e}(n|H_i) \underline{e}^T(n|H_i)] [C_\tau^T(n|H_i)]^{-1} \quad i = 0, 1 \quad (6.3.31b)$$

$$= E \{ C_\tau^{-1}(n|H_i) \underline{e}(n|H_i) \underline{e}^T(n|H_i) [C_\tau^T(n|H_i)]^{-1} \} \quad i = 0, 1 \quad (6.3.31c)$$

$$= E [\underline{I} \quad \quad \quad \underline{I}^T(n|H_i)] \quad i = 0, 1 \quad (6.3.31d)$$

where

$$\underline{I}(n|H_i) = C_\tau^{-1}(n|H_i) \underline{e}(n|H_i) \quad i = 0, 1, \quad (6.3.32a)$$

$$\underline{I}^T(n|H_i) = [\underline{I}_A^T(n|H_i) \quad \underline{I}_B^T(n|H_i)] \quad i = 0, 1 \quad (6.3.32b)$$

and I_{2J} is a $2J \times 2J$ identity matrix. The diagonal form of eq(6.3.31d) implies that $\mathbf{z}(n|H_i)$ is uncorrelated across channels and quadrature components. Using the quadrature form for $\mathbf{z}(n|H_i)$ defined in eq(6.3.32) in (6.3.31d), we obtain

$$E[\mathbf{z}_A(n|H_i)\mathbf{z}_A^T(n|H_i)] = E[\mathbf{z}_B(n|H_i)\mathbf{z}_B^T(n|H_i)] = I_J \quad i = 0,1 \quad (6.3.33a)$$

and

$$E[\mathbf{z}_A(n|H_i)\mathbf{z}_B^T(n|H_i)] = E[\mathbf{z}_B(n|H_i)\mathbf{z}_A^T(n|H_i)] = [0] \quad i = 0,1 \quad (6.3.33b)$$

where I_J is a $J \times J$ identity matrix. Thus, the variances of the quadrature components of $\tau_j(n|H_i)$ $i=0,1$ and $j=1,2,\dots,J$ are unity. From eq(6.3.30), we obtain

$$K_r^{-1}(n|n-1, H_i) = [C_\tau^T(n|H_i)]^{-1} C_\tau^{-1}(n|H_i) \quad i = 0,1 \quad (6.3.34)$$

and

$$|K_r(n|n-1, H_i)| = |C_\tau(n|H_i)| |C_\tau^T(n|H_i)| \quad i = 0,1 \quad (6.3.35a)$$

$$= \prod_{j=1}^J c_{\tau_{A,jj}}(n|H_i) c_{\tau_{B,jj}}(n|H_i) \prod_{j=1}^J c_{\tau_{A,jj}}(n|H_i) c_{\tau_{B,jj}}(n|H_i) \quad (6.3.35b)$$

$$= \prod_{j=1}^J [c_{\tau_{A,jj}}(n|H_i)]^2 [c_{\tau_{B,jj}}(n|H_i)]^2 \quad i = 0,1. \quad (6.3.35c)$$

Using eq(6.3.34), the quadratic terms in eq(6.3.16b) become

$$\begin{aligned} \mathbf{z}^T(n|H_i) [K_r(n|n-1, H_i)]^{-1} \mathbf{z}(n|H_i) &= \\ &= \mathbf{z}^T(n|H_i) [C_\tau^T(n|H_i)]^{-1} C_\tau^{-1}(n|H_i) \mathbf{z}(n|H_i) \quad i = 0,1 \end{aligned} \quad (6.3.36a)$$

$$= \mathbf{z}^T(n|H_i) \mathbf{z}(n|H_i) \quad i = 0,1 \quad (6.3.36b)$$

$$= [\mathbf{z}_A^T(n|H_i) \quad \mathbf{z}_B^T(n|H_i)] \begin{bmatrix} \mathbf{z}_A(n|H_i) \\ \mathbf{z}_B(n|H_i) \end{bmatrix} \quad i = 0,1 \quad (6.3.36c)$$

$$= \mathbf{z}_A^T(n|H_i) \mathbf{z}_A(n|H_i) + \mathbf{z}_B^T(n|H_i) \mathbf{z}_B(n|H_i) \quad i = 0,1 \quad (6.3.36d)$$

$$= \sum_{j=1}^J [\tau_{jA}^2(n|H_i) + \tau_{jB}^2(n|H_i)] \quad i = 0,1 \quad (6.3.36e)$$

where eq(6.3.32a) is used to obtain (6.3.36b). Using eqs(6.3.35c) and (6.3.36e) in (6.3.16b), we obtain

$$\ln \Lambda_{H_1, H_0}^Q = \frac{1}{2} \sum_{j=1}^J \sum_{n=1}^N \left[\ln \frac{[c_{\tau A_{jj}}(n|H_0)]^2}{[c_{\tau A_{jj}}(n|H_1)]^2} + \ln \frac{[c_{\tau B_{jj}}(n|H_0)]^2}{[c_{\tau B_{jj}}(n|H_1)]^2} + \right. \\ \left. [\tau_{jA}^2(n|H_0) + \tau_{jB}^2(n|H_0)] - [\tau_{jA}^2(n|H_1) + \tau_{jB}^2(n|H_1)] \right]. \quad (6.3.37)$$

As in the previous section, when eqs(6.3.28) hold, we have (Appendix J)

$$\sum_{n=1}^N \left[\ln \frac{[c_{\tau A_{jj}}(n|H_0)]^2}{[c_{\tau A_{jj}}(n|H_1)]^2} + \ln \frac{[c_{\tau B_{jj}}(n|H_0)]^2}{[c_{\tau B_{jj}}(n|H_1)]^2} \right] = 2 \sum_{n=1}^N \left[\ln \frac{|c_{\alpha_{jj}}(n|H_0)|^2}{|c_{\alpha_{jj}}(n|H_1)|^2} \right] \quad (6.3.38a)$$

and

$$\sum_{j=1}^J [\tau_{jA}^2(n|H_i) + \tau_{jB}^2(n|H_i)] = 2 \sum_{j=1}^J |\alpha_j(n|H_i)|^2 \quad (6.3.38b)$$

so that eq(6.3.37) reduces to eq(6.1.38).

6.4. SYSTEM ARCHITECTURE FOR THE LIKELIHOOD RATIO EVALUATION

In this section, we describe an implementation of the system to evaluate the likelihood ratios presented above. The practical implementation of systems to evaluate these expressions involves several assumptions. In this dissertation, it is assumed that the signal and non-white clutter processes defined in chapter 3 are

modelled as autoregressive processes embedded in additive white Gaussian noise. We note, however, that the addition of these underlying processes leads to an autoregressive moving average (ARMA) observation process. Thus, a filtering strategy for 'whitening' the observation data as discussed previously in this chapter and section 2.4, should employ ARMA filters. However, algorithms involving estimates of the ARMA parameters to update the filter weights are computationally intensive and require further investigation in terms of stability issues. A more practical approach, which is used in this work, approximates the received ARMA observation process by an AR process. Although an AR process of infinite order is required to exactly model an ARMA process, an adequate approximation can be made with an AR process of 'sufficiently high' order [36]. As a result, the observation processes in the remainder of this work are treated as though they are AR processes. The system architecture for the evaluation of the likelihood ratios can then be realized with the filtering methods described in section 2.4; ie., with prediction error filters using tapped delay lines and lattice structures.

In the investigation conducted in this dissertation and reported in the next section, the signal and clutter processes are synthesized as AR processes (see chapter 4) and added to white noise. In order to determine an approximate AR model for the received ARMA process, it is first necessary to determine the model order. In general, the order is larger under hypothesis H_1 due to presence of the signal [16]. Once the orders under H_0 and H_1 have been selected, the coefficients for the AR models which approximate the ARMA processes under the two hypotheses can be obtained. These, in turn, determine the prediction error filter coefficients. In chapter 7, two cases are considered depending upon whether the parameters of the underlying processes are known 'a priori' or must be estimated from the data.

6.4.1. System Architecture for Gaussian Processes

The likelihood ratios of eqs(6.1.26) and (6.1.38) are presented again for easy reference; ie.,

$$\ln \Lambda_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ \ln \frac{\frac{\sigma_{\gamma_j}^2(n|H_0)}{2}}{\sigma_{\gamma_j}(n|H_1)} + \frac{|\gamma_j(n|H_0)|^2}{\sigma_{\gamma_j}(n|H_0)} - \frac{|\gamma_j(n|H_1)|^2}{\sigma_{\gamma_j}(n|H_1)} \right\} \quad (6.4.1a)$$

$$= \sum_{j=1}^J \sum_{n=1}^N \ln \Lambda(\gamma_j, n) \quad (6.4.1b)$$

where

$$\ln \Lambda(\gamma_j, n) = \left\{ \ln \frac{\frac{\sigma_{\gamma_j}^2(n|H_0)}{2}}{\sigma_{\gamma_j}(n|H_1)} + \frac{|\gamma_j(n|H_0)|^2}{\sigma_{\gamma_j}(n|H_0)} - \frac{|\gamma_j(n|H_1)|^2}{\sigma_{\gamma_j}(n|H_1)} \right\} \quad (6.4.1c)$$

and

$$\ln \Lambda_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ \ln \frac{|c_{\alpha_{jj}}(n|H_0)|^2}{|c_{\alpha_{jj}}(n|H_1)|^2} + |\alpha_j(n|H_0)|^2 - |\alpha_j(n|H_1)|^2 \right\}. \quad (6.4.2a)$$

$$= \sum_{j=1}^J \sum_{n=1}^N \ln \Lambda(\alpha_j, n) \quad (6.4.2b)$$

where

$$\ln \Lambda(\alpha_j, n) = \left\{ \ln \frac{|c_{\alpha_{jj}}(n|H_0)|^2}{|c_{\alpha_{jj}}(n|H_1)|^2} + |\alpha_j(n|H_0)|^2 - |\alpha_j(n|H_1)|^2 \right\} \quad (6.4.2c)$$

A block diagram of the system architecture that implements the likelihood ratio of eq (6.4.1) is shown in Fig. 6.4.1. This is the multichannel extension of the implementation reported in [16,23]. The lower branch is used to compute the

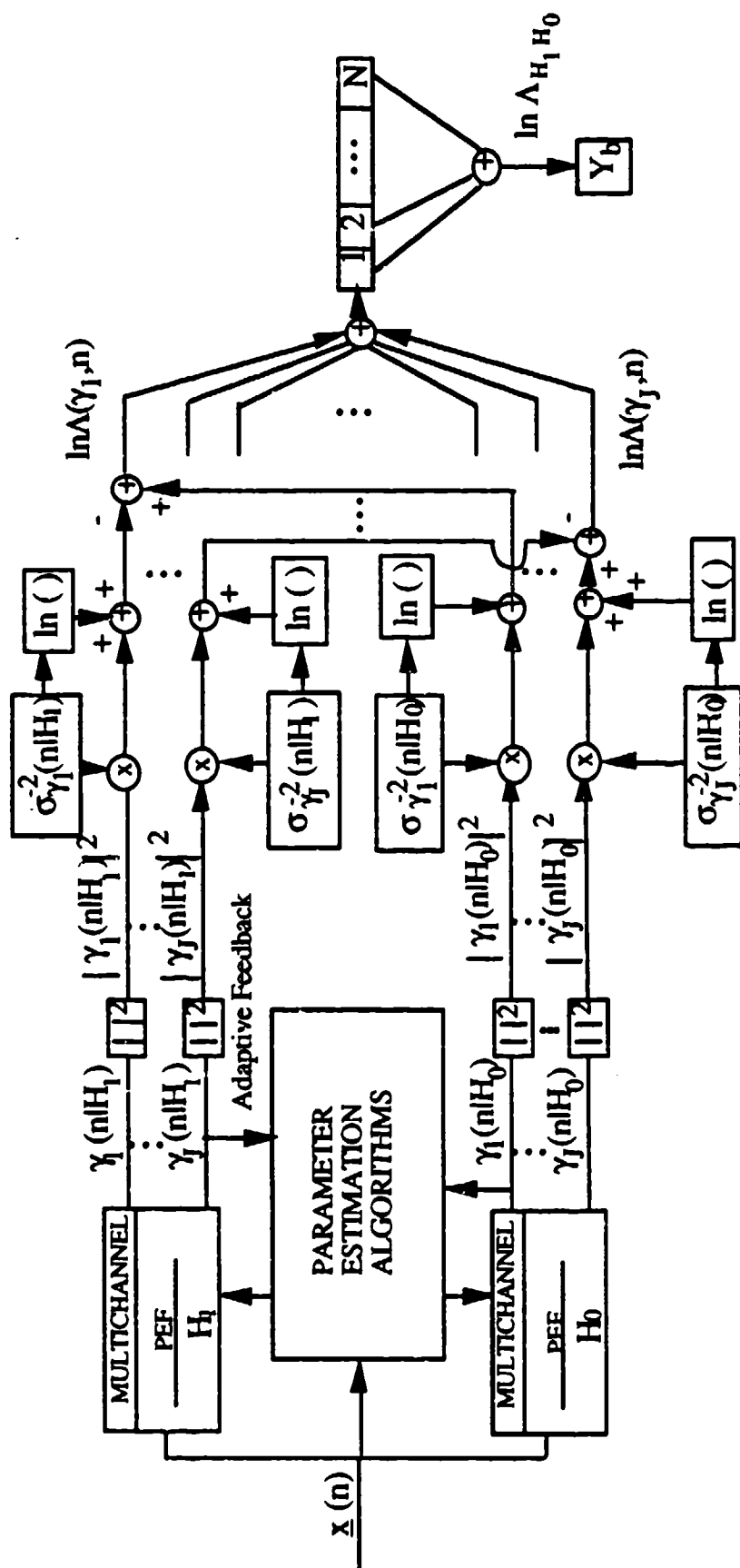


Figure 6.4.1 Multichannel Likelihood Ratio Architecture

second term and the numerator of the first term in eq(6.4.1), while the upper branch determines the third term and the denominator of the first term. The quantity Y_b is the threshold for the decision procedure and is discussed in Appendix K. As noted above (and discussed in more detail in chapter 7), the specific choice of the prediction error filter structure will depend on the assumed underlying model of the observation processes [20]. A multichannel forward prediction error filter using a tapped delay line architecture is shown in Fig. 6.4.2. A lattice structure for the single channel case, utilizing both forward and backward coefficients, is shown in Fig. 6.4.3. An extension of the lattice filter to the multichannel case was shown in Fig. 2.4.4 and is again shown in Fig. 6.4.4. Since these realizations are based on AR processes, whereas the actual observations are from ARMA processes, the order of the realizations shown in Figs. 6.4.2 through 6.4.4 should be infinite. In practice, a finite order is used resulting in approximations to the innovations processes.

6.4.2. Implications of the Estimation Procedure

We now consider various implementations depending upon the amount of 'a priori' statistical knowledge available regarding the observation processes under the two hypotheses. This knowledge is utilized to determine both the prediction error filter coefficients and error variances for each hypothesis. In this subsection, we will consider approximations to the likelihood ratios which depend on the extent of our knowledge regarding these 'a priori' statistics. For the scalar case, similar approximations are presented by Zhang and Haykin [20].

Case 1

In this case, the statistical properties of the observation processes under both hypotheses are considered to be known 'a priori'. Treating the observation processes as AR processes, the filter coefficients for F_0 and F_1 as well as the error variances are known exactly and eqs(6.4.1) and (6.4.2) provide an exact determination of the likelihood ratio. In this case, the first term in each likelihood ratio is a constant and can be incorporated into the threshold, Y_b . And so, these equations reduce to

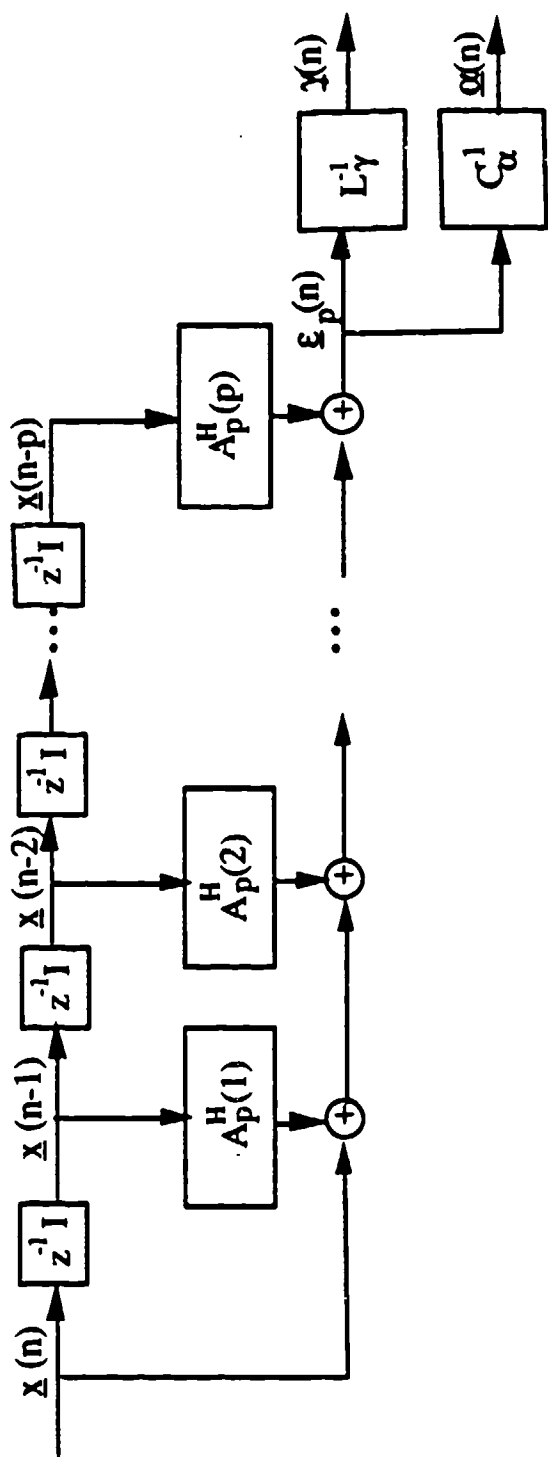


Figure 6.4.2 Multichannel Prediction Error Filter

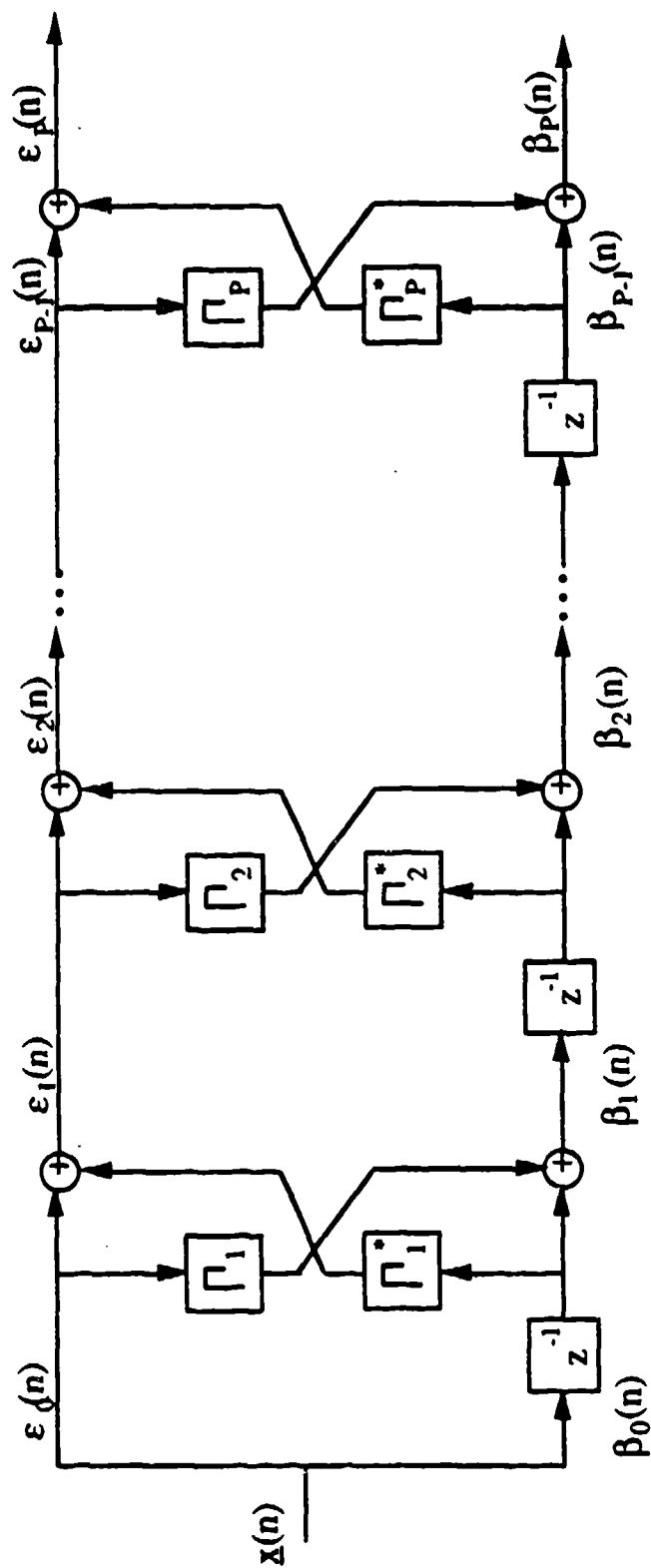


Figure 6.4.3 Single Channel Lattice Filter

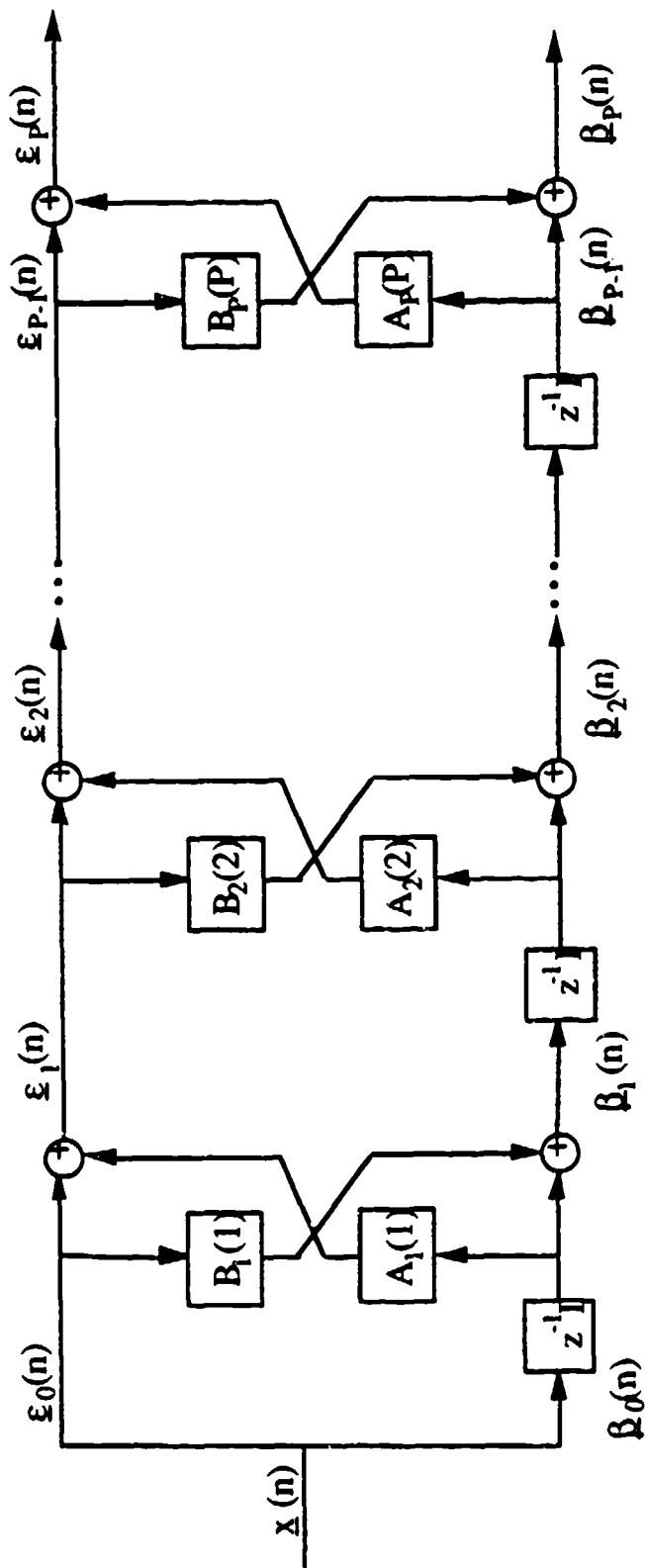


Figure 6.4.4 Multichannel Lattice Filter

$$\ln \Lambda'_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ \frac{|\gamma_j(n|H_0)|^2}{\sigma_{\gamma_j(n|H_0)}^2} - \frac{|\gamma_j(n|H_1)|^2}{\sigma_{\gamma_j(n|H_1)}^2} \right\} \quad (6.4.3)$$

and

$$\ln \Lambda'_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ |\alpha_j(n|H_0)|^2 - |\alpha_j(n|H_1)|^2 \right\}, \quad (6.4.4)$$

respectively. In this case, when hypothesis H_0 is true (ie., no signal present), filter F_0 provides MMSE vector outputs while filter F_1 produces much larger error residuals. Thus, the likelihood ratios decrease due to the increase in the second terms in eqs(6.4.3) and (6.4.4). When H_1 is true, the situation reverses and the likelihood ratios increase as the first terms increase.

Case 2A

In this case, no 'a priori' information is available to obtain the prediction error filter coefficients. However, complete knowledge is assumed regarding the prediction error variance terms. This will be the situation when large amounts of 'a priori' data are used to obtain the error variances. The prediction error filter coefficients for F_0 and F_1 are obtained by processing the received data as will be explained in section 6.4.3. In this case, the likelihood ratios become

$$\ln \Lambda_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ \frac{|\hat{\gamma}_j(n|H_0)|^2}{\sigma_{\gamma_j(n|H_0)}^2} - \frac{|\hat{\gamma}_j(n|H_1)|^2}{\sigma_{\gamma_j(n|H_1)}^2} \right\} \quad (6.4.5)$$

$$\ln \Lambda_{H_1, H_0} = \sum_{j=1}^J \sum_{n=1}^N \left\{ |\hat{\alpha}_j(n|H_0)|^2 - |\hat{\alpha}_j(n|H_1)|^2 \right\} \quad (6.4.6)$$

where $\hat{\cdot}$ denotes the estimated values. We again note that the architecture in Fig. 6.4.1 utilizes two prediction error filters (PEF) implemented in parallel with each designed to be an optimal estimator under the given hypothesis. However, in this case, only one hypothesis is true and only one set of observation processes are available which must be utilized to set the weights of both filters. Thus, if

both filters are of equal order, their coefficients (and consequently their outputs) will be equivalent. However, filter F_1 , designed for the hypothesis H_1 data, is chosen to be of higher order than that of F_0 . This is justified on the basis that the underlying process under H_1 is approximated by a higher order AR process due to the presence of the random signal [16,23]. When H_1 is true, the F_1 filter designed for the H_1 processes adapts to the new process. However, the F_0 filter designed for the H_0 process produces a larger error output since this lower order filter now operates on a higher order AR process. Therefore, it cannot adapt to the underlying process coefficients. As a result, the first summations in eqs(6.4.5) and (6.4.6) increase. Since this term provides a positive contribution to the likelihood ratio, it is the mechanism which raises the likelihood above a predetermined threshold under H_1 . On the other hand, when H_0 is true and there is a large amount of data available, the output from the F_1 filter is likely to be close to that from the F_0 filter. This result occurs because the additional coefficients of the F_1 filter are likely to be negligible, while the remaining coefficients are likely to be close to those of the F_0 filter.

Case 2B

In this case, as in case 2A, observation data is used to estimate the prediction error filter coefficients while complete knowledge is assumed regarding the prediction error variance terms. Whereas, the filter coefficients were estimated in case 2A using one set of available observation data, they are estimated in case 2B using data available under both hypotheses. This procedure will enable us to utilize large quantities of data to accurately determine the coefficients under both hypotheses. The likelihood ratios for this case are the same as those for case 2A. However, in the limit as the estimated coefficients (for both hypotheses) approach the true parameters of the processes, the likelihood ratio approaches that of case 1.

6.4.3. Filter Coefficient Determination

The filter coefficients can be determined via several approaches. For wide-sense stationary processes, the p th order filter coefficients $A_p^H(k)$ and forward prediction error covariance matrix $[\Sigma_f]_p$ could be estimated via the

Levinson-Wiggins-Robinson (LWR), the Strand-Nuttall and the Vieira-Morf methods [7,8,9,10,29]. For the LWR method, estimates of the correlation sequence are obtained by recursive update. Since this method utilizes correlation sequence estimates, it is anticipated to yield less accurate estimates of the coefficients than the Strand-Nuttall or Vieira-Morf approaches for the same reasons as noted by Burg [35] in the single channel case; ie. the unbiased estimate of the correlation matrix may not be positive semi-definite and therefore physically unrealizable. On the other hand, the biased estimate may yield inaccurate estimates, especially for limited data. The Strand-Nuttall method (the multichannel generalization of the Burg algorithm), however, estimates the filter coefficients directly from the data thus bypassing the requirement to first estimate the correlation matrix. The reflection coefficients for the single channel or multi-channel cases are estimated by this algorithm and applied to the lattice filter structures shown in Figs. 6.4.4 and 6.4.5, respectively. From these estimates, the autoregressive coefficient matrices $A_M^H(k)$ [as well as the backward coefficients] can be determined and used as the filter coefficients (see chapter 2).

The additional second stage multichannel filter coefficients $[L_{\gamma_p}]^{-1}$ and $[C_{\alpha_p}]^{-1}$ discussed in chapter 2 must also be determined in order to use the simplified form of eqs(6.1.26) and (6.1.38), respectively. These coefficients complete the spatial whitening of the $\underline{x}(n)$ process via eq(6.1.16) and (6.1.29), respectively. An LDL^H decomposition of the error covariance matrix $[\Sigma_f]_p = E[\underline{x}(n)\underline{x}^H(n)]$ yields the lower triangular unit diagonal L_γ matrix which is an estimate of L_u . The inverse of L_{γ_p} is the matrix coefficient that we desire in the second stage of the filter. Similarly, a Cholesky decomposition CC^H of $[\Sigma_f]_p$ provides the matrix C_{α_p} .

Finally, we point out that coefficients for the more general linear estimator $\hat{\underline{x}}(n)$ proposed in section 6.3 could be obtained using the same form as algorithms proposed for wide-sense jointly stationary complex baseband processes; ie., algorithms such as the Strand-Nuttall and Vieira-Morf could be used but extended to the quadrature form. Although the matrices become $2J \times 2J$ rather than $J \times J$, the resulting likelihood ratios are capable of processing the more general class of Gaussian processes with unconstrained quadrature components. We note that for

this class of processes, the narrowband bandpass processes could be non-stationary.

In this chapter, we initially derived a multichannel likelihood ratio for complex Gaussian processes. The approach was model-based and utilized multichannel innovations processes. Two generalized likelihood ratios were considered using the LDL^H and Cholesky CC^H decompositions on the error covariance matrix to whiten across channels. These likelihood ratios were then extended to consider the more general class of Gaussian processes with unconstrained quadrature components. In this development, we proposed an estimator capable of whitening not only the complex processes, but also their quadrature components. The implementation of these likelihood ratios is described in section 6.4. In this dissertation, the parameter estimation is performed directly on the wide-sense stationary baseband processes as shown in Fig.6.4.1. We note, however, that this implementation architecture is amenable to adaptive estimators which would use the feedback from the error vectors. Finally, we pointed out that coefficients for the more general linear estimator $\hat{p}(n)$ proposed in section 6.3 could be obtained using the same form as algorithms proposed for wide-sense jointly stationary complex baseband processes.

CHAPTER 7

DETECTION PERFORMANCE EVALUATION: PROCEDURE AND NUMERICAL RESULTS

7.1. GENERAL DESCRIPTION

In this section, we describe the procedure used to implement the detection scheme described in section 6.4 for performance evaluation. We again indicate that this implementation involves several assumptions. First, we note that we are modeling the signal and clutter as autoregressive (AR) processes of specific orders. For sufficiently high orders, the AR models can yield suitable approximations of the true physical processes observed in practice. However, when these AR processes are added to a white noise process to form the received process (also referred to as the observation process), an ARMA process results. Since our implementation is based on an AR model for the observation processes, it is necessary to approximate the received ARMA observation process as an AR process. As noted previously, an ARMA process can be modeled as closely as desired by an AR model of 'sufficient' order [36]. These AR models enable us to implement prediction error filters with tapped delay lines and lattice structures in the detection architecture as described in sections 2.4 and 6.4.

In the numerical investigation conducted here, the signal and clutter processes are synthesized as AR processes (see chapter 4) and added to white noise. In order to determine the approximate AR model for the received ARMA process, it is first necessary to determine the model order. In general, the order is larger under hypothesis H_1 due to the presence of the signal [16]. Once the orders under H_0 or H_1 have been selected, the coefficients for the AR model which approximate the ARMA process under each hypothesis can be determined. These in turn determine the prediction error filter coefficients. Two cases will be considered depending upon whether the parameters of the underlying processes are known 'a priori' or must be estimated from the data.

In the next two subsections, we describe the procedures used to determine the filter orders and coefficients. In the final subsections, the Monte-Carlo simulation procedures for detection performance evaluation are described. The procedures presented in this chapter are illustrated by generating two channel vector processes (each of order two) using the method described in chapter IV.

7.2. 'KNOWN' UNDERLYING PROCESSES

In this case, it is assumed that the models of the underlying signal, clutter and noise processes are known 'a priori'. Since the signal and clutter are AR processes and the additive Gaussian noise is white, this amounts to knowing the variances of each process, as well as the orders and AR coefficients of the AR processes. Even so, it is necessary to determine the orders and AR coefficients of the AR models used to approximate the received ARMA processes under both H_0 and H_1 . The procedure employed to accomplish this is described next.

Ideally, it is expected that the probability of detection will monotonically increase to some asymptotic value as the AR orders in the implementation of section 6.4 are increased, thereby resulting in better approximations to the received ARMA process. Our procedure given below is to successively try higher orders until negligible improvements in detection performance result. Specifically, we utilize the following empirical method:

- (i) Observation data processes for both the H_0 and H_1 hypotheses are generated using a sufficiently large number of time samples N_{TC} and realizations N_{RC} to reduce the variance associated with the coefficient estimates to an acceptable level. The notation TC and RC are used to denote that these time samples and realizations are used for the purpose of determining the coefficients. The ergodicity considerations discussed in chapter 5 are used to select appropriate values for N_{TC} and realizations N_{RC} .
- (ii) Both sets of observation data are now processed by the Strand-Nuttall algorithm to obtain an estimate of the matrix coefficient for each realization of the N_{TC} time samples. The estimates of the matrix coefficients are then averaged over the N_{RC} realizations. These computations are performed for models of order $p = 2, 4, 8, 12, 16$ and 20 . In general, these various model orders would be considered for both prediction error filters shown in Fig.6.4.1. In this dissertation, however, we only consider the signal in additive Gaussian white noise case. Since the $A(k)$ coefficients for the H_0 filter are null matrices in this case, we need only consider the order of the H_1 filter.
- (iii) The AR coefficients for the model of order p are used to determine the coefficients of the p th order prediction error filters shown in Fig.6.4.1. The probability of detection is evaluated using N_T time samples for each

case using the procedure to be described in section 7.3. For the signal in white noise problem, the six cases of the filters for orders 2, 4, 8, 12, 16 and 20 for the H_1 filter are used in the likelihood ratio implementation in section 6.4 and the filter orders are selected at the value for which detection performance is sufficiently close to its asymptotic value.

7.2.1. Signal in Additive White Noise Problem

For the signal in white noise problem, filter coefficients of value zero are used in the F_0 filter since the MMSE coefficient matrix for an uncorrelated, zero-mean multichannel white noise process is a null matrix. For the F_1 filter, we determine the filter order and coefficients using the empirical method outlined above. As a first example, we consider a case with relatively high temporal and cross-channel correlation. For this case, the parameters were $\lambda_{11}=\lambda_{22}=0.9$, $|\rho_{12}|=0.99$ and $S/N=+3\text{dB}$ using $\sigma_{s_{11}}^2=\sigma_{s_{22}}^2=2$ and $\sigma_{w_{11}}^2=\sigma_{w_{22}}^2=1$. The specific values of $N_{TC}=10,000$ and $N_{RC}=100$ were selected in our process synthesis procedure described in chapter 4. The detection performance for the two-channel signal in additive white noise case for $N_T=10$ was evaluated using the Monte-Carlo procedure to be described later. The probability of detection is plotted in Fig.7.2.1 versus the six filter orders of F_1 for the two false alarm probability values of 6.93×10^{-2} and 6.93×10^{-4} . As noted, for a filter order greater than four, negligible differences in the values of the probability of detection were obtained. Even so, an F_1 filter of order eight was selected for the simulations involving this case.

We now consider a second example and emphasize the procedure used to determine the filter coefficients and variances used in the likelihood ratio calculation. In this example, we consider the signal in additive white noise where the signal parameters are $\sigma_{s_{11}}^2=\sigma_{s_{22}}^2=0.3162$, $\lambda_{11}=\lambda_{22}=0.9$, $|\rho_{12}|=0.99$ and

$S/N=-5\text{dB}$. The additive white noise vector process considered here is uncorrelated across channels with a variance of unity on each channel. The first step in this procedure is to synthesize $N_{RC}=100$ realizations each with $N_{TC}=10,000$ observation time samples under hypothesis H_1 . Figs. 7.2.2a and 7.2.2c show the corresponding time-averaged autocorrelation functions for each

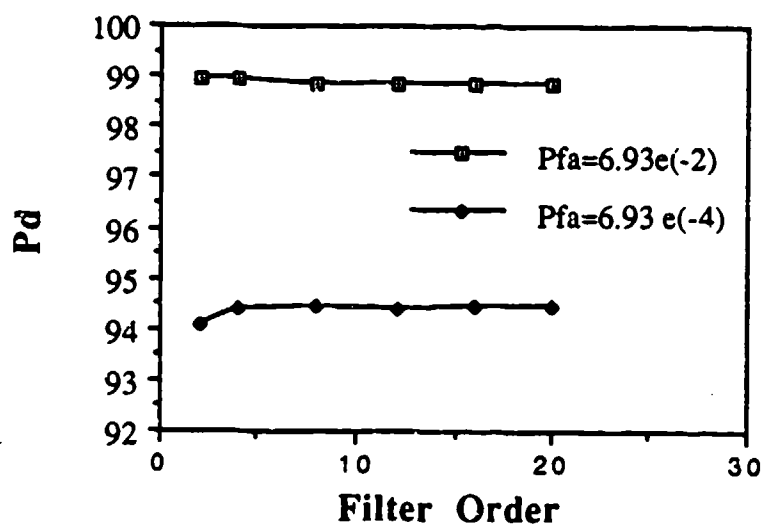
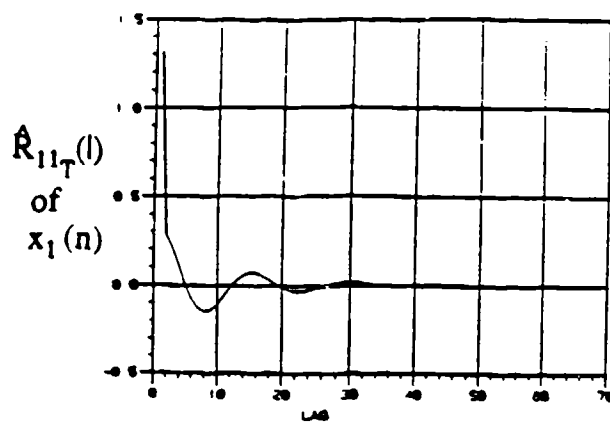
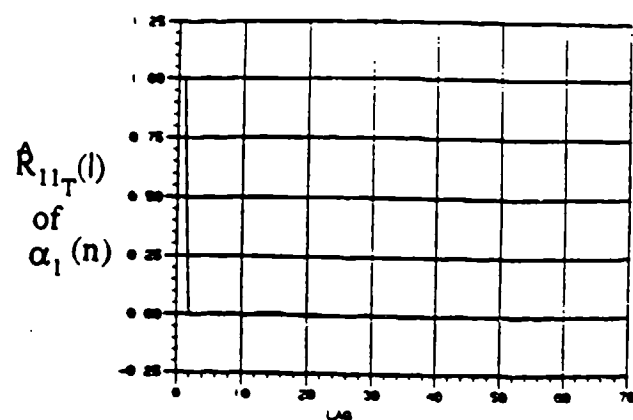


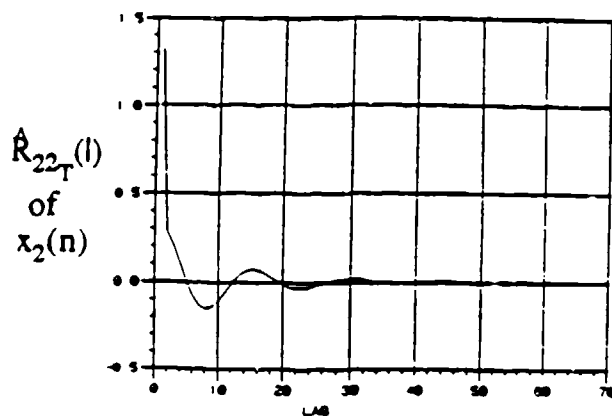
Fig 7.2.1 Probability of detection for a signal in additive white noise versus transversal prediction error filter orders for F_1 .



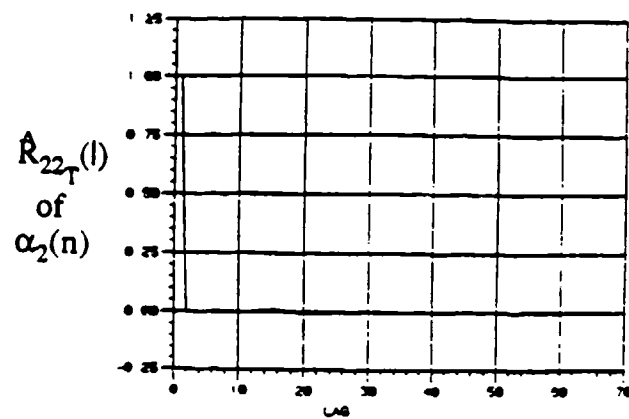
a



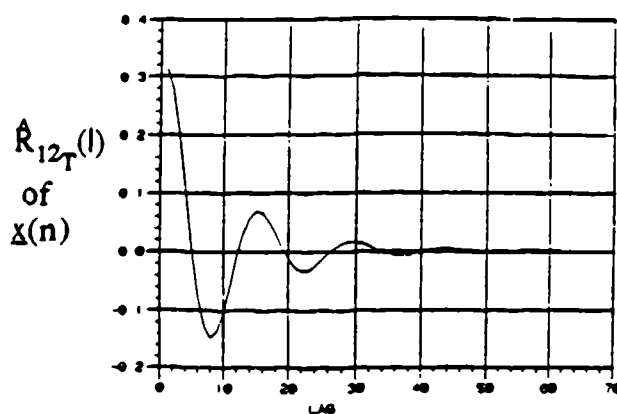
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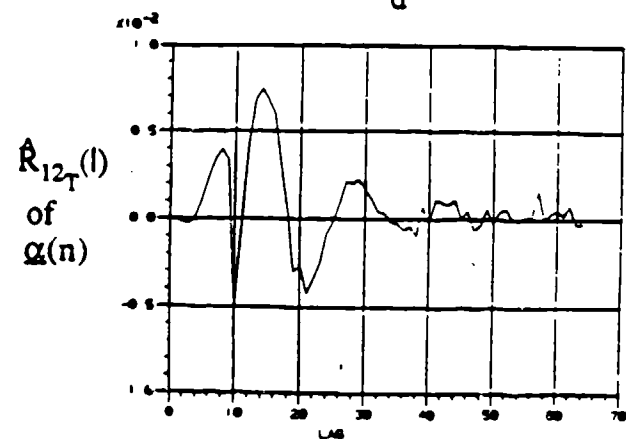
c



d



e



f

Fig. 7.2.2 Auto- and cross-correlation functions for the prewhitened $\underline{x}(n)$ and the whitened $\underline{\alpha}(n)$ processes consisting of an AR(2) signal in additive white noise; a.) $\hat{R}_{11T}(l)$ for $x_1(n)$ b.) $\hat{R}_{11T}(l)$ for $\alpha_1(n)$ c.) $\hat{R}_{22T}(l)$ for $x_2(n)$ d.) $\hat{R}_{22T}(l)$ for $\alpha_2(n)$ e.) $\hat{R}_{12T}(l)$ for $\underline{x}(n)$ f.) $\hat{R}_{12T}(l)$ for $\underline{\alpha}(n)$.

channel of the synthesized $\underline{x}(n|H_1)$ process, while Fig. 7.2.2e shows the cross-correlation function. Based on the results discussed previously, we approximate the multichannel ARMA process under hypothesis H_1 with a multichannel AR process of order eight. We now use the synthesized observation data generated under H_1 to estimate these coefficients. These coefficients are obtained using the Strand-Nuttall estimation algorithm with the above 10,000 time samples for each of the 100 realizations. We then average the coefficients over the 100 realizations. The resulting estimates of the $A_p(k)$ $k=1,2,\dots,8$ matrix coefficients are the coefficients used in the first stage of the F_1 filter to obtain the $\underline{\epsilon}(n|H_1)$ error vector. The second stage coefficient used to decorrelate $\underline{\epsilon}(n|H_1)$ across the channels is then obtained using a Cholesky decomposition CC^H of the estimated forward error covariance matrix to obtain C_α^{-1} (see chapter 6). The observation data $\underline{x}(n|H_1)$ is now used as an input to the two stage filter to obtain $\underline{\alpha}(n|H_1)$ using C_α^{-1} . Figs. 7.2.2b and 7.2.2d show the autocorrelation functions of the vector $\underline{\alpha}(n|H_1)$ for each channel in this example. The autocorrelation functions for each channel indicate that the $\underline{\alpha}(n|H_1)$ processes are temporally whitened, unit variance processes. This result agrees with the analytical development leading to the likelihood ratio expressed by eq(6.1.38). The cross-correlation function for the two channels is shown in Fig. 7.2.2f and indicates that the cross-channel correlation has been significantly reduced. We note that the results in this example illustrate the fact that the F_1 filter of order eight performs the desired whitening of the observation data received under hypothesis H_1 , when using the coefficients estimated with the large data base described above. The filter coefficients and error variances obtained by this procedure are treated as the 'known' values of the AR process which in turn is a close approximation to the true ARMA process.

For the signal in white noise problem, the observation data under H_0 is totally uncorrelated white noise both in time and across channels. Thus, the parameter coefficients for this process are null matrices.

7.3. Monte-Carlo Procedure for the 'Known' Underlying Process Case

In this section, we describe the detection performance evaluation procedure used once the filter coefficients and error variances have been determined using the method presented in the previous section. First, a sequence of N_T data samples is generated for each realization under both hypotheses. Under H_0 , noise processes are generated without the signal while under H_1 , the signal process is included. In the 'known' case, we note that the value of N_T used in the detection procedure is not necessarily the value N_{TC} used to determine the coefficients. Each sequence of N_T samples is then used to compute a single value of the likelihood ratio. The detection threshold level Y_b is determined by computing the likelihood ratio for N_R realizations of the H_0 observation data where the N_T data samples are used in each realization. These values are rank ordered and a threshold level determined to establish a specified false alarm probability. Specifically, if $X(H_i)$ is the random variable representing the number of times that the threshold is exceeded under hypothesis H_i $i=0,1$ obtained from N_R independent realizations, then the probability of exceeding a threshold can be estimated using

$$\hat{p}_i = \frac{X(H_i)}{N_R} \quad i = 0,1. \quad (7.3.1)$$

And so, under hypothesis H_0 , the estimate of the false alarm probability becomes

$$\hat{p}_{fa} = \hat{p}_0 = \frac{X(H_0)}{N_R}. \quad (7.3.2)$$

Similarly, the estimated probability of detection $\hat{p}_d$ is expressed as

$$\hat{p}_d = \hat{p}_1 = \frac{X(H_1)}{N_R}. \quad (7.3.3)$$

For the 'known' parameter case, detection performance is computed in section 7.6.1.1 using two-channels ($J=2$) consisting of $N_T = 10$ time samples plus

eight additional samples to pre-fill the F_1 filter and $N_R = 20,000$ realizations for the Monte-Carlo procedure. The F_1 and F_0 filter coefficients are obtained as discussed in the previous section.

Motivated by a discussion in [79], we next determine the standard deviation associated with the estimated probability of false alarm, $\hat{P}_{fa}$, and probability of detection, $\hat{P}_d$, when using the Monte-Carlo procedure. Let $A(H_i)$ denote the event that the threshold is exceeded under hypothesis H_i $i=0,1$. We define the probability of $A(H_i)$ as $P[A(H_i)]=p_i$ and $P[\bar{A}(H_i)]=1-p_i$ where $\bar{A}(H_i)$ is the complementary event; ie., the event that the threshold was not exceeded. For $i = 0$ and 1 , p_0 and p_1 are the probabilities of false alarm P_{fa} , and detection P_d , respectively. The sample space consists of all possible sequences $\{a_1, a_2, \dots, a_{N_R}\}$, where each a_j $j=1, 2, \dots, N_R$ is either $A(H_i)$ or $\bar{A}(H_i)$ depending on whether $A(H_i)$ or $\bar{A}(H_i)$ occurred on the j th repetition of $\mathcal{E}$. We also assume that $P[A(H_i)]=p_i$ remains constant for all repetitions. Since $X(H_i)$ is the number of times that the event $A(H_i)$ occurred under hypothesis H_i , $X(H_i)$ is a binomial random variable with parameters N_R and p_i . The probability that $X(H_i) = D$ is expressed as

$$P[X(H_i)=D] = \binom{N_R}{D} (p_i)^D (1-p_i)^{N_R-D} \quad i=0,1 \quad D=0,1,\dots,N_R \quad (7.3.4)$$

where D is the number of times the threshold is exceeded. The mean and variance of $X(H_i)$ are expressed as

$$E[X(H_i)] = N_R p_i \quad i=0,1 \quad (7.3.5a)$$

and

$$V[X(H_i)] = N_R p_i (1-p_i) \quad i=0,1, \quad (7.3.5b)$$

respectively. Using eqs(7.3.1) and (7.3.5), the mean, variance, and standard deviation of $\hat{p}_i$ are expressed respectively as

$$E[\hat{p}_i] = E[X(H_i)/N_R] = E[X(H_i)]/N_R = p_i \quad i = 0,1 \quad (7.3.6a)$$

$$V[\hat{p}_i] = V[X(H_i)/N_R] = \frac{V[X(H_i)]}{(N_R)^2} = \frac{p_i(1-p_i)}{N_R} \quad i = 0,1 \quad (7.3.6b)$$

and

$$\sigma_{\hat{p}_i} = \sqrt{V[X(H_i)/N_R]} = \sqrt{\frac{p_i(1-p_i)}{N_R}} \quad i = 0,1 \quad (7.3.6c)$$

where $i=0$ and 1 provide the appropriate expression for the probability of false alarm and detection, respectively. Eq(7.3.6a) indicates that $\hat{p}_i$ expressed in eq(7.3.1) is an unbiased estimate of p_i . From eq(7.3.6c), we obtain for N_R the expression given by

$$N_R = \left[\frac{1-p_i}{p_i} \right] \frac{1}{(\sigma_{\hat{p}_i}/p_i)^2} \quad (7.3.7)$$

where the ratio $\sigma_{\hat{p}_i}/p_i$ is referred to as the 'coefficient of variation' [84].

Eq(7.3.7) enables us to determine the number of realizations in the Monte-Carlo procedure that are required to achieve a specific value of this ratio for the estimate of P_{fa} or P_d . As an example, under hypothesis H_0 , we have from eq(7.3.7) with $p_0 = P_{fa}$

$$N_R = \left[\frac{1-P_{fa}}{P_{fa}} \right] \frac{1}{(\sigma_{\hat{p}_0}/P_{fa})^2} \quad (7.3.8)$$

For $\sigma_{\hat{p}_0}/P_{fa} = 0.30$ and $P_{fa} \ll 1$, we note that [79]

$$N_R = 11/P_{fa} \quad (7.3.9)$$

while for $\sigma_{\hat{p}_0}/P_{fa} = 0.05$

$$N_R = 400/P_{fa} \quad (7.3.10)$$

We can also use eq(7.3.6c) to obtain

$$\sigma_{\hat{P}_0}/P_{fa} = \left[\frac{1-P_{fa}}{N_R P_{fa}} \right]^{\frac{1}{2}}. \quad (7.3.11)$$

Using $N_R=20,000$ realizations, the ratio expressed by eq(7.3.11) is 2.59% when $P_{fa}=0.0693$, while for $P_{fa}=6.93 \times 10^{-4}$, we obtain 26.85%. Under hypothesis H_1 , we have $p_1 = P_d$. In this case, eq(7.3.6c) can be used to determine

$$\sigma_{\hat{P}_1}/P_d = \left[\frac{1-P_d}{N_R P_d} \right]^{\frac{1}{2}}. \quad (7.3.12)$$

With $N_R=20,000$ realizations, the ratio expressed by eq(7.3.12) is 0.236% when $P_d=0.90$. The expressions derived above enable us to determine confidence levels associated with the detection results. Eqs(7.3.11) and (7.3.12) are used in section 7.6 to place error bars on the detection results. In the case of processes with 'known' parameters, the standard deviation of the detection results are due to use of the finite number of realizations N_R in the Monte-Carlo procedure. As we will now discuss, the standard deviation of the detection performance results associated with the case of 'unknown' process parameters, is also a function of the errors associated with the estimates of the parameters of the observation process as will be discussed in the next section.

In the implementation of the Monte-Carlo method for N_R realizations of the observation data with N_T samples in each realization are generated under H_0 . This data is used to calculate N_R likelihood ratios which are then rank ordered. We then select the upper d_0 values from the rank ordered likelihood ratios so that the estimated probability of false alarm $\hat{P}_{fa}$ can be calculated as

$$\hat{P}_{fa} = \frac{d_0}{N_R}. \quad (7.3.13)$$

The threshold is now set between the values of the likelihood ratio corresponding to the d_0 th value and the next lower value. Thus, d_0 values of the detection statistic will lie above the threshold providing an estimate of the desired

false alarm probability. Similarly, N_R independent realizations under hypothesis H_1 are used to compute N_R likelihood ratios with the H_1 data. The estimated probability of detection $\hat{P}_d$ is now computed by counting threshold crossings for the threshold obtained above.

We note, however, that $\hat{P}_{fa}$ and, therefore, the threshold Y_b are random variables due to the randomness associated with the Monte-Carlo method. We must, therefore, consider the effect of the error associated with the estimated threshold $\hat{Y}_b$ on the probability of detection P_d . Denoting the standard deviation of $\hat{Y}_b$ as σ_{Y_b} and using eq(K.1.47) from Appendix K to obtain P_d for the model I(INCOH) signal, the tolerance about P_d can be expressed as

$$P_d \pm \delta = 1 - I_\Gamma \left[\frac{Y_b \pm \sigma_{Y_b}}{\sqrt{N-1}}, (N-2) \right] + \left[1 + \frac{1}{N(S/N)_{avg}} \right]^{N-1} \cdot \left\{ \exp[-(Y_b \pm \sigma_{Y_b})/(1 + N(S/N)_{avg})] \cdot I_\Gamma \left[\frac{Y_b \pm \sigma_{Y_b}}{\sqrt{N-1}[1 + 1/N(S/N)_{avg}]}, (N-2) \right] - 1 \right\} \quad (7.3.14)$$

where $I_\Gamma(u,s)$ is the incomplete Gamma function defined by eq(K.1.44) and δ denotes the error tolerance in P_d corresponding to a shift of σ_{Y_b} in the threshold Y_b . Similarly, for the model II(INCOH) signal, eq(K.2.9b) yields

$$P_d \pm \delta = 1 - I_\Gamma \left[\frac{Y_b \pm \sigma_{Y_b}}{\sqrt{N}[1 + (S/N)_{avg}]}, (N-1) \right]. \quad (7.3.15)$$

7.4. Monte-Carlo Procedure for 'Unknown' Underlying Processes

For the detection results presented in this dissertation, the prime emphasis is placed on validation of the likelihood ratio implementation using the case of the 'known' underlying processes. Detection results for the case of 'unknown' underlying processes are not included here. However, it is worthwhile to discuss the procedure for this case. For processes with 'unknown' parameters, we make no 'a priori' assumptions regarding the values of the underlying process parameters. For this case, the generalized likelihood ratio is implemented since we are estimating these parameters using the available observation data. However, it is often assumed that the model orders of the processes and the prediction error variances under each hypothesis are 'known' and could be selected using the procedure described previously.

The generalized likelihood ratio is usually evaluated using parameter estimation algorithms based on time-averaging methods. For the 'unknown' parameter case, particular emphasis is given to obtaining detection performance limits which result due to the errors associated with the time-averaged estimates. In chapter 5, analytic expressions are derived for the variance of the time-averaged auto- and cross-correlation function estimators using ergodic considerations and compared to computed results. The results reveal that the error variances associated with these estimates are related not only to the extent of the observation window, but also to 'fundamental process characteristics' such as the temporal and cross-channel correlation as well as the variance of the processes. Thus, detection performance limits for a generalized likelihood ratio using estimation methods will also depend directly on these underlying process characteristics. In future analyses, the detection results will be evaluated to reflect the errors associated with the AR coefficient estimates. Specifically, a confidence bound will be determined for each detection result using signal processes containing various amounts of temporal and cross-channel correlation.

For the case of 'unknown' process parameters, the detection procedure is modified from that of the 'known' parameter case in order to compute the confidence bounds on the detection results. As in the 'known' parameter case, a sequence of N_T data samples is generated under each hypothesis, H_0 and H_1 . In this case, however, the first set (ie., one realization) containing N_T observation data time samples under H_0 is used to estimate the coefficients for both filters F_0 and F_1 using their respective filter orders. As noted in section 6.4, the order of

the filter F_0 is lower than that of F_1 since the addition of the signal process increases the order of the process. Several procedures could now be used to determine the threshold setting depending upon the objectives of the detection analysis. This could again be accomplished using 'a priori' data. However, a procedure more congruous with this 'unknown' parameter case would be to use N_{Rt} realizations (which are independent of the data set used to estimate the coefficients) to compute N_{Rt} likelihood ratios. Again, these values are rank ordered and a threshold level determined to establish a specified false alarm probability. The same procedure is then used with a data set obtained under H_1 ; ie., the coefficients for both F_0 and F_1 are computed using a single data set containing the signal process. Then, N_{Rd} independent realizations are used to compute N_{Rd} likelihood ratios. Finally the number of threshold crossings d_i are computed to determine the probability of detection P_d using eq(7.3.3). This result represents the first detection computation obtained using one set of observation data to estimate the filter coefficients. This procedure is repeated for N_d detection runs using an independent realization of data samples to estimate the coefficients. The N_d detections are then utilized to determine the error variance of the detection results due to the errors in the estimation procedure. The results could be determined for processes with various levels of temporal and cross-channel correlation.

7.5. INCOHERENT DETECTION

As noted in section 6.2, when no coefficients are used in the prediction error filters of the likelihood ratio implementation, the detection results obtained correspond to those of incoherent detection; ie., the observation data is processed without regard to the phase associated with the processes. In this case, we simply perform the likelihood ratio calculation without whitening the $\underline{x}(n|H_i)$ $i=0,1$ data by the prediction error filters. These results will be discussed further in the next section.

7.6. DETECTION PERFORMANCE RESULTS USING MONTE-CARLO SIMULATION

7.6.1 Introduction

In this section, we evaluate the performance of the multichannel detection scheme using the likelihood ratio implementation architecture described in section 6.4. The input observation data processes are generated using the process synthesis procedure described in section 4.7. Detection performance results are determined not only as a function of channel signal-to-noise ratio, but also in terms of the temporal and cross-channel correlation of the processes. As described in section 4.7, these quantities are parametrically controlled in the process synthesis procedure. For example, the temporal correlation on the i th channel is denoted by the parameter λ_{ii} . As $\lambda_{ii} \rightarrow 0$, the process becomes uncorrelated sample-to-sample. On the other hand, as $\lambda_{ii} \rightarrow 1$, the process becomes highly correlated in time. Likewise, the magnitude of the cross-channel correlation is denoted by $|\rho_{ij}|$. Thus, signal and clutter processes can be simulated over a wide range of temporal and cross-channel correlation values using our simulation procedure. Two extreme cases occur when $\lambda_{ii} \approx |\rho_{ij}| \approx 0$ and $\lambda_{ii} \approx |\rho_{ij}| \approx 1$ for all $i, j = 1, 2, \dots, J$. In the former case, all of the J channel processes are uncorrelated both in time and space (ie., across channels). In the latter, they are highly correlated in both of these dimensions. In the next subsection, we describe the characteristics of signal models as presented in the literature and show their relationship to the processes synthesized in this dissertation. As we will discuss below, analytical expressions [13,88] for probability of detection performance for these cases are available. Furthermore, the analytical expressions can be modified for the two limiting cases of multichannel process correlation (see Appendix K for details). In subsection 7.6.4, these analytical expressions provide the bounds for the experimental detection results considered here.

7.6.2 Models of Signal Processes

In this subsection, we describe the signal process models for a single channel representing the two limiting cases of temporal correlation which have been used in the literature [13, 72, 88] to obtain analytic detection results. In Appendix K, we extend these results to the multichannel case.

In specific applications such as the radar problem, a transmitted waveform consisting of N_T pulses (called a pulse train) is used to probe the observation space for each scan of the radar. The echo returns thus constitute the observation processes and contain a modulation in terms of the amplitude and phase resulting from the interaction with the physical environment. For these problems, we distinguish between the amplitude and phase characteristics of the transmitted waveform and those of the echo return.

A transmitted waveform with statistically independent random phases on each pulse is defined as an incoherent pulse train. In Appendix K, we consider the optimum receiver for this pulse train. In the radar problem, such a waveform results due to transmitter phase instabilities and thermal noise. A transmitted pulse train which is assumed to have an exactly known phase relationship pulse-to-pulse is defined as a coherent pulse train. We distinguish, however, between the 'known coherent pulse train' where the initial phase of the first pulse is assumed to be known and the 'coherent pulse train' waveform in which it is an unknown and modeled as a uniformly distributed random variable [13]. In subsection 7.6.4, it is this latter waveform that we will consider for the coherent detection results.

In the radar problem, the transmitted waveform impinges upon a desired target and the background environment. The resulting echo return, therefore, consists of a signal and clutter return, respectively. In addition, there is the presence of additive white noise. The scattering of electromagnetic radiation from several scattering centers on either the target or the background results in the superposition of the echo returns and is defined in the radar terminology as scintillation. These fluctuating signals introduce a statistical variation in the detection process. The rate of these fluctuating signals is dependent upon the target/clutter dynamics as well as the sampling rate or pulse repetition frequency (prf) of the radar; ie., for slow maneuvering targets and/or high radar prf, the fluctuation rate is low. For targets exhibiting high rate of change maneuvers and/or radars with low prf, the pulse-to-pulse returns are more uncorrelated. The two extreme cases of signal temporal correlation have been designated as 'slow' and 'fast' fluctuating targets [72]. The 'slow' fluctuating model is characterized as a process with total amplitude correlation pulse-to-pulse while the 'fast' fluctuating model represents a totally uncorrelated pulse-to-pulse process.

Assuming that the k th received pulse is a narrowband bandpass signal $s_k(t)$, it can be expressed as [13]

$$s_k(t) = A_k p(t - \tau_k) \cos[\omega_c(t - \tau_k) + \theta_k(t - \tau_k) + \phi_k] \quad (7.6.1)$$

where A_k is the signal amplitude of the k th pulse, $p(t)$ is a unit amplitude rectangular gating function of duration T , $\theta_k(t)$ is the k th signal pulse phase modulation, τ_k is the signal echo delay of the k th pulse, and ϕ_k is the initial phase of the k th pulse. For a given pulse, we can set τ_k equal to zero without loss of generality, so that eq(7.6.1) becomes

$$s_k(t) = A_k p(t) \cos[\omega_c t + \theta_k(t) + \phi_k]. \quad (7.6.2)$$

Eq(7.6.2) can also be written as

$$s_k(t) = \text{Re}\{\mathfrak{I}_k(t) \exp[j(\omega_c t)]\} \quad (7.6.3a)$$

$$= \text{Re}\{[s_{Ik}(t) + js_{Qk}(t)] \exp[j(\omega_c t)]\} \quad (7.6.3b)$$

$$= s_{Ik}(t) \cos(\omega_c t) - s_{Qk}(t) \sin(\omega_c t) \quad (7.6.3c)$$

where

$$s_{Ik}(t) = A_k \cos[\theta_k(t) + \phi_k] \quad (7.6.4a)$$

$$s_{Qk}(t) = A_k \sin[\theta_k(t) + \phi_k] \quad (7.6.4b)$$

and the symbol $\sim$ is used in this section to denote the complex form. We also note that the amplitude A_k can be expressed as

$$A_k = \sqrt{s_{Ik}^2 + s_{Qk}^2}. \quad (7.6.5)$$

For narrowband processes, the quadrature components $s_{Ik}(t)$ and $s_{Qk}(t)$ are uncorrelated [3]. In the special case where these quantities are zero-mean bivariate Gaussian processes, they are also statistically independent. It can be

shown that A_k and ϕ_k are statistically independent random variables where A_k is Rayleigh distributed and ϕ_k is uniformly distributed. We now designate model I(INCOH) as the signal model where all amplitudes A_k $k=1,2,\dots,N_T$ in a pulse train are equal but vary randomly from pulse-train to pulse-train according to the Rayleigh distribution and the initial phase ϕ_k of the k th pulse is a statistically independent, uniformly distributed random variable. This model is used to describe the case where the phase of the echo returns are randomized due to phase instabilities associated with the transmitted waveform. We designate model II(INCOH) as the signal model where both the amplitudes and phases of the echo pulses are statistically independent random variables with Rayleigh and uniform distributions, respectively. Model II(INCOH), therefore, represents a signal with totally uncorrelated amplitude and phase pulse-to-pulse on each realization. And so, models I(INCOH) and model II(INCOH) provide the two extremes on the amplitude correlation of the processes.

Finally, for a coherent transmitted pulse train, we define the model I(COH) as the signal model where the amplitudes over the entire pulse train are totally correlated random variables as are the initial phases. However, they are uncorrelated between pulse trains with Rayleigh and uniform distributions, respectively. A summary of these models is shown in Table 7.1.

In the process synthesis procedure, the temporal correlation of the i th channel process is controlled by the temporal correlation coefficient λ_{ii} . For λ_{ii} approaching zero or unity, both the real and imaginary amplitudes become either uncorrelated or correlated in time, respectively. Since the phase of the process is related to the ratio of these amplitudes through the inverse tangent, as expressed by eq(4.7.1.1), λ_{ii} also controls the pulse-to-pulse phase correlation. Thus, for $\lambda_{ii} \rightarrow 0$, we approach model II(INCOH), while for $\lambda_{ii} \rightarrow 1$, we approach model I(COH).

In the next subsection, we consider the equivalence between the signal-to-noise ratio defined in the literature in conjunction with the analytical detection expressions that are used in this dissertation.

	Model I(INCOH)	Model I(COH)	Model II(INCOH)
Amplitude PDF	Rayleigh	Rayleigh	Rayleigh
Initial Phase PDF	uniform	uniform	uniform
pulse-to-pulse amplitude	correlated slow fluctuation	correlated slow fluctuation	uncorrelated fast fluctuation
pulse-to-pulse phase	uncorrelated (uniform distr)	totally correlated	uncorrelated (uniform distr)
ϕ_1 of pulse train	random	random	random
correlation between pulse trains	uncorrelated	uncorrelated	uncorrelated

Table 7.1 Summary of Signal Model Characteristics

7.6.3 Signal-to-Noise Ratio

We now relate the signal-to-noise ratio defined in the literature [13,88] for the analytical detection curves used in the next subsection and that defined in subsection 4.3.2 for the baseband processes. Dropping the subscript k in eq(7.6.2), we consider the bandpass sinusoidal signal with random amplitude A expressed as

$$s(t) = Ap(t)\cos[\omega_c t + \theta(t) + \phi]. \quad (7.6.6)$$

The signal energy E_p in the pulse of duration T is defined as

$$E_p = \int_0^T s^2(t) dt \quad (7.6.7a)$$

$$= A^2 \int_0^T \cos^2[\omega_c t + \theta(t) + \phi] dt \quad (7.6.7b)$$

$$= \frac{A^2}{2} \int_0^T \{ 1 + \cos[2\omega_c t + 2\theta(t) + 2\phi] \} dt \quad (7.6.7c)$$

$$= \frac{A^2}{2} \int_0^T dt + \frac{A^2}{2} \int_0^T \cos[2\omega_c t + 2\theta(t) + 2\phi] dt. \quad (7.6.7d)$$

Since the function $\theta(t)$ is slowly varying and ϕ is a constant and the contribution from the double frequency term is recognized to be negligible with respect to that from the DC term, eq(7.6.7d) reduces to

$$E_p \approx \frac{A^2 T}{2} \quad (7.6.8)$$

The single pulse power signal-to-noise ratio, commonly used in the literature, is expressed as [72,74]

$$(S/N)_p \equiv \frac{\mathcal{R}_p}{2} \quad (7.6.9)$$

where

$$\mathcal{R}_p \equiv \frac{E_p}{N_0/2}. \quad (7.6.10)$$

Using eq(7.6.8b) and the definition of the signal-to-noise ratio $\mathcal{R}_p$ in eq(7.6.9), we obtain

$$(S/N)_p \equiv \frac{\mathcal{R}_p}{2} = \frac{1}{2} \left[\frac{\frac{1}{2} A^2 T}{N_0/2} \right] \quad (7.6.11a)$$

$$= \frac{\frac{1}{2} A^2}{N_0 B} = \frac{\text{avg power of signal during duration } T}{\text{avg noise power in bandwidth } B} \quad (7.6.11b)$$

where the filter bandwidth $B \approx 1/T$. Averaging eq(7.6.11b) with respect to A , we obtain

$$(S/N)_{\text{avg}} \equiv \frac{\bar{\mathcal{R}}_p}{2} \approx \frac{\frac{1}{2} E[A^2]}{N_0 B}. \quad (7.6.12)$$

We now consider the signal-to-noise ratio for the baseband processes. We define the variance σ_s^2 of the sampled complex baseband signal process as

$$\sigma_s^2 \equiv E[\tilde{s}(n)\tilde{s}^*(n)] \quad (7.6.13a)$$

$$= E[s_I^2(n)] + E[s_Q^2(n)]. \quad (7.6.13b)$$

Using eqs(7.6.4) in (7.6.13b) with the subscript k dropped,

$$\sigma_s^2 = E[A^2] \cos^2[\theta(nT_s) + \phi] + E[A^2] \sin^2[\theta(nT_s) + \phi] \quad (7.6.14a)$$

$$= E[A^2] \quad (7.6.14b)$$

where T_s is the time duration between samples (ie., pulses). For the noise variance σ_w^2 , we have

$$\sigma_w^2 = E[\tilde{w}(n)\tilde{w}^*(n)] \quad (7.6.15a)$$

$$= E[w_I^2(n)] + E[w_Q^2(n)] \quad (7.6.15b)$$

$$= \sigma_{w_I}^2 + \sigma_{w_Q}^2. \quad (7.6.15c)$$

For an ideal bandpass filter of bandwidth B , the autocorrelation function of the noise quadrature components can be shown to be

$$R_{w_I}(l) = R_{w_Q}(l) = N_0 B \text{sinc}(Bl) \quad (7.6.16)$$

so that at $l=0$, we have

$$\sigma_{w_I}^2 = R_{w_I}(0) = N_0 B \quad (7.6.17a)$$

and

$$\sigma_{w_Q}^2 = R_{w_Q}(0) = N_0 B. \quad (7.6.17b)$$

Using eqs(7.6.17) in eq(7.6.15c), we have

$$\sigma_w^2 = 2N_0 B. \quad (7.6.18)$$

Finally, using eq(7.6.14b) and (7.6.18), we obtain

$$(S/N) \equiv \sigma_s^2 / \sigma_w^2 = \frac{E[A^2]}{2N_0 B} \quad (7.6.19a)$$

$$= \frac{\frac{1}{2} E[A^2]}{N_0 B}. \quad (7.6.19b)$$

We note that eqs(7.6.12) and (7.6.19b) are identical. In subsection 7.6.4, the synthesized processes used as inputs to the detection architecture have a

signal-to-noise ratio as defined by eq(7.6.19a). The analytic detection curves to which these results are compared have a signal-to-noise ratio defined by eq(7.6.12). As discussed here, the two definitions are equivalent.

We now consider the case of a coherent pulse train that is exactly known except for ϕ_1 ; ie., the phase relations between pulses in the train are assumed to be known exactly, but ϕ_1 on each train is unknown. This condition is met for the model I(COH) process. We again note that in the process synthesis procedure described in section 4.7, it is this case which is approached as the temporal correlation is increased.

The signal-to-noise ratio for the case where N_T pulses are coherently integrated can be obtained as follows. We first note that the signal-to-noise ratio defined above is that for single-pulse detection. Let us denote this ratio for channel j as $(S/N)_{j,1}$ where the subscript j indicates the channel and the 1 indicates single pulse. The effective signal-to-noise ratio for an ideal coherent multiple pulse system on channel j using N_T pulses is [88,89]

$$(S/N)_{j,N_T} = N_T (S/N)_{j,1} \quad (7.6.20)$$

where N_T is the number of pulses coherently integrated. This equation will be used in the next subsection to determine the model I(COH) detection curve.

7.6.4. Signal Detection in Additive White Noise

In this section, detection results are computed using the Monte-Carlo procedure and compared to available analytical detection curves for the models described in section 7.6.2. Two-channel ($J=2$) complex baseband correlated Gaussian signal vectors are synthesized and added to a complex Gaussian white noise vector. The noise vector consists of uncorrelated channel processes unless specifically stated. All signal processes are autoregressive AR(2) with order two. The parameters for the signal processes used in this section are listed in Table 7.6.1. These observation data processes are used as an input to the likelihood implementation architecture described in section 6.4 using the Chole'sky decomposition $C_\alpha C_\alpha^H$ in the second filtering stage (see Fig.6.4.2). The prediction error filter coefficients for the tapped delay line filter F_1 are determined from

the H_1 hypothesis data as described in section 7.2.1 using the forward coefficients from the Strand-Nuttal algorithm[†]; ie., for the 'known' parameter case, a large number of data samples is used to determine the prediction error filter coefficients in an 'a priori' manner. As noted in section 7.2.1, the order of the F_1 filter used in these analyses was eight; ie., the matrices $A_p(k)$ for $k=1,2,\dots,8$ and C_α^{-1} are used. Since the hypothesis H_0 data consists of uncorrelated white noise, the MMSE coefficient matrix for this process is a null matrix. Therefore, $A_p(k)$ for the F_0 filter is a null matrix coefficient and the second stage coefficient matrix is an identity matrix.

For the signal parameters listed in section A of Table 7.6.1, the computed detection results are plotted in Figs. 7.6.1 and 7.6.2 for probability of false alarm levels of 0.0693 and 6.93×10^{-4} , respectively. We note that these false alarm values were selected using Fehner numbers $n'=10$ and 1000, respectively, where $P_{fa}=\ln 2/n'$ [88]. These detection results were computed with $J=2$ channels, $N_T=10$ observation samples, and equal signal-to-noise ratios on each channel; ie., $(S/N)=(S/N)_1=(S/N)_2$. The Monte-Carlo procedure used $N_R=20,000$ realizations as discussed in section 7.3. We also plot the analytical probability of detection curves from Meyer and Mayer [88] for the case I(INCOH), case I(COH) and case II(INCOH) signals described in section 7.6.2. The model I(COH) detection curve for the coherent integration of N_T pulses is determined using the detection curves [88] for $(S/N)_{j,N_T}$ rather than $(S/N)_{j,1}$ as discussed at the end of section 7.6.3. In Appendix K, these curves are shown to provide bounds on the detection results presented here using $JN_T=20$.

Examination of Figs. 7.6.1 and 7.6.2 reveals that the analytical detection curves in each of these two figures have a point at which the curves intersect. For example, in Fig. 7.6.1, the model I(COH) and model II(INCOH) curves intersect at $S/N = -1\text{dB}$. We have purposely selected processes with S/N values above and below these intersection points in order to investigate the effect of the temporal and cross-channel correlation on detection performance in both regions. We anticipate that processes uncorrelated both temporally and across channels will lie closer to the model II(INCOH) bound while those with high correlation will lie closer to the model I(COH) bound.

[†] The software for the Strand-Nuttal algorithm is available in [17] and was used here.

section	case	λ_{11}	λ_{22}	$ \rho_{12} $ a, b, c	$(S/N)_1$	$(S/N)_2$
A	1 a,b,c	0.1	0.1	0,0.5,0.99	-5dB	-5dB
	2 a,b,c	0.9	0.9	0,0.5,0.99	-5dB	-5dB
	3 a,b,c	0.1	0.1	0,0.5,0.99	+3dB	+3dB
	4 a,b,c	0.9	0.9	0,0.5,0.99	+3dB	+3dB
	5 a,b,c	0.999	0.999	0,0.5,0.99	+3dB	+3dB
B	6 a,b,c	0.1	0.1	0,0.5,0.99	+3dB	+3dB
	7 a,b,c	0.1	0.1	0,0.5,0.99	+3dB	0dB
	8 a,b,c	0.1	0.1	0,0.5,0.99	+3dB	-3dB
	9 a,b,c	0.1	0.1	0,0.5,0.99	+3dB	-6dB
	10a,b,c	0.1	0.1	0,0.5,0.99	+3dB	-9dB
	11a,b,c	0.1	0.1	0,0.5,0.99	+3dB	-20dB
	12a,b,c	0.1	0.1	0,0.5,0.99	+3dB	-30dB
	13a,b,c	0.1	0.1	0,0.5,0.99	+3dB	-40dB
	14a,b,c	0.9	0.9	0,0.5,0.99	+3dB	+3dB
	15a,b,c	0.9	0.9	0,0.5,0.99	+3dB	0dB
	16a,b,c	0.9	0.9	0,0.5,0.99	+3dB	-3dB
	17a,b,c	0.9	0.9	0,0.5,0.99	+3dB	-6dB
	18a,b,c	0.9	0.9	0,0.5,0.99	+3dB	-9dB
	19a,b,c	0.9	0.9	0,0.5,0.99	+3dB	-20dB
	20a,b,c	0.9	0.9	0,0.5,0.99	+3dB	-30dB
	21a,b,c	0.9	0.9	0,0.5,0.99	+3dB	-40dB
	22a,b,c	0.999	0.999	0,0.5,0.99	+3dB	+3dB
	23a,b,c	0.999	0.999	0,0.5,0.99	+3dB	0dB
	24a,b,c	0.999	0.999	0,0.5,0.99	+3dB	-3dB
	25a,b,c	0.999	0.999	0,0.5,0.99	+3dB	-6dB
	26a,b,c	0.999	0.999	0,0.5,0.99	+3dB	-9dB
	27a,b,c	0.999	0.999	0,0.5,0.99	+3dB	-20dB
	28a,b,c	0.999	0.999	0,0.5,0.99	+3dB	-30dB
	29a,b,c	0.999	0.999	0,0.5,0.99	+3dB	-40dB

Table 7.6.1 The AR(2) Signal Parameters used in the detection analyses.

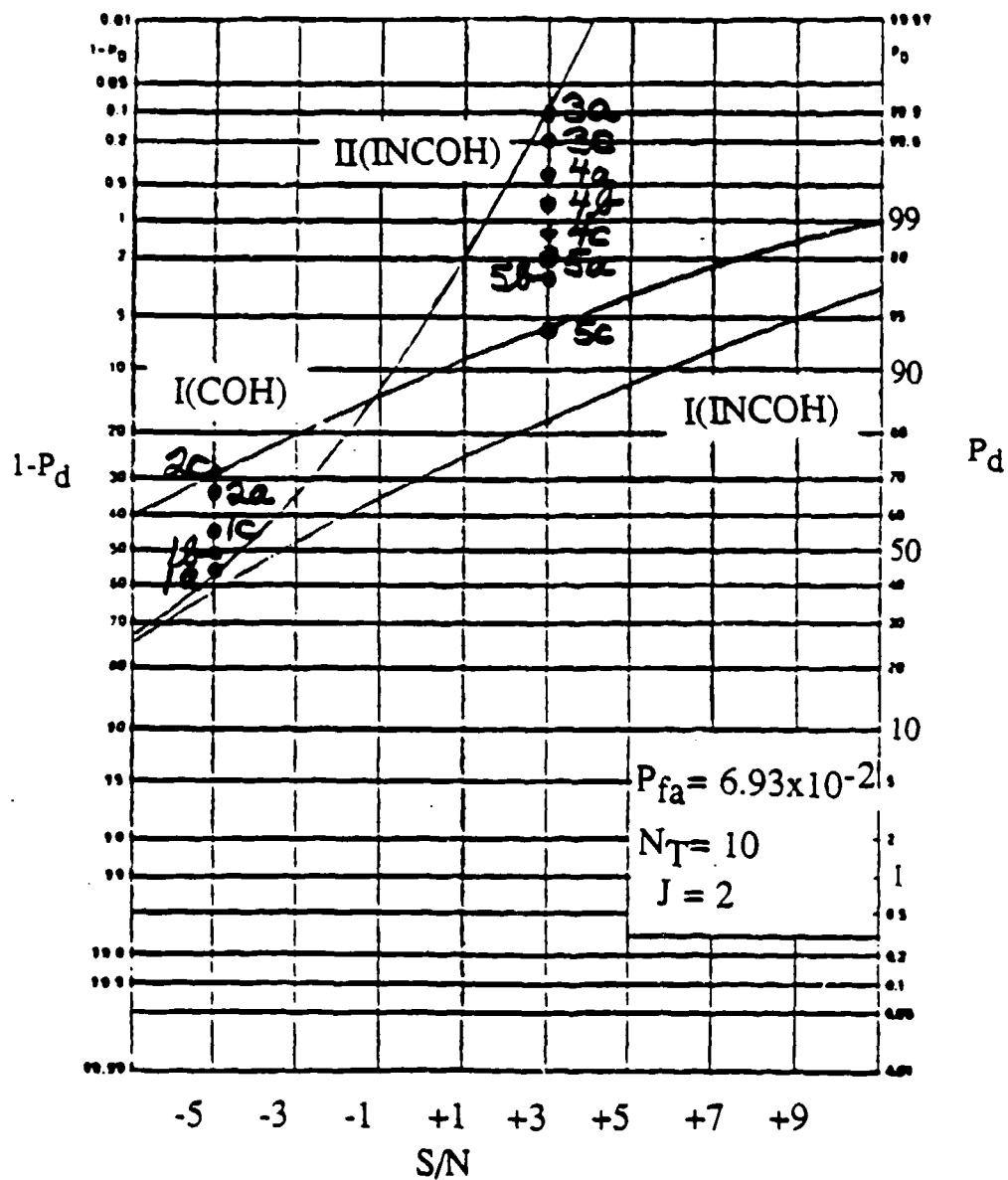


Fig. 7.6.1 Probability of detection versus S/N for a Gaussian signal plus Gaussian white noise 'with' F_1 filter coefficients; $S/N = (S/N)_1 = (S/N)_2$, $P_{fa} = 0.0693$, $N = 10$, $J = 2$.

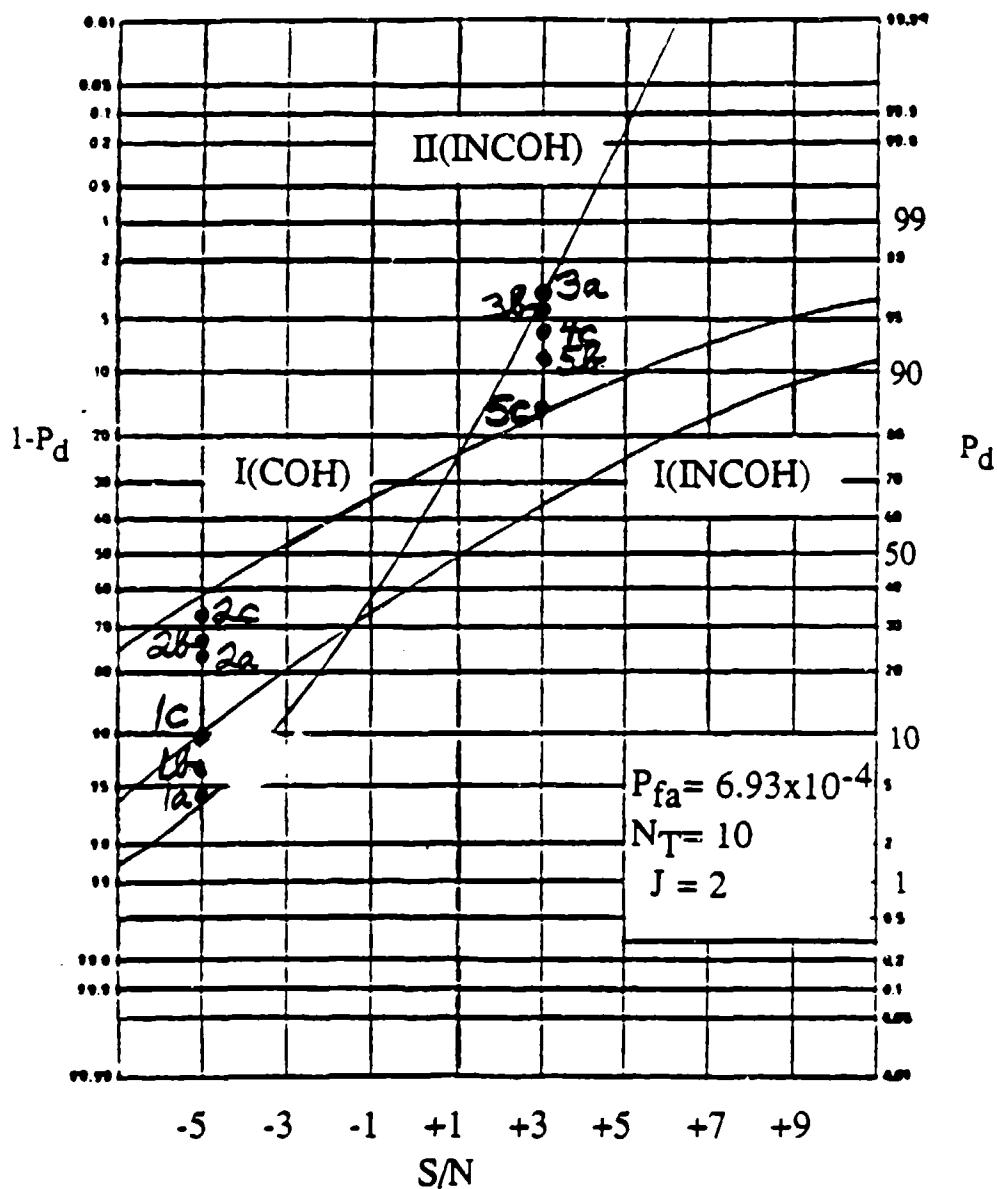


Fig. 7.6.2 Probability of detection versus S/N for a Gaussian signal plus Gaussian white noise 'with' F_1 filter coefficients; $S/N=(S/N)_1=(S/N)_2$, $P_{fa}=6.93 \times 10^{-4}$, $N=10$, $J=2$.

In Appendix K, we show that the analytical detection performance curves associated with the models described in subsection 7.6.2 for the single channel case can be extended to obtain bounds on the detection results for multichannel detection. Specifically, if the number of pulses N_T is replaced with JN_T , the resulting detection curves describe the limiting cases of either totally correlated or uncorrelated processes in both time and space (ie., across channels). For example, with $J=2$ and $N_T=10$, the analytical detection curves for 20 pulses can be used for the limiting cases of correlation denoted for $\lambda_{ii}=|\rho_{12}|=0$ and $\lambda_{ii}=|\rho_{12}|=1$. Finally, we note in Appendix K that these analytic curves are approximate for the low signal-to-noise ratio case.

In Table 7.6.1, signal cases 1a and 3a have temporal correlation coefficients $\lambda_{11}=\lambda_{22}=0.1$ and cross-correlation coefficient magnitude $|\rho_{12}|=0.0$. These cases therefore approximate the model II(INCOH) signal model. At the other extreme, cases 2c and 5c for which $\lambda_{11}=\lambda_{22}=0.999$ and $|\rho_{12}|=0.99$ approximate the model I(COH) signal model. As shown in Fig. 7.6.1 for $P_{fa}=0.0693$, P_d for cases 1a and 3a lie closest to the model II(INCOH) curve at S/N values of -5dB and +3dB, respectively. On the other hand, P_d for cases 2c and 5c lie closer to the model I(COH) curve. Consistent with the cross-over of the analytic curves, note that the relative detection performance has reversed for these cases; ie., for the low S/N value of -5dB, the more highly correlated process of case 2c has a higher P_d than the uncorrelated process of case 1a. Whereas at the higher S/N value of +3dB, the more highly correlated process of case 5c has a lower P_d than the corresponding process of case 3a. We also note that the case corresponding to processes with intermediate correlations, lie at P_d values commensurate with their correlation values. The results when $P_{fa}=6.93 \times 10^{-4}$ are shown in Fig. 7.6.2. The plotted values in both of these Figures are tabulated in Table 7.6.2 including the sample standard deviation σ_{n-1} computed using four independent detection trials.

We again call attention to the the fact that cases 1a and 3a approximate the model II(INCOH) signal while cases 2c and 5c approximate model I(COH) signal. We now consider the effect of not utilizing the phase information contained in the detection scheme for these processes. Since the pulse-to-pulse phase for the model II(INCOH) signal is uniformly distributed over zero to 2π radians, phase sensitive detection provides no improvement in performance. Alternatively, the

case	λ_{11}	λ_{22}	$ \rho_{12} $	S/N	P_{fa}	P_d	$\pm\sigma_{n-1}$
1a	0.1	0.1	0.0	-5dB	0.0693	0.430	± 0.006
1b	0.1	0.1	0.5	-5dB	0.0693	0.464	
1c	0.1	0.1	0.99	-5dB	0.0693	0.555	± 0.005
2a	0.9	0.9	0.0	-5dB	0.0693	0.687	
2b	0.9	0.9	0.5	-5dB	0.0693	0.691	
2c	0.9	0.9	0.99	-5dB	0.0693	0.712	± 0.001
3a	0.1	0.1	0.0	+3dB	0.0693	0.999	± 0.0003
3b	0.1	0.1	0.5	+3dB	0.0693	0.999	
3c	0.1	0.1	0.99	+3dB	0.0693	0.998	
4a	0.9	0.9	0.0	+3dB	0.0693	0.996	
4b	0.9	0.9	0.5	+3dB	0.0693	0.995	
4c	0.9	0.9	0.99	+3dB	0.0693	0.989	± 0.0006
5a	0.999	0.999	0.0	+3dB	0.0693	0.981	
5b	0.999	0.999	0.5	+3dB	0.0693	0.977	
5c	0.999	0.999	0.99	+3dB	0.0693	0.938	± 0.001
1a	0.1	0.1	0.0	-5dB	6.93×10^{-4}	0.046	± 0.003
1b	0.1	0.1	0.5	-5dB	6.93×10^{-4}	0.063	
1c	0.1	0.1	0.99	-5dB	6.93×10^{-4}	0.098	± 0.024
2a	0.9	0.9	0.0	-5dB	6.93×10^{-4}	0.244	
2b	0.9	0.9	0.5	-5dB	6.93×10^{-4}	0.260	
2c	0.9	0.9	0.99	-5dB	6.93×10^{-4}	0.319	± 0.003
3a	0.1	0.1	0.0	+3dB	6.93×10^{-4}	0.970	± 0.001
3b	0.1	0.1	0.5	+3dB	6.93×10^{-4}	0.968	
3c	0.1	0.1	0.99	+3dB	6.93×10^{-4}	0.981	
4a	0.9	0.9	0.0	+3dB	6.93×10^{-4}	0.966	
4b	0.9	0.9	0.5	+3dB	6.93×10^{-4}	0.959	
4c	0.9	0.9	0.99	+3dB	6.93×10^{-4}	0.942	± 0.002
5a	0.999	0.999	0.0	+3dB	6.93×10^{-4}	0.931	
5b	0.999	0.999	0.5	+3dB	6.93×10^{-4}	0.921	
5c	0.999	0.999	0.99	+3dB	6.93×10^{-4}	0.847	± 0.001

Table 7.6.2 Tabulated Detection Results for Figs 7.6.1 and 7.6.2.

effect of not utilizing the phase in the detection scheme causes no degradation in performance. On the other hand, the model I(COH) signal contains total pulse-to-pulse phase correlation. In this case, the effect of not utilizing the phase in the detection scheme causes the detection performance to degrade to that of the model I(INCOH) signal. Finally, we note that the processing method which does not utilize the pulse-to-pulse phase is the 'incoherent' integration method. The detection results for the special case when the filter coefficients are not utilized in F_0 and F_1 filters correspond to those of 'incoherent' pulse integration. Alternatively, when the coefficients are utilized in the prediction error filters, the detection results correspond to those of 'coherent' pulse integration.

In order to compare detection performance for incoherent integration to that of coherent integration, the filter coefficients of the prediction error filters F_0 and F_1 are disabled; ie., the $A_p(k)$ coefficients are set to null matrices and C_α^{-1} is set to an identity matrix allowing the observation data to pass through the filters unaltered. The detection results for this case are shown in Figs. 7.6.3 and 7.6.4 for $P_{fa}=0.0693$ and 6.93×10^{-4} , respectively. We note that the computed detection results now range between the model I(INCOH) and model II(INCOH) curves as noted above. At -5dB, a comparison of these two figures again indicates a reversal in the detection performance as a function of the temporal and cross-channel correlation. This result is consistent with the fact that at $P_{fa}=6.93 \times 10^{-4}$, the uncorrelated signal model described by the model II(INCOH) curve falls below the model I(INCOH) curve, while for $P_{fa}=0.0693$, it lies above it. The results are also displayed in Table 7.6.3.

From the results presented in these figures, we conclude for the case of 'known' coefficients that the detection results computed using the likelihood ratio implementation described in section 6.4 are bounded by the analytic detection curves [88]. Second, the detection performance of the processes at a specific signal-to-noise ratio is shown to be a function of both the temporal and cross-channel correlation. Furthermore, this functional dependence varies with signal-to-noise ratio and is consistent with the cross-over point in the analytic curves. We will discuss the significance of this result in the conclusions under section 8.1. Again, we note that the detection results for the special case when the filter coefficients are not utilized in F_0 and F_1 correspond to those of 'incoherent' pulse

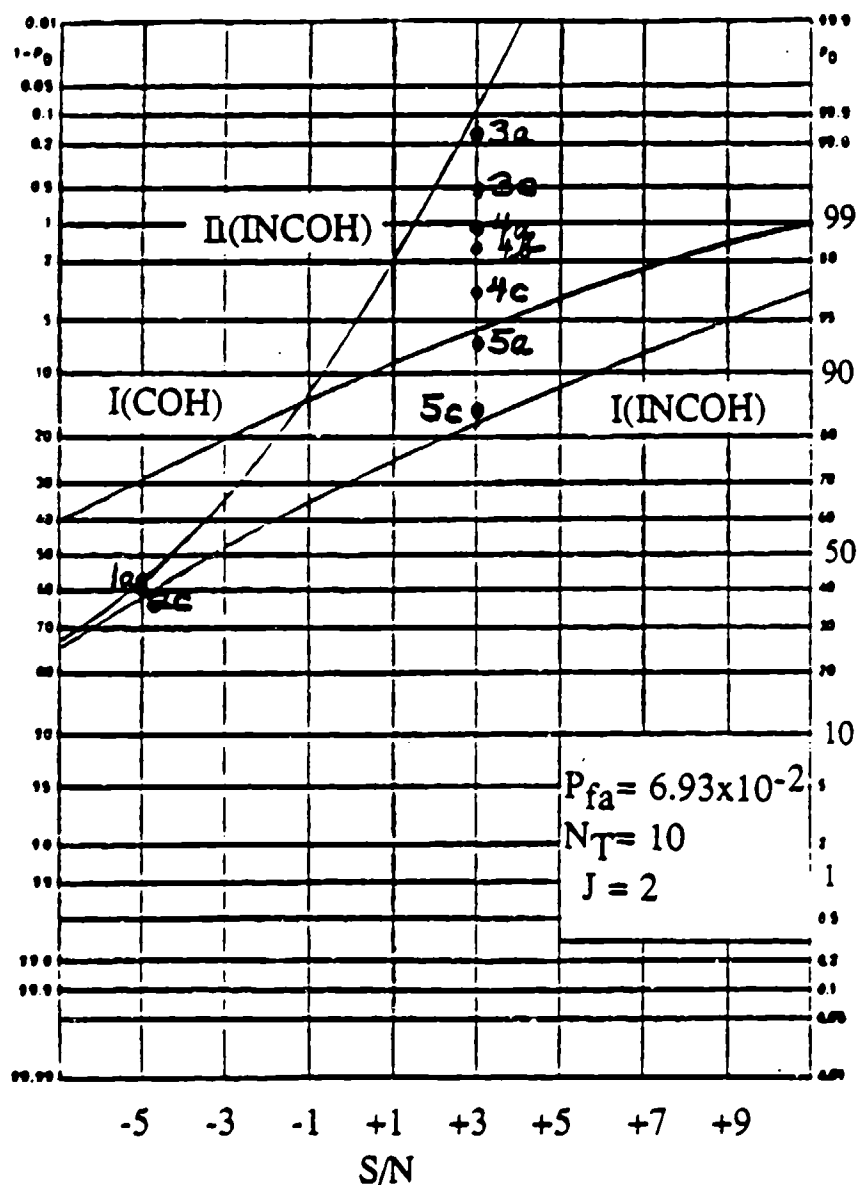


Fig. 7.6.3 Probability of detection versus S/N for the Gaussian signal plus Gaussian white noise 'without' F_1 filter coefficients; $S/N=(S/N)_1=(S/N)_2$, $P_{fa}=0.0693$, $N=10$, $J=2$.

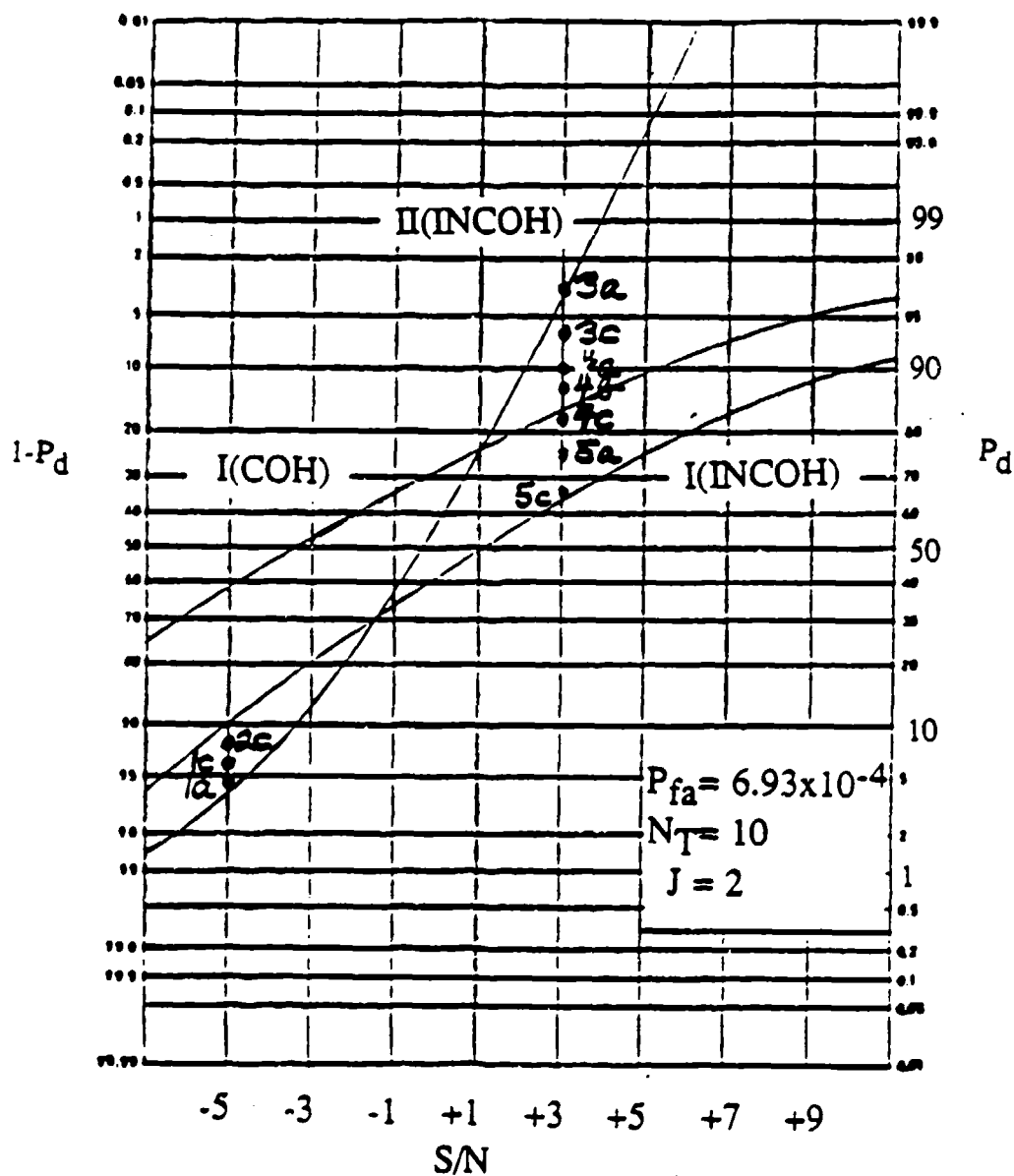


Fig. 7.6.4 Probability of detection versus S/N for the Gaussian signal plus Gaussian white noise 'without' F_1 filter coefficients; $S/N=(S/N)_1=(S/N)_2$, $P_{fa}=6.93 \times 10^{-4}$, $N=10$, $J=2$.

case	λ_{11}	λ_{22}	$ \rho_{12} $	S/N	P_{fa}	P_d
1a	0.1	0.1	0.0	-5dB	0.0693	0.428
1b	0.1	0.1	0.5	-5dB	0.0693	0.428
1c	0.1	0.1	0.99	-5dB	0.0693	0.421
2a	0.9	0.9	0.0	-5dB	0.0693	0.426
2b	0.9	0.9	0.5	-5dB	0.0693	0.422
2c	0.9	0.9	0.99	-5dB	0.0693	0.414
3a	0.1	0.1	0.0	+3dB	0.0693	0.998
3b	0.1	0.1	0.5	+3dB	0.0693	0.998
3c	0.1	0.1	0.99	+3dB	0.0693	0.995
4a	0.9	0.9	0.0	+3dB	0.0693	0.989
4b	0.9	0.9	0.5	+3dB	0.0693	0.983
4c	0.9	0.9	0.99	+3dB	0.0693	0.955
5a	0.999	0.999	0.0	+3dB	0.0693	0.932
5c	0.999	0.999	0.99	+3dB	0.0693	0.841
1a	0.1	0.1	0.0	+3dB	6.93×10^{-4}	0.049
1b	0.1	0.1	0.5	-5dB	6.93×10^{-4}	0.050
1c	0.1	0.1	0.99	-5dB	6.93×10^{-4}	0.054
2a	0.9	0.9	0.0	-5dB	6.93×10^{-4}	0.058
2b	0.9	0.9	0.5	-5dB	6.93×10^{-4}	0.061
2c	0.9	0.9	0.99	-5dB	6.93×10^{-4}	0.069
3a	0.1	0.1	0.0	-5dB	6.93×10^{-4}	0.971
3b	0.1	0.1	0.5	+3dB	6.93×10^{-4}	0.965
3c	0.1	0.1	0.99	+3dB	6.93×10^{-4}	0.940
4a	0.9	0.9	0.0	+3dB	6.93×10^{-4}	0.902
4b	0.9	0.9	0.5	+3dB	6.93×10^{-4}	0.882
4c	0.9	0.9	0.99	+3dB	6.93×10^{-4}	0.821
5a	0.999	0.999	0.0	+3dB	6.93×10^{-4}	0.761
5c	0.999	0.999	0.99	+3dB	6.93×10^{-4}	0.654

Table 7.6.3 Tabulated Detection Results for Figs. 7.6.3 and 7.6.4.

integration. Alternatively, when the coefficients are utilized in the prediction error filters, the detection results correspond to those of 'coherent' pulse integration.

Previously, we considered the situation in which S/N was the same on both channels. We now extend the investigation to consider the situation in which S/N remains the same on one channel but is lowered on the other channel. In this case, the overall detection performance will degrade from the previous cases.

In the simulation results that follow, $(S/N)_1$ was held fixed at +3dB while $(S/N)_2$ was varied from +3dB to values ranging as low as -40dB. In each case, two channels each with ten observations were considered (ie., $J=2$ and $N_T=10$). Again, we note that no cross-channel correlation is considered on the noise processes. The coherent detection results obtained 'with' the filter coefficients are shown in Fig. 7.6.5 and 7.6.6 for $P_{fa}=0.0693$ and 6.93×10^{-4} , respectively, and $\lambda_{11}=\lambda_{22}=0.1$. The values are also displayed for signal cross-channel correlation coefficients $|\rho_{12}|=0, 0.5$, and 0.99 . We also show the P_d value for channel 1 alone using $N_T=10$. This last result provides the basis for a comparison between the two-channel detection results using $(S/N)_2 < (S/N)_1$ and the results obtained with channel 1 alone. The tabulated results are displayed in Table 7.6.4. Again, the sample standard deviations were computed with four independent trials. In Table 7.6.5, we focus our attention on cases 6a and 13a. We note that these processes have low temporal correlation and no cross-channel correlation. Therefore, they closely approximate the model II (INCOH) process. We note that the single channel values are in excellent agreement with their corresponding analytical values. The detection results for case 6a with $(S/N)_1=(S/N)_2=+3\text{dB}$ are also in excellent agreement with the analytic values. Finally, the case 13a process with $(S/N)_1=+3\text{dB}$ and $(S/N)_2=-40\text{dB}$ is nearly equal to that of the single channel case for which $(S/N)_1=+3\text{dB}$ and $N_T=10$. These results indicate that the two-channel case shows improved detection performance over that of the channel 1 case alone. Furthermore, as $(S/N)_2$ decreases, the two-channel detection results for these cases approach that of channel 1 alone.

For $\lambda_{11}=\lambda_{22}=0.9$, the corresponding detection results are shown in Figs. 7.6.7 and 7.6.8 for $P_{fa}=0.0693$ and 6.93×10^{-4} , respectively. The plotted values in both of these figures are tabulated in Table 7.6.6. It is interesting to note that a cross-over in the detection curves occurs in each figure, but is slightly more

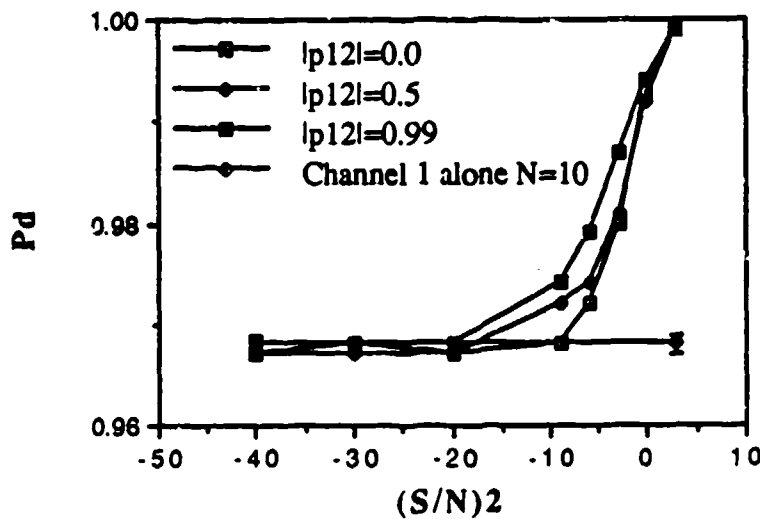


Figure 7.6.5 Probability of detection results as a function of $(S/N)^2$ and $|p_{12}|$ for $P_{fa} = 0.0693$, $(S/N)_1 = +3\text{dB}$ and $\lambda_{11} = \lambda_{22} = 0.1$.

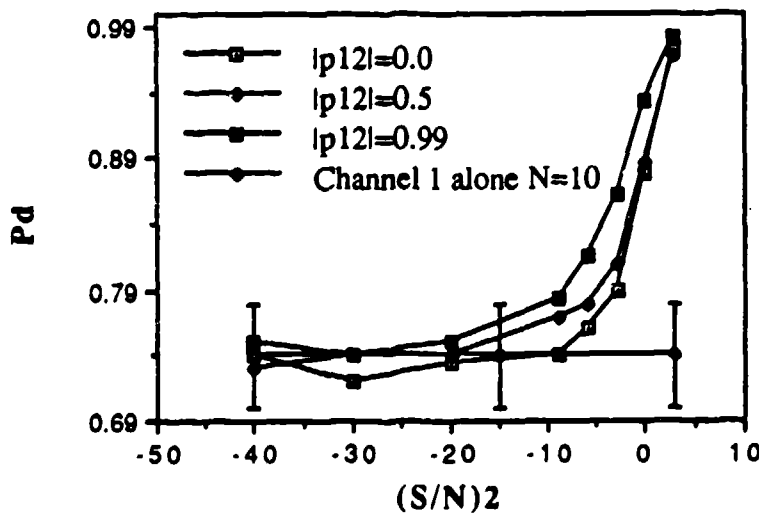


Figure 7.6.6 Probability of detection results as a function of $(S/N)^2$ and $|p_{12}|$ for $P_{fa} = 6.93 \times 10^{-4}$, $(S/N)_1 = +3\text{dB}$ and $\lambda_{11} = \lambda_{22} = 0.1$.

		P_d	P_d	P_d	P_{fa}
case		a	b	c	
	$(S/N)_2$	$ p_{12} =0$	$ p_{12} =0.5$	$ p_{12} =0.99$	
6	+3dB	0.999	0.999	0.999	0.0693
7	0dB	0.993	0.992	0.994	0.0693
8	-3dB	0.980	0.981 ± 0.002	0.987	0.0693
9	-6dB	0.972 ± 0.001	0.974 ± 0.001	0.979	0.0693
10	-9dB	0.968 ± 0.001	0.972 ± 0.003	0.974	0.0693
11	-20dB	0.967 ± 0.001	0.967 ± 0.001	0.968 ± 0.01	0.0693
12	-30dB	0.968 ± 0.001	0.967 ± 0.001	0.968 ± 0.001	0.0693
13	-40dB	0.968 ± 0.001	0.967 ± 0.001	0.967 ± 0.001	0.0693
6	+3dB	0.971	0.968	0.967	6.93×10^{-4}
7	0dB	0.879	0.887	0.933	6.93×10^{-4}
8	-3dB	0.789	0.80 ± 0.02	0.862	6.93×10^{-4}
9	-6dB	0.76 ± 0.02	0.78 ± 0.02	0.815	6.93×10^{-4}
10	-9dB	0.74 ± 0.02	0.77 ± 0.03	0.783	6.93×10^{-4}
11	-20dB	0.74 ± 0.01	0.76 ± 0.03	0.75 ± 0.02	6.93×10^{-4}
12	-30dB	0.72 ± 0.01	0.73 ± 0.02	0.74 ± 0.02	6.93×10^{-4}
13	-40dB	0.74 ± 0.01	0.73 ± 0.01	0.75 ± 0.03	6.93×10^{-4}

Table 7.6.4 Probability of detection results as a function of $(S/N)_2$ and $|p_{12}|$ for $\lambda_{11}=\lambda_{22}=0.1$.

case	N_T	J	$(S/N)_1$	$(S/N)_2$	P_d (experimental)	P_d (analytical)	P_{fa}
					$\lambda=0.1$ $ p_{12} =0$	model II(INCOH)	
6a	10	2	+3dB	+3dB	0.999	0.999	0.0693
channel 1	10	1	+3dB	-	0.968 ± 0.001	0.968	0.0693
13a	10	2	+3dB	-40dB	0.968 ± 0.001		0.0693
6a	10	2	+3dB	+3dB	0.971	0.971	6.93×10^{-4}
channel 1	10	1	+3dB	-	0.74 ± 0.04	0.740	6.93×10^{-4}
13a	10	2	+3dB	-40dB	0.74 ± 0.01		6.93×10^{-4}

Table 7.6.5 Probability of detection for $\lambda_{11}=\lambda_{22}=0.1$ and $|p_{12}|=0.0$ with corresponding analytical results for the model II(INCOH) process.

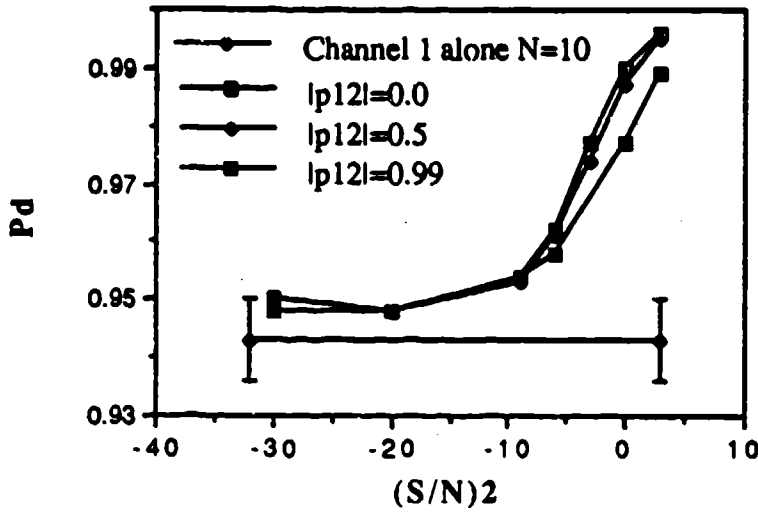


Figure 7.6.7 Probability of detection results as a function of $(S/N)_2$ and $|p_{12}|$ for $P_{fa}=0.0693$, $(S/N)_1=+3\text{dB}$ and $\lambda_{11}=\lambda_{22}=0.9$.

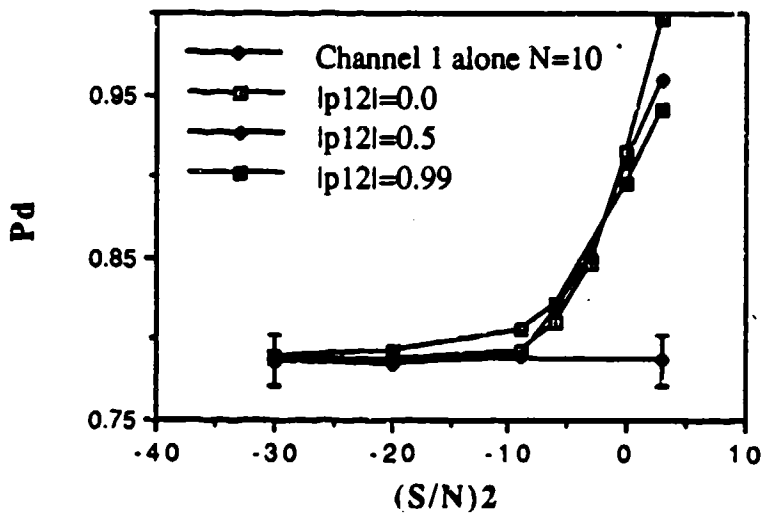


Figure 7.6.8 Probability of detection results as a function of $(S/N)_2$ and $|p_{12}|$ for $P_{fa}=6.93 \times 10^{-4}$, $(S/N)_1=+3\text{dB}$ and $\lambda_{11}=\lambda_{22}=0.9$.

		P_d	P_d	P_d	P_{fa}
case		a	b	c	
	$(S/N)_2$	$ \rho_{12} =0$	$ \rho_{12} =0.5$	$ \rho_{12} =0.99$	
14	+3dB	0.996	0.995	0.989	0.0693
15	0dB	0.990	0.987	0.977	0.0693
16	-3dB	0.977	0.974	0.987	0.0693
17	-6dB	0.962	0.961	0.958	0.0693
18	-9dB	0.954	0.953	0.954	0.0693
19	-20dB	0.948	0.948 ± 0.002	0.948 ± 0.002	0.0693
20	-30dB	0.948 ± 0.004	0.948 ± 0.001	0.948	0.0693
14	+3dB	0.966	0.959	0.941	6.93×10^{-4}
15	0dB	0.915	0.909	0.895	6.93×10^{-4}
16	-3dB	0.847	0.851	0.862	6.93×10^{-4}
17	-6dB	0.810	0.818	0.822	6.93×10^{-4}
18	-9dB	0.793	0.788	0.806	6.93×10^{-4}
19	-20dB	0.788	0.785 ± 0.002	0.792 ± 0.007	6.93×10^{-4}
20	-30dB	0.788 ± 0.005	0.786 ± 0.003	0.788 ± 0.008	6.93×10^{-4}

Table 7.6.6a Probability of detection results as a function of $(S/N)_2$ and $|\rho_{12}|$ for $N_T=10$, $J=2$, $(S/N)_1=+3\text{dB}$ and $\lambda_{11}=\lambda_{22}=0.9$.

case	N_T	J	$(S/N)_1$	$(S/N)_2$	P_d (experimental)	P_{fa}
					$\lambda=0.9$	
channel 1	10	1	+3dB	-	0.943 ± 0.007	0.0693
20a	10	2	+3dB	-30dB	0.948 ± 0.004	0.0693
channel 1	10	1	+3dB	-	0.787 ± 0.016	6.93×10^{-4}
20a	10	2	+3dB	-30dB	0.788 ± 0.005	6.93×10^{-4}

Table 7.6.6b Probability of detection for case 20a and channel 1 alone with $(S/N)_1=+3\text{dB}$ and $\lambda_{11}=0.9$.

evident in Fig. 7.6.8. These results are consistent with those shown in Figs. 7.6.1 to 7.6.4; ie., at the higher S/N values, the uncorrelated signal case has a higher P_d than the correlated signal case. Alternatively, at the lower S/N values, the more correlated signal case has the higher P_d . The one exception (using only a single trial) is the detection result for case 16c and is noted in this table. We will comment on this result at the end of this section, but simply note that case 16c has high temporal and cross-channel correlation. Finally, we note that the two-channel detection results for Fig. 7.6.7 appear to asymptotically approach a detection result slightly above the single channel case. This result would imply that the additional information from the channel 2 data consisting primarily of the additive white noise process provides a detection performance improvement. The sample standard deviations for the detection results, however, preclude such a conclusion.

In Figures 7.6.9 and 7.6.10, we present detection results for cases 22c through 29c where $\lambda_{11}=\lambda_{22}=0.999$ and $|\rho_{12}|=0.99$. The tabulated values are displayed in Table 7.6.7. We note that these processes have very high temporal and cross-channel correlation and therefore closely approximate the model I(COH) process. Specifically, case 22c with $(S/N)_1=(S/N)_2=+3\text{dB}$ should approximate (but lie slightly above) the lower bound provided by the model I(COH) detection curve for $JN_T=20$ pulses at a +3dB signal-to-noise ratio. In table 7.6.8, case 22c shows good agreement with the analytic values. Although the result at $P_{fa}=0.0693$ fell 0.1% lower than the analytic curve, it is within the experimental error limits. The detection results for the case 29c process with $(S/N)_2=-40\text{dB}$ is nearly equal to, but slightly higher than that of the single channel case. This result again indicates that the two-channel detection results approach the single channel case in the limit as the signal-to-noise ratio on one channel diminishes. As in the detection results for Fig. 7.6.7, we again note that the two-channel detection results appear to asymptotically approach a value slightly above the single channel case. Again, this result would imply that the additional information from the channel 2 data, consisting primarily of the additive white noise process, provides a detection performance improvement. The limited number of repeated trials (four), however, precludes such a conclusion.

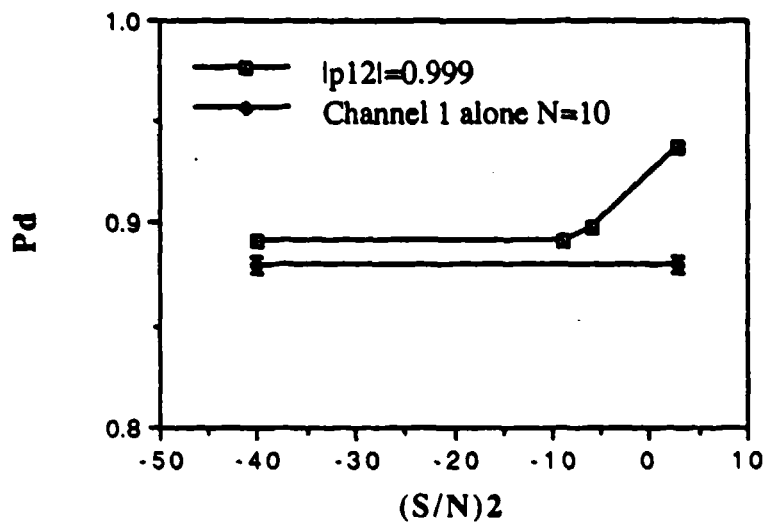


Figure 7.6.9 Probability of detection results as a function of $(S/N)^2$ for $|p_{12}|=0.99$, $(S/N)_1=+3\text{dB}$ and $\lambda_{11}=\lambda_{22}=0.999$, $P_{fa}=0.0693$.

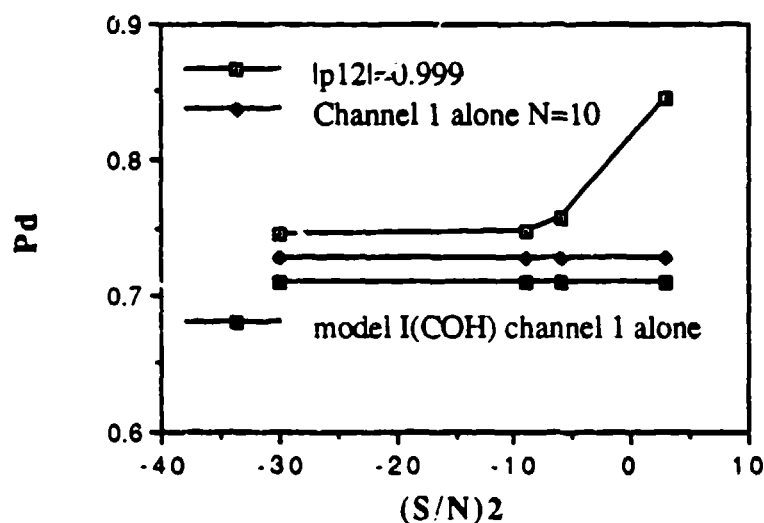


Figure 7.6.10 Probability of detection results as a function of $(S/N)^2$ for $|p_{12}|=0.99$, $(S/N)_1=+3\text{dB}$ and $\lambda_{11}=\lambda_{22}=0.999$, $P_{fa}=6.93 \times 10^{-4}$.

case	$(S/N)_1$	$(S/N)_2$	P_d	P_{fa}
22c	+3dB	+3dB	0.937	0.0693
23c	+3dB	0dB	inst	0.0693
24c	+3dB	-3dB	inst	0.0693
25c	+3dB	-6dB	0.898	0.0693
26c	+3dB	-9dB	0.892	0.0693
27c	+3dB	-20dB	inst	0.0693
28c	+3dB	-30dB	inst	0.0693
29c	+3dB	-40dB	0.891 ± 0.002	0.0693
22c	+3dB	+3dB	0.846	6.93×10^{-4}
23c	+3dB	0dB	inst	6.93×10^{-4}
24c	+3dB	-3dB	inst	6.93×10^{-4}
25c	+3dB	-6dB	0.758	6.93×10^{-4}
26c	+3dB	-9dB	0.747	6.93×10^{-4}
27c	+3dB	-20dB	inst	6.93×10^{-4}
28c	+3dB	-30dB	inst	6.93×10^{-4}
29c	+3dB	-40dB	0.746 ± 0.006	6.93×10^{-4}

Table 7.6.7 Probability of detection results as a function of $(S/N)_2$ for $\lambda_{11}=\lambda_{22}=0.999$ and $|\rho_{12}|=0.99$.

case	N_T	J	$(S/N)_1$	$(S/N)_2$	P_d (experimental)	P_d (analytical)	P_{fa}
					$\lambda=0.999$ $ \rho_{12} =0.99$	model I(COH)	
22c	10	2	+3dB	+3dB	0.937	0.938	0.0693
channel 1	10	1	+3dB	-	0.880 ± 0.004	0.879	0.0693
29c	10	2	+3dB	-40dB	0.891 ± 0.002		0.0693
22c	10	2	+3dB	+3dB	0.846	0.840	6.93×10^{-4}
channel 1	10	1	+3dB	-	0.728 ± 0.007	0.710	6.93×10^{-4}
29c	10	2	+3dB	-40dB	0.746 ± 0.006		6.93×10^{-4}

Table 7.6.8 Probability of detection for $\lambda_{11}=\lambda_{22}=0.999$ and $|\rho_{12}|=0.99$ with corresponding analytical results for the model I(COH) process.

Finally, we note that for cases 23c, 24c, 27c and 28c, instabilities occurred. We will comment on these results later, but again indicate that these instabilities have occurred for processes with high temporal and cross-channel correlation.

The final detection problem that we will consider is a two channel signal plus white noise problem in which the channels of the white noise are correlated with $|\rho_{w_{12}}|=0.99$. The subscript w denotes noise. The detection results are presented in Tables 7.6.9 and 7.6.10 for $P_{fa}=0.0693$ and 6.93×10^{-4} , respectively.

In each of these tables, we also present the results previously obtained without noise channel correlation. As noted previously, at $(S/N)_1=(S/N)_2=+3\text{dB}$, increasing cross-channel correlation on the signal process caused P_d to decrease.

At this signal-to-noise ratio, our results indicate that, the increased cross-channel correlation on the noise process causes P_d to increase. However, at $(S/N)_1=(S/N)_2=-5\text{dB}$, the opposite trend is noted; ie., with increasing cross-channel correlation on the signal, P_d increases while with increasing cross-channel correlation on the noise, P_d decreased. The large number of instabilities are again noted.

We now comment on the instabilities which were noted in our detection results. These occurrences are sometimes noted to be detection values which may deviate slightly from analytic values or from adjacent experimental values on a detection curve. On the other hand, they were sometimes noted to be values which gave either unity or zero detection probability and, in some cases, prevented the operation of the software. As noted previously, these unstable results occur most often for the case of processes with high temporal and/or cross-channel correlation. In this dissertation, we have used the Cholesky decomposition method both in the process synthesis procedure described in section 4.5 and in the detection implementation procedure for eq(6.1.38). As we have noted previously, when the processes are highly correlated, the elements of the Cholesky matrix used in the synthesis method become very small. As a result, these matrices become ill-conditioned. In future work, we will give further consideration to the LDL^H decomposition in terms of the stability problem. Second, we again point out that our approach to the detection problem using 'known' coefficients is an approximation based on our use of a large amount of 'a priori' data. We must therefore consider that, in some cases, the variance associated with the Strand-Nuttall estimator used to obtain the F_1 and F_0 filter

case	(S/N) ₁	(S/N) ₂	P _D correlated channel noise	P _D uncorrelated channel noise
3a	+3dB	+3dB	inst.	0.999±0.0003
3b	+3dB	+3dB	inst.	0.999
3c	+3dB	+3dB	0.999	0.998
4c	+3dB	+3dB	0.996	0.989±0.001
5c	+3dB	+3dB	0.980	0.938
2c	-5dB	-5dB	0.689	0.712±0.001
1c	-5dB	-5dB	0.440	0.555±0.005
1b	-5dB	-5dB	inst.	0.464
1a	-5dB	-5dB	inst.	0.430

Table 7.6.9 Detection results for the case of correlated channel noise with $|\rho_{w_{12}}|=0.99$ and $P_{fa}=0.0693$.

case	(S/N) ₁	(S/N) ₂	P _D correlated channel noise	P _D uncorrelated channel noise
3a	+3dB	+3dB	inst.	0.970±0.001
3b	+3dB	+3dB	inst.	0.968
4c	+3dB	+3dB	0.966	0.942±0.002
5c	+3dB	+3dB	0.927	0.847±0.001
2c	-5dB	-5dB	0.220	0.319±0.003
1c	-5dB	-5dB	0.043	0.098±0.024
1b	-5dB	-5dB	inst.	0.063
1a	-5dB	-5dB	inst.	0.046±0.003

Table 7.6.10 Detection results for the case of correlated channel noise with $|\rho_{w_{12}}|=0.99$ and $P_{fa}=6.93 \times 10^{-4}$.

coefficients may have been large enough to cause inaccuracies. These considerations are addressed more thoroughly in chapter 5. Furthermore, Nuttall [10] has noted that $[\Sigma_f]_{p-1}$ and $[\Sigma_b]_{p-1}$ have not been shown to be non-negative definite. Since the inverse of these matrices are used as the weights in the Strand-Nuttall algorithm [see eqs(2.5.60) and (2.5.61)], instabilities could result due to the lack of positive semi-definiteness. Finally, as we noted in sections 7.3 and 7.4, the Monte-Carlo procedure itself causes some inaccuracies in the detection results due to the finite number of trials that are used. However, the instabilities noted here are greater than those due to the Monte-Carlo procedure.

In this chapter, we have presented the procedure used to evaluate the detection performance of the likelihood ratios presented in chapter 6. In order to compare our results with available analytic detection curves, we have focused our attention on the Gaussian signal in additive Gaussian white noise detection problem where the observation processes have 'known' parameters. First, detection results were presented as a function of signal-to-noise ratio and probability of false alarm for signal processes containing various amounts of temporal and cross-channel correlation. Second, we considered the detection performance for the case of a two-channel process in which the signal-to-noise ratio on one of the channels decreased. Finally, we considered the case of additive white noise containing cross-channel correlation.

CHAPTER 8

SUMMARY, CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

8.1 SUMMARY AND CONCLUSIONS

In this section, we present a summary of the contributions made in this thesis, as well as several conclusions. First, an approach has been developed for the synthesis of multichannel autoregressive (AR) random processes. The procedure allows for the control of temporal and cross-channel correlation sequences of synthesized processes subject to specific constraints for correlation sequences. The resulting synthesized processes provide a 'fit' in a minimum mean squared error (MMSE) sense to the correlation functions which are specified in terms of temporal and cross-channel correlation parameters of the processes. Computer simulation results are presented showing the case of a two channel AR process with various temporal and cross-channel correlation. A method is also suggested to generalize the synthesized outputs to obtain Gaussian processes with unconstrained quadrature components where the assumptions associated with stationarity on the associated bandpass processes are relaxed. These processes provide an input to the likelihood ratio implementation to obtain detection results as a function of channel signal-to-noise as well as the process correlation parameters.

As a second contribution, the condition for ergodicity of auto- and cross-channel correlation functions expressed in terms of fundamental process characteristics is derived. Specifically, analytic expressions are developed for the variance of time-averaged correlation functions for discrete, complex baseband processes. These expressions provide a measure which can be used to specify the window size of the observation interval required to achieve a specific value of this variance. A unique aspect of this development is determination of the functional dependence of these expressions in terms of the process characteristics such as the process variance as well as the temporal and cross-channel correlation. In addition, the analytic expressions are simplified for the special case where the associated bandpass processes are wide-sense jointly stationary (see Appendices N and O). Analytic expressions are validated using the multichannel process synthesis method noted above. This analysis is of particular significance for the case of 'unknown' process parameters where the generalized likelihood

ratio is evaluated using parameter estimation algorithms based on time-averaging methods. In this case, detection performance limits result due to the errors associated with the time-averaged estimates.

Third, model-based likelihood ratios are developed for the multichannel binary detection problem using error vector processes. These ratios are derived assuming wide-sense stationarity of the baseband Gaussian processes. However, a more general likelihood ratio applicable for non-stationary bandpass processes is also developed. In the special case of wide-sense stationary bandpass processes, one of the likelihood ratios is shown to reduce to the multichannel generalization of the innovations-based detection algorithm (IBDA) presented by Metford, Haykin, and Taylor [24] for scalar processes.

Fourth, the likelihood ratio developed for the non-stationary bandpass processes utilized a more powerful estimator than previously noted. This estimator is capable of whitening not only the complex observation processes, but also their quadrature components. Furthermore, we point out that coefficients for the more general linear estimator $\hat{\mathbf{x}}(n)$ proposed in section 6.3 could be obtained using the same form as algorithms proposed for wide-sense jointly stationary complex baseband processes; ie., algorithms such as the Strand-Nuttall and Vieira-Morf could be used but extended to the quadrature form. Although the matrices become $2J \times 2J$ rather than $J \times J$, the resulting likelihood ratios are capable of processing the more general class of Gaussian processes with unconstrained quadrature components. We note that for this class of processes, the narrowband bandpass processes could be non-stationary.

Detection performance is obtained for the special case where the underlying processes are assumed to have 'known' autoregressive process parameters in additive white noise. The procedure used to obtain these parameters consisted of performing estimates with large amounts of 'a priori' data. Results for two-channel signal vectors containing various temporal and cross-channel correlation are obtained using a Monte-Carlo procedure. These results are compared to known optimal detection curves for validation. It was observed that at high signal-to-noise ratio, the probability of detection decreased with increasing signal temporal and cross-channel correlation while at low values, it increased. These results offer insight regarding multichannel processing methods that should be utilized to improve detection performance. Specifically, at high signal-to-noise ratios, channel diversity (which implies uncorrelated

channel processes) offers the potential for improved detection performance. However, at very low signal-to-noise ratios, the processing should utilize multiple processes which contain correlated signal information provided the noise processes are uncorrelated across the channels.

Detection performance was also examined for the case where one of the channels was degraded in terms of its signal-to-noise ratio. It was determined that the detection results converged to that of the channel with the higher signal-to-noise ratio as the performance of the second channel diminished. Finally, detection results were obtained for the case where the additive white noise contained cross-channel correlation. It was determined that the detection results for increased cross-channel noise correlation had the opposite functional dependence as that noted for the signal; ie., at high signal-to-noise ratio, the probability of detection increased with increasing noise cross-channel correlation while at low values, it decreased. Thus, at low signal-to-noise ratios it is important to reduce the cross-channel noise correlation.

Several instabilities were noted in our detection results. These occurrences are sometimes noted to be detection values which may deviate slightly from analytic values or from adjacent experimental values on a detection curve. On the other hand, they were sometimes noted to be values which gave either unity or zero detection probability and, in some cases, prevented the operation of the software. As noted previously, these unstable results occur most often for the case of processes with high temporal and/or cross-channel correlation. In this dissertation, we have used the Cholesky decomposition method both in the process synthesis procedure described in section 4.5 and in the detection implementation procedure for eq(6.1.38). As we have noted previously, when the processes are highly correlated, the elements of the Cholesky matrix used in the synthesis method become very small. As a result, these matrices become ill-conditioned. In future work, we will give further consideration to the LDL^H decomposition in terms of the stability problem. Second, we again point out that our approach to the detection problem using 'known' coefficients is an approximation based on our use of a large amount of 'a priori' data. We must therefore consider that, in some cases, the variance associated with the Strand-Nuttall estimator used to obtain the F_1 and F_0 filter coefficients may have been large enough to cause inaccuracies. These considerations are addressed more thoroughly in chapter 5.

Furthermore, Nuttall [10] has noted that $[\Sigma_f]_{p-1}$ and $[\Sigma_b]_{p-1}$ have not been shown to be non-negative definite. Since the inverse of these matrices are used as the weights in the Strand-Nuttall algorithm [see eqs(2.5.60) and (2.5.61)], instabilities could result due to the lack of positive semi-definiteness. Finally, as we noted in sections 7.3 and 7.4, the Monte-Carlo procedure itself causes some inaccuracies in the detection results due to the finite number of trials that are used. However, the instabilities noted here are greater than those due to the Monte-Carlo procedure.

8.2 SUGGESTIONS FOR FUTURE WORK

8.2.1 Multichannel Modeling and Model Order Selection

The detection approach evaluated in this dissertation is model-based; ie., the underlying signal and clutter noise processes are assumed to be characterized by parametric models such as autoregressive processes. This approach leads to a realizable architecture which can be implemented for the detection of random signals in noise and is amenable for adaptive processing methods. However, detection performance is linked to the validity of the model representation of the observation processes [20]. Furthermore, the model representation is highly problem dependent; ie., the characterization of the observation processes by an appropriate model must be tailored to the specific problem being addressed. For example, the models applicable for processes observed by a pulse-Doppler radar may be significantly different than those observed by a wideband high resolution radar.

In the introduction, we noted several analyses involving the modeling of processes with time series for a wide range of applications. In the radar problem, however, a limited number of investigations regarding the characterization of clutter processes using autoregressive models has been conducted [26,41,42,79,80,91,92,93]. Furthermore, we are not aware of any parametric modeling analyses for multichannel radar applications where multichannel indicates waveform diversity such as multiple center frequencies and polarization. Noting the similarity between one-dimensional multichannel and two-dimensional single channel methods [11,12], extensions could be made to range cross-range imaging analyses [94].

The validation of such models in specific applications therefore constitutes an essential area for further research. Such an investigation must include a data collection program capable of providing a sufficient data base to obtain high confidence on the parameter estimates. A quantitative measure on the data base window size requirements could be assessed with the ergodicity analysis described in chapter 5.

As noted in section 6.4, determination of the model orders which fit the observation data processes under both hypotheses is important in the detection implementation procedure; ie., it enables the determination of the filter orders in the detection architecture. Model order selection methods for AR and ARMA

models has been given extensive consideration for single channel processes [36,63,66,67,96-114]. The extension of these analyses to the multichannel case requires further investigation, however.

We also note an interesting and potentially useful feature of multichannel AR processes which may be significant in the modeling as well as the detection analyses. In this dissertation, we have used underlying AR models to describe the signal and clutter processes. However, as noted in previous chapters, autoregressive moving average (ARMA) models could be used to model the summation of AR models of lower order [36] or the summation of an AR model plus additive white noise [95]. Nuttal [10], however, has also noted that a multichannel AR process contains channel elements which are in themselves single channel ARMA processes. This is due to the increased degrees of freedom of the multichannel AR process [17]. It therefore appears that, under certain conditions, multichannel processes containing a summation of underlying AR processes and white noise may be very accurately modeled as a multichannel AR process with relatively low order. This result is significant since it implies that lower order multichannel filters may be implemented in the detection architecture. The conditions under which this reduced order modeling may be achieved is an area for further investigation.

8.2.2 Analytic Detection Approach for Multichannel Processes

The analytic-numeric integration method of Schwartz [115] for the single channel case provides expressions for the probabilities of detection P_d and false alarm P_{fa} . These equations (also discussed in [23]) are expressed as

$$P_d = 1 - \frac{1}{\pi} \int_0^{\infty} \text{Re} \left\{ \frac{(-1/j\xi) [\exp(-j\xi T) - 1]}{\det[I - 2j\xi(R_d + R_s)^*R^*]} d\xi \right\} \quad (8.2.1)$$

and

$$P_{fa} = 1 - \frac{1}{\pi} \int_0^{\infty} \text{Re} \left\{ \frac{(-1/j\xi) [\exp(-j\xi T) - 1]}{\det[I - 2j\xi(R_d)^*R^*]} d\xi \right\} \quad (8.2.2)$$

where T is the threshold and the single channel correlation matrices are expressed as

$$R_d = R_c + R_w \quad (8.2.3)$$

$$R = [R_d^{-1} - (R_d + R_s)^{-1}] \quad (8.2.4)$$

$$R_s = E[\underline{s} \underline{s}^H] \quad (8.2.5)$$

$$R_c = E[\underline{c} \underline{c}^H] \quad (8.2.6)$$

The extension of these equations to the multichannel case can be made using the correlation matrices discussed in section 4.3.

8.2.3 Multichannel Algorithm Development

Detection performance for the 'unknown' signal case using estimation procedures in the generalized multichannel likelihood ratios developed here is critically dependent upon the multichannel algorithms [6-11,25,29,32,60,128-137]. The continued development and practical implementation of multichannel algorithms constitutes an essential research area for further development. Algorithm performance must be assessed in terms of computational complexity, efficiency with limited data, number of operations per iteration, numerical stability, implementation into structural processing architectures, and suitability for VLSI implementation. Consideration has been given to the use of modular processing architectures for the implementation of adaptive multichannel lattice algorithms [130,131]. In this approach, a modular decomposition is applied to multichannel lattice recursions to transform them to single-channel (scalar) recursions. Levi-Ari [131] has noted that the following properties are realized through this approach:

1. Only scalar computations are required.
2. The architecture is modular, regular (all cells are identical) and requires only local communication. It is, therefore, well suited for VLSI implementation.
3. Computations are performed in parallel and information flows between cells in a pipeline fashion. Thus, the throughput is independent of the number of channels.
4. The architecture is expandable. The number of channels can be increased by connecting together several identical subsystems.

He also notes that 'the principle of modular decomposition that is used to derive this architecture is remarkably universal. Exact-least-squares lattice algorithms obey this principle because they all share an inherently modular geometric interpretation [130]. The scope of applicability of the decomposition principle extends, however, much beyond this particular class of algorithms, including, for instance, systolic array architectures for convolution and matrix multiplication [141,142]'.

Consideration has also been given to the use of Gram-Schmidt orthogonalization predictor realizations to implement algorithms for filter coefficient (weight) updating [140-142]. The Gram-Schmidt orthogonalization procedure has particular significance in the interpretation and realization of innovations processes [59]. Block-processing algorithms have also been considered for high speed processing applications [143-151]. These methods are useful when the cost associated with updates at each data point is too high [151]. Investigation of these methods in terms of numerical accuracy, complexity, efficiency, as well as suitability for implementation into processing architectures remains a viable area for further research.

8.2.4 Detection with Unconstrained Gaussian Quadrature Components

In this dissertation, we have also considered the more general class of Gaussian processes containing correlated quadrature components which are not restricted to the conditions

$$R_{xx}^{\Pi}(l) = R_{xx}^{QQ}(l) \quad (8.2.7a)$$

and

$$R_{xx}^{IQ}(l) = -R_{xx}^{QI}(l). \quad (8.2.7b)$$

A synthesis procedure is presented in section 4.6 and a likelihood ratio is developed in section 6.3 for these more general processes. Implementation architectures and detection results, however, were presented for the more restrictive case of processes satisfying eqs(8.2.8). Detection results for the more general case would provide a valuable extension of the results presented here.

We again note that the more general case involves processes with non-stationary bandpass processes.

8.2.5 Detection Performance versus Estimator Error Variance

In many signal processing applications it is necessary to estimate parameters of a random process. Typically, time-averaging procedures are employed for this purpose. For example, algorithms such as the Levinson, Burg and Strand-Nuttall recursion methods are used to estimate autoregressive (AR) parameters of the processes. It is often noted, however, that unacceptably large deviations in the estimates can result if a sufficient observation window size is not utilized; ie., for a given observation sample size, the variance of the estimates can be very large. Such a condition results in unacceptably low confidence levels associated with the quality of the estimate. In these estimation problems, what is often overlooked, is the explicit dependence of these deviations not only on the observation window size, but also on 'fundamental' characteristics of the processes such as the temporal correlation, variance and (in certain cases) the cross-correlation of the processes. More specifically, for a given window size, these characteristics of the processes determine fundamental limits on the quality of the estimates. To our knowledge, no simplified procedure has been suggested which enables one to quantitatively determine the functional relationship between the quality of a given estimator and these process characteristics.

In chapter 5, we developed a procedure in which time-averaged estimates and their variances are expressed not only in terms of the observation sample size time window, but also on the fundamental process parameters which limit estimation performance. Specifically, the ergodicity conditions for the time-averaged complex auto- and cross-correlation function estimators are considered. The unique aspect of this development is the simplified procedure for determining the functional dependence of the ergodic assumption in terms of the 'fundamental' characteristics. The resulting analytic expressions determine the signal processing requirements to achieve a specified variance of the estimator. These expressions provide much more insight regarding the functional role of the process characteristics in determining the ergodic behavior than previously noted. Furthermore, the analytic expressions are derived for the general case of Gaussian processes with unconstrained quadrature components where the usual assumptions associated with the correlation of the quadrature components are

relaxed. Validation of the analytic expressions was presented using the multichannel synthesis method described in chapter 4.

We suggest the extension of this approach to the estimation algorithms utilized in the detection implementation architecture presented here. Specifically, for the case of 'unknown' process parameters, detection performance should be evaluated using the parameter estimation algorithms based on the time-averaging methods. Particular emphasis should focus on the detection performance limits resulting from the errors associated with these estimates; ie., the error variance associated with the parameter estimates will result in a corresponding error variance associated with the detection results.

8.2.6 Detection Performance with Multipath, Feedback and Model-Mismatch

In this section, we note two investigations which could be conducted to consider the robustness of the detection method considered in this dissertation. First, detection in the presence of multipath clutter or system feedback could be evaluated using a minor extension of the process synthesis procedure described in section 4.5. This would involve modifications of the procedure described in block form by Fig. 4.5.1. As an example, we could consider the system shown in Fig 8.2.1.

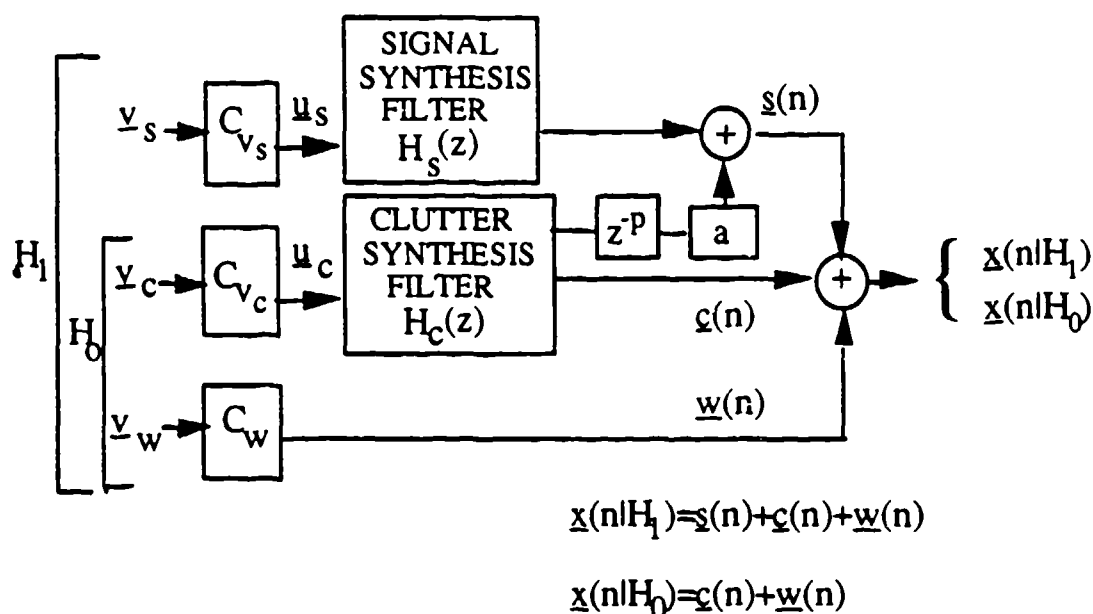


Fig 8.2.1 Process Synthesis Procedure involving Feedback and Multipath.

We note in this case that the signal contains a delayed and weighted clutter sample in an additive manner, thereby including the effect of multipath. The delay is controlled by 'p' and the weight by 'a'. Replacing the addition operation with multiplication would enable the modeling of multiplicative noise processes. Processes synthesized as shown in Fig. 8.2.1 now provide an input to the detection implementation architectures. Performance could be evaluated in terms of the clutter weighting and delay. Multiple delays can be modeled by additional delay elements in the synthesis network.

Second, the robustness of the model-based detection method could be evaluated in terms of model mismatch to the observation data processes. An example of such a mismatch could consist of the case where lower model orders are incorrectly assumed for the observation processes under either (or both) hypotheses. This would cause one to use lower filter orders on either (or both) of the F_0 and F_1 filters in the detection implementation architecture. Detection performance evaluations as a function of the filter orders could be conducted.

8.2.7 Sequential Detection for Discrimination

The theory of sequential binary hypothesis testing was developed by Wald [152,153] and generalized to the multiple hypothesis case by Reed [154]. In [155], Selin considered the detection of a deterministic signal in Gaussian noise. In this work, Karhunen-Loève basis functions were used to obtain a set of uncorrelated and, therefore, independent observations. This approach was motivated by the fact that independent observations greatly simplify the sequential test. The extension of the sequential approach using linear mean-square prediction to obtain causally transformed innovations processes is summarized in the tutorial by Therrien [39] noting the pioneering work of Schweppe [1] and Selin [156]. Application areas are noted in [157-160]. In this approach, the classifier structure involves a two step procedure involving linear prediction followed by a comparison test using the accumulated prediction errors. Following the discussion by Therrien [39],

Since linear prediction can often be performed on just a short history of the process, a great economy of storage and computation

time can result. Since the classifier compares the accumulated prediction errors, discrimination is based on the entire set of observations and not just on the number of observations used for linear prediction. Further, if the process is stationary, the classifier is open-ended. That is, if a sufficiently long prediction interval is taken, the filters used for linear prediction are approximately time invariant and the sequential test can continue for as long as necessary without continual modification of the filter parameters.

The recursive classifier structure, though ideal for sequential discrimination, is not limited to that application. The structure can also be used to implement fixed statistical tests such as those of the Bayes or Neyman-Pearson type and it shares some common features with estimator/correlator receiver realizations [1,51,54,19 Part III]. Further, although the optimal properties will not hold, the classifier can be applied to signatures with non-Gaussian distributions and several interpretations are possible. In particular, the classifier can be described in the time domain in terms of autoregressive time series models [36], Gauss-Markov state models [161], and innovations concepts [54]. In the frequency domain it can be related to the so-called data adaptive methods of spectral estimation [162].

The extension of the approach outlined in [39] to the multichannel case remains an area for further investigation. The multichannel approach presented in this dissertation should readily extend to the use of multiple processes in the sequential detection procedure. Thus, a practical methodology can be established to utilize information from multiple processes in the discrimination problem.

8.2.8 Detection using Wideband Radar Waveforms

A limited number of radar detection analyses have been conducted using wideband transmitter waveforms [116-118]. These analyses utilize the compressed radar pulse to over-resolve the target signal source along the slant range axis and assess detection performance over range cells using incoherent integration.

The use of the multichannel detection method presented in this dissertation could provide a very useful approach for analyses involving wideband radar systems. In this case, we could view each channel to consist of a distinct frequency interval of the transmitted spectral bandwidth. For closely spaced frequency intervals, partial correlation may exist between channels. The

multichannel detection methods presented here utilize these channel processes to achieve integration gain in terms of both temporal and cross-channel correlation. The multichannel approach offers the potential to efficiently utilize all of the information contained in the spectral bandwidth of the received data processes since the processing is coherent in both of these dimensions. However, computational complexity will also increase as the number of channels increases.

8.2.9 Detection using Multiple Sensors

The use of multiple sensors to improve detection performance has been given considerable attention in recent years [119-125]. For sensors distributed in location, practical constraints such as communication channel bandwidth dictate that local decisions be made at each sensor with subsequent detection decisions performed at the fusion center. Thus, thresholding is performed at each sensor to make a hypothesis selection. The transmission of this binary decision can therefore be communicated at a much lower bit rate. This detection scheme is referred to as the decentralized method. Unfortunately, the practical advantages of the decentralized scheme are achieved at the expense of optimality in detection performance. Researchers have often noted 'insignificant' loss of performance [119] when compared to the case of centralized processing; ie., the case where all the observation data is available for processing so that a single threshold is established. However, these analyses have not considered the more general case of correlated sensor observations. Furthermore, they do not compare their results to the case where coherent processing is used across the sensor observations. In this dissertation, the multichannel approach involves a 'centralized' processing method. Not only does this approach offer the potential for optimality in the Gaussian detection problem, it also enables the implementation of coherent processing across channels as well as in time. Viewing each channel as a sensor process, coherent processing across sensors cannot be achieved in the decentralized approach. On the other hand, the centralized processing approach often involves considerable computational complexity in addition to the increased transmission data rate requirements. It would therefore appear that the multichannel approach as applied to the sensor fusion problem would be most applicable to the situation where sensor suites are situated locally and sufficient computational resources are available.

An interesting dimension of the multichannel approach to multiple sensor applications would involve consideration of dissimilar sensors. In the radar problem, this might consist of radars operating at widely separated frequencies each of which may be operating with multiple polarization vectors. In earthquake analyses, the use of sensors such as seismometers, shearometers, tiltometers, etc. could all be used to determine a 'change' in the observation processes. The prime feature of the likelihood ratio developed here is the capability to transform these dissimilar multichannel processes to error residuals. These error residuals in turn are utilized to measure the change in the processes characterized by a given set of parameters (hypothesis H_0) to another set (hypothesis H_1). Thus, the multichannel approach offers the potential to utilize all of the information from these distinct sensor processes in the detection problem. Furthermore, if correlation does exist among these 'distinct' processes, the multichannel method will utilize this additional information in the detection scheme. In the case where no cross-channel correlation is present, the likelihood ratio implementation scheme reduces to the special case using diagonal matrix coefficients.

In the radar problem, the use of dissimilar sensors presents practical problems such as the determination of a common surveillance volume, multiple signal sources, differences in data rates, and missing data.

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APPENDIX A

MMSE CRITERION APPLIED TO THE MULTICHANNEL LINEAR PREDICTOR

In this appendix, we first consider the multichannel minimum mean-squared error (MMSE) criterion for a linear predictor. Under this criterion, we show that the coefficients of the linear predictor satisfy the normal equations. Second, we show that the resulting error vector is uncorrelated with past observations. Finally, we show that the error vector is a statistically independent white noise process. We will follow the derivation presented by [Nuttall 76a] using our notation. From eqs(2.4.1) and (2.4.2), we have

$$\underline{\varepsilon}_p(n) = \underline{x}(n) - \hat{\underline{x}}_p(n|n-1, \dots, n-p) \quad (\text{A.1a})$$

$$= \underline{x}(n) + \sum_{k=1}^p \underline{A}_p^H(k) \underline{x}(n-k) \quad (\text{A.1b})$$

$$= \underline{x}(n) + (\underline{A})_p^H \underline{x}_{n-1, n-p} \quad (\text{A.1c})$$

where

$$\hat{\underline{x}}_p(n|n-1, \dots, n-p) = - \sum_{k=1}^p \underline{A}_p^H(k) \underline{x}(n-k) \quad (\text{A.2})$$

$$\underline{x}_{n-1, n-p}^T = [\underline{x}^T(n-1) \underline{x}^T(n-2) \dots \underline{x}^T(n-p)] \quad (\text{A.3})$$

and the vector of matrices $(\underline{A})_p^H$ is the same as that defined in eq(2.4.3a), but without the identity matrix $\underline{I}$, so that

$$(\underline{A})_p^H = [\underline{A}_p^H(1) \underline{A}_p^H(2) \dots \underline{A}_p^H(p)]. \quad (\text{A.4})$$

Let

$$E[\underline{x}(n) \underline{x}_{n-1, n-p}^H] \equiv \underline{C} \quad (\text{A.5})$$

and

$$E[\underline{x}_{n-1, n-p} \underline{x}_{n-1, n-p}^H] = [\tilde{\underline{R}}_{xx}]_p \equiv \underline{Q}. \quad (\text{A.6})$$

With respect to dimensions, $(\Delta)^H$ is $J \times J_p$, $\underline{x}_{n-1,n-p}$ is $J_p \times 1$, C is $J \times J_p$ and Q is $J_p \times J_p$. We note that $Q^H = Q$ and $\underline{V}^H Q \underline{V} > 0$ for any $J_p \times 1$ vector $\underline{V} \neq 0$ when Q is positive definite. Now, the $J \times J$ error covariance matrix is expressed as

$$E[\underline{\epsilon}_p(n) \underline{\epsilon}_p^H(n)] = E\{[\underline{x}(n) + (\Delta)_p^H \underline{x}_{n-1,n-p}][\underline{x}^H(n) + \underline{x}_{n-1,n-p}^H (\Delta)_p]\} \quad (A.7a)$$

$$= E[\underline{x}(n) \underline{x}^H(n)] + (\Delta)_p^H C^H + C (\Delta)_p + (\Delta)_p^H Q (\Delta)_p \quad (A.7b)$$

$$= R_{xx}(0) - C Q^{-1} C^H + [(\Delta)_p^H + C Q^{-1}] Q [(\Delta)_p^H + C Q^{-1}]^H. \quad (A.7c)$$

Let the $J \times J_p$ matrix $[(\Delta)_p^H + C Q^{-1}]$ be expressed as

$$[(\Delta)_p^H + C Q^{-1}] = \begin{bmatrix} \underline{V}_1^H \\ \underline{V}_2^H \\ \dots \\ \underline{V}_J^H \end{bmatrix} \quad (A.8)$$

where $\underline{V}_j$ is a $J_p \times 1$ vector. We now denote the last term in eq(A.7c) as the $J \times J$ matrix

$$L = [(\Delta)_p^H + C Q^{-1}] Q [(\Delta)_p^H + C Q^{-1}]^H = [l_{mj}] \quad 1 \leq m, j \leq J \quad (A.9a)$$

$$= \begin{bmatrix} \underline{V}_1^H \\ \underline{V}_2^H \\ \dots \\ \underline{V}_J^H \end{bmatrix} Q [\underline{V}_1 \ \underline{V}_2 \ \dots \ \underline{V}_J] \quad (A.9b)$$

where the complex scalar l_{mj} is now expressed as

$$l_{mj} = \underline{V}_m^H Q \underline{V}_j. \quad (A.10)$$

The real quantity $l_{jj} = \underline{V}_j^H Q \underline{V}_j > 0$ for any $\underline{V}_j \neq \underline{0}$, since Q is Hermitian and provided Q is positive definite; the minimum value of l_{jj} is zero and is attained if and only if $\underline{V}_j = \underline{0}$. Therefore, $\text{tr} L$ is minimized, attaining value zero, by the choice $\underline{V}_j = \underline{0}$, $1 \leq j \leq J$. Using eq(A.7c), $\text{tr} E[\underline{\epsilon}_p(n) \underline{\epsilon}_p^H(n)] = E[\underline{\epsilon}_p^H(n) \underline{\epsilon}_p(n)]$ is minimized by the choice of $(\underline{A})_p^H$ as

$$(\underline{A})_p^H = -CQ^{-1} \quad (\text{A.11a})$$

or

$$Q(\underline{A})_p = -C^H \quad (\text{A.11b})$$

since the leading two terms in eq(A.7) are independent of $(\underline{A})_p^H$. Eq(A.11b) can be written in expanded form using eqs(A.2), (A.5) and (A.6), so that

$$\begin{bmatrix} R_{xx}(0) & R_{xx}(1) & R_{xx}(2) & \dots & R_{xx}(p-1) \\ R_{xx}(-1) & R_{xx}(0) & R_{xx}(1) & \dots & R_{xx}(p-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{xx}(2-p) & R_{xx}(3-p) & R_{xx}(4-p) & \dots & R_{xx}(1) \\ R_{xx}(1-p) & R_{xx}(2-p) & R_{xx}(3-p) & \dots & R_{xx}(0) \end{bmatrix} \begin{bmatrix} A_p(1) \\ A_p(2) \\ \dots \\ A_p(p-1) \\ A_p(p) \end{bmatrix} = - \begin{bmatrix} R_{xx}(-1) \\ R_{xx}(-2) \\ \dots \\ R_{xx}(1-p) \\ R_{xx}(-p) \end{bmatrix}. \quad (\text{A.12})$$

Eq(A.12) contains the normal equations (not to be confused with its augmented form) for a prediction error filter of order p and can also be expressed as

$$\sum_{k=1}^p R_{xx}(k-m) A_p(k) = -R_{xx}(-m) \quad m = 1, 2, \dots, p \quad (\text{A.13})$$

or

$$\sum_{k=1}^p A_p^H(k) R_{xx}(m-k) = -R_{xx}(m) \quad m = 1, 2, \dots, p. \quad (\text{A.14})$$

We now show that the error vector $\underline{\epsilon}_p(n)$ is uncorrelated with past observation vectors when eqs(A.14) hold. Eq (A.14) can be expressed as

$$\sum_{k=1}^p A_p^H(k) E[\underline{x}(n-k) \underline{x}^H(n-m)] = -E[\underline{x}(n) \underline{x}^H(n-m)] \quad m = 1, 2, \dots, p. \quad (A.15)$$

Rearranging eq(A.15) and interchanging the order of expectation and summation, we can obtain

$$E \left\{ \underline{x}(n) + \sum_{k=1}^p A_p^H(k) \underline{x}(n-k) \right\} \underline{x}^H(n-m) = [0] \quad m = 1, 2, \dots, p. \quad (A.16)$$

From eq(A.16), it follows that

$$E[\underline{\epsilon}_p(n) \underline{x}^H(n-m)] = [0] \quad m = 1, 2, \dots, p. \quad (A.17)$$

Eq(A.17) is the principle of orthogonality. It indicates that the MMSE vector $\underline{\epsilon}_p(n)$ is uncorrelated with past data. We can also show that the MMSE vector process $\underline{\epsilon}_p(n)$ is a white noise process. Following a similar, although more general discussion than that presented by Kailath [57], let us consider the observation process $\underline{x}(n)$ such that

$$\underline{x}(n) = \underline{s}(n) + \underline{c}(n) + \underline{w}(n) \quad (A.18a)$$

$$= \underline{z}(n) + \underline{w}(n) \quad (A.18b)$$

where

$$\underline{z}(n) = \underline{s}(n) + \underline{c}(n), \quad (A.18c)$$

$\underline{s}(n)$ and $\underline{c}(n)$ are the signal and non-white clutter processes, respectively, and $\underline{w}(n)$ is a zero-mean, white noise vector. We will assume that the signal is uncorrelated with future clutter samples, but not with past values of the clutter.

In addition, we assume $\mathbf{z}(n)$ is uncorrelated with future white noise samples, but not necessarily with its past values. We now consider the estimate $\hat{\mathbf{x}}(n|n-1)$ such that

$$\hat{\mathbf{x}}(n|n-1) = E[\mathbf{x}(n)|n-1] \quad (\text{A.19a})$$

$$= E[\mathbf{z}(n) + \mathbf{w}(n)|n-1] \quad (\text{A.19b})$$

$$= \hat{\mathbf{z}}(n|n-1) + \hat{\mathbf{w}}(n|n-1) \quad (\text{A.19c})$$

Since $\hat{\mathbf{w}}(n|n-1)$ is the mean of the conditional density function $p_w(\mathbf{w}|n-1)$, and since the mean of $p_w(\mathbf{w}|n-1) = p_w(\mathbf{w})$ is zero, then with $\hat{\mathbf{w}}(n|n-1) = \mathbf{0}$, eq(A.19c) reduces to

$$\hat{\mathbf{x}}(n|n-1) = \hat{\mathbf{z}}(n|n-1) \quad (\text{A.19d})$$

so that

$$E[\boldsymbol{\varepsilon}_p(n)\boldsymbol{\varepsilon}_p^H(k)] = E\{[\mathbf{x}(n) - \hat{\mathbf{x}}(n|n-1)][\mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1)]^H\} \quad (\text{A.20a})$$

$$= E\{[\tilde{\mathbf{z}}(n) + \mathbf{w}(n)][\tilde{\mathbf{z}}(k) + \mathbf{w}(k)]^H\} \quad (\text{A.20b})$$

$$= E[\tilde{\mathbf{z}}(n)\tilde{\mathbf{z}}^H(k)] + E[\mathbf{w}(n)\tilde{\mathbf{z}}^H(k)] \\ + E[\tilde{\mathbf{z}}(n)\mathbf{w}^H(k)] + E[\mathbf{w}(n)\mathbf{w}^H(k)]. \quad (\text{A.20c})$$

where

$$\tilde{\mathbf{z}}(n) = \mathbf{z}(n) - \hat{\mathbf{z}}(n|n-1) \quad (\text{A.21})$$

and the notation $\sim$ denotes error in this section. For $n > k$,

$$E[\mathbf{w}(n)\tilde{\mathbf{z}}^H(k)] = [0] \quad (\text{A.22})$$

so that

$$E[\boldsymbol{\varepsilon}_p(n)\boldsymbol{\varepsilon}_p^H(k)] = E[\tilde{\mathbf{z}}(n)\tilde{\mathbf{z}}^H(k)] + E[\tilde{\mathbf{z}}(n)\mathbf{w}^H(k)] + E[\mathbf{w}(n)\mathbf{w}^H(k)]. \quad (\text{A.23})$$

Since $\hat{\mathbf{z}}(k|k-1)$ is a linear combination of past data up to time $(k-1)$, the orthogonality principle results in $E[\tilde{\mathbf{z}}(n)\hat{\mathbf{z}}^H(k|k-1)] = [0]$. Hence,

$$E[\tilde{\mathbf{z}}(n)\tilde{\mathbf{z}}^H(k)] = E\{\tilde{\mathbf{z}}(n)[\mathbf{z}(k) - \hat{\mathbf{z}}(k|k-1)]^H\} \quad (\text{A.24a})$$

$$= E[\tilde{\mathbf{z}}(n)\mathbf{z}^H(k)] \quad n > k \quad (\text{A.24b})$$

so that

$$E[\mathbf{\varepsilon}_p(n)\mathbf{\varepsilon}_p^H(k)] = E[\tilde{\mathbf{z}}(n)\mathbf{z}^H(k)] + E[\tilde{\mathbf{z}}(n)\mathbf{w}^H(k)] + E[\mathbf{w}(n)\mathbf{w}^H(k)]. \quad (\text{A.25a})$$

$$= E[\tilde{\mathbf{z}}(n)\mathbf{z}^H(k)] + E[\mathbf{w}(n)\mathbf{w}^H(k)]. \quad (\text{A.25b})$$

Again, the orthogonality principle yields $E[\tilde{\mathbf{z}}(n)\mathbf{z}^H(k)] = [0]$ for $n > k$, so that

$$E[\mathbf{\varepsilon}_p(n)\mathbf{\varepsilon}_p^H(k)] = E[\mathbf{w}(n)\mathbf{w}^H(k)] = [0] \quad n > k. \quad (\text{A.26})$$

For $n < k$,

$$E[\tilde{\mathbf{z}}(n)\mathbf{w}^H(k)] = [0] \quad (\text{A.27})$$

so that from eq(20.c)

$$E[\mathbf{\varepsilon}_p(n)\mathbf{\varepsilon}_p^H(k)] = E[\tilde{\mathbf{z}}(n)\tilde{\mathbf{z}}^H(k)] + E[\mathbf{w}(n)\tilde{\mathbf{z}}^H(k)] + E[\mathbf{w}(n)\mathbf{w}^H(k)]. \quad (\text{A.28})$$

But,

$$E[\tilde{\mathbf{z}}(n)\tilde{\mathbf{z}}^H(k)] = E\{[\mathbf{z}(n) - \hat{\mathbf{z}}(n|n-1)]\tilde{\mathbf{z}}^H(k)\} \quad (\text{A.29a})$$

$$= E[\mathbf{z}(n)\tilde{\mathbf{z}}^H(k)] \quad n < k. \quad (\text{A.29b})$$

Now,

$$E[\mathbf{\varepsilon}_p(n)\mathbf{\varepsilon}_p^H(k)] = E[\mathbf{z}(n)\tilde{\mathbf{z}}^H(k)] + E[\mathbf{w}(n)\tilde{\mathbf{z}}^H(k)] + E[\mathbf{w}(n)\mathbf{w}^H(k)]. \quad (\text{A.30a})$$

$$= E[\underline{x}(n) \tilde{\underline{z}}^H(k)] + E[\underline{w}(n) \underline{w}^H(k)]. \quad (\text{A.30b})$$

Again, the orthogonality principle yields $E[\underline{x}(n) \tilde{\underline{z}}^H(k)] = [0]$ for $n < k$, so that

$$E[\underline{\varepsilon}_p(n) \underline{\varepsilon}_p^H(k)] = E[\underline{w}(n) \underline{w}^H(k)] = [0] \quad n < k. \quad (\text{A.31})$$

At $k=n$,

$$\begin{aligned} E[\underline{\varepsilon}_p(n) \underline{\varepsilon}_p^H(n)] &= E[\tilde{\underline{z}}(n) \tilde{\underline{z}}^H(n)] + E[\underline{w}(n) \tilde{\underline{z}}^H(n)] \\ &\quad + E[\tilde{\underline{z}}(n) \underline{w}^H(n)] + E[\underline{w}(n) \underline{w}^H(n)]. \end{aligned} \quad (\text{A.32})$$

But, noting that $\hat{\underline{z}}(k|k-1)$ is a linear combination of past data up to time $(n-1)$, we have

$$E[\underline{w}(n) \tilde{\underline{z}}^H(n)] = E\{ \underline{w}(n) [\underline{z}(n) - \hat{\underline{z}}(n|n-1)]^H \} \quad (\text{A.33a})$$

$$= E[\underline{w}(n) \underline{z}^H(n)] \equiv R_{wz}(0) \quad (\text{A.33b})$$

and

$$E[\tilde{\underline{z}}(n) \underline{w}^H(n)] = E[\underline{z}(n) \underline{w}^H(n)] \equiv R_{zw}(0) = R_{wz}^H(0). \quad (\text{A.34})$$

And so

$$E[\underline{\varepsilon}_p(n) \underline{\varepsilon}_p^H(k)] = \begin{cases} R_{\tilde{\underline{z}}\tilde{\underline{z}}}(0) + R_{wz}(0) + R_{wz}^H(0) + R_{ww}(0) & k=n \\ [0] & k \neq n. \end{cases} \quad (\text{A.35})$$

where

$$R_{\tilde{\underline{z}}\tilde{\underline{z}}}(0) = E[\tilde{\underline{z}}(n) \tilde{\underline{z}}^H(n)]. \quad (\text{A.36})$$

Eq(A.35) indicates that $\mathbf{e}_p(n)$ is a white noise process. The covariance matrices in eq(A.35) can be examined further. When the zero-mean additive white noise $\mathbf{w}(n)$ is uncorrelated with the signal and clutter processes, then

$$\mathbf{R}_{wz}(0) = \mathbf{R}_{zw}(0) = [0] \quad (\text{A.37})$$

so that eq(A.35) becomes

$$\mathbf{E}[\mathbf{e}_p(n)\mathbf{e}_p^H(n)] = \begin{cases} \mathbf{R}_{zz}(0) + \mathbf{R}_{ww}(0) & k=n \\ [0] & k \neq n. \end{cases} \quad (\text{A.38})$$

The covariance matrix $\mathbf{R}_{zz}(0) = \mathbf{E}[\tilde{\mathbf{z}}(n)\tilde{\mathbf{z}}^H(n)]$ can also be considered further as

$$\mathbf{R}_{zz}(0) = \mathbf{E}[\tilde{\mathbf{z}}(n)\tilde{\mathbf{z}}^H(n)] = \mathbf{E}\{[\tilde{\mathbf{s}}(n) + \tilde{\mathbf{c}}(n)][\tilde{\mathbf{s}}(n) + \tilde{\mathbf{c}}(n)]^H\} \quad (\text{A.39a})$$

$$= \mathbf{E}[\tilde{\mathbf{s}}(n)\tilde{\mathbf{s}}^H(n)] + \mathbf{E}[\tilde{\mathbf{s}}(n)\tilde{\mathbf{c}}^H(n)] + \mathbf{E}[\tilde{\mathbf{c}}(n)\tilde{\mathbf{s}}^H(n)] + \mathbf{E}[\tilde{\mathbf{c}}(n)\tilde{\mathbf{c}}^H(n)] \quad (\text{A.39b})$$

$$= \mathbf{R}_{ss}(0) + \mathbf{R}_{sc}(0) + \mathbf{R}_{cs}(0) + \mathbf{R}_{cc}(0) \quad (\text{A.39c})$$

where

$$\mathbf{R}_{ss}(0) = \mathbf{E}[\tilde{\mathbf{s}}(n)\tilde{\mathbf{s}}^H(n)] \quad (\text{A.40a})$$

$$\mathbf{R}_{sc}(0) = \mathbf{E}[\tilde{\mathbf{s}}(n)\tilde{\mathbf{c}}^H(n)] \quad (\text{A.40b})$$

$$\mathbf{R}_{cs}(0) = \mathbf{E}[\tilde{\mathbf{c}}(n)\tilde{\mathbf{s}}^H(n)] \quad (\text{A.40c})$$

$$\mathbf{R}_{cc}(0) = \mathbf{E}[\tilde{\mathbf{c}}(n)\tilde{\mathbf{c}}^H(n)]. \quad (\text{A.40d})$$

Finally, although the MMSE vector $\mathbf{e}_p(n)$ has been shown to be temporally white, note that its components can be correlated across the channels.

APPENDIX B

DERIVATION OF EQS(2.5.74) AND (2.5.84)

In this appendix, we derive eqs(2.5.74) for $J=2$ and (2.5.84) for $J=3$.
Using $J=2$ in eq(2.5.71a), we have

$$P = (-1)^0 \mathcal{A}_0 \mathcal{C} \mathcal{B} + (-1)^1 \mathcal{A}_1 \mathcal{C} \mathcal{B}^0 = \mathcal{C} \mathcal{B} - \mathcal{A}_1 \mathcal{C} \quad (\text{B.1a})$$

$$= \mathcal{C} \mathcal{B} - (\mathcal{A} \mathcal{A}_0 + \alpha_1 I) \mathcal{C} \quad (\text{B.1b})$$

$$= \mathcal{C} \mathcal{B} - [\mathcal{A} - \text{tr}(\mathcal{A} \mathcal{A}_0) I] \mathcal{C} \quad (\text{B.1c})$$

$$= \mathcal{C} \mathcal{B} - [\mathcal{A} - \text{tr}(\mathcal{A}) I] \mathcal{C} \quad (\text{B.1d})$$

which is eq((2.5.74a). Using $J=2$ in eq(2.5.71b)

$$Q = (-1)^0 \alpha_0 \mathcal{B}^2 + (-1)^1 \alpha_1 \mathcal{B}^1 + (-1)^2 \alpha_2 \mathcal{B}^0 \quad (\text{B.2a})$$

$$= \mathcal{B} \mathcal{B} + \text{tr}(\mathcal{A} \mathcal{A}_0) \mathcal{B} - \frac{1}{2} \text{tr}(\mathcal{A} \mathcal{A}_1) I \quad (\text{B.2b})$$

From eq(2.5.73a)

$$\mathcal{A}_1 = \mathcal{A} \mathcal{A}_0 + \alpha_1 I. \quad (\text{B.3})$$

Using eq(B.3) in (B.2b)

$$Q = \mathcal{B} \mathcal{B} + \text{tr}(\mathcal{A}) \mathcal{B} - \frac{1}{2} \text{tr}[\mathcal{A}(\mathcal{A} \mathcal{A}_0 + \alpha_1 I)] I. \quad (\text{B.4})$$

From eq(2.5.73b)

$$\mathcal{A}_0 = I \quad (\text{B.5})$$

and from eq(2.5.72a)

$$\alpha_1 = -\text{tr}(\mathcal{A}). \quad (\text{B.6})$$

Using these equations in (B.4)

$$Q = \mathcal{B} \mathcal{B} + \text{tr}(\mathcal{A}) \mathcal{B} - \frac{1}{2} \text{tr}\{\mathcal{A}[\mathcal{A} - \text{tr}(\mathcal{A}) I]\} I \quad (\text{B.7})$$

which is eq(2.5.74b). Using $J=3$ in eq(2.5.71a), we have

$$P = (-1)^0 \mathcal{A}_0 \mathcal{C} \mathcal{B}^2 + (-1)^1 \mathcal{A}_1 \mathcal{C} \mathcal{B} + (-1)^2 \mathcal{A}_2 \mathcal{C} \mathcal{B}^0. \quad (\text{B.8})$$

Using eq(2.5.73a), eq(B.8) reduces to

$$P = \mathcal{C} \mathcal{B}^2 - [\mathcal{A} \mathcal{A}_0 + \alpha_1 I] \mathcal{C} \mathcal{B} + [\mathcal{A} \mathcal{A}_1 + \alpha_2 I] \mathcal{C} \quad (\text{B.9a})$$

$$= \mathcal{C} \mathcal{B}^2 - [\mathcal{A} - \text{tr}(\mathcal{A}) I] \mathcal{C} \mathcal{B} + \{ \mathcal{A} [\mathcal{A} - \text{tr}(\mathcal{A}) I] - \frac{1}{2} \text{tr}(\mathcal{A} \mathcal{A}_1) I \} \mathcal{C} \quad (\text{B.9b})$$

$$= \mathcal{C} \mathcal{B}^2 - [\mathcal{A} - \text{tr}(\mathcal{A}) I] \mathcal{C} \mathcal{B} + \{ \mathcal{A}^2 - \mathcal{A} \text{tr}(\mathcal{A}) - \frac{1}{2} \text{tr}[\mathcal{A}^2 - \mathcal{A} \text{tr}(\mathcal{A})] I \} \mathcal{C} \quad (\text{B.9c})$$

$$= \mathcal{C} \mathcal{B}^2 - [\mathcal{A} - \text{tr}(\mathcal{A}) I] \mathcal{C} \mathcal{B} + \{ \mathcal{A}^2 - \mathcal{A} \text{tr}(\mathcal{A}) - \frac{1}{2} \text{tr}(\mathcal{A}^2) I + \frac{1}{2} \text{tr}[\mathcal{A} \text{tr}(\mathcal{A})] I \} \mathcal{C}. \quad (\text{B.9d})$$

From eq(2.5.77b)

$$\mathcal{B}^2 = (\text{tr} \mathcal{B}) \mathcal{B} - (\det \mathcal{B}) I \quad (\text{B.10})$$

with a similar expression for $\mathcal{A}^2$ so that

$$\begin{aligned} P = & \mathcal{C} [(\text{tr} \mathcal{B}) \mathcal{B} - (\det \mathcal{B}) I] - [\mathcal{A} - (\text{tr} \mathcal{A}) I] \mathcal{C} \mathcal{B} \\ & + \{ (\text{tr} \mathcal{A}) \mathcal{A} - (\det \mathcal{A}) I - \mathcal{A} (\text{tr} \mathcal{A}) - \frac{1}{2} \text{tr}[(\text{tr} \mathcal{A}) \mathcal{A}] I \\ & + \frac{1}{2} \text{tr}[(\det \mathcal{A}) I] I + \frac{1}{2} \text{tr}[\mathcal{A} \text{tr}(\mathcal{A})] I \} \mathcal{C}. \end{aligned} \quad (\text{B.11})$$

Since

$$\text{tr}[(\det \mathcal{A}) I] = [J(\det \mathcal{A}) I] = 3(\det \mathcal{A}) I, \quad \text{for } J=3, \quad (\text{B.12})$$

then

$$P = \mathcal{C} [(\text{tr} \mathcal{B}) \mathcal{B} - (\det \mathcal{B}) I] - \mathcal{A} \mathcal{C} \mathcal{B} + [\text{tr} \mathcal{A}] \mathcal{C} \mathcal{B} + \{ -(\det \mathcal{A}) I + \frac{3}{2} (\det \mathcal{A}) I \} \mathcal{C} \quad (\text{B.13a})$$

$$= [\text{tr} \mathcal{A} + \text{tr} \mathcal{B}] \mathcal{C} \mathcal{B} - \mathcal{C} (\det \mathcal{B}) - \mathcal{A} \mathcal{C} \mathcal{B} + \frac{1}{2} (\det \mathcal{A}) \mathcal{C} \quad (\text{B.13b})$$

$$= [\text{tr} \mathcal{A} + \text{tr} \mathcal{B}] \mathcal{C} \mathcal{B} + [\frac{1}{2} (\det \mathcal{A}) - (\det \mathcal{B})] \mathcal{C} - \mathcal{A} \mathcal{C} \mathcal{B} \quad (\text{B.13c})$$

which is eq(2.5.84a). Similarly, eq(2.5.71b) for $J=3$ yields

$$Q = (-1)^0 \alpha_0 \mathcal{B}^3 + (-1)^1 \alpha_1 \mathcal{B}^2 + (-1)^2 \alpha_2 \mathcal{B}^1 + (-1)^3 \alpha_3 \mathcal{B}^0 \quad (\text{B.14a})$$

$$= \mathcal{B}^3 - [-\text{tr}(\mathcal{A} \mathcal{A}_0)] \mathcal{B}^2 + [-\frac{1}{2} \text{tr}(\mathcal{A} \mathcal{A}_1)] \mathcal{B} - [-\frac{1}{3} \text{tr}(\mathcal{A} \mathcal{A}_2)] I \quad (\text{B.14b})$$

$$= \mathbf{B}^3 + [\text{tr}(\mathbf{A})]\mathbf{B}^2 - \frac{1}{2} \text{tr}\{\mathbf{A}[\mathbf{A}\mathbf{A}_0 + \alpha_1 \mathbf{I}]\}\mathbf{B} \\ + \frac{1}{3} \text{tr}\{\mathbf{A}[\mathbf{A}\mathbf{A}_1 - \frac{1}{2} \text{tr}(\mathbf{A}\mathbf{A}_1)\mathbf{I}]\}\mathbf{I} \quad (\text{B.14c})$$

$$= \mathbf{B}^3 + [\text{tr}(\mathbf{A})]\mathbf{B}^2 - \frac{1}{2} \text{tr}(\mathbf{A}^2)\mathbf{B} - \frac{1}{2} \text{tr}\{\mathbf{A}[-\text{tr}(\mathbf{A})]\}\mathbf{B} \\ + \frac{1}{3} \text{tr}\{\mathbf{A}^2[\mathbf{A}\mathbf{A}_0 + \alpha_1 \mathbf{I}]\}\mathbf{I} - \frac{1}{6} \text{tr}\{\mathbf{A} \text{tr}\{\mathbf{A}[\mathbf{A}\mathbf{A}_0 + \alpha_1 \mathbf{I}]\}\mathbf{I}\} \quad (\text{B.14d})$$

$$= \mathbf{B}^3 + [\text{tr}(\mathbf{A})]\mathbf{B}^2 - \frac{1}{2} \text{tr}(\mathbf{A}^2)\mathbf{B} + \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{B} + \frac{1}{3} \text{tr}(\mathbf{A}^3)\mathbf{I} \\ + \frac{1}{3} \text{tr}\{\mathbf{A}^2[-\text{tr}\mathbf{A}]\}\mathbf{I} - \frac{1}{6} \text{tr}\{\mathbf{A}[\text{tr}(\mathbf{A}^2)]\}\mathbf{I} - \frac{1}{6} \text{tr}\{\mathbf{A} \text{tr}[\mathbf{A}(-\text{tr}\mathbf{A})]\}\mathbf{I} \quad (\text{B.14e})$$

$$= \mathbf{B}^3 + [\text{tr}(\mathbf{A})]\mathbf{B}^2 - \frac{1}{2} \text{tr}(\mathbf{A}^2)\mathbf{B} + \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{B} + \frac{1}{3} \text{tr}(\mathbf{A}^3)\mathbf{I} \\ - \frac{1}{3} \text{tr}(\mathbf{A}^2)\text{tr}(\mathbf{A})\mathbf{I} - \frac{1}{6} \text{tr}(\mathbf{A})\text{tr}(\mathbf{A}^2)\mathbf{I} + \frac{1}{6} [\text{tr}\mathbf{A}]^3 \mathbf{I} \quad (\text{B.14f})$$

$$= \mathbf{B}^3 + [\text{tr}(\mathbf{A})]\mathbf{B}^2 - \frac{1}{2} \text{tr}(\mathbf{A}^2)\mathbf{B} + \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{B} + \frac{1}{3} \text{tr}(\mathbf{A}^3)\mathbf{I} \\ - \frac{1}{2} \text{tr}(\mathbf{A}^2)\text{tr}(\mathbf{A})\mathbf{I} + \frac{1}{6} [\text{tr}\mathbf{A}]^3 \mathbf{I}. \quad (\text{B.14g})$$

However,

$$\mathbf{B}^2 = (\text{tr}\mathbf{B})\mathbf{B} - (\det\mathbf{B})\mathbf{I} \quad (\text{B.15})$$

with a similar expression for $\mathbf{A}^2$ so that

$$\mathbf{Q} = (\text{tr}\mathbf{B})\mathbf{B}^2 - (\det\mathbf{B})\mathbf{B} + (\text{tr}\mathbf{A})\mathbf{B}^2 - \frac{1}{2} \text{tr}[(\text{tr}\mathbf{A})\mathbf{A} - (\det\mathbf{A})\mathbf{I}]\mathbf{B} + \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{B} \\ + \frac{1}{3} \text{tr}\{\mathbf{A}[(\text{tr}\mathbf{A})\mathbf{A} - (\det\mathbf{A})\mathbf{I}]\} - \frac{1}{2} \text{tr}[(\text{tr}\mathbf{A})\mathbf{A} - (\det\mathbf{A})\mathbf{I}]\text{tr}(\mathbf{A})\mathbf{I} \\ + \frac{1}{6} [\text{tr}\mathbf{A}]^3 \mathbf{I}. \quad (\text{B.16a})$$

$$= [\text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})]\mathbf{B}^2 - (\det\mathbf{B})\mathbf{B} - \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{B} + \frac{1}{2} \text{tr}[(\det\mathbf{A})\mathbf{I}]\mathbf{B} + \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{B} \\ + \frac{1}{3} (\text{tr}\mathbf{A})\text{tr}(\mathbf{A}^2)\mathbf{I} - \frac{1}{3} \text{tr}\{(\det\mathbf{A})\mathbf{I}\}\mathbf{I} - \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{I} + \frac{1}{2} \text{tr}[(\det\mathbf{A})\mathbf{I}]\text{tr}(\mathbf{A})\mathbf{I} \\ + \frac{1}{6} [\text{tr}\mathbf{A}]^3 \mathbf{I}. \quad (\text{B.16b})$$

Using eq(2.5.79b)

$$\mathbf{Q} = [\text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})]\mathbf{B}^2 - (\det\mathbf{B})\mathbf{B} - \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{B} + \frac{1}{2} \text{tr}[(\det\mathbf{A})\mathbf{I}]\mathbf{B} + \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{B} \\ + \frac{1}{3} (\text{tr}\mathbf{A})[(\text{tr}\mathbf{A})^2 - \text{tr}(\det\mathbf{A})]\mathbf{I} - \frac{1}{3} \text{tr}\{(\det\mathbf{A})\mathbf{I}\}\mathbf{I} - \frac{1}{2} (\text{tr}\mathbf{A})^2 \mathbf{I} \\ + \frac{1}{2} \text{tr}[(\det\mathbf{A})\mathbf{I}]\text{tr}(\mathbf{A})\mathbf{I} + \frac{1}{6} [\text{tr}\mathbf{A}]^3 \mathbf{I}. \quad (\text{B.17a})$$

$$\begin{aligned}
&= [\text{tr}(\mathcal{A}) + \text{tr}(\mathcal{B})]\mathcal{B}^2 - (\det \mathcal{B})\mathcal{B} + \frac{1}{2} \text{tr}[(\det \mathcal{A})\mathcal{I}]\mathcal{B} + \frac{1}{6} (\text{tr} \mathcal{A})\text{tr}(\det \mathcal{A})\mathcal{I} \\
&\quad - \frac{1}{3} \text{tr}(\det \mathcal{A})\mathcal{I} - \frac{1}{2}(\text{tr} \mathcal{A})^2\mathcal{I} + \frac{1}{2} [\text{tr} \mathcal{A}]^3\mathcal{I}
\end{aligned} \tag{B.17b}$$

Using eqs(B.12) and (B.15) in (B.17b) and collecting terms

$$\begin{aligned}
Q &= [\text{tr} \mathcal{A} + \text{tr} \mathcal{B}][(\text{tr} \mathcal{B})\mathcal{B} - (\det \mathcal{B})\mathcal{I}] - (\det \mathcal{B})\mathcal{B} + \frac{3}{2} (\det \mathcal{A})\mathcal{B} + \frac{1}{2} (\text{tr} \mathcal{A})(\det \mathcal{A})\mathcal{I} \\
&\quad - (\det \mathcal{A})\mathcal{I} - \frac{1}{2}(\text{tr} \mathcal{A})^2\mathcal{I} + \frac{1}{2} [\text{tr} \mathcal{A}]^3\mathcal{I}
\end{aligned} \tag{B.18}$$

which is eq(2.5.84b).

APPENDIX C

DERIVATION OF CONSTRAINT ON $|\rho_{gij}|$ FOR A GAUSSIAN SHAPED CROSS-CORRELATION FUNCTION

In this appendix, we derive the constraint expressed by eq(4.4.4). We consider the specific example of the Gaussian shaped cross-correlation function. Let us define $\mathcal{K}_{gij}$ such that

$$\mathcal{K}_{gij} = |\rho_{gij}| \sigma_{gii} \sigma_{gjj} \quad (C.1)$$

and use this expression in eq(4.3.6) so that

$$R_{gij}(l) = \frac{\mathcal{K}_{gij} (\lambda_{gij})^{(l+|gij|)^2}}{(\lambda_{gij})^{|gij|}} \exp[j\theta_{gij}(l)]. \quad (C.2)$$

With reference to eq(4.3.3), note that

$$|R_{gji}(l)| = |R_{gji}^*(l)| \quad (C.3a)$$

and

$$\theta_{gji}(l) = -\theta_{gij}(-l). \quad (C.3b)$$

It follows that

$$R_{gji}(l) = \frac{\mathcal{K}_{gij} (\lambda_{gij})^{(l+|gij|)^2}}{(\lambda_{gij})^{|gij|}} \exp[-j\theta_{gij}(-l)]. \quad (C.4)$$

Inserting eqs(C.2) and (C.4) in constraint eq(4.4.3), we obtain

$$\left[\frac{\kappa_{g_{ij}}(\lambda_{g_{ij}})^2(l-l_{g_{ij}})^2}{(\lambda_{g_{ij}})^2 l_{g_{ij}}} \exp[j\theta_{g_{ij}}(l)] + \frac{\kappa_{g_{ij}}(\lambda_{g_{ij}})^2(-l+l_{g_{ij}})^2}{(\lambda_{g_{ij}})^2 l_{g_{ij}}} \exp[-j\theta_{g_{ij}}(l)] \right]^2 \leq 4\sigma_{g_{ii}}^2 \sigma_{g_{jj}}^2 \quad (C.5)$$

Noting that $(-l + l_{g_{ij}})^2 = (l - l_{g_{ij}})^2$, we have

$$\frac{4(\kappa_{g_{ij}})^2(\lambda_{g_{ij}})^2(l-l_{g_{ij}})^2}{(\lambda_{g_{ij}})^2 l_{g_{ij}}} \cos^2[\theta_{g_{ij}}(l)] \leq 4\sigma_{g_{ii}}^2 \sigma_{g_{jj}}^2 \quad (C.6)$$

so that from eq(C.1),

$$\frac{|\rho_{g_{ij}}|^2(\lambda_{g_{ij}})^2(l-l_{g_{ij}})^2}{(\lambda_{g_{ij}})^2 l_{g_{ij}}} \cos^2[\theta_{g_{ij}}(l)] \leq 1. \quad (C.7)$$

The most stringent condition for this constraint equation occurs when $l = l_{g_{ij}}$, so that

$$|\rho_{g_{ij}}|^2 \leq \frac{(\lambda_{g_{ij}})^2 l_{g_{ij}}^2}{\cos^2[\theta_{g_{ij}}(l_{g_{ij}})]} \quad (C.8)$$

Taking the square root of both sides of this equation, we have the constraint

$$|\rho_{g_{ij}}| \leq \pm \frac{(\lambda_{g_{ij}}) l_{g_{ij}}}{|\cos[\theta_{g_{ij}}(l_{g_{ij}})]|} \quad (C.9)$$

Since $|p_{g_{ij}}|$ is positive, we have

$$0 \leq |p_{g_{ij}}| \leq \frac{(\lambda_{g_{ij}})^2 |g_{ij}|}{|\cos[\theta_{g_{ij}}(l_{g_{ij}})]|} . \quad (C.10)$$

APPENDIX D DERIVATION OF EQ(5.2.3)

In this appendix, we derive the expression for the variance of $\hat{R}_{iiT}(l, N)$ expressed by eq(5.2.3). Consider the biased time-averaged estimate of the autocorrelation function; ie.,

$$\hat{R}_{iiT_b}(l, N_T) = \begin{cases} \frac{1}{N_T} \sum_{n=0}^{N_T-l-1} x_i(n) x_i^*(n-l) & 0 \leq l \leq N_T-1 \\ \frac{1}{N_T} \sum_{n=0}^{N_T-|l|-1} x_i^*(n) x_i(n-|l|) & -(N_T-1) \leq l \leq 0. \end{cases} \quad (D.1)$$

where the symbol $\wedge$ in this discussion designates the quantity as an estimate. Let

$$\phi(n, l) = x_i(n) x_i^*(n - l). \quad (D.2)$$

Assuming stationarity, the covariance of $\phi(n, l)$ can be expressed as

$$C_{\phi\phi}(k, l) = E[\{ \phi(n, l) - E[\phi(n, l)] \} \{ \phi^*(n-k, l) - E[\phi^*(n-k, l)] \}] \quad (D.3a)$$

$$= E[\phi(n, l) \phi^*(n-k, l)] - E[\phi(n, l)] E[\phi^*(n-k, l)] \\ - E[\phi(n, l)] E[\phi^*(n-k, l)] + E[\phi(n, l)] E[\phi^*(n-k, l)] \quad (D.3b)$$

$$= R_{\phi\phi}(k, l) - E[\phi(n, l)] E[\phi^*(n-k, l)] \quad (D.3c)$$

where

$$R_{\phi\phi}(k, l) = E[\phi(n, l) \phi^*(n - k, l)]. \quad (D.4)$$

Assuming stationarity, we have from eq(D.2),

$$E[\phi(n,l)] = R_{ii}(l) \quad (D.5a)$$

and

$$E[\phi^*(n-k,l)] = R_{ii}^*(l) \quad (D.5b)$$

so that

$$C_{\phi\phi}(k,l) = R_{\phi\phi}(k,l) - |R_{ii}(l)|^2 \quad (D.6a)$$

$$= E[x_i(n)x_i^*(n-l)x_i^*(n-k)x_i(n-l-k)] - |R_{ii}(l)|^2. \quad (D.6b)$$

Now consider the variance of the complex estimate $\hat{R}_{iiT}(l,N)$ which can be expressed as

$$V_{ii}(l,N) = E \left\{ [\hat{R}_{iiT}(l,N) - E[\hat{R}_{iiT}(l,N)]] [\hat{R}_{iiT}^*(l,N) - E[\hat{R}_{iiT}^*(l,N)]] \right\} \quad (D.7a)$$

$$= E[\hat{R}_{iiT}(l,N)\hat{R}_{iiT}^*(l,N)] - E[\hat{R}_{iiT}(l,N)]E[\hat{R}_{iiT}^*(l,N)]. \quad (D.7b)$$

Using eq(D.1) for positive and negative l , we have

$$\begin{aligned} \hat{R}_{iiT_b}(l,N_T)\hat{R}_{iiT_b}^*(l,N_T) &= \\ &= \begin{cases} \frac{1}{N_T^2} \sum_{n=0}^{N_T-l-1} \sum_{p=0}^{N_T-l-1} x_i(n)x_i^*(n-l)x_i^*(p)x_i(p-l) & 0 \leq l \leq N_T - 1 \\ \frac{1}{N_T^2} \sum_{n=0}^{N_T-|l|-1} \sum_{p=0}^{N_T-|l|-1} x_i(n)x_i(n-|l|)x_i^*(p)x_i^*(p-|l|) & -(N_T - 1) \leq l \leq 0 \end{cases} \end{aligned} \quad (D.8)$$

so that

$$E[\hat{R}_{iiT_b}(l, N_T) \hat{R}_{iiT_b}^*(l, N_T)] =$$

$$= \begin{cases} \frac{1}{N_T^2} \sum_{n=0}^{N_T-l-1} \sum_{p=0}^{N_T-l-1} E[x_i(n) x_i^*(n-l) x_i(p) x_i^*(p-l)] & 0 \leq l \leq N_T - 1 \\ \frac{1}{N_T^2} \sum_{n=0}^{N_T-||l|-1} \sum_{p=0}^{N_T-||l|-1} E[x_i^*(n) x_i(n-||l|) x_i(p) x_i^*(p-||l|)] & -(N_T-1) \leq l \leq 0. \end{cases} \quad (D.9)$$

Also,

$$E[\hat{R}_{iiT_b}(l, N_T)] = \begin{cases} \frac{1}{N_T} \sum_{n=0}^{N_T-l-1} R_{ii}(l) & 0 \leq l \leq N_T - 1 \\ \frac{1}{N_T} \sum_{n=0}^{N_T-||l|-1} R_{ii}(l) & -(N_T-1) \leq l \leq 0 \end{cases} \quad (D.10)$$

so that

$$E[\hat{R}_{iiT_b}(l, N_T)] E[\hat{R}_{iiT_b}^*(l, N_T)] =$$

$$= \begin{cases} \frac{1}{N_T^2} \sum_{n=0}^{N_T-l-1} \sum_{p=0}^{N_T-l-1} |R_{ii}(l)|^2 & 0 \leq l \leq N_T \\ \frac{1}{N_T^2} \sum_{n=0}^{N_T-||l|-1} \sum_{p=0}^{N_T-||l|-1} |R_{ii}(l)|^2 & -(N_T-1) \leq l \leq 0. \end{cases} \quad (D.11)$$

Using eqs(D.9) and (D.11) in the expression

$$V_{B_{ii}}(l, N_T) = E[\hat{R}_{iiT_b}(l, N_T) \hat{R}_{iiT_b}^*(l, N_T)] - E[\hat{R}_{iiT_b}(l, N_T)] E[\hat{R}_{iiT_b}^*(l, N_T)] \quad (D.12)$$

we obtain

$$\begin{aligned}
V_{B_{ii}}(l, N_T) = & \left\{ \begin{aligned} & \frac{1}{N_T^2} \sum_{n=0}^{N_T-l-1} \sum_{p=0}^{N_T-l-1} \{ E[x_i(n)x_i^*(n-l)x_i^*(p)x_i(p-l)] - |R_{ii}(l)|^2 \} \\ & \text{for } 0 \leq l \leq N_T - 1 \\ & \frac{1}{N_T^2} \sum_{n=0}^{N_T-|l|-1} \sum_{p=0}^{N_T-|l|-1} \{ E[x_i^*(n)x_i(n-|l|)x_i^*(p)x_i(p-|l|)] - |R_{ii}(l)|^2 \} \\ & \text{for } -(N_T-1) \leq l \leq 0. \end{aligned} \right. \quad (D.13)
\end{aligned}$$

Using eq(D.6b) in (D.13)

$$V_{B_{ii}}(l, N_T) = \left\{ \begin{aligned} & \frac{1}{N_T^2} \sum_{n=0}^{N_T-l-1} \sum_{p=0}^{N_T-l-1} C_{\phi\phi}(n-p, l) \quad 0 \leq l \leq N_T - 1 \\ & \frac{1}{N_T^2} \sum_{n=0}^{N_T-|l|-1} \sum_{p=0}^{N_T-|l|-1} C_{\phi\phi}(n-p, |l|) \quad -(N_T-1) \leq l \leq 0. \end{aligned} \right. \quad (D.14)$$

We now let $k = n - p$ where

$$-(N_T - l - 1) \leq k \leq N_T - l - 1 \quad \text{for } 0 \leq l \leq N_T - 1 \quad (D.15a)$$

$$-(N_T - |l| - 1) \leq k \leq N_T - |l| - 1 \quad \text{for } -(N_T-1) \leq l \leq 0. \quad (D.15b)$$

We also note that eq(D.15b) is equivalent to eq(D.15a) for negative values of l so that

$$\begin{aligned}
V_{B_{ii}}(l, N_T) &= \\
&= \begin{cases} \frac{1}{N_T^2} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} [N_T - |l| - |k|] C_{\phi\phi}(k, l) & 0 \leq l \leq N_T - 1 \\ \frac{1}{N_T^2} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} [N_T - |l| - |k|] C_{\phi\phi}^*(k, |l|) & -(N_T-1) \leq l \leq 0. \end{cases} \quad (D.16)
\end{aligned}$$

However, for negative lag l , we have

$$C_{\phi\phi}^*(k, |l|) = C_{\phi\phi}(k, l) \quad (D.17)$$

so that after dividing the bracketed factor by one of the N_T terms in the denominator

$$V_{B_{ii}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l| + |k|}{N_T} \right] C_{\phi\phi}(k, l) \quad (D.18)$$

for both positive and negative values of l . In Appendix G, we show that the imaginary terms in $C_{\phi\phi}(k, l)$ cancel when summed over positive and negative values of k so that

$$V_{B_{ii}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l| + |k|}{N_T} \right] \text{Re} \{ C_{\phi\phi}(k, l) \}. \quad (D.19)$$

APPENDIX E DERIVATION OF EQ(5.2.4)

In this Appendix, we derive eq(5.2.4). From eq(5.2.2a)

$$E[\phi(n,l)] = R_{ii}(l) \quad (E.1a)$$

and

$$E[\phi^*(n-k,l)] = R_{ii}^*(l) \quad (E.1b)$$

so that from eq(5.2.2d)

$$C_{\phi\phi}(k,l) = R_{\phi\phi}(k,l) - |R_{ii}(l)|^2. \quad (E.2)$$

And so, eq(5.2.3c) becomes

$$V_{B_{ii}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T} \right] [R_{\phi\phi}(k,l) - |R_{ii}(l)|^2]. \quad (E.3)$$

We now consider

$$R_{\phi\phi}(k,l) = E[\phi(n,l)\phi^*(n-k,l)] \quad (E.4a)$$

$$= E[x_i(n)x_i^*(n-l)x_i^*(n-k)x_i(n-l-k)]. \quad (E.4b)$$

For processes with zero-mean, jointly stationary Gaussian quadrature components $x_{iI}(n)$ and $x_{iQ}(n)$, eq(E.4b) can be expressed as [see Appendix F]

$$R_{\phi\phi}(k,l) = |R_{ii}(l)|^2 + |R_{ii}(k)|^2 + F_{ii}(l,k) \quad (E.5)$$

where

$$\begin{aligned}
F_{ii}(l,k) = & \{ R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k) \} \{ R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k) \} \\
& + \{ R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k) \} \{ R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k) \} \\
& - j \{ R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k) \} \{ R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k) \} \\
& + j \{ R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k) \} \{ R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k) \}.
\end{aligned} \tag{E.6}$$

Using eq(E.5) in (E.3), we have

$$V_{B_{ii}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-||l|-1)}^{N_T-||l|-1} \left[1 - \frac{||l|+|k|}{N_T} \right] [|R_{ii}(k)|^2 + F_{ii}(l,k)]. \tag{E.7}$$

By examination of eq(E.6), we note that the imaginary terms in eq(E.7) sum to zero. This can be seen by first noting that $F_{ii}(l,k)$ is real for $k=0$. We also note that imaginary terms evaluated with negative values of k serve to cancel the corresponding imaginary terms for positive values of k . And so, only the real part of the function $F_{ii}(l,k)$ contributes to the $V_{B_{ii}}(l, N_T)$ function. Therefore,

$$V_{B_{ii}}(l, N_T) = \frac{1}{N_T} \sum_{k=-(N_T-||l|-1)}^{N_T-||l|-1} \left[1 - \frac{||l|+|k|}{N_T} \right] [|R_{ii}(k)|^2 + \text{Re}\{F_{ii}(l,k)\}] \tag{E.8}$$

which is eq(5.2.4) in the text. In Appendix G, it is shown that

$$F_{ii}(l,k) = E[x_i(n)x_i(n-l-k)]E[x_i^*(n-l)x_i^*(n-k)]. \tag{E.9}$$

APPENDIX F DERIVATION OF EQ(E.5) IN APPENDIX E

In this appendix, we derive eq(E.5) of Appendix E. Consider eq(E.4b) expressed as

$$R_{\phi\phi}(k,l) = E[x_i(n)x_i^*(n-l)x_i^*(n-k)x_i(n-l-k)]. \quad (F.1)$$

In the special case where the process $x_i(n)$ is complex Gaussian, then [167]

$$\begin{aligned} R_{\phi\phi}(k,l) &= E[x_i(n)x_i^*(n-l)]E[x_i^*(n-k)x_i(n-l-k)] \\ &+ E[x_i(n)x_i^*(n-k)]E[x_i^*(n-l)x_i(n-l-k)]. \end{aligned} \quad (F.2)$$

where use has been made of the fact that $E[x_i(n)x_i(k)] = 0$ for $n \neq k$. However, we do not wish to constrain this discussion to this restrictive case. Rather, we wish to consider the more general case of a Gaussian process $x_i(n)$ with unconstrained quadrature components. We therefore consider

$$x_i(n) = x_{iI}(n) + j x_{iQ}(n) \quad (F.3)$$

where the processes $x_{iI}(n)$ and $x_{iQ}(n)$ are jointly Gaussian. Using eq(F.3) in (F.1), we obtain

$$\begin{aligned} R_{\phi\phi}(k,l) &= E \{ [x_{iI}(n) + j x_{iQ}(n)][x_{iI}(n-l) - j x_{iQ}(n-l)] \\ &\quad \cdot [x_{iI}(n-k) - j x_{iQ}(n-k)][x_{iI}(n-l-k) + j x_{iQ}(n-l-k)] \} \end{aligned} \quad (F.4a)$$

$$\begin{aligned} &= E \{ [x_{iI}(n)x_{iI}(n-l) + x_{iQ}(n)x_{iQ}(n-l) + jx_{iQ}(n)x_{iI}(n-l) - jx_{iI}(n)x_{iQ}(n-l)] \\ &\quad \cdot [x_{iI}(n-k)x_{iI}(n-l-k) + x_{iQ}(n-k)x_{iQ}(n-l-k) \\ &\quad + jx_{iI}(n-k)x_{iQ}(n-l-k) - jx_{iQ}(n-k)x_{iI}(n-l-k)] \} \end{aligned} \quad (F.4b)$$

$$\begin{aligned}
&= E[x_{iI}(n)x_{iI}(n-l)x_{iI}(n-k)x_{iI}(n-l-k)] + E[x_{iQ}(n)x_{iQ}(n-l)x_{iI}(n-k)x_{iI}(n-l-k)] \\
&+ E[x_{iI}(n)x_{iI}(n-l)x_{iQ}(n-k)x_{iQ}(n-l-k)] + E[x_{iQ}(n)x_{iQ}(n-l)x_{iQ}(n-k)x_{iQ}(n-l-k)] \\
&- E[x_{iQ}(n)x_{iI}(n-l)x_{iI}(n-k)x_{iQ}(n-l-k)] + E[x_{iI}(n)x_{iQ}(n-l)x_{iI}(n-k)x_{iQ}(n-l-k)] \\
&+ E[x_{iQ}(n)x_{iI}(n-l)x_{iQ}(n-k)x_{iI}(n-l-k)] - E[x_{iI}(n)x_{iQ}(n-l)x_{iQ}(n-k)x_{iI}(n-l-k)] \\
&+ jE[x_{iI}(n)x_{iI}(n-l)x_{iI}(n-k)x_{iQ}(n-l-k)] - jE[x_{iI}(n)x_{iI}(n-l)x_{iQ}(n-k)x_{iI}(n-l-k)] \\
&+ jE[x_{iQ}(n)x_{iQ}(n-l)x_{iI}(n-k)x_{iQ}(n-l-k)] - jE[x_{iQ}(n)x_{iQ}(n-l)x_{iQ}(n-k)x_{iI}(n-l-k)] \\
&+ jE[x_{iQ}(n)x_{iI}(n-l)x_{iI}(n-k)x_{iI}(n-l-k)] + jE[x_{iQ}(n)x_{iI}(n-l)x_{iQ}(n-k)x_{iQ}(n-l-k)] \\
&- jE[x_{iI}(n)x_{iQ}(n-l)x_{iI}(n-k)x_{iI}(n-l-k)] - jE[x_{iI}(n)x_{iQ}(n-l)x_{iQ}(n-k)x_{iQ}(n-l-k)]
\end{aligned}
\tag{F.4c}$$

For Gaussian, zero-mean quadrature components, eq(F.4c) can be expressed as

$$\begin{aligned}
R_{\phi\phi}(k,l) &= [R_{ii}^{II}(l)]^2 + [R_{ii}^{II}(k)]^2 + R_{ii}^{II}(l+k)R_{ii}^{II}(k-l) \\
&+ R_{ii}^{QQ}(l)R_{ii}^{II}(l) + [R_{ii}^{QI}(k)]^2 + R_{ii}^{QI}(l+k)R_{ii}^{QI}(k-l) \\
&+ R_{ii}^{II}(l)R_{ii}^{QQ}(l) + [R_{ii}^{IQ}(k)]^2 + R_{ii}^{IQ}(l+k)R_{ii}^{IQ}(k-l) \\
&+ [R_{ii}^{QQ}(l)]^2 + [R_{ii}^{QQ}(k)]^2 + R_{ii}^{QQ}(l+k)R_{ii}^{QQ}(k-l) \\
&- R_{ii}^{QI}(l)R_{ii}^{IQ}(l) - R_{ii}^{QI}(k)R_{ii}^{IQ}(k) - R_{ii}^{QQ}(l+k)R_{ii}^{II}(k-l) \\
&+ [R_{ii}^{IQ}(l)]^2 + R_{ii}^{II}(k)R_{ii}^{QQ}(k) + R_{ii}^{IQ}(l+k)R_{ii}^{QI}(k-l) \\
&+ [R_{ii}^{QI}(l)]^2 + R_{ii}^{QQ}(k)R_{ii}^{II}(k) + R_{ii}^{QI}(l+k)R_{ii}^{IQ}(k-l) \\
&- R_{ii}^{IQ}(l)R_{ii}^{QI}(l) - R_{ii}^{IQ}(k)R_{ii}^{QI}(k) - R_{ii}^{II}(l+k)R_{ii}^{QQ}(k-l) \\
&+ j \{ R_{ii}^{II}(l)R_{ii}^{IQ}(l) + R_{ii}^{II}(k)R_{ii}^{IQ}(k) + R_{ii}^{IQ}(l+k)R_{ii}^{II}(k-l) \} \\
&- j \{ R_{ii}^{II}(l)R_{ii}^{QI}(l) + R_{ii}^{IQ}(k)R_{ii}^{II}(k) + R_{ii}^{II}(l+k)R_{ii}^{IQ}(k-l) \}
\end{aligned}$$

$$\begin{aligned}
& +j \{ R_{ii}^{QQ}(l)R_{ii}^{IQ}(l) + R_{ii}^{QI}(k)R_{ii}^{QQ}(k) + R_{ii}^{QQ}(l+k)R_{ii}^{QI}(k-l) \} \\
& -j \{ R_{ii}^{QQ}(l)R_{ii}^{QI}(l) + R_{ii}^{QQ}(k)R_{ii}^{QI}(k) + R_{ii}^{QI}(l+k)R_{ii}^{QQ}(k-l) \} \\
& +j \{ R_{ii}^{QI}(l)R_{ii}^{II}(l) + R_{ii}^{QI}(k)R_{ii}^{II}(k) + R_{ii}^{QI}(l+k)R_{ii}^{II}(k-l) \} \\
& +j \{ R_{ii}^{QI}(l)R_{ii}^{QQ}(l) + R_{ii}^{QQ}(k)R_{ii}^{IQ}(k) + R_{ii}^{QQ}(l+k)R_{ii}^{IQ}(k-l) \} \\
& -j \{ R_{ii}^{IQ}(l)R_{ii}^{II}(l) + R_{ii}^{II}(k)R_{ii}^{QI}(k) + R_{ii}^{II}(l+k)R_{ii}^{QI}(k-l) \} \\
& -j \{ R_{ii}^{IQ}(l)R_{ii}^{QQ}(l) + R_{ii}^{IQ}(k)R_{ii}^{QQ}(k) + R_{ii}^{IQ}(l+k)R_{ii}^{QQ}(k-l) \}
\end{aligned} \tag{F.5}$$

where we note that the first two terms in each parenthesis for the imaginary terms cancel. Since

$$R(l) = [R_{ii}^{II}(l) + R_{ii}^{QQ}(l)] + j [R_{ii}^{QI}(l) - R_{ii}^{IQ}(l)] \tag{F.6}$$

then

$$\begin{aligned}
|R(l)|^2 &= [R_{ii}^{II}(l)]^2 + 2R_{ii}^{II}(l)R_{ii}^{QQ}(l) + [R_{ii}^{QQ}(l)]^2 \\
&\quad + [R_{ii}^{QI}(l)]^2 - 2R_{ii}^{QI}(l)R_{ii}^{IQ}(l) + [R_{ii}^{IQ}(l)]^2
\end{aligned} \tag{F.7}$$

and similarly for $|R(k)|^2$ so that

$$R_{\phi\phi}(k) = |R_{ii}(l)|^2 + |R_{ii}(k)|^2 + F_{ii}(l,k) \tag{F.8}$$

where

$$\begin{aligned}
F_{ii}(l,k) = & \{ R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k) \} \{ R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k) \} \\
& + \{ R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k) \} \{ R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k) \} \\
& - j \{ R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k) \} \{ R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k) \} \\
& + j \{ R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k) \} \{ R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k) \} \quad (F.9)
\end{aligned}$$

as noted in eq(E.6) of Appendix E.

APPENDIX G DERIVATION OF EQ(E.9) IN APPENDIX E

In this appendix, we show that the term $F_{ii}(l,k)$ in eq(E.9) is

$$F_{ii}(l,k) = E[x_i(n)x_i(n-l-k)]E[x_i^*(n-l)x_i^*(n-k)]. \quad (G.1)$$

Expressing the process $x_i(n)$ in terms of its quadrature components

$$E[x_i(n)x_i(n-l-k)] = E\{[x_{iI}(n) + jx_{iQ}(n)][x_{iI}(n-l-k) + jx_{iQ}(n-l-k)]\} \quad (G.2a)$$

$$= \{R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k)\} + j\{R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k)\} \quad (G.2b)$$

and

$$E[x_i^*(n-l)x_i^*(n-k)] = E\{[x_{iI}(n-l) - jx_{iQ}(n-l)][x_{iI}(n-k) - jx_{iQ}(n-k)]\} \quad (G.3a)$$

$$= \{R_{ii}^{II}(k-l) - R_{ii}^{QQ}(k-l)\} - j\{R_{ii}^{QI}(k-l) + R_{ii}^{IQ}(k-l)\} \quad (G.3b)$$

But

$$R_{ii}^{II}(k-l) = R_{ii}^{II}(l-k) \quad (G.4a)$$

$$R_{ii}^{QQ}(k-l) = R_{ii}^{QQ}(l-k) \quad (G.4b)$$

$$R_{ii}^{QI}(k-l) = R_{ii}^{IQ}(l-k) \quad (G.4c)$$

$$R_{ii}^{IQ}(k-l) = R_{ii}^{QI}(l-k) \quad (G.4d)$$

so that eq(G.3b) becomes

$$\begin{aligned}
E[x_i^*(n-l)x_i^*(n-k)] &= \\
&= \{R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k)\} - j\{R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k)\}. \quad (G.5)
\end{aligned}$$

Substituting eqs(G.2b) and (G.5) into (G.1), we obtain

$$\begin{aligned}
F_{ii}(l,k) &= \{R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k)\} \{R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k)\} \\
&\quad + \{R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k)\} \{R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k)\} \\
&\quad - j \{R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k)\} \{R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k)\} \\
&\quad + j \{R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k)\} \{R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k)\}. \quad (G.6)
\end{aligned}$$

In Appendix E, we noted that when eq(G.6) was used in eq(E.7), the imaginary terms summed to zero. This was shown by first noting that $F_{ii}(l,k)$ is real for $k=0$. We also note that imaginary terms evaluated with negative values of k serve to cancel the corresponding imaginary terms for positive values of k . And so, only the real part of the function $F_{ii}(l,k)$ contributes to the function $V_{B_{ii}}(l)$. If the corresponding bandpass processes are stationary and narrowband, then (see ref [3], section III.E.2.a),

$$R_{ii}^{II}(\alpha) = R_{ii}^{QQ}(\alpha) \quad \text{all } \alpha \quad (G.7a)$$

and

$$R_{ii}^{QI}(\alpha) = -R_{ii}^{IQ}(\alpha). \quad \text{all } \alpha. \quad (G.7b)$$

Using eqs(G.7) in (G.6), we obtain $F_{ii}(l,k) \approx 0$. In this case, eq(5.2.4) reduces to eq(5.2.8).

APPENDIX H CONSIDERATION OF THE TERMS $F_{ij}(l,k)$ AND $R_{ii}(k)R_{jj}^*(k)$ IN EQS(5.2.19) AND (5.2.20)

In this appendix, we consider the terms $F_{ij}(l,k)$ and $R_{ii}(k)R_{jj}^*(k)$ in eq(5.2.19). Consider,

$$F_{ij}(l,k) = E[x_i(n)x_j(n-l-k)]E[x_j^*(n-l)x_i^*(n-k)]. \quad (H.1)$$

Expressing the processes $x_i(n)$ and $x_j(n)$ in quadrature form, we have

$$E[x_i(n)x_j(n-l-k)] = E\{[x_{iI}(n) + jx_{iQ}(n)][x_{jI}(n-l-k) + jx_{jQ}(n-l-k)]\} \quad (H.2a)$$

$$= \{R_{ij}^{II}(l+k) - R_{ij}^{QQ}(l+k)\} + j\{R_{ij}^{QI}(l+k) + R_{ij}^{IQ}(l+k)\} \quad (H.2b)$$

and

$$E[x_j^*(n-l)x_i^*(n-k)] = E\{[x_{jI}(n-l) - jx_{jQ}(n-l)][x_{iI}(n-k) - jx_{iQ}(n-k)]\} \quad (H.3a)$$

$$= \{R_{ji}^{II}(k-l) - R_{ji}^{QQ}(k-l)\} - j\{R_{ji}^{QI}(k-l) + R_{ji}^{IQ}(k-l)\}. \quad (H.3b)$$

But

$$R_{ji}^{II}(k-l) = R_{ij}^{II}(l-k) \quad (H.4a)$$

$$R_{ji}^{QQ}(k-l) = R_{ij}^{QQ}(l-k) \quad (H.4b)$$

$$R_{ji}^{QI}(k-l) = R_{ij}^{IQ}(l-k) \quad (H.4c)$$

$$R_{ji}^{IQ}(k-l) = R_{ij}^{QI}(l-k) \quad (H.4d)$$

so that eq(H.3b) becomes

$$\begin{aligned}
E[x_j^*(n-l)x_i^*(n-k)] &= \\
&= \{R_{ij}^{II}(l-k) - R_{ij}^{QQ}(l-k)\} - j\{R_{ij}^{QI}(l-k) + R_{ij}^{IQ}(l-k)\}. \quad (H.5)
\end{aligned}$$

Substituting eqs(H.2b) and (H.5) into (H.1), we have

$$\begin{aligned}
F_{ij}(l,k) &= \{R_{ij}^{II}(l+k) - R_{ij}^{QQ}(l+k)\} \{R_{ij}^{II}(l-k) - R_{ij}^{QQ}(l-k)\} \\
&\quad + \{R_{ij}^{QI}(l+k) + R_{ij}^{IQ}(l+k)\} \{R_{ij}^{QI}(l-k) + R_{ij}^{IQ}(l-k)\} \\
&\quad - j \{R_{ij}^{II}(l+k) - R_{ij}^{QQ}(l+k)\} \{R_{ij}^{QI}(l-k) + R_{ij}^{IQ}(l-k)\} \\
&\quad + j \{R_{ij}^{II}(l-k) - R_{ij}^{QQ}(l-k)\} \{R_{ij}^{QI}(l+k) + R_{ij}^{IQ}(l+k)\}. \quad (H.6)
\end{aligned}$$

By examination of eq(H.6), we note that the imaginary terms sum to zero when $F_{ij}(l,k)$ is substituted into eq(5.2.19). We now consider the term $R_{ii}(k)R_{jj}^*(k)$ in eq(5.2.19); ie.,

$$R_{ii}(k)R_{jj}^*(k) = E[x_i(n)x_i^*(n-k)]E[x_j^*(n-l)x_j(n-l-k)]. \quad (H.7)$$

Now

$$\begin{aligned}
E[x_i(n)x_i^*(n-k)] &= \\
&= E\{[x_{iI}(n) + jx_{iQ}(n)][x_{iI}(n-k) - jx_{iQ}(n-k)]\} \quad (H.8a)
\end{aligned}$$

$$= \{R_{ii}^{II}(k) + R_{ii}^{QQ}(k)\} + j\{R_{ii}^{QI}(k) - R_{ii}^{IQ}(k)\} \quad (H.8b)$$

and

$$\begin{aligned}
E[x_j^*(n-l)x_j(n-l-k)] &= \\
&= E\{[x_{jI}(n-l) - jx_{jQ}(n-l)][x_{jI}(n-l-k) + jx_{jQ}(n-l-k)]\} \quad (H.9a)
\end{aligned}$$

$$= \{ R_{jj}^{II}(k) + R_{jj}^{QQ}(k) \} - j \{ R_{jj}^{QI}(k) - R_{jj}^{IQ}(k) \}. \quad (H.9b)$$

And so

$$\begin{aligned} R_{ii}(k)R_{jj}^*(k) &= \{ R_{ii}^{II}(k) + R_{ii}^{QQ}(k) \} \{ R_{jj}^{II}(k) + R_{jj}^{QQ}(k) \} \\ &\quad + \{ R_{ii}^{QI}(k) - R_{ii}^{IQ}(k) \} \{ R_{jj}^{QI}(k) - R_{jj}^{IQ}(k) \} \\ &\quad - j \{ R_{ii}^{II}(k) + R_{ii}^{QQ}(k) \} \{ R_{jj}^{QI}(k) - R_{jj}^{IQ}(k) \} \\ &\quad + j \{ R_{ii}^{II}(k) + R_{ii}^{QQ}(k) \} \{ R_{ii}^{QI}(k) - R_{ii}^{IQ}(k) \}. \end{aligned} \quad (H.10)$$

Consider the imaginary terms

$$\begin{aligned} \text{Im} \{ R_{ii}(k)R_{jj}^*(k) \} &= -j R_{ii}^{II}(k)R_{jj}^{QI}(k) - j R_{ii}^{QQ}(k)R_{jj}^{QI}(k) + j R_{ii}^{II}(k)R_{jj}^{IQ}(k) \\ &\quad + j R_{ii}^{QQ}(k)R_{jj}^{IQ}(k) + j R_{ii}^{II}(k)R_{ii}^{QI}(k) + j R_{ii}^{QQ}(k)R_{ii}^{QI}(k) \\ &\quad - j R_{ii}^{II}(k)R_{ii}^{IQ}(k) - j R_{ii}^{QQ}(k)R_{ii}^{IQ}(k). \end{aligned} \quad (H.11)$$

We now recall that

$$R_{ii}^{II}(k) = R_{ii}^{II}(-k) \quad (H.12a)$$

$$R_{ii}^{QQ}(k) = R_{ii}^{QQ}(-k) \quad (H.12b)$$

$$R_{ii}^{QI}(k) = R_{ii}^{IQ}(-k) \quad (H.12c)$$

$$R_{ii}^{IQ}(k) = R_{ii}^{QI}(-k) \quad (H.12d)$$

with equivalent expressions for the j channel processes. When $R_{ii}(k)R_{jj}^*(k)$ is used in the summation of eq(5.2.19), we recognize that the terms in eq(H.11) will cancel when the positive and negative k values are determined. For example, the

first term in this equation becomes $-j R_{ii}^{\Pi}(k)R_{jj}^{IQ}(k)$ when k is negative. This term will cancel the third term in the equation for k positive. Similarly, the other terms cancel. It is concluded that all of the imaginary terms in eq(5.2.19) sum to zero and, therefore, need not be considered.

Finally, we note that if the corresponding bandpass processes are jointly stationary and narrowband, then (see Appendix O),

$$R_{ij}^{\Pi}(\alpha) = R_{ij}^{QQ}(\alpha) \quad (H.13a)$$

and

$$R_{ij}^{QI}(\alpha) = -R_{ij}^{IQ}(\alpha). \quad (H.13b)$$

Using these equations in eq(H.6), we obtain $F_{ij}(l,k) = 0$. In this case, eq(5.2.19) reduces to eq(5.2.20).

APPENDIX I

SIMPLIFICATION OF EQ(6.3.27) IN THE SPECIAL CASE OF STATIONARY BANDPASS PROCESSES

In this appendix, we prove that the likelihood ratio of eq(6.3.27) reduces to that of eq(6.1.26) for the special case where the auto- and cross-correlation sequences of the quadrature components satisfy the specific constraints given by eq(6.3.28). From eqs(6.3.26) and (6.1.25), we have

$$\sum_{j=1}^J \left[\frac{[\psi_{jA}(n|H_i)]^2}{\sigma_{\psi_{jA}}^2} + \frac{[\psi_{jB}(n|H_i)]^2}{\sigma_{\psi_{jB}}^2} \right] = \underline{e}^T(n|H_i) [K_r(n|n-1, H_i)]^{-1} \underline{e}(n|H_i) \quad i = 0, 1 \quad (I.1)$$

and

$$\sum_{j=1}^J \frac{|\gamma_j(n|H_i)|^2}{\sigma_{\gamma_j}^2(n|H_i)} = \underline{\epsilon}^H(n|H_i) [K_x(n|n-1, H_i)]^{-1} \underline{\epsilon}(n|H_i) \quad i = 0, 1 \quad (I.2)$$

where

$$K_r(n|n-1, H_i) = \begin{bmatrix} R_e^{II}(0|H_i) & R_e^{IQ}(0|H_i) \\ R_e^{QI}(0|H_i) & R_e^{QQ}(0|H_i) \end{bmatrix} \quad i = 0, 1 \quad (I.3)$$

and

$$[K_x(n|n-1, H_i)] = R_\epsilon^{II}(0|H_i) + R_\epsilon^{QQ}(0|H_i) + j \{ R_\epsilon^{QI}(0|H_i) - R_\epsilon^{IQ}(0|H_i) \} \quad i = 0, 1. \quad (I.4)$$

For the special case where

$$R_x^{II}(\|H_i) = R_x^{QQ}(\|H_i) \quad i = 0, 1 \quad (I.5a)$$

and

$$R_x^{QI}(0|H_i) = - R_x^{IQ}(0|H_i) \quad i = 0,1. \quad (I.5b)$$

In Appendix M, we show that when eqs(I.5) hold, we have

$$R_e^{\Pi}(0|H_i) = R_e^{QQ}(0|H_i) \quad i = 0,1 \quad (I.6a)$$

$$R_e^{QI}(0|H_i) = - R_e^{IQ}(0|H_i) \quad i = 0,1 \quad (I.6b)$$

and the identically equivalent relations

$$R_\varepsilon^{\Pi}(0|H_i) = R_\varepsilon^{QQ}(0|H_i) \quad i = 0,1 \quad (I.7a)$$

$$R_\varepsilon^{QI}(0|H_i) = - R_\varepsilon^{IQ}(0|H_i) \quad i = 0,1. \quad (I.7b)$$

In this case, we have from eqs(I.3) and (I.4)

$$K_r(n|n-1, H_i) = \begin{bmatrix} R_e^{\Pi}(0|H_i) & -R_e^{QI}(0|H_i) \\ R_e^{QI}(0|H_i) & R_e^{\Pi}(0|H_i) \end{bmatrix} \quad i = 0,1 \quad (I.8)$$

and

$$[K_x(n|n-1, H_i)] = 2R_\varepsilon^{\Pi}(0|H_i) + j \ 2R_\varepsilon^{QI}(0|H_i) \quad i = 0,1. \quad (I.9)$$

Expanding eq(I.1), we have

$$\sum_{j=1}^J \frac{[\psi_{jA}(n|H_i)]^2}{\sigma_{\psi_{jA}}} + \frac{[\psi_{jB}(n|H_i)]^2}{\sigma_{\psi_{jB}}} = \underline{e}^T(n|H_i) [K_r(n|n-1, H_i)]^{-1} \underline{e}(n|H_i) \quad i = 0,1 \quad (I.10a)$$

$$= [\underline{e}_i^T(n|H_i) \quad \underline{e}_Q^T(n|H_i)] \begin{bmatrix} R_e^{\Pi}(0|H_i) & R_e^{IQ}(0|H_i) \\ R_e^{QI}(0|H_i) & R_e^{QQ}(0|H_i) \end{bmatrix}^{-1} \begin{bmatrix} \underline{e}_i(n|H_i) \\ \underline{e}_Q(n|H_i) \end{bmatrix} \quad i = 0,1 \quad (I.10b)$$

$$= [\underline{e}_i^T(n|H_i) \quad \underline{e}_Q^T(n|H_i)] \begin{bmatrix} K_{r11} & K_{r12} \\ K_{r21} & K_{r22} \end{bmatrix} \begin{bmatrix} \underline{e}_i(n|H_i) \\ \underline{e}_Q(n|H_i) \end{bmatrix} \quad i = 0,1 \quad (I.10b)$$

where

$$\begin{bmatrix} K_{r11} & K_{r12} \\ K_{r21} & K_{r22} \end{bmatrix} = \begin{bmatrix} R_e^{\Pi}(0|H_i) & R_e^{IQ}(0|H_i) \\ R_e^{QI}(0|H_i) & R_e^{QQ}(0|H_i) \end{bmatrix}^{-1} \quad i = 0,1. \quad (I.11)$$

and, from the matrix inversion lemma [95, pg. 25]

$$K_{r11} = \{ R_e^{\Pi}(0) \} - [R_e^{IQ}(0)][R_e^{QQ}(0)]^{-1}[R_e^{QI}(0)]^{-1} \quad (I.12a)$$

$$K_{r12} = -\{ R_e^{\Pi}(0) \} - [R_e^{IQ}(0)][R_e^{QQ}(0)]^{-1}[R_e^{QI}(0)]^{-1} [R_e^{IQ}(0)][R_e^{QQ}(0)]^{-1} \quad (I.12b)$$

$$K_{r21} = -\{ R_e^{QQ}(0) \} - [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{IQ}(0)]^{-1} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} \quad (I.12c)$$

$$K_{r22} = \{ R_e^{QQ}(0) \} - [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{IQ}(0)]^{-1}. \quad (I.12d)$$

In the special case where eqs(I.7) hold

$$K_{r11} = \{ R_e^{\Pi}(0) \} + [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{QI}(0)]^{-1} \quad (I.13a)$$

$$\begin{aligned} K_{r12} &= \{ R_e^{\Pi}(0) \} + [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{QI}(0)]^{-1} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} \\ &= K_{r11} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} \end{aligned} \quad (I.13b)$$

$$\begin{aligned} K_{r21} &= -\{ R_e^{\Pi}(0) \} + [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{QI}(0)]^{-1} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} \\ &= -K_{r11} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} = -K_{r12} \end{aligned} \quad (I.13c)$$

$$K_{r_{22}} = \{ R_e^{\Pi}(0) + [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{QI}(0)] \}^{-1} = K_{r_{11}}. \quad (I.13d)$$

And so

$$\begin{aligned} \sum_{j=1}^J \frac{[\psi_{jA}(n|H_i)]^2}{\sigma_{\psi_{jA}}} + \frac{[\psi_{jB}(n|H_i)]^2}{\sigma_{\psi_{jB}}} &= \\ &= \underline{\epsilon}_I^T(n|H_i) K_{r_{11}} \underline{\epsilon}_I(n|H_i) - \underline{\epsilon}_I^T(n|H_i) K_{r_{21}} \underline{\epsilon}_Q(n|H_i) \\ &\quad + \underline{\epsilon}_Q^T(n|H_i) K_{r_{21}} \underline{\epsilon}_I(n|H_i) + \underline{\epsilon}_Q^T(n|H_i) K_{r_{11}} \underline{\epsilon}_Q(n|H_i). \end{aligned} \quad (I.14)$$

Expanding eq(I.2), we have

$$\sum_{j=1}^J \frac{|\gamma_j(n|H_i)|^2}{\sigma_{\gamma_j(n|H_i)}} = \underline{\epsilon}^H(n|H_i) [K_x(n|n-1, H_i)]^{-1} \underline{\epsilon}(n|H_i) \quad i = 0, 1 \quad (I.15a)$$

$$= [\underline{\epsilon}_I^T(n|H_i) - j \underline{\epsilon}_Q^T(n|H_i)] [K_x(n|n-1, H_i)]^{-1} [\underline{\epsilon}_I(n|H_i) + j \underline{\epsilon}_Q(n|H_i)] \quad i=0, 1. \quad (I.15b)$$

We now expand the matrix $[K_x(n|n-1, H_i)]^{-1}$. Let

$$A = A_I + jA_Q \quad (I.16a)$$

and

$$A^{-1} = B_I + jB_Q. \quad (I.16b)$$

Then

$$A^{-1} A = [B_I + jB_Q] [A_I + jA_Q] = I. \quad (I.17)$$

It follows that

$$B_I A_I - B_Q A_Q = I \quad (I.18a)$$

$$B_Q A_I + B_I A_Q = 0 \quad (I.18b)$$

$$B_Q A_I = - B_I A_Q \quad (I.18c)$$

and

$$B_Q = - B_I A_Q A_I^{-1}. \quad (I.18d)$$

Using eq(I.18d) in (I.18a), we obtain

$$B_I A_I + B_I A_Q A_I^{-1} A_Q = B_I [A_I + A_Q A_I^{-1} A_Q] = I \quad (I.19)$$

so that

$$B_I = [A_I + A_Q A_I^{-1} A_Q]^{-1} \quad (I.20a)$$

and

$$B_Q = - [A_I + A_Q A_I^{-1} A_Q]^{-1} A_Q A_I^{-1}. \quad (I.20b)$$

Let

$$A = [K_x(nln-1, H_i)]. \quad (I.21)$$

From eq(I.9)

$$A_I = 2R_{\epsilon}^{II}(OH_i) \quad (I.22a)$$

and

$$A_Q = 2R_{\epsilon}^{QI}(OH_i) \quad (I.22b)$$

so that

$$B_I = [A_I + A_Q A_I^{-1} A_Q]^{-1} = \frac{1}{2} \{ R_{\epsilon}^{II}(OH_i) + R_{\epsilon}^{QI}(OH_i) [R_{\epsilon}^{II}(OH_i)]^{-1} R_{\epsilon}^{QI}(OH_i) \}^{-1} \quad (I.23a)$$

and

$$B_Q = - [A_I + A_Q A_I^{-1} A_Q]^{-1} A_Q A_I^{-1} \quad (I.23b)$$

$$= - \frac{1}{2} \{ R_\varepsilon^\Pi(0|H_i) + R_\varepsilon^{QI}(0|H_i) [R_\varepsilon^\Pi(0|H_i)]^{-1} R_\varepsilon^{QI}(0|H_i) \}^{-1} R_\varepsilon^{QI}(0|H_i) [R_\varepsilon^\Pi(0|H_i)]^{-1} \quad (I.23c)$$

$$= - B_I R_\varepsilon^{QI}(0|H_i) [R_\varepsilon^\Pi(0|H_i)]^{-1} \quad (I.23d)$$

Since eq(I.15) is real, only the real terms from the right hand side of eq(I.15b) need be considered. It follows that

$$\sum_{j=1}^J \frac{|y_j(n|H_i)|^2}{2 \sigma_{y_j}^2(n|H_i)} = \varepsilon_I^T(n|H_i) B_I \varepsilon_I(n|H_i) - \varepsilon_I^T(n|H_i) B_Q \varepsilon_Q(n|H_i) + \varepsilon_Q^T(n|H_i) B_Q \varepsilon_I(n|H_i) + \varepsilon_Q^T(n|H_i) B_I \varepsilon_Q(n|H_i). \quad (I.24)$$

In Appendix M, we show that when eqs(I.5) hold, the estimators $\hat{\varepsilon}_I(n|n-1, H_i)$ and $\hat{\varepsilon}_Q(n|n-1, H_i)$ are equivalent so that

$$\varepsilon_I(n|H_i) = \hat{\varepsilon}_I(n|H_i) \quad i = 0, 1 \quad (I.25a)$$

$$\varepsilon_Q(n|H_i) = \hat{\varepsilon}_Q(n|H_i) \quad i = 0, 1 \quad (I.25b)$$

$$R_\varepsilon^\Pi(0|H_i) = \hat{R}_\varepsilon^\Pi(0|H_i) \quad i = 0, 1 \quad (I.25c)$$

$$R_\varepsilon^{QI}(0|H_i) = \hat{R}_\varepsilon^{QI}(0|H_i) \quad i = 0, 1. \quad (I.25d)$$

Using these relations in eqs(I.13) and (I.23),

$$B_I = \frac{1}{2} K_{r_{11}} \quad (I.26a)$$

and

$$B_Q = \frac{1}{2} K_{r_{21}}. \quad (I.26b)$$

It follows from eq(I.14) and (I.24) that

$$\sum_{j=1}^J \left[\frac{[\psi_{jA}(n|H_i)]^2}{\sigma_{\psi_{jA}}^2} + \frac{[\psi_{jB}(n|H_i)]^2}{\sigma_{\psi_{jB}}^2} \right] = 2 \sum_{j=1}^J \frac{|\gamma_j(n|H_i)|^2}{\sigma_{\gamma_j}^2(n|H_i)} \quad (I.27)$$

We now consider the determinant of eq(I.3). From eq(6.3.23)

$$|K_r(n|n-1, H_i)| = |L_\psi(n|H_i)| |D_\psi(n|H_i)| |L_\psi^T(n|H_i)| \quad i = 0,1 \quad (I.28a)$$

$$= |D_\psi(n|H_i)| \quad i = 0,1 \quad (I.28b)$$

$$= \prod_{j=1}^J d_{jj_A}^\psi(n|H_i) d_{jj_B}^\psi(n|H_i) \quad i = 0,1 \quad (I.28c)$$

$$= \prod_{j=1}^J \sigma_{\psi_{jA}}^2(n|H_i) \sigma_{\psi_{jB}}^2(n|H_i) \quad i = 0,1. \quad (I.28d)$$

Also, from eqs(6.1.22)

$$|K_x(n|n-1, H_i)| = |L_\gamma(n|H_i)| |D_\gamma(n|H_i)| |L_\gamma^H(n|H_i)| \quad i=0,1 \quad (I.29a)$$

$$= |D_\gamma(n|H_i)| \quad (I.29b)$$

$$= \prod_{j=1}^J d_{\gamma_{jj}}(n|H_i) = \prod_{j=1}^J \sigma_{\gamma_j}^2(n|H_i) \quad i=0,1. \quad (I.29c)$$

In Appendix N, we show that in the special case where eqs(I.7) hold,

$$|K_r(n|n-1, H_i)| = \left(\frac{1}{2}\right)^{2J} |K_x(n|n-1, H_i)|^2. \quad (I.30)$$

And so,

$$\frac{|K_r(n|n-1, H_0)|}{|K_r(n|n-1, H_1)|} = \frac{|K_x(n|n-1, H_0)|^2}{|K_x(n|n-1, H_1)|^2}. \quad (I.31)$$

In this case, using eqs(I.28d) and (I.29c) in (I.31), we obtain

$$\frac{\prod_{j=1}^J \sigma_{\psi_{jA}}^2(n|H_0) \sigma_{\psi_{jB}}^2(n|H_0)}{\prod_{j=1}^J \sigma_{\psi_{jA}}^2(n|H_1) \sigma_{\psi_{jB}}^2(n|H_1)} = \frac{\prod_{j=1}^J \sigma_{\gamma_j}^4(n|H_0)}{\prod_{j=1}^J \sigma_{\gamma_j}^4(n|H_1)} . \quad (I.32)$$

Taking the natural logarithm of both sides of eq(I.32), we obtain

$$\sum_{j=1}^J \left[\ln \frac{\sigma_{\psi_{jA}}^2(n|H_0)}{\sigma_{\psi_{jA}}^2(n|H_1)} + \ln \frac{\sigma_{\psi_{jB}}^2(n|H_0)}{\sigma_{\psi_{jB}}^2(n|H_1)} \right] = 2 \sum_{j=1}^J \left[\ln \frac{\sigma_{\gamma_j}^2(n|H_0)}{\sigma_{\gamma_j}^2(n|H_1)} \right] . \quad (I.33)$$

Using eqs(I.27) and (I.33), eq(6.3.27) reduces to eq(6.1.26).

APPENDIX J **SIMPLIFICATION OF EQ(6.3.36) IN THE SPECIAL CASE OF** **STATIONARY BANDPASS PROCESSES**

In the special case where eqs(6.3.28) hold, we show in this appendix that the likelihood ratio of eq(6.3.37) reduces to that of eq(6.1.38). From eqs(6.3.36) and (6.1.37), we have

$$\sum_{j=1}^J [\tau_{jA}^2(n|H_i) + \tau_{jB}^2(n|H_i)] = \underline{\epsilon}^T(n|H_i) [K_r(n|n-1, H_i)]^{-1} \underline{\epsilon}(n|H_i) \quad i = 0, 1 \quad (J.1)$$

and

$$\sum_{j=1}^J |\alpha_j(n|H_i)|^2 = \underline{\epsilon}^H(n|H_i) [K_x(n|n-1, H_i)]^{-1} \underline{\epsilon}(n|H_i) \quad i = 0, 1 \quad (J.2)$$

where

$$K_r(n|n-1, H_i) = \begin{bmatrix} R_e^{\Pi}(0|H_i) & R_e^{IQ}(0|H_i) \\ R_e^{QI}(0|H_i) & R_e^{QQ}(0|H_i) \end{bmatrix} \quad i = 0, 1 \quad (J.3)$$

and

$$[K_x(n|n-1, H_i)] = R_e^{\Pi}(0|H_i) + R_e^{QQ}(0|H_i) + j \{ R_e^{QI}(0|H_i) - R_e^{IQ}(0|H_i) \} \quad i = 0, 1. \quad (J.4)$$

As shown Appendix I, for the special case where

$$R_x^{\Pi}(\|H_i) = R_x^{QQ}(\|H_i) \quad i = 0, 1 \quad (J.5a)$$

and

$$R_x^{QI}(\|H_i) = - R_x^{IQ}(\|H_i) \quad i = 0, 1. \quad (J.5b)$$

we have

$$K_r(n|n-1, H_i) = \begin{bmatrix} R_e^{\Pi}(0|H_i) & -R_e^{QI}(0|H_i) \\ R_e^{QI}(0|H_i) & R_e^{\Pi}(0|H_i) \end{bmatrix} \quad i = 0, 1 \quad (J.6)$$

and

$$[K_x(n|n-1, H_i)] = 2R_e^{\Pi}(0|H_i) + j 2R_e^{QI}(0|H_i) \quad i = 0, 1. \quad (J.7)$$

Expanding eq(J.1), and following the same discussion as presented in Appendix I, we obtain

$$\sum_{j=1}^J [\tau_{jA}^2(n|H_i) + \tau_{jB}^2(n|H_i)] = \mathbf{e}^T(n|H_i) [K_r(n|n-1, H_i)]^{-1} \mathbf{e}(n|H_i) \quad i = 0, 1 \quad (J.8a)$$

$$\begin{aligned} &= \mathbf{e}_I^T(n|H_i) K_{r_{11}} \mathbf{e}_I(n|H_i) - \mathbf{e}_I^T(n|H_i) K_{r_{21}} \mathbf{e}_Q(n|H_i) \\ &\quad + \mathbf{e}_Q^T(n|H_i) K_{r_{21}} \mathbf{e}_I(n|H_i) + \mathbf{e}_Q^T(n|H_i) K_{r_{11}} \mathbf{e}_Q(n|H_i). \end{aligned} \quad (J.8b)$$

where

$$K_{r_{11}} = \{ R_e^{\Pi}(0) + [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{QI}(0)] \}^{-1} \quad (J.9a)$$

$$\begin{aligned} K_{r_{12}} &= \{ R_e^{\Pi}(0) + [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{QI}(0)] \}^{-1} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} \\ &= K_{r_{11}} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} \end{aligned} \quad (J.9b)$$

$$\begin{aligned} K_{r_{21}} &= - \{ R_e^{\Pi}(0) + [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{QI}(0)] \}^{-1} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} \\ &= - K_{r_{11}} [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1} \end{aligned} \quad (J.9c)$$

$$K_{r_{22}} = \{ R_e^{\Pi}(0) + [R_e^{QI}(0)][R_e^{\Pi}(0)]^{-1}[R_e^{QI}(0)] \}^{-1} = K_{r_{11}}. \quad (J.9d)$$

Expanding eq(J.2), we have

$$\sum_{j=1}^J |\alpha_j(n|H_i)|^2 = \underline{\epsilon}^H(n|H_i) [K_x(n|n-1, H_i)]^{-1} \underline{\epsilon}(n|H_i) \quad i = 0, 1 \quad (J.10a)$$

$$= [\underline{\epsilon}_I^T(n|H_i) - j\underline{\epsilon}_Q^T(n|H_i)] [K_x(n|n-1, H_i)]^{-1} [\underline{\epsilon}_I(n|H_i) + j\underline{\epsilon}_Q(n|H_i)] \quad i=0, 1. \quad (J.10b)$$

Since eq(J.10) is real, only the real terms from the right hand side of eq(J.10b) need be considered. It follows that

$$\begin{aligned} \sum_{j=1}^J |\alpha_j(n|H_i)|^2 = & \underline{\epsilon}_I^T(n|H_i) B_I \underline{\epsilon}_I(n|H_i) - \underline{\epsilon}_I^T(n|H_i) B_Q \underline{\epsilon}_Q(n|H_i) \\ & + \underline{\epsilon}_Q^T(n|H_i) B_Q \underline{\epsilon}_I(n|H_i) + \underline{\epsilon}_Q^T(n|H_i) B_I \underline{\epsilon}_Q(n|H_i). \end{aligned} \quad (J.11)$$

where B_I and B_Q are defined in eqs(I.23). However, as pointed out in Appendices I and M, when eqs(J.5) hold, the estimators $\hat{\underline{\epsilon}}(n|n-1, H_i)$ and $\hat{\underline{\epsilon}}(n|n-1, H_i)$ are equivalent so that

$$\underline{\epsilon}_I(n|H_i) = \underline{\epsilon}_I(n|H_i) \quad i = 0, 1 \quad (J.12a)$$

$$\underline{\epsilon}_Q(n|H_i) = \underline{\epsilon}_Q(n|H_i) \quad i = 0, 1 \quad (J.12b)$$

$$R_{\epsilon}^{\Pi}(0|H_i) = R_{\epsilon}^{\Pi}(0|H_i) \quad i = 0, 1 \quad (J.12c)$$

$$R_{\epsilon}^{QI}(0|H_i) = R_{\epsilon}^{QI}(0|H_i) \quad i = 0, 1. \quad (J.12d)$$

In Appendix I, we also showed that

$$B_I = \frac{1}{2} K_{r_{11}} \quad (J.13a)$$

and

$$B_Q = \frac{1}{2} K_{r_{21}}. \quad (J.13b)$$

It follows from eqs(J.8) and (J.11) that

$$\sum_{j=1}^J [\tau_{jA}^2(n|H_i) + \tau_{jB}^2(n|H_i)] = 2 \sum_{j=1}^J |\alpha_j(n|H_i)|^2 \quad (J.14)$$

We now consider the determinant of eq(J.3). From eq(6.3.35)

$$|K_r(n|n-1, H_i)| = |C_\tau(n|H_i)| |C_\tau^T(n|H_i)| \quad i = 0, 1 \quad (J.15a)$$

$$= \prod_{j=1}^J c_{\tau A_{jj}}(n|H_i) c_{\tau B_{jj}}(n|H_i) \prod_{j=1}^J c_{\tau A_{jj}}(n|H_i) c_{\tau B_{jj}}(n|H_i) \quad (J.15b)$$

$$= \prod_{j=1}^J [c_{\tau A_{jj}}(n|H_i)]^2 [c_{\tau B_{jj}}(n|H_i)]^2 \quad i = 0, 1. \quad (J.15c)$$

Also, from eqs(6.1.36)

$$|K_x(n|n-1, H_i)| = |C_\alpha(n|H_i)| |C_\alpha^H(n|H_i)| \quad i = 0, 1 \quad (J.16a)$$

$$= \prod_{j=1}^J c_{\alpha jj}(n|H_i) \prod_{j=1}^J c_{\alpha jj}^*(n|H_i) \quad i = 0, 1 \quad (J.16b)$$

$$= \prod_{j=1}^J |c_{\alpha jj}(n|H_i)|^2 \quad i = 0, 1. \quad (J.16c)$$

In Appendix N, we show that in the special case where eqs(J.7) hold,

$$|K_r(n|n-1, H_i)| = \left(\frac{1}{2}\right)^{2J} |K_x(n|n-1, H_i)|^2. \quad (J.17)$$

And so,

$$\frac{|K_r(n|n-1, H_0)|}{|K_r(n|n-1, H_1)|} = \frac{|K_x(n|n-1, H_0)|^2}{|K_x(n|n-1, H_1)|^2}. \quad (J.18)$$

In this case, using eqs(J.15c) and (J.16c) in (J.18), we obtain

$$\frac{\prod_{j=1}^J [c_{\tau A_{jj}}(n|H_0)]^2 [c_{\tau B_{jj}}(n|H_0)]^2}{\prod_{j=1}^J [c_{\tau A_{jj}}(n|H_1)]^2 [c_{\tau B_{jj}}(n|H_1)]^2} = \frac{\prod_{j=1}^J |c_{\alpha_{jj}}(n|H_0)|^4}{\prod_{j=1}^J |c_{\alpha_{jj}}(n|H_1)|^4}. \quad (J.19)$$

Taking the natural logarithm of both sides yields

$$\sum_{n=1}^N \left[\ln \frac{[c_{\tau A_{jj}}(n|H_0)]^2}{[c_{\tau A_{jj}}(n|H_1)]^2} + \ln \frac{[c_{\tau B_{jj}}(n|H_0)]^2}{[c_{\tau B_{jj}}(n|H_1)]^2} \right] = 2 \sum_{n=1}^N \left[\ln \frac{|c_{\alpha_{jj}}(n|H_0)|^2}{|c_{\alpha_{jj}}(n|H_1)|^2} \right]. \quad (J.20)$$

Using eqs(J.14) and (J.20), eq(6.3.37) reduces to eq(6.1.38).

APPENDIX K ANALYTIC EXPRESSIONS FOR MODEL I(INCOH) AND MODEL II(INCOH)

In this appendix, we consider the development of the analytic expressions for the model I(INCOH) and model II(INCOH) signals in additive white noise. In addition, the method used to obtain coherent integration curves for the model I(COH) signal are described. Finally, extensions are made to the multichannel case.

K.1 Model I(INCOH) Signal in White Noise

We now determine the probability of detection for the model I(INCOH) signal in additive white Gaussian noise problem. Let us initially consider a pulse train consisting of N pulses in which the signal components are identical except for random initial phases which are statistically independent from pulse-to-pulse. Following DiFranco and Rubin [13], we designate $r_i\sqrt{\mathcal{R}_p}$ as the i th sampled value of the matched filter envelope output where $\mathcal{R}_p = 2E_p/N_0$ is the peak signal-to-noise ratio, E_p is the signal energy of each pulse, and the two sided noise spectral density is $N_0/2$ (watts/Hz). Assuming a uniform probability density for the phase, the probability density of r_i for a single observation containing a sine-wave signal plus white Gaussian noise is a Rician density function expressed as [13]

$$p(r_i | \mathcal{R}_p) = \begin{cases} r_i \exp[-(r_i^2 + \mathcal{R}_p)/2] I_0(r_i \sqrt{\mathcal{R}_p}) & r_i \geq 0 \\ 0 & r_i < 0 \end{cases} \quad (\text{K.1.1})$$

where $I_0(\cdot)$ is the modified zeroth order Bessel function of the first kind. We note that the sampling instant corresponds to the time at which the single-pulse matched filter output peaks due to the i th signal pulse. The sinusoidal signal input can be expressed as

$$s(t) = A_p(t) \cos[\omega_c t + \theta(t) + \phi] \quad (\text{K.1.2})$$

where $p(t)$ is a unit amplitude rectangular gating function of duration T , ω_c is the carrier angular frequency, $\theta(t)$ is the signal phase modulation and ϕ is the unknown initial random phase which is uniformly distributed. We now define the signal energy E_p in the pulse of duration T as

$$E_p = \int_0^T s^2(t) dt \quad (K.1.3a)$$

$$= A^2 \int_0^T \cos^2[\omega_c t + \theta(t) + \phi] dt \quad (K.1.3b)$$

$$= \frac{A^2}{2} \int_0^T \{ 1 + \cos[2\omega_c t + 2\theta(t) + 2\phi] \} dt. \quad (K.1.3c)$$

The function $\theta(t)$ is slowly varying and ϕ is a constant for a specific pulse train so that

$$E_p = \frac{A^2}{2} \int_0^T dt + \frac{A^2}{2} \int_0^T \cos[2\omega_c t + c] dt \quad (K.1.4a)$$

$$= \frac{A^2 T}{2} \quad (K.1.4b)$$

where the constant c is used to represent $2[\theta(t) + \phi]$ and the second integration is negligible due to the rapid fluctuations of the $2\omega_c$ frequency term. Using eq(K.1.4b) and the definition of the signal-to-noise ratio $\mathcal{R}_p$

$$\mathcal{R}_p = \frac{2E_p}{N_0} = \frac{A^2 T}{N_0} = \frac{A^2}{\psi_0} \quad (K.1.5)$$

where $\psi_0 = N_0/T$ is the matched filter output noise power for a pulse of duration T . For a filter bandwidth $B = 1/T$, we have

$$\psi_0 = N_0/T = N_0 B. \quad (K.1.6)$$

We now consider an equivalent density distribution to that expressed by eq(K.1.1) in order to determine relationships between the various parameters defined above. Let us define a new random variable v_i such that

$$r_i = \frac{v_i}{\psi_0^{1/2}} \quad (\text{K.1.7})$$

and consider the probability density $p(v_i|\mathcal{R}_p)$ as

$$p(v_i|\mathcal{R}_p) = \frac{p(r_i|\mathcal{R}_p)}{|J|} \Big|_{r_i=v_i/\psi_0^{1/2}} \quad (\text{K.1.8})$$

where the Jacobian J is

$$J = \frac{dv_i}{dr_i} = \psi_0^{1/2}. \quad (\text{K.1.9})$$

Using eqs(K.1.1), (K.1.5) and (K.1.9) in (K.1.8), we have the new density function expressed as

$$p(v_i|A) = \begin{cases} \frac{v_i}{\psi_0} \exp[-(v_i^2 + A^2)/2\psi_0] I_0(v_i A/\psi_0) & v_i \geq 0 \\ 0 & v_i < 0. \end{cases} \quad (\text{K.1.10})$$

We note that $r_i \sqrt{\mathcal{R}_p}$, the sampled matched filter envelope output, is $v_i A/\psi_0$, in the new notation. In [13], it is shown that the Bayesian detection test strategy for N observations of the non-fluctuating, incoherent pulse-train reduces to the threshold comparison test which in our transformed notation is expressed as

$$\exp(-NA^2/2\psi_0) \prod_{i=1}^N I_0(v_i A/\psi_0^{1/2}) \begin{cases} \geq T_1 & \text{(signal present)} \\ < T_1 & \text{(signal absent)} \end{cases} \quad (\text{K.1.11})$$

where T_1 is the threshold. For known N and $A/\psi_0^{1/2}$, an equivalent test strategy is

$$\prod_{i=1}^N I_0(v_i A/\psi_0^{1/2}) \begin{cases} \geq T_1 \exp(NA^2/2\psi_0) = T_2 & \text{(signal present)} \\ < T_2 & \text{(signal absent).} \end{cases} \quad (\text{K.1.12})$$

Since $I_0(x)$ is a monotonically increasing function of x , $\ln[I_0(x)]$ is also monotonically increasing so that an alternate test is

$$\sum_{i=1}^N \ln[I_0(v_i A/\psi_0^{1/2})] \begin{cases} \geq T_3 & \text{(signal present)} \\ < T_3 & \text{(signal absent).} \end{cases} \quad (\text{K.1.13})$$

where $T_3 = \ln T_2$. The leading terms in the power series expansion of $I_0(x)$ are

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n} n! n!} = 1 + x^2/4 + x^4/64 + \dots \quad (\text{K.1.14})$$

Therefore, for small x

$$\ln I_0(x) \approx x^2/4. \quad x \ll 1 \quad (\text{K.1.15})$$

For a small signal-to-noise ratio such that $x = v_i A/\psi_0^{1/2} \ll 1$, eq(K.1.13) can be expressed as

$$\sum_{i=1}^N v_i^2 \begin{cases} \geq 4T_3\psi_0/A^2 = T_4 & \text{(signal present)} \\ < T_4 & \text{(signal absent).} \end{cases} \quad (\text{K.1.16})$$

Thus, for small x , a square law detector provides nearly optimal performance. For $x > 1$, $I_0(x)$ can be approximated by the asymptotic expansion [89]

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}} \left(1 + \frac{1}{8x} + \dots \right) \quad x > 1 \quad (\text{K.1.17a})$$

and

$$\ln I_0(x) = x - \frac{1}{2} \ln 2\pi x + \ln(1 + 1/8x + \dots) \quad x > 1. \quad (\text{K.1.17b})$$

For a large signal-to-noise ratio such that $x = v_i A / \psi_0^{1/2} \gg 1$, the first term on the RHS of eq(K.1.17b) is the dominant term. Therefore, eq(K.1.13) can be reduced to

$$\sum_{i=1}^N v_i \begin{cases} \geq T_3 \psi_0^{1/2} / A = T_5 & (\text{signal present}) \\ < T_5 & (\text{signal absent}) \end{cases} \quad (\text{K.1.18})$$

which indicates that a linear envelope detector is nearly optimal for $x \gg 1$. We now return to the test statistic expressed by eq(K.1.16) for the small signal case and define the test statistic by [73,74]

$$y = \sum_{i=1}^N v_i^2. \quad (\text{K.1.19})$$

Defining a related test statistic $Y = y/2\psi_0$ and random variable z_i such that

$$z_i = v_i^2 / 2\psi_0 \quad (\text{K.1.20})$$

we have

$$Y = \frac{1}{2} \sum_{i=1}^N v_i^2 / \psi_0 = \sum_{i=1}^N z_i. \quad (\text{K.1.21})$$

In the discussion which follows, we consider the characteristic function of the probability density distribution. In order to utilize available Fourier integral pairs, we now consider another transformation of eq(K.1.10). Using the transformation of variables defined by eq(K.1.20), eq(K.1.10) can be transformed as

$$p(z_i|A) = \begin{cases} \frac{p(v_i|A)}{|J|} \Big|_{v_i=\sqrt{2z_i}\psi_0^{1/2}} & z_i \geq 0 \\ 0 & z_i < 0. \end{cases} \quad (\text{K.1.22})$$

where J is the Jacobian. In this case,

$$|J| = \left| \frac{dz_i}{dv_i} \right| = v_i/\psi_0 \quad r_i \geq 0 \quad (\text{K.1.23})$$

so that

$$p(z_i|A) = \begin{cases} \exp[-(z_i + A^2/2\psi_0)] I_0[\sqrt{2z_i} A/\psi_0^{1/2}] & z_i \geq 0 \\ 0 & z_i < 0. \end{cases} \quad (\text{K.1.24})$$

The random variable Y is the sum of N statistically independent random variables z_i . The characteristic function of the random variable Y is expressed in terms of the characteristic functions of the individual random variables $C_{z_i}(\xi|A)$

as

$$C_Y(\xi|A) = \prod_{i=1}^N C_{z_i}(\xi|A). \quad (\text{K.1.25})$$

where the characteristic function for z_i is expressed as

$$C_{z_i}(\xi|A) = \int_{-\infty}^{\infty} \exp(j\xi z_i) p(z_i|A) dz_i. \quad (\text{K.1.26})$$

Substituting eq(K.1.24) into (K.1.26) yields

$$C_{z_i}(\xi|A) = \int_0^{\infty} \exp(j\xi z_i) \exp[-(z_i + A^2/2\psi_0)] I_0(\sqrt{2z_i} A/\psi_0^{1/2}) dz_i. \quad (\text{K.1.27})$$

Using integral pair 655.1 in [75], we obtain

$$C_{z_i}(\xi|A) = \frac{\exp[jA^2\xi/2\psi_0(1-j\xi)]}{1-j\xi} \quad (K.1.28)$$

Substituting eq(K.1.28) into (K.1.25) yields

$$C_Y(\xi|A) = \prod_{i=1}^N \frac{\exp[jA^2\xi/2\psi_0(1-j\xi)]}{1-j\xi} \quad (K.1.29a)$$

$$= \frac{\exp[jNA^2\xi/2\psi_0(1-j\xi)]}{(1-j\xi)^N} \quad (K.1.29b)$$

As noted above, the signal amplitude A for the model I(INCOH) target model is constant on each pulse train but varies randomly from pulse-train to pulse-train with Rayleigh probability density

$$p(A) = \begin{cases} \frac{A}{A_0^2} \exp[-A^2/A_0^2] & A \geq 0 \\ 0 & A < 0 \end{cases} \quad (K.1.30)$$

where the mean square value of A is $2A_0^2$; ie.,

$$\overline{A^2} = \int_0^{\infty} A^2 p(A) dA \quad (K.1.31a)$$

$$= \frac{1}{A_0^2} \int_0^{\infty} A^3 \exp[-A^2/A_0^2] dA \quad (K.1.31b)$$

$$= \frac{1}{2A_0^2} \int_0^{\infty} \alpha \exp[-\alpha/2A_0^2] d\alpha \quad (K.1.31c)$$

$$= 2A_0^2 \quad (K.1.31d)$$

where $\alpha=A^2$ and eq(K.1.30) was used to obtain (K.1.31b). The single pulse power signal-to-noise ratio, commonly used in the radar literature, is expressed as [72,74]

$$(S/N)_p = \frac{A^2}{2\psi_0} = \frac{\mathcal{R}_p}{2} \quad (\text{K.1.32})$$

so that averaging with respect to A yields using eq(K.1.31d)

$$(S/N)_{\text{avg}} = \overline{\frac{A^2}{2\psi_0}} \quad (\text{K.1.33a})$$

$$= A_0^2/\psi_0 = \frac{\bar{\mathcal{R}}_p}{2} . \quad (\text{K.1.33b})$$

Averaging eq(K.1.29b) with respect to A using (K.1.30), we obtain

$$C_Y(\xi) = \overline{C_Y(\xi|A)}_A = \int_0^\infty C_Y(\xi|A)p(A)dA \quad (\text{K.1.34a})$$

$$= \frac{1}{(1-j\xi)^N} \int_0^\infty \frac{A}{A_0} \exp[-A^2 g^2] dA \quad (\text{K.1.34b})$$

where

$$g^2 = \frac{1}{2A_0^2} - \frac{jN\xi}{2\psi_0(1-j\xi)} . \quad (\text{K.1.35})$$

Evaluation of eq(K.1.34b) yields

$$C_Y(\xi) = \frac{1}{2g^2 A_0^2 (1-j\xi)^N} . \quad (\text{K.1.36})$$

Using eq(K.1.35) in (K.1.36), we have

$$C_Y(\xi) = \frac{1}{A_0^2 \left[\frac{1}{A_0^2} - \frac{j\xi N}{\psi_0(1-j\xi)} \right] (1-j\xi)^N} \quad (\text{K.1.37a})$$

$$= \frac{1}{\left[1 - \frac{j\xi N A_0^2 / \psi_0}{(1-j\xi)} \right] (1-j\xi)^N} \quad (\text{K.1.37b})$$

$$= \frac{1}{(1-j\xi - j\xi N A_0^2 / \psi_0)(1-j\xi)^{N-1}} \quad (\text{K.1.37c})$$

$$= \frac{1}{\{ 1 - j\xi [1 + N(S/N)_{\text{avg}}] \} (1-j\xi)^{N-1}} \quad (\text{K.1.37d})$$

where eq(K.1.33b) was used to obtain eq(K.1.37d). From eq(K.1.37d), we can obtain

$$[1 - j\xi(1 + N(S/N)_{\text{avg}})] C_Y(\xi) = \frac{1}{(1-j\xi)^{N-1}}. \quad (\text{K.1.38})$$

Rearranging the terms of eq(K.1.38) yields

$$C_Y(\xi) = \frac{1}{(1-j\xi)^{N-1}} + j\xi[1 + N(S/N)_{\text{avg}}] C_Y(\xi). \quad (\text{K.1.39})$$

Taking the anticharacteristic function of eq(K.1.39) term by term yields

$$p(Y) = \frac{Y^{N-2} \exp(-Y)}{(N-2)!} - (1 + N(S/N)_{\text{avg}}) \frac{dp(Y)}{dY}. \quad (\text{K.1.40})$$

In the absence of a signal, $(S/N)_{\text{avg}}=0$ so that eq(K.1.40) yields

$$\frac{dp(Y)}{dY} + p(Y) = \frac{Y^{N-2} \exp(-Y)}{(N-2)!}. \quad (\text{K.1.41})$$

The solution of eq(K.1.41) is

$$p(Y) = \frac{Y^{N-1} \exp(-Y)}{(N-1)!}. \quad (\text{K.1.42})$$

Denoting the threshold as Y_b , the false alarm probability is obtained as

$$P_{fa} = \int_{Y_b}^{\infty} p(Y|H_0) dY \quad (K.1.43a)$$

$$= 1 - \int_0^{Y_b} p(Y) dY \quad (K.1.43b)$$

$$= 1 - \int_0^{Y_b} \frac{Y^{N-1} \exp(-Y)}{(N-1)!} dY \quad (K.1.43c)$$

$$= 1 - I_{\Gamma} \left[\frac{Y_b}{\sqrt{N}}, (N-1) \right] \quad (K.1.43d)$$

The integral in eq(K.1.43c) is the incomplete gamma function, $I_{\Gamma}(Y_b/\sqrt{N}, N-1)$. where

$$I_{\Gamma}(u, s) = \int_0^{u\sqrt{1+s}} \frac{\exp(-v) v^s}{s!} dv \quad (K.1.44)$$

and results have been tabulated in [76]. Eq(K.1.40) can be used to obtain the probability of detection using

$$P_d = \int_{Y_b}^{\infty} p(Y) dY \quad (K.1.45a)$$

$$= 1 - \int_0^{Y_b} p(Y) dY \quad (K.1.45b)$$

$$= 1 - \int_0^{Y_b} \frac{Y^{N-2} \exp(-Y)}{(N-2)!} dY + (1 + N(S/N)_{avg}) [p(Y)] \Big|_0^{Y_b}. \quad (K.1.45c)$$

The integral in eq(K.1.45c) is the incomplete gamma function, $I_{\Gamma}(Y_b/\sqrt{N-1}, N-2)$.

An alternative expression for $p(Y)$ similar to that derived in [13] is

$$p(Y) = \frac{\left[1 + \frac{1}{N(S/N)_{\text{avg}}}\right]^{N-2} \exp[-Y/(1+N(S/N)_{\text{avg}})]}{N(S/N)_{\text{avg}}} I_{\Gamma} \left[\frac{Y}{\sqrt{N-1}(1+1/N(S/N)_{\text{avg}})}, (N-2) \right] \quad (\text{K.1.46})$$

Using eq(K.1.46) in (K.1.45c), we obtain

$$P_d = 1 - I_{\Gamma} \left[\frac{Y_b}{\sqrt{N-1}}, (N-2) \right] + \left[1 + \frac{1}{N(S/N)_{\text{avg}}} \right]^{N-1} \exp[-Y_b/(1 + N(S/N)_{\text{avg}})] \\ \cdot I_{\Gamma} \left[\frac{Y_b}{\sqrt{N-1}[1+1/N(S/N)_{\text{avg}}]}, (N-2) \right]. \quad (\text{K.1.47})$$

K.2. Model II(INCOH) Signal in White Noise

We now develop the probability of detection expression for the model II(INCOH) signal in additive Gaussian white noise problem. In [13], it is noted that the receiver structure for this case is the same as that for the nonfluctuating pulse-train. Therefore, the probability of false alarm is the same as that expressed by eq(K.1.43d). The probability of detection will be evaluated using eq(K.1.45b). In this case, however, each pulse amplitude in a given pulse-train is a random variable with a Rayleigh density distribution expressed by eq(K.1.30). We therefore consider the density function $p(Y)$ where Y is defined in eq(K.1.21) and express it as

$$p(Y) = \overline{p(Y|\underline{A})}_{\underline{A}} \quad (\text{K.2.1})$$

where the averaging is over $\underline{A}$ and $\underline{A}$ represents the set of statistically independent random amplitudes $(A_1, A_2, \dots, A_N)$ in each pulse train and each pulse. The

characteristic function $\overline{C_Y(\xi)}_{\underline{A}}$ of the sum random variable $Y = \sum_{i=1}^N z_i$ is related to that of the random variable z_i , $i=1,2,\dots,N$ by

$$\overline{C_Y(\xi)}_{\underline{A}} = \prod_{i=1}^N \overline{C_{z_i}(\xi)}_{A_i}. \quad (\text{K.2.2})$$

Since $\overline{C_{z_i}(\xi)}_{A_i}$ is the same for all i ,

$$\overline{C_Y(\xi)}_{\Delta} = [\overline{C_{z_i}(\xi)}_{A_i}]^N. \quad (K.2.3)$$

Using eq(K.1.37d) with $N=1$ and using the subscript A_i to denote that the average is over the i th pulse amplitude, we obtain

$$\overline{C_{z_i}(\xi)}_{A_i} = \frac{1}{1 - j\xi[1 + (S/N)_{avg}]}. \quad (K.2.4)$$

Substituting eq(K.2.4) into (K.2.3)

$$\overline{C_Y(\xi)}_{\Delta} = \frac{1}{\{1 - j\xi[1 + (S/N)_{avg}]\}^N} \quad (K.2.5a)$$

$$= \frac{1}{[1 + (S/N)_{avg}]^N \{1/[1 + (S/N)_{avg}] - j\xi\}^N}. \quad (K.2.5b)$$

Using Fourier transform pair 431 in [75], the anticharacteristic function of eq(K.2.5b) is found to be

$$p(Y) = \frac{1}{[1 + (S/N)_{avg}]^N (N-1)!} Y^{N-1} \exp[-Y/(1 + (S/N)_{avg})]. \quad (K.2.6)$$

Using eq(K.2.6) in (K.1.45b) provides

$$P_d = 1 - \int_0^{Y_b} p(Y) dY \quad (K.2.7a)$$

$$= 1 - \frac{1}{(N-1)!} \int_0^{Y_b} \left(\frac{Y}{1 + (S/N)_{avg}} \right)^{N-1} \exp[-Y/(1 + (S/N)_{avg})] \frac{dY}{1 + (S/N)_{avg}} \quad (K.2.7b)$$

Using

$$x = \frac{Y}{1 + (S/N)_{avg}} \quad (K.2.8)$$

eq(K.2.7b) becomes

$$P_d = 1 - \int_0^{Y_b/(1+(S/N)_{avg})} [x^{N-1} \exp(-x) / (N-1)!] dx \quad (K.2.9a)$$

$$= 1 - I_\Gamma \left[\frac{Y_b}{\sqrt{N}(1+(S/N)_{avg})}, (N-1) \right]. \quad (K.2.9b)$$

K.3 Model I(COH) Signal in Noise using Coherent Pulse Train Detection

In this section, we consider the case of a coherent pulse train that is exactly known except for initial phase; ie., the phase relations between pulses in the train are assumed to be known exactly, but the initial phase on each train is unknown. This condition is met for the model I(COH) signal. We also note in the process synthesis procedure described in chapter 4, it is this case which is approached as the temporal correlation is increased.

The detection curve for this case can be obtained as follows. We first note that the signal-to-noise ratio defined above is that for single-pulse detection. Let us define this ratio for channel j (see next section) as $(S/N)_{j,1}$. The effective signal-to-noise ratio for an ideal coherent multiple pulse system on channel j using N_T pulses is [88,89]

$$(S/N)_{j,N_T} = N_T (S/N)_{j,1} \quad (K.3.1)$$

where N_T is the number of pulses coherently integrated. Thus, the model I(COH) detection curve can be plotted by first determining the single-pulse detection probability for $(S/N)_{j,1}$ and then using the detection probabilities corresponding to $(S/N)_{j,N_T}$. These analytic curves are used in chapter 7 to compare to the results obtained using the coherent pulse train detection implementation described in chapter 6.0.

K.4 Multichannel Post-Detection Integration

The case of multichannel post-detection integration has been considered in the context of frequency diverse channels [78] for the model I(INCOH) signal on each channel and no correlation between channels. This is a special case in this dissertation with total temporal correlation on each channel i and no cross-channel correlation between all $i, j = 1, 2, \dots, J$. In this section, we review this detection problem. In addition, we also consider the problems for the case II(INCOH) model with no channel correlation, as well as the model I(INCOH) signal with total correlation between channels. These last two models provide the extremes in detection performance for multichannel post-detection integration and will be used to bound the detection results presented in section 7.6.

K.4.1. Correlated Signal, Uncorrelated Channels

In this subsection, we briefly outline the approach presented in [78]. In that paper, an exact closed-form solution is derived for the output probability density function and the detection probability for a given false alarm probability and signal-to-noise ratio when all JN pulses are square law detected and incoherently integrated. Each of the JN pulses has independent Gaussian noise, and each pulse train of N pulses has an independent Gaussian signal, giving a total of J independent signals in all JN pulses. In [78], Vannicola considers representing eq(K.1.21) in our notation as

$$Y = \frac{1}{2} \sum_{i=1}^{JN} v_i^2 / \psi_0 = \sum_{i=1}^{JN} z_i. \quad (K.4.1)$$

where the probability density of Y under hypothesis H_0 is

$$P(Y; H_0) = \frac{Y^{JN-1} e^{-Y}}{(JN-1)!} \quad (K.4.2)$$

so that the false alarm probability becomes

$$P_{fa} = \int_{Y_b}^{\infty} P(Y; H_0) dY = \int_{Y_b}^{\infty} \frac{Y^{JN-1} e^{-Y}}{(JN-1)!} dY. \quad (K.4.3)$$

This integral has been evaluated in [77] for $1 \leq JN \leq 150$ and false-alarm probabilities equal to 10^{-m} , where m is an integer ranging between 1 and 12.

In [78], the probability of detection is derived starting with the characteristic function of the signal out of the N pulse integrator for a single channel using an alternate form of eq(K.1.37d). In our notation, this equation is expressed as

$$\overline{C_{Y_1}(\xi)}_{A_1} = \frac{(-1)^N}{(1 + N(S/N)_{avg}) [-1/(1+(S/N)_{avg}) + j\xi] (-1+j\xi)^{N-1}} \quad (K.4.4)$$

Adding the integrator outputs for the J channels, we have

$$Y = \sum_{i=1}^J Y_i \quad (K.4.5)$$

where Y_i is independent of Y_k , $i \neq k$. The characteristic function of Y is

$$\overline{C_Y(\xi)}_{\Delta_1, \Delta_2, \dots, \Delta_J} = \frac{(-1)^{JN}}{[1+N(S/N)_{avg}]^J \{-1/[1+(S/N)_{avg}] + j\xi\}^J (-1+j\xi)^{J(N-1)}} \quad (K.4.6)$$

Fourier transforming eq(K.4.6), Vannicola [78] obtains

$$\begin{aligned} p(Y) = \frac{(-\beta)^J e^{\beta Y}}{(1+\beta)^{JN-1}} & \left\{ \frac{[(1+\beta)Y]^{J-1}}{(J-1)!} + \sum_{j=0}^{J-2} (-1)^{J-1-j} \frac{(JN-2-j)!}{(JN-J-1)!(J-1-j)!} \right. \\ & \cdot \frac{[(1+\beta)Y]^j}{(j)!} \cdot (-1)^{J-1} e^{-(1+\beta)Y} \\ & \cdot \sum_{k=0}^{JN-J-1} \frac{(JN-2-k)!}{(JN-J-1-k)!(J-1)!} \frac{[(1+\beta)Y]^k}{(k)!} \end{aligned} \quad (K.4.7)$$

where $\beta = -1/[1+N(S/N)_{avg}]$. Using eq(K.4.7) in eq(K.1.45b),

$$P_d = 1 - \frac{(\beta)^{J-1}}{(1+\beta)^{JN-1}} \left\{ \sum_{j=0}^{J-1} \frac{(JN-2-j)!}{(JN-J-1)!(J-1-j)!} \frac{I_\Gamma((1+j), -\beta Y_b)}{j!} + \beta \sum_{k=0}^{JN-J-1} (1+\beta)^k \frac{(JN-2-k)!}{(JN-J-1-k)!(J-1)!} \frac{I_\Gamma((1+k), Y_b)}{k!} \right\}. \quad (K.4.8)$$

where $I_\Gamma(\cdot)$ is the incomplete gamma function. Probability of detection curves are shown in [78]. We note that these results describe the special case in this dissertation for incoherent integration of a multichannel Gaussian signal with $\lambda_{jj}=1$ $j=1,2,\dots,J$ and $\rho_{ij}=0$ in additive Gaussian white noise; ie., the signals are case I (INCOH) on each channel and are uncorrelated across channels. Finally, as noted in [78], computational difficulties were noted in the evaluation of eq(K.4.8). In Appendix L, an alternative derivation is suggested which may overcome these limitations.

K.4.2. Uncorrelated Signal, Uncorrelated Channels

For the case where the pulses on each channel are described by model II (INCOH), we consider the form of eq(K.2.5b) for each of the j channels, so that

$$\overline{C_{Y_j}(\xi)_{\Delta_j}} = \frac{1}{[1 + (S/N)_{avg}]^N \{ 1/[1 + (S/N)_{avg}] - j\xi \}^N}. \quad (K.4.9)$$

Adding the integrator outputs for the J channels, we have,

$$Y = \sum_{j=1}^J Y_j \quad (K.4.10)$$

where Y_j is independent of Y_k , $j \neq k$. The characteristic equation of Y is

$$\overline{C_Y(\xi)_{\Delta_j}} = \frac{1}{[1 + (S/N)_{avg}]^{JN} \{ 1/[1 + (S/N)_{avg}] - j\xi \}^{JN}} \quad (K.4.11)$$

where Δ_j contains the JN amplitudes. Again, using Fourier pair 431 in [75], the anticharacteristic function of eq(K.4.11) is

$$p(Y) = \frac{1}{[1 + (S/N)_{\text{avg}}]^{JN} (JN-1)!} Y^{JN-1} \exp\{-Y/[1+(S/N)_{\text{avg}}]\}. \quad (\text{K.4.12})$$

Using eq(K.4.12) in (K.4.44b) provides

$$P_d = 1 - \int_0^{Y_b} p(Y) dY \quad (\text{K.4.13a})$$

$$= 1 - \frac{1}{(JN-1)!} \int_0^{Y_b} \left(\frac{Y}{1+(S/N)_{\text{avg}}} \right)^{JN-1} \exp[-Y/(1+(S/N)_{\text{avg}})] \frac{dY}{1+A_0^2}. \quad (\text{K.4.13b})$$

Using $x = Y/[1+(S/N)_{\text{avg}}]$, eq(K.4.13b) becomes

$$P_d = 1 - \int_0^{Y_b/[1+(S/N)_{\text{avg}}]} [x^{JN-1} \exp(-x) / (JN-1)!] dx \quad (\text{K.4.14a})$$

$$= 1 - I_\Gamma \left[\frac{Y_b}{\sqrt{JN} [1+(S/N)_{\text{avg}}]}, (JN-1) \right]. \quad (\text{K.4.14b})$$

A comparison of eqs(K.4.14b) and (K.2.9b) indicates that in the special case where the pulses are uncorrelated on each channel as well as across each channel, the detection curves for the single channel model II(INCOH) can be used for the corresponding multichannel case with N replaced by JN . This result will be used in section 7.6.4 to provide a performance curve for a detection problem with $J=2$, $N=10$ by using an available single channel model II(INCOH) detection curve for 20 pulses.

K.4.3. Correlated Signals, Correlated Channels

For the case where the JN pulses on all the channels are described by the model I(INCOH) signal, we can reconsider the non-fluctuating pulse train of section K.1 to contain JN signals with constant amplitude and random phases on each realization. In this case, eq(K.1.21) again becomes

$$Y = \frac{1}{2} \sum_{i=1}^{JN} v_i^2 / \psi_0 = \sum_{i=1}^{JN} z_i. \quad (K.4.15)$$

Following the same discussion as presented in that section, but using JN to replace N , eq(K.1.47) is modified as

$$P_d = 1 - I_\Gamma \left[\frac{Y_b}{\sqrt{JN-1}}, (JN-2) \right] + \left[1 + \frac{1}{JN(S/N)_{avg}} \right]^{JN-1} \left\{ \exp[-Y_b/(1 + JN(S/N)_{avg})] \right. \\ \left. \cdot I_\Gamma \left[\frac{Y_b}{\sqrt{JN-1}(1+1/JN(S/N)_{avg})}, (JN-2) \right] - 1 \right\} \quad (K.4.16)$$

Similar to the previous case, eq(K.4.16) reveals that the detection curves for the single channel model I(INCOH) can be used for the corresponding multichannel case with N replaced by JN . Thus, as noted previously, available detection curves can be used for this limiting case.

APPENDIX L ALTERNATIVE DERIVATION FOR MULTICHANNEL P_d ANALYTIC EXPRESSIONS

In [78], Vannicola derives the derivation of the probability of detection using eq(K.4.4). In this appendix, we consider the alternative expression of eq(K.1.37d) for the i th channel; ie.,

$$C_{Y_i}(\xi) = \overline{C_{Y_i}(\xi|A_i)}_{A_i} = \frac{1}{\{1 - j\xi[1 + N(S/N)_{avg}]\} (1 - j\xi)^{N-1}}. \quad (L.1)$$

Adding the integrator outputs for the J channels, we have

$$Y = \sum_{i=1}^J Y_i \quad (L.2)$$

where Y_i is independent of Y_k , $i \neq k$. The characteristic function of Y is

$$C_Y(\xi) = \prod_{i=1}^J C_{Y_i}(\xi) \quad (L.3a)$$

$$= \frac{1}{\{1 - j\xi[1 + N(S/N)_{avg}]\}^J (1 - j\xi)^{J(N-1)}}. \quad (L.3b)$$

This equation enables us to obtain

$$\{1 - j\xi[1 + N(S/N)_{avg}]\}^J \overline{C_Y(\xi)}_{\Delta} = \frac{1}{(1 - j\xi)^{J(N-1)}}. \quad (L.4)$$

Let $\alpha = [1 + N(S/N)_{avg}]$ and consider

$$[1 - j\xi\alpha]^J = \sum_{k=0}^J \begin{bmatrix} J \\ k \end{bmatrix} (-j\xi\alpha)^{J-k} \quad (L.5a)$$

$$= 1 + \sum_{k=0}^{J-1} \begin{bmatrix} J \\ k \end{bmatrix} (-j\xi\alpha)^{J-k}. \quad (L.5b)$$

where

$$[J]_k = \frac{J(J-1)(J-2) \dots (J-k+1)}{k!} \quad (L.6)$$

Using eq(L.5b) in (L.4) and rearranging terms

$$\overline{C_Y(\xi)_\Delta} = \frac{1}{(1 - j\xi)^{J(N-1)}} - \sum_{k=0}^{J-1} [J]_k (-j\xi\alpha)^{J-k} \overline{C_Y(\xi)_\Delta} \quad (L.7a)$$

$$= \frac{1}{(1 - j\xi)^{J(N-1)}} - \sum_{k=0}^{J-1} [J]_k (\alpha)^{J-k} (-j\xi)^{J-k} \overline{C_Y(\xi)_\Delta} \quad (L.7b)$$

The anticharacteristic function of eq(L.7b) taken term by term is

$$p(Y) = \frac{Y^{J(N-1)-1} e^{-Y}}{[J(N-1)-1]!} - \sum_{k=0}^{J-1} [J]_k (\alpha)^{J-k} \frac{d^{J-k} p(Y)}{dY^{J-k}} \quad (L.8)$$

where

$$(-j\xi)^{J-k} \overline{C_Y(\xi)_\Delta} \xleftrightarrow{F.T.} \frac{d^{J-k} p(Y)}{dY^{J-k}} \quad (L.9)$$

Eq(L.8) is a non-homogeneous, linear differential equation with constant coefficients that depend upon α , J and k . The solution of this equation is to be used in eqs(L.7a) to obtain P_d .

APPENDIX M CONSIDERATION OF THE ERROR VECTORS $\underline{\epsilon}_p(n)$ AND $\underline{\epsilon}_p(n)$

In this appendix, we first consider the error vectors $\underline{\epsilon}(n)$ and $\underline{\epsilon}(n)$ discussed in chapter 6.0 and show the conditions under which they are equivalent. Second, we generalize the presentation of the normal equations derived in Appendix A to the case of processes with unconstrained quadrature components. Finally, we also derive the form of the augmented normal equations for the unconstrained quadrature component case.

M.1 CONDITIONS FOR THE EQUIVALENCE OF $\underline{\epsilon}_p(n)$ AND $\underline{\epsilon}_p(n)$

The $2J \times 1$ real forward error vector $\underline{\epsilon}_p(n)$ resulting from a filter of order p is expressed as

$$\underline{\epsilon}_p(n) = \underline{r}(n) - \hat{\underline{r}}_p(n|n-1, \dots, n-p) \quad (M.1.1)$$

where

$$\underline{\epsilon}_p(n) = \begin{bmatrix} \epsilon_{pI}(n) \\ \epsilon_{pQ}(n) \end{bmatrix} \quad (M.1.2a)$$

and

$$\underline{r}(n) = \begin{bmatrix} r_I(n) \\ r_Q(n) \end{bmatrix}. \quad (M.1.2b)$$

On the other hand, the complex $J \times 1$ error vector $\underline{\epsilon}_p(n)$ is expressed as

$$\underline{\epsilon}_p(n) = \underline{x}(n) - \hat{\underline{x}}_p(n|n-1, \dots, n-p) \quad (M.1.3)$$

where

$$\hat{\underline{x}}_p(n|n-1, \dots, n-p) = - \sum_{k=1}^p A_p^H(k) \underline{x}(n-k). \quad (M.1.4a)$$

and

$$\underline{x}(n) = \underline{x}_I(n) + j \underline{x}_Q(n). \quad (M.1.4b)$$

In Appendix N, we consider the probability density functions for both the conventional complex Gaussian random vector and the general Gaussian vector with unconstrained quadrature components. We showed that the latter is a more general density distribution which is equal to the former under the special conditions that

$$R_{x_{jk}}^{II}(l) = R_{x_{jk}}^{QQ}(l) \quad \text{all } j, k=1,2,\dots,J \quad (M.1.5a)$$

and

$$R_{x_{jk}}^{IQ}(l) = -R_{x_{jk}}^{QI}(l) \quad \text{all } j, k=1,2,\dots,J. \quad (M.1.5b)$$

Specifically, when eqs(M.1.5) hold,

$$p_{\underline{x}}(\underline{x}'_{1,N}) = p_{\underline{r}}(\underline{r}'_{1,N}) \quad (M.1.6)$$

where the quantities $\underline{x}'_{1,N}$ and $\underline{r}'_{1,N}$ denote the values assumed by the complex and real random vectors, $\underline{x}_{1,N}$ and $\underline{r}_{1,N}$ (see Appendix N). In this special case, therefore, the processes $\underline{x}_{1,N}$ and $\underline{r}_{1,N}$ are identical. And so, the vectors $\underline{r}(n)$ and $\underline{x}(n)$ denoted in eqs(M.1) and (M.1.3), respectively, and their associated linear estimates $\hat{\underline{r}}_p(n|n-1,\dots,n-p)$ and $\hat{\underline{x}}_p(n|n-1,\dots,n-p)$ are identical. As a result, the error vectors $\underline{e}_p(n)$ and $\underline{\varepsilon}_p(n)$ are also identical; ie.,

$$\underline{e}_p(n) = \underline{\varepsilon}_p(n) \quad \text{when } R_{x_{jk}}^{II}(l) = R_{x_{jk}}^{QQ}(l) \quad \text{and} \quad R_{x_{jk}}^{IQ}(l) = -R_{x_{jk}}^{QI}(l) \\ \text{all } j, k=1,2,\dots,J. \quad (M.1.7)$$

M.2 THE PREDICTION ERROR FILTER NORMAL EQUATIONS FOR PROCESSES WITH UNCONSTRAINED QUADRATURE COMPONENTS

From eqs(6.3.10), we have

$$\mathbf{e}_p(n) = \mathbf{r}(n) - \hat{\mathbf{r}}_p(n|n-1, \dots, n-p) \quad (\text{M.2.1a})$$

$$= \mathbf{r}(n) + \sum_{k=1}^p \mathcal{A}_p^T(k) \mathbf{r}(n-k) \quad (\text{M.2.1b})$$

$$= \mathbf{r}(n) + (\mathcal{A})_p^T \mathbf{r}_{n-1, n-p} \quad (\text{M.2.1c})$$

where $\hat{\mathbf{r}}_p(n|n-1, \dots, n-p)$ is obtained using a linear combination of p past data samples, and is defined as

$$\hat{\mathbf{r}}_p(n|n-1, \dots, n-p) = - \sum_{k=1}^p \mathcal{A}_p^T(k) \begin{bmatrix} \mathbf{x}_I(n-k) \\ \mathbf{x}_Q(n-k) \end{bmatrix} \quad (\text{M.2.2})$$

$$\mathbf{r}_{n-1, n-p}^T = [\mathbf{r}^T(n-1) \mathbf{r}^T(n-2) \dots \mathbf{r}^T(n-p)]. \quad (\text{M.2.3})$$

The vector of matrices $(\mathcal{A})_p^T$ is defined without the identity matrix $\mathbf{I}$, so that

$$(\mathcal{A})_p^H = [\mathbf{A}_p^T(1) \mathbf{A}_p^T(2) \dots \mathbf{A}_p^T(p)] \quad (\text{M.2.4})$$

where the $2J \times 2J$ block matrix $\mathcal{A}_p^T(k)$ is expressed in terms of its $J \times J$ block elements such that

$$\mathcal{A}_p^T(k) = \begin{bmatrix} \mathbf{A}_{p11}^T(k) & \mathbf{A}_{p21}^T(k) \\ \mathbf{A}_{p12}^T(k) & \mathbf{A}_{p22}^T(k) \end{bmatrix} \quad (\text{M.2.5})$$

and the numerical subscripts on the $J \times J$ block matrix elements of eq(M.2.5) designate their position in the matrix. In expanded form, eq(M.2.1) can be written as

$$\mathbf{e}_p(n) = \begin{bmatrix} \mathbf{e}_{pI}(n) \\ \mathbf{e}_{pQ}(n) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_I(n) \\ \mathbf{x}_Q(n) \end{bmatrix} + \sum_{k=1}^p \begin{bmatrix} A_{p11}^T(k) & A_{p21}^T(k) \\ A_{p12}^T(k) & A_{p22}^T(k) \end{bmatrix} \begin{bmatrix} \mathbf{x}_I(n-k) \\ \mathbf{x}_Q(n-k) \end{bmatrix}. \quad (\text{M.2.6})$$

Let

$$E[\mathbf{I}(n) \mathbf{I}_{n-1,n-p}^T] \equiv \mathbf{C} \quad (\text{M.2.7})$$

and

$$E[\mathbf{I}_{n-1,n-p} \mathbf{I}_{n-1,n-p}^T] = [\tilde{\mathbf{R}}_\pi]_p \equiv \mathbf{Q}. \quad (\text{M.2.8})$$

With respect to dimensions, $(\mathbf{A})_p^T$ is $2J \times 2Jp$, $\mathbf{I}_{n-1,n-p}$ is $2Jp \times 1$, $\mathbf{C}$ is $2J \times 2Jp$ and $\mathbf{Q}$ is $2Jp \times 2Jp$. We note that $\mathbf{Q}^T = \mathbf{Q}$ and $\mathbf{V}^T \mathbf{Q} \mathbf{V} > 0$ for any $2Jp \times 1$ vector $\mathbf{V} \neq \mathbf{0}$ when $\mathbf{Q}$ is positive definite. Now, the $2J \times 2J$ error covariance matrix is expressed as

$$E[\mathbf{e}_p(n) \mathbf{e}_p^T(n)] = E\{[\mathbf{I}(n) + (\mathbf{A})_p^T \mathbf{I}_{n-1,n-p}][\mathbf{I}^T(n) + \mathbf{I}_{n-1,n-p}^T (\mathbf{A})_p]\} \quad (\text{M.2.9a})$$

$$= E[\mathbf{I}(n) \mathbf{I}^T(n)] + (\mathbf{A})_p^T \mathbf{C}^T + \mathbf{C} (\mathbf{A})_p + (\mathbf{A})_p^T \mathbf{Q} (\mathbf{A})_p \quad (\text{M.2.9b})$$

$$= \mathbf{R}_\pi(0) - \mathbf{C} \mathbf{Q}^{-1} \mathbf{C}^T + [(\mathbf{A})_p^T + \mathbf{C} \mathbf{Q}^{-1}] \mathbf{Q} [(\mathbf{A})_p^T + \mathbf{C} \mathbf{Q}^{-1}]^T. \quad (\text{M.2.9c})$$

Let the $2J \times 2Jp$ matrix $[(\mathbf{A})_p^T + \mathbf{C} \mathbf{Q}^{-1}]$ be expressed as

$$[(\mathcal{A})_p^T + CQ^{-1}] = \begin{bmatrix} \mathcal{V}_1^T \\ \mathcal{V}_2^T \\ \dots \\ \mathcal{V}_J^T \end{bmatrix} \quad (\text{M.2.10})$$

where $\mathcal{V}_j$ is a $2J \times 1$ vector. We now denote the last term in eq(M.2.9c) as the $2J \times 2J$ matrix

$$\mathcal{L} = [(\mathcal{A})_p^T + CQ^{-1}] Q [(\mathcal{A})_p^T + CQ^{-1}]^T = [\mathcal{L}_{mj}] \quad 1 \leq m, j \leq J \quad (\text{M.2.11a})$$

$$= \begin{bmatrix} \mathcal{V}_1^T \\ \mathcal{V}_2^T \\ \dots \\ \mathcal{V}_J^T \end{bmatrix} Q [\mathcal{V}_1 \ \mathcal{V}_2 \ \dots \ \mathcal{V}_J] \quad (\text{M.2.11b})$$

where the real scalar $\mathcal{L}_{mj}$ is now expressed as

$$\mathcal{L}_{mj} = \mathcal{V}_m^T Q \mathcal{V}_j. \quad (\text{M.2.12})$$

The real quantity $\mathcal{L}_{jj} = \mathcal{V}_j^T Q \mathcal{V}_j > 0$ for any $\mathcal{V}_j \neq \mathbf{0}$, since Q is Hermitian and provided Q is positive definite; the minimum value of $\mathcal{L}_{jj}$ is zero and is attained if and only if $\mathcal{V}_j = \mathbf{0}$. Therefore, $\text{tr} \mathcal{L}$ is minimized, attaining value zero, by the choice $\mathcal{V}_j = \mathbf{0}$, $1 \leq j \leq J$. Using eq(M.2.9c), $\text{tr} E[\mathbf{e}_p(n) \mathbf{e}_p^T(n)] = E[\mathbf{e}_p^T(n) \mathbf{e}_p(n)]$ is minimized by the choice of $(\mathcal{A})_p^T$ as

$$(\mathbf{A})_p^T = -\mathbf{CQ}^{-1} \quad (\text{M.2.13a})$$

or

$$(\mathbf{A})_p^T \mathbf{Q} = -\mathbf{C} \quad (\text{M.2.13b})$$

since the leading two terms in eq(M.2.9) are independent of $(\mathbf{A})_p^T$. Eq(M.2.13b) can be written in expanded form using eqs(M.2.4), (M.2.7) and (M.2.8), so that

$$\sum_{k=1}^p \begin{bmatrix} A_{p11}^T(k) & A_{p21}^T(k) \\ A_{p12}^T(k) & A_{p22}^T(k) \end{bmatrix} \begin{bmatrix} R_{xx}^{II}(m-k) & R_{xx}^{IQ}(m-k) \\ R_{xx}^{QI}(m-k) & R_{xx}^{QQ}(m-k) \end{bmatrix} = - \begin{bmatrix} R_{xx}^{II}(m) & R_{xx}^{IQ}(m) \\ R_{xx}^{QI}(m) & R_{xx}^{QQ}(m) \end{bmatrix} \quad (\text{M.2.14a})$$

$m = 1, 2, \dots, p$

or

$$\sum_{k=1}^p \mathbf{A}_p^T(k) R_{\pi\pi}(m-k) = -R_{\pi\pi}(m) \quad m = 1, 2, \dots, p. \quad (\text{M.2.14b})$$

Eq(M.2.14) contains the normal equations for a prediction error filter of order p using Gaussian processes with unconstrained quadrature components.

M.3 PRINCIPLE OF ORTHOGONALITY FOR GAUSSIAN PROCESSES WITH UNCONSTRAINED QUADRATURE COMPONENTS

We now show that the error vector $\mathbf{e}_p(n)$ is uncorrelated with past observation vectors when eqs(M.2.14) hold. Eq (M.2.14) can be expressed as

$$\sum_{k=1}^p \mathbf{A}_p^T(k) E[\mathbf{r}(n-k) \mathbf{r}^T(n-m)] = -E[\mathbf{r}(n) \mathbf{r}^T(n-m)] \quad m = 1, 2, \dots, p. \quad (\text{M.3.1})$$

Rearranging eq(M.3.1) and interchanging the order of expectation and summation, we can obtain

$$E\left\{\mathbf{e}(n) + \sum_{k=1}^p \mathbf{A}_p^T(k) \mathbf{e}(n-k)\right\} \mathbf{e}^T(n-m) = [0] \quad m = 1, 2, \dots, p. \quad (\text{M.3.2})$$

From eq(M.2.1b), it follows that

$$E[\mathbf{e}_p(n) \mathbf{e}^T(n-m)] = [0] \quad m = 1, 2, \dots, p. \quad (\text{M.3.3})$$

In expanded form, eq(M.3.3) becomes

$$\begin{bmatrix} E[\mathbf{e}_I(n) \mathbf{x}_I^T(n-l)] & E[\mathbf{e}_I(n) \mathbf{x}_Q^T(n-l)] \\ E[\mathbf{e}_Q(n) \mathbf{x}_I^T(n-l)] & E[\mathbf{e}_Q(n) \mathbf{x}_Q^T(n-l)] \end{bmatrix} = \begin{bmatrix} [0] & [0] \\ [0] & [0] \end{bmatrix} \quad m = 1, 2, \dots, p. \quad (\text{M.3.4})$$

Eqs(M.3.3) and (M.3.4) denote the orthogonality principle for the concatenated quadrature components. It indicates that when the error vector of eq(M.2.1) is a minimum mean-squared error(MMSE) vector, the quadrature components of the error vector are uncorrelated with those of the past observation processes.

Following a similar discussion as presented in Appendix A leading to eq(A.35), we can obtain

$$E[\mathbf{e}_p(n) \mathbf{e}_p^T(n)] = \begin{cases} [\mathbf{R}_{zz}(0)]^q + [\mathbf{R}_{wz}(0)]^q + [\mathbf{R}_{wz}^H(0)]^q + [\mathbf{R}_{ww}(0)]^q & k=n \\ [0] & k \neq n. \end{cases} \quad (\text{M.3.5})$$

where the superscript q denotes the quadrature form of the correlation matrices. Eq(M.3.5) indicates that the current MMSE quadrature components are white noise vector processes. We also note that this condition holds when the

coefficients of the estimator satisfy the concatenated quadrature form of the normal equations.

M.4 THE AUGMENTED PREDICTION ERROR FILTER NORMAL EQUATIONS FOR PROCESSES WITH UNCONSTRAINED QUADRATURE COMPONENTS

Following a presentation similar to that in section 2.4.1, we first consider eq(M.2.1c) written in matrix form as

$$\mathbf{e}_p(n) = \mathbf{A}_p^T \mathbf{L}_{n,n-p} \quad (\text{M.4.1})$$

where

$$\mathbf{A}_p^T = [\mathbf{I} \quad (\mathbf{A}_p)^T] \quad (\text{M.4.2a})$$

$$\mathbf{L}_{n,n-p}^T = [\mathbf{L}^T(n) \mathbf{L}^T(n-1) \dots \mathbf{L}^T(n-p)] \quad (\text{M.4.2b})$$

$$\mathbf{L}^T(k) = [r_1(k) \ r_2(k) \dots r_J(k)]. \quad (\text{M.4.2c})$$

Post-multiply both sides of eq(M.4.1) by $\mathbf{L}_{n,n-p}^T$ and take the expectation operation so that

$$E[\mathbf{e}_p(n) \mathbf{L}_{n,n-p}^T] = E[\mathbf{A}_p^T \mathbf{L}_{n,n-p} \mathbf{L}_{n,n-p}^T] \quad (\text{M.4.3a})$$

$$= \mathbf{A}_p^T E[\mathbf{L}_{n,n-p} \mathbf{L}_{n,n-p}^T] \quad (\text{M.4.3b})$$

$$= \mathbf{A}_p^T [\mathbf{R}_{xx}]_{p+1} \quad (\text{M.4.3c})$$

where the subscript $p+1$ denotes the dimension of the correlation matrix in terms of the $J \times J$ blocks and $[\mathbf{R}_{xx}]_{p+1}$ is defined as

$$[\tilde{\mathbf{R}}_{\mathbf{x}}]_{p+1} = E[\mathbf{x}_{n,n-p} \mathbf{x}_{n,n-p}^T]. \quad (\text{M.4.4})$$

Again, $\sim$ denotes time order reversal of the vector $\mathbf{x}_{n,n-p}$. Using eq(M.4.2), we now expand the LHS of eq(M.4.3a) as

$$E[\mathbf{e}_p(n) \mathbf{x}_{n,n-p}^T] = E\{\mathbf{e}_p(n) [\mathbf{x}^T(n) \mathbf{x}^T(n-1) \dots \mathbf{x}^T(n-p)]\} \quad (\text{M.4.5a})$$

$$= \{ E[\mathbf{e}_p(n) \mathbf{x}^T(n)] E[\mathbf{e}_p(n) \mathbf{x}^T(n-1)] \dots E[\mathbf{e}_p(n) \mathbf{x}^T(n-p)] \} \quad (\text{M.4.5b})$$

and determine the coefficients of $\mathbf{A}_p^T$ subject to the condition that $\mathbf{e}_p(n)$ is a minimum mean-squared error process. Under this condition, the orthogonality principle holds (see Section M.3): i.e.,

$$E[\mathbf{e}_p(n) \mathbf{x}^T(n-k)] = [0] \quad 1 \leq k \leq p \quad (\text{M.4.6})$$

so that eq(M.4.5b) becomes

$$E[\mathbf{e}_p(n) \mathbf{x}_{n,n-p}^T] = \{ E[\mathbf{e}_p(n) \mathbf{x}^T(n)] [0] [0] \dots [0] \} \quad (\text{M.4.7})$$

where $[0]$ is a $2J \times 2J$ null matrix. Solving eq(M.2.1b) for $\mathbf{x}(n)$, we obtain

$$\mathbf{x}(n) = \mathbf{e}_p(n) + \sum_{k=1}^p \mathbf{A}_p^T(k) \mathbf{x}(n-k) \quad (\text{M.4.8a})$$

so that

$$E[\mathbf{e}_p(n) \mathbf{x}^T(n)] = E[\mathbf{e}_p(n) \mathbf{e}_p^T(n)] - \sum_{k=1}^p E[\mathbf{e}_p(n) \mathbf{x}^T(n-k)] \mathbf{A}_p(k). \quad (\text{M.4.8b})$$

Again, using eq(M.4.6), the summation term in eq(M.4.8b) is a null matrix. Substituting the remaining term into eq(M.4.7) yields

$$E[\mathbf{e}_p(n) \mathbf{e}_{n,n-p}^T] = \{ E[\mathbf{e}_p(n) \mathbf{e}_p^T(n)] [0] [0] \dots [0] \} \quad (\text{M.4.9a})$$

$$= \{ [\Sigma_f]_{p_r}^T [0] [0] \dots [0] \} \quad (\text{M.4.9b})$$

where

$$[\Sigma_f]_{p_r} = E[\mathbf{e}_p(n) \mathbf{e}_p^T(n)] = [\Sigma_f]_{p_r}^T \quad (\text{M.4.10})$$

is the $2J \times 2J$ forward prediction error covariance matrix for the p th order filter expressed in concatenated quadrature form. Since eqs(M.4.3c) and (M.4.19b) are identical expressions, equality of the RHS of these equations provides

$$\mathbf{A}_p^T [\tilde{\mathbf{R}}_{\Pi}]_{p+1} = \{ [\Sigma_f]_{p_r}^T [0] [0] \dots [0] \}. \quad (\text{M.4.11})$$

Taking the transpose of eq(M.4.11) and noting that $[\tilde{\mathbf{R}}_{\Pi}]$ is equal to its transpose, we obtain

$$[\tilde{\mathbf{R}}_{\Pi}]_{p+1} \mathbf{A}_p = [\Sigma_f]_{p_r} \mathbf{I} \quad (\text{M.4.12})$$

where

$$\mathbf{I} = \{ \mathbf{I} [0] [0] \dots [0] \} \quad (\text{M.4.13})$$

and $\mathbf{I}$ is a $2J \times 2J$ identity matrix. Eq(M.4.12) is the augmented form of the normal equations for a forward multichannel prediction error filter with Gaussian unconstrained quadrature components. Thus, $\mathbf{A}_p$ contains the minimum mean-squared error (MMSE) matrix coefficients for a linear prediction error filter of order p .

APPENDIX N RELATIONSHIP BETWEEN THE DENSITY FUNCTIONS OF THE COMPLEX GAUSSIAN PROCESS AND THE GENERAL GAUSSIAN PROCESS WITH UNCONSTRAINED QUADRATURE COMPONENTS

In this appendix, we follow a presentation developed by Weiner [165] which considers the probability density functions for both the conventional complex Gaussian random vector and the general Gaussian vector with unconstrained quadrature components. We will show that the latter is a more general density distribution which is equal to the former under special conditions. A discussion of this subject area has also been addressed by Goodman [166]. In the discussion presented here, we will generalize that of [165] to the multichannel case.

The general case of JN complex random variables is now considered where J and N denote the number of channels and time samples, respectively. Using the definitions from section 6.3, let us consider a complex baseband process $x_i(n)$ for channel i such that

$$x_i(n) = x_{iI}(n) + j x_{iQ}(n) \quad i = 1, 2, \dots, J \quad n = 1, 2, \dots, N \quad (\text{N.1})$$

where $x_{iI}(n)$ and $x_{iQ}(n)$ are the in-phase and quadrature components of the process $x_i(n)$, respectively. We now define the vectors

$$\mathbf{x}_{I1,N}^T = [x_{I1}^T(1) \ x_{I1}^T(2) \ \dots \ x_{I1}^T(N)] \quad (\text{N.2a})$$

and

$$\mathbf{x}_{Q1,N}^T = [x_{Q1}^T(1) \ x_{Q1}^T(2) \ \dots \ x_{Q1}^T(N)] \quad (\text{N.2b})$$

where

$$\mathbf{x}_{I1}^T(k) = [x_{1I}(k) \ x_{2I}(k) \ \dots \ x_{JI}(k)] \quad k = 1, 2, \dots, N \quad (\text{N.3a})$$

$$\mathbf{x}_{Q1}^T(k) = [x_{1Q}(k) \ x_{2Q}(k) \ \dots \ x_{JQ}(k)] \quad k = 1, 2, \dots, N \quad (\text{N.3b})$$

with initial conditions

$$\underline{x}_{I1,1} = \underline{x}_I(1) \quad (\text{N.4a})$$

$$\underline{x}_{Q1,1} = \underline{x}_Q(1). \quad (\text{N.4b})$$

Eqs(N.2) and (N.3) enable us to write the complex vectors $\underline{x}(n)$ and $\underline{x}_{1,N}$ denoted by

$$\underline{x}(n) = \underline{x}_I(n) + j \underline{x}_Q(n) \quad n = 1, 2, \dots, N \quad (\text{N.5a})$$

and

$$\underline{x}_{1,N} = \underline{x}_{I1,N} + j \underline{x}_{Q1,N} \quad (\text{N.5b})$$

We now introduce the means of the above quantities, such that

$$\bar{x}_i(n) = E[x_i(n)] \quad n = 1, 2, \dots, N \quad (\text{N.6a})$$

$$\bar{\underline{x}}_I^T(k) = E[\underline{x}_I^T(k)] \quad k = 1, 2, \dots, N \quad (\text{N.6b})$$

$$\bar{\underline{x}}_Q^T(k) = E[\underline{x}_Q^T(k)] \quad k = 1, 2, \dots, N \quad (\text{N.6c})$$

$$\bar{\underline{x}}_{I1,N}^T = E[\underline{x}_{I1,N}^T] \quad (\text{N.6d})$$

$$\bar{\underline{x}}_{Q1,N}^T = E[\underline{x}_{Q1,N}^T] \quad (\text{N.6e})$$

and

$$\bar{\underline{x}}_{1,N} = E[\underline{x}_{1,N}]. \quad (\text{N.6f})$$

By definition, the $JN \times JN$ covariance matrix of the complex vector $\underline{x}_{1,N}$ is expressed as

$$K_{\underline{x}\underline{x}} = E\{[\underline{x}_{1,N} - \bar{\underline{x}}_{1,N}][\underline{x}_{1,N} - \bar{\underline{x}}_{1,N}]^H\}. \quad (\text{N.7})$$

Its inverse is denoted as $Q_{\underline{x}\underline{x}}$. Specifically,

$$Q_{\underline{x}\underline{x}} = [K_{\underline{x}\underline{x}}]^{-1}. \quad (N.8)$$

Finally, let $\underline{x}'(n)$ denote the value assumed by the random vector $\underline{x}(n)$, $n=1,2,\dots,N$, and define the vectors

$$[\underline{x}'(n)]^T = [x'_1(n) \ x'_2(n) \ \dots \ x'_j(n)] \quad (N.9a)$$

and

$$[\underline{x}'_{1,N}]^T = \{ [\underline{x}'(1)]^T [\underline{x}'(2)]^T \dots [\underline{x}'(N)]^T \}. \quad (N.9b)$$

The complex vector $\underline{x}_{1,N}$ is defined to be Gaussian if and only if its density function is of the form

$$p_{\underline{x}}(\underline{x}'_{1,N}) = \frac{1}{[(\pi)^{JN} |K_{\underline{x}\underline{x}}|]} \exp \left\{ - [\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}]^H Q_{\underline{x}\underline{x}} [\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}] \right\}. \quad (N.10)$$

Alternatively, one could focus on the real random vectors expressed in eq(N.2). It is convenient to combine these two vectors into the single concatenated vector

$$[\underline{l}_{1,N}]^T = [\underline{x}_{1,N}^T \ \underline{q}_{1,N}^T]. \quad (N.11)$$

The mean vector of $[\underline{l}_{1,N}]$ is given by

$$[\bar{\underline{l}}_{1,N}]^T = [\bar{\underline{x}}_{1,N}^T \ \bar{\underline{q}}_{1,N}^T]. \quad (N.12)$$

The $JN \times JN$ covariance matrices pertaining to the quadrature components are given by

$$K_{\underline{x}\underline{x}}^{\Pi} = E\{[\underline{x}_{I1,N} - \bar{\underline{x}}_{I1,N}][\underline{x}_{I1,N} - \bar{\underline{x}}_{I1,N}]^T\} \quad (N.13a)$$

$$K_{\underline{x}\underline{x}}^{QQ} = E\{[\underline{x}_{Q1,N} - \bar{\underline{x}}_{Q1,N}][\underline{x}_{Q1,N} - \bar{\underline{x}}_{Q1,N}]^T\} \quad (N.13b)$$

$$K_{\underline{x}\underline{x}}^{OI} = E\{[\underline{x}_{Q1,N} - \bar{\underline{x}}_{Q1,N}][\underline{x}_{I1,N} - \bar{\underline{x}}_{I1,N}]^T\} \quad (N.13c)$$

$$K_{\underline{x}\underline{x}}^{IQ} = E\{[\underline{x}_{I1,N} - \bar{\underline{x}}_{I1,N}][\underline{x}_{Q1,N} - \bar{\underline{x}}_{Q1,N}]^T\}. \quad (N.13d)$$

It follows that the $2JN \times 2JN$ covariance matrix of $[\underline{I}_{1,N}]$ is

$$K_{\underline{I}} = E\{[\underline{I}_{1,N} - \bar{\underline{I}}_{1,N}][\underline{I}_{1,N} - \bar{\underline{I}}_{1,N}]^T\} \quad (N.14a)$$

$$= \begin{bmatrix} K_{\underline{x}\underline{x}}^{\Pi} & K_{\underline{x}\underline{x}}^{IQ} \\ K_{\underline{x}\underline{x}}^{OI} & K_{\underline{x}\underline{x}}^{QQ} \end{bmatrix}. \quad (N.14b)$$

Let the inverse of $K_{\underline{I}}$ be denoted by $Q_{\underline{I}}$; ie.,

$$Q_{\underline{I}} = [K_{\underline{I}}]^{-1}. \quad (N.15)$$

Observe that the $2JN \times 2JN$ real matrices $K_{\underline{I}}$ and $Q_{\underline{I}}$ should not be confused with the $JN \times JN$ complex matrices $K_{\underline{x}\underline{x}}$ and $Q_{\underline{x}\underline{x}}$ which were defined in eqs(N.7) and (N.8). Finally, let $[\underline{I}'_{1,N}]$ denote the values assumed by the random variable $[\underline{I}_{1,N}]$ with quadrature components $[\underline{x}'_{I1,N}]$ and $[\underline{x}'_{Q1,N}]$ such that

$$[\underline{I}'_{1,N}]^T = \{[\underline{x}'_{I1,N}]^T [\underline{x}'_{Q1,N}]^T\}. \quad (N.16)$$

The real vector $\underline{I}_{1,N}$ is defined to be Gaussian if and only if its density function is of the form

$$p_{\mathbf{r}}(\mathbf{r}_{1,N}') = \frac{1}{[(2\pi)^{2JN} |\mathbf{K}_{\mathbf{r}}|]^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} [\mathbf{r}_{1,N}' - \bar{\mathbf{r}}_{1,N}]^T \mathbf{Q}_{\mathbf{r}} [\mathbf{r}_{1,N}' - \bar{\mathbf{r}}_{1,N}] \right\} \quad (\text{N.17})$$

By definition, $p_{\mathbf{r}}(\mathbf{r}_{1,N}')$ is the $2JN$ -dimensional multivariate density function for the real random variables $x_{1I}(k)$, $x_{2I}(k)$, ..., $x_{JI}(k)$, $x_{1Q}(k)$, $x_{2Q}(k)$, ..., $x_{JQ}(k)$ for $k = 1, 2, \dots, N$. Also, since we have from eq(N.5b)

$$\mathbf{x}_{1,N} = \mathbf{x}_{I1,N} + j \mathbf{x}_{Q1,N}, \quad (\text{N.18})$$

it follows that

$$p_{\mathbf{x}}(\mathbf{x}_{1,N}') = p_{\mathbf{x}_I \mathbf{x}_Q}([\mathbf{x}_{I1,N}', [\mathbf{x}_{Q1,N}']]). \quad (\text{N.19})$$

Thus, the NJ -dimensional multivariate density function for the complex random variables $x_1(k)$, $x_2(k)$, ..., $x_J(k)$, for $k = 1, 2, \dots, N$ should also be equal to the $2JN$ -dimensional multivariate density function for the real random variables $x_1(k)$, $x_2(k)$, ..., $x_J(k)$, for $k = 1, 2, \dots, N$. This gives rise to the question, "Must certain conditions hold if $p_{\mathbf{x}}(\mathbf{x}_{1,N}')$ given by eq(N.10) is to equal $p_{\mathbf{r}}(\mathbf{r}_{1,N}')$, as given by eq(N.17)?" By inspection of the two equations, it is apparent that

$$p_{\mathbf{x}}(\mathbf{x}_{1,N}') = p_{\mathbf{r}}(\mathbf{r}_{1,N}') \quad (\text{N.20})$$

provided the following two equalities are met:

$$1.) \quad \frac{1}{[(\pi)^{JN} |\mathbf{K}_{\mathbf{x}}|]} = \frac{1}{[(2\pi)^{2JN} |\mathbf{K}_{\mathbf{r}}|]^{\frac{1}{2}}} \quad (\text{N.21})$$

$$2.) \quad -[\mathbf{x}_{1,N}' - \bar{\mathbf{x}}_{1,N}]^H \mathbf{Q}_{\mathbf{x}} [\mathbf{x}_{1,N}' - \bar{\mathbf{x}}_{1,N}] = -\frac{1}{2} [\mathbf{r}_{1,N}' - \bar{\mathbf{r}}_{1,N}]^T \mathbf{Q}_{\mathbf{r}} [\mathbf{r}_{1,N}' - \bar{\mathbf{r}}_{1,N}]. \quad (\text{N.22})$$

Equality 1

Squaring eq(N.21) and rearranging terms, equality 1 is equivalent to

$$[\det(K_{\underline{xx}})]^2 = (2)^{2JN} \det(K_{\underline{xx}}) \quad (N.23)$$

where the notation $\det(A)$ replaces $|A|$. For any square matrix A , it is known that

$$\det(A^T) = \det(A). \quad (N.24)$$

Therefore,

$$\det(K_{\underline{xx}}) = \det(K_{\underline{xx}}^T). \quad (N.25)$$

However, as pointed out previously, $K_{\underline{xx}}$ is Hermitian. It follows that

$$K_{\underline{xx}}^T = K_{\underline{xx}}^* \quad (N.26)$$

where $*$ denotes the complex conjugation relation. Consequently,

$$\det(K_{\underline{xx}}) = \det(K_{\underline{xx}}^*) \quad (N.27)$$

and the left side of eq(N.23) can be expressed as

$$[\det(K_{\underline{xx}})]^2 = \det(K_{\underline{xx}}) \det(K_{\underline{xx}}^*). \quad (N.28)$$

Making use of eq(N.5b) and (N.13) in (N.7) and substituting the results into eq(N.28), we obtain

$$[\det(K_{\underline{xx}})]^2 = \det[(K_{\underline{xx}}^{\text{II}} + K_{\underline{xx}}^{\text{QQ}}) + j(K_{\underline{xx}}^{\text{QI}} - K_{\underline{xx}}^{\text{IQ}})] \\ \cdot \det[(K_{\underline{xx}}^{\text{II}} + K_{\underline{xx}}^{\text{QQ}}) - j(K_{\underline{xx}}^{\text{QI}} - K_{\underline{xx}}^{\text{IQ}})]. \quad (\text{N.29})$$

Attention is now focused on the RHS of eq(N.23). It is necessary to express $\det(K_{\underline{II}})$ in terms of its submatrices $K_{\underline{xx}}^{\text{II}}$, $K_{\underline{xx}}^{\text{QQ}}$, $K_{\underline{xx}}^{\text{QI}}$, and $K_{\underline{xx}}^{\text{IQ}}$. For this purpose, the following properties of determinants are useful:

Property 1

Given square matrices A, B, and C, such that $C = AB$, then

$$\det(C) = \det(A)\det(B). \quad (\text{N.30})$$

Property 2

Having partitioned a square matrix into triangular form, where the submatrices are all square and of the same order, then the determinant of the matrix equals the product of the submatrices on the principal diagonal. For example, let O, E, F, and G be square matrices of the same order where [0] is a null matrix. Then

$$\det\left\{\begin{bmatrix} E & F \\ [0] & G \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} E & [0] \\ F & G \end{bmatrix}\right\} = \det(E)\det(G). \quad (\text{N.31})$$

With properties 1 and 2 in mind, let us attempt to decompose $K_{\underline{II}}$ in the form shown below.

$$K_{\underline{II}} = \begin{bmatrix} K_{\underline{xx}}^{\text{II}} & K_{\underline{xx}}^{\text{IQ}} \\ K_{\underline{xx}}^{\text{QI}} & K_{\underline{xx}}^{\text{QQ}} \end{bmatrix} = \begin{bmatrix} I & -jI \\ [0] & D \end{bmatrix} \begin{bmatrix} A & [0] \\ B & I \end{bmatrix} \begin{bmatrix} I & jI \\ [0] & C \end{bmatrix} \quad (\text{N.32})$$

where $[0]$, I , A , B , C , and D are all $JN \times JN$ matrices, $[0]$ is a null matrix and I is an identity matrix. Recall that

$$\det(I) = 1. \quad (N.33)$$

If the decomposition of eq(N.32) is possible, then

$$\det(K_{II}) = \det(D)\det(A)\det(C) \quad (N.34)$$

Carrying out the matrix product on the RHS of eq(N.32), we have

$$K_{II} = \begin{bmatrix} K_{XX}^{II} & K_{XX}^{IQ} \\ K_{XX}^{QI} & K_{XX}^{QQ} \end{bmatrix} = \begin{bmatrix} A-jB & B+j(A-C) \\ DB & D(C+jB) \end{bmatrix}. \quad (N.35)$$

Equating the appropriate submatrices, there results

$$K_{XX}^{II} = A - j B \quad (N.36a)$$

$$K_{XX}^{IQ} = B + j (A - C) \quad (N.36b)$$

$$K_{XX}^{QI} = DB \quad (N.36c)$$

$$K_{XX}^{QQ} = D(C + j B). \quad (N.36d)$$

Solution of eqs(N.36) for the matrices A , B , C , and D yields

$$A = K_{\Pi}^{\Pi} + j (K_{\Delta\Delta}^{\Pi} + j K_{\Delta\Delta}^{IQ}) [K_{\Delta\Delta}^{QQ} - j K_{\Delta\Delta}^{QI}]^{-1} (K_{\Delta\Delta}^{QI}) \quad (N.37a)$$

$$B = (K_{\Delta\Delta}^{\Pi} + j K_{\Delta\Delta}^{IQ}) [K_{\Delta\Delta}^{QQ} - j K_{\Delta\Delta}^{QI}]^{-1} (K_{\Delta\Delta}^{QI}) \quad (N.37b)$$

$$C = (K_{\Delta\Delta}^{\Pi} + j K_{\Delta\Delta}^{IQ}) \quad (N.37c)$$

$$D = [K_{\Delta\Delta}^{QQ} - j K_{\Delta\Delta}^{QI}] [K_{\Delta\Delta}^{\Pi} + j K_{\Delta\Delta}^{IQ}]^{-1}. \quad (N.37d)$$

Using eqs(N.37) in eq(N.34), it follows that

$$\begin{aligned} \det(K_{\Pi\Pi}) &= \det \{ [K_{\Delta\Delta}^{QQ} - j K_{\Delta\Delta}^{QI}] [K_{\Delta\Delta}^{\Pi} + j K_{\Delta\Delta}^{IQ}]^{-1} \} \\ &\quad \cdot \det \{ K_{\Delta\Delta}^{\Pi} + j [K_{\Delta\Delta}^{\Pi} + j K_{\Delta\Delta}^{IQ}] [K_{\Delta\Delta}^{QQ} - j K_{\Delta\Delta}^{QI}]^{-1} (K_{\Delta\Delta}^{QI}) \} \\ &\quad \cdot \det \{ [K_{\Delta\Delta}^{\Pi} + j K_{\Delta\Delta}^{IQ}] \}. \end{aligned} \quad (N.38)$$

Examination of eqs(N.29) and (N.38) reveals that equality 1, as expressed by eq(N.23), will not be met in general. The equality will hold provided

$$K_{\Delta\Delta}^{\Pi} = K_{\Delta\Delta}^{QQ} \quad (N.39a)$$

and

$$K_{\Delta\Delta}^{QI} = - K_{\Delta\Delta}^{IQ}. \quad (N.39b)$$

Then eq(N.29) becomes

$$[\det(K_{\Delta\Delta})]^2 = \det[2(K_{\Delta\Delta}^{\Pi} + j K_{\Delta\Delta}^{QI})] \det[2(K_{\Delta\Delta}^{\Pi} - j K_{\Delta\Delta}^{QI})]. \quad (N.40)$$

Given an NxN matrix A and a constant k, such that B=kA, it is known that

$$\det(B) = \det(kA) = k^N \det(A). \quad (N.41)$$

It follows that eq(N.40) reduces to

$$[\det(K_{\underline{xx}})]^2 = (2)^{2JN} \det[(K_{\underline{xx}}^{\text{II}} + j K_{\underline{xx}}^{\text{QI}})] \det[(K_{\underline{xx}}^{\text{II}} - j K_{\underline{xx}}^{\text{QI}})]. \quad (N.42)$$

On the other hand, eq(N.38) simplifies to

$$\det(K_{\underline{xx}}) = \det[(K_{\underline{xx}}^{\text{II}} + j K_{\underline{xx}}^{\text{QI}})] \det[(K_{\underline{xx}}^{\text{II}} - j K_{\underline{xx}}^{\text{QI}})]. \quad (N.43)$$

It is concluded, therefore, that equality 1 holds provided the condition of eqs(N.39) are valid. Whether these conditions are sufficient for equality 2 to also hold is now investigated.

Equality 2

With reference to eq(N.22), equality 2 requires that

$$[\underline{x}_{1,N}' - \bar{\underline{x}}_{1,N}]^H Q_{\underline{xx}} [\underline{x}_{1,N}' - \bar{\underline{x}}_{1,N}] = \frac{1}{2} [\underline{L}_{1,N}' - \bar{\underline{L}}_{1,N}]^T Q_{\underline{xx}} [\underline{L}_{1,N}' - \bar{\underline{L}}_{1,N}]. \quad (N.44)$$

To check the validity of this equality for the conditions under which equality 1 holds, the matrices $Q_{\underline{xx}}$ and $Q_{\underline{xx}}$ need to be determined for the special case of eqs(N.39). We can express the covariance matrix on $\underline{x}_{1,N}$ as

$$K_{\underline{xx}} = (K_{\underline{xx}}^{\text{II}} + K_{\underline{xx}}^{\text{QQ}}) + j (K_{\underline{xx}}^{\text{QI}} - K_{\underline{xx}}^{\text{IQ}}). \quad (N.45)$$

For the special case of eqs(N.39), K_{xx} simplifies to

$$K_{xx} = 2(K_{xx}^{II} + j K_{xx}^{QI}). \quad (N.46)$$

Similarly, the covariance matrix on $\underline{x}_{1,N}$, which is given in eq(N.14)

$$K_{\underline{x}} = \begin{bmatrix} K_{xx}^{II} & K_{xx}^{IQ} \\ K_{xx}^{QI} & K_{xx}^{QQ} \end{bmatrix}. \quad (N.47a)$$

reduces to

$$K_{\underline{x}} = \begin{bmatrix} K_{xx}^{II} & -K_{xx}^{QI} \\ K_{xx}^{QI} & K_{xx}^{II} \end{bmatrix}. \quad (N.47b)$$

Determination of Q_{xx} is facilitated by trying to find matrices A_1 and A_2 such that

$$Q_{xx} = A_1 - j A_2. \quad (N.48)$$

Because Q_{xx} is the inverse matrix of K_{xx} , it follows from eq(N.46) that

$$K_{xx} Q_{xx} = 2(K_{xx}^{II} + j K_{xx}^{QI})(A_1 - j A_2) \quad (N.49a)$$

$$= 2[(K_{xx}^{II} A_1 + K_{xx}^{QI} A_2) + j (K_{xx}^{QI} A_1 - K_{xx}^{II} A_2)] = I \quad (N.49b)$$

where I is an identity matrix. Equating imaginary parts,

$$2(K_{xx}^{QI} A_1 - K_{xx}^{II} A_2) = [0]. \quad (N.50)$$

The solution for A_2 yields

$$A_2 = [K_{xx}^{II}]^{-1} K_{xx}^{QI} A_1. \quad (N.51)$$

Equating the real parts of eq(49b), we have

$$2(K_{xx}^{II} A_1 + K_{xx}^{QI} A_2) = 1. \quad (N.52)$$

Substitution of eq(N.51) into (N.52), the solution for A_1 results in

$$A_1 = \frac{1}{2} (K_{xx}^{II} + K_{xx}^{QI} [K_{xx}^{II}]^{-1} K_{xx}^{QI})^{-1}. \quad (N.53)$$

Using eq(N.51), it follows that

$$A_2 = [K_{xx}^{II}]^{-1} K_{xx}^{QI} A_1 \quad (N.54a)$$

$$= \frac{1}{2} [K_{xx}^{II}]^{-1} K_{xx}^{QI} (K_{xx}^{II} + K_{xx}^{QI} [K_{xx}^{II}]^{-1} K_{xx}^{QI})^{-1} \quad (N.54b)$$

$$= \frac{1}{2} ([K_{xx}^{QI}]^{-1} K_{xx}^{II})^{-1} (K_{xx}^{II} + K_{xx}^{QI} [K_{xx}^{II}]^{-1} K_{xx}^{QI})^{-1} \quad (N.54c)$$

$$= \frac{1}{2} [(K_{xx}^{II} + K_{xx}^{QI} [K_{xx}^{II}]^{-1} K_{xx}^{QI}) ((K_{xx}^{QI})^{-1} K_{xx}^{II})]^{-1} \quad (N.54d)$$

$$= \frac{1}{2} [K_{xx}^{QI} + K_{xx}^{II} [K_{xx}^{QI}]^{-1} K_{xx}^{II}]^{-1}. \quad (N.54e)$$

From matrix theory, it is known that

$$(A^{-1})^T = (A^T)^{-1}. \quad (N.55)$$

Consequently,

$$A_1^T = \frac{1}{2} \{ (K_{xx}^{II})^T + (K_{xx}^{OI})^T ([K_{xx}^{II}]^{-1})^T (K_{xx}^{OI})^T \}^{-1}. \quad (N.56)$$

However, from eqs(N.39)

$$(K_{xx}^{II})^T = K_{xx}^{II} \quad (N.57a)$$

$$(K_{xx}^{OI})^T = K_{xx}^{IO} = -K_{xx}^{OI}. \quad (N.57b)$$

It is concluded that

$$A_1^T = \frac{1}{2} \{ K_{xx}^{II} + K_{xx}^{OI} [K_{xx}^{II}]^{-1} K_{xx}^{OI} \}^{-1} = A_1. \quad (N.58)$$

A_1 , therefore, is a symmetric matrix. Similarly,

$$A_2^T = \frac{1}{2} \{ (K_{xx}^{OI})^T + (K_{xx}^{II})^T ([K_{xx}^{OI}]^{-1})^T (K_{xx}^{II})^T \}^{-1} \quad (N.59a)$$

$$= -\frac{1}{2} \{ K_{xx}^{OI} + K_{xx}^{II} [K_{xx}^{OI}]^{-1} K_{xx}^{II} \} = -A_2. \quad (N.59b)$$

It is seen that A_2 is a skew symmetric matrix. And so, from eq(N.48)

$$Q_{xx} = A_1 - j A_2. \quad (N.60a)$$

$$= \frac{1}{2} \{ K_{xx}^{II} + K_{xx}^{OI} [K_{xx}^{II}]^{-1} K_{xx}^{OI} \}^{-1} - j \frac{1}{2} \{ K_{xx}^{OI} + K_{xx}^{II} [K_{xx}^{OI}]^{-1} K_{xx}^{II} \} \quad (N.60b)$$

Having found Q_{xx} , attention is now devoted to finding Q_{xx} . For this purpose, let Q_{xx} be partitioned such that

$$Q_{\Pi} = [K_{\Pi}]^{-1} = \begin{bmatrix} A_3 & A_4 \\ A_5 & A_6 \end{bmatrix} \quad (N.61)$$

where the submatrices A_3 , A_4 , A_5 , and A_6 are each of dimension $JN \times JN$. From eq(N.14b), we have

$$K_{\Pi} = \begin{bmatrix} K_{XX}^{II} & K_{XX}^{IQ} \\ K_{XX}^{QI} & K_{XX}^{QQ} \end{bmatrix}. \quad (N.62)$$

Using the matrix inversion lemma [95, pg. 25], eqs(N.61) and (N.62) yield

$$A_3 = \{ K_{XX}^{II} + K_{XX}^{QI} [K_{XX}^{II}]^{-1} K_{XX}^{QI} \}^{-1} \quad (N.63a)$$

$$A_4 = \{ K_{XX}^{QI} + K_{XX}^{II} [K_{XX}^{QI}]^{-1} K_{XX}^{II} \}^{-1} \quad (N.63b)$$

$$A_5 = - \{ K_{XX}^{QI} + K_{XX}^{II} [K_{XX}^{QI}]^{-1} K_{XX}^{II} \}^{-1} \quad (N.63c)$$

$$A_6 = \{ K_{XX}^{II} + K_{XX}^{QI} [K_{XX}^{II}]^{-1} K_{XX}^{QI} \}^{-1}. \quad (N.63d)$$

With reference to the expressions for A_1 and A_2 , it is seen that

$$A_3 = 2A_1 \quad (N.63a)$$

$$A_4 = 2A_2 \quad (N.63b)$$

$$A_5 = -2A_2 \quad (N.63c)$$

$$A_6 = 2A_1. \quad (N.63d)$$

It is concluded that

$$Q_{\Pi} = 2 \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}. \quad (N.64)$$

Having determined the matrices Q_{xx} and Q_{xx} , it is now possible to evaluate the quadratic forms in eq(N.44). Recall that $[\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}]$ is a complex JN -dimensional vector given by

$$[\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}] = (\underline{x}'_{I1,N} - \bar{\underline{x}}_I)_{1,N} + j(\underline{x}'_{Q1,N} - \bar{\underline{x}}_Q)_{1,N}. \quad (N.65)$$

Therefore,

$$\begin{aligned} & [\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}]^H Q_{xx} [\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}] = \\ & = [(\underline{x}'_{I1,N} - \bar{\underline{x}}_I)_{1,N} - j((\underline{x}'_{Q1,N} - \bar{\underline{x}}_Q)_{1,N})]^T [A_1 - jA_2] [(\underline{x}'_{I1,N} - \bar{\underline{x}}_I)_{1,N} - j(\underline{x}'_{Q1,N} - \bar{\underline{x}}_Q)_{1,N}]^T \end{aligned} \quad (N.66a)$$

$$\begin{aligned} & = (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_1 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) - j(\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_1 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) \\ & \quad + (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_2 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) - j(\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_2 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) \\ & \quad + j(\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_1 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) + (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_1 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) \\ & \quad - j(\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_2 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) - (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_2 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}). \end{aligned} \quad (N.66b)$$

Observe that each term in eq(N.66b) is a scalar quantity. Because A_1 is symmetric and the transpose of a scalar equals itself,

$$(\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_1 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) = (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_1^T (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) \quad (N.67a)$$

$$= (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_1 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}). \quad (N.67b)$$

Similarly, because A_2 is skew symmetric,

$$(\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_2 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) = (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_2^T (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) \quad (N.68a)$$

$$= -(\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_2 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) \quad (N.68b)$$

and

$$(\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_2 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) = (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_2^T (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) \quad (N.69a)$$

$$= -(\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_2 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}). \quad (N.69b)$$

Since a scalar can equal its negative value only if the scalar is zero, it follows that

$$(\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_2 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) = 0 \quad (N.70a)$$

and

$$(\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_2 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) = 0. \quad (N.70b)$$

Upon substitution of eqs(N.67) and (N.70) into eq(N.66), the imaginary terms cancel and we are left with

$$\begin{aligned} [\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}]^H Q_{xx} [\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}] &= \\ &= (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_1 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}) + (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N})^T A_2 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) \\ &\quad + (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_1 (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N}) - (\underline{x}'_{Q1,N} - \bar{\underline{x}}_{Q1,N})^T A_2 (\underline{x}'_{I1,N} - \bar{\underline{x}}_{I1,N}). \end{aligned} \quad (N.71)$$

To evaluate the quadratic form on the right hand side of eq(N.44), recall that $[\underline{x}'_{1,N} - \bar{\underline{x}}_{1,N}]$ is the $2JN$ -dimensional vector given by

$$[\underline{x}_{1,N}' - \bar{\underline{x}}_{1,N}] = \begin{bmatrix} (\underline{x}_{I1,N}' - \bar{\underline{x}}_{I1,N}) \\ (\underline{x}_{Q1,N}' - \bar{\underline{x}}_{Q1,N}) \end{bmatrix}. \quad (\text{N.72})$$

As a result,

$$\begin{aligned} \frac{1}{2} [\underline{x}_{1,N}' - \bar{\underline{x}}_{1,N}]^T Q_{\underline{x}} [\underline{x}_{1,N}' - \bar{\underline{x}}_{1,N}] &= \\ &= \frac{1}{2} [(\underline{x}_{I1,N}' - \bar{\underline{x}}_{I1,N})^T (\underline{x}_{Q1,N}' - \bar{\underline{x}}_{Q1,N})^T] 2 \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \begin{bmatrix} (\underline{x}_{I1,N}' - \bar{\underline{x}}_{I1,N}) \\ (\underline{x}_{Q1,N}' - \bar{\underline{x}}_{Q1,N}) \end{bmatrix} \end{aligned} \quad (\text{N.73a})$$

$$\begin{aligned} &= (\underline{x}_{I1,N}' - \bar{\underline{x}}_{I1,N})^T A_1 (\underline{x}_{I1,N}' - \bar{\underline{x}}_{I1,N}) + (\underline{x}_{I1,N}' - \bar{\underline{x}}_{I1,N})^T A_2 (\underline{x}_{Q1,N}' - \bar{\underline{x}}_{Q1,N}) \\ &\quad - (\underline{x}_{Q1,N}' - \bar{\underline{x}}_{Q1,N})^T A_2 (\underline{x}_{I1,N}' - \bar{\underline{x}}_{I1,N}) + (\underline{x}_{Q1,N}' - \bar{\underline{x}}_{Q1,N})^T A_1 (\underline{x}_{Q1,N}' - \bar{\underline{x}}_{Q1,N}). \end{aligned} \quad (\text{N.73b})$$

Comparison of eq(N.73b) with (N.71) reveals that equality 2 is valid under the constraints of eq(N.45).

In summary, by definition, the probability density function of a complex Gaussian random vector is given in eq(N.10) whereas that for a real Gaussian random vector is given by eq(N.17). Although both can be interpreted as the $2JN$ -dimensional multivariate density function for the real random variables $x_{1I}(k)$, $x_{2I}(k)$, ..., $x_{JI}(k)$, $x_{1Q}(k)$, $x_{2Q}(k)$, ..., $x_{JQ}(k)$ for $k = 1, 2, \dots, N$, they are not equal, in general. However, for the special case in which

$$K_{\Delta\Delta}^{\Pi} = K_{\Delta\Delta}^{QQ} \quad (N.74a)$$

and

$$K_{\Delta\Delta}^{QI} = -K_{\Delta\Delta}^{IQ}, \quad (N.74b)$$

it is true that

$$p_{\Delta}(\Delta_{1,N}') = p_I(I_{1,N}'). \quad (N.75)$$

Wide-sense stationarity of the bandpass processes is of particular interest. As shown in Appendix O, eqs(N.74) are satisfied for zero-mean, wide-sense stationary bandpass processes. Hence, the probability density functions $p_{\Delta}(\Delta_{1,N}')$ and $p_I(I_{1,N}')$ are identical for a wide-sense stationary bandpass random process. When the conditions in eq(N.74) do not apply, $p_I(I_{1,N}')$ represents the most general situation since

$$p_{\Delta}(\Delta_{1,N}') \neq p_I(I_{1,N}'). \quad (N.76)$$

APPENDIX O CONDITIONS FOR NARROWBAND STATIONARY BANDPASS PROCESSES

In this Appendix, we derive the conditions required to obtain narrowband stationary bandpass processes. In section O.1, we derive these conditions for the single channel case while in O.2, we consider the multichannel case. Although the presentation in O.1 was independently derived in [3], a very similar discussion appears in [164]. However, we are not aware of a presentation as described in O.2.

O.1. SINGLE CHANNEL CASE

We now consider a real, narrowband, bandpass process $n_i(t)$ such that

$$n_i(t) = \text{Re}[x_i(t)\exp(j2\pi f_{c_i}t)] \quad (\text{O.1.1})$$

where $x_i(t)$ is the complex baseband process previously defined in eq(3.2.2) as

$$x_i(t) = x_{iI}(t) + jx_{iQ}(t). \quad (\text{O.1.2})$$

The quantities $x_{iI}(t)$ and $x_{iQ}(t)$ are the real-valued low pass quadrature components. Using eq(O.1.2) in (O.1.1), the process $n_i(t)$ can be expressed in canonical form as

$$n_i(t) = x_{iI}(t)\cos(2\pi f_{c_i}t) - x_{iQ}(t)\sin(2\pi f_{c_i}t). \quad (\text{O.1.3})$$

Taking the Hilbert transform of eq(O.1.3) and recognizing that the quadrature components are low pass, we obtain

$$\hat{n}_i(t) = x_{iI}(t)\sin(2\pi f_{c_i}t) + x_{iQ}(t)\cos(2\pi f_{c_i}t). \quad (\text{O.1.4})$$

Eqs(O.1.3) and (O.1.4) can now be used to solve for the quadrature components resulting in

$$x_{iI}(t) = n_i(t)\cos(2\pi f_{c_i}t) + \hat{n}_i(t)\sin(2\pi f_{c_i}t) \quad (\text{O.1.5a})$$

and

$$x_{iQ}(t) = \hat{n}_i(t)\cos(2\pi f_{c_i}t) - n_i(t)\sin(2\pi f_{c_i}t). \quad (\text{O.1.5b})$$

In this section, we will determine relationships between the auto- and cross-correlation functions of the quadrature components of the process $x_i(t)$ under the assumption that $n_i(t)$ is stationary. In this case the correlation function of $n_i(t)$ is $R_{n_i n_i}(\tau)$ while the power spectral density is $S_{n_i n_i}(f)$ centered about $\pm f_{c_i}$.

Dropping the redundant subscript notation, we consider the Hilbert transform of $n(t)$ as

$$\hat{n}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n(\lambda)}{t-\lambda} d\lambda. \quad (\text{O.1.6})$$

The cross-correlation function $R_{N\hat{N}}(\tau)$ is expressed as

$$R_{n\hat{n}}(\tau) = E[n(t)\hat{n}(t-\tau)]. \quad (\text{O.1.7})$$

Substituting eq (O.1.6) into (O.1.7)

$$R_{n\hat{n}}(\tau) = E \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n(t)n(\lambda)}{t-\tau-\lambda} d\lambda \right]. \quad (\text{O.1.8})$$

Interchanging the expectation and integration

$$R_{nn}^{\wedge}(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{E[n(t)n(\lambda)]}{t-\tau-\lambda} d\lambda \quad (O.1.9)$$

Assuming wide-sense stationarity on the bandpass process,

$$R_{nn}(t-\lambda) = E[n(t)n(\lambda)] \quad (O.1.10)$$

so that

$$R_{nn}^{\wedge}(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{nn}(t-\lambda)}{t-\tau-\lambda} d\lambda \quad (O.1.11)$$

Now let $\alpha = t - \lambda$ so that $d\lambda = -d\alpha$ and

$$R_{nn}^{\wedge}(\tau) = \frac{1}{\pi} \int_{\infty}^{-\infty} \frac{R_{nn}(\alpha)}{\alpha-\tau} (-d\alpha) \quad (O.1.12a)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{nn}(\alpha)}{\alpha-\tau} d\alpha \quad (O.1.12b)$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{nn}(\alpha)}{\tau-\alpha} d\alpha = -\hat{R}_{nn}(\tau). \quad (O.1.12c)$$

Similarly,

$$R_{nn}^{\Delta}(\tau) = E[\hat{n}(t)n(t-\tau)] \quad (O.1.13a)$$

$$= E \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n(\lambda)n(t-\tau)}{t-\lambda} d\lambda \right] \quad (O.1.13b)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{E[n(\lambda)n(t-\tau)]}{t-\lambda} d\lambda \quad (\text{O.1.13c})$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{nn}(\lambda-t+\tau)}{t-\lambda} d\lambda. \quad (\text{O.1.13d})$$

Let $\alpha = \lambda-t+\tau$ so that $d\lambda = d\alpha$ and

$$R_{nn}^{\wedge}(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{nn}(\alpha)}{\tau-\alpha} d\alpha = \hat{R}_{nn}(\tau) \quad (\text{O.1.14})$$

and so noting the equality of eqs(O.1.12c) and (O.1.14)

$$R_{nn}^{\wedge}(\tau) = -R_{nn}^{\wedge}(\tau). \quad (\text{O.1.15})$$

Also, at $\tau = 0$ the function $R_{nn}(\alpha)/\alpha$ is odd so that

$$R_{nn}^{\wedge}(0) = 0. \quad (\text{O.1.16})$$

We now consider the correlation functions associated with the quadrature components defined in eq(O.1.5a) and (O.1.5b)

$$R_{ii}^{\Pi}(\tau) = E[x_{ii}(t)x_{ii}(t-\tau)] \quad (\text{O.1.17a})$$

$$\begin{aligned} &= E[n(t)n(t-\tau)]\cos(2\pi f_c t)\cos[2\pi f_c(t-\tau)] \\ &\quad + E[n(t)\hat{n}(t-\tau)]\cos(2\pi f_c t)\sin[2\pi f_c(t-\tau)] \\ &\quad + E[\hat{n}(t)n(t-\tau)]\sin(2\pi f_c t)\cos[2\pi f_c(t-\tau)] \\ &\quad + E[\hat{n}(t)\hat{n}(t-\tau)]\sin(2\pi f_c t)\sin[2\pi f_c(t-\tau)]. \end{aligned} \quad (\text{O.1.17b})$$

Let

$$A = 2\pi f_c t \quad (\text{O.1.18a})$$

$$B = 2\pi f_c (t - \tau). \quad (\text{O.1.18b})$$

Using the identities

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad (\text{O.1.18c})$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \quad (\text{O.1.18d})$$

we obtain

$$\cos(2\pi f_c t) \cos[2\pi f_c (t - \tau)] = \cos(2\pi f_c \tau) - \sin A \sin B \quad (\text{O.1.18e})$$

$$\cos(2\pi f_c t) \sin[2\pi f_c (t - \tau)] = \cos A \sin B \quad (\text{O.1.18f})$$

$$\sin(2\pi f_c t) \cos[2\pi f_c (t - \tau)] = \sin A \cos B \quad (\text{O.1.18g})$$

$$\sin(2\pi f_c t) \sin[2\pi f_c (t - \tau)] = \sin A \sin B. \quad (\text{O.1.18h})$$

Using these relations and eqs(O.1.14) and (O.1.15) in (O.1.17b)

$$\begin{aligned} R_{ii}^{\Pi}(\tau) &= R_{nn}(\tau) [\cos(2\pi f_c \tau) - \sin A \sin B] \\ &\quad - \hat{R}_{nn}(\tau) \cos A \sin B + \hat{R}_{nn}(\tau) \sin A \cos B \\ &\quad + R_{nn}^{\wedge\wedge}(\tau) \sin A \sin B \end{aligned} \quad (\text{O.1.19a})$$

$$\begin{aligned} &= R_{nn}(\tau) [\cos(2\pi f_c \tau) - \sin A \sin B] \\ &\quad + \hat{R}_{nn}(\tau) [\sin A \cos B - \cos A \sin B] \\ &\quad + R_{nn}^{\wedge\wedge}(\tau) \sin A \sin B \end{aligned} \quad (\text{O.1.19b})$$

$$\begin{aligned} &= R_{nn}(\tau) [\cos(2\pi f_c \tau) - \sin A \sin B] \\ &\quad + \hat{R}_{nn}(\tau) \sin(2\pi f_c \tau) \\ &\quad + R_{nn}^{\wedge\wedge}(\tau) [\sin A \sin B]. \end{aligned} \quad (\text{O.1.19c})$$

In Appendix P and [3], we show that

$$R_{nn}^{\wedge\wedge}(\tau) = + R_{nn}(\tau) \quad (\text{O.1.20})$$

so that

$$R_{ii}^{\Pi}(\tau) = R_{nn}(\tau)\cos(2\pi f_c\tau) + \hat{R}_{nn}(\tau)\sin(2\pi f_c\tau). \quad (O.1.21)$$

Similarly,

$$R_{ii}^{QQ}(\tau) = R_{nn}(\tau)\cos(2\pi f_c\tau) + \hat{R}_{nn}(\tau)\sin(2\pi f_c\tau) \quad (O.1.22)$$

so that

$$\boxed{R_{ii}^{\Pi}(\tau) = R_{ii}^{QQ}(\tau).} \quad (O.1.23)$$

We now consider

$$R_{ii}^{IQ}(\tau) = E[n_I(t)n_Q(t-\tau)] \quad (O.1.24a)$$

$$\begin{aligned} &= E[n(t)\hat{n}(t-\tau)]\cos(2\pi f_c t)\cos[2\pi f_c(t-\tau)] \\ &\quad + E[\hat{n}(t)\hat{n}(t-\tau)]\sin(2\pi f_c t)\cos[2\pi f_c(t-\tau)] \\ &\quad - E[n(t)n(t-\tau)]\cos(2\pi f_c t)\sin[2\pi f_c(t-\tau)] \\ &\quad - E[\hat{n}(t)n(t-\tau)]\sin(2\pi f_c t)\sin[2\pi f_c(t-\tau)] \end{aligned} \quad (O.1.24b)$$

$$\begin{aligned} &= R_{n\hat{n}}(\tau)\cos A\cos B + R_{\hat{n}\hat{n}}(\tau)\sin A\cos B \\ &\quad - R_{nn}(\tau)\cos A\sin B - R_{\hat{n}n}(\tau)\sin A\sin B \end{aligned} \quad (O.1.24c)$$

$$\begin{aligned} &= R_{n\hat{n}}(\tau)\cos A\cos B + R_{nn}(\tau)\sin A\cos B \\ &\quad - R_{nn}(\tau)\cos A\sin B - R_{\hat{n}n}(\tau)\sin A\sin B \end{aligned} \quad (O.1.24d)$$

where eq(O.1.20) was used to in the second term above. From eqs.(O.1.12c) and (O.1.14)

$$R_{n\hat{n}}(\tau) = -\hat{R}_{nn}(\tau) \quad (O.1.25a)$$

$$R_{\hat{n}n}(\tau) = \hat{R}_{nn}(\tau) \quad (O.1.25b)$$

so that

$$R_{ij}^{IQ}(\tau) = R_{nn}(\tau)[\sin A \cos B - \cos A \sin B] \\ - \hat{R}_{nn}(\tau)[\cos A \cos B + \sin A \sin B] \quad (\text{O.1.26a})$$

$$= R_{nn}(\tau) \sin(A - B) - \hat{R}_{nn}(\tau) \cos(A - B) \quad (\text{O.1.26b})$$

$$= R_{nn}(\tau) \sin(2\pi f_c \tau) - \hat{R}_{nn}(\tau) \cos(2\pi f_c \tau). \quad (\text{O.1.26c})$$

Similarly,

$$-R_{ij}^{QI}(\tau) = R_{nn}(\tau) \sin(2\pi f_c \tau) - \hat{R}_{nn}(\tau) \cos(2\pi f_c \tau) \quad (\text{O.1.27d})$$

so that

$$\boxed{R_{ij}^{IQ}(\tau) = -R_{ij}^{QI}(\tau).} \quad (\text{O.1.27e})$$

We also note that since $R_{nn}(\tau)$ is even and $\hat{R}_{nn}(\tau)$ is odd, then eq(O.1.26c) indicates that $R_{ij}^{IQ}(\tau)$ is odd. It was noted at the end of chapter 3, that the stationary narrowband process is a special case which results in odd cross-correlation quadrature components $R_{ij}^{IQ}(\tau)$ and $R_{ij}^{QI}(\tau)$.

O.2. MULTICHANNEL CASE

We now consider two real, narrowband, bandpass processes $n_i(t)$ and $n_j(t)$ defined in eq(O.1.1) and develop properties similar to those developed in the previous section. Specifically, we will determine relationships between the cross-correlation functions of the quadrature components involving $x_i(t)$ and $x_j(t)$; i.e., $R_{ij}^{II}(\tau)$, $R_{ij}^{QQ}(\tau)$, $R_{ij}^{QI}(\tau)$ and $R_{ij}^{IQ}(\tau)$. The bandpass processes are expressed as

$$n_i(t) = \text{Re}\{x_i(t) \exp(j2\pi f_c t)\} \quad (\text{O.2.1a})$$

and

$$n_j(t) = \text{Re}\{x_j(t) \exp(j2\pi f_c t)\}. \quad (\text{O.2.1b})$$

In section 4.3.2, however, we suggested that each channel process can be translated to a common reference frequency f_{c_R} . And so, f_{c_i} and f_{c_j} in (O.2.1) can be replaced with f_{c_R} . Eqs(O.2.1) can then be expressed as

$$n_i(t) = \text{Re}\{x_i(t)\exp(j2\pi f_{c_R}t)\} \quad (\text{O.2.2a})$$

and

$$n_j(t) = \text{Re}\{x_j(t)\exp(j2\pi f_{c_R}t)\}. \quad (\text{O.2.2b})$$

Using the quadrature form for $x_i(t)$ and $x_j(t)$ expressed in eq(3.2.2), we obtain the canonical forms for the above equations as

$$n_i(t) = x_{iI}(t)\cos(2\pi f_{c_R}t) - x_{iQ}(t)\sin(2\pi f_{c_R}t) \quad (\text{O.2.3a})$$

and

$$n_j(t) = x_{jI}(t)\cos(2\pi f_{c_R}t) - x_{jQ}(t)\sin(2\pi f_{c_R}t). \quad (\text{O.2.3b})$$

The Hilbert transform of $n_j(t)$ is expressed as

$$\hat{n}_j(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n_j(\lambda)}{t-\lambda} d\lambda. \quad (\text{O.2.4})$$

Assuming wide-sense joint stationarity between the two bandpass processes, the cross-correlation function $R_{n_i\hat{n}_j}(\tau)$ is expressed as

$$R_{n_i\hat{n}_j}(\tau) = E[n_i(t)\hat{n}_j(t-\tau)]. \quad (\text{O.2.5})$$

Substituting eq (O.2.4) into (O.2.5)

$$R_{n_i\hat{n}_j}(\tau) = E\left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n_i(t)n_j(\lambda)}{t-\lambda} d\lambda\right]. \quad (\text{O.2.6})$$

Interchanging the expectation and integration

$$R_{n_i n_j}(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{E[n_i(t) n_j(\lambda)]}{t - \tau - \lambda} d\lambda. \quad (O.2.7)$$

We now consider,

$$R_{n_i n_j}(t - \lambda) = E[n_i(t) n_j(\lambda)] \quad (O.2.8)$$

so that

$$R_{n_i n_j}(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{n_i n_j}(t - \lambda)}{t - \tau - \lambda} d\lambda. \quad (O.2.9)$$

Now let $\alpha = t - \lambda$ so that $d\lambda = -d\alpha$ and

$$R_{n_i n_j}(\tau) = \frac{1}{\pi} \int_{\infty}^{-\infty} \frac{R_{n_i n_j}(\alpha)}{\alpha - \tau} (-d\alpha) \quad (O.2.10a)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{n_i n_j}(\alpha)}{\alpha - \tau} d\alpha \quad (O.2.10b)$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{n_i n_j}(\alpha)}{\tau - \alpha} d\alpha \quad (O.2.10c)$$

$$= -\hat{R}_{n_i n_j}(\tau). \quad (O.2.10d)$$

Similarly,

$$R_{\hat{n}_i n_j}(\tau) = E[\hat{n}_i(t) n_j(t - \tau)] \quad (O.2.11a)$$

$$= E \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n_i(\lambda) n_j(t-\tau)}{t-\lambda} d\lambda \right] \quad (O.2.11b)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{E[n_i(\lambda) n_j(t-\tau)]}{t-\lambda} d\lambda \quad (O.2.11c)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{n_i n_j}(\lambda-t+\tau)}{t-\lambda} d\lambda. \quad (O.2.11d)$$

Let $\alpha = \lambda - t + \tau$ so that $d\lambda = d\alpha$ and eq(O.2.11d) becomes

$$R_{\hat{n}_i \hat{n}_j}(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{n_i n_j}(\alpha)}{\tau - \alpha} d\alpha \quad (O.2.12a)$$

$$= \hat{R}_{n_i n_j}(\tau). \quad (O.2.12b)$$

From eqs(O.2.10d) and (O.2.12b), we have

$$R_{\hat{n}_i \hat{n}_j}(\tau) = -R_{\hat{n}_j \hat{n}_i}(\tau). \quad (O.2.13)$$

We now consider the cross-correlation functions associated with the quadrature components of the $x_i(t)$ and $x_j(t)$ processes defined in eqs(O.2.5a).

First, consider

$$R_{ij}^{\Pi}(\tau) = E[x_{iI}(t) x_{jI}(t-\tau)]. \quad (O.2.14)$$

Using eqs(O.2.3a) and the corresponding Hilbert transforms of eq(O.2.4), we solve for

$$x_{iI}(t) = n_i(t)\cos(2\pi f_{c_R} t) + \hat{n}_i(t)\sin(2\pi f_{c_R} t) \quad (\text{O.2.15a})$$

and

$$x_{iQ}(t) = \hat{n}_i(t)\cos(2\pi f_{c_R} t) - n_i(t)\sin(2\pi f_{c_R} t). \quad (\text{O.2.15b})$$

Using eq(O.2.15a) and the corresponding equation for $x_{jI}(t)$ in eq(O.2.14), we obtain

$$\begin{aligned} R_{ij}^{II}(\tau) = & E[n_i(t)n_j(t-\tau)]\cos(2\pi f_{c_R} t)\cos[2\pi f_{c_R} (t-\tau)] \\ & + E[n_i(t)\hat{n}_j(t-\tau)]\cos(2\pi f_{c_R} t)\sin[2\pi f_{c_R} (t-\tau)] \\ & + E[\hat{n}_i(t)n_j(t-\tau)]\sin(2\pi f_{c_R} t)\cos[2\pi f_{c_R} (t-\tau)] \\ & + E[\hat{n}_i(t)\hat{n}_j(t-\tau)]\sin(2\pi f_{c_R} t)\sin[2\pi f_{c_R} (t-\tau)]. \end{aligned} \quad (\text{O.2.16})$$

We now define

$$A = 2\pi f_{c_R} t \quad (\text{O.2.17a})$$

and

$$B = 2\pi f_{c_R} (t-\tau). \quad (\text{O.2.17b})$$

Let us recall the identities

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad (\text{O.2.18a})$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B. \quad (\text{O.2.18b})$$

Using eq(O.2.18a), we have

$$\cos(2\pi f_{c_R} t)\cos[2\pi f_{c_R} (t-\tau)] = \cos(2\pi f_{c_R} \tau) - \sin A \sin B. \quad (\text{O.2.19})$$

From the identities defined above,

$$\cos(2\pi f_{c_R} t) \sin[2\pi f_{c_R} (t-\tau)] = \cos A \sin B \quad (\text{O.2.20a})$$

$$\sin(2\pi f_{c_R} t) \cos[2\pi f_{c_R} (t-\tau)] = \sin A \cos B \quad (\text{O.2.20b})$$

$$\sin(2\pi f_{c_R} t) \sin[2\pi f_{c_R} (t-\tau)] = \sin A \sin B. \quad (\text{O.2.20c})$$

Eq(O.2.16) can now be written

$$\begin{aligned} R_{ij}^{\Pi}(\tau) &= R_{n_i n_j}(\tau) [\cos(2\pi f_{c_R} \tau) - \sin A \sin B] \\ &\quad + R_{\hat{n}_i \hat{n}_j}(\tau) \cos A \sin B \\ &\quad + R_{\hat{n}_i n_j}(\tau) \sin A \cos B \\ &\quad + R_{\hat{n}_i \hat{n}_j}(\tau) \sin A \sin B \end{aligned} \quad (\text{O.2.21a})$$

$$\begin{aligned} &= R_{n_i n_j}(\tau) [\cos(2\pi f_{c_R} \tau) - \sin A \sin B] \\ &\quad + \hat{R}_{n_i n_j}(\tau) [\sin A \cos B - \cos A \sin B] \\ &\quad + R_{\hat{n}_i \hat{n}_j}(\tau) \sin A \sin B \end{aligned} \quad (\text{O.2.21b})$$

where we have used eqs(O.2.10d) and (O.2.12b) to obtain (O.2.21b). In Appendix P and [3], we show that

$$R_{\hat{n}_i \hat{n}_j}(\tau) = R_{n_i n_j}(\tau). \quad (\text{O.2.22})$$

Using this equation and the identity expressed in eq(O.2.18b),

$$R_{ij}^{\Pi}(\tau) = R_{n_i n_j}(\tau) \cos(2\pi f_{c_R} \tau) + \hat{R}_{n_i n_j}(\tau) \sin(2\pi f_{c_R} \tau). \quad (\text{O.2.23})$$

Similarly, using the relation $\sin(A-B) = -\sin(B-A)$, we obtain

$$R_{ij}^{QQ}(\tau) = R_{n_i n_j}(\tau) \cos(2\pi f_{c_R} \tau) + \hat{R}_{n_i n_j}(\tau) \sin(2\pi f_{c_R} \tau) \quad (O.2.24)$$

so that

$$\boxed{R_{ij}^{\Pi}(\tau) = R_{ij}^{QQ}(\tau).} \quad (O.2.25)$$

Next, we consider

$$R_{ij}^{IQ}(\tau) = E[x_{iI}(t)x_{jQ}(t-\tau)]. \quad (O.2.26)$$

Using eqs(O.2.15a) and the j channel equivalent of eq(O.2.15b) in (O.2.26), we obtain

$$\begin{aligned} R_{ij}^{IQ}(\tau) &= E[n_i(t)\hat{n}_j(t-\tau)]\cos(2\pi f_{c_R} t)\cos[2\pi f_{c_R}(t-\tau)] \\ &\quad + E[\hat{n}_i(t)\hat{n}_j(t-\tau)]\sin(2\pi f_{c_R} t)\cos[2\pi f_{c_R}(t-\tau)] \\ &\quad - E[n_i(t)n_j(t-\tau)]\cos(2\pi f_{c_R} t)\sin[2\pi f_{c_R}(t-\tau)] \\ &\quad - E[\hat{n}_i(t)n_j(t-\tau)]\sin(2\pi f_{c_R} t)\sin[2\pi f_{c_R}(t-\tau)] \end{aligned} \quad (O.2.27a)$$

$$\begin{aligned} &= R_{n_i \hat{n}_j}(\tau) \cos A \cos B + R_{\hat{n}_i \hat{n}_j}(\tau) \sin A \cos B \\ &\quad - R_{n_i n_j}(\tau) \cos A \sin B - R_{\hat{n}_i n_j}(\tau) \sin A \sin B \end{aligned} \quad (O.2.27b)$$

$$\begin{aligned} &= R_{n_i \hat{n}_j}(\tau) \cos A \cos B + R_{n_i n_j}(\tau) \sin A \cos B \\ &\quad - R_{\hat{n}_i n_j}(\tau) \cos A \sin B - R_{\hat{n}_i \hat{n}_j}(\tau) \sin A \sin B \end{aligned} \quad (O.2.27c)$$

where eq(O.2.22) was used in the second term above. Using eqs.(O.2.10d) and (O.2.12b) in (O.2.27c)

$$R_{ij}^{IQ}(\tau) = R_{n_i n_j}(\tau)[\sin A \cos B - \cos A \sin B] \\ - \hat{R}_{n_i n_j}(\tau)[\cos A \cos B + \sin A \sin B] \quad (O.2.28a)$$

$$= R_{n_i n_j}(\tau) \sin(A - B) - \hat{R}_{n_i n_j}(\tau) \cos(A - B) \quad (O.2.28b)$$

$$= R_{n_i n_j}(\tau) \sin(2\pi f_{c_R} \tau) - \hat{R}_{n_i n_j}(\tau) \cos(2\pi f_{c_R} \tau). \quad (O.2.28c)$$

Similarly,

$$R_{ij}^{QI}(\tau) = E[x_{iQ}(t)x_{jI}(t-\tau)] \quad (O.2.29a)$$

$$= E[\hat{n}_i(t)n_j(t-\tau)]\cos(2\pi f_{c_R} t)\cos[2\pi f_{c_R}(t-\tau)] \\ + E[\hat{n}_i(t)\hat{n}_j(t-\tau)]\cos(2\pi f_{c_R} t)\sin[2\pi f_{c_R}(t-\tau)] \\ - E[n_i(t)n_j(t-\tau)]\sin(2\pi f_{c_R} t)\cos[2\pi f_{c_R}(t-\tau)] \\ - E[n_i(t)\hat{n}_j(t-\tau)]\sin(2\pi f_{c_R} t)\sin[2\pi f_{c_R}(t-\tau)] \quad (O.2.29b)$$

$$= R_{\hat{n}_i n_j}(\tau) \cos A \cos B + R_{\hat{n}_i \hat{n}_j}(\tau) \cos A \sin B \\ - R_{n_i n_j}(\tau) \sin A \cos B - R_{n_i \hat{n}_j}(\tau) \sin A \sin B \quad (O.2.29c)$$

$$= R_{\hat{n}_i n_j}(\tau) \cos A \cos B + R_{n_i n_j}(\tau) \cos A \sin B \\ - R_{n_i n_j}(\tau) \sin A \cos B - R_{n_i \hat{n}_j}(\tau) \sin A \sin B \quad (O.2.29d)$$

$$= \hat{R}_{n_i n_j}(\tau) [\cos A \cos B + \sin A \sin B] \\ - R_{n_i n_j}(\tau) [\sin A \cos B - \cos A \sin B] \quad (O.2.29e)$$

$$= \hat{R}_{n_i n_j}(\tau) \cos(2\pi f_{c_R} \tau) - R_{n_i n_j}(\tau) \sin(2\pi f_{c_R} \tau) \quad (O.2.29f)$$

so that

$$\boxed{R_{ij}^{IQ}(\tau) = -R_{ij}^{QI}(\tau).} \quad (O.2.30)$$

APPENDIX P DERIVATION OF EQS(O.1.20) AND (O.2.22)

In this appendix, we prove that

$$R_{\hat{n}_i \hat{n}_j}(\tau) = R_{n_i n_j}(\tau) \quad \text{for } i=j \text{ and } i \neq j \quad (\text{P.1})$$

where the symbol $\wedge$ in this discussion refers to the Hilbert transform. Consider

$$R_{\hat{n}_i \hat{n}_j}(\tau) = E[\hat{n}_i(t) \hat{n}_j(t-\tau)] \quad (\text{P.2a})$$

$$= E \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{n}_i(t) n_j(\lambda)}{t-\tau-\lambda} d\lambda \right] \quad (\text{P.2b})$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{E[\hat{n}_i(t) n_j(\lambda)]}{t-\tau-\lambda} d\lambda. \quad (\text{P.2c})$$

But

$$E[\hat{n}_i(t) n_j(\lambda)] = E \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{n_i(\phi) n_j(\lambda)}{t-\phi} d\phi \right] \quad (\text{P.3a})$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{E[n_i(\phi) n_j(\lambda)]}{t-\phi} d\phi \quad (\text{P.3b})$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{n_i n_j}(\phi-\lambda)}{t-\phi} d\phi \quad (\text{P.3c})$$

Let

$$\alpha = t - \lambda \quad (\text{P.4a})$$

and

$$\beta = \phi - \lambda \quad (\text{P.4b})$$

so that

$$E[\hat{n}_i(t)n_j(\lambda)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R_{n_i n_j}(\beta)}{t-\beta-\lambda} d\beta \quad (\text{P.5a})$$

$$= \hat{R}_{n_i n_j}(t-\lambda) = \hat{R}_{n_i n_j}(\alpha). \quad (\text{P.5b})$$

Using eq(P.5b) in (P.2c), we have

$$R_{\hat{n}_i \hat{n}_j}(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{R}_{n_i n_j}(\alpha)}{\alpha-\tau} (-d\alpha) \quad (\text{P.6a})$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{R}_{n_i n_j}(\alpha)}{\tau-\alpha} d\alpha \quad (\text{P.6b})$$

$$= \hat{R}_{n_i n_j}(\tau) = R_{n_i n_j}(\tau). \quad (\text{P.6c})$$

These results also hold when $i = j$.

APPENDIX Q DERIVATION OF EQ(4.3.2a)

In this appendix, we derive eq(4.3.2a). Using $l=0$ in eq(4.3.1b) together with the definition

$$R_{gij}(0) = (\rho_{gij}) \sigma_{gii} \sigma_{gjj}, \quad (Q.1)$$

we obtain

$$R_{gij}(0) = (\rho_{gij}) \sigma_{gii} \sigma_{gjj} = K_{gij} f_g(\lambda_{gij}, |l_{gij}|) \Big|_{l=0} \exp\{j\theta_{gij}(0)\}. \quad (Q.2)$$

Solving eq(Q.2) for the normalizing coefficient

$$K_{gij} = \frac{(\rho_{gij}) \sigma_{gii} \sigma_{gjj}}{f_g(\lambda_{gij}, |l_{gij}|) \Big|_{l=0}} \exp(-j\theta_{gij}(0)) \quad (Q.3)$$

so that eq(4.3.1b) becomes

$$R_{gij}(l) = \frac{(\rho_{gij}) \sigma_{gii} \sigma_{gjj} f_g(\lambda_{gij}, |l_{gij}|)}{f_g(\lambda_{gij}, |l_{gij}|) \Big|_{l=0}} \exp\{j[\theta_{gij}(l) - \theta_{gij}(0)]\}. \quad (Q.4a)$$

$$= \frac{|\rho_{gij}| \sigma_{gii} \sigma_{gjj} f_g(\lambda_{gij}, |l_{gij}|)}{f_g(\lambda_{gij}, |l_{gij}|) \Big|_{l=0}} \exp\{j[\theta_{gij}(l)]\} \quad g = s, c. \quad (Q.4b)$$

where, by definition [3],

$$\rho_{gij} = |\rho_{gij}| \exp[j\theta_{gij}(0)] \quad (Q.5)$$

We also have

$$R_{gij}(l) = [R_{gij}(-l)]^* \quad (Q.6a)$$

$$= \frac{(\rho_{gij})^* \sigma_{gii} \sigma_{gjj} f_g(\lambda_{gij}, -l - l_{gij})}{f_g(\lambda_{gij}, l - l_{gij})|_{l=0}} \exp\{-j[\theta_{gij}(-l) - \theta_{gij}(0)]\}. \quad (Q.6b)$$

$$= \frac{|\rho_{gij}| \sigma_{gii} \sigma_{gjj} f_g(\lambda_{gij}, -l - l_{gij})}{f_g(\lambda_{gij}, l - l_{gij})|_{l=0}} \exp\{-j[\theta_{gij}(-l)]\} \quad g = s, c. \quad (Q.6c)$$

Eqs(Q.4) and (Q.6) provide us with a useful description of the cross-correlation function in terms of the cross-correlation coefficient ρ_{gij} , the standard deviations σ_{ii} and σ_{jj} of the channel i and j processes, respectively, and the temporal cross-correlation coefficient, λ_{gij} .

For the autocorrelation function ($i=j$), we have

$$|\rho_{gii}| = 1 \quad (Q.7)$$

since any given channel process is totally correlated with itself at zero lag. Also,

$$\theta_{gii}(0) = 0 \quad (Q.8)$$

since $\theta_{gii}(l)$ is an odd function of l . Eq(Q.4b) for the autocorrelation function now becomes

$$R_{gii}(l) = \frac{\sigma_{gii}^2 f_g(\lambda_{gii}, l)}{f_g(\lambda_{gii}, l)|_{l=0}} \exp(j[\theta_{gii}(l)]) \quad (Q.9)$$

where we again note that $l_{gii}=0$ for $i=j$ since the autocorrelation function peaks at lag zero. Furthermore, since the function $f_g(\cdot)$ for the autocorrelation function has a peak value of unity at $l=0$, eq.(Q.9) reduces to

$$R_{gii}(l) = \sigma_{gii}^2 f_g(\lambda_{gii}, l) \exp(j\theta_{gii}(l)) \quad g = s, c. \quad (Q.10)$$

At $l = 0$, eq.(Q.10) becomes

$$R_{g_{ii}}(0) = \sigma_{g_{ii}}^2 \quad g=s,c \quad (Q.11)$$

which is, as expected, the variance of the zero mean, channel i process.

APPENDIX R SPECTRA FOR COMPLEX CORRELATION FUNCTIONS

In this appendix, we derive relationships between the auto- and cross-spectra and their associated complex auto- and cross-correlation functions. In particular, we consider the shaping forms of the correlation functions discussed in section 4.3.

R.1. GENERAL RELATIONSHIPS

The power spectral density $P_{ii}(f)$ of the continuous random process $x_i(t)$ is defined as the Fourier transform of its autocorrelation function such that

$$P_{ii}(f) = \int_{-\infty}^{\infty} R_{ii}(\tau) \exp(-j2\pi f\tau) d\tau \quad (R.1)$$

while for the discrete processes $x_i(n)$, we use the discrete Fourier transform

$$P_{ii}(f) = T \sum_{l=-\infty}^{\infty} R_{ii}(l) \exp(-j2\pi f l T) \quad (R.2)$$

where $x_i(t)$ is assumed to be bandlimited to $\pm \frac{1}{2T}$ Hz, and $P_{ii}(f)$ is periodic in frequency with period $\frac{1}{T}$ Hz.

The cross-power spectrum $P_{ij}(f)$ of two processes $x_i(t)$ and $x_j(t)$ is similarly expressed in terms of the cross-correlation functions such that for continuous processes

$$P_{ij}(f) = \int_{-\infty}^{\infty} R_{ij}(\tau) \exp(-j2\pi f\tau) d\tau. \quad (R.3)$$

Assuming $x_i(t)$ and $x_j(t)$ to be bandlimited to $\pm \frac{1}{2T}$ Hz, the discrete Fourier transform for the discrete processes results in the cross-power spectrum

$$P_{ij}(f) = T \sum_{l=-\infty}^{\infty} R_{ij}(l) \exp(-j2\pi f l T). \quad (R.4)$$

Using the property of autocorrelation functions; i.e.,

$$R_{ij}(\tau) = R_{ij}^*(-\tau) \quad (R.5)$$

In eq (R.1), we have

$$P_{ij}(f) = \int_{-\infty}^{\infty} R_{ij}^*(-\tau) \exp(-j2\pi f \tau) d\tau \quad (R.6a)$$

$$= \int_{-\infty}^{\infty} R_{ij}(\tau) \exp(j2\pi f \tau) d\tau = [P_{ij}(f)]^* \quad (R.6b)$$

where the last equality results by changing the variable τ to $-\tau$ and appropriately changing the direction of the integration. Similarly, using eq (R.5) in eq (R.2) provides the discrete time version as

$$P_{ij}(f) = T \sum_{l=-\infty}^{\infty} R_{ij}^*(-l) \exp(-j2\pi f l T) \quad (R.7a)$$

$$= T \sum_{l=-\infty}^{\infty} R_{ij}(l) \exp(+j2\pi f l T) = [P_{ij}(f)]^*. \quad (R.7b)$$

Equations (R.6b) and (R.7b) indicate that the autospectra are real. Since, in general

$$R_{ij}(\tau) = R_{ij}^*(-\tau) \quad (R.8a)$$

and

$$R_{ij}(l) = R_{ij}^*(-l) \quad (R.8b)$$

the cross-spectra is, in general, complex. From the Fourier inversion formula, we have

$$R_{ij}(\tau) = \int_{-\infty}^{\infty} P_{ij}(f) \exp(j2\pi f\tau) df \quad (R.9a)$$

and

$$R_{ij}(\tau) = \int_{-\infty}^{\infty} P_{ij}(f) \exp(j2\pi f\tau) df \quad (R.9b)$$

for continuous time processes; the inverse discrete time Fourier transform yields

$$R_{ij}(l) = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} P_{ij}(f) \exp(j2\pi fTl) df \quad (R.10a)$$

and

$$R_{ij}(l) = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} P_{ij}(f) \exp(j2\pi fTl) df \quad (R.10b)$$

At $\tau = 0$, eqs (R.9a) and (R.9b) become

$$R_{ij}(0) = \int_{-\infty}^{\infty} P_{ij}(f) df \quad (R.11a)$$

and

$$R_{ij}(0) = \int_{-\infty}^{\infty} P_{ij}(f) df \quad (R.11b)$$

for continuous time processes; for $l=0$, eqs (R.10a) and (R.10b) become

$$R_{11}(0) = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} P_{11}(f) df \quad (\text{R.12a})$$

and

$$R_{11}(0) = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} P_{11}(f) df \quad (\text{R.12b})$$

for discrete time processes.

R.2 QUADRATURE COMPONENT FORM

We now consider the continuous complex autocorrelation function in terms of its quadrature components, such that

$$R_{11}(\tau) = R_{C11}(\tau) + jR_{D11}(\tau). \quad (\text{R.13})$$

Using eq (R.13) in eq (R.1), we have

$$P_{11}(f) = \int_{-\infty}^{\infty} R_{C11}(\tau) \exp(-j2\pi f\tau) d\tau + j \int_{-\infty}^{\infty} R_{D11}(\tau) \exp(-j2\pi f\tau) d\tau. \quad (\text{R.14})$$

But since $R_{C11}(\tau)$ is even and $R_{D11}(\tau)$ is odd (see section 3.2), eq (R.14) becomes

$$P_{11}(f) = \int_{-\infty}^{\infty} R_{C11}(\tau) \cos(2\pi f\tau) d\tau + \int_{-\infty}^{\infty} R_{D11}(\tau) \sin(2\pi f\tau) d\tau \quad (\text{R.15a})$$

$$= P_{C11}(f) + P_{D11}(f) \quad (\text{R.15b})$$

Eq(R.15b) indicates that the autospectrum is real, as noted previously. Since the first integral is an even function of f while the second is odd, the resulting spectrum is not, in general, symmetric about $f=0$. In the case where

$P_{D_{ij}}(f) = 0$, the spectrum is even. This results when $R_{D_{ij}}(\tau) = 0$, so that $R_{ij}(\tau)$ is real.

R.3 SPECTRAL DISTRIBUTION USING THE CORRELATION SHAPING FUNCTIONS

We now present a discussion which will help to clarify the role of the temporal correlation coefficient λ_{ij} introduced in section 4.3 to shape the autocorrelation function.

R.3.a. Spectrum of the Gaussian Shaped Autocorrelation Function

In this case, we consider a real autocorrelation function for a continuous-time process on channel i and eqs(4.3.8) are used with the subscript notation g dropped. Specifically,

$$\theta_{ij}(\tau) = 0 \quad (R.16a)$$

and

$$f(l_{ij}, \tau) = l_{ij}^{\tau^2} = \exp(-2\pi^2 \mu_{ij}^2 \tau^2) \quad (R.16b)$$

where

$$l_{ij} = \exp(-2\pi^2 \mu_{ij}^2). \quad (R.17)$$

With these expressions

$$R_{ij}(\tau) = \sigma_{s_{ij}}^2 \exp(-2\pi^2 \mu_{ij}^2 \tau^2) \quad (R.18)$$

is a real, Gaussian shaped autocorrelation function. Taking the derivative of eq(R.1) with respect to f and eq(R.9a) with respect to τ , we have

$$\dot{P}_{ij}(f) = \int_{-\infty}^{\infty} [-j2\pi\tau R_{ij}(\tau)] \exp(-j2\pi f\tau) d\tau \quad (R.19a)$$

and

$$\dot{R}_{ii}(\tau) = \int_{-\infty}^{\infty} [j2\pi f P_{ii}(f)] \exp(+j2\pi f \tau) df. \quad (R.19b)$$

Therefore,

$$-j2\pi\tau R_{ii}(\tau) \xleftrightarrow{F.T} \dot{P}_{ii}(f) \quad (R.20a)$$

and

$$\dot{R}_{ii}(\tau) \xleftrightarrow{F.T} j2\pi f P_{ii}(f) \quad (R.20b)$$

where $\xleftrightarrow{F.T}$ denotes the Fourier transform pair. Taking the derivative of eq(R.18) with respect to τ

$$\dot{R}_{ii}(\tau) = -4\pi^2 \mu_{ii}^2 \tau R_{ii}(\tau) \quad (R.21)$$

so that from eq. (R.20b)

$$-4\pi^2 \mu_{ii}^2 \tau R_{ii}(\tau) \xleftrightarrow{F.T} j2\pi f P_{ii}(f). \quad (R.22)$$

Dividing both sides of (R.22) by $-j2\pi\mu_{ii}^2$ yields

$$-j2\pi\tau R_{ii}(\tau) \xleftrightarrow{F.T} -\left[\frac{f}{\mu_{ii}}\right] P_{ii}(f). \quad (R.23)$$

Using eq. (R.20a) and eq.(R.23)

$$\dot{P}_{ii}(f) = -\left[\frac{f}{\mu_{ii}}\right] P_{ii}(f). \quad (R.24)$$

Solving this equation for $P_{ii}(f)$, we obtain

$$P_{ii}(f) = P_{ii}(0) \exp(-f^2/2\mu_{ii}^2) \quad (R.25)$$

where it is now observed that μ_{ii}^2 is the variance of the Gaussian power spectral density function. Eq(R.25) indicates that the real, Gaussian autocorrelation function results in a symmetric, Gaussian power spectral density (PSD).

For the discrete time case, $\tau \rightarrow lT$ where T is the sample period, so that

$$R_{ii}(l) = \sigma_{s_{ii}}^2 \exp(-2\pi^2 \mu_{ii}^2 T^2 l^2) \quad (R.26a)$$

$$= \sigma_{s_{ii}}^2 \lambda_{ii}^{l^2} \quad (R.26b)$$

where

$$\lambda_{ii} = \exp(-2\pi^2 \mu_{ii}^2 T^2). \quad (R.26c)$$

From eq(R.26b), we see that as the variance of the power spectrum μ_{ii}^2 ranges from zero to infinity, λ_{ii} goes from one to zero, respectively. For $\lambda_{ii}=0$, $R_{ii}(l)$ is a discrete delta function $\delta(l)$ and $P_{ii}(f)$ is a white noise spectrum. When $\lambda_{ii} = 1$, $R_{ii}(l)$ denotes the case of total temporal correlation with a periodic line spectrum for $P_{ii}(f)$ which results from the Poisson sum formula using $R_{ii}(l)=\sigma_{s_{ii}}^2$ in eq(R.2). For λ_{ii} ranging from zero to unity, all values of temporal correlation are obtained. Thus, λ_{ii} is a measure of the temporal correlation between consecutive samples of the random processes [23].

R.3.b. Spectrum of the Exponential Shaped Autocorrelation Function

In this section, we consider a real exponential autocorrelation function for a continuous time process on channel i . For this case, we have

$$\theta_{ii}(\tau) = 0 \quad (R.27a)$$

$$f(l_{ii}, \tau) = l_{ii}^{|\tau|} = \exp(-2\pi\kappa_{ii} |\tau|) \quad (\text{R.27b})$$

where, in this case

$$l_{ii} = \exp(-2\pi\kappa_{ii}). \quad (\text{R.28})$$

With these expressions, we have

$$R_{ii}(\tau) = \sigma_{s_{ii}}^2 \exp(-2\pi\kappa_{ii} |\tau|) \quad (\text{R.29})$$

The power spectral density is determined by considering $R_{ii}(\tau)$ as the superposition of two functions such that

$$R_{ii}(\tau) = \sigma_{s_{ii}}^2 l_{ii}^{\tau} u(\tau) + \sigma_{s_{ii}}^2 l_{ii}^{-\tau} u(-\tau) \quad (\text{R.30})$$

where $u(\tau)$ is the unit step function. And so, the power spectral density is expressed as

$$P_{ii}(f) = \int_0^{\infty} \sigma_{s_{ii}}^2 \exp(-2\pi\kappa_{ii}\tau) \exp(-j2\pi f\tau) d\tau + \int_{-\infty}^0 \sigma_{s_{ii}}^2 \exp(+2\pi\kappa_{ii}\tau) \exp(-j2\pi f\tau) d\tau \quad (\text{R.31a})$$

$$= \frac{\sigma_{s_{ii}}^2}{(2\pi\kappa_{ii}) + j(2\pi f)} + \frac{\sigma_{s_{ii}}^2}{(2\pi\kappa_{ii}) - j(2\pi f)} \quad (\text{R.31b})$$

$$= \frac{2\sigma_{s_{ii}}^2 (2\pi\kappa_{ii})}{(2\pi\kappa_{ii})^2 + (2\pi f)^2} \quad (\text{R.31c})$$

The peak value of this function occurs at $f=0$ so that

$$P_{ii}(0) = \frac{\sigma_{s_{ii}}^2}{\pi \kappa_{ii}}. \quad (R.32)$$

Also

$$P_{ii}(\kappa_{ii}) = \frac{\sigma_{s_{ii}}^2}{2\pi \kappa_{ii}} \quad (R.33)$$

so that the 3-dB down points occur at $f = \kappa_{ii}$ in the exponential case. The inflection point of $P_{ii}(f)$ occurs at $f = \kappa_{ii} \frac{\sqrt{3}}{3}$.

A similar presentation can be made for the discrete-time case using

$$f(\lambda_{ii}, l) = \lambda_{ii}^{|l|} = \exp(-2\pi \kappa_{ii} T |l|) \quad (R.34)$$

where, in this case

$$\lambda_{ii} = \exp(-2\pi \kappa_{ii} T). \quad (R.35)$$