AD-A241 410

NAVSWC TR 91-325

PUSHOUTS AND GRAPH RECONSTRUCTION

BY A.D. PARKS
STRATEGIC SYSTEMS DEPARTMENT

JUNE 1991

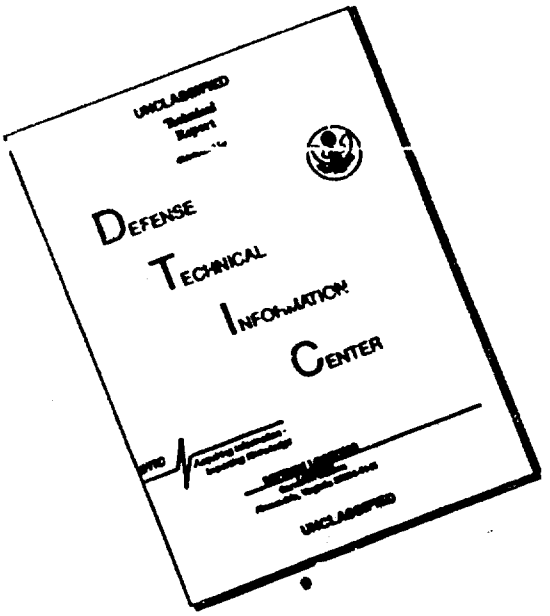
DTIC
ELECTE
OCT.10.1991
S B D

Approved for public release; distribution is unlimited.

91-12654

NAVAL SURFACE WARFARE CENTER
Dahlgren, Virginia 22448-5000 • Silver Spring, Maryland 20903-5000

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

NAVSWC TR 91-325

PUSHOUTS AND GRAPH RECONSTRUCTION

BY A.D. PARKS
STRATEGIC SYSTEMS DEPARTMENT

JUNE 1991

Approved for public release; distribution is unlimited.

NAVAL SURFACE WARFARE CENTER
Dahlgren, Virginia 22448-5000 • Silver Spring, Maryland 20903-5000

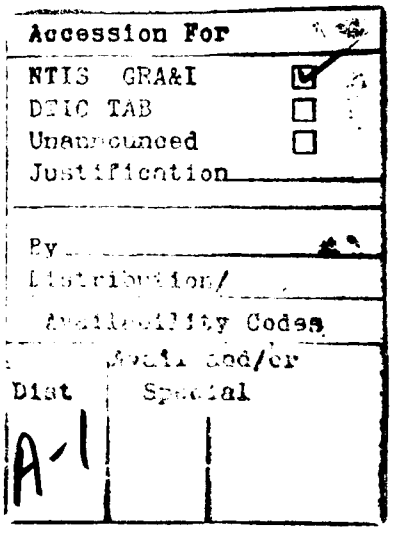
FOREWORD

This report describes a certain "tangential" theoretical result obtained from work performed in the Strategic Systems (K) Department at the Naval Surface Warfare Center (NAVSWC) as part of an independent research grant entitled "A Category Theoretic Approach to Relational Database Schemes."

This report has been reviewed and approved by Ted Sims, Space Sciences Branch Head, and James L. Sloop, Space and Surface Systems Division Head.

Approved by:

R. L. SCHMIDT, Head
Strategic Systems Department



CONTENTS

	Page
PRELIMINARIES	1
CARD DIAGRAMS AND CONJUGATE σ-PAIRS	2
MAIN RESULTS	3
APPLICATIONS	5
REMARKS	6
REFERENCES	7
DISTRIBUTION	(1)

PRELIMINARIES

A *graph* $G = (V, E)$ is a finite nonempty set V of *vertices* and a set E of *edges* that are unordered pairs of elements of V. The *order* p of G is $|V|$, and the *size* q of G is $|E|$. The edge $e = \{u, v\} \in E$ *joins* the *adjacent vertices* u and v (denoted $u \sim v$), and is *incident* to both u and v.

The *neighborhood* $\mathbf{n}(v)$ of a vertex v is the set of vertices adjacent to v. The number $d(v)$ of edges incident to v is the *degree* or *valency* of v. When $d(v) = 1$, then v is an *end-vertex* of G. The *valency-sequence* $\mathbf{vs}(v)$ of a vertex v is the non-decreasing sequence of valencies of neighbors of v. The *reduced valency-sequence* $\mathbf{rvs}(v)$ is obtained by subtracting one from each term of $\mathbf{vs}(v)$. A $\mathbf{vs}(u)$ for $u \in V(G - v)$ is *v-excluded* if $v \in \mathbf{n}(u)$ in G.

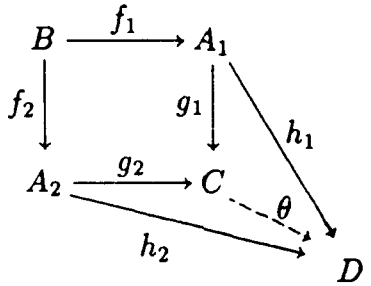
Ordinary graphs are those with order at least three. A *vertex-deleted subgraph* (or card) $G_v = G - v$ is the subgraph of G obtained by removing v and all edges incident to v. A *null graph* N_p has order p and size $q = 0$. A *star graph* $K_{1,s}$ is one that is isomorphic to the quotient graph obtained by identifying s edges at one common central vertex.

The category **Graph** has finite, simple, and undirected graphs for objects, and graph homomorphisms for morphisms. Here, $f : G \to H$ is a morphism when $f(u) \sim f(v)$ whenever $u \sim v$ (f is not necessarily onto).

In any category, a *solution* of the diagram

$$A_2 \xleftarrow{f_2} B \xrightarrow{f_1} A_1 \qquad (\Delta)$$

is an object C and morphisms g_1, g_2 such that $g_1 f_1 = g_2 f_2$. A *pushout* of (Δ) is a solution (C, g_1, g_2) with the additional property that for any solution (D, h_1, h_2), there exists a unique morphism $\theta : C \to D$ making the following diagram commute [1,p.33]:



If (C, g_1, g_2) and (D, h_1, h_2) are pushouts of (Δ), then θ is an isomorphism [1,p.16]. This is the *universal property* of pushouts.

If O_1 and O_2 are objects in some category, then their *coproduct* is an object $O_1 \coprod O_2$ together with morphisms $h_k : O_k \to O_1 \coprod O_2$, $k = 1, 2$, called *injections*, with the property

that for any object X and any morphisms $m_k : O_k \to X$, $k = 1, 2$, there is a unique morphism $\psi : O_1 \coprod O_2 \to X$ such that $m_k = \psi h_k$, $k = 1, 2$ [3,p.58]. A coproduct in **Graph** is just the disjoint union of graphs together with the inclusion maps [2,p.128]. Specifically, if G and H are graphs, then $G \coprod H$ has as vertex set the disjoint union $V(G) \bigvee V(H)$; edge set the disjoint union $E(G) \bigvee E(H)$; and injections (imbeddings) $V(G) \hookrightarrow V(G) \bigvee V(H)$ and $V(H) \hookrightarrow V(G) \bigvee V(H)$.

An *automorphism* of a graph G is an isomorphism of G onto itself, i.e., a permutation of V which preserves adjacency. Collectively, the automorphisms of G (under composition) form the *automorphism* group $Aut(G)$ of G. Subsets V' and V'' of V are *α-equivalent* whenever there is an $\alpha \in Aut(G)$ mapping V' into V''. If for each pair, $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$, of k-tuples of vertices of G there is an automorphism α of G such that $\alpha(u_i) = v_i$, $i = 1, \ldots, k$, then the graph G is *k-transitive*.

A *hypomorphism* of a graph G onto a graph H is a bijection $\sigma : V(G) \to V(H)$ such that for every $v \in V(G)$, G_v is isomorphic to ($\simeq$) $H_{\sigma(v)}$. Graphs G and H are *hypomorphic* whenever a hypomorphism of G onto H exists. If G and H are hypomorphic ordinary graphs, then they are both connected or disconnected; they have the same number of vertices and the same number of edges; they have the same degree sequence; and corresponding vertices ($v \leftarrow \sigma(v)$) have the same valency-sequence [4].

A graph G is *reconstructible* if every graph hypomorphic to G is isomorphic to G. If $z \in V(G)$, then a graph H is a *z-reconstruction* of G when $V(G) = V(H)$, $G - z = H - z$, and G is hypomorphic to H.

CARD DIAGRAMS AND CONJUGATE σ-PAIRS

We consider only ordinary graphs. Let G be such a graph with vertex v. Let $K = K_{1,d(v)}$ be the subgraph of G induced by the edges incident to v; $N = N_{d(v)} = V(G_v) \cap V(K)$; and define the *$G_v$ card diagram*

$$G_v \overset{i}{\hookleftarrow} N \overset{j}{\hookrightarrow} K, \qquad (*)$$

where the inclusions i and j are clearly homomorphisms ($E(N) = \emptyset$).

The proof of the following Lemma is obvious.

Lemma 1. *If $\sigma : G \to H$ is a hypomorphism, then for each $(*)$ there is an induced $H_{\sigma(v)}$ card diagram*

$$H_{\sigma(v)} \overset{i'}{\hookleftarrow} N' \overset{j'}{\hookrightarrow} K', \qquad (\#)$$

where $N' = N_{d(\sigma(v))}$, $K' = K_{1,d(\sigma(v))}$, and i', j' are inclusions.

For a hypomorphism $\sigma : G \to H$, the set $P(\sigma)$ contains p *σ-pairs* $\{i, i'\}$ where i and i' are corresponding inclusions appearing, respectively, in the G_v and $H_{\sigma(v)}$ card diagrams. If there are isomorphisms ϕ and ψ such that diagram (@) commutes, then call i and i'

σ-conjugate.

$$\begin{array}{ccc} G_v & \xleftarrow{\quad i \quad} & N \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ H_{\sigma(v)} & \xleftarrow{\quad i' \quad} & N' \end{array} \qquad (@)$$

Lemma 2. *The inclusions i and i' are σ-conjugate if, and only if, there is an isomorphism $\phi : G_v \to H_{\sigma(v)}$ such that $im(\phi i) = im(i')$.*

PROOF: ($\Leftarrow$) Let $\psi = (i')^{-1}\phi i$. ($\Rightarrow$) Obvious. ∎

The now obvious proofs of the two following corollaries are omitted.

Corollary 3. *Let σ, i, i', be as in Lemma 2. Suppose $\phi : G_v \to H_{\sigma(v)}$ is an isomorphism. If there is an automorphism α of G such that $im(i)$ and $im(\phi^{-1}i')$ are α-equivalent, then i and i' are σ-conjugate.*

Corollary 4. *With the same hypothesis as Lemma 2, if i and i' are onto, then they are σ-conjugate.*

The next Lemma also has an easy proof.

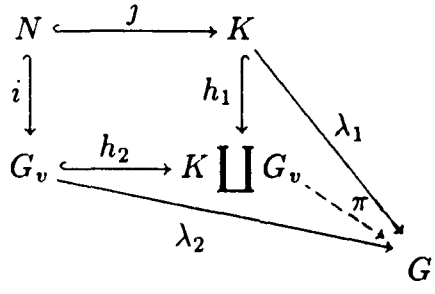
Lemma 5. *Given any isomorphism $\psi : N \to N'$, there is an isomorphism $\gamma : K \to K'$ such that $\gamma j = j'\psi$.*

MAIN RESULTS

Continuing with the same notation as above, let

$$\pi : K \coprod G_v \to G \quad \text{and} \quad \pi' : K' \coprod H_{\sigma(v)} \to H$$

be the projective homomorphisms which map the vertices and edges of the coproduct onto G and H. For example, the restriction of π to K is just an inclusion map. Next consider the diagram

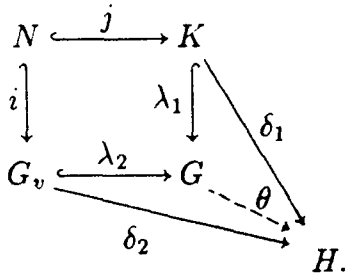


(1)

where the homomorphisms $\lambda_1 = \pi h_1$ and $\lambda_2 = \pi h_2$. Similarly, the π' morphism induces a corresponding diagram with λ_1' and λ_2'.

Lemma 6. *$(G, \lambda_1, \lambda_2)$ is a pushout of $(*)$ in* **Graph**.

PROOF: It is easy to show that $(G, \lambda_1, \lambda_2)$ is a solution to $(*)$. To see that $(G, \lambda_1, \lambda_2)$ is a pushout, consider the diagram



(2)

Given that (H, δ_1, δ_2) is a solution, one can define the desired morphism

$$\theta = (\delta_1 \coprod \delta_2) \circ \pi^{-1}$$

where $\delta_1 \coprod \delta_2 : K \coprod G_v \to H$ exists because $K \coprod G_v$ is a coproduct. ∎

Theorem 7. *An ordinary graph G is reconstructible if, and only if, for each hypomorph H of G there are homomorphisms δ_1 and δ_2 such that (H, δ_1, δ_2) is a pushout of $(*)$ for some card of G.*

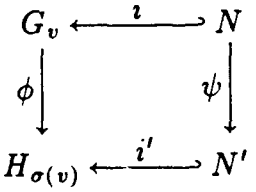
PROOF: If G is reconstructible, then we can choose θ in diagram (2) to be an isomorphism and define

$$\delta_1 = \theta\lambda_1 \quad \text{and} \quad \delta_2 = \theta\lambda_2.$$

Since $(G, \lambda_1, \lambda_2)$ is a pushout of $(*)$, we see (H, δ_1, δ_2) is a pushout of $(*)$. The converse follows from the universal property of pushouts of $(*)$, i.e., any two are isomorphic. ∎

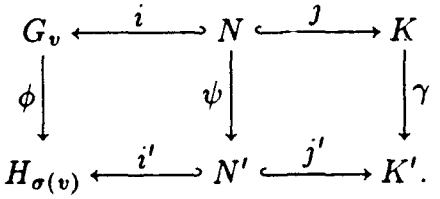
Theorem 8. *Let G be an ordinary graph. If for each hypomorphism σ, $P(\sigma)$ contains a conjugate σ-pair, then G is reconstructible.*

PROOF: If i,i' are σ-conjugate, then there exists isomorphisms ϕ and ψ such that



(3)

is commutative. Also, it follows from Lemma 5 that there exists an isomorphism γ which extends the diagram (3) to the commutative diagram:



(4)

It follows from a proof analogous to that of Lemma 6, that for the inclusions $\delta_1 : K' \hookrightarrow H$, $\delta_2 : H_{\sigma(v)} \hookrightarrow H$, the triple (H, δ_1, δ_2) is a pushout of the bottom row of diagram (4). With this and the isomorphisms in diagram (4), we conclude that (H, δ_1, δ_2) is a pushout of $(*)$. But since H is an arbitrary hypomorph of G, we apply Theorem 7 and conclude that G is reconstructible. ∎

APPLICATIONS

Assuming all graphs are ordinary, applications of Theorem 8, and the lemmas and corollaries of the second section, yield the following:

Theorem 9. *Let G be a graph of order p. If there is a $v \in V(G)$ such that $d_G(v) = p - 1$, then G is reconstructible.*

PROOF: Diagram (@) commutes for each hypomorph H, since $d_H(\sigma(v)) = p - 1$ and i, i' are onto. ∎

Theorem 10. *A graph G which has a $d_G(v)$-transitive G_v card is reconstructible.*

PROOF: Observe that $\mathrm{im}(i)$ and $\mathrm{im}(\phi^{-1}i')$ are α-equivalent in diagram (@) for each hypomorph. ∎

Theorem 11. *A graph G is reconstructible if every hypomorph of G is a z-reconstruction of G.*

PROOF: For each hypomorph, ϕ and ψ are identities and diagram (@) commutes. ∎

The following two theorems are equivalent to the "good neighbor" theorems found in [4,pp. 216–217].

Theorem 12. *A graph is reconstructible if it has a vertex with a unique reduced valency sequence.*

PROOF: A unique $\mathbf{rvs}_G(v)$ implies the corresponding set

$$\{\, d_{G_v}(w) \,|\, w \in \mathbf{n}_G(v) \,\}$$

is unique in G_v. Since ϕ preserves this uniqueness, diagram (@) commutes for each hypomorph. ∎

Theorem 13. *A graph is reconstructible if it has a vertex v each neighbor of which has a unique, possibly v-excluded, valency sequence.*

PROOF: Observe that

$$\mathbf{vs}_{G_v}(w), \quad \text{where} \quad w \in \mathbf{n}_G(v),$$

is preserved by ϕ and the diagram (@) commutes for each hypomorph. ∎

REMARKS

The Reconstruction Conjecture, which asserts that every ordinary graph can be uniquely reconstructed from its deck of vertex deleted subgraphs, is generally regarded as one of the foremost unsolved problems in graph theory. Here we have cast the reconstruction problem in a category theoretic framework and demonstrated the ease with which certain reconstruction results can be obtained when viewed from this perspective. We note that while seeming of little practical value, it is possible that such results may prove useful in areas such as distributed information synthesis.

REFERENCES

1. T. S. Blyth,"Categories, " Longman Group Limited, Essex, England, 1986.

2. P. Hell, *An Introduction to the Category of Graphs*, Annals New York Academy of Sciences (1979), 120–136.

3. P. J. Hilton and U. Stammbach," A Course in Homological Algebra," Springer-Verlag, New York, (1971).

4. C. St. J. A. Nash-Williams, *The Reconstruction Problem*, In: L. Beineke and R. Wilson, eds.,"Selected Topics in Graph Theory," Academic Press, London, 1978, pp. 205–236.

DISTRIBUTION

	Copies
ATTN CODE 3T COMMANDER NAVAL WEAPONS CENTER CHINA LAKE CA 93555-6001	1
ATTN CODE 014 COMMANDER NAVAL OCEAN SYSTEMS CENTER SAN DIEGO CA 91252-5000	1
ATTN CODE 101 OFFICER IN CHARGE NEW LONDON LABORATORY DETACHMENT NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CT 06320	1
ATTN CODE 01B COMMANDER NAVAL AIR DEVELOPMENT CENTER WARMINSTER PA 18974-5000	1
ATTN CODE 011 OFFICER IN CHARGE ANNAPOLIS LABORATORY DAVID TAYLOR RESEARCH CENTER ANNAPOLIS MD 21402-1198	1
ATTN CODE 30 OFFICER IN CHARGE OFFICE FOR APPLIED TECHNOLOGY OFFICE OF CHIEF OF NAVAL RESEARCH 800 N QUINCY ST ARLINGTON VA 22204	1
ATTN CODE PD 60 INFORMATION MANAGEMENT SYSTEMS PROGRAM DIRECTORATE SPACE AND NAVAL WARFARE SYSTEMS COMMAND WASHINGTON D C 20363-5100	1
ATTN INFORMATION SCIENCE AND TECHNOLOGY OFFICE DEFENSE ADVANCED RESEARCH PROJECTS AGENCY 1400 WILSON BLVD ROSYLN VA 22201	1
ATTN CODE 5510 WARFARE SYSTEMS AND SENSORS RESEARCH DIRECTORATE NAVY CENTER FOR APPLIED RESEARCH IN ARTIFICIAL INTELLIGENCE NAVAL RESEARCH LABORATORY WASHINGTON D C 20375	1
ATTN R.E. NANCE SYSTEMS RESEARCH CENTER VIRGINIA POLYTECHNIC INSTITUTE & STATE UNIVERSITY BLACKSBURG VA 24061	1
DEFENSE TECHNICAL INFORMATION CENTER CAMERON STATION ALEXANDRIA VA 22314	12

DISTRIBUTION (CONTINUED)

	Copies		Copies
		U	1
ATTN GIFT AND EXCHANGE DIVISION	4	U04	1
LIBRARY OF CONGRESS			
WASHINGTON DC 20540			

INTERNAL DISTRIBUTION

C	1
D	1
D2	1
D4	1
E231	3
E32 GIDEP	1
G	1
H	1
J	1
K	1
K02 FALLIN	1
K07 HUGHEY	1
K10	1
K104 FELL, DIDONATO	2
K106 BLANTON	1
K107 HILL	1
K12	1
K13	1
K13 PARKS	15
K14	1
K40	1
K50	1
N	1
N35 TATE, RIEDL	2
N305 GREEN	1
R	1
R04	1
R44 SZU	1
R44 ZIEN	1

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	June 1991	Final

4. TITLE AND SUBTITLE
Pushouts and Graph Reconstruction

5. FUNDING NUMBERS

6. AUTHOR(S)
A.D. Parks

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Surface Warfare Center (K13)
Dahlgren, VA 22448-5000

8. PERFORMING ORGANIZATION REPORT NUMBER
NAVSWC TR 91-325

9. SPONSORING/MONITORING AGENCY NAME(S) AND

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

Reconstructibility of a graph G is shown to depend on the existence of morphisms that make each hypomorph of G a pushout of some card diagram $G_v \leftarrow N \rightarrow K$, N a null graph and K a star graph. These morphisms $i : N \rightarrow G_v$ and $j : N \rightarrow K$ are shown to exist for each hypomorph of G when there are isomorphisms ψ, ϕ such that for some card G_v, $\phi \circ i = i' \circ \psi$, i' a hypomorphically induced monomorphism. This result relates reconstructibility to certain symmetry properties of cards and subsumes the *good neighbor* theorems.

14. SUBJECT TERMS
Graph Reconstruction, Category Theory, Pushouts

15. NUMBER OF PAGES
16

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

GENERAL INSTRUCTIONS FOR COMPLETING SF 298

The Report Documentation Page (RDP) is used in announcing and cataloging reports. It is important that this information be consistent with the rest of the report, particularly the cover and its title page. Instructions for filling in each block of the form follow. It is important to ***stay within the lines*** to meet ***optical scanning requirements.***

Block 1. Agency Use Only *(Leave blank)*.

Block 2. Report Date. Full publication date including day, month, and year, if available (e.g. 1 Jan 88). Must cite at least the year.

Block 3. Type of Report and Dates Covered. State whether report is interim, final, etc. If applicable, enter inclusive report dates (e.g. 10 Jun 87 - 30 Jun 88).

Block 4. Title and Subtitle. A title is taken from the part of the report that provides the most meaningful and complete information. When a report is prepared in more than one volume, repeat the primary title, add volume number, and include subtitle for the specific volume. On classified documents enter the title classification in parentheses.

Block 5. Funding Numbers. To include contract and grant numbers; may include program element number(s), project number(s), task number(s), and work unit number(s). Use the following labels:

C	- Contract	**PR**	- Project
G	- Grant	**TA**	- Task
PE	- Program Element	**WU**	- Work Unit Accession No.

BLOCK 6. Author(s). Name(s) of person(s) responsible for writing the report, performing the research, or credited with the content of the report. If editor or compiler, this should follow the name(s).

Block 7. Performing Organization Name(s) and address(es). Self-explanatory.

Block 8. Performing Organization Report Number. Enter the unique alphanumeric report number(s) assigned by the organization performing the report.

Block 9. Sponsoring/Monitoring Agency Name(s) and Address(es). Self-explanatory.

Block 10. Sponsoring/Monitoring Agency Report Number. *(If Known)*

Block 11. Supplementary Notes. Enter information not included elsewhere such as: Prepared in cooperation with...; Trans. of...; To be published in.... When a report is revised, include a statement whether the new report supersedes or supplements the older report.

Block 12a. Distribution/Availability Statement. Denotes public availability or limitations. Cite any availability to the public. Enter additional limitations or special markings in all capitals (e.g. NOFORN, REL, ITAR).

DOD	- See DoDD 5230.24, "Distribution Statements on Technical Documents."
DOE	- See authorities.
NASA	- See Handbook NHB 2200.2
NTIS	- Leave blank

Block 12b. Distribution Code.

DOD	- Leave blank.
DOE	- Enter DOE distribution categories from the Standard Distribution for Unclassified Scientific and Technical Reports.
NASA	- Leave blank.
NTIS	- Leave blank.

Block 13. Abstract. Include a brief ***(Maximum 200 words)*** factual summary of the most significant information contained in the report.

Block 14. Subject Terms. Keywords or phrases identifying major subjects in the report.

Block 15. Number of Pages. Enter the total number of pages.

Block 16. Price Code. Enter appropriate price code ***(NTIS only)***

Block 17.-19. Security Classifications. Self-explanatory. Enter U.S. Security Classification in accordance with U.S. Security Regulations (i.e., UNCLASSIFIED). If form contains classified information, stamp classification on the top and bottom of this page.

Block 20. Limitation of Abstract. This block must be completed to assign a limitation to the abstract. Enter either UL (unlimited or SAR (same as report). An entry in this block is necessary if the abstract is to be limited. If blank, the abstract is assumed to be unlimited.