

AD-A241 421

115 PAGE (When Data Entered)

## DOCUMENTATION PAGE

READ INSTRUCTIONS  
BEFORE COMPLETING FORM

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

## 4. TITLE (and Subtitle)

On the Lagrangian Description of Vorticity

## 5. TYPE OF REPORT &amp; PERIOD COVERED

## 6. PERFORMING C.O. REPORT NUMBER

## 7. AUTHOR(s)

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## 8. CONTRACT OR GRANT NUMBER(s)

N00014-86-K-0057

## 9. PERFORMING ORGANIZATION NAME AND ADDRESS

Department of Mechanical Engineering  
University of California  
Berkeley, CA 9472010. PROGRAM ELEMENT, PROJECT, TASK  
AREA & WORK UNIT NUMBERS

R&amp;T 4322-534

## 11. CONTROLLING OFFICE NAME AND ADDRESS

Mechanics Division, Fluid Mechanics Program  
Office of Naval Research  
800 N. Quincy St., Arlington, VA 22217

## 12. REPORT DATE

8/91

## 13. NUMBER OF PAGES

14

## 14. MONITORING AGENCY NAME &amp; ADDRESS (if different from Controlling Office)

## 15. SECURITY CLASS. (of this report)

15a. DECLASSIFICATION/DOWNGRADING  
SCHEDULE

## 16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

## 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20; if different from Report)

## 18. SUPPLEMENTARY NOTES

## 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Vorticity, Lagrangian description, Beltrami vorticity vector, relation  
between acceleration and vorticity, circulation (Lagrangian and Eulerian  
forms), vorticity theorems.

## 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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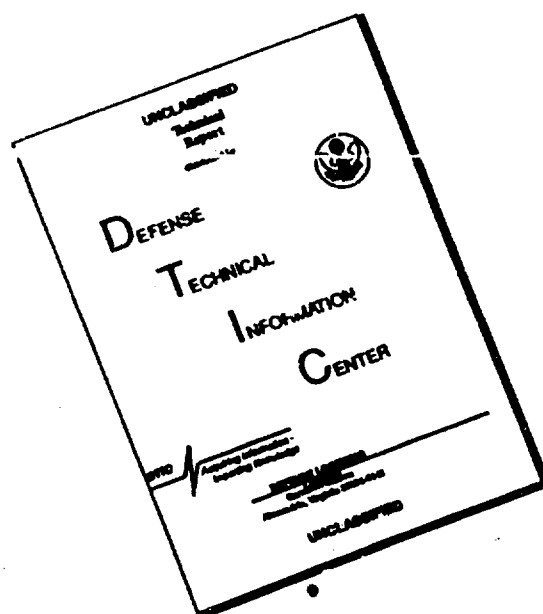
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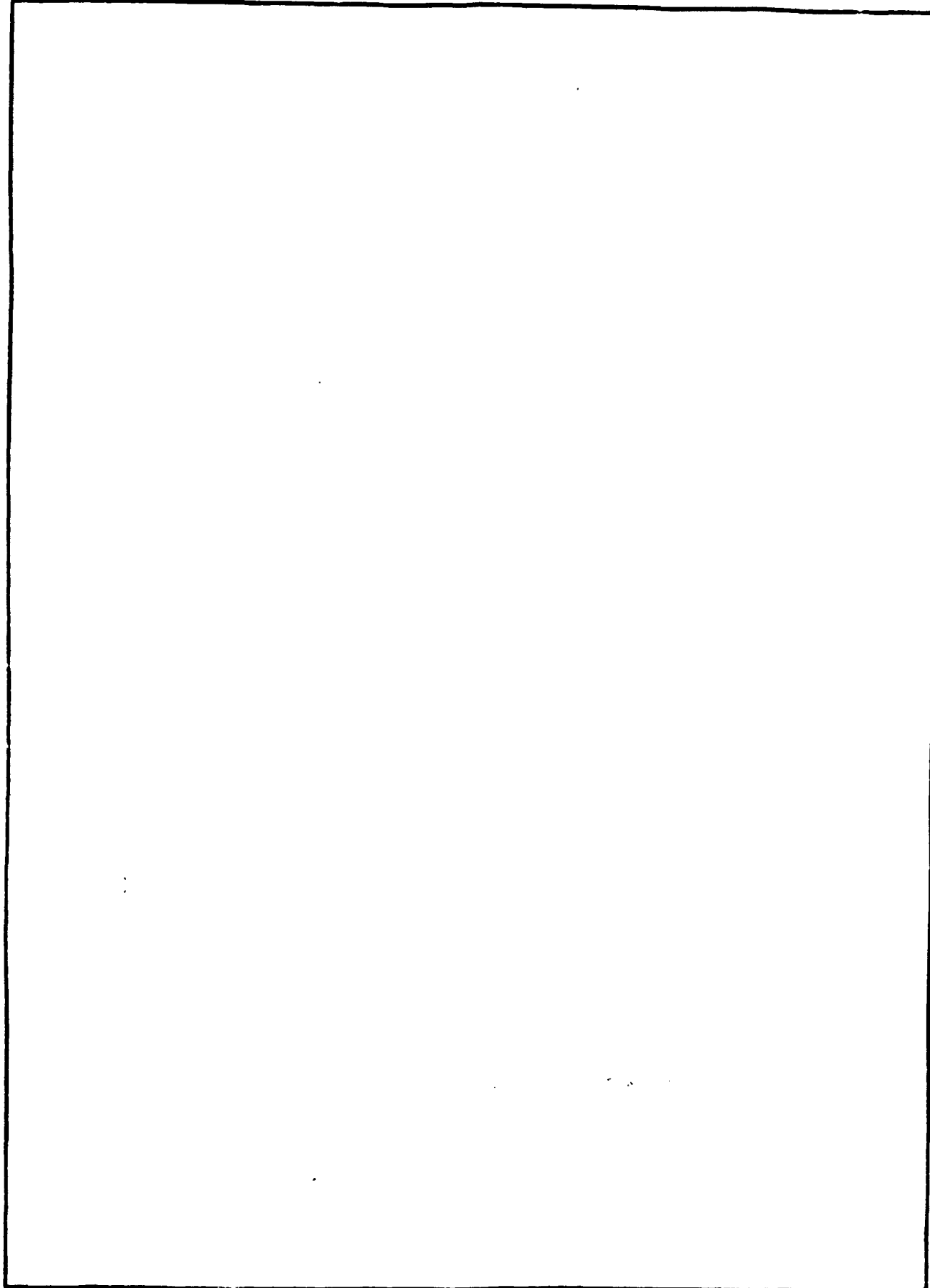
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## *On the Lagrangian Description of Vorticity*

J. CASEY & P. M. NAGHDI

*Communicated by D. OWEN*

### **Abstract**

This paper exploits a Lagrangian form of the vorticity field that is based on a formula of Beltrami, the merits of which do not seem to have been fully appreciated. A number of new results are readily derived by using the Lagrangian description, and the classical vorticity theorems are also included. The results, being purely kinematical, apply to all deformable media (including viscous fluids) and may also be of value in computations.

### **1. Introduction**

Although it has been rarely discussed, the Lagrangian description of vorticity does offer some advantages over the conventional (Eulerian) description. Thus, for example, TRUESDELL<sup>1</sup> has shown that the division of vorticity transport into convective and diffusive mechanisms becomes especially transparent when a Lagrangian viewpoint is adopted. Furthermore, as will become apparent below, the treatment of conditions for the materiality of vortex-lines in a deforming continuum is particularly straightforward when phrased in Lagrangian terms. Because of these and other reasons of a more general nature, it seems to us that a Lagrangian description of vorticity should be more fully investigated than it has been.

Since the history of the subject is quite tangled, it is desirable to provide some background information. In the context of classical hydrodynamics, CAUCHY showed that under certain circumstances the current vorticity vector  $\omega$  at a fluid particle could be explicitly written in terms of its initial value<sup>2</sup>  $\omega_0$ . One consequence

<sup>1</sup> See sect. 84 of TRUESDELL's monograph [1] and also an earlier paper by him [2].

<sup>2</sup> See our eqn. (3.25), in which  $F$  signifies the deformation gradient tensor and  $J$  is its determinant. CAUCHY's formula, presented in 1815, appears as eqns. (16) on p. 40 of [3]. As TRUESDELL has pointed out in his discussion [1, p. 173ff.], the significance of Cauchy's formula was not appreciated in the decades following its publication. A notable exception is STOKES, who became increasingly aware of its importance (see [4, p. 108] and [5]). A discussion of Cauchy's formula is given in LAMB [6, p. 205] and it is treated also in SERRIN [7, p. 152], TRUESDELL & TOUPIN [8, p. 421], and BATCHELOR [9, p. 276].

of Cauchy's formula is that vortex-lines in a circulation-preserving flow are carried as material lines. In contrast, in a more general motion, the vortex-lines no longer move as material lines, and Cauchy's formula no longer holds. Nevertheless, it is still possible to have an expression of the same form as Cauchy's in order to associate with the current vorticity field a *time-dependent* field  $\omega^\circ$  defined on the reference configuration of the continuum. To our knowledge, BELTRAMI [10] was the first to give such a formula, and it appears to have gone unnoticed ever since.<sup>3</sup> Beltrami's development is rather convoluted, but he definitely has a Lagrangian vorticity vector and he also deduces a few of its properties. In the Lagrangian representation of vorticity which TRUESDELL presents in [1], Beltrami's vorticity vector is not explicitly introduced, but a large number of results are derived in Lagrangian form.<sup>4</sup> In the present paper, we employ Beltrami's formula as the cornerstone for the construction of the vorticity field in Lagrangian form. The discussion is purely kinematical and therefore applies to all deformable media including, for example, viscoelastic and elastic-plastic materials, as well as viscous fluids.

The contents of the paper are as follows. In Section 2, we briefly review the Eulerian description of the vorticity field. It should be noted that in the Eulerian description the curl of the acceleration is a relatively complicated function of the vorticity in that it also involves the rate of deformation tensor (see (2.12) and (2.9)<sub>1,2</sub>). The Lagrangian representation, which is introduced in Section 3, furnishes a relationship between acceleration and the vorticity that could not be simpler: the curl of the acceleration is the material derivative of the (time-dependent) Beltrami vorticity vector (see (3.19)<sub>1</sub> and (3.17)<sub>1</sub>). Many theoretical results follow directly from this, and it may also be of computational value. In Section 4, Eulerian and Lagrangian representations for circulation are discussed, and the time-evolution of vortex-tubes is analyzed. In Section 5, we present necessary and sufficient conditions for vortex-lines to be transported as material lines. Also included in Sections 4 and 5 are the connections between the present results and classical ones, especially the vorticity theorems of HELMHOLTZ [11] and KELVIN [12].<sup>5</sup>

## 2. Preliminaries. Eulerian description

Consider a deformable continuum  $\mathcal{B}$  moving in three-dimensional space and let  $X$  be a typical particle of  $\mathcal{B}$ . Also, let  $X$  and  $x$ , respectively, denote the position occupied by  $X$  in a fixed reference configuration  $\kappa^\circ$  of  $\mathcal{B}$  and in the present configuration  $\kappa$  at time  $t$ . The motion of  $\mathcal{B}$  is described by the mapping  $x = \chi(X, t)$ . The particle velocity  $v$ , deformation gradient  $F$ , determinant of  $F$ , and right

<sup>3</sup> The formula is represented by eqns. (15) of [10]. These correspond to eqn. (3.2)<sub>1</sub> below, when written in its equivalent form  $J\omega = F\omega^\circ$ .

<sup>4</sup> Although TRUESDELL cites [10] in sect. 84 of [1], it seems that he overlooked the value of Beltrami's eqns. (15).

<sup>5</sup> Various statements and proofs of these theorems can be found in [1, 6, 7, 8, 9], and the reader may also wish to consult [13, Sect. II.13] and [14, Sect. 11]. A discussion of the continuing usefulness of the classical results can be found in [15].

Cauchy-Green measure of deformation are

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t}, \quad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad J = \det \mathbf{F} > 0, \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}, \quad (2.1)$$

while the velocity gradient, rate of deformation tensor, and vorticity tensor are given by

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \quad \mathbf{D} = \mathbf{L}_{\text{sym}} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \quad \mathbf{W} = \mathbf{L}_{\text{skw}} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T). \quad (2.2)$$

We recall that

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \quad \dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1}\mathbf{L}, \quad \dot{J} = J \operatorname{tr} \mathbf{D} = J \operatorname{div} \mathbf{v}. \quad (2.3)$$

The vorticity vector  $\boldsymbol{\omega}$  and tensor  $\mathbf{W}$  satisfy the relations

$$\boldsymbol{\omega} = -\varepsilon[\mathbf{W}] = \operatorname{curl} \mathbf{v}, \quad \mathbf{W} = -\frac{1}{2}\varepsilon\boldsymbol{\omega}, \quad 2\mathbf{W}\mathbf{c} = \boldsymbol{\omega} \times \mathbf{c}, \quad (2.4a)$$

where  $\varepsilon$  is the permutation tensor and  $\mathbf{c}$  is an arbitrary vector. Relative to a fixed orthonormal basis  $\{\mathbf{e}_i; i = 1, 2, 3\}$ , (2.4a)<sub>1,2,3</sub> have the representations

$$\omega_i = -\varepsilon_{ijk} W_{jk} = \varepsilon_{ijk} v_{k,j}, \quad W_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k, \quad (2.4b)$$

where the usual convention of summation over repeated indices has been employed,

and where  $(\cdot)_{,j} = \frac{\partial(\cdot)}{\partial x_j}$ . We observe that

$$\mathbf{W}\boldsymbol{\omega} = \mathbf{0}, \quad \mathbf{L}\boldsymbol{\omega} = \mathbf{D}\boldsymbol{\omega} = \mathbf{L}^T\boldsymbol{\omega}, \quad \operatorname{div} \boldsymbol{\omega} = 0. \quad (2.5)$$

A motion is *irrotational* at time  $t$  if and only if  $\boldsymbol{\omega}$ , or equivalently  $\mathbf{W}$ , vanishes at time  $t$  for each particle  $X$  of  $\mathcal{B}$ .

The acceleration and its spatial gradient are

$$\mathbf{a} = \dot{\mathbf{v}}, \quad \mathbf{A} = \frac{\partial \mathbf{a}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad (2.6)$$

and it is clear that

$$\begin{aligned} \dot{\mathbf{L}} &= \mathbf{A} - \mathbf{L}^2 = \mathbf{A} - \mathbf{D}^2 - \mathbf{W}^2 - \mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}, \\ \dot{\mathbf{D}} &= \mathbf{A}_{\text{sym}} - \mathbf{D}^2 - \mathbf{W}^2, \quad \dot{\mathbf{W}} = \mathbf{A}_{\text{skw}} - \mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}, \end{aligned} \quad (2.7)$$

where the symmetric and skew parts of  $\mathbf{A}$  are defined by formulae of the type (2.2)<sub>3,5</sub>.

Recalling the formula<sup>6</sup> for the convected rate  $\hat{\mathbf{W}}$  of  $\mathbf{W}$  and noting that  $\mathbf{W}^T \mathbf{W} = -\mathbf{W}^2$ , we have

$$\hat{\mathbf{W}} = \dot{\mathbf{W}} + \mathbf{L}^T \mathbf{W} + \mathbf{W} \mathbf{L} = \mathbf{A}_{\text{skw}}. \quad (2.8)$$

<sup>6</sup> See TRUESDELL & NOLL [16, p. 97]. Note that a tensor does not have to be *objective* for a formula of the type (2.8) to apply.

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Just as with the skew tensor  $W$ , we may uniquely associate a vector  $x$  with the skew part of the acceleration gradient. Thus,

$$x = -e[A_{ikw}] = \text{curl } a, \quad A_{ikw} = -\frac{1}{2} \epsilon_{ik} x. \quad (2.9)$$

Expressing  $v$  as a function of the spatial variable  $x$  and  $t$ , one readily obtains

$$a = \frac{\partial v}{\partial t} + \omega \times v + \frac{1}{2} \text{grad } (v \cdot v), \quad (2.10)$$

and hence

$$x = \frac{\partial \omega}{\partial t} + \text{curl } (\omega \times v) = \frac{\partial \omega}{\partial t} + \text{div } (\omega \otimes v - v \otimes \omega), \quad (2.11)$$

where  $\otimes$  denotes the tensor product of vectors<sup>7</sup> and  $\text{div } (\omega \otimes v) = (\omega_i v_j)_{,j} e_i$ . Consequently, in view of (2.5)<sub>2,4</sub>,

$$x = \dot{\omega} + \omega \text{ div } v - D\omega. \quad (2.12)$$

If, for an arbitrary vector  $u$ , one defines the *Truesdell rate*<sup>8</sup> of  $u$  by

$$\dot{u} = \dot{u} + \{(\text{tr } D) I - L\} u, \quad (2.13)$$

it then becomes clear that

$$x = \dot{\omega}. \quad (2.14)$$

### 3. Lagrangian description

Recall that the *adjugate*  $F^*$  of  $F$  satisfies the relations

$$F^* = J(F^{-1})^T, \quad (Fa) \times (Fb) = F^*(a \times b), \quad \det F^* = J^2 > 0, \quad (3.1)$$

for any vectors  $a, b$ . We define time-dependent vectors  $\omega^\circ$  and  $x^\circ$  by<sup>9</sup>

$$\omega^\circ = JF^{-1}\omega = \pi^{-1}\{\omega\}, \quad (3.2)$$

$$x^\circ = \pi^{-1}\{x\}. \quad (3.3)$$

As will quickly become evident from its properties, the vector  $\omega^\circ$  provides a Lagrangian description of vorticity. In component form, the relationships between

<sup>7</sup> Recall that  $(a \otimes b)c = ab \cdot c$  for any vectors  $a, b, c$ .

<sup>8</sup> This terminology is introduced to reflect certain properties which the rate  $\dot{u}$  has in common with the Truesdell rate of an objective tensor (see CASEY & NAGHDI [17, p. 357]).

<sup>9</sup> The operation  $\pi^{-1}$  is known as a *Piola transformation*. Operations of this type commonly appear in transformation formulae connecting Lagrangian and Eulerian descriptions of fields (e.g., the stress field). The vorticity vector  $\omega^\circ$  (actually,  $\frac{1}{2}\omega^\circ$ ) was introduced by BELTRAMI [10], who used a procedure somewhat different from that given here.

$\omega^\circ$  and  $\omega$  are

$$\omega_A^\circ = JX_{A,i}\omega_i, \quad \omega_i = \frac{1}{J}x_{i,A}\omega_A^\circ, \quad (3.4)$$

where  $\omega_A^\circ$  are the components of  $\omega^\circ$  relative to a fixed orthonormal basis  $E_A$  (which may or may not coincide with  $e_i$ ). Similar equations hold between the components  $\alpha_i$  and  $\alpha_A^\circ$ .

Observing the identity,

$$(JX_{A,i})_{,A} = 0, \quad \text{or} \quad \text{Div } F^* = 0, \quad (3.5)$$

we deduce from (3.4)<sub>1</sub> that

$$\text{Div } \omega^\circ = \omega_{A,A}^\circ = JX_{A,i}\omega_{i,A} = J \text{div } \omega. \quad (3.6)$$

Hence, in view of (2.5)<sub>4</sub>,

$$\text{Div } \omega^\circ = 0. \quad (3.7)$$

Letting

$$W^\circ = -\frac{1}{2}\varepsilon\omega^\circ, \quad (3.8)$$

we note that formulae of the type (2.4a)<sub>1,4</sub> and (2.5)<sub>1</sub> also hold for  $W^\circ$  and  $\omega^\circ$ . For any vector  $c$ , we have

$$\omega \times c = \left(\frac{1}{J}F\omega^\circ\right) \times F(F^{-1}c) = \frac{1}{J}F^*(\omega^\circ \times (F^{-1}c)), \quad (3.9)$$

where use has been made of (3.2)<sub>1</sub> and (3.1)<sub>2</sub>. Then, applying (2.4a)<sub>4</sub> and its analogue for the Lagrangian quantities, and invoking (3.1)<sub>1</sub>, we find that

$$W^\circ = F^T W F. \quad (3.10)$$

We also observe that irrotational motions are characterized by the vanishing of  $\omega^\circ$  (or equivalently,  $W^\circ$ ).

In anticipation of later results, we transform the velocity and acceleration fields by means of the formulae<sup>10</sup>

$$v^\circ = F^T v, \quad a^\circ = F^T a, \quad (3.11)$$

and also let

$$L^\circ = \frac{\partial v^\circ}{\partial X}, \quad A^\circ = \frac{\partial a^\circ}{\partial X}. \quad (3.12)$$

It can then be easily verified that

$$\begin{aligned} L^\circ &= F^T L F + x_{i,AB}v_i E_A \otimes E_B, \\ A^\circ &= F^T A F + x_{i,AB}a_i E_A \otimes E_B, \\ \dot{v}_A^\circ &= v_{i,A}v_i + a_A^\circ, \\ \dot{L}_{AB}^\circ &= (\dot{v}_A^\circ)_{,B} = v_{i,AB}v_i + v_{i,A}v_{i,B} + A_{AB}^\circ. \end{aligned} \quad (3.13)$$

<sup>10</sup> It is worth pointing out that the transformation (3.11), could have been introduced prior to (3.2)<sub>1</sub>. The connection between these two fields will appear in (3.15)<sub>4</sub>.



Breaking  $L^\circ$  and  $A^\circ$  in (3.13)<sub>1,2</sub> into their symmetric and skew-symmetric parts, we obtain

$$\begin{aligned} L_{sym}^\circ &= F^T D F + x_{i,AB} v_i E_A \otimes E_B, & L_{skw}^\circ &= F^T W F = W^\circ, \\ A_{sym}^\circ &= F^T A_{sym} F + x_{i,AB} a_i E_A \otimes E_B, & A_{skw}^\circ &= F^T A_{skw} F, \end{aligned} \quad (3.14)$$

where use has been made of (2.2)<sub>2,4</sub> and equations of the type (2.2)<sub>3,5</sub> and of (3.10).

It follows from (3.11)<sub>1</sub>, (2.2)<sub>1,2,4</sub>, (3.14)<sub>3</sub>, and a formula of the type (2.4b)<sub>1</sub> that

$$\begin{aligned} \text{Curl } v^\circ &= \varepsilon_{IJK} v_{K,J}^\circ E_I = \varepsilon_{IJK} x_{i,K} x_{j,J} W_{ij} E_I \\ &= \varepsilon_{IJK} W_{KJ}^\circ E_I \\ &= \omega^\circ. \end{aligned} \quad (3.15)$$

Hence, in view of (2.4a)<sub>2</sub> and (3.2)<sub>2</sub>,

$$\text{Curl } v^\circ = \pi^{-1} \{\text{curl } v\}. \quad (3.16)$$

Likewise,

$$\text{Curl } a^\circ = x^\circ = \pi^{-1} \{\text{curl } a\}. \quad (3.17)$$

Next, we deduce from (3.13)<sub>3</sub> that

$$\begin{aligned} \varepsilon_{IJK} (\dot{v}_K^\circ)_{,J} &= \varepsilon_{IJK} v_{i,KJ} v_i + \varepsilon_{IJK} v_{i,J} v_{i,K} + \varepsilon_{IJK} a_{K,J}^\circ \\ &= \varepsilon_{IJK} a_{K,J}^\circ, \end{aligned} \quad (3.18a)$$

or

$$\text{Curl } \dot{v}^\circ = \text{Curl } a^\circ. \quad (3.18b)$$

It follows from (3.15)<sub>4</sub>, (3.18b), (3.17)<sub>1</sub>, (3.3), and (2.14) that

$$\dot{\omega}^\circ = x^\circ = \pi^{-1} \{\dot{\omega}\}, \quad \dot{W}^\circ = A_{skw}^\circ, \quad (3.19)$$

where a formula of the type (2.9)<sub>3</sub> has been used in the last step.

Suppose that at some time  $t = t_0$ , the body occupies its reference configuration. Let the corresponding value of  $\omega^\circ$  (and, hence, also that of  $\omega$ ) be  $\omega_0^\circ$ . It then follows from (3.19)<sub>1</sub> and (3.2)<sub>1</sub> that the vorticity vector  $\omega$  at time  $t$  can be written as<sup>11</sup>

$$\omega = \frac{1}{J} F \left\{ \omega_0^\circ + \int_{t_0}^t x^\circ dt \right\}. \quad (3.20)$$

*The d'Alembert-Euler condition.* The condition that

$$\alpha = 0 \quad (3.21)$$

<sup>11</sup> TRUESDELL [1, Sect. 85] obtained (3.20) by another method and called it the "basic vorticity formula". He interpreted it as representing the convection and diffusion of vorticity, through the first and second terms, respectively, on its right-hand side. In this connection, see also [2].

for each particle and for all  $t$ , is called the *d'Alembert-Euler condition*.<sup>12</sup> In view of (3.3), (2.14), (2.9)<sub>3,1</sub>, and (3.14)<sub>4</sub>, each of the following conditions is equivalent to (3.21):

$$\mathbf{x}^\circ = \mathbf{0}, \quad \dot{\boldsymbol{\omega}} = \mathbf{0}, \quad A_{ikw} = 0, \quad A_{ikw}^\circ = 0. \quad (3.22)$$

It is then obvious from (3.19)<sub>1</sub> that the d'Alembert-Euler condition is met if and only if the time-dependent vector  $\boldsymbol{\omega}^\circ$  retains the value  $\boldsymbol{\omega}_0^\circ$  which it had at time  $t_0$ , i.e.,

$$\boldsymbol{\omega}^\circ = \boldsymbol{\omega}_0^\circ, \quad (3.23)$$

and equivalently if and only if

$$\mathbf{W}^\circ = \mathbf{W}_0^\circ, \quad (3.24)$$

i.e., the value of  $\mathbf{W}^\circ$  at  $t = t_0$ . Furthermore, it follows from (3.2)<sub>1</sub> and (3.23) that a necessary and sufficient condition for (3.21) to hold is that

$$J\boldsymbol{\omega} = F\boldsymbol{\omega}_0^\circ. \quad (3.25)$$

Equation (3.25) is *Cauchy's vorticity formula*. It is essential to observe the following difference between (3.25) and (3.2)<sub>1</sub>: in the former  $\boldsymbol{\omega}_0^\circ$  is a time-independent vector, whereas in the latter  $\boldsymbol{\omega}^\circ$  is, in general, time-dependent.

An important well-known result (*The Lagrange-Cauchy theorem*) can be immediately deduced from (3.25): *Suppose that the d'Alembert-Euler condition is satisfied. If a motion is irrotational at any one time, then it is irrotational for all time.* For, if  $\boldsymbol{\omega}(t^*) = \mathbf{0}$  for some given instant  $t^*$ , then, by virtue of (3.25),  $\boldsymbol{\omega}_0^\circ = \mathbf{0}$ . Hence, again invoking (3.25), we find that  $\boldsymbol{\omega}(t) = \mathbf{0}$  for all  $t$ .

#### 4. Circulation: Eulerian and Lagrangian Forms

Consider a simple closed curve  $\mathcal{C}$ , which we shall refer to as a *circuit*, in the present configuration  $\kappa$ , and let  $\mathcal{C}^\circ$  denote its inverse image (also a circuit) in the reference configuration  $\kappa^\circ$ . The circulation around  $\mathcal{C}$  has the representations

$$\Gamma = \tilde{\Gamma}(\mathcal{C}, t) = \int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{x} = \int_{\mathcal{C}} \mathbf{v}^\circ \cdot d\mathbf{X} = \hat{\Gamma}(\mathcal{C}^\circ, t), \quad (4.1)$$

where (3.11)<sub>1</sub> has been used. Applying Stokes's theorem to (4.1)<sub>2,3</sub>, and recalling (2.4a)<sub>2</sub> and (3.15)<sub>4</sub>, we obtain

$$\Gamma = \int_{\mathcal{S}} \boldsymbol{\omega} \cdot \mathbf{n} \, da = \int_{\mathcal{S}^\circ} \boldsymbol{\omega}^\circ \cdot \mathbf{N} \, dA, \quad (4.2)$$

where  $\mathcal{S}$  is any surface bounded by  $\mathcal{C}$ ,  $\mathbf{n}$  is a unit normal to  $\mathcal{S}$  (chosen to be compatible with the orientation of  $\mathcal{C}$ ),  $\mathcal{S}^\circ$  is the inverse image of  $\mathcal{S}$  in  $\kappa^\circ$ , and  $\mathbf{N}$  is a unit normal to  $\mathcal{S}^\circ$ . It is clear from (4.2)<sub>1</sub> that a motion is irrotational at time  $t$  if and only if  $\Gamma = 0$  for every reducible<sup>13</sup> circuit  $\mathcal{C}$  that can be drawn in the present

<sup>12</sup> See TRUESDELL [1, p. 87].

<sup>13</sup> i.e., one that can be continuously shrunk to a point of the present configuration while remaining throughout in this configuration.

configuration of the continuum (*Kelvin's Kinematical Theorem*). A similar statement involving  $\mathcal{C}^\circ$  follows from (4.2)<sub>2</sub>.

If we now fix attention on the particles that compose  $\mathcal{C}^\circ$ , then clearly a material curve is described as time progresses and its image in  $\kappa$  is  $\mathcal{C}$ . The rate of change of  $\Gamma$  for this fixed set of particles is given by

$$\dot{\Gamma} = \frac{\partial \hat{\Gamma}}{\partial t}(\mathcal{C}^\circ, t) = \int_{\mathcal{C}^\circ} \mathbf{a} \cdot d\mathbf{x} = \int_{\mathcal{C}^\circ} \mathbf{a}^\circ \cdot d\mathbf{X}, \quad (4.3)$$

where (3.1)<sub>2</sub> has been used in deriving the last step.

Again, applying Stokes's theorem to the integrals in (4.3)<sub>3,4</sub> and recalling (2.9)<sub>2</sub>, (2.14), (3.17)<sub>1</sub>, and (3.19)<sub>1</sub>, we find that

$$\dot{\Gamma} = \int_{\mathcal{C}^\circ} \mathbf{x} \cdot \mathbf{n} \, da = \int_{\mathcal{C}^\circ} \dot{\boldsymbol{\omega}} \cdot \mathbf{n} \, da = \int_{\mathcal{C}^\circ} \mathbf{x}^\circ \cdot \mathbf{N} \, dA = \int_{\mathcal{C}^\circ} \dot{\boldsymbol{\omega}}^\circ \cdot \mathbf{N} \, dA. \quad (4.4)$$

A motion is *circulation-preserving* if and only if  $\dot{\Gamma} = 0$ , for all  $t$ , and for every reducible circuit  $\mathcal{C}^\circ$  that can be drawn in the reference configuration of the continuum. It is obvious from (4.4)<sub>1</sub> and (3.21) that a motion is circulation-preserving if and only if the d'Alembert-Euler condition is satisfied [1, Sect. 46].

For  $\boldsymbol{\omega} \neq 0$ , a *vortex-line in the present configuration* is a line which is everywhere tangent to the vector  $\boldsymbol{\omega}$ ; it may be called an  $\omega$ -line. It passes through a definite set of particles<sup>14</sup> in the configuration  $\kappa$  at time  $t$ . Correspondingly, since  $d\mathbf{X} = \mathbf{F}^{-1} d\mathbf{x}$ , it is clear from (3.2)<sub>1</sub> that the line joining this same set of particles in  $\kappa^\circ$  is tangent to the vorticity vector  $\boldsymbol{\omega}^\circ$ , so that it is an  $\omega^\circ$ -line.

In the present configuration, draw any reducible circuit  $\mathcal{C}$  whose tangent at every point is not parallel to the vorticity vector at that point. The  $\omega$ -lines passing through  $\mathcal{C}$  then form a vortex-tube<sup>15</sup> in  $\kappa$ ; let us call it an  $\omega$ -tube. Transporting this tube back into  $\kappa^\circ$ , we obtain an  $\omega^\circ$ -tube there (Figure 1).

Let  $\omega (> 0)$  and  $\omega^\circ (> 0)$  be the magnitudes of  $\boldsymbol{\omega}$  and  $\boldsymbol{\omega}^\circ$ , respectively, and write

$$\boldsymbol{\omega} = \omega \mathbf{m}, \quad \boldsymbol{\omega}^\circ = \omega^\circ \mathbf{m}^\circ. \quad (4.5)$$

Recall the formulae

$$\lambda \mathbf{m} = \mathbf{F} \mathbf{m}^\circ, \quad \lambda^2 = \mathbf{m}^\circ \cdot \mathbf{C} \mathbf{m}^\circ, \quad (4.6)$$

where  $\lambda > 0$  is the stretch of a line-element lying along the unit vector  $\mathbf{m}^\circ$  in  $\kappa^\circ$  and where use has been made of (2.1)<sub>5</sub>.<sup>16</sup> It follows from (3.2)<sub>1</sub>, (4.5)<sub>1,2</sub>, and (4.6):

<sup>14</sup> Of course, a moment later, the vortex-line and the line through the particles will in general no longer be identical *i.e.*, the vortex-lines are not, in general, *material* lines. Nevertheless, the above construction is useful. Conditions under which the vortex-lines are material will be established in Sect. 5.

<sup>15</sup> Several slightly different definitions of a vortex-tube have been given in the literature. The circuit is sometimes taken to be infinitely small (as in HELMHOLTZ's [11] original definition and also in LAMB [6]). Reducibility is often only implicit, and occasionally not made part of the definition at all (see MEYER [18]). Here, we have adopted the definition given in BATCHELOR [9].

<sup>16</sup> The scalar function  $\lambda$  is a function of the variables  $(\mathbf{X}, t, \mathbf{m}^\circ)$  in its Lagrangian representation and a function of  $(\mathbf{x}, t, \mathbf{m})$  in its Eulerian representation.

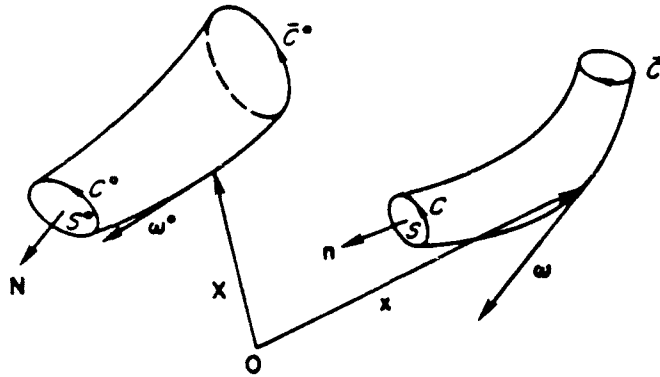


Fig. 1. Lagrangian and Eulerian vortex-tubes

that

$$\omega = \frac{\lambda}{J} \omega^0. \quad (4.7)$$

As indicated in Figure 1, let  $\mathcal{S}^0, \mathcal{S}$  be tube sections bounded by circuits  $\mathcal{C}^0, \mathcal{C}$ , respectively, and let  $N, n$  be unit normals vector fields to these surfaces<sup>17</sup>. Then, remembering that

$$\gamma n = F^* N, \quad (4.8)$$

where  $\gamma (= da/dA)$  is the area-stretch, and again using (3.2)<sub>1</sub>, we have

$$\omega \cdot n = \frac{1}{\gamma} \omega^0 \cdot N. \quad (4.9)$$

Let a volume element  $dV$  in  $\kappa^0$  be constructed with its base  $dA$  on the section  $\mathcal{S}^0$ , and with its sides of length  $dL$  composed of  $\omega^0$ -lines. With these notations,  $dV = dL dA m^0 \cdot N$ , and its image in  $\kappa$  is the volume element  $dv = dl da m \cdot n$ . Since  $dv = J dV$ , it is then clear that

$$m \cdot n = \frac{J}{\lambda \gamma} m^0 \cdot N. \quad (4.10)$$

If  $\mathcal{S}^0$  is a normal cross-section ( $N = m^0$ ) of the  $\omega^0$ -tube element, then  $m \cdot n = J/\lambda \gamma$ .

Using (2.5)<sub>4</sub> and applying the divergence theorem to a region in the present configuration bounded by an  $\omega$ -tube and two cross-sections  $\mathcal{S}$  and  $\bar{\mathcal{S}}$ , we obtain

**Helmholtz's First Vorticity Theorem.**

$$\int_{\mathcal{S}} \omega \cdot n da = \int_{\bar{\mathcal{S}}} \omega \cdot n da, \quad (4.11)$$

<sup>17</sup> The typeface in the figures differs somewhat from that in the text.

i.e., the strength of an  $\omega$ -tube is the same at all cross-sections of the tube at time  $t$  [6].

Equivalently, in view of (4.2)<sub>1</sub>, we have Kelvin's form of this theorem<sup>18</sup>:

$$\tilde{\Gamma}(\mathcal{C}, t) = \tilde{\Gamma}(\bar{\mathcal{C}}, t), \quad (4.12)$$

where the circuit  $\bar{\mathcal{C}}$  is the boundary of the section  $\bar{\mathcal{S}}$ . It is clear that the theorem also holds for  $\omega^\circ$ -tubes:

$$\int_{\mathcal{S}} \omega^\circ \cdot N dA = \int_{\bar{\mathcal{S}}} \omega^\circ \cdot N dA, \quad \hat{\Gamma}(\mathcal{C}^\circ, t) = \hat{\Gamma}(\bar{\mathcal{C}}^\circ, t). \quad (4.13)$$

In addition, it should be observed that since the vector fields  $\mathbf{x}$  and  $\mathbf{x}^\circ$  also are divergence-free, Helmholtz's First Theorem also holds when expressed with reference to tubes of these vector fields.

As was mentioned previously, vortex-tubes are not, in general, material tubes. Consider, for instance, the set of particles that lie on an  $\omega$ -tube at time  $t$  (Figure 2). Then, at some later time  $t'$ , the particles which were on the curves  $\mathcal{C}(t)$  and  $\bar{\mathcal{C}}(t)$  will now lie on curves  $\mathcal{C}(t')$  and  $\bar{\mathcal{C}}(t')$ , and the particles on the tube  $MNPQ$  will be deformed into the tube  $M'N'P'Q'$ . However, at time  $t'$ , the vortex-tube through  $\mathcal{C}(t')$  could be  $M'N'P''Q''$  rather than  $M'N'P'Q'$ , and the vortex-tube through  $\bar{\mathcal{C}}(t')$  could be  $P'Q'M''N''$ .

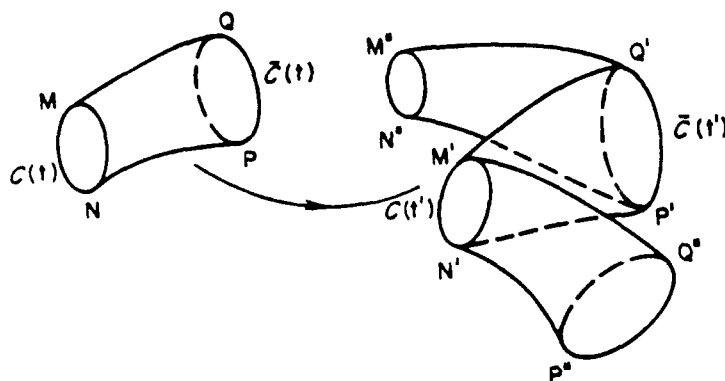


Fig. 2. The particles which lie on a vortex-tube  $MNPQ$  at time  $t$  do not in general lie on a vortex tube at a subsequent time  $t'$ .

We now proceed to examine the rate at which vortex-lines change for a given particle. To this end, we first take the material derivative of (4.6)<sub>1</sub> and employ (2.3)<sub>1</sub> and (2.5)<sub>1</sub> to obtain

$$\dot{\lambda} \mathbf{m} + \lambda \dot{\mathbf{m}} = \lambda D \mathbf{m} + F \mathbf{m}^\circ. \quad (4.14)$$

<sup>18</sup> See KELVIN [12]. For historical comments, see TRUESDELL [1].

Since  $\mathbf{m}$  is a unit vector, it follows from (4.14) that

$$\begin{aligned}\frac{\dot{\lambda}}{\lambda} &= \mathbf{m} \cdot D\mathbf{m} + \frac{\mathbf{m}^\circ \cdot C\dot{\mathbf{m}}^\circ}{\mathbf{m}^\circ \cdot C\dot{\mathbf{m}}^\circ}, \\ \dot{\mathbf{m}} &= \left\{ D - \left( \mathbf{m} \cdot D\mathbf{m} + \frac{1}{\lambda^2} \mathbf{m}^\circ \cdot C\dot{\mathbf{m}}^\circ \right) I \right\} \mathbf{m} + \frac{1}{\lambda} F\dot{\mathbf{m}}^\circ,\end{aligned}\quad (4.15)$$

where use has been made of (2.1)<sub>3</sub> and (4.6)<sub>1,2</sub>.

By virtue of (4.7),

$$\frac{\dot{\omega}}{\omega} - \frac{\dot{\omega}^\circ}{\omega^\circ} = \frac{\dot{\lambda}}{\lambda} - \frac{\dot{J}}{J}. \quad (4.16)$$

Also, in view of (4.5)<sub>1</sub>, (2.14), and (2.13),

$$\mathbf{x} = \dot{\omega}\mathbf{m} + \omega\dot{\mathbf{m}} \quad (4.17)$$

and hence, by (3.3),

$$\mathbf{x}^\circ = \dot{\omega}\pi^{-1}\{\dot{\mathbf{m}}\} + \omega\pi^{-1}\{\dot{\mathbf{m}}\}. \quad (4.18)$$

It then follows from (4.18) that

$$\frac{\omega}{\omega^\circ} \pi^{-1}\{\dot{\mathbf{m}}\} = \dot{\mathbf{m}}^\circ - \left( \frac{\dot{\omega}}{\omega} - \frac{\dot{\omega}^\circ}{\omega^\circ} \right) \mathbf{m}^\circ, \quad (4.19)$$

where (3.19)<sub>1</sub>, (4.5)<sub>2</sub>, (4.6)<sub>1</sub>, and (4.7) have been utilized.

## 5. Material vortex-lines

A vector-field  $\mathbf{u}$  is *material* if and only if

$$\mathbf{u}(\mathbf{x}, t) = F\mathbf{u}^\circ(\mathbf{X}), \quad (5.1)$$

where  $\mathbf{u}(\mathbf{x}, t)$  is the value of  $\mathbf{u}$  at a point  $\mathbf{x}$  in the present configuration, and  $\mathbf{u}^\circ$  is the value of  $\mathbf{u}$  at  $\mathbf{X}$  in the reference configuration, the vectors  $\mathbf{x}$  and  $\mathbf{X}$  being related through the motion  $\chi$  of the continuum. We emphasize that  $\mathbf{u}^\circ$  in (5.1) is independent of time.

It is obvious from (5.1) and (2.3)<sub>1</sub> that if  $\mathbf{u}$  is material, then

$$\dot{\mathbf{u}} = L\mathbf{u}. \quad (5.2)$$

Let us now consider (2.12); it is clear that in general the vorticity field is not material. (But, in the special case in which the motion is isochoric ( $J = 1$ ) and the d'Alembert-Euler condition (3.21) is met, it can be seen from (3.25) that the vorticity field is material.) Of wider significance is the case in which only the vortex-lines are material.

**Theorem I.** Suppose  $\omega \neq 0$ . Each of the following conditions (a), (b), (c), (d) is necessary and sufficient for the  $\omega$ -lines to be material:

$$\begin{aligned} (a) \quad \dot{m}^\circ &= 0, & (c) \quad x^\circ &= \frac{\dot{\omega}^\circ}{\omega^\circ} \omega^\circ, \\ (b) \quad \dot{m} &= -\left(\frac{\dot{\omega}}{\omega} - \frac{\dot{\omega}^\circ}{\omega^\circ}\right) m, & (d) \quad x &= \dot{\omega} = \frac{\dot{\omega}^\circ}{\omega^\circ} \omega. \end{aligned} \quad (5.3)$$

**Proof.** First observe that the four conditions in (5.3) are all equivalent to one another. Thus, by (4.19), (4.6)<sub>1</sub>, and (4.7), condition (5.3a) is equivalent to condition (5.3b). Utilizing (4.5)<sub>2</sub> and (3.19)<sub>1</sub>, we see that conditions (5.3a) and (5.3c) are equivalent to each other. Finally, by using the transformations (3.2)<sub>2</sub> and (3.3) and their inverses, we can establish the equivalency of (5.3c) and (5.3d).

(i) *Necessity.* By the definition (5.1), the vector  $m^\circ$  in (4.6)<sub>1</sub> must be independent of time. Therefore, condition (5.3a) holds. Hence also do (5.3b), (5.3c), and (5.3d).

(ii) *Sufficiency.* If condition (5.3a) holds, then

$$m^\circ = m_0^\circ, \quad (5.4)$$

a time-independent unit vector. Substituting (5.4) into (4.6)<sub>1</sub>, we obtain

$$\lambda m = F m_0^\circ, \quad (5.5)$$

which implies that the  $\omega$ -lines are material. Since we have already shown that each of the conditions (5.3b), (5.3c), and (5.3d) is equivalent to (5.3a), they also must individually imply (5.5).

**Corollary I (Helmholtz's Second Vorticity Theorem).** *If the d'Alembert-Euler condition is satisfied, the vortex-lines are material lines.*

An equivalent statement is: *If the motion is circulation-preserving, the vortex-lines are material lines.* To prove the corollary, simply observe that (3.22)<sub>1</sub> is equivalent to the d'Alembert-Euler condition, and consequently, in view of (3.19)<sub>1</sub> and (4.5)<sub>2</sub>,  $\dot{m}^\circ = 0$ . Therefore, condition (5.3a) is satisfied, and the result follows from the theorem.

It is of interest to note that in the present case (i.e., when  $\alpha^\circ = 0$ ), (5.3b) yields

$$\dot{m} = -\frac{\dot{\omega}}{\omega} m. \quad (5.6)$$

**Corollary II (Helmholtz's Third Vorticity Theorem).** *If the d'Alembert-Euler condition is satisfied, a time-independent strength can be associated with each vortex-tube.*

Equivalently, we may replace the hypothesis of this corollary with the hypothesis that the motion be circulation-preserving. The proof is straightforward:

Helmholtz's Second Theorem implies that the vortex-tubes are material, and (4.4)<sub>1</sub> now tells us that  $\dot{\Gamma} = 0$ . Therefore, in view of (4.1)<sub>4</sub>,

$$\Gamma = \hat{\Gamma}(\mathcal{C}^0). \quad (5.7)$$

Hence, recalling Helmholtz's First Theorem, we see that a time-independent strength can be associated with each vortex-tube.

The upshot of the three Helmholtz Theorems is that if the d'Alembert-Euler condition is satisfied, then each vortex-tube moves as a material surface, and with it is associated a scalar measure  $\Gamma$  which varies neither along the tube at a given value of  $t$ , nor as a function of  $t$  for a given tube. Thus, once the d'Alembert-Euler condition is satisfied,  $\Gamma$  is a *constant* for each vortex-tube.

**Corollary III.** *If the  $\omega$ -lines are material and if*

$$\dot{\omega}^0 = 0, \quad (5.8)$$

*then the d'Alembert-Euler condition is satisfied.*

This follows immediately from (5.3a), (4.5)<sub>2</sub>, (3.19)<sub>1</sub>, and (3.22)<sub>1</sub>.

**Corollary IV.** *In a circulation-preserving flow, no stretching of vortex-lines occurs if and only if  $J\omega$  is time-independent.*<sup>19</sup>

In a circulation-preserving flow,  $\alpha = 0$ , and hence by (3.23),  $\omega^0 = \omega_0^0$ . Also, the vortex-lines are material by Corollary I. Setting  $\lambda = 1$  in (4.7), we then see that  $J\omega = \omega_0^0$ , which is time-independent.

**Theorem II.** *Suppose  $\omega \neq 0$ . Each of the following conditions (a), (b), (c), (d) is necessary and sufficient for the  $\omega$ -lines to be material:*

$$\begin{aligned} (a) \quad m^0 \times \dot{m}^0 &= 0, & (b) \quad \omega^0 \times \dot{\omega}^0 &= 0, \\ (c) \quad \omega \times x &= 0, & (d) \quad m \times \dot{m} &= 0. \end{aligned} \quad (5.9)$$

**Proof.** It follows from (3.2)<sub>1</sub>, (3.3), and (3.1)<sub>1,2</sub> that

$$\omega \times x = \frac{1}{J} (F^{-1})^T (\omega^0 \times x^0). \quad (5.10)$$

Therefore,  $\omega \times x = 0$  if and only if  $\omega^0 \times x^0 = 0$ . By virtue of (3.19)<sub>1</sub> and (4.5)<sub>2</sub>, the latter can occur if and only if  $m^0 \times \dot{m}^0 = 0$ , which can occur if, and (since  $m^0$  is a unit vector) only if,  $\dot{m}^0 = 0$ . Also, in view of (4.17) and (4.5)<sub>1</sub>,  $\omega \times x = 0$  if and only if  $m \times \dot{m} = 0$ . Consequently, each of the conditions in (5.9) is equivalent to (5.3a), which is a necessary and sufficient condition for the  $\omega$ -lines to be material.<sup>20</sup>

<sup>19</sup> [1, p. 180].

<sup>20</sup> The condition (5.9c) is given by TRUESDELL [1, Sect. 45], who employs a different argument.



*Acknowledgment.* The work of J. C. was supported by the National Science Foundation under Grant MSS-9019932, and that of P.M.N. was supported by the U.S. Office of Naval Research under Contract N00014-86-K-0057, Work Unit 4322-534, with the University of California at Berkeley.

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(Received September 26, 1990)