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**RESIDUAL STRESSES AND
THERMO-MECHANICAL BEHAVIOR
OF METAL-MATRIX COMPOSITES**

**Final Report
(DRAFT)**

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(U) A plastic-flow code was used to calculate microscopic stress-temperature histories in metal-matrix composites during a cooling and reheating cycle. Stress results are given for graphite/aluminum, tungsten/aluminum and silicon-carbide aluminum composites. For the graphite/aluminum material a parametric study was carried out on the effect of variations in various fiber properties on the calculated stresses. A computer code is presented for calculating the thermo-elastic properties of cross-plyed laminates from those of the individual plies on from the fiber and matrix properties appropriate to each ply. Modifications to a code (PRUFC) developed previously for computing the properties of a unidirectionally reinforced composite from those of the fiber and matrix components are described. A formally exact two-dimensional Fourier-series solution is presented for determining the stresses and deformation of a plate subjected to an arbitrary, transverse temperature distribution. Numerical results are given for several temperature profiles, including a discontinuous step-function distribution.			
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1. INTRODUCTION

One of the primary tasks in the present effort was the use of the S-CUBED plastic-flow code to calculate for typical metal-matrix composites the residual microscopic stresses which would result from given fabrication and/or processing cycles. Such calculations have been done for graphite/aluminum, tungsten/aluminum and silicon-carbide/aluminum unidirectionally reinforced fibrous composites. In particular, a rather extensive series of parametric calculations was performed for the graphite/aluminum case to determine the changes in the aluminum matrix residual stresses which would result from variations and/or uncertainties in the thermoelastic properties of the graphite fibers.

The S-CUBED plastic-flow code, which uses the concentric-cylinder approximation for unidirectional composites, is described in detail in Ref. 1. Briefly, the equations of elastic-plastic flow are integrated numerically as the composite is cooled from an initial stress-free state and then reheated. The fibers are assumed to remain perfectly elastic throughout the cooling and heating cycle, and they may be treated as being anisotropic-transversely isotropic. The matrix material is assumed to be isotropic and is allowed to undergo plastic flow when its stress state reaches the yield surface. The magnitude of the yield stress may depend both on the temperature and the degree of plastic flow.

Results of the parametric study of a graphite/aluminum-2024 composite are given in Section 2, where stress-temperature histories are given for the axial stress, hoop stress, and fiber-matrix interfacial radial stress. The temperature cycle used for these calculations is a cooling from a consolidation temperature of 930°F to a minimum temperature of -240°F and then reheating until the aluminum matrix again reaches the yield surface. Results are given both for the T0 temper (slow cooling) and the T4 temper (rapid quench) thermoelastic properties and yield strength models used for 6061 and 2024 aluminums are detailed in Appendix A.

Results of two calculations on a tungsten/aluminum composite are given in Section 3, and similar calculations on a silicon carbide/aluminum composite are described in Section 4.

In Section 5, a computational method is presented for determining the elastic moduli and thermal expansion characteristics of cross-plyed laminates from the properties of the individual unidirectionally reinforced composite plies. Calculated results as a function of lay-up angle are given for a laminate made up of graphite/aluminum layers. A computer code has been constructed for treating a laminate with plies in up to three different directions. The properties of the individual plies may be calculated directly from the fiber and matrix properties if desired. The code also provides values for the intra-ply thermal stress derivatives about an initial stress-free state. A FORTRAN listing of the code is given in Appendix B.

A consideration of numerical results from a finite-element code calculation as given in a recent report by Hashin and Humphreys (Ref. 2) has led to a revision of the PRUFC code. This code was recently developed at S-CUBED for calculating the properties of unidirectional fibrous composites from those of the constituents (Ref. 3). In Section 6, various ways of obtaining the transverse properties of a composite within the context of the concentric-cylinder approximation used in the PRUFC code are examined and compared with the results of a more nearly exact (but much more expensive to run) finite-element code computation. A comparison of residual stresses calculated with the S-CUBED plastic-flow code (concentric-cylinder approximation) with those obtained from the finite-element method is also given in Section 6. The agreement is quite good.

In Section 7, a two-dimensional Fourier-series solution is presented for the stresses and thermal deformations in an elastic plate subjected to an arbitrary transverse temperature distribution.

A summary and some general conclusions are offered in Section 8.

2. GRAPHITE/ALUMINUM PARAMETER STUDY

A series of calculations was performed to determine the sensitivity of the matrix residual stresses to variations in various fiber properties. Eighteen different sets of graphite fiber properties were used. For each set of fiber properties, results were obtained for a slow cooling (T0 temper) and a rapid quench (T4 temper) of the 2024 aluminum matrix from an initial consolidation temperature of 930°F to a minimum temperature of -240°F. The fiber properties are tabulated in Table 2.1, and the thermoelastic properties and yield-stress models used for the 2024 aluminum are summarized in Appendix A. The room temperature thermoelastic properties of the composite as calculated with the latest version of the PRUFC code (Section 6) for each set of fiber properties are given in Table 2.2, the values being appropriate to the case where the aluminum matrix is not on its yield surface.

2.1 CALCULATIONAL RESULTS

Plots of the volume-averaged (matrix) axial stress, volume-averaged hoop stress, and the radial stress at the fiber-matrix interface as a function of temperature are given as Figures 2.1 through 2.42. The temperature cycle is a reheating from -240°F after a slow (T0) or fast (T4) cooldown from an initial consolidation temperature of 930°F. For this high initial temperature, the matrix is completely yielded after only a few degrees (<30°F) of cooling, and the changes in the stress state upon further cooling are due primarily to the variation of yield strength with temperature and plastic flow. Upon reheating from -240°F, however, the matrix immediately drops below the yield surface; and the variation in stress with temperature is then governed by the (reversible) equations of thermoelasticity. As the reheating continues, the matrix will eventually again reach its yield surface, as indicated by the open circles on the reheating curves. The plots are

TABLE 2.1

FIBER PROPERTIES USED IN THE PARAMETRIC RESIDUAL-STRESS STUDY
OF A GRAPHITE/ALUMINUM 2024 UNIDIRECTIONALLY REINFORCED COMPOSITE

Calc. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
n_f^*	0.45	**	=	=	=	=	=	=	=	=	=	=	=	=	0.37	=	0.30	0.60
E_x (Msi)	1	=	=	=	=	=	=	=	=	=	=	=	1.5	2.0	1.25	=	1.5	=
E_z (Msi)	35	=	=	=	=	=	=	=	=	=	50	100	50	=	35	=	50	=
ν_{xy}^+	0.3	=	=	=	=	=	=	=	0.4	0.5	0.4	=	=	=	0.45	=	0.4	=
ν_{xz}	0.3	=	=	=	=	=	0.4	0.5	0.4	=	=	=	=	=	0.41	=	0.4	=
$\alpha_x (10^{-6} \text{ } ^\circ\text{F}^{-1})$	5	=	=	10	15	20	15	=	=	=	=	=	=	=	2.7	15	=	=
$\alpha_z (10^{-6} \text{ } ^\circ\text{F}^{-1})$	0.5	-1	-1.5	-1	=	=	=	=	=	=	=	=	=	=	-0.57	=	-1	=

* n_f is the fiber volume fraction.

** The "equal" sign indicates a value the same as that in the preceding column.

† For the Poisson's ratios, the second subscript is the direction of the applied stress (fibers parallel to the z-axis).

TABLE 2.2

ROOM-TEMPERATURE G/AL 2024 COMPOSITE PROPERTIES AS CALCULATED
WITH THE PRUFC CODE (ELASTIC MATRIX) FOR EACH SET OF
FIBER PROPERTIES AS GIVEN IN TABLE 2.1

Calc. No	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Vol. Frac.	0.45	=	=	=	=	=	=	=	=	=	=	=	=	=	0.37	=	0.30	0.60
E_x (Msi)	4.75	=	=	=	=	=	=	4.74	4.75	4.78	=	4.82	5.32	5.81	5.83	=	6.89	4.05
E_z (Msi)	21.53	=	=	=	=	=	=	21.55	21.53	=	28.28	50.78	28.28	=	19.57	=	22.36	34.21
ν_{xy}	0.300	=	=	=	=	=	=	0.298	0.296	0.322	0.349	0.329	0.334	0.338	0.336	=	=	0.330
ν_{xz}	0.326	=	=	=	=	=	=	0.340	0.355	0.341	0.343	0.341	=	0.345	0.347	0.342	=	0.339
α_x ($10^{-6} \text{ } ^\circ\text{F}^{-1}$)	13.69	13.98	14.08	14.70	15.43	16.15	15.38	15.33	15.34	15.30	15.64	16.05	15.54	15.46	13.12	14.97	15.15	15.68
α_z ($10^{-6} \text{ } ^\circ\text{F}^{-1}$)	3.72	2.62	2.26	2.61	=	2.60	=	2.59	2.60	=	1.74	0.53	1.74	=	3.76	3.82	3.41	0.64
G_{xy} (Mxi)	1.83	=	=	=	=	=	=	=	1.80	1.77	1.80	=	1.99	2.17	2.18	=	2.58	1.52

terminated at this point due to the limitations of the present version of the S-CUBED plastic-flow code. Physically, if the reheating were continued beyond the yield point, the slope of the stress-temperature curves would exhibit a discontinuous sharp decrease in absolute magnitude with a subsequent return to zero stress at the melt point. As long as the temperature at the upper yield point is not exceeded, subsequent cooling will cause the stresses to retrace the original heating path from -240°F . (The possible effect of creep on the matrix residual stresses is not treated here.)

The effect on the matrix stress of a variation of the axial coefficient of thermal expansion for the fibers is shown in Figures 2.1 through 2.6. Neither the axial, hoop, nor interfacial radial stress shows a marked variation for the range $\alpha_z = 0.5 \times 10^{-6} \text{ }^{\circ}\text{F}^{-1}$ to $\alpha_z = -1.5 \times 10^{-6} \text{ }^{\circ}\text{F}^{-1}$ for the fiber expansion coefficient. The reversible, elastic portion of the axial stress shows the greatest dependency.

The effect of a variation in the fibers transverse coefficient of thermal expansion on the matrix stresses is depicted in Figures 2.7 through 2.12. The axial stress does not exhibit a strong dependency, but the hoop and interfacial radial stresses show a marked variation. Perhaps it is not surprising that as the fiber's transverse expansion coefficient approaches that of the matrix, the hoop and radial residual stresses become small. Here it should be pointed out that a negative (compressive) interfacial radial stress would increase the effective fiber-matrix bond strength; hence a moderately large negative value might be desirable.

A variation of the fiber's Poisson's ratios, as shown in Figures 2.13 through 2.18, has only a minute effect on all three of the calculated stress-temperature histories.

An increase in the fiber's axial elastic modulus from 50 Msi to 100 Msi, Figures 2.19 through 2.24, does produce a more rapid decrease in the matrix axial stress upon reheating from -240°F but

has a completely negligible effect on the hoop and interfacial radial stress-temperature histories.

A parametric variation in the transverse elastic modulus of the fibers leads to the calculated results for the matrix stresses plotted in Figures 2.25 through 2.30. Doubling the fiber modulus from 1 Msi to 2 Msi has a negligible effect on the axial matrix stresses, and the effect on the calculated hoop and interfacial radial stresses is also small.

For the calculations discussed above, the fiber volume fraction in all cases is $n_f = 0.45$. In Figures 2.31 through 2.36, results are plotted for a fiber volume fraction of $n_f = 0.37$ for two values of the fiber's transverse thermal expansion coefficient. As was the case with Calculations 2, 4 and 6, the larger value of the transverse expansion coefficient produces a significant decrease in the magnitudes of the hoop stresses and interfacial radial stresses.

Results for two different fiber volume fractions of $n_f = 0.30$ and $n_f = 0.60$ are plotted in Figures 2.37 through 2.42. The residual stresses are somewhat higher in magnitude for the larger volume fraction.

2.2 SUMMARY AND CONCLUSIONS

In general, for all of the fiber parameter variations discussed above, the calculations indicate that the axial matrix stress at room temperature after a rapid quench from 930°F to -240°F (T4 temper) will lie in the range from about 30 ksi to 40 ksi (tensile) and that the matrix will remain elastic up to about 600°F upon reheating. The residual hoop stress at room temperature in all cases is tensile and ranges from about 10 ksi to 20 ksi in magnitude if the fiber's transverse coefficient of thermal expansion is taken to be less than or equal to that of the matrix ($\sim 12.4 \times 10^{-6} \text{ } ^\circ\text{F}^{-1}$ for 2024 aluminum at room temperature). Further, the interfacial

radial stress between the fiber and matrix is compressive as long as the fiber's transverse expansion coefficient does not exceed that of the matrix, the magnitude ranging from about 2 ksi to 10 ksi over the range of material parameters used in these calculations. This would appear to be a significant result, since a compressive interfacial stress is desirable as an aid in maintaining the integrity of the fiber-matrix bond. This bond is critical, since it is the source of the (localized) shear forces which are ultimately responsible for equilibrating axial strain between the fiber and matrix components.

For the slow cool to -240°F (T0 temper) the axial residual stresses at room temperature for all cases considered here are very close to zero, a situation which would appear to be desirable if the composite is to be subjected to subsequent mechanical loading. Because of the reduced yield strength of the T0 temper with respect to that of the T4, however, the matrix will again reach its yield point after reheating to between 200°F and 300°F . Even though the axial stress at room temperature is much lower for the T0 temper than the T4, the hoop stress will still be about 5 ksi to 10 ksi in tension for the range of fiber parameters considered. As long as the fiber's transverse expansion coefficient is less than that of the matrix, the interfacial radial stresses would still be compressive according to the calculations, the magnitudes being in the range from 2 ksi to about 8 ksi.

A room temperature axial residual stress of 30 ksi to 40 ksi in the matrix as calculated for the T4 temper would be undesirable if the composite were to be subjected to large tensile loads. This is due to the fact that the matrix would reach the yield surface, with a consequent degradation of the axial modulus, at a lower applied load than it would if the matrix stress were initially near zero. For compressive loads about the ambient state, however, an initial tensile residual stress might be useful. In any case, it is not immediately apparent how the room temperature residual stress for the T4 temper might be reduced. Continued heating beyond the

yield point of $\sim 600^{\circ}\text{F}$ would result in a stress temperature path that must ultimately pass through zero stress at the melt point. A recooling from any point on this path segment would result in a (reversible) elastic curve for the axial stress that would lie to the right of the initial heating curve from -240°F , and consequently the room temperature axial residual stress would be higher than it was before. This might be alleviated however, if creep of the matrix material was taken into account.

Some reduction of the axial residual stress would be achieved by cooling below -240°F initially. No explicit calculation was done, but it appears from an extrapolation of the curves presented here that cooling to $\sim -320^{\circ}\text{F}$ (liquid nitrogen) would reduce the axial residual stress by about 5 ksi or 10 ksi.

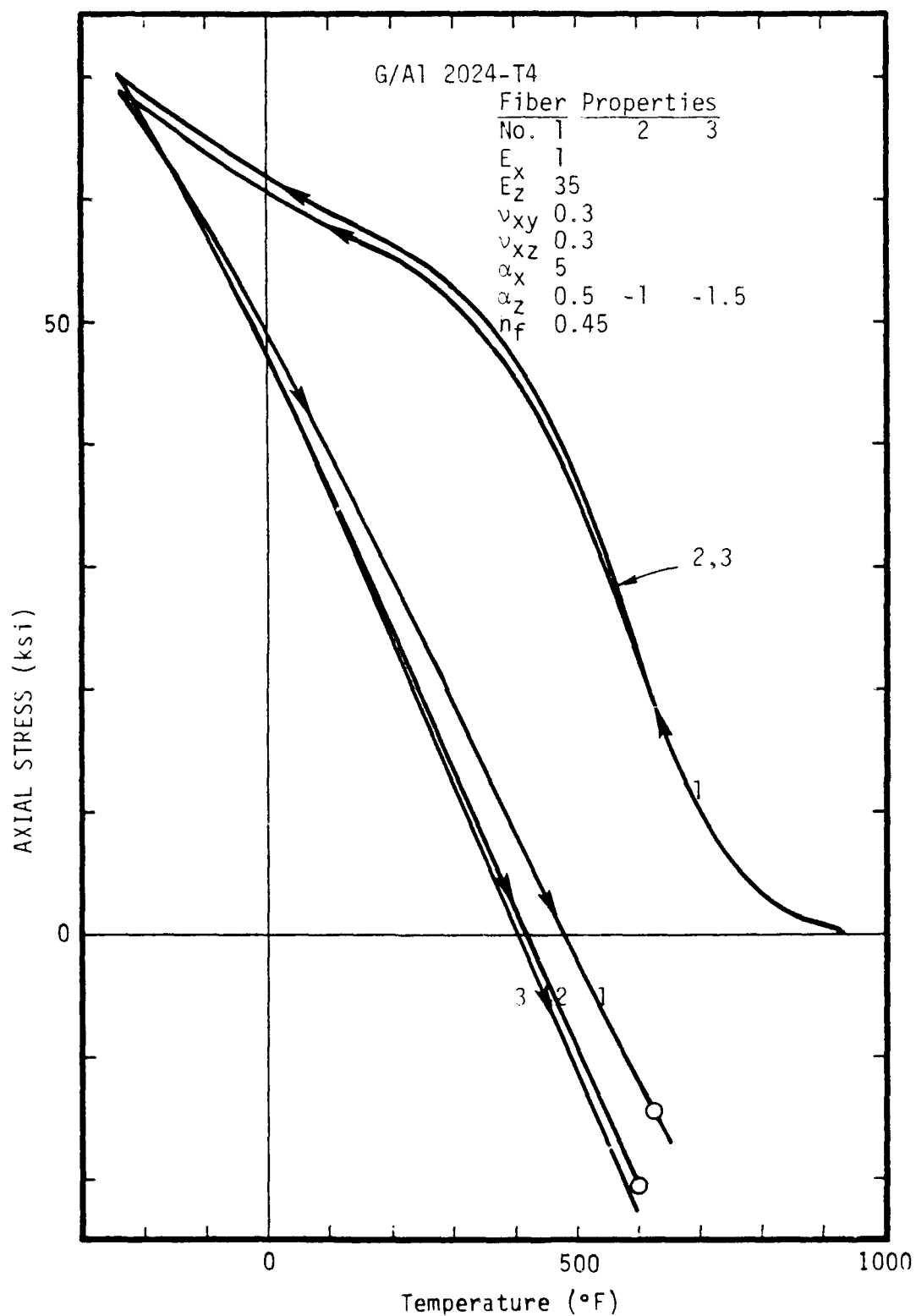


Figure 2.1. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

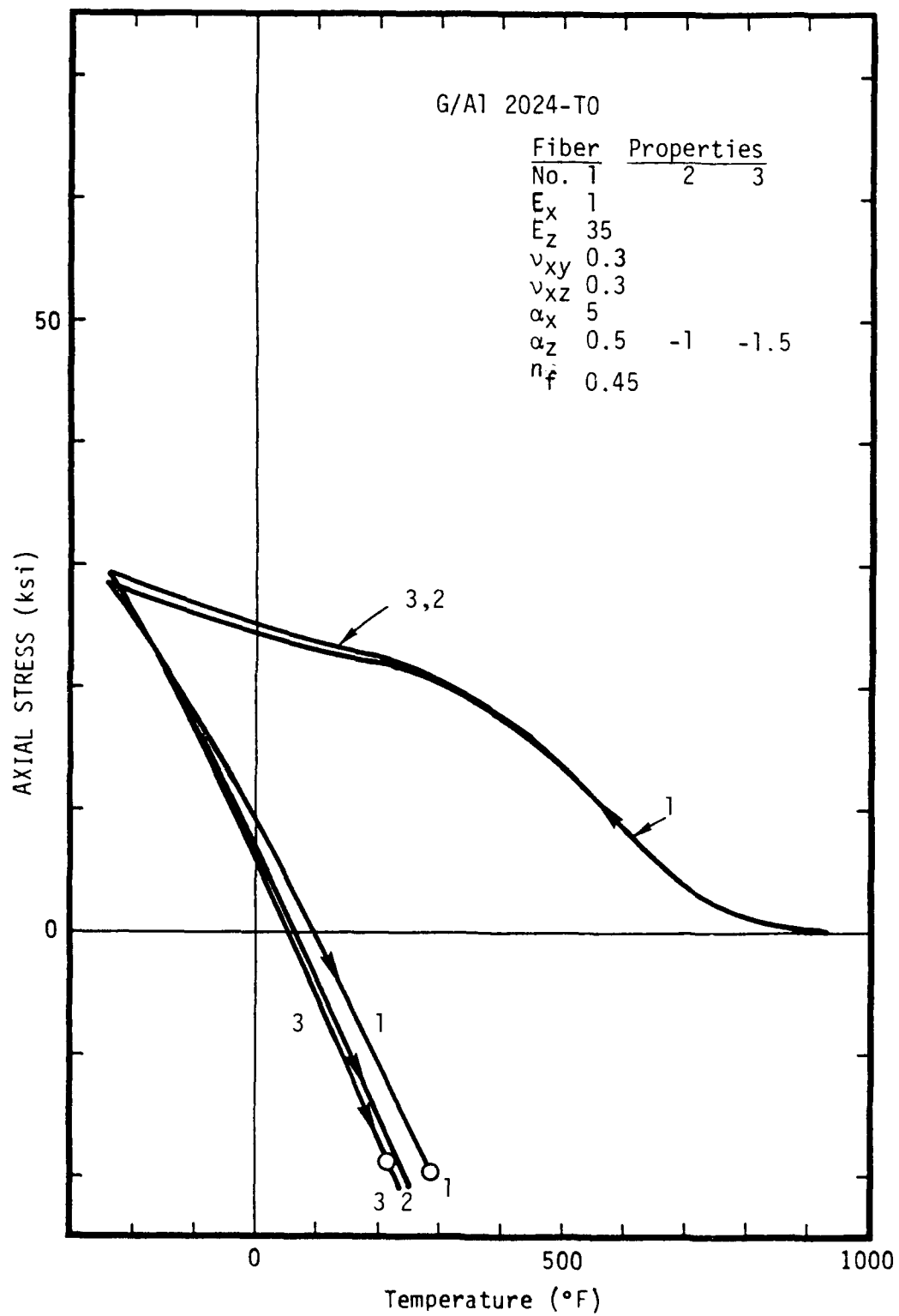


Figure 2.2. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

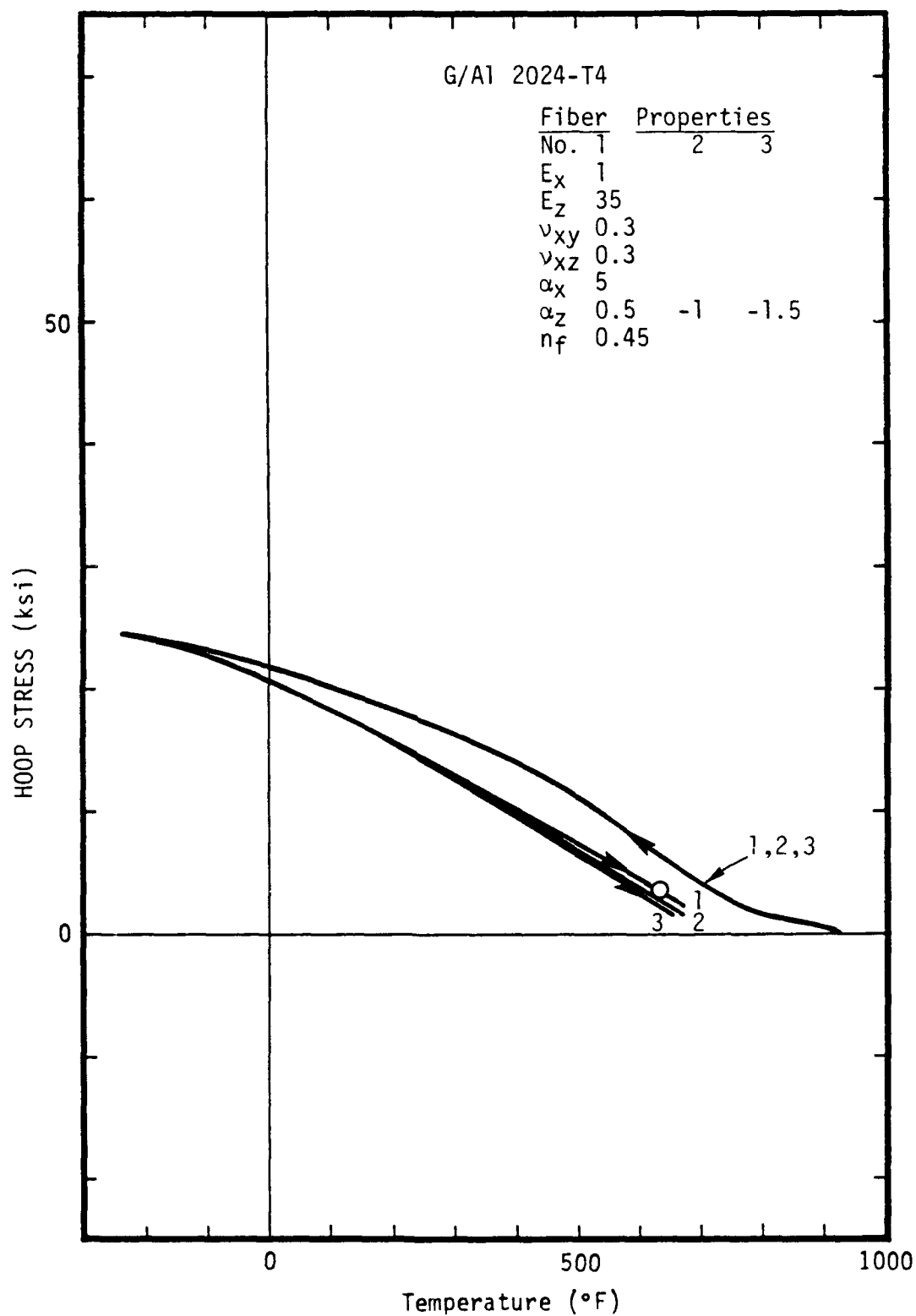


Figure 2.3. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

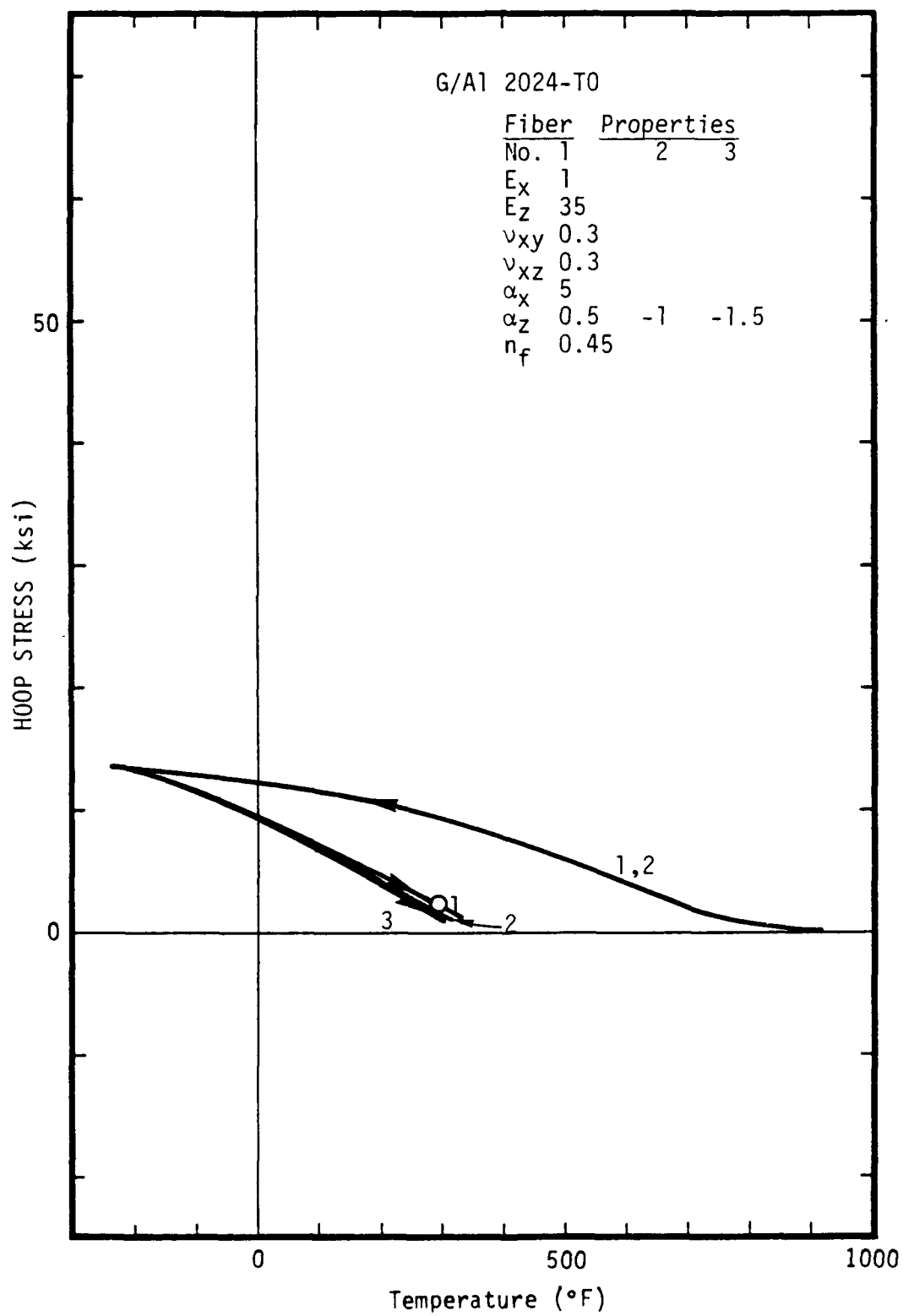


Figure 2.4. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

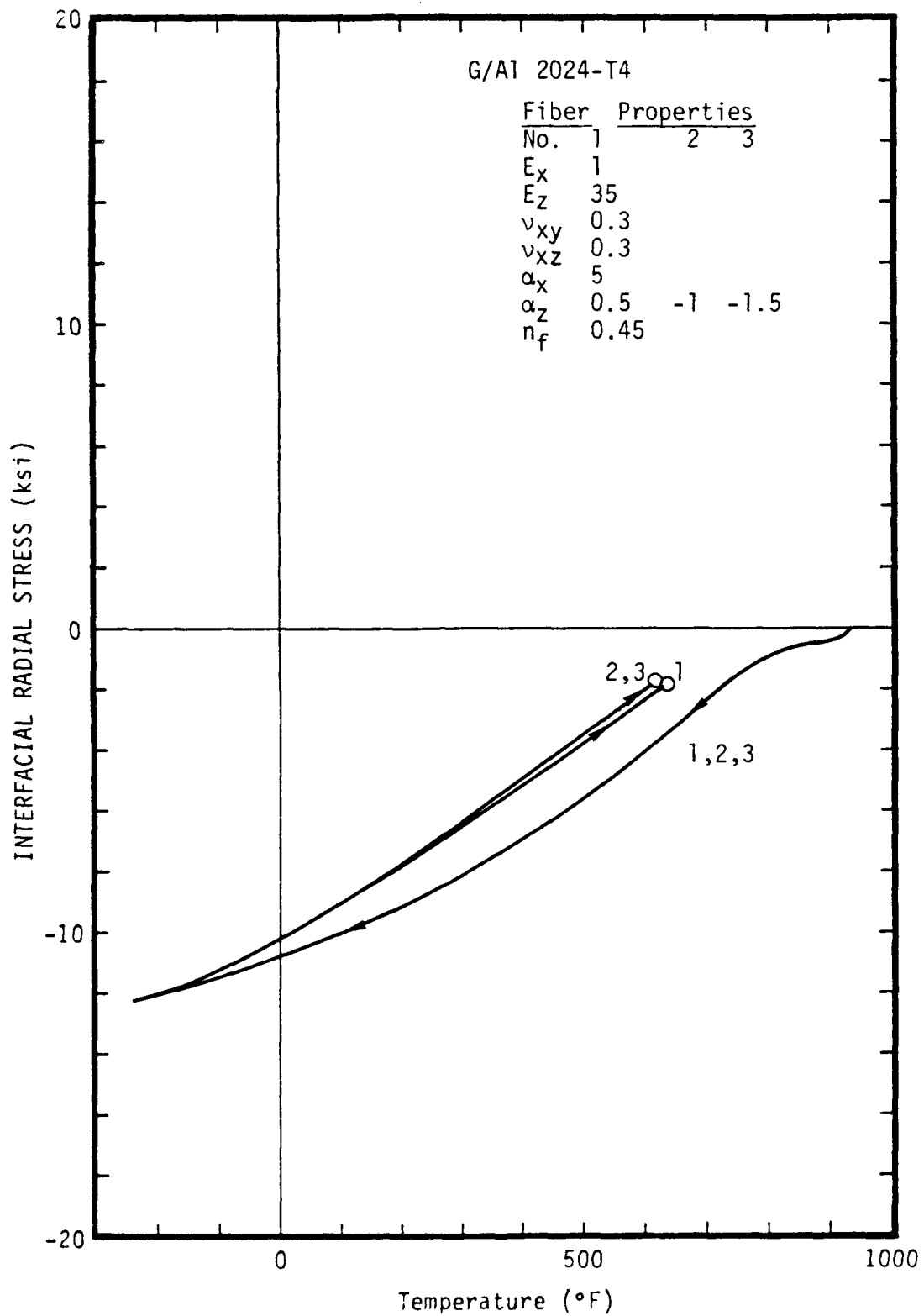


Figure 2.5. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (rapid cooling).

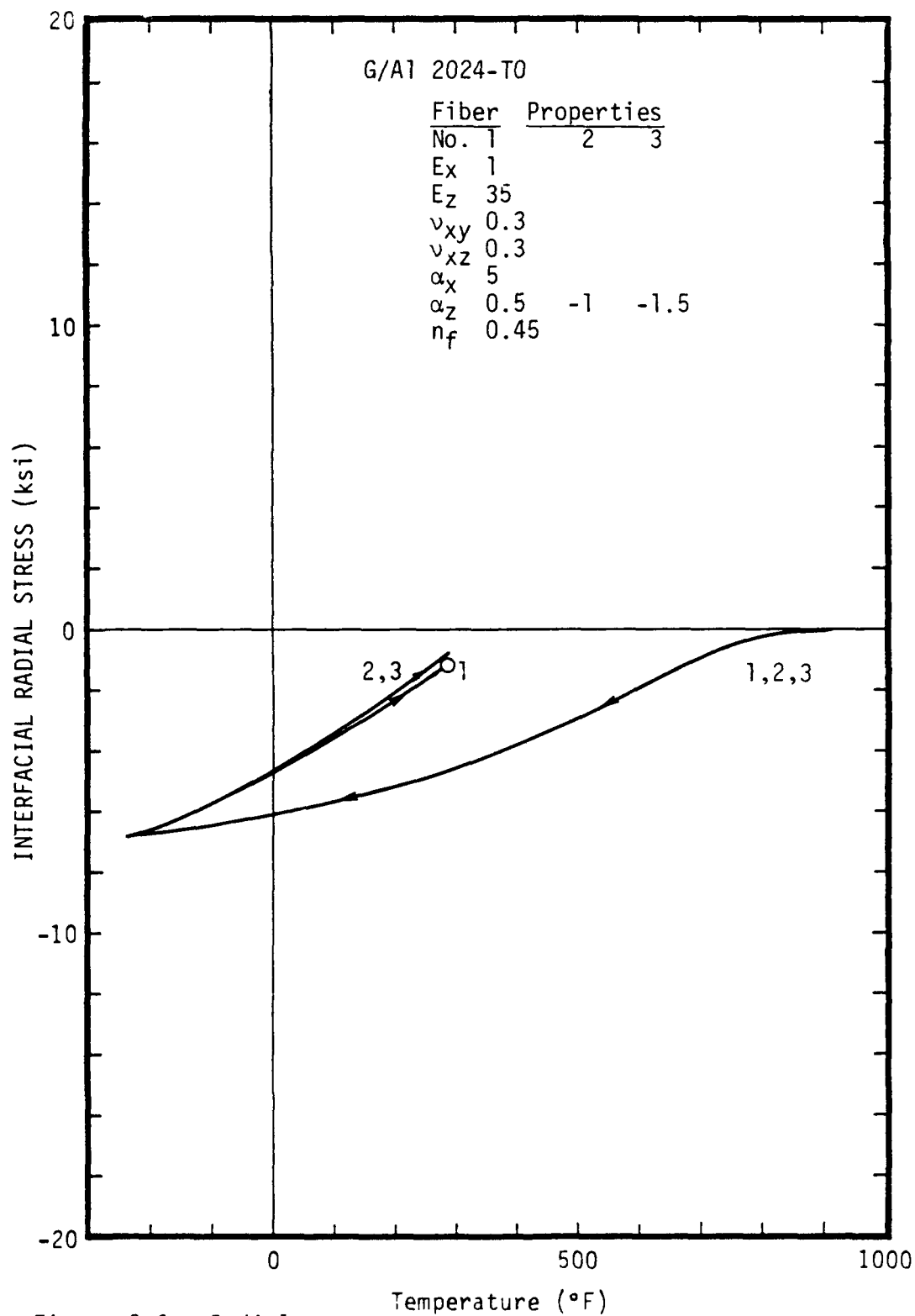


Figure 2.6. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (slow cooling).

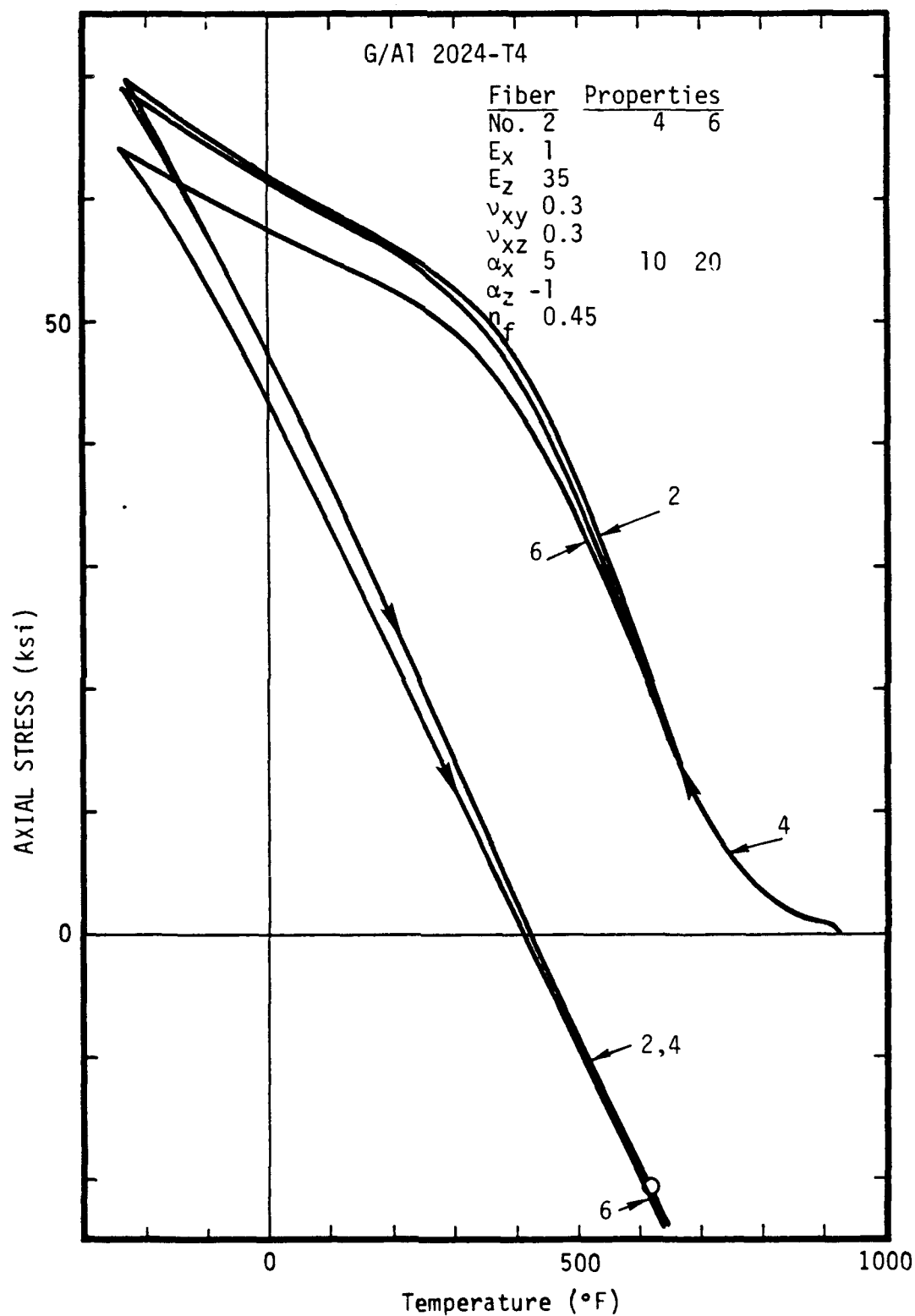


Figure 2.7. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

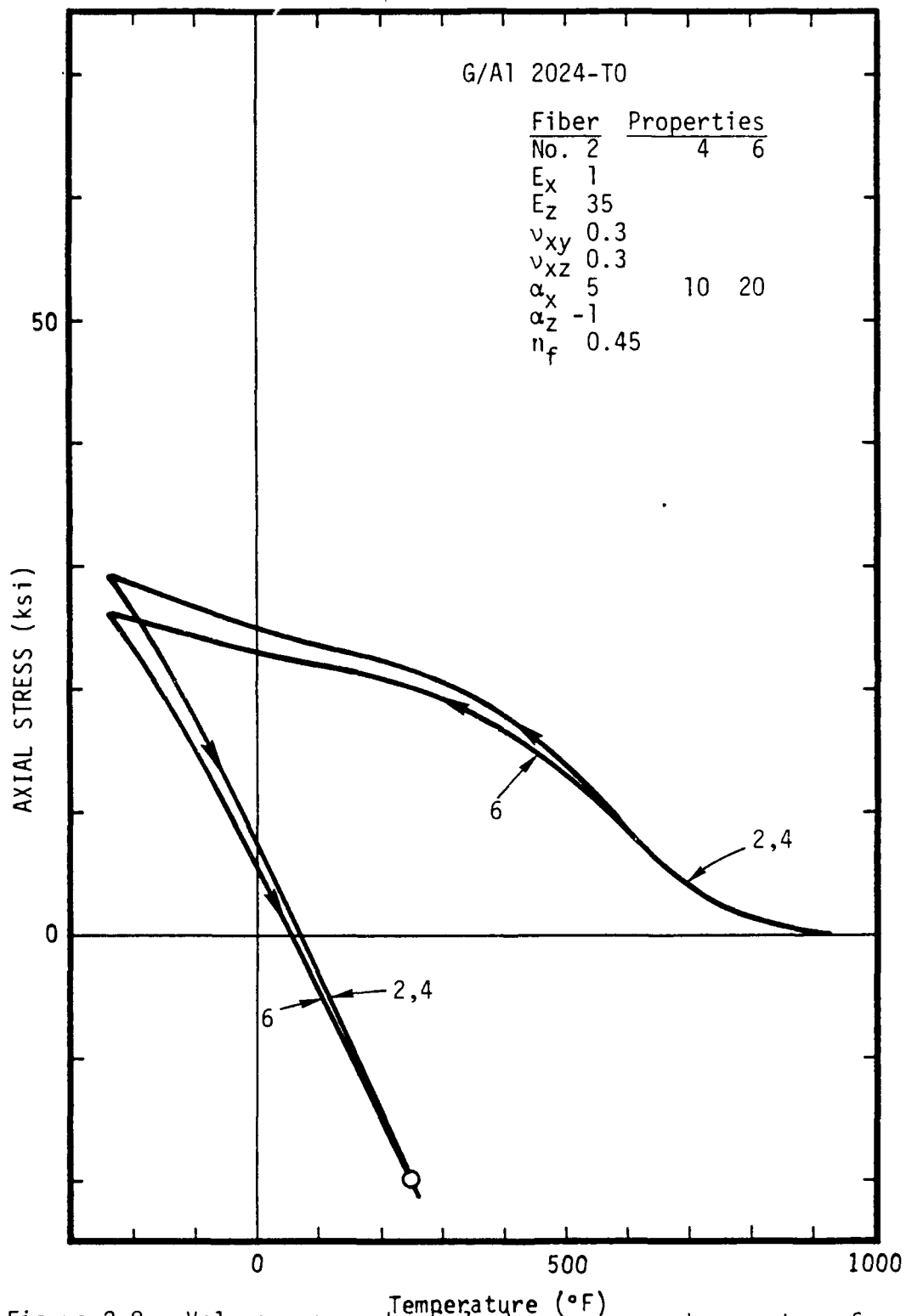


Figure 2.8. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

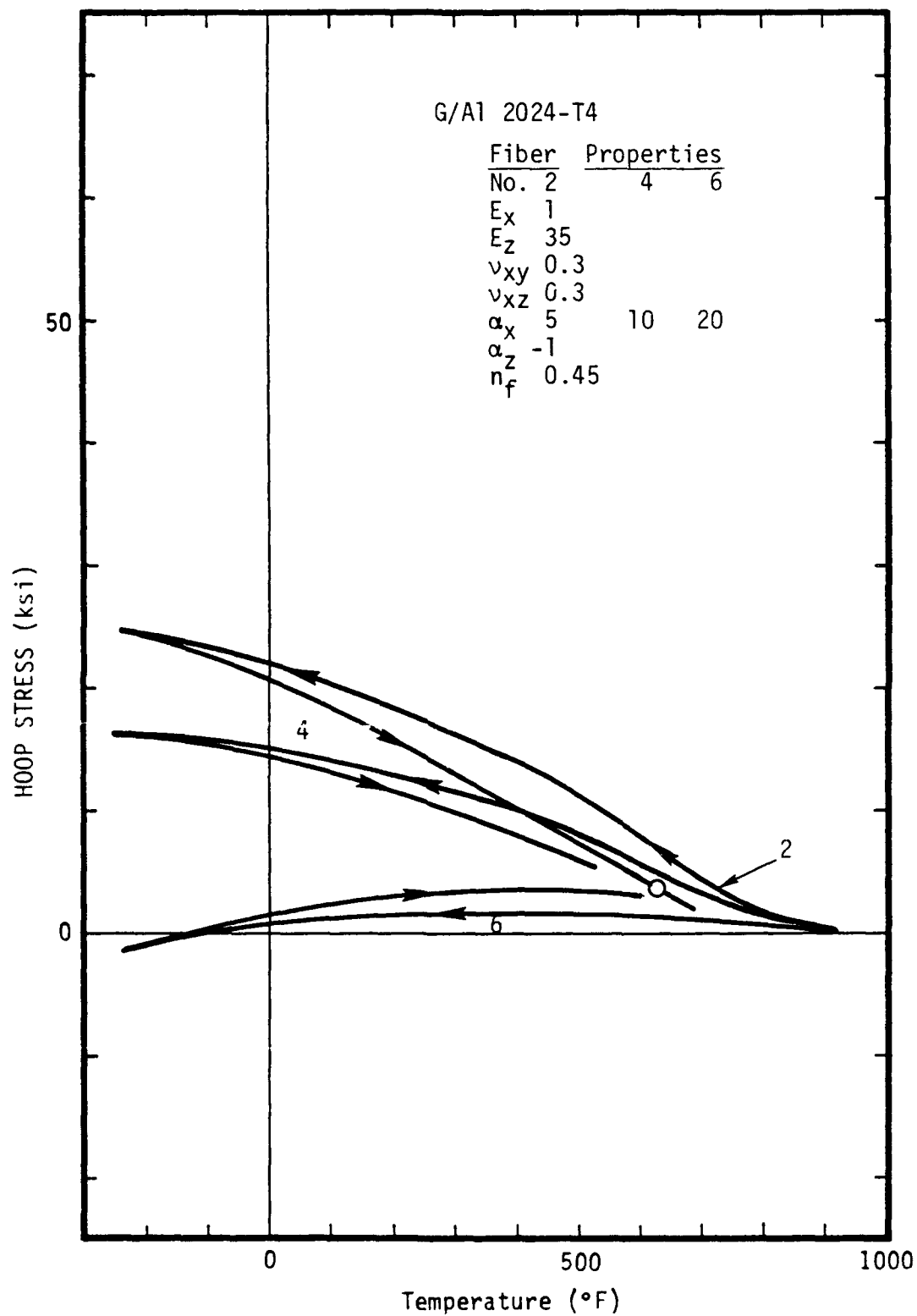


Figure 2.9. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

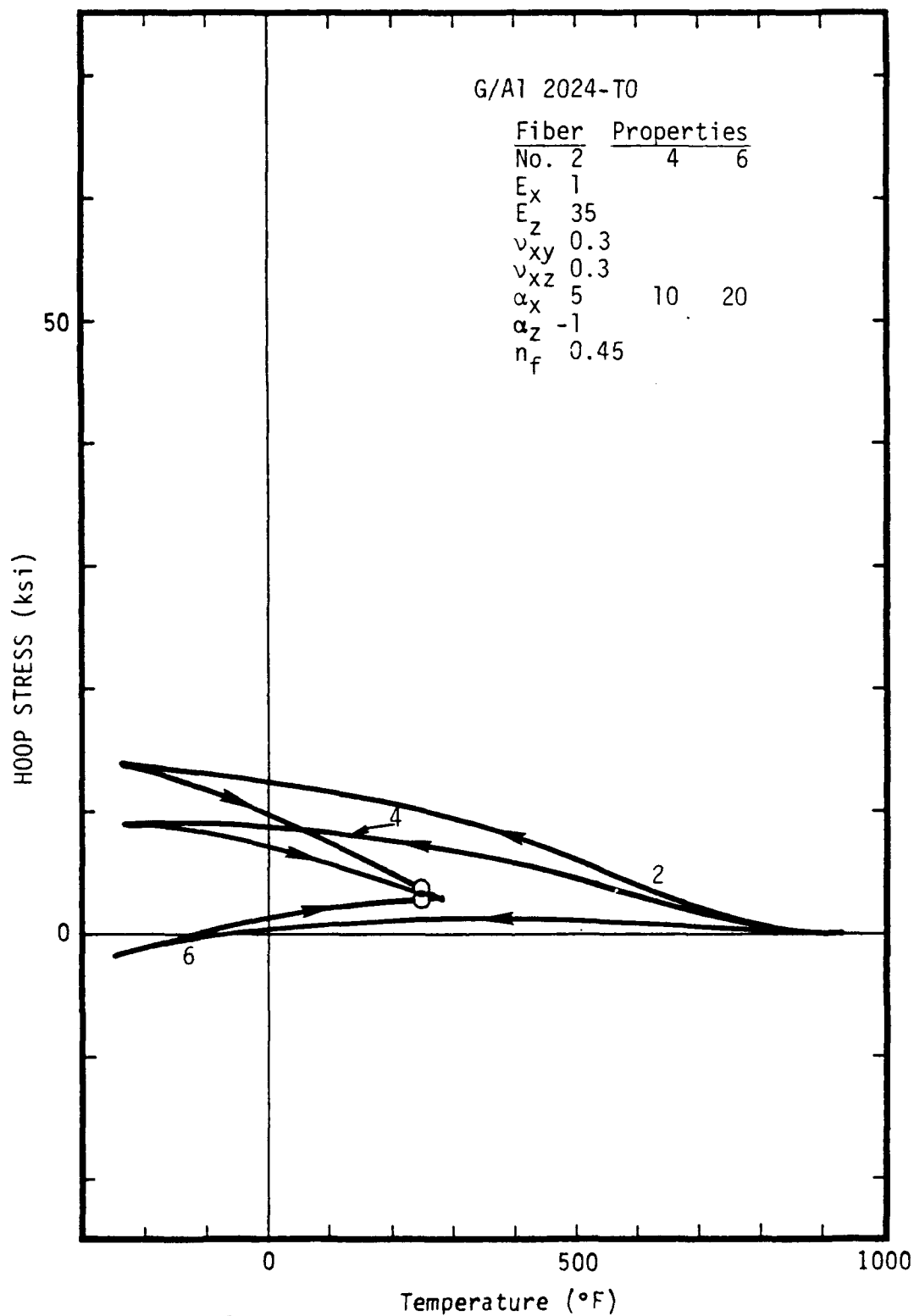


Figure 2.10. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

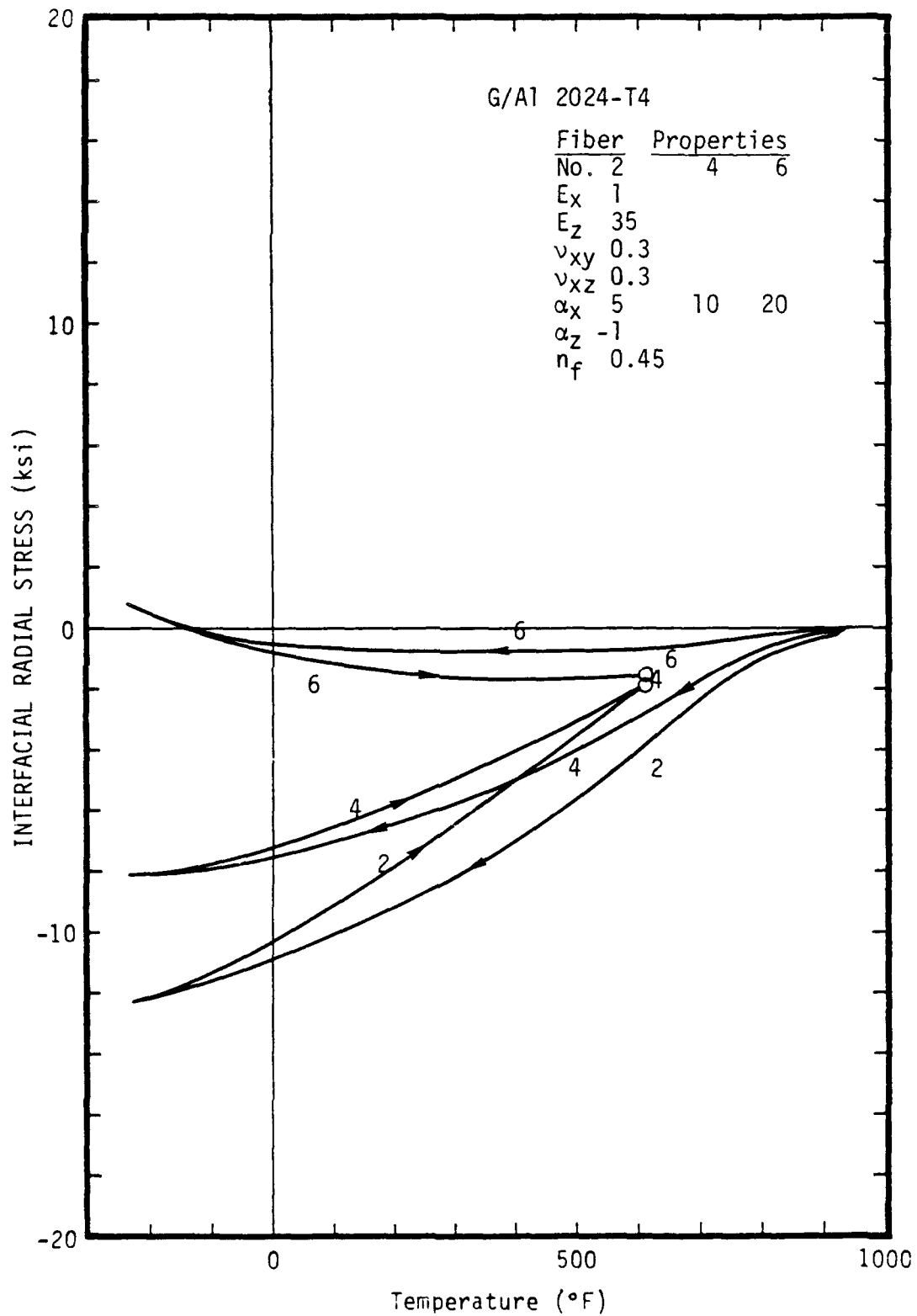


Figure 2.11. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (rapid cooling).

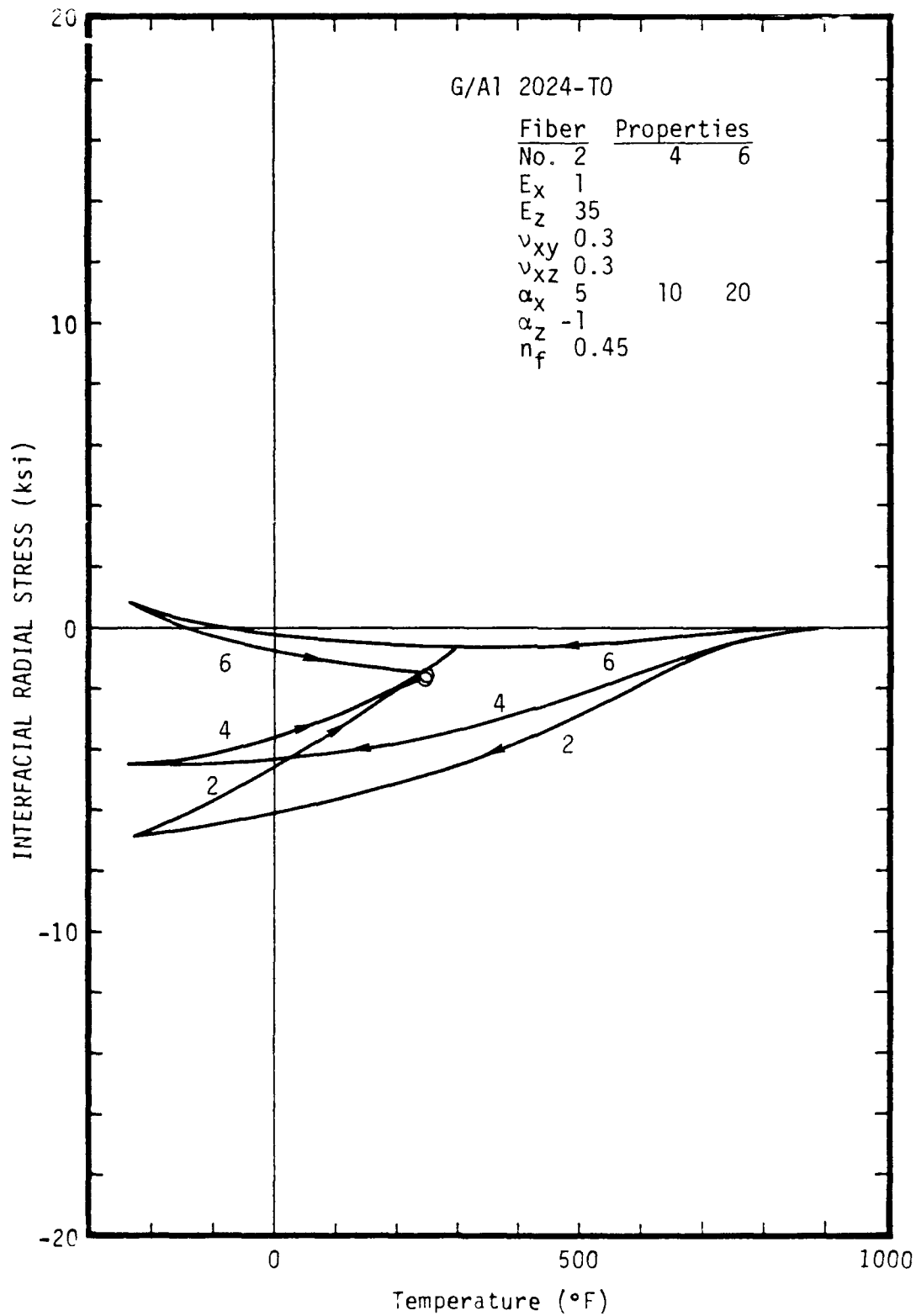


Figure 2.12. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (slow cooling).

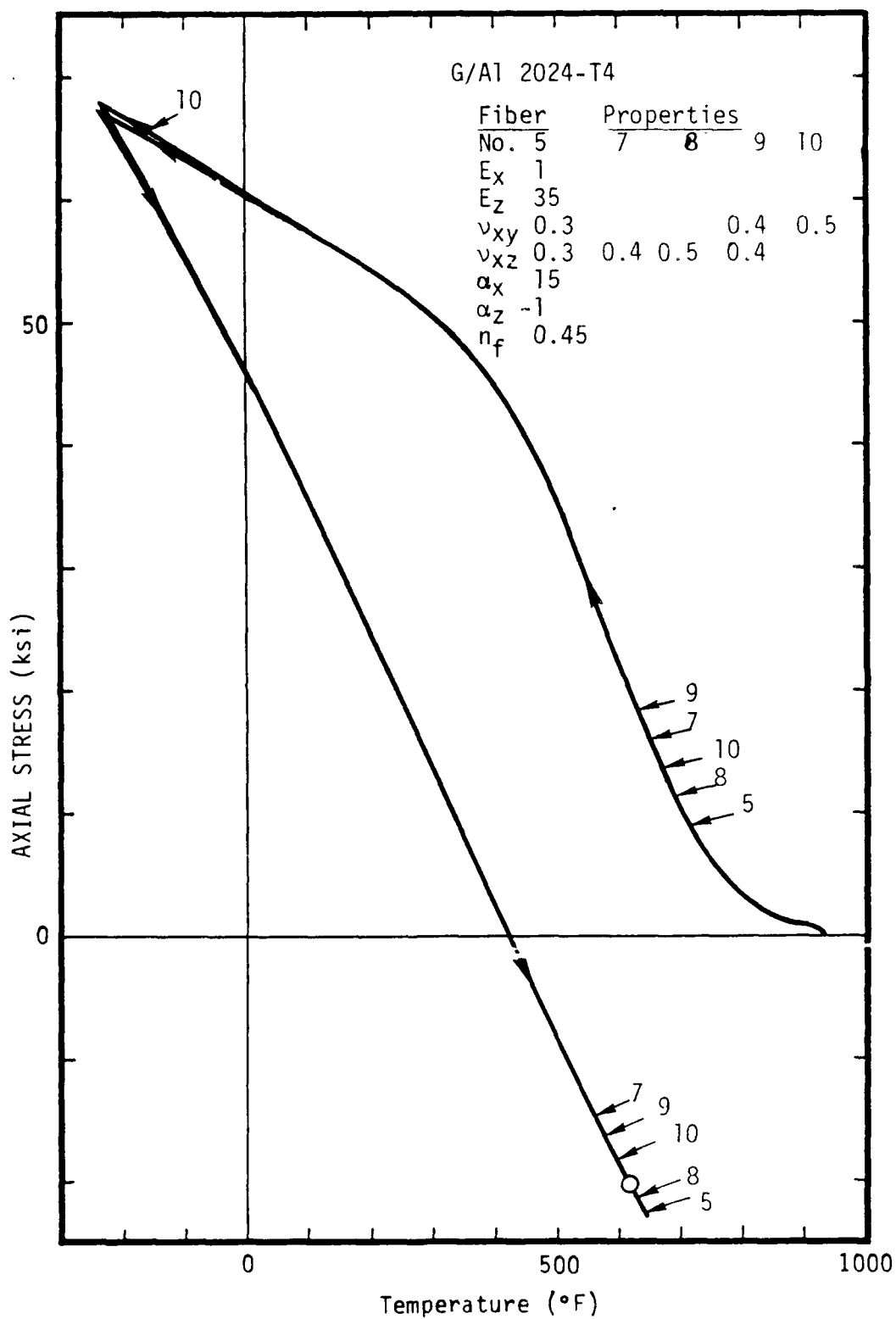


Figure 2.13. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

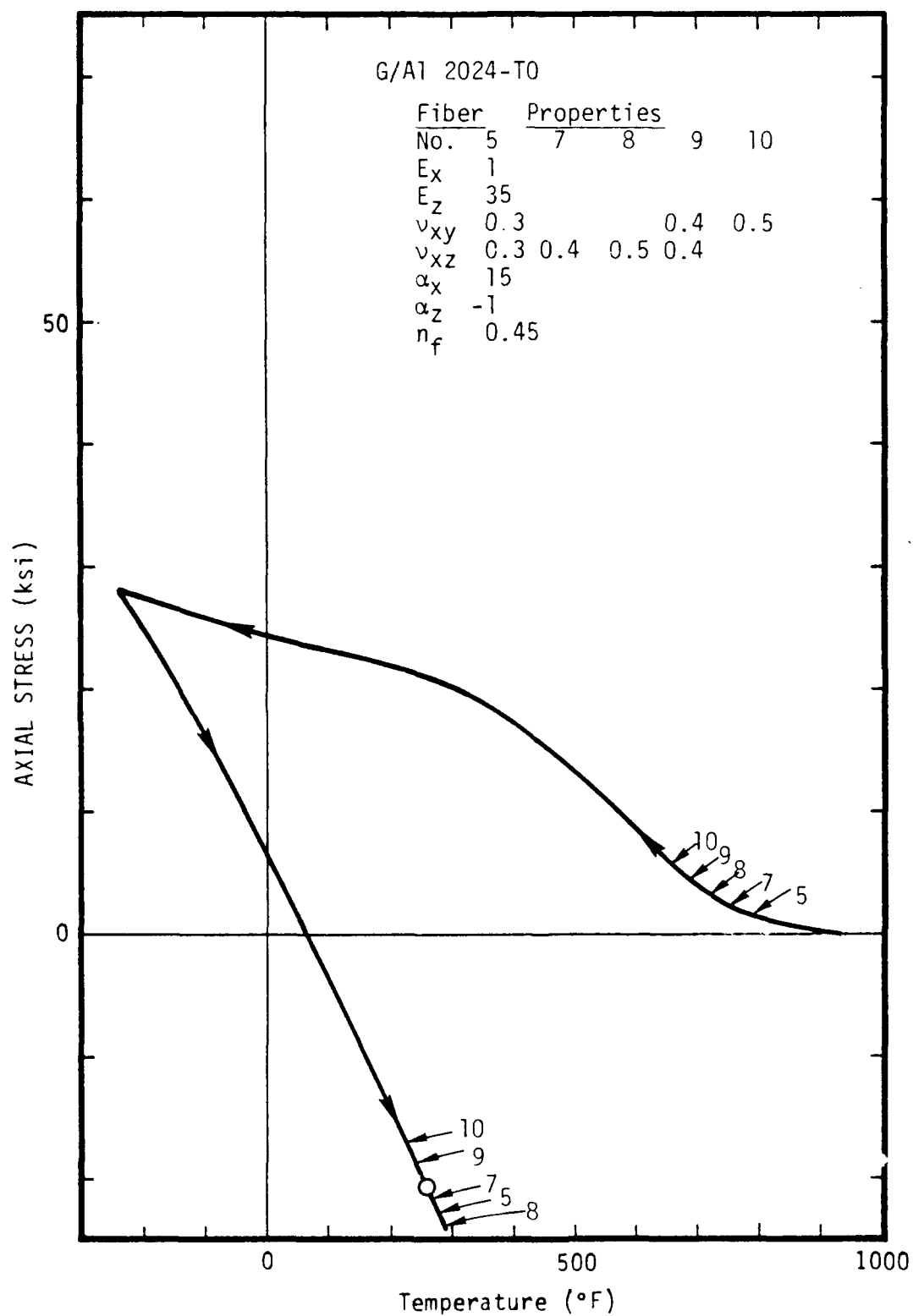


Figure 2.14. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

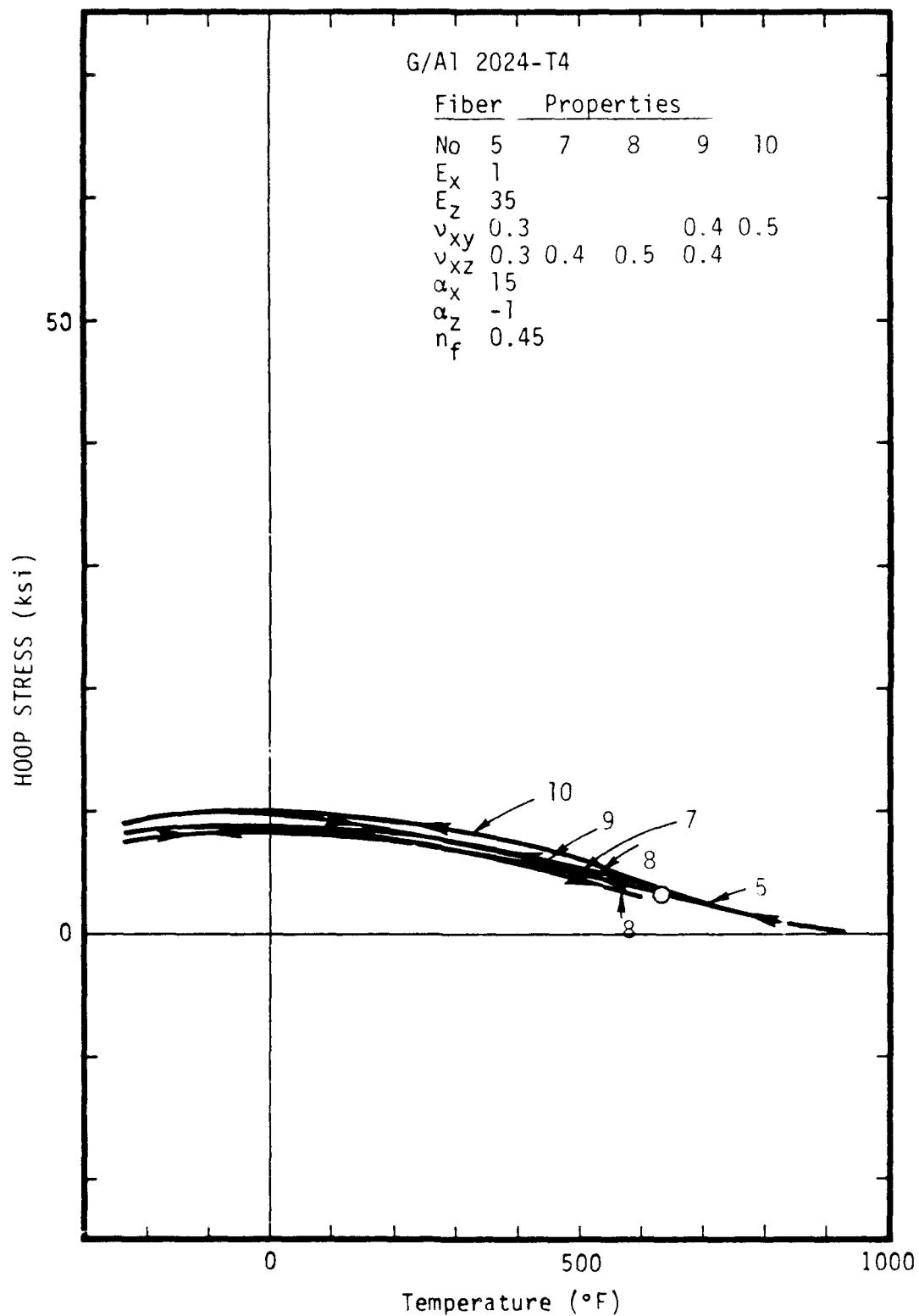


Figure 2.15. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

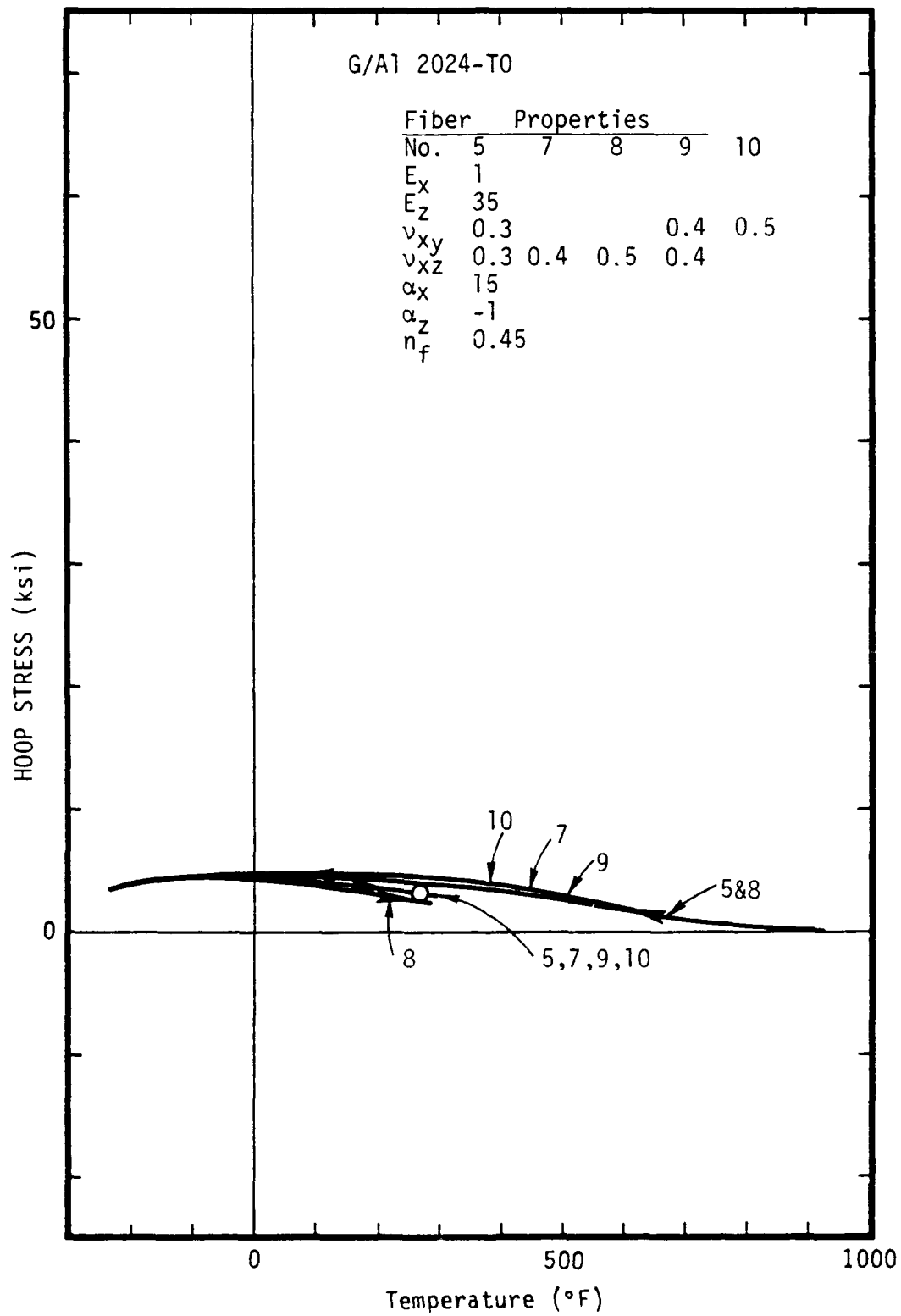


Figure 2.16. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

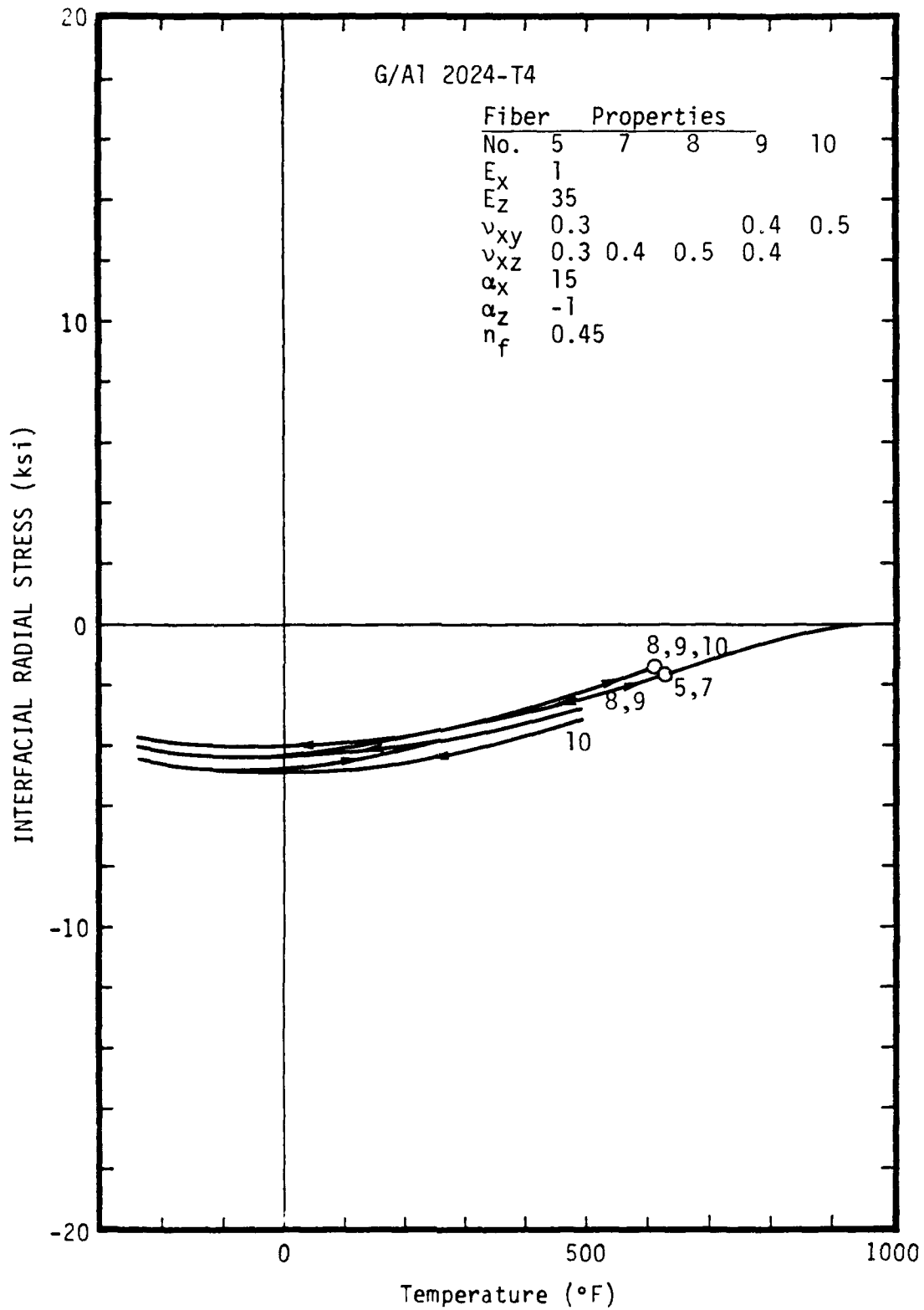


Figure 2.17. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (rapid cooling).

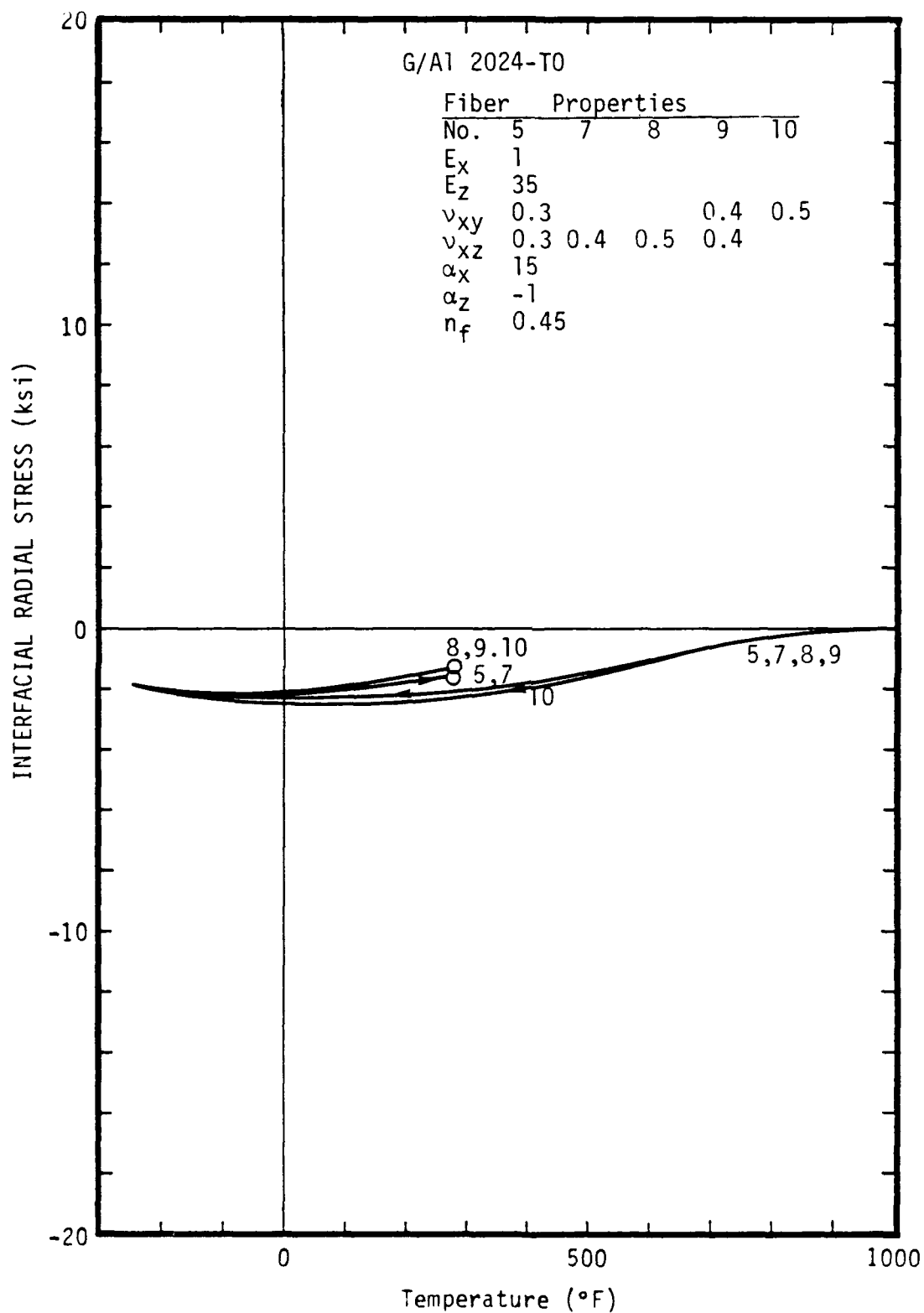


Figure 2.18. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (slow cooling).

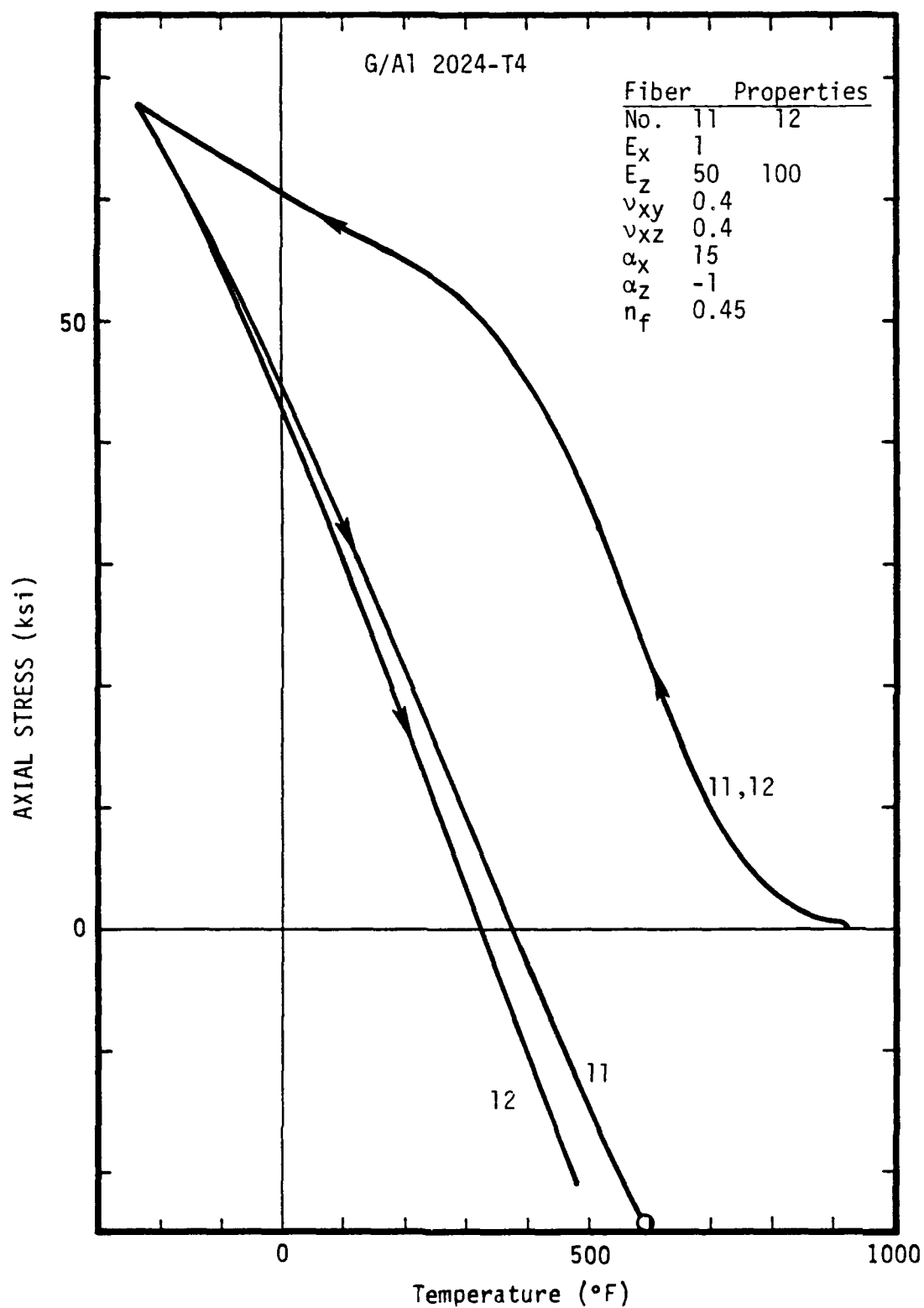


Figure 2.19. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

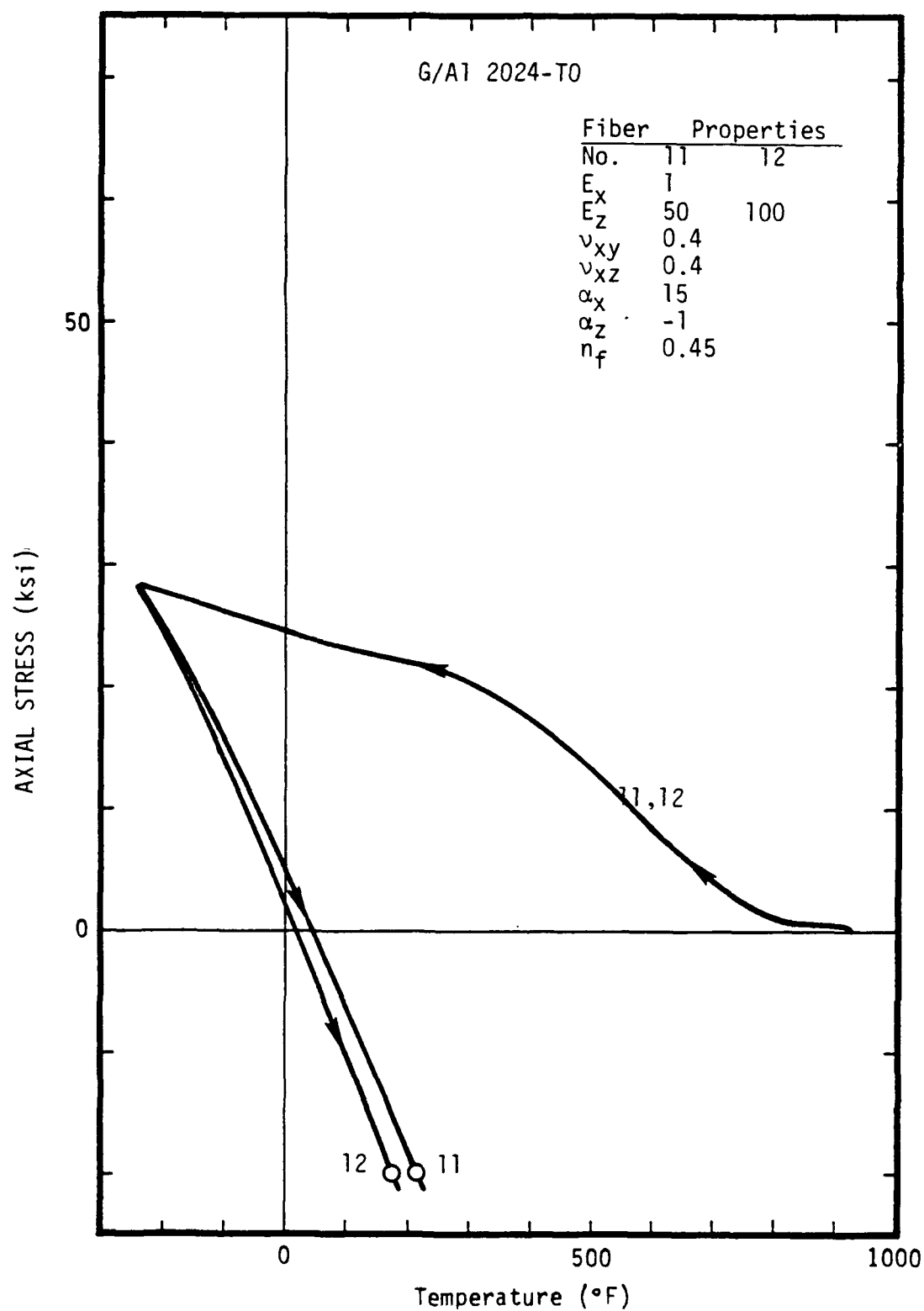


Figure 2.20. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

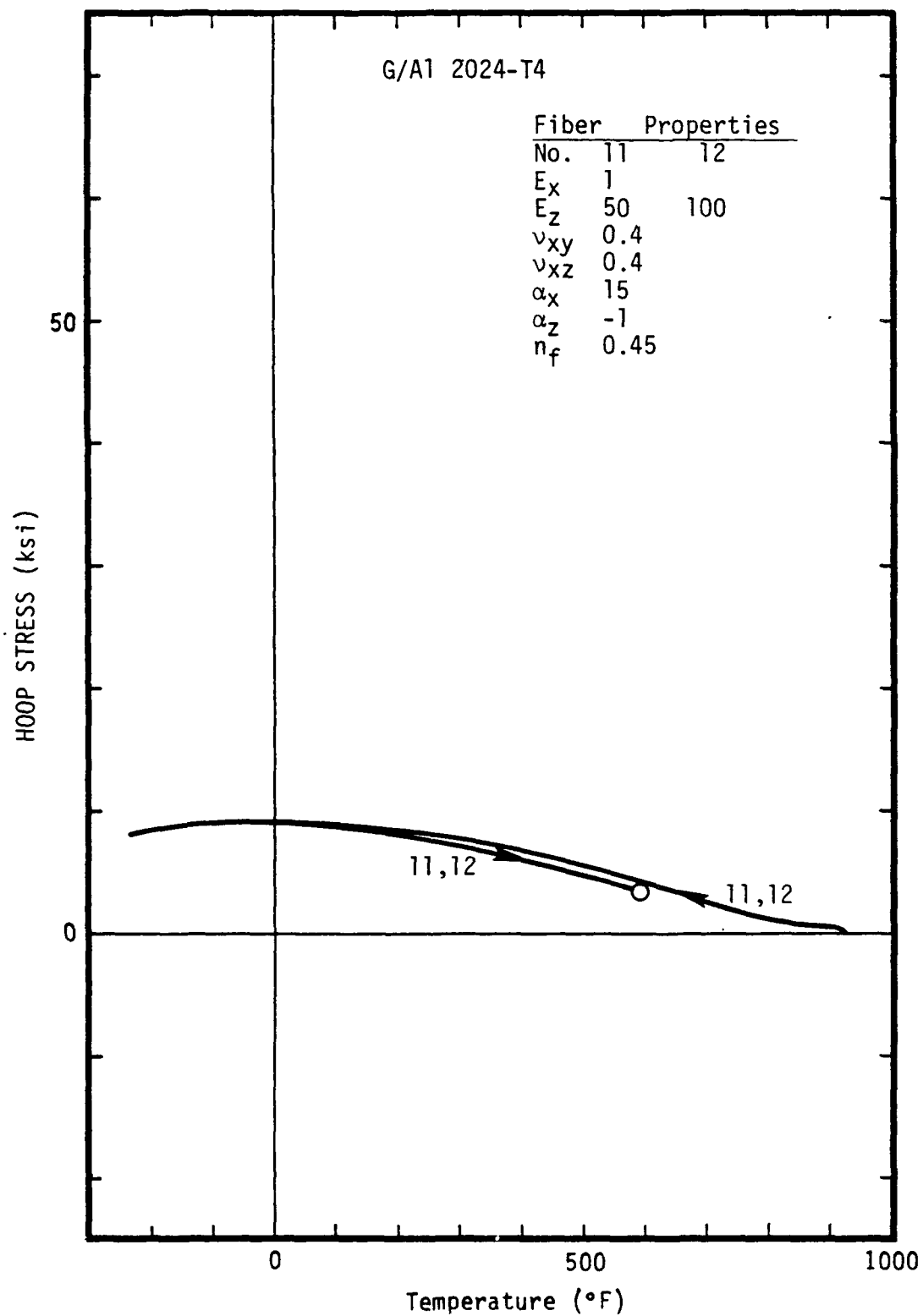


Figure 2.21. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

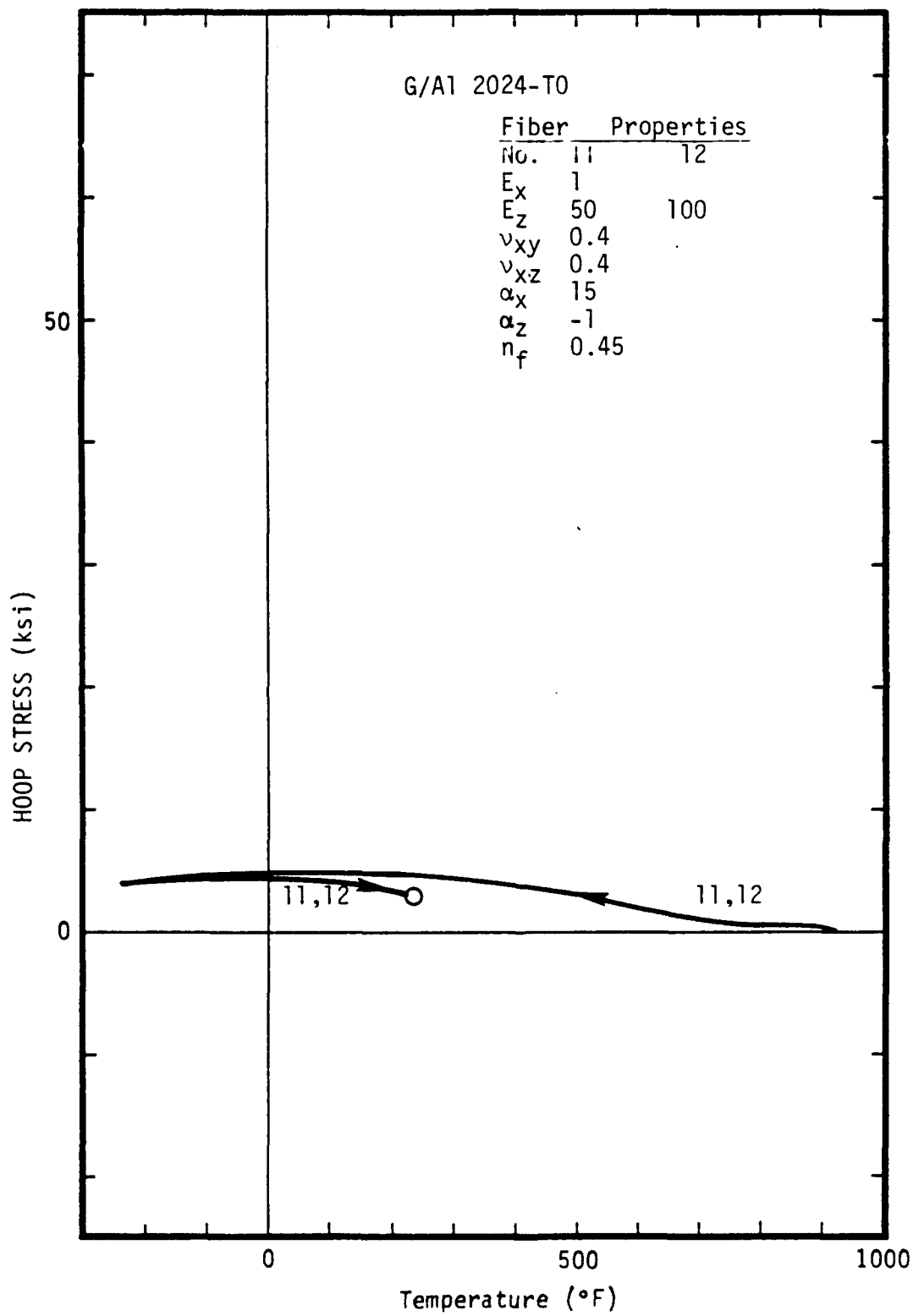


Figure 2.22. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

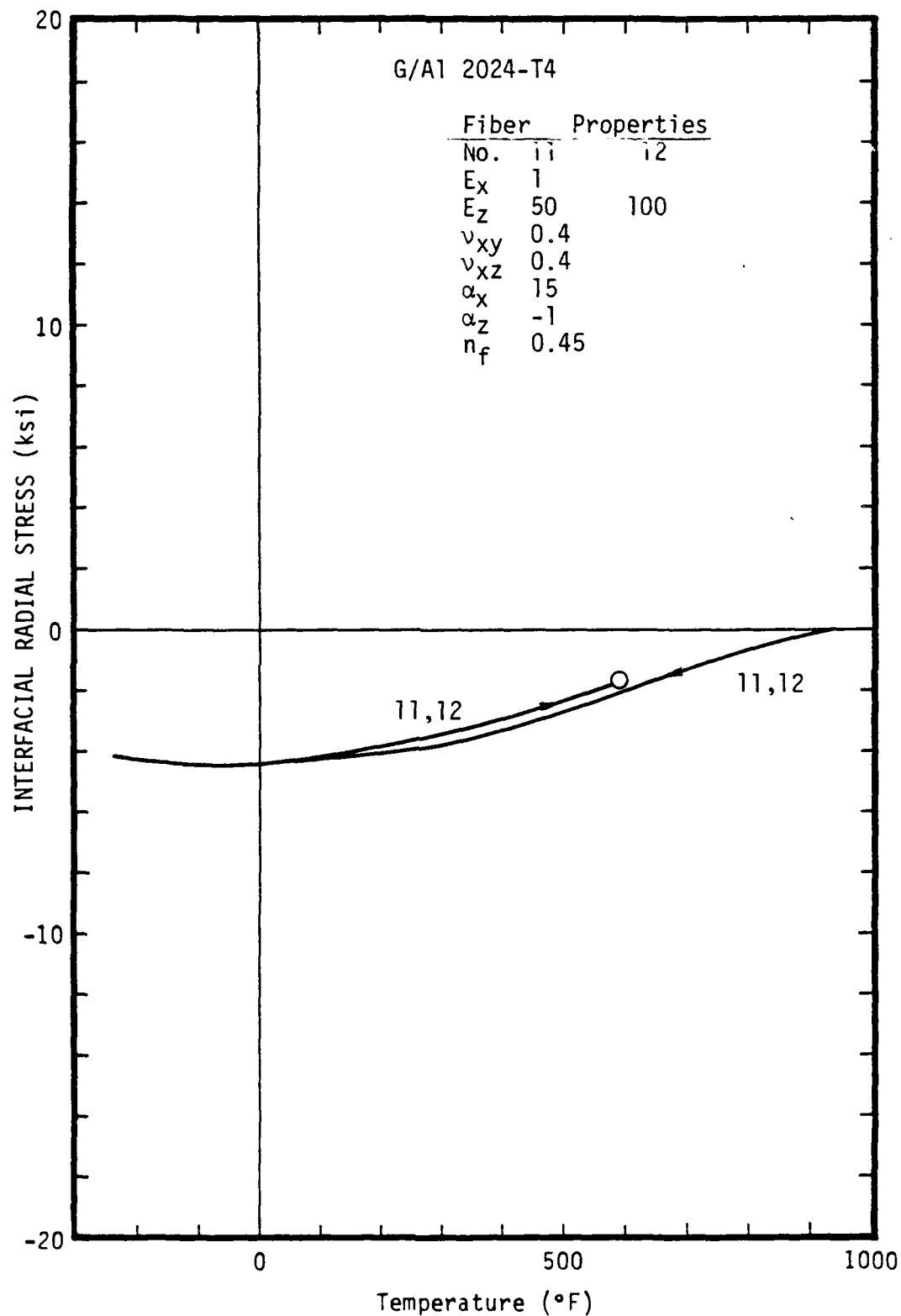


Figure 2.23. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (rapid cooling).

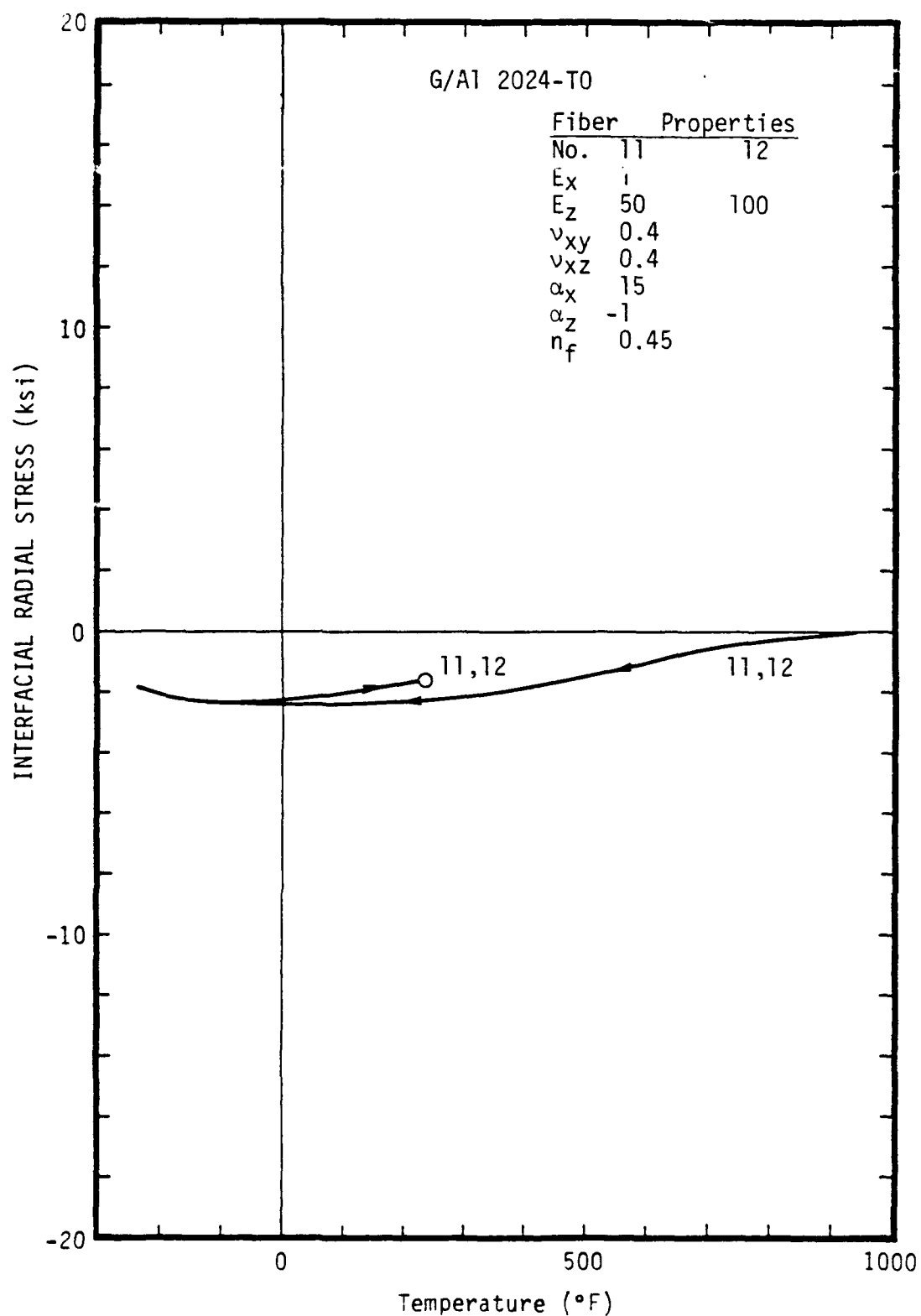


Figure 2.24. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (slow cooling).

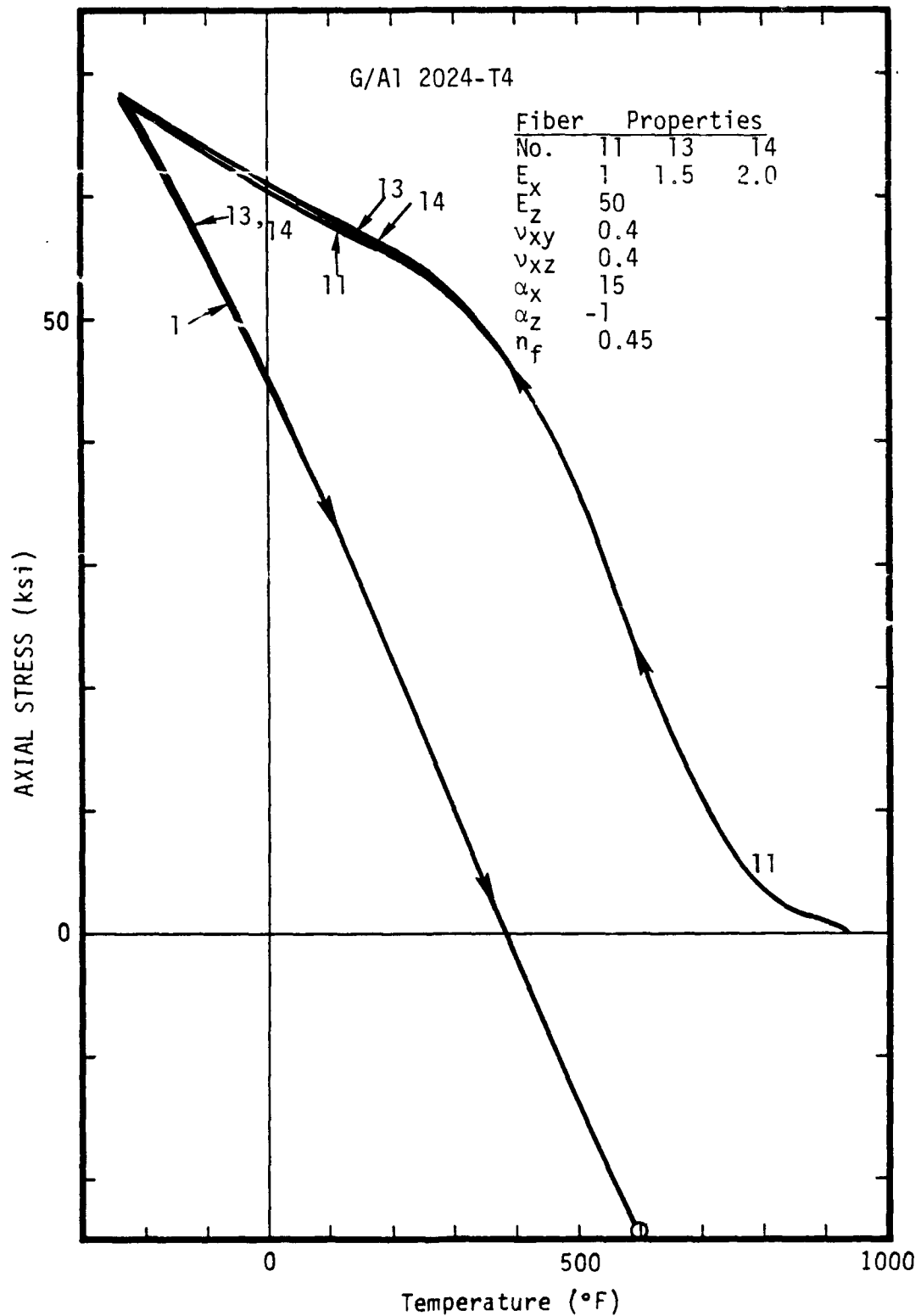


Figure 2.25. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

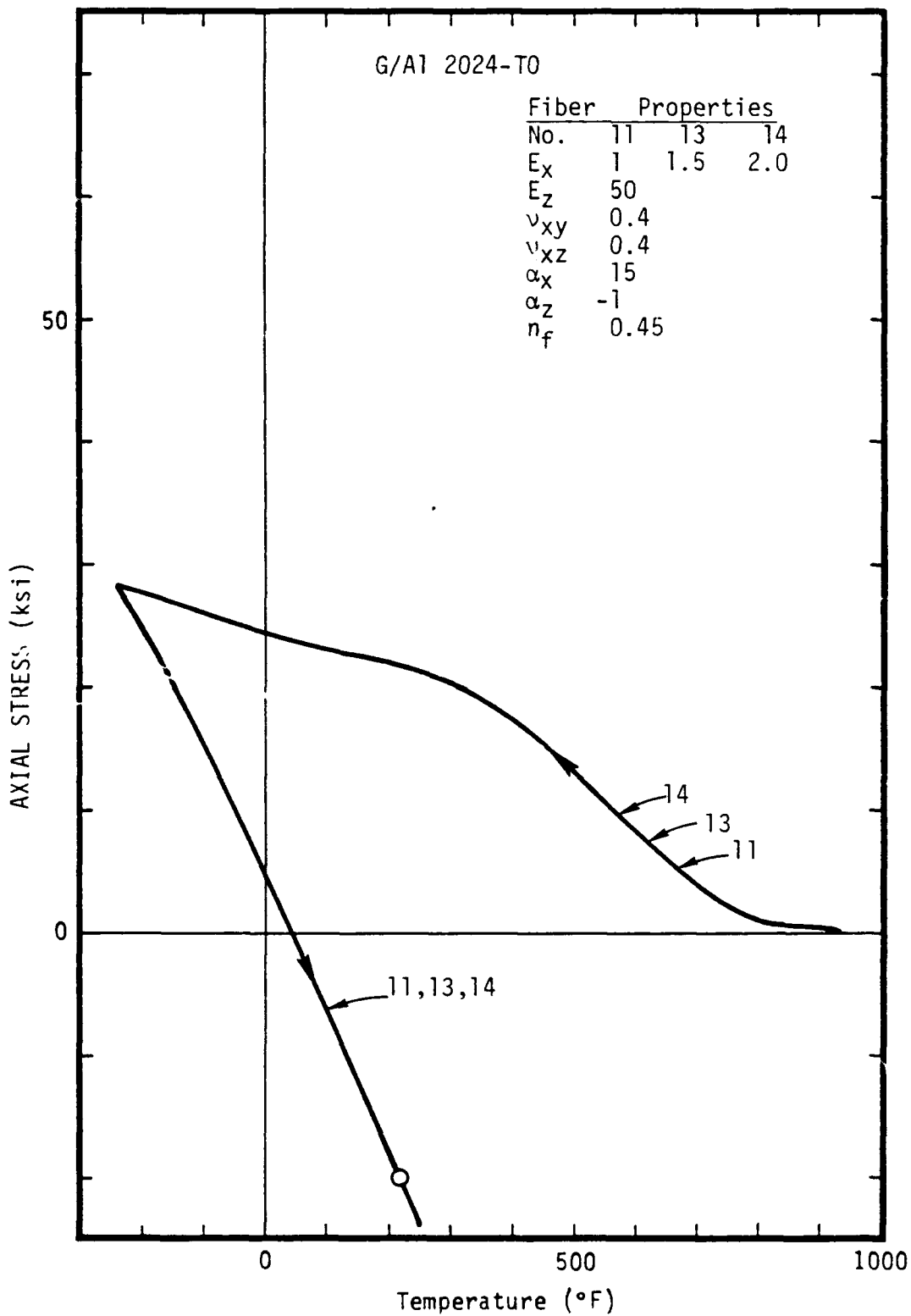


Figure 2.26. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

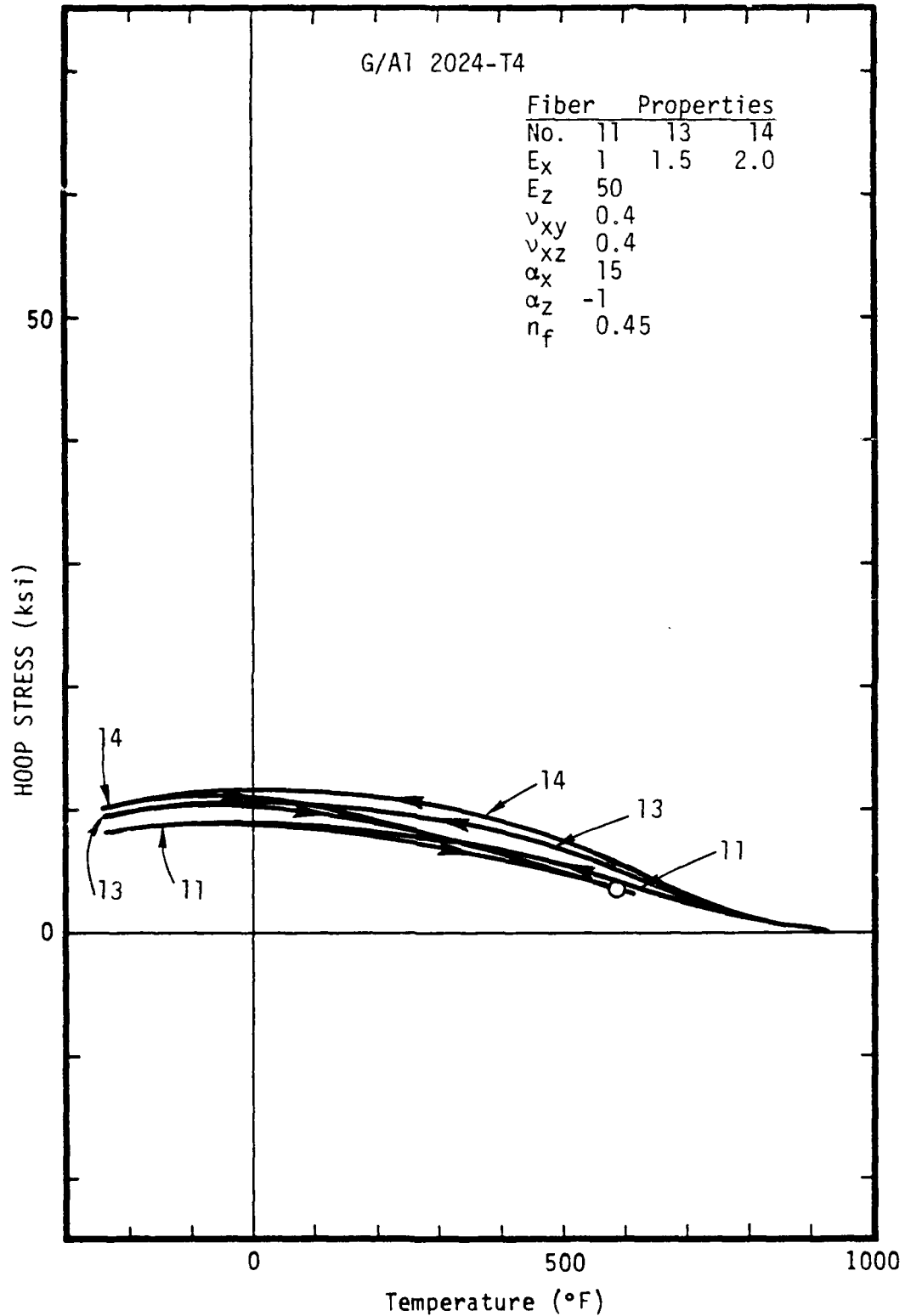


Figure 2.27. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

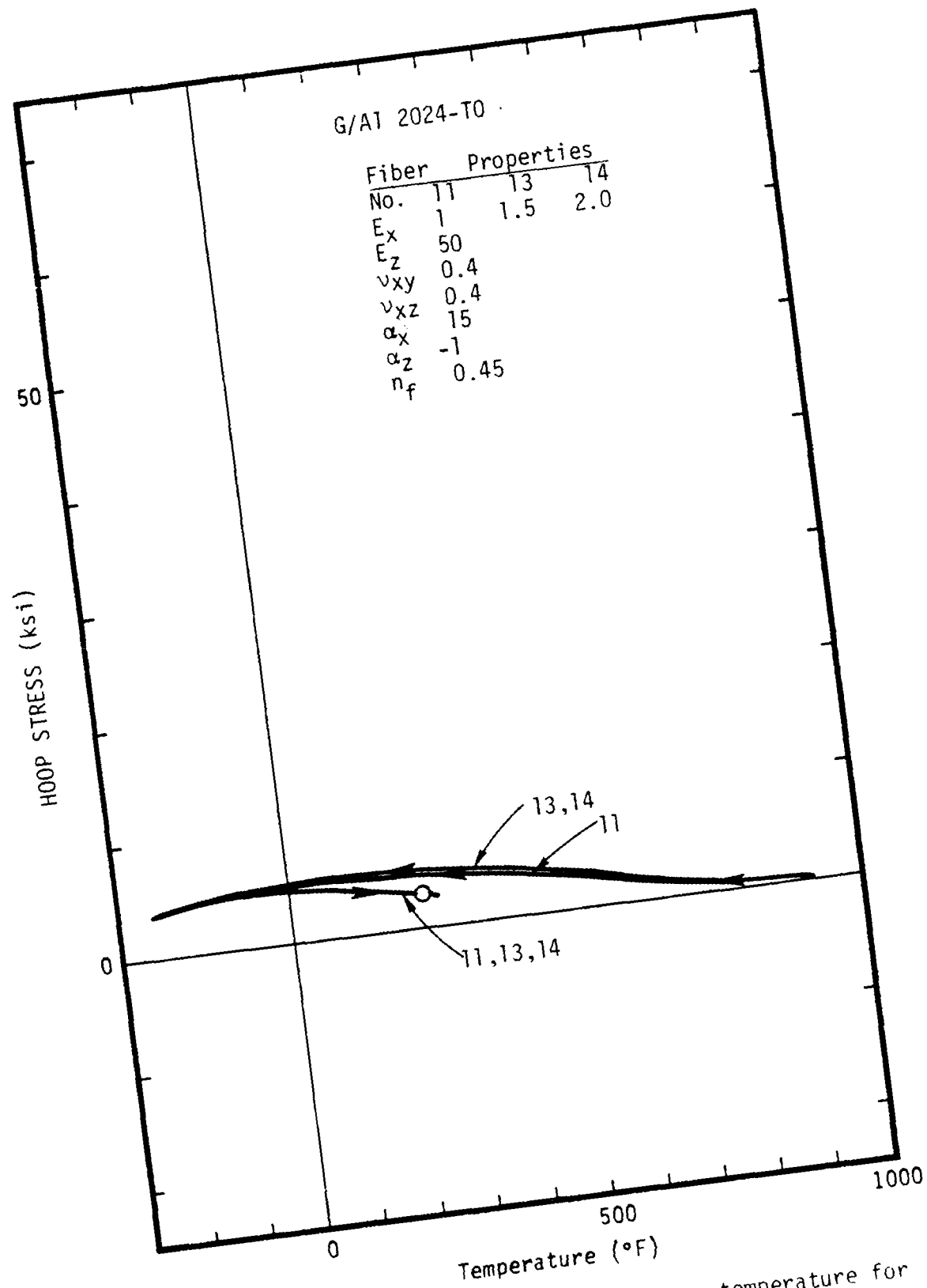


Figure 2.28. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

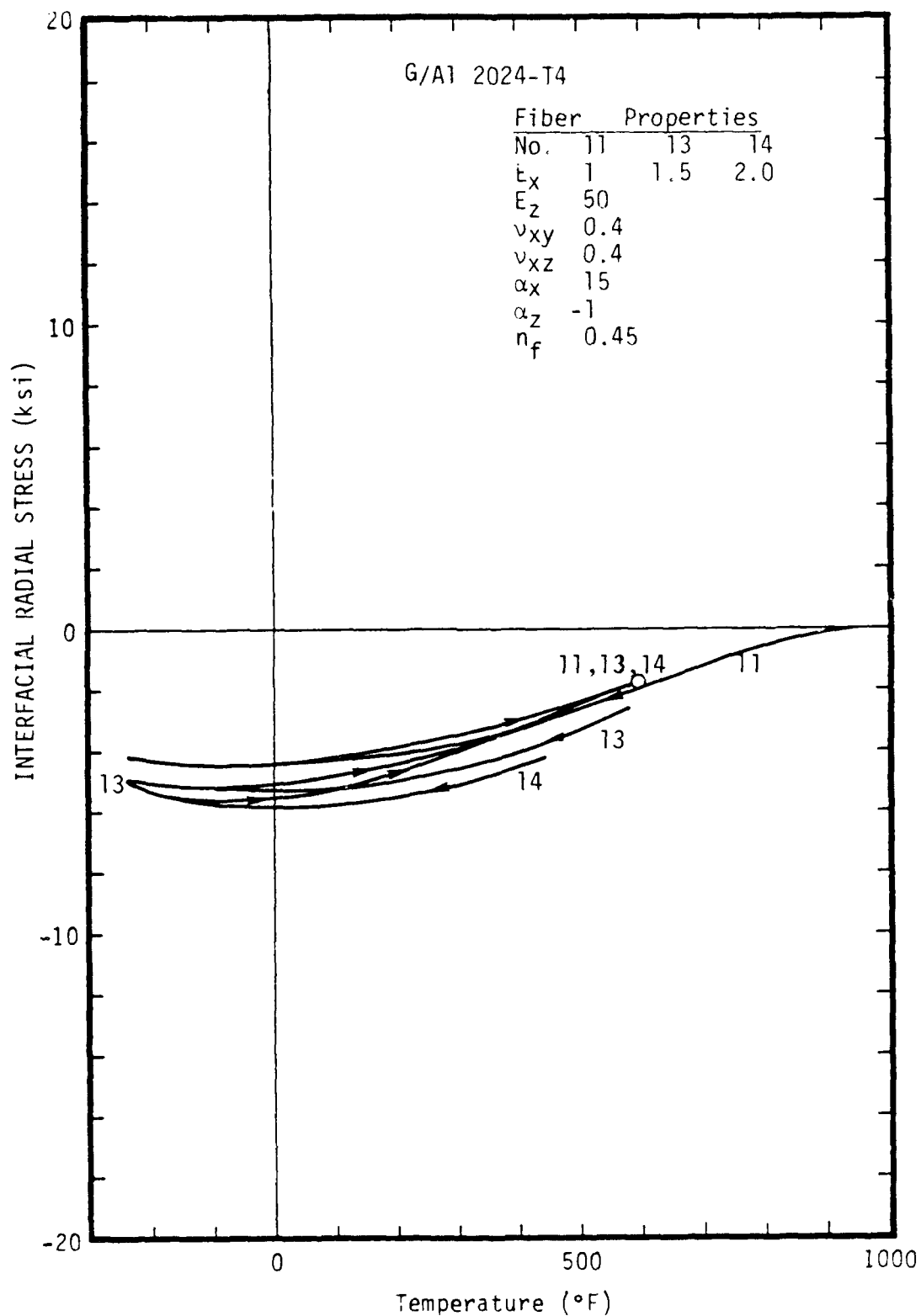


Figure 2.29. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (rapid cooling).

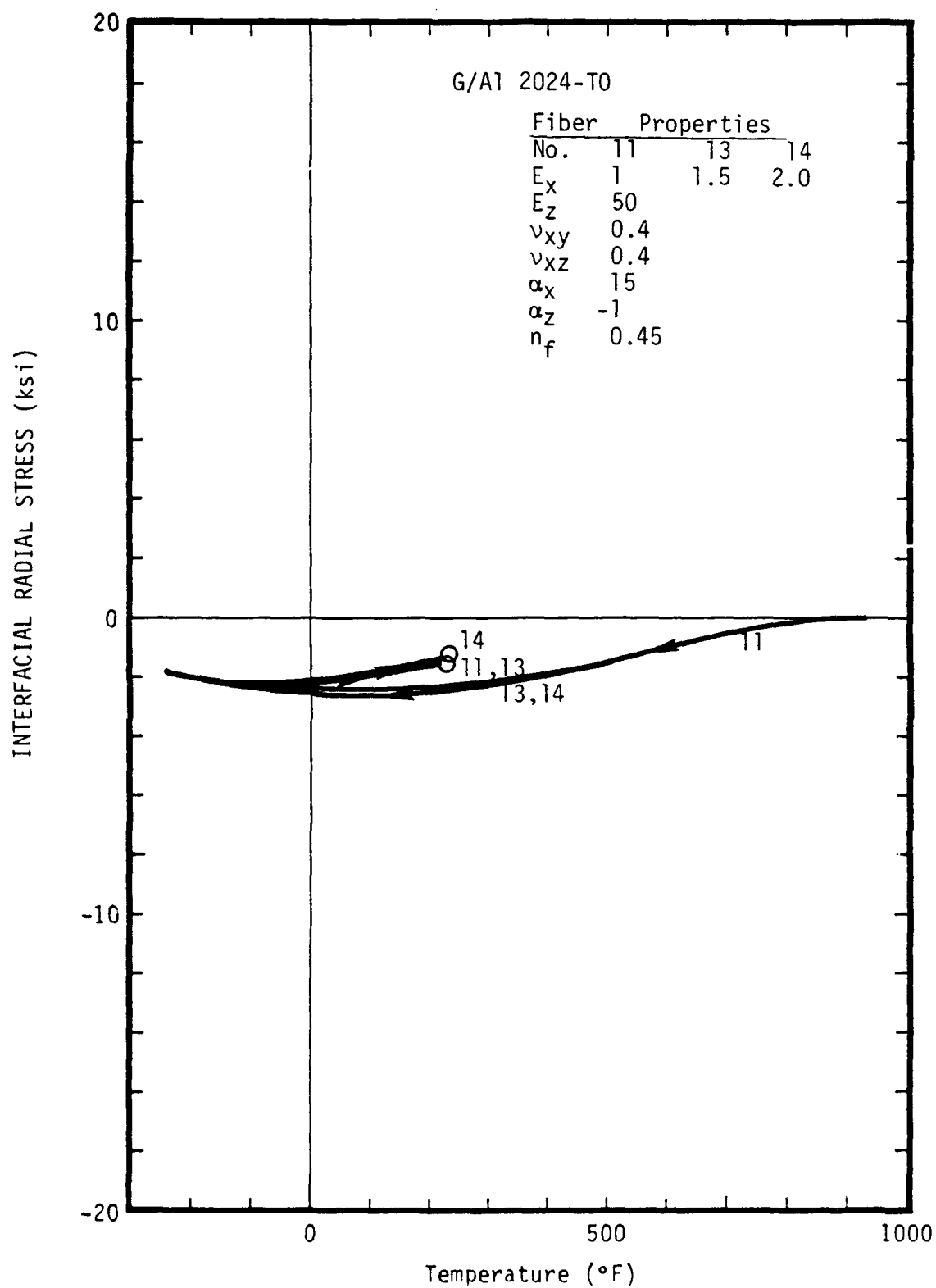


Figure 2.30. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (slow cooling).

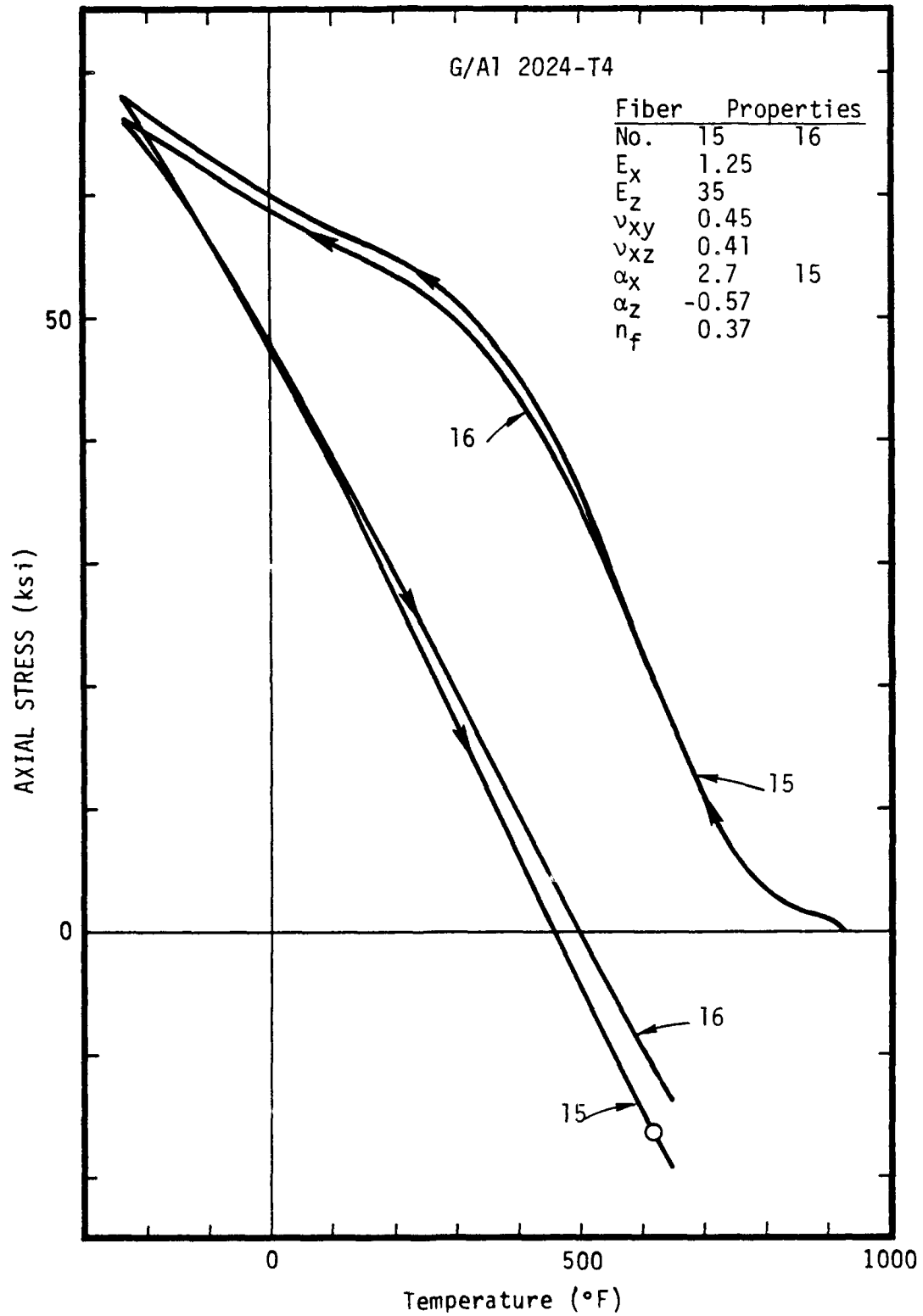


Figure 2.31. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

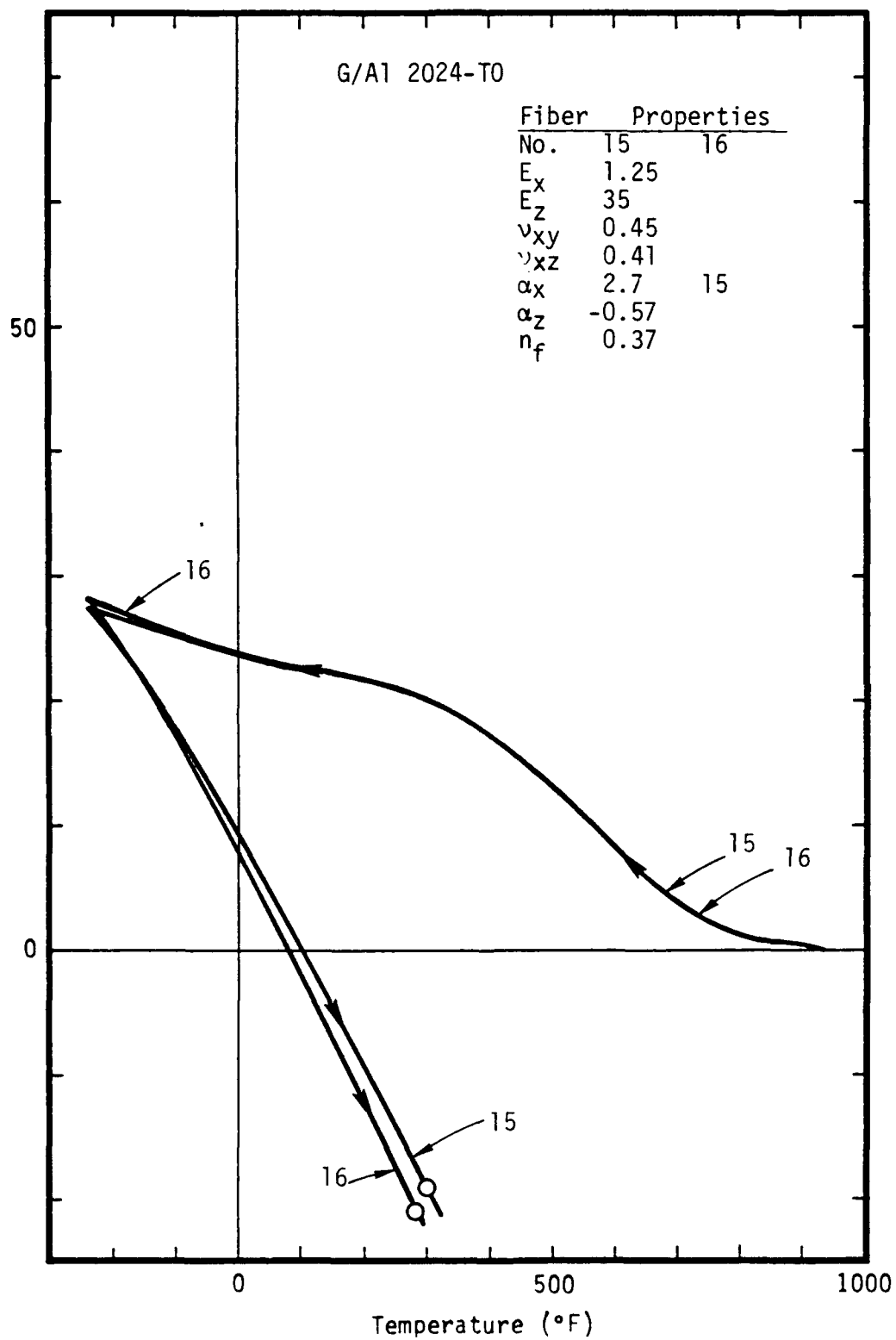


Figure 2.32. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

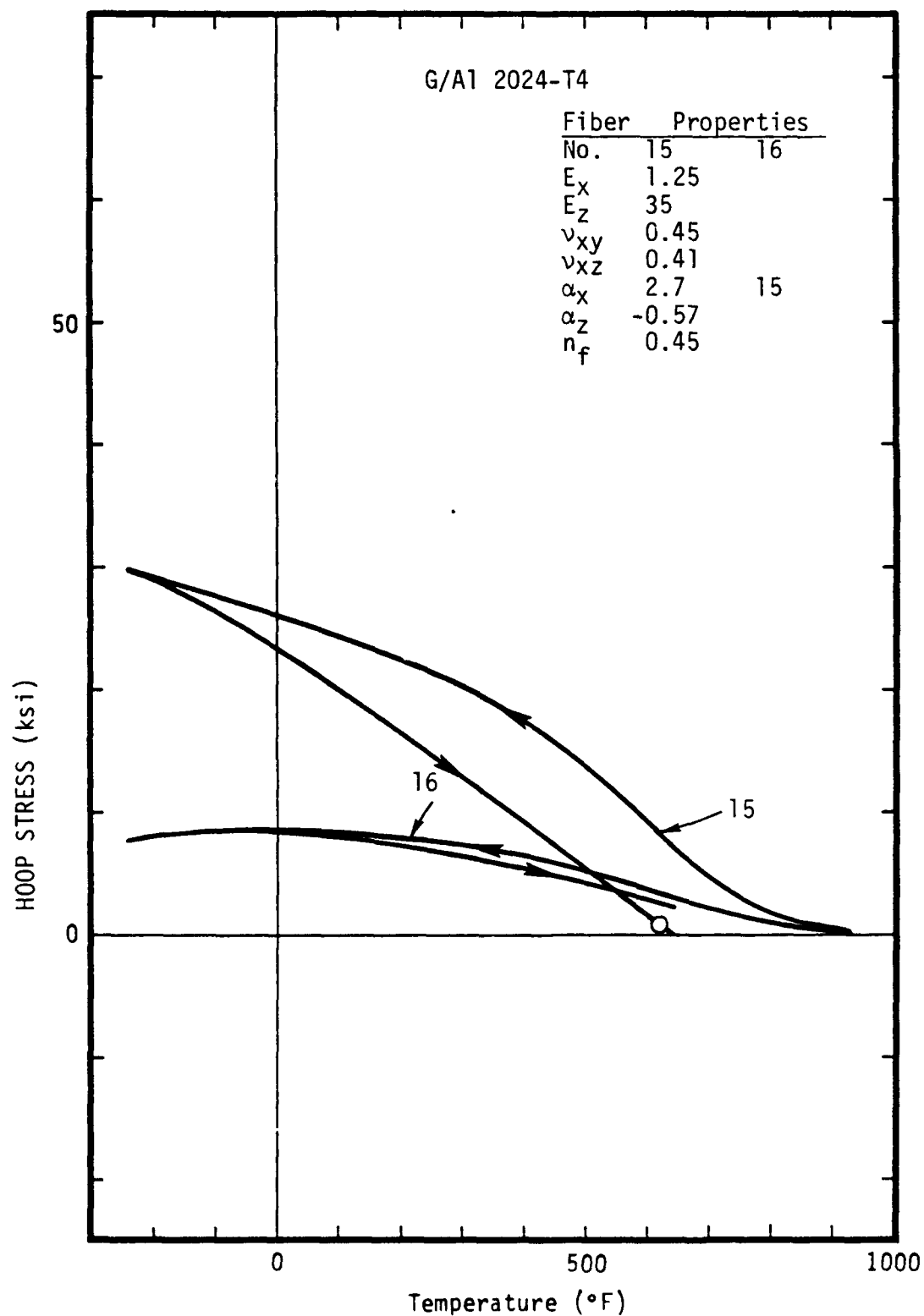


Figure 2.33. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

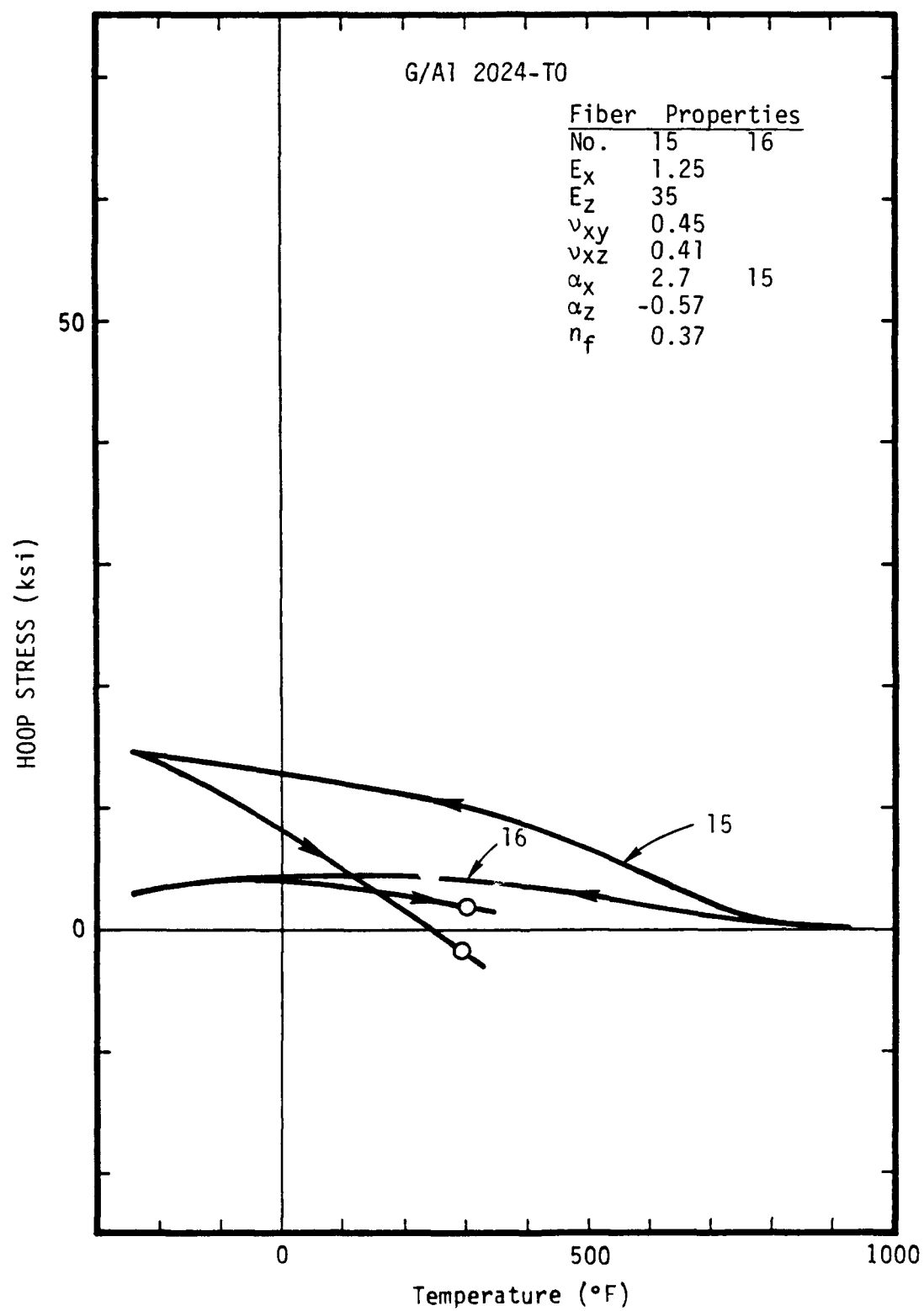


Figure 2.34. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

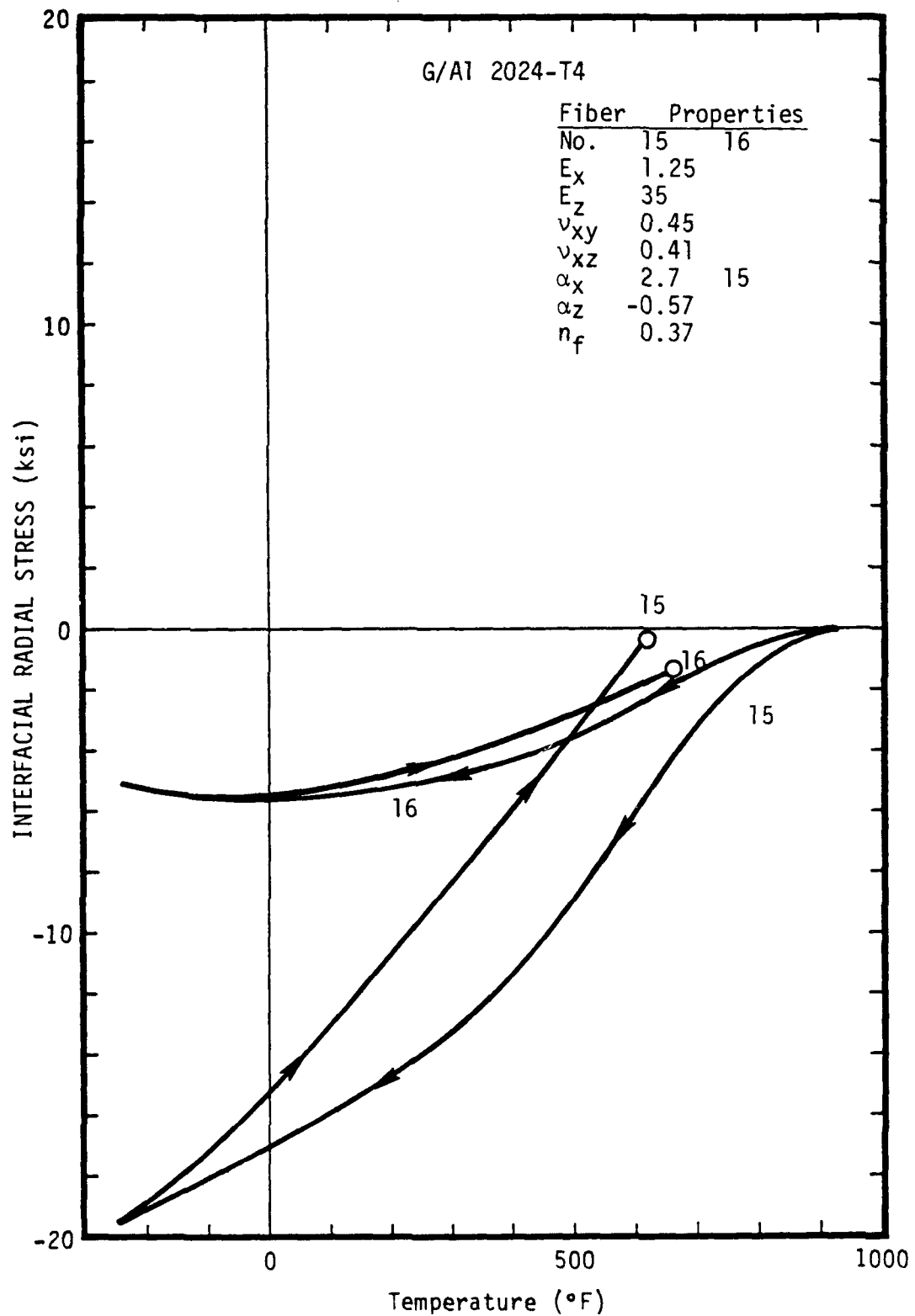


Figure 2.35. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (rapid cooling).

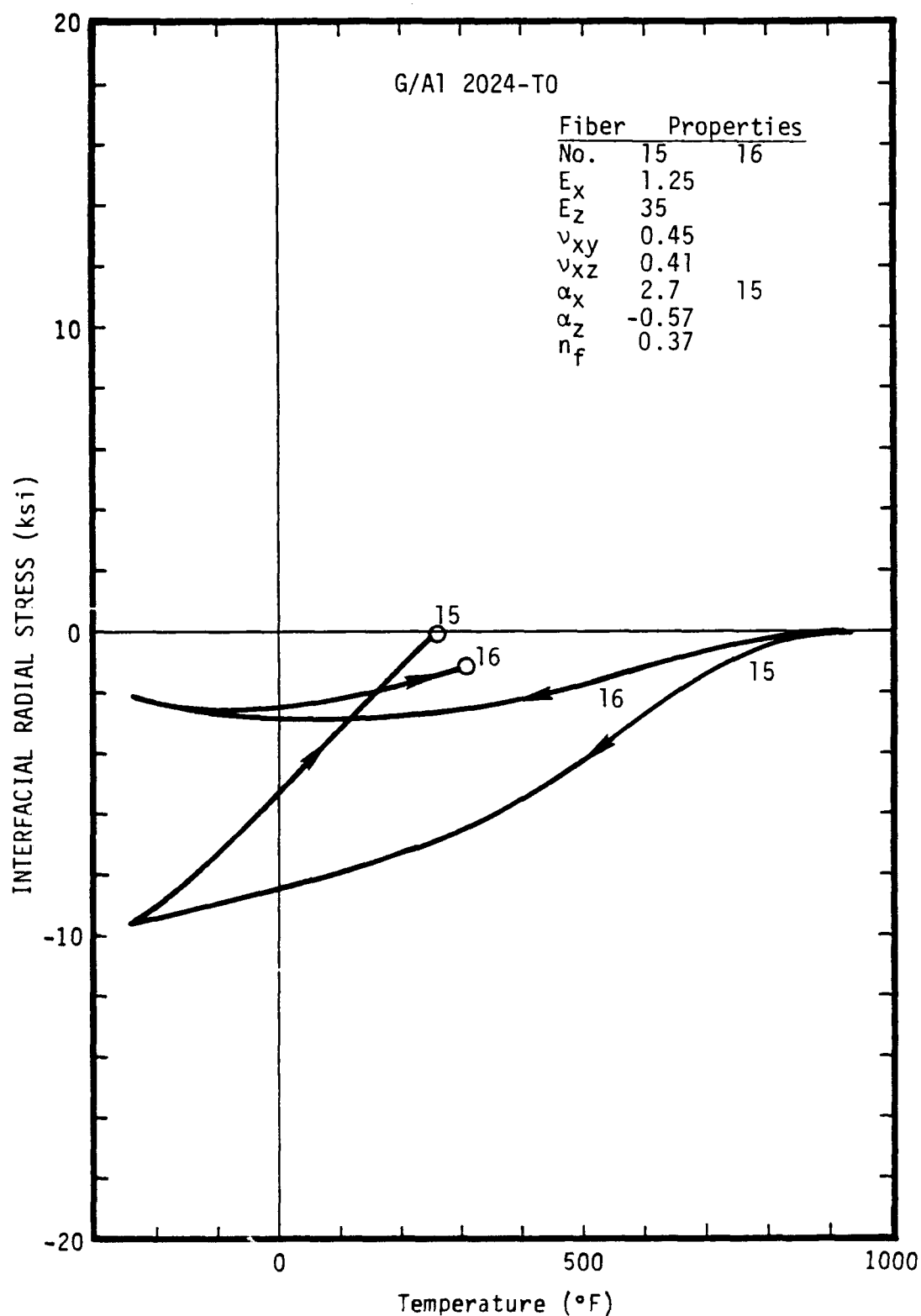


Figure 2.36. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (slow cooling).

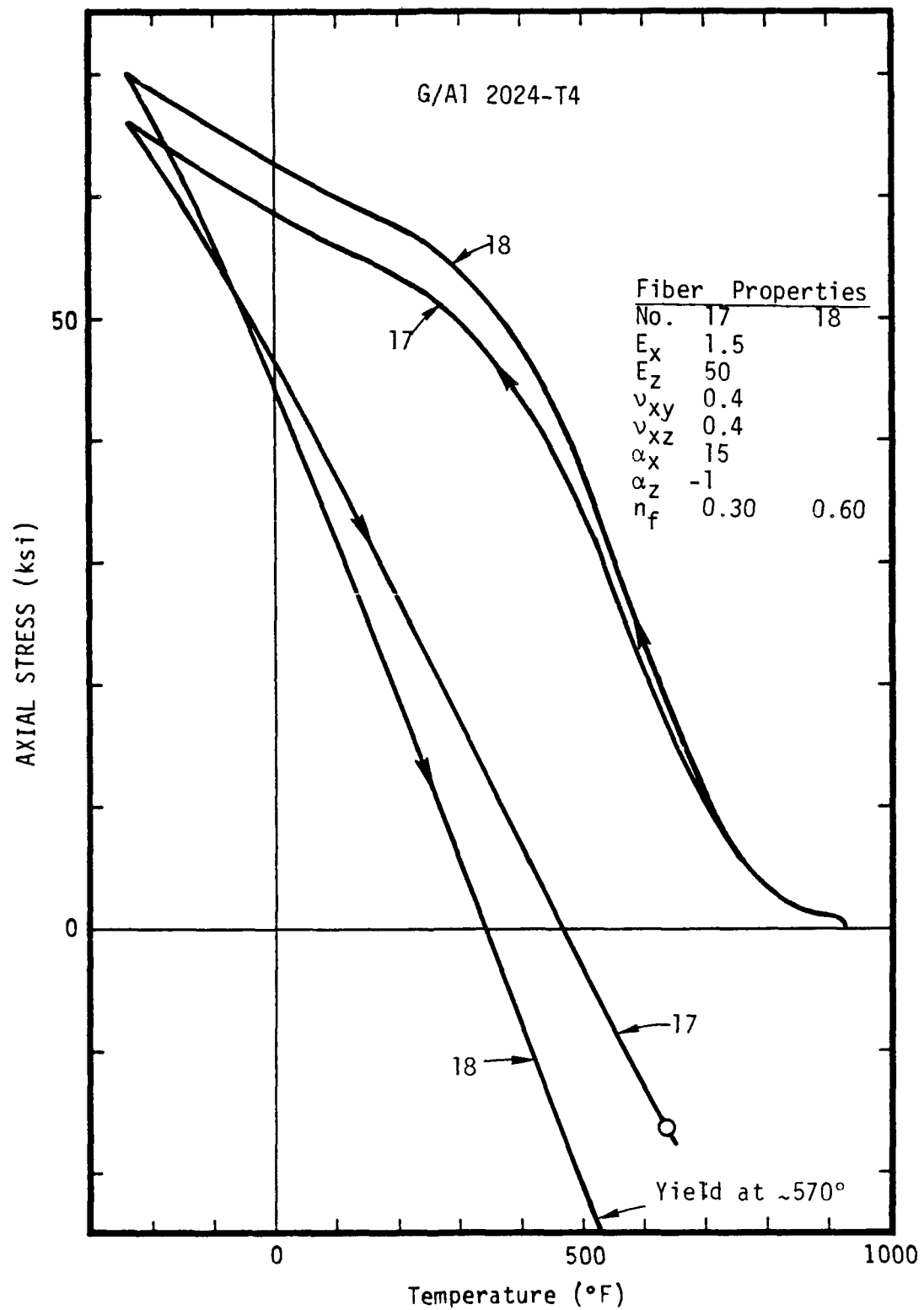


Figure 2.37. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

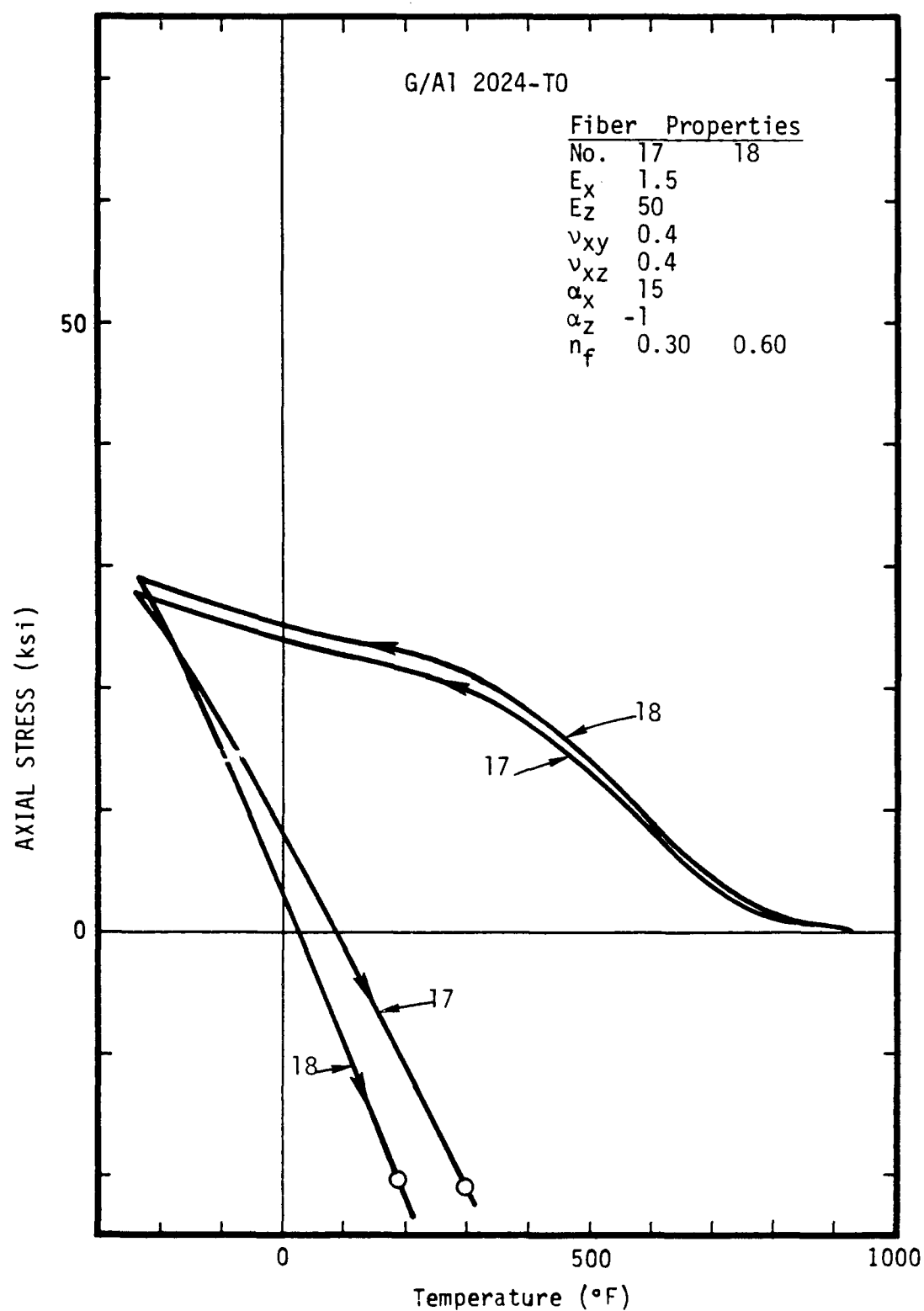


Figure 2.38. Volume-averaged axial stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

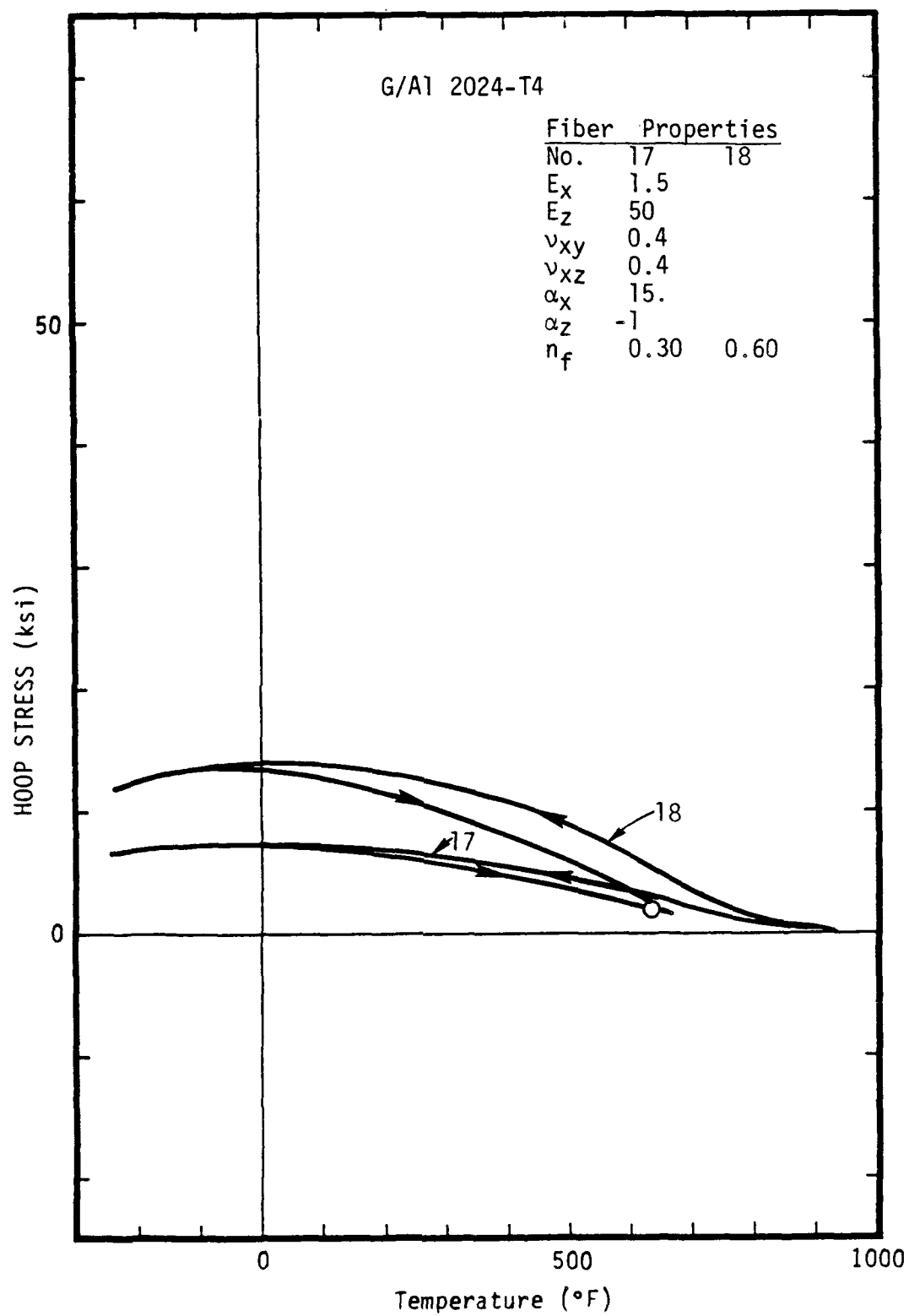


Figure 2.39. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (rapid cooling).

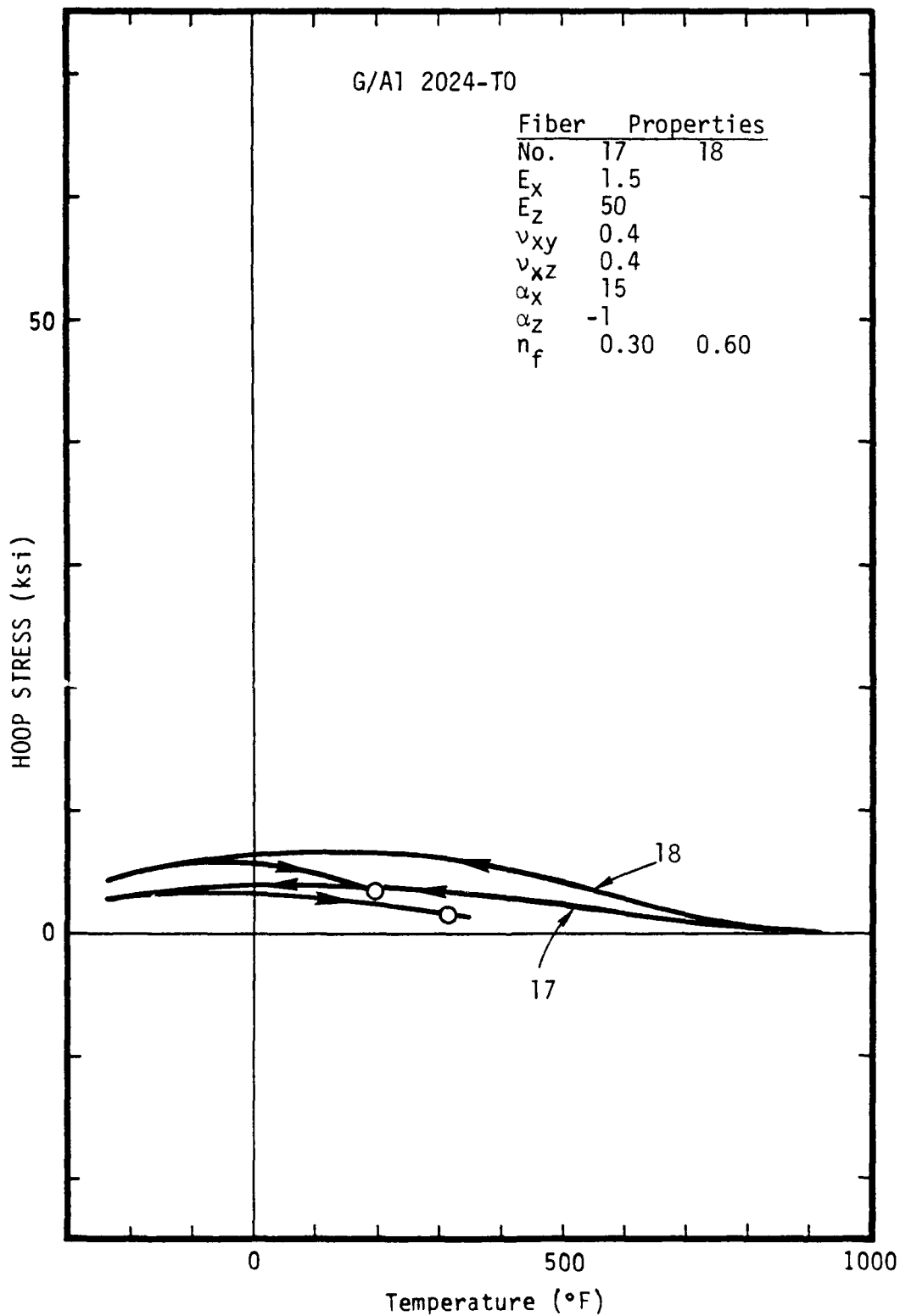


Figure 2.40. Volume-averaged hoop stress versus temperature for the matrix of a graphite/aluminum composite (slow cooling).

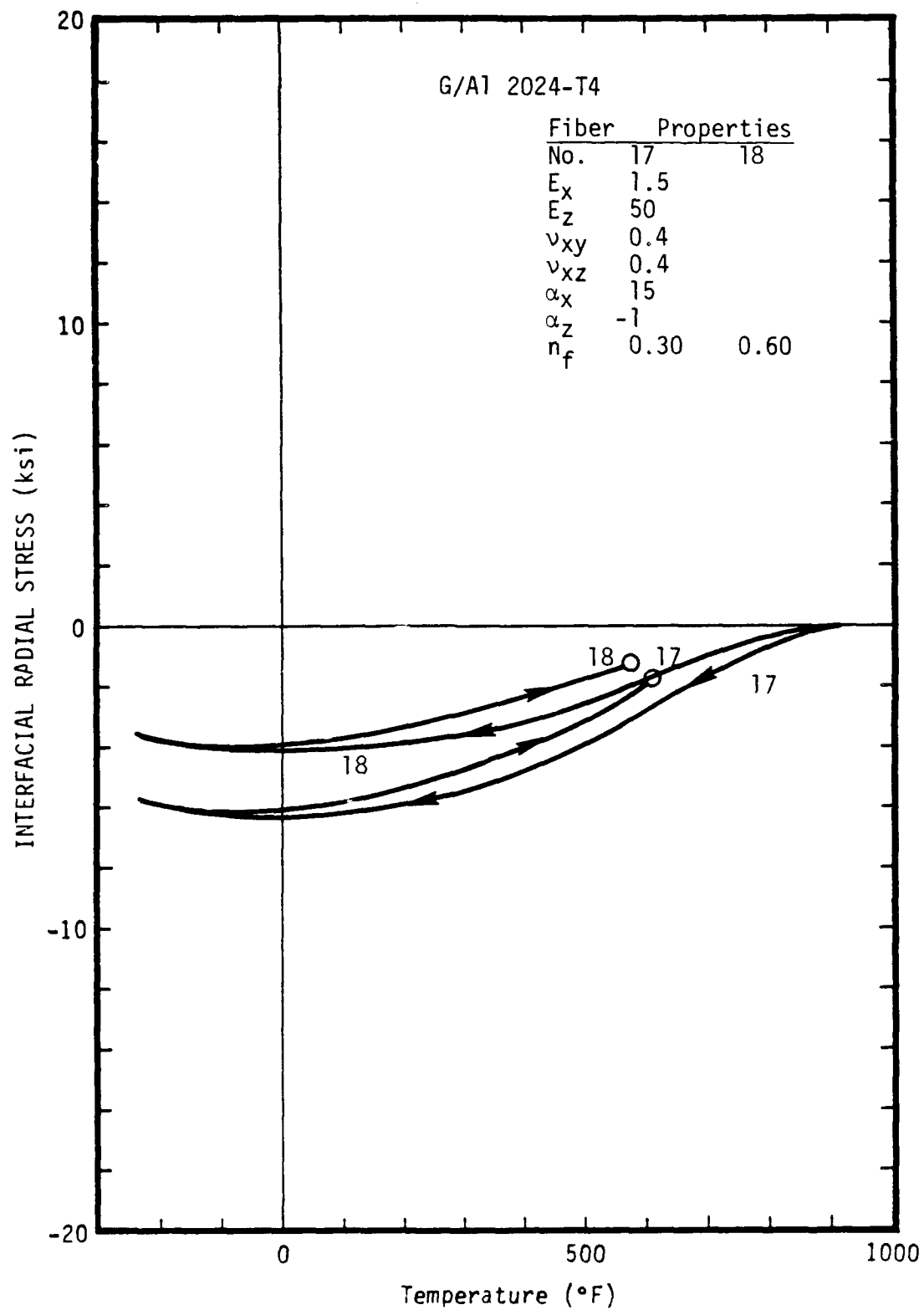


Figure 2.41. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (rapid cooling).

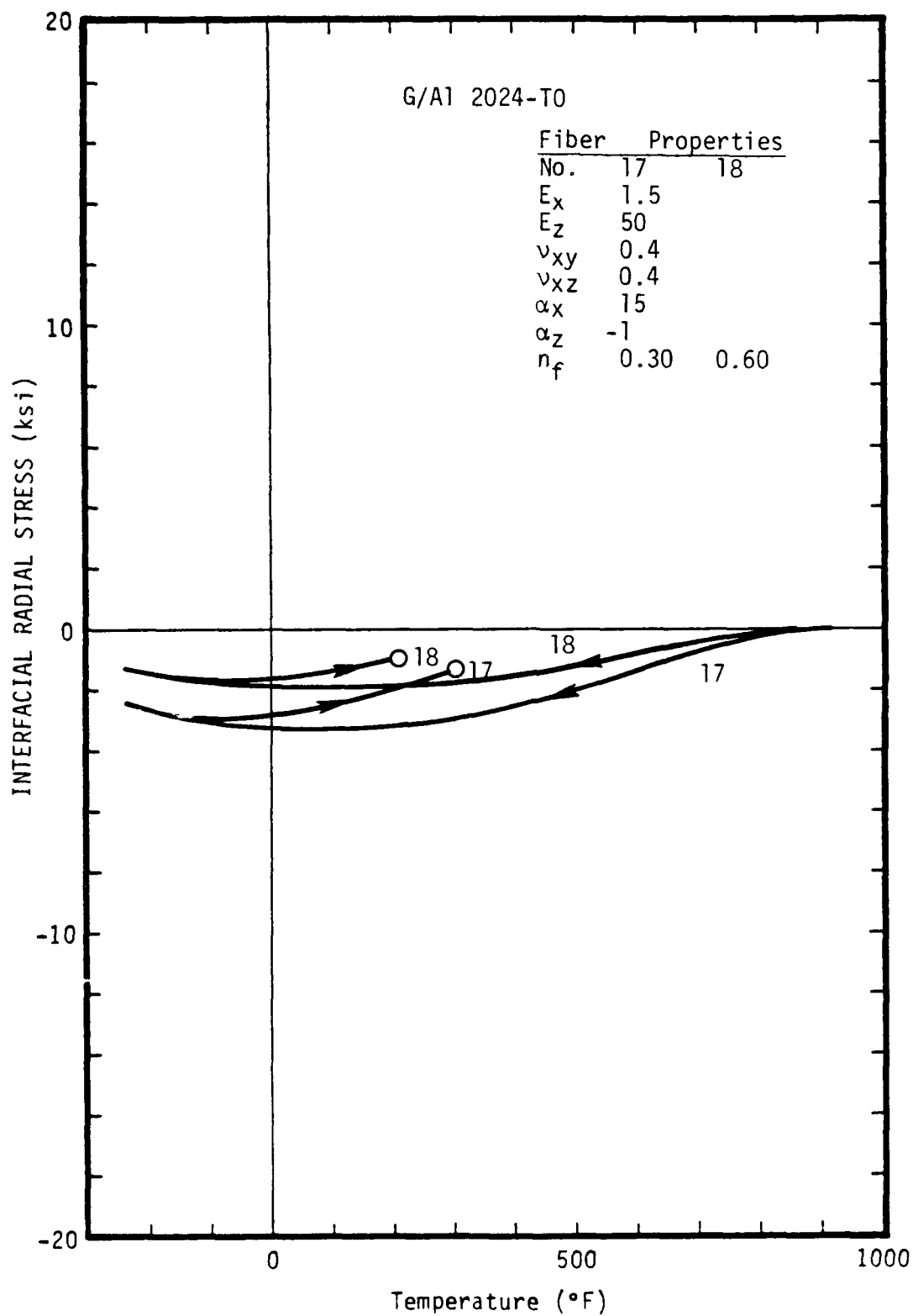


Figure 2.42. Radial stress versus temperature at the fiber-matrix interface of a graphite/aluminum composite (slow cooling).

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3. TUNGSTEN/ALUMINUM RESIDUAL STRESSES

Calculated axial and hoop stresses (volume averaged) in the aluminum matrix are given in Figures 3.1 and 3.2 for a composite which is 45 percent by volume of tungsten fibers and in Figures 3.3 and 3.4 for a 55 percent fiber loading. The temperature cycle is a cooling from a consolidation temperature of 930°F to a minimum temperature of -240°F followed by a reheating. The open circles on the reheating curves mark the temperature at which the stress state in the matrix again reaches the yield surface. As was the case for the graphite/aluminum calculations in Section 2, the matrix reaches the yield surface after only a few degrees of cooling (< 50°F) from the initial value of 930°F. The subsequent increase in stress is due to the temperature and plastic-flow dependence of the yield strength.

The material properties and yield-strength models for the 2024 aluminum are given in Appendix A. The tungsten fibers are assumed to be elastic over the entire temperature range, the following values being used for the tungsten's elastic modulus, Poisson's ratio and linear coefficient of thermal expansion:

$$E = 59 \text{ Msi}$$

$$\nu = 0.28$$

$$\alpha = 2.5 \times 10^{-6} \text{ in/in/}^{\circ}\text{F} .$$

The properties of the composite as calculated with the PRUFC code are given in Table 3.1 below.

TABLE 3.1 CALCULATED THERMOELASTIC PROPERTIES
OF A TUNGSTEN/AL2024 COMPOSITE

n_f	0.45	0.55
E_x (Msi)	17.74	20.69
E_z (Msi)	32.33	37.18
ν_{xy}	0.434	0.428
ν_{xz}	0.305	0.300
$\alpha_z (10^{-6} \text{ } ^\circ\text{F}^{-1})$	8.48	7.34
$\alpha_z (10^{-6} \text{ } ^\circ\text{F}^{-1})$	4.34	3.82
G_{xy} (Msi)	6.19	7.24
G_{xz} (Msi)	7.63	8.98

For the T0 temper (slow cooling from 930°F) the residual axial and hoop stresses are close to zero after reheating to room temperature from -240°F. Because of the low yield strength, however, the matrix will again reach the yield surface at about 150°F. For the T4 temper (rapid quench), the higher yield strength allows the matrix to remain elastic up to about 500°F. The high value of the axial residual stress at room temperature, however, would be undesirable if the composite is to be subjected to large tensile loads.

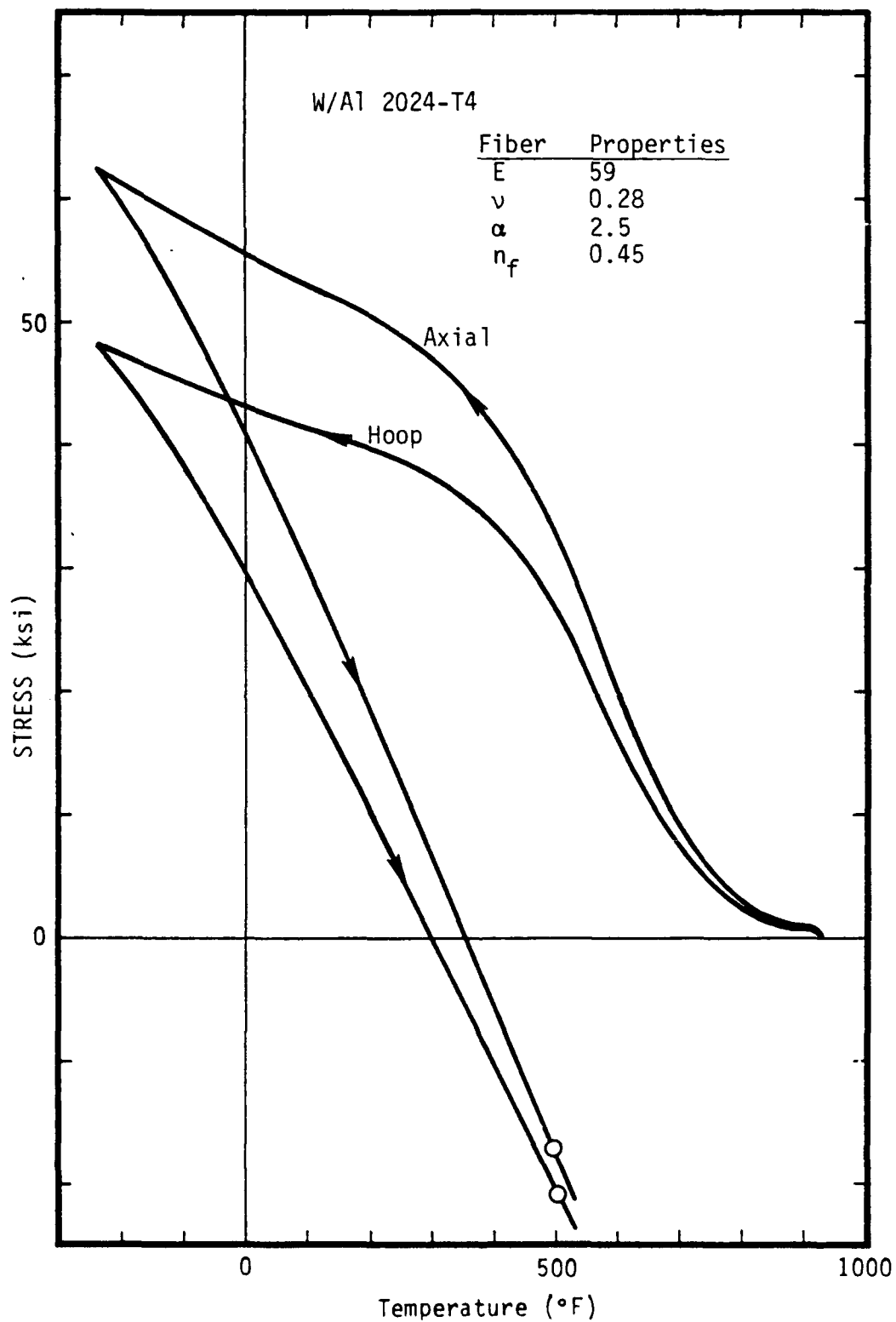


Figure 3.1. Volume-averaged axial and hoop stresses versus temperature for the matrix of a tungsten/aluminum composite (rapid cooling).

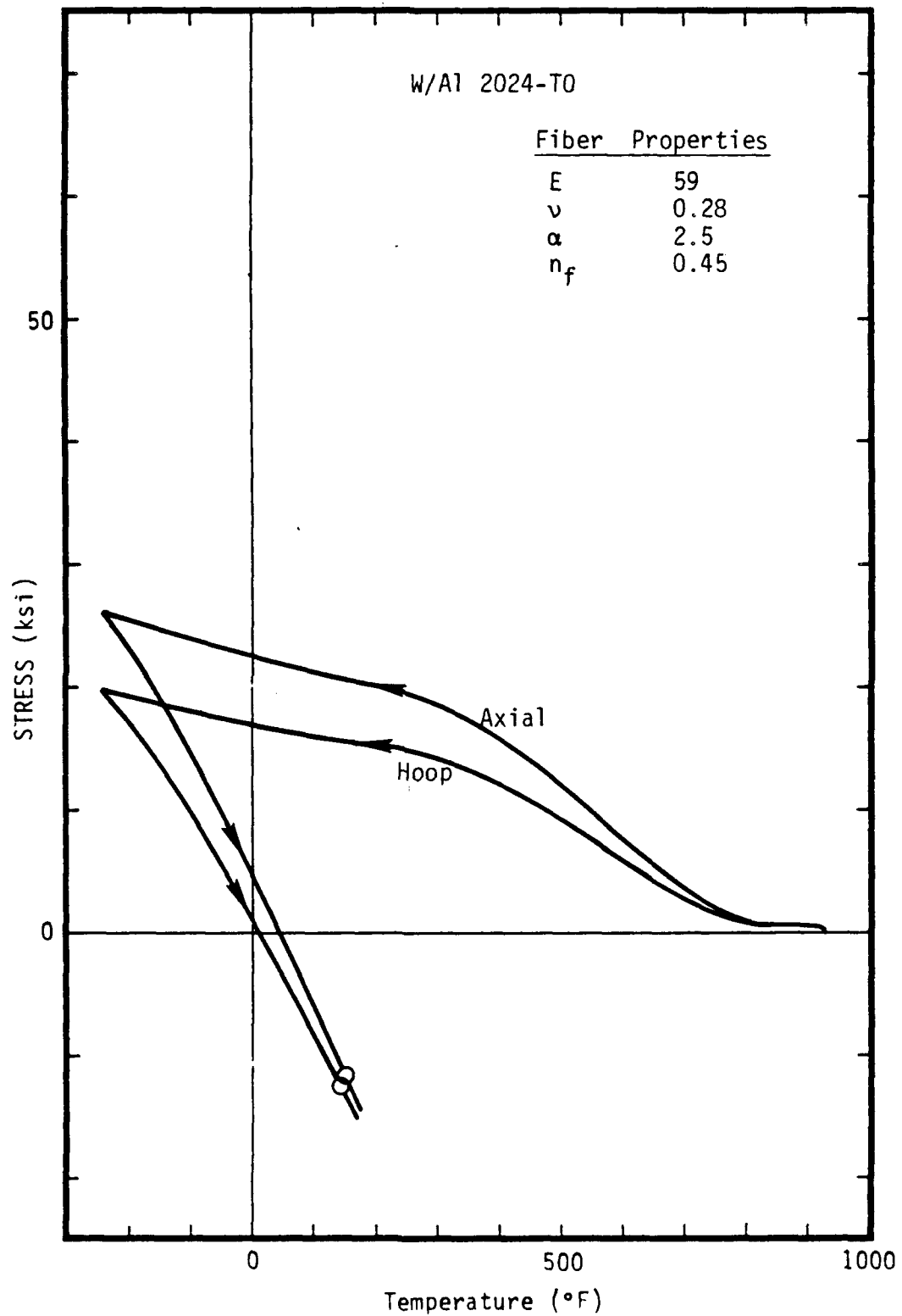


Figure 3.2. Volume-averaged axial and hoop stresses versus temperature for the matrix of a tungsten/aluminum composite (slow cooling).

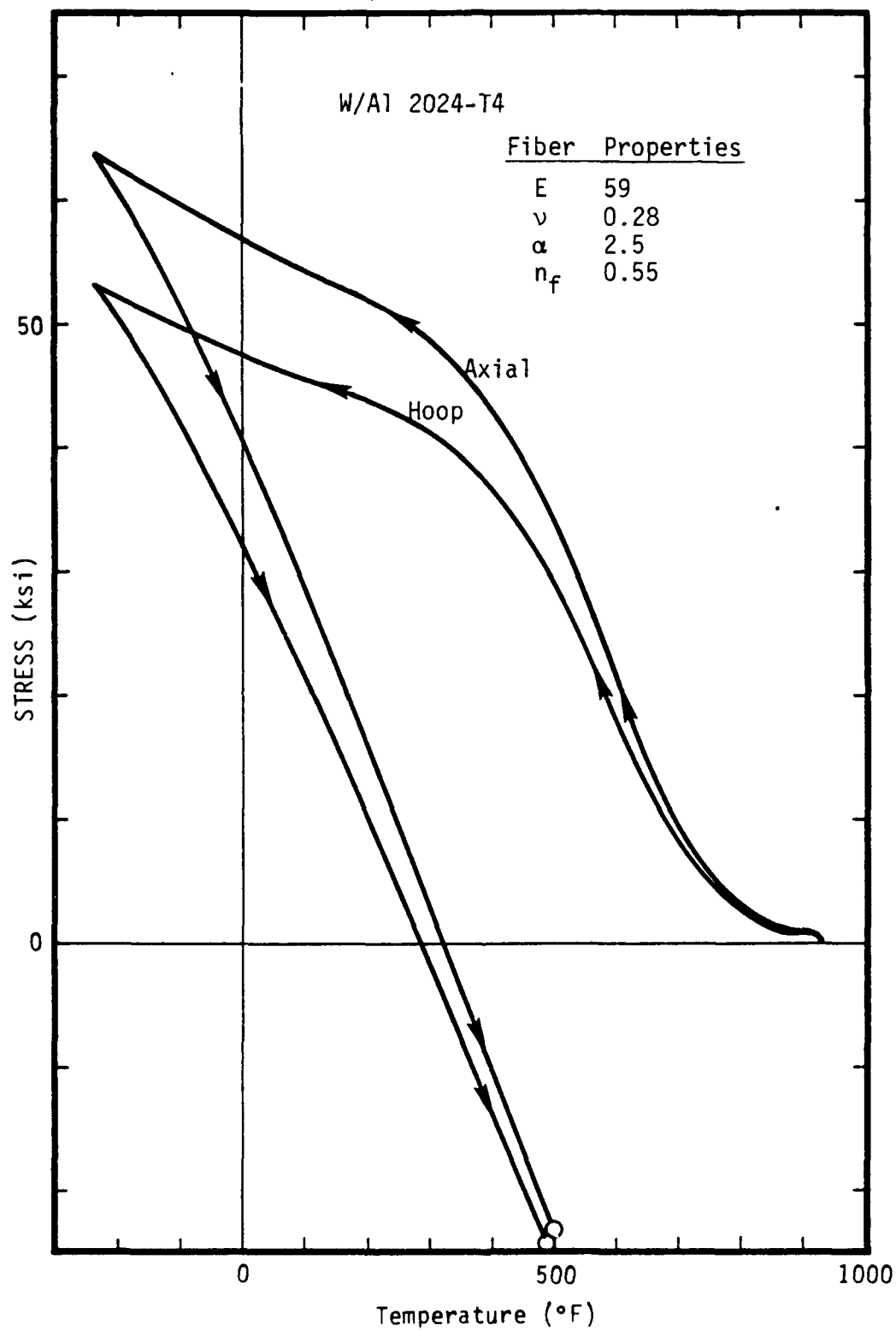


Figure 3.3. Volume-averaged axial and hoop stresses versus temperature for the matrix of a tungsten/aluminum composite (rapid cooling).

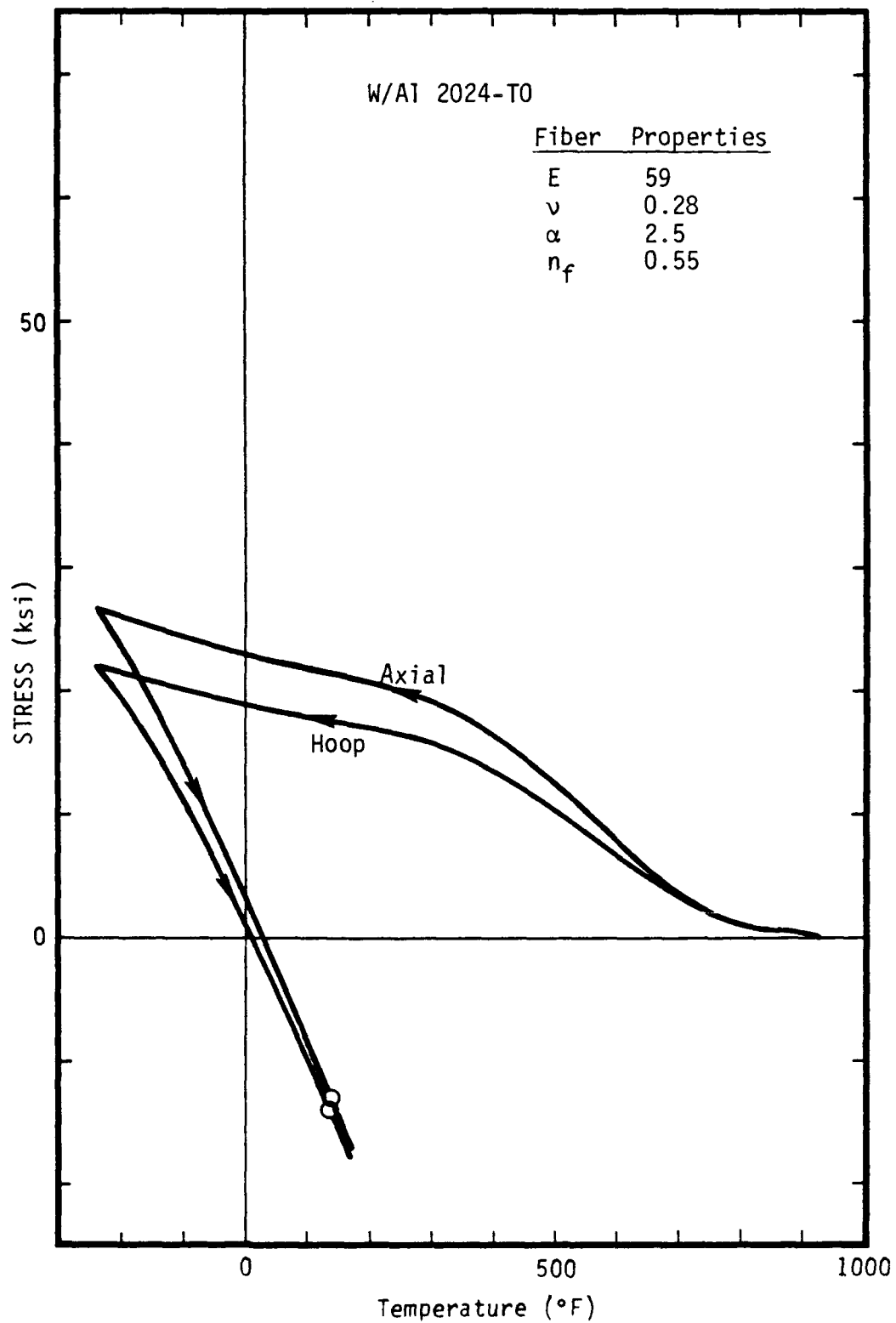


Figure 3.4. Volume-averaged axial and hoop stresses versus temperature for the matrix of a tungsten/aluminum composite (slow cooling).

4. SILICON CARBIDE/ALUMINUM RESIDUAL STRESSES

Several calculations have been done to determine the magnitudes of the residual stresses to be expected in a SiC/Al composite as a result of cooling from the consolidation to ambient temperature. The effect of these stresses on the elastic properties of unidirectional panels and cross-ply laminates is also considered. The results of this section have been submitted previously as an Interim Progress Report to the Naval Research Laboratory (Ref.4).

4.1 FIBER PROPERTIES

For bulk SiC, which we assume to be isotropic, the elastic (Young's) modulus and shear modulus were taken as (Ref. 5):

$$E = 68 \text{ Msi}$$

$$G = 27 \text{ Msi}$$

The coefficient of thermal expansion was assumed to be

$$\alpha = 2.42 \times 10^{-6} \text{ in/in/}^{\circ}\text{F}^{-1}$$

which, from Ref 5, is appropriate for the temperature range considered here.

The SiC fibers, however, are not completely bulk material, but apparently contain a core of essentially zero strength material (carbon) whose diameter is about 1.3×10^{-3} in. as compared with an outer fiber diameter of about 5.6×10^{-3} in. The volume fraction of the carbon core is thus 0.054. The overall properties of a fiber were determined with the PRUFC Code (Ref. 3), in which the sheath of the concentric cylinder model was assigned properties appropriate to the bulk SiC, and the core was assigned elastic moduli a factor of 10^{-3} smaller (hollow cylinder approximation). The resulting calculated properties of the fiber, which now becomes anisotropic-transversely isotropic, are:

$$E_A = 64.33 \text{ Msi}$$

$$E_T = 60.56 \text{ Msi}$$

$$\nu_{TT} = 0.2400$$

$$\nu_{TA} = 0.2600$$

$$G_{AT} = 24.25 \text{ Msi}$$

$$\alpha_A = \alpha_T = 2.42 \times 10^{-6} \text{ } ^\circ\text{F}^{-1} ,$$

where the Poisson's ratio ν_{TT} refers to the transverse contraction for an applied transverse stress, and ν_{TA} refers to the transverse contraction for an applied axial stress. As expected, because of the small volume fraction of the core, the net fiber properties differ only slightly from those of the bulk SiC.

The material properties and yield-strength models for the aluminum 6061 matrix are given in Appendix A.

4.2 RESIDUAL STRESS RESULTS

Average axial stresses in the aluminum matrix of a SiC/Al composite are shown in Figure 4.1. Results are given for fiber volume fractions of 0.35 and 0.50 and for both the T0 and T4 aluminum tempers. The assumed thermal cycle process is a cooling from an initially stress-free state at 700°F to room temperature and then reheating.

As mentioned previously, the code uses a concentric-cylinder approximation to model the composite. For initial cooling from the stress-free state, both the inner cylinder (fiber) and the outer (matrix) are elastic. As cooling continues, the matrix material will in general reach its yield point, after which the Reuss equations for plastic flow (see, for example, Ref. 6) are integrated numerically to determine the distribution of stress in the outer (matrix) cylinder. In the code's present form the fiber is treated as linearly elastic over the entire temperature range.

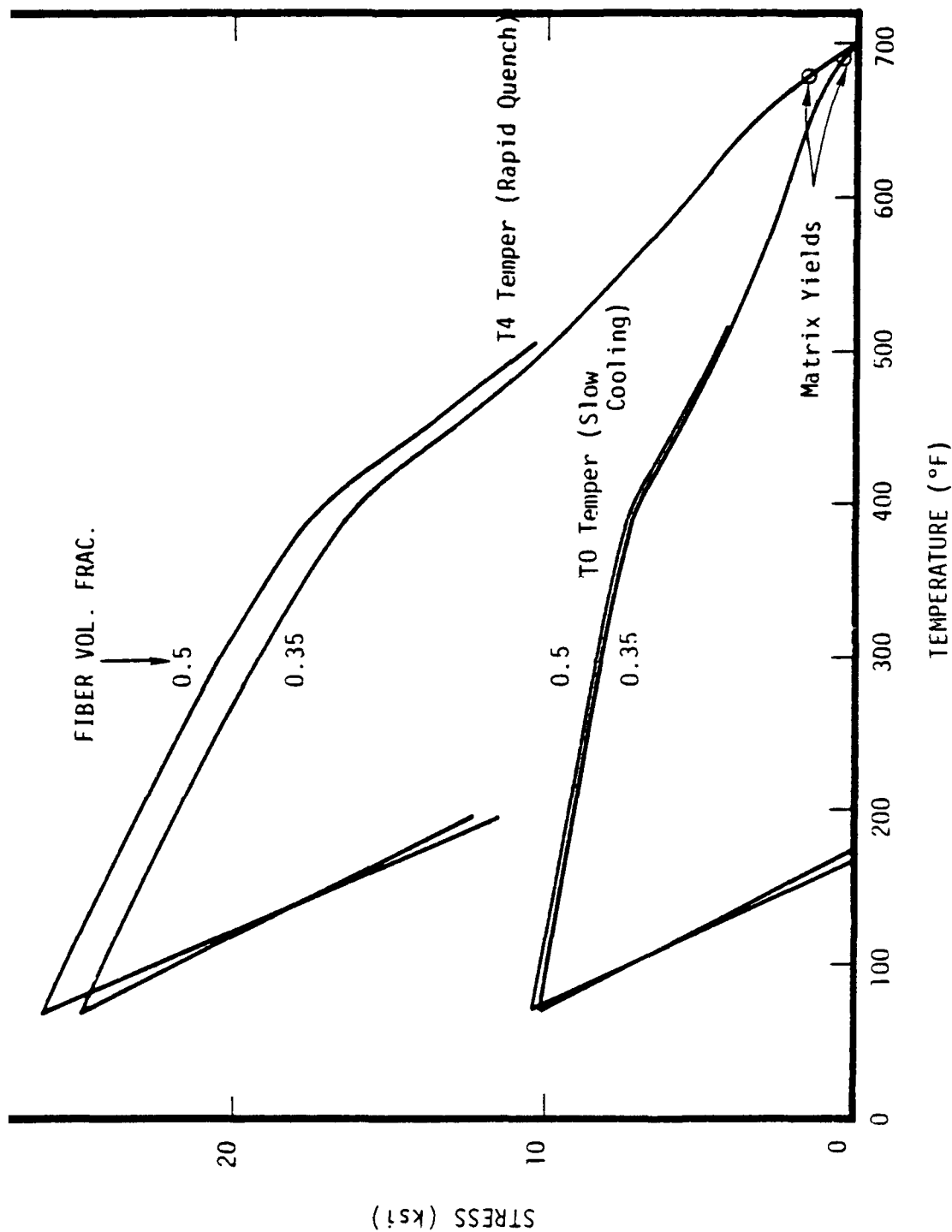


Figure 4.1. Calculated average axial stresses in the matrix of a SiC/Al 6061 composite as a function of temperature for different fiber volume fractions and matrix temperatures. The assumed thermal cycle is a cooling to 70°F from a consolidation temperature of 700°F, and then a reheating from 70°F to indicate the rate of decrease of the residual stress.

For the cases depicted in Figure 4.1, about 20 degrees or less of cooling from the initial temperature of 700°F is sufficient to bring the aluminum matrix to its yield surface. The additional increase in axial stress for further cooling results from two causes: the increase of the matrix yield stress itself as a function of decreasing temperature, and a further increase due to plastic-flow work hardening. At the lowest temperature (70°F) the axial stress is a maximum, and its magnitude depends strongly on the effective temper of the aluminum. The calculation suggests that the maximum residual stress due to rapid quenching (T4 temper) would be over twice that of the slow-cooled or annealed state. If the composite is reheated from room temperature, the matrix material will fall off the yield surface, and as shown in Figure 4.1, the rate of decrease of stress in the elastic regime will be quite rapid. For the annealed case, a temperature increase of about 100°F would be sufficient to reduce the residual stresses to approximately zero.

It is anticipated that A357 aluminum casting alloy will also be used in the fabrication of these composites. No residual stress calculations for this material have been done as yet because we have been unable to locate yield strength information for the T0 and T4 tempers. No large differences with respect to the 6061 matrix are expected however, since the yield strength of the A357 alloy in the T6 temper is comparable to that of 6061-T6, and other mechanical properties also are not significantly different.

4.3 ELASTIC PROPERTIES OF A UNIAXIALLY REINFORCED LAMINATE

Since the coefficient of thermal expansion for the aluminum matrix is greater than that of the SiC fibers, the residual stresses in the matrix after cool-down will be tensile in the axial direction. (The calculated hoop stresses in the concentric cylinder model are also tensile, but the radial stress is compressive, so that there should be little tendency for debonding to occur between the matrix and fibers.) If the composite is not subjected to an

overall tensile axial stress, the matrix will remain on the yield surface and the effective axial elastic modulus will be lower than it would be in the absence of residual stresses.

The plastic-flow code as described in Ref. 1 has been augmented so that it is now possible to calculate the stresses as a function of axial strain following the cool-down procedure. The calculated overall axial stress as a function of axial strain at 70°F is plotted in Figure 4.2 for the 6061-T0 yield model after cooling from 700°F. Also shown is the straight-line stress-strain curve for an elastic matrix as calculated with the PRUFC code, and an experimental stress-strain curve obtained from AVCO (Ref. 7), the volume fraction of fibers in this case being 0.45. The experimental curve changes slope at about 90 ksi, and the slope of the higher-stress segment is in good agreement with that of the curve corresponding to the yielded matrix. The lower-stress portion of the experimental curve however, is in agreement with the stress-strain curve as calculated using an elastic matrix.

The overall axial stress for the composite is related to that in the fibers and matrix by

$$\sigma_z = n_f \sigma_{zf} + (1 - n_f) \sigma_{zm} .$$

For this material one has, to good approximation,

$$\sigma_{zf} = \frac{E_f}{E_m} \sigma_{zm}$$

for the fiber stress in terms of that in the matrix, where E_f and E_m are the respective fiber and matrix axial elastic moduli. For an elastic matrix of modulus $E_m = 10$ Msi, and an overall composite axial stress of 90 ksi, the matrix would experience a stress of 26 ksi, a value close to the nominal yield stress of 6061 aluminum in the T4 temper. On the other hand, if the break in the experimental curve at 90 ksi is to be interpreted as the transition point from an

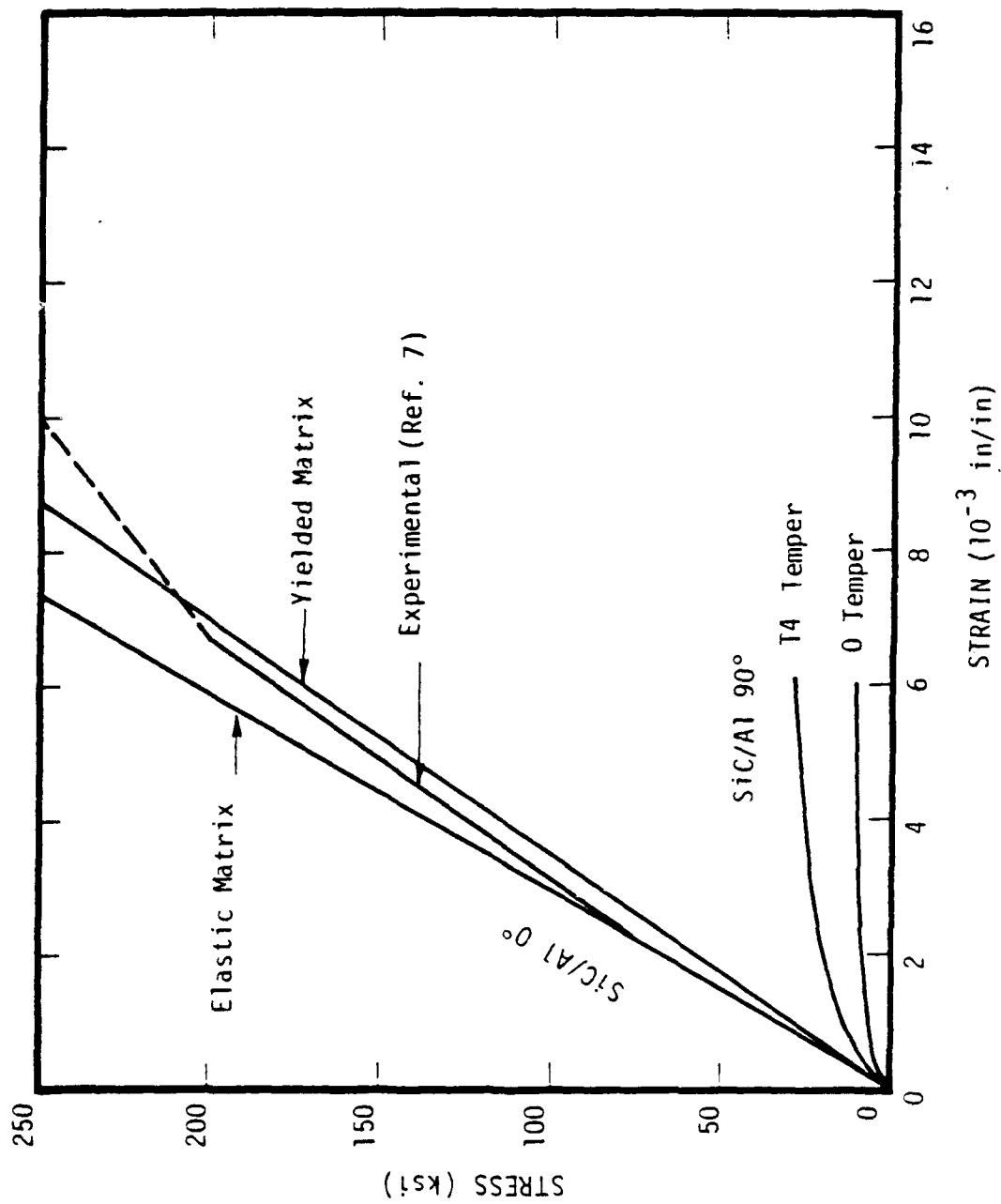


Figure 4.2. Comparison of calculated and experimental axial stress-strain curves for a 0.45 fiber volume-fraction SiC/Al 6061 composite. The curve for the yielded matrix corresponds to the T0 temper model of Figure 4.1.

elastic to a plastic matrix, then the initial state of the matrix would have to be essentially stress-free, in marked disagreement with the calculated axial residual stress as shown in Figure 4.1.

The slope of the calculated stress-strain curve corresponding to a yielded matrix is 29.3 Msi, which is close to the value

$$n_f E_f = (0.45)(64.33) = 28.9 \text{ Msi} ,$$

which one would expect for the composite elastic modulus if the matrix modulus were zero. This result is to be expected, since at the yield surface the effective modulus of the matrix is approximately equal to the local slope of the yield-strength versus strain curve, which typically ranges from several hundred ksi to about 1 Msi (see Figures A.3 and A.4 of Appendix A). Since this slope for either the T0 or T4 temper is small as compared with the 64 Msi modulus of the fiber, the choice of tempers has a negligible effect on the calculated value of the effective axial modulus for the composite if the matrix is indeed on the yield surface.

The concentric-cylinder approximation used in the S-CUBED plastic-flow code does not lend itself to the calculation of the transverse stress-strain relations if the matrix is on the yield surface. From the PRUFC code, however, for an elastic matrix the calculated transverse modulus for the composite is

$$E_T = 17.1 \text{ Msi}$$

for a fiber volume fraction of 0.45. The mixture result for a simple series combination of the fiber and matrix materials,

$$\frac{1}{E_T} = \frac{0.45}{60.56} + \frac{0.55}{10}$$

gives the result

$$E_T^* = 16.0 \text{ Msi} ,$$

quite close to the PRUFC result, so that the simple series combination might be expected to give a useful approximation to the transverse behavior, even if the matrix has yielded.

Note here that if the matrix material is at the yield surface due to the cool-down residual stresses, then the dominant residual stress is tension in the axial direction. Consequently, an applied transverse tension will pull the matrix off the yield surface and the initial transverse behavior will be elastic. Under an applied transverse compressive stress, the matrix would stay on the yield surface and flow plastically, resulting in a lower apparent modulus for the composite.

For the simple series combination, the overall strain is given by

$$\epsilon = n_f \epsilon_f + (1 - n_f) \epsilon_m$$

in the transverse direction, or

$$\epsilon = n_f \frac{\sigma}{E_f} + (1 - n_f) \epsilon_m(\sigma) .$$

The above relation was used to obtain the estimates for the 90 degree orientation composites as plotted in Figure 4.2.

4.4 LAMINATE PROPERTIES

The code constructed at S-CUBED (Appendix B) may be used to compute the elastic properties of a laminate with plies oriented in up to three different arbitrary directions. As input one may use either the elastic and thermal properties of each type of ply, or the volume fractions and individual properties of the fiber and matrix components for each type of ply. In the latter case the code computes the ply properties as per the method used in the PRUFC

code. With the fiber and matrix properties as input, the code was used to determine the elastic behavior of a $0^\circ/90^\circ/90^\circ/0^\circ$ laminate with each ply consisting of a 0.45 fiber volume-fraction SiC/Al composite. The resulting slope of the initial elastic segment is indicated in Figure 4.3, both for an initial stress-free state and for the case where the matrix material is at the yield surface due to cool-down residual stresses. For the 90° plies, the initial behavior, as discussed above, will be elastic even though the matrix is initially on the yield surface. It turns out that for these orientations, the resulting elastic modulus of the composite laminate is very nearly the same as that of a simple combination of parallel springs. Thus, with equal volume fractions of the 0° and 90° plies, the laminate modulus corresponding to an elastic matrix is

$$(34.5 + 17.1)/2 = 25.8 \text{ Msi},$$

and for the yielded matrix it drops to

$$(29.3 + 17.1)/2 = 23.2 \text{ Msi}$$

as indicated in Figure 4.3. Eventually, the 90° plies, which had initially dropped off the yield surface, will once again reach the yield surface. Using an elastic-perfectly plastic model with a yield stress of 21 ksi (T4 temper), this is found to occur at an overall stress of about 28 ksi, at which point the slope of the stress-strain curve will drop to about

$$29.3/2 = 14.6 \text{ ksi}$$

The resulting stress-strain relationship agrees well with a projected estimate from AVC0, as shown in Figure 4.3.

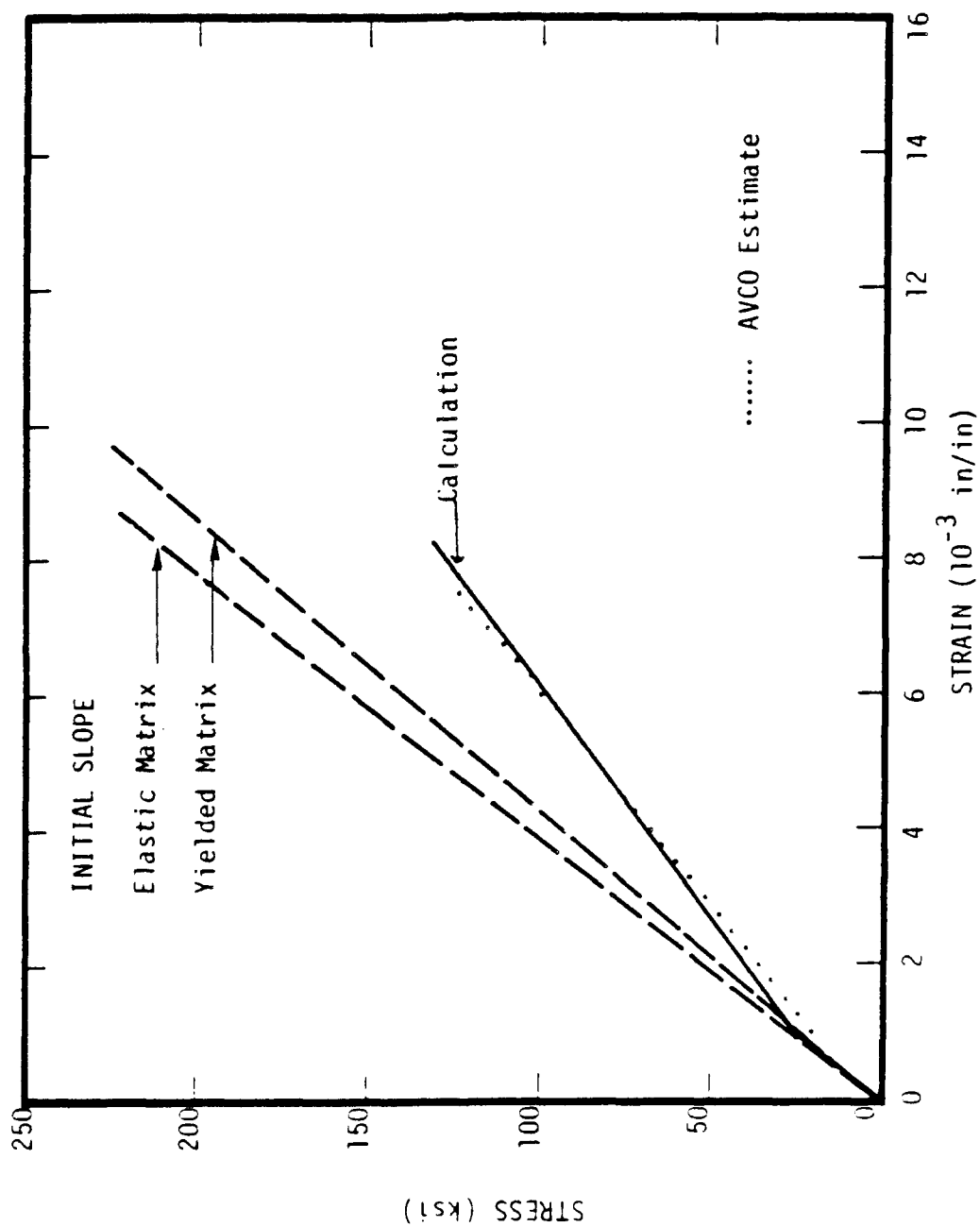


Figure 4.3. Calculated stress-strain relations for a $0^{\circ}/90^{\circ}/90^{\circ}/0^{\circ}$ laminate consisting of 0.45 fiber volume-fraction SiC/Al plies. Also shown is a projected estimate from AVCO (Ref. 7).

5. THERMOELASTIC PROPERTIES OF CROSS-PLY LAMINATES

Computational machinery is presented for determining the elastic moduli and thermal expansion characteristics of cross-ply laminates from the properties of the individual unidirectionally reinforced layers. Explicit formulas are given for the elastic constants and thermal expansion coefficients for a laminate with equal volume fractions of plies oriented in two directions. The computer code (Appendix B) in its present form may be used for plies oriented in up to three different directions. The code also provides the stress-temperature derivatives within each type of ply, appropriate to a uniform heating from an initial stress-free state.

5.1 TRANSFORMATION OF ELASTIC CONSTANTS

As a first step, it is necessary to obtain expressions for the elastic constants of a fiber-reinforced material in a coordinate system rotated with respect to that defined by the direction of the fibers. The stress-strain relations for a fiber-reinforced material assume the simplest form in the fiber coordinate system (FCS), i.e., that coordinate system in which one of the axes is in the direction of the fibers. If we take the z-axis as being parallel to the fibers, then the stresses are given in terms of the strains by the matrix relation

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{bmatrix} \quad (5.1)$$

i.e.,

$$\begin{aligned}\sigma_{xx} &= C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} + C_{13}\epsilon_{zz} , \\ \sigma_{yz} &= 2C_{44}\epsilon_{yz} , \text{ etc.},\end{aligned}\tag{5.2}$$

where the strains are defined by

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)\tag{5.3}$$

In what follows, the subscripts 1, 2, 3 will be used interchangeably with x, y, z, respectively, to denote the coordinate axes. In the usual case, where the material is not only orthotropic but also transversely isotropic,

$$C_{66} = \frac{1}{2} (C_{11} - C_{12}) .\tag{5.4}$$

For the laminate analysis, we require the elastic constants of the ply in the rotated coordinate system $x'y'z'$ as shown in Figure 5.1, obtained by a positive rotation ($\theta > 0$) about the x, x' axis, which is perpendicular to the face of the ply. The elastic constants C_{pq} are, of course, components of the fourth rank tensor, C_{ijmn} , which relates the stresses to the strains,

$$\sigma_{ij} = C_{ijmn}\epsilon_{mn} ,\tag{5.5}$$

using the summation convention $m, n = 1, 2, 3$ over repeated indices.

In general, the transformation to a new coordinate system is given by

$$C'_{pqkr} = l_{pi} l_{qj} l_{km} l_{rn} C_{ijmn} ,\tag{5.6}$$

where the l_{ij} relate the components of a vector in the primed coordinate system to those in the unprimed; and in the present case (Figure 5.1), these quantities are given by the matrix

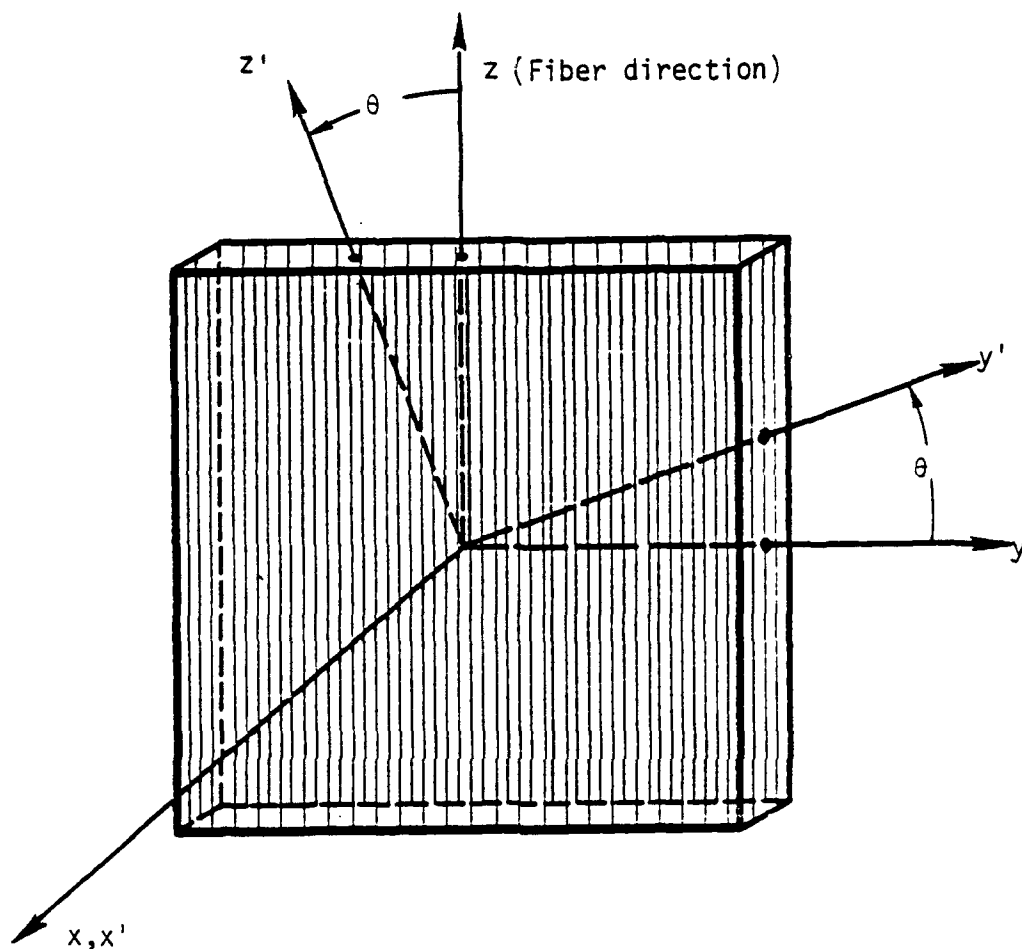


Figure 5.1 Coordinate system $x'y'z'$ used for laminate analysis.

$$\begin{bmatrix} l_{ij} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad (5.7)$$

In the FCS, the nonvanishing components of the tensor C_{ijkl} are given in terms of the elastic constants defined in Eq. (5.1) by

$$\begin{aligned} C_{11} &= C_{1111} = C_{2222} \\ C_{12} &= C_{1122} = C_{2211} \\ C_{13} &= C_{1133} = C_{2233} = C_{3311} = C_{3322} \\ C_{33} &= C_{3333} \\ C_{44} &= C_{2323} = C_{2332} = C_{3223} = C_{3232} \\ &= C_{1313} = C_{1331} = C_{3113} = C_{3131} \\ C_{66} &= C_{1212} = C_{1221} = C_{2112} = C_{2121} \end{aligned} \quad (5.8)$$

Thus, the calculation of a typical component in the new coordinate system requires the consideration of 21 terms. For example,

$$\begin{aligned} C'_{3333} &= l_{31}l_{31}l_{31}l_{31}C_{1111} + l_{32}l_{32}l_{32}l_{32}C_{2222} \\ &+ l_{31}l_{31}l_{32}l_{32}C_{1122} + l_{32}l_{32}l_{31}l_{31}C_{2211} \\ &+ \dots + \dots + \dots \\ &+ l_{32}l_{31}l_{31}l_{32}C_{2112} + l_{32}l_{31}l_{32}l_{31}C_{2121} \end{aligned} \quad (5.9)$$

Many of the terms are zero, however, because of the simple form of the transformation matrix, Eq. (5.7).

The result is that the stresses and strains in the primed coordinate system of Figure 5.1 are related by (note that the matrix is not symmetric)

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{yz} \\ \sigma'_{xz} \\ \sigma'_{xy} \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & 2C'_{14} & 0 & 0 \\ C'_{12} & C'_{22} & C'_{23} & 2C'_{24} & 0 & 0 \\ C'_{13} & C'_{23} & C'_{33} & 2C'_{34} & 0 & 0 \\ C'_{14} & C'_{24} & C'_{34} & 2C'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C'_{55} & 2C'_{56} \\ 0 & 0 & 0 & 0 & 2C'_{56} & 2C'_{66} \end{bmatrix} \begin{bmatrix} \epsilon'_{xx} \\ \epsilon'_{yy} \\ \epsilon'_{zz} \\ \epsilon'_{yz} \\ \epsilon'_{xz} \\ \epsilon'_{xy} \end{bmatrix} \quad (5.10)$$

where the new elastic constants are given in terms of the old by

$$\begin{aligned}
 C'_{11} &= C_{11} \\
 C'_{12} &= C_{12} \cos^2 \theta + C_{13} \sin^2 \theta \\
 C'_{13} &= C_{12} \sin^2 \theta + C_{13} \cos^2 \theta \\
 C'_{14} &= -C_{12} \sin \theta \cos \theta + C_{13} \sin \theta \cos \theta \\
 C'_{22} &= C_{11} \cos^4 \theta + C_{33} \sin^4 \theta + 2C_{13} \sin^2 \theta \cos^2 \theta \\
 &\quad + 4C_{44} \sin^2 \theta \cos^2 \theta \\
 C'_{23} &= (C_{11} + C_{33}) \sin^2 \theta \cos^2 \theta + C_{13} (\sin^4 \theta + \cos^4 \theta) \\
 &\quad - 4C_{44} \sin^2 \theta \cos^2 \theta \\
 C'_{24} &= -C_{11} \sin \theta \cos^3 \theta + C_{13} (\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta) \\
 &\quad + C_{33} \sin^3 \theta \cos \theta + 2C_{44} (\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta) \\
 C'_{33} &= C_{33} \cos^4 \theta + C_{11} \sin^4 \theta + 2C_{13} \sin^2 \theta \cos^2 \theta \\
 &\quad + 4C_{44} \sin^2 \theta \cos^2 \theta \\
 C'_{34} &= -C_{11} \sin^3 \theta \cos \theta + C_{33} \sin \theta \cos^3 \theta \\
 &\quad + C_{13} (\sin^3 \theta \cos \theta - \sin \theta \cos^3 \theta) \\
 &\quad + 2C_{44} (\sin^3 \theta \cos \theta - \sin \theta \cos^3 \theta)
 \end{aligned} \quad (5.11)$$

$$\begin{aligned}
C'_{44} &= (C_{11} + C_{33}) \sin^2 \theta \cos^2 \theta - 2C_{13} \sin^2 \theta \cos^2 \theta \\
&\quad + C_{44} (\cos^2 \theta - \sin^2 \theta)^2 \\
C'_{55} &= C_{44} \cos^2 \theta + C_{66} \sin^2 \theta \\
C'_{56} &= (C_{44} - C_{66}) \sin \theta \cos \theta \\
C'_{66} &= C_{66} \cos^2 \theta + C_{44} \sin^2 \theta
\end{aligned}$$

The above results have been checked against those given by Hashin^[8] for an initial coordinate system in which the fibers are parallel to the x-axis.

The description in the primed coordinate system is completed by a consideration of the thermal stress parameters. In the fiber coordinate system the thermal stresses are given by

$$\begin{aligned}
\sigma_{xx}^{(T)} &= \sigma_{yy}^{(T)} = -\gamma_1 \Delta T, \\
\sigma_{zz}^{(T)} &= -\gamma_3 \Delta T,
\end{aligned} \tag{5.12}$$

where ΔT is the temperature increment, and the stress-temperature coupling coefficients are given in terms of the linear coefficients of thermal expansion by

$$\begin{aligned}
\gamma_1 &= (C_{11} + C_{12})\alpha_x + C_{13}\alpha_z \\
\gamma_3 &= 2C_{13}\alpha_x + C_{33}\alpha_z
\end{aligned} \tag{5.13}$$

The nonvanishing components of this second rank tensor in the primed coordinate system are

$$\begin{aligned}
\gamma'_{11} &= \gamma_1 \\
\gamma'_{22} &= \gamma_1 \cos^2 \theta + \gamma_3 \sin^2 \theta \\
\gamma'_{33} &= \gamma_1 \sin^2 \theta + \gamma_3 \cos^2 \theta \\
\gamma'_{23} &= \gamma'_{32} = (\gamma_3 - \gamma_1) \sin \theta \cos \theta
\end{aligned} \tag{5.14}$$

In summary, the stresses are given in terms of the strains in the rotated, or laminate coordinate system (LCS), by

$$\begin{aligned}
 \sigma_{11} &= C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{13}\epsilon_{33} + 2C_{14}\epsilon_{23} - \gamma_{11}\Delta T \\
 \sigma_{22} &= C_{12}\epsilon_{11} + C_{22}\epsilon_{22} + C_{23}\epsilon_{33} + 2C_{24}\epsilon_{23} - \gamma_{22}\Delta T \\
 \sigma_{33} &= C_{13}\epsilon_{11} + C_{23}\epsilon_{22} + C_{33}\epsilon_{33} + 2C_{34}\epsilon_{23} - \gamma_{33}\Delta T \\
 \sigma_{23} &= C_{14}\epsilon_{11} + C_{24}\epsilon_{22} + C_{34}\epsilon_{33} + 2C_{44}\epsilon_{23} - \gamma_{23}\Delta T \\
 \sigma_{13} &= 2C_{55}\epsilon_{13} + 2C_{56}\epsilon_{12} \\
 \sigma_{12} &= 2C_{56}\epsilon_{13} + 2C_{66}\epsilon_{12} ,
 \end{aligned} \tag{5.15}$$

where, for convenience, the primes have been omitted. If we consider the stresses as a six component column matrix, then the above relations can be written in matrix form as

$$\sigma = C\epsilon - \gamma \Delta T , \tag{5.16}$$

and the inverse relations giving the strains in terms of the stresses are

$$\epsilon = C^{-1}\sigma + C^{-1}\gamma \Delta T ,$$

or equivalently as

$$\epsilon = S\sigma + \alpha \Delta T , \tag{5.18}$$

i.e.,

$$\begin{aligned}
 \epsilon_{11} &= S_{11}\sigma_{11} + S_{12}\sigma_{22} + S_{13}\sigma_{33} + 2S_{14}\sigma_{23} + \alpha_{11}\Delta T \\
 \epsilon_{22} &= S_{12}\sigma_{11} + S_{22}\sigma_{22} + S_{23}\sigma_{33} + 2S_{24}\sigma_{23} + \alpha_{22}\Delta T \\
 \epsilon_{33} &= S_{13}\sigma_{11} + S_{23}\sigma_{22} + S_{33}\sigma_{33} + 2S_{34}\sigma_{23} + \alpha_{33}\Delta T \\
 \epsilon_{23} &= S_{14}\sigma_{11} + S_{24}\sigma_{22} + S_{34}\sigma_{33} + 2S_{44}\sigma_{23} + \alpha_{23}\Delta T \\
 \epsilon_{13} &= 2S_{55}\sigma_{13} + 2S_{56}\sigma_{12} \\
 \epsilon_{12} &= 2S_{56}\sigma_{13} + 2S_{66}\sigma_{12}
 \end{aligned} \tag{5.19}$$

where the inverse S of the matrix C is easily obtained using the computer (note again that the matrix S and the matrix C are not symmetric when the strains are defined by Eq. (5.3)).

5.2 THERMAL EXPANSION COEFFICIENTS

Here we shall be concerned with the thermal expansion coefficients of a laminate containing equal volume fractions of plies with fibers oriented in one or the other of two directions. The z -axis of the laminate coordinate system is taken as the bisector of the angle between the fiber directions, as indicated in Figure 5.2.

The elastic constants in the LCS of a ply with fibers in the z_1 direction are given directly by the relations (5.11); and the elastic constants of a ply with fibers along z_2 may be obtained from the connections (5.11) by replacing $\sin\theta$ by $-\sin\theta$, i.e.,

$$\begin{aligned} c_{14}^{(2)} &= -c_{14}^{(1)} \\ c_{24}^{(2)} &= -c_{24}^{(1)} \\ c_{34}^{(2)} &= -c_{34}^{(1)} \\ c_{56}^{(2)} &= -c_{56}^{(1)} \\ \gamma_{23}^{(2)} &= -\gamma_{23}^{(1)} \\ \alpha_{23}^{(2)} &= -\alpha_{23}^{(1)} \end{aligned} \quad , \quad (5.20)$$

all the other parameters being equal. If the laminate thickness (x -direction) is very much less than the lateral dimensions (y, z directions), then away from the edges the shear stresses between plies will vanish, and the appropriate boundary conditions for determining the thermal expansion coefficients are

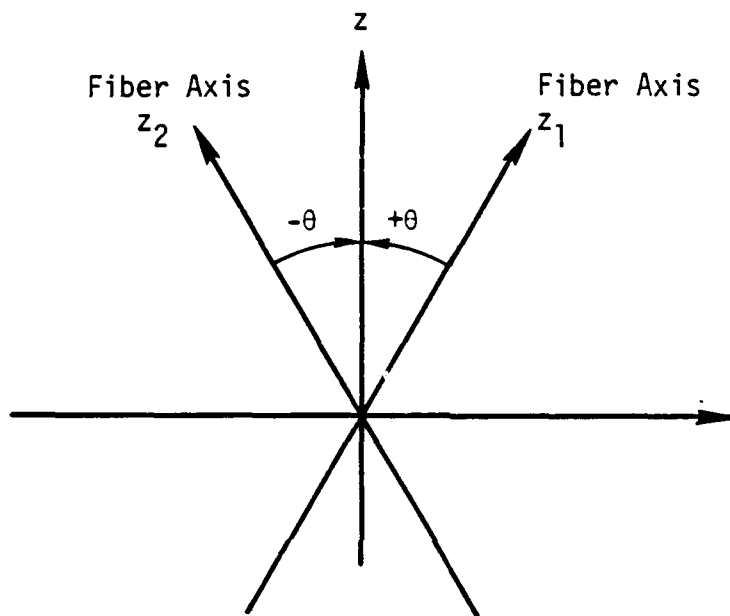


Figure 5.2. Laminate geometry.

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} = 0 \quad (5.21)$$

$$\epsilon_{zz}^{(1)} = \epsilon_{zz}^{(2)} \quad (5.22)$$

$$\epsilon_{yy}^{(1)} = \epsilon_{yy}^{(2)} \quad (5.23)$$

$$\epsilon_{yz}^{(1)} = \epsilon_{yz}^{(2)} \quad (5.24)$$

and (equal volume fractions),

$$\sigma_{zz}^{(1)} + \sigma_{zz}^{(2)} = 0 \quad (5.25)$$

$$\sigma_{yy}^{(1)} + \sigma_{yy}^{(2)} = 0 \quad (5.26)$$

$$\sigma_{yz}^{(1)} + \sigma_{yz}^{(2)} = 0 \quad (5.27)$$

The conditions (5.22), (5.23), and (5.24) give the three equations,

$$\begin{aligned} (S_{23}^{(1)} + S_{23}^{(2)})\sigma_{yy}^{(1)} + (S_{33}^{(1)} + S_{33}^{(2)})\sigma_{zz}^{(1)} \\ + 2(S_{34}^{(1)} + S_{34}^{(2)})\sigma_{yz}^{(1)} + (\alpha_{33}^{(1)} - \alpha_{33}^{(2)})\Delta T = 0 \end{aligned} \quad (5.28)$$

$$\begin{aligned} (S_{22}^{(1)} + S_{22}^{(2)})\sigma_{yy}^{(1)} + (S_{23}^{(1)} + S_{23}^{(2)})\sigma_{zz}^{(1)} \\ + 2(S_{24}^{(1)} + S_{24}^{(2)})\sigma_{yz}^{(1)} + (\alpha_{22}^{(1)} - \alpha_{22}^{(2)})\Delta T = 0 \end{aligned} \quad (5.29)$$

$$\begin{aligned} (S_{24}^{(1)} + S_{24}^{(2)})\sigma_{yy}^{(1)} + (S_{34}^{(1)} + S_{34}^{(2)})\sigma_{zz}^{(1)} \\ + 2(S_{44}^{(1)} + S_{44}^{(2)})\sigma_{yz}^{(1)} + (\alpha_{23}^{(1)} - \alpha_{23}^{(2)})\Delta T = 0 \end{aligned} \quad (5.30)$$

where use has been made of (5.21), (5.25), (5.26), and (5.27). Since

$$S_{34}^{(2)} = -S_{34}^{(1)}, \alpha_{33}^{(2)} = \alpha_{33}^{(1)}, \text{ etc.,}$$

the first two relations (5.28) and (5.29) give the intuitively obvious result

$$\sigma_{yy}^{(1)} = \sigma_{zz}^{(1)} = \sigma_{yy}^{(2)} = \sigma_{zz}^{(2)} = 0 \quad , \quad (5.31)$$

and the third condition (5.30) gives

$$4S_{44}^{(1)} \sigma_{yz}^{(1)} + 2\alpha_{23}^{(1)} \Delta T = 0 \quad . \quad (5.32)$$

The z-component of the thermal strain is just

$$\epsilon_{zz} = \epsilon_{zz}^{(1)} = 2S_{34}^{(1)} \sigma_{yz}^{(1)} + \alpha_{33}^{(1)} \Delta T \quad , \quad (5.33)$$

which with the use of (5.32) gives the desired result

$$\alpha_z = \frac{\epsilon_{zz}}{\Delta T} = - \frac{S_{34}^{(1)}}{S_{44}^{(1)}} \alpha_{23}^{(1)} + \alpha_{33}^{(1)} \quad . \quad (5.34)$$

Similarly,

$$\epsilon_{yy} = 2S_{24}^{(1)} \sigma_{yz}^{(1)} + \alpha_{22}^{(1)} \Delta T \quad (5.35)$$

which with the use of (5.32) gives

$$\alpha_y = \frac{\epsilon_{yy}}{\Delta T} = - \frac{S_{24}^{(1)}}{S_{44}^{(1)}} \alpha_{23}^{(1)} + \alpha_{22}^{(1)} \quad ; \quad (5.36)$$

and finally, one obtains,

$$\alpha_x = - \frac{S_{14}^{(1)}}{S_{44}^{(1)}} \alpha_{23}^{(1)} + \alpha_{11}^{(1)} \quad (5.37)$$

Remember that in the above expressions the S_{ij} and α_{ij} are elements of the tensors in the laminate coordinate system.

The elements S_{ij} may be obtained in two ways: by computing the elements of the matrix C and taking the inverse, or by direct computation from the elements in the fiber coordinate system using the transformation Eqs. (5.11)

For example,

$$\begin{aligned}
 S'_{34} = & -S_{11}\sin^3\theta\cos\theta + S_{33}\sin\theta\cos^3\theta \\
 & + S_{13}(\sin^3\theta\cos\theta - \sin\theta\cos^3\theta) \\
 & + 2S_{44}(\sin^3\theta\cos\theta - \sin\theta\cos^3\theta) ,
 \end{aligned} \tag{5.38}$$

where

$$\begin{aligned}
 S_{11} &= 1/E_x = S_{22} \\
 S_{12} &= -\nu_{xy}/E_x = S_{21} \\
 S_{13} &= -\nu_{xz}/E_z = S_{31} = S_{23} = S_{32} \\
 S_{33} &= 1/E_z \\
 S_{44} &= 1/(4C_{44}) = 1/(4G_A) \\
 S_{66} &= 1/(4C_{66}) = 1/(4G_T)
 \end{aligned} \tag{5.39}$$

are the nonvanishing components in the FCS in terms of the Young's moduli E_x , E_z and the Poisson's ratios ν_{xy} , ν_{xz} (contraction, force). The quantities G_A , G_T are the standard axial and transverse shear moduli.

5.3 ELASTIC PARAMETERS

The elastic (Young's) modulus in the z -direction for the laminate is obtained by applying the conditions

$$\epsilon_{zz}^{(1)} = \epsilon_{zz}^{(2)} = \epsilon_0, \quad (\Delta T = 0) \tag{5.40}$$

$$\epsilon_{yy}^{(1)} = \epsilon_{yy}^{(2)} \tag{5.41}$$

$$\epsilon_{yz}^{(1)} = \epsilon_{yz}^{(2)} \tag{5.42}$$

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} = 0 \tag{5.43}$$

$$\sigma_{yy}^{(1)} + \sigma_{yy}^{(2)} = 0 \tag{5.44}$$

$$\sigma_{yz}^{(1)} + \sigma_{yz}^{(2)} = 0 \quad (5.45)$$

where (5.44) and (5.45) imply equal volume fractions of the two types of ply.

The condition (5.41) gives

$$\begin{aligned} S_{22}^{(1)} \sigma_{yy}^{(1)} + S_{23}^{(1)} \sigma_{zz}^{(1)} + 2S_{24}^{(1)} \sigma_{23}^{(1)} \\ = S_{22}^{(2)} \sigma_{yy}^{(2)} + S_{23}^{(2)} \sigma_{zz}^{(2)} + 2S_{24}^{(2)} \sigma_{23}^{(2)} \end{aligned} \quad (5.46)$$

and (5.42) gives

$$\begin{aligned} S_{24}^{(1)} \sigma_{yy}^{(1)} + S_{34}^{(1)} \sigma_{zz}^{(1)} + 2S_{44}^{(1)} \sigma_{23}^{(1)} \\ = S_{24}^{(2)} \sigma_{yy}^{(2)} + S_{34}^{(2)} \sigma_{zz}^{(2)} + 2S_{44}^{(2)} \sigma_{23}^{(2)} \end{aligned} \quad (5.47)$$

where use has been made of (5.43). With the use of (5.44) and (5.45) the above relations become

$$\begin{aligned} (S_{22}^{(1)} + S_{22}^{(2)}) \sigma_{yy}^{(1)} + S_{23}^{(1)} \sigma_{zz}^{(1)} - S_{23}^{(2)} \sigma_{zz}^{(2)} \\ + 2(S_{24}^{(1)} + S_{24}^{(2)}) \sigma_{23}^{(1)} = 0 \end{aligned} \quad (5.48)$$

$$\begin{aligned} (S_{24}^{(1)} + S_{24}^{(2)}) \sigma_{yy}^{(1)} + S_{34}^{(1)} \sigma_{zz}^{(1)} - S_{34}^{(2)} \sigma_{zz}^{(2)} \\ + 2(S_{44}^{(1)} + S_{44}^{(2)}) \sigma_{23}^{(1)} = 0 \end{aligned} \quad (5.49)$$

and (5.40) gives the two relations

$$S_{23}^{(1)} \sigma_{yy}^{(1)} + S_{33}^{(1)} \sigma_{zz}^{(1)} + 2S_{34}^{(1)} \sigma_{23}^{(1)} = \epsilon_0 \quad (5.50)$$

$$- S_{23}^{(2)} \sigma_{yy}^{(1)} + S_{33}^{(2)} \sigma_{zz}^{(2)} - 2S_{34}^{(2)} \sigma_{23}^{(1)} = \epsilon_0 \quad (5.51)$$

The relations (5.48), (5.49), (5.50), and (5.51) constitute a set of four equations in the four variables $\sigma_{yy}^{(1)}$, $\sigma_{zz}^{(1)}$, $\sigma_{zz}^{(2)}$, and $\sigma_{23}^{(1)}$.

Equation (5.48) simplifies to

$$2S_{22}^{(1)}\sigma_{yy}^{(1)} + S_{23}^{(1)}(\sigma_{zz}^{(1)} - \sigma_{zz}^{(2)}) = 0, \quad (5.52)$$

and (5.49) reduces to

$$S_{34}^{(1)}(\sigma_{zz}^{(1)} + \sigma_{zz}^{(2)}) + 4S_{44}^{(1)}\sigma_{23}^{(1)} = 0. \quad (5.53)$$

The relation (5.51) is also

$$-S_{23}^{(1)}\sigma_{yy}^{(1)} + S_{33}^{(1)}\sigma_{zz}^{(2)} + 2S_{34}^{(1)}\sigma_{23}^{(1)} = \epsilon_0 \quad (5.54)$$

which when subtracted from (5.50) gives

$$2S_{23}^{(1)}\sigma_{yy}^{(1)} + S_{33}^{(1)}(\sigma_{zz}^{(1)} - \sigma_{zz}^{(2)}) = 0, \quad (5.55)$$

and a consideration of (5.52) and (5.55) gives the results,

$$\begin{aligned} \sigma_{yy}^{(1)} &= \sigma_{yy}^{(2)} = 0 \\ \sigma_{zz}^{(1)} &= \sigma_{zz}^{(2)}, \end{aligned} \quad (5.56)$$

which, it must be admitted, seem very appropriate.

The relation (5.53) then becomes

$$2S_{34}^{(1)}\sigma_{zz}^{(1)} + 4S_{44}^{(1)}\sigma_{23}^{(1)} = 0 \quad (5.57)$$

and (5.54) is now

$$S_{33}^{(1)}\sigma_{zz}^{(1)} + 2S_{34}^{(1)}\sigma_{23}^{(1)} = \epsilon_0 \quad (5.58)$$

The above two relations yield the desired result,

$$\frac{\sigma_{zz}^{(1)}}{\epsilon_0} = S_{33}^{(1)} - \frac{S_{34}^{(1)2}}{S_{44}^{(1)}} = 1 \quad (5.59)$$

or

$$\frac{1}{E_z} = S_{33}^{(1)} - \frac{S_{34}^{(1)2}}{S_{44}^{(1)}} \quad (5.60)$$

for the reciprocal of Young's modulus E_z .

The elastic modulus in the y -direction follows from the conditions

$$\epsilon_{yy}^{(1)} = \epsilon_{yy}^{(2)} = \epsilon_0 \quad (5.61)$$

$$\epsilon_{zz}^{(1)} = \epsilon_{zz}^{(2)} \quad (5.62)$$

$$\epsilon_{yz}^{(1)} = \epsilon_{yz}^{(2)} \quad (5.63)$$

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} = 0 \quad (5.64)$$

and for equal volume fractions,

$$\sigma_{zz}^{(1)} + \sigma_{zz}^{(2)} = 0 \quad (5.65)$$

$$\sigma_{yz}^{(1)} + \sigma_{yz}^{(2)} = 0 \quad (5.66)$$

Doing the same algebra as above for E_z gives the desired relation

$$\frac{1}{E_y} = S_{22}^{(1)} - \frac{S_{24}^{(1)2}}{S_{44}^{(1)}} \quad (5.67)$$

for the reciprocal of the Young's modulus.

In a similar fashion, one obtains for the elastic modulus in the direction perpendicular to the laminate,

$$\frac{1}{E_x} = S_{11}^{(1)} - \frac{S_{14}^{(1)2}}{S_{44}^{(1)}} \quad (5.68)$$

The corresponding Poisson's ratios are easily obtained from the above results. For the case of an applied stress in the z direction,

$$\epsilon_{xx} = S_{13}\sigma_{zz} + 2S_{14}\sigma_{23} \quad (5.69)$$

$$\epsilon_{zz} = S_{33}\sigma_{zz} + 2S_{34}\sigma_{23}$$

which with the use of (5.57) gives

$$\nu_{xz} = - \frac{S_{13}S_{44} - S_{14}S_{34}}{S_{33}S_{44} - S_{34}^2} \quad (5.70)$$

Likewise,

$$\epsilon_{yy} = S_{23}\sigma_{zz} + 2S_{24}\sigma_{23} \quad (5.71)$$

and

$$\nu_{yz} = - \frac{S_{23}S_{44} - S_{24}S_{34}}{S_{33}S_{44} - S_{34}^2} \quad (5.72)$$

In order to obtain an expression for the in-plane shear modulus, we apply the conditions:

$$\epsilon_{yz}^{(1)} = \epsilon_{yz}^{(2)} = \epsilon_0 \quad (5.73)$$

$$\epsilon_{zz}^{(1)} = \epsilon_{zz}^{(2)} \quad (5.74)$$

$$\epsilon_{yy}^{(1)} = \epsilon_{yy}^{(2)} \quad (5.75)$$

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} = 0 \quad (5.76)$$

$$\sigma_{zz}^{(1)} + \sigma_{zz}^{(2)} = 0 \quad (5.77)$$

$$\sigma_{yy}^{(1)} + \sigma_{yy}^{(2)} = 0 \quad (5.78)$$

The condition (5.74) yields with the use of (5.76), (5.77), and (5.78),

$$S_{23}\sigma_{yy}^{(1)} + S_{33}\sigma_{zz}^{(1)} + S_{34}(\sigma_{yz}^{(1)} + \sigma_{yz}^{(2)}) = 0 \quad (5.79)$$

and (5.75) gives

$$S_{22}\sigma_{yy}^{(1)} + S_{23}\sigma_{zz}^{(1)} + S_{24}(\sigma_{yz}^{(1)} + \sigma_{yz}^{(2)}) = 0 \quad (5.80)$$

The superscript is omitted in the above expression if the quantity has the same numerical value for both types of ply (θ and $-\theta$ fiber orientation).

From (5.73) we obtain two relations

$$S_{24}\sigma_{yy}^{(1)} + S_{34}\sigma_{zz}^{(1)} + 2S_{44}\sigma_{yz}^{(1)} = \epsilon_0 \quad (5.81)$$

$$S_{24}\sigma_{yy}^{(2)} + S_{34}\sigma_{zz}^{(2)} + 2S_{44}\sigma_{yz}^{(2)} = \epsilon_0 \quad (5.82)$$

Subtracting (5.82) from (5.81) leads to the conclusion that

$$\sigma_{yz}^{(1)} = \sigma_{yz}^{(2)} \quad (5.83)$$

whereas adding (5.82) and (5.81) gives the useful relation,

$$S_{24}\sigma_{yy}^{(1)} + S_{34}\sigma_{zz}^{(1)} + 2S_{44}\sigma_{yz}^{(1)} = \epsilon_0 \quad (5.84)$$

which together with (5.79) and (5.80) form a set of three simultaneous linear equations in the variables $\sigma_{yy}^{(1)}$, $\sigma_{zz}^{(1)}$, and $\sigma_{yz}^{(1)}$. These may be solved numerically to obtain the in-plane shear modulus

$$G_{44}^{(eff)} = \frac{\sigma_{yz}^{(1)}}{2\epsilon_0} \quad (5.85)$$

5.4 NUMERICAL RESULTS

Calculated results for several parameters as a function of the lay-up angle θ (Figure 5.2) are presented for a laminate whose plies consist of 30 percent by volume graphite fibers in a 2024 aluminum

matrix. The input parameters for the fibers and matrix, as given in Table 5.1 below, are the same as those used by Hashin and Humphreys⁽²⁾ in a recent report.

TABLE 5.1
INPUT ELASTIC CONSTANTS FOR FIBER AND MATRIX

Elastic Constant	T-50 Graphite Fiber	2024-T4 Al Matrix
E_z (GPa)	388.2	72.4
E_x (GPa)	7.6	72.4
C_{44} (GPa)	14.9	27.2
C_{66} (GPa)	2.6	27.2
ν_{xz} **	0.41	0.33
ν_{xy}	0.45	0.33
α_z ($^{\circ}\text{C}^{-1}$)	-0.68×10^{-6}	22.5×10^{-6}
α_x ($^{\circ}\text{C}^{-1}$)	9.74×10^{-6}	22.5×10^{-6}

**The subscripts are (contraction, force).

The PRUFC code^[3] was used to calculate the elastic properties of a ply in the fiber coordinate system, with the results as given in Table 5.2 (the PRUFC code was subsequently revised as discussed in Section 6).

The linear thermal expansion coefficients for the laminate, as calculated with Eqs. (5.34), (5.36), and (5.37) are plotted as a function of ply angle in Figure 5.3. It will be noted that the axial thermal expansion has a minimum at about 20 deg (40 deg total angle between fiber directions). It is also comforting to note that the calculated results exhibit the required symmetry about 45 degrees.

TABLE 5.2
INPUT ELASTIC PROPERTIES FOR A FIBER REINFORCED PLY
(30 percent by volume of fibers)

Elastic Constant	Value
E_z (GPa)	167.18
E_x (GPa)	37.11
C_{44} (GPa)	22.82
C_{66} (GPa)	12.68
ν_{xz}	0.33840
ν_{xy}	0.46294
α_z ($^{\circ}\text{C}^{-1}$)	6.29×10^{-6}
α_x ($^{\circ}\text{C}^{-1}$)	25.66×10^{-6}
C_{11} (GPa)	50.85
C_{12} (GPa)	25.48
C_{13} (GPa)	25.83
C_{33} (GPa)	184.66

The elastic moduli along the three laminate axes are plotted in Figure 5.4, and two Poisson's ratios are given in Figure 5.5 as a function of lay-up angle. The in-plane shear modulus for the laminate is shown in Figure 5.6.

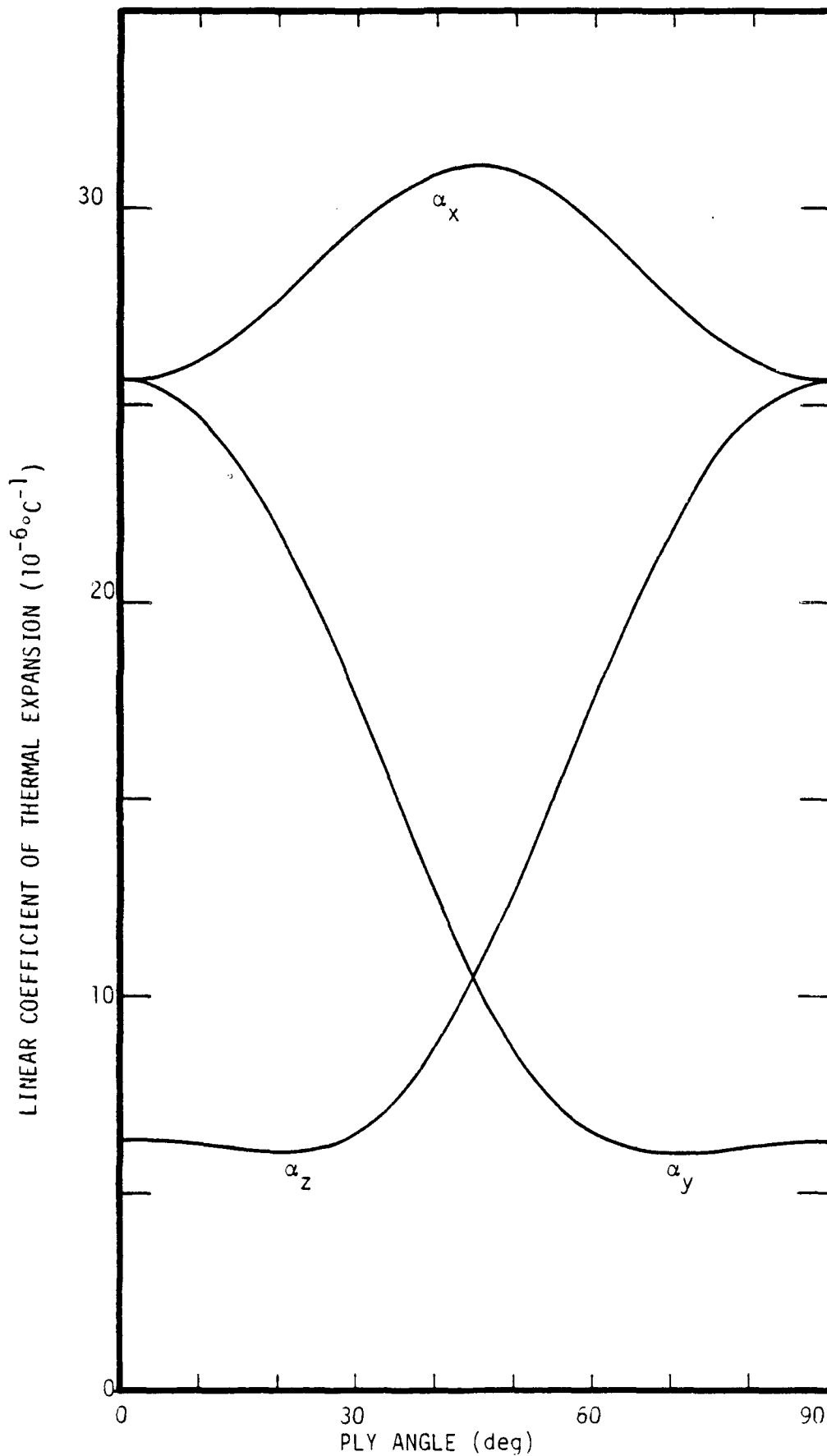


Figure 5.3. Calculated linear thermal expansion coefficients for a cross-ply metal-matrix laminate as a function of lay-up angle.

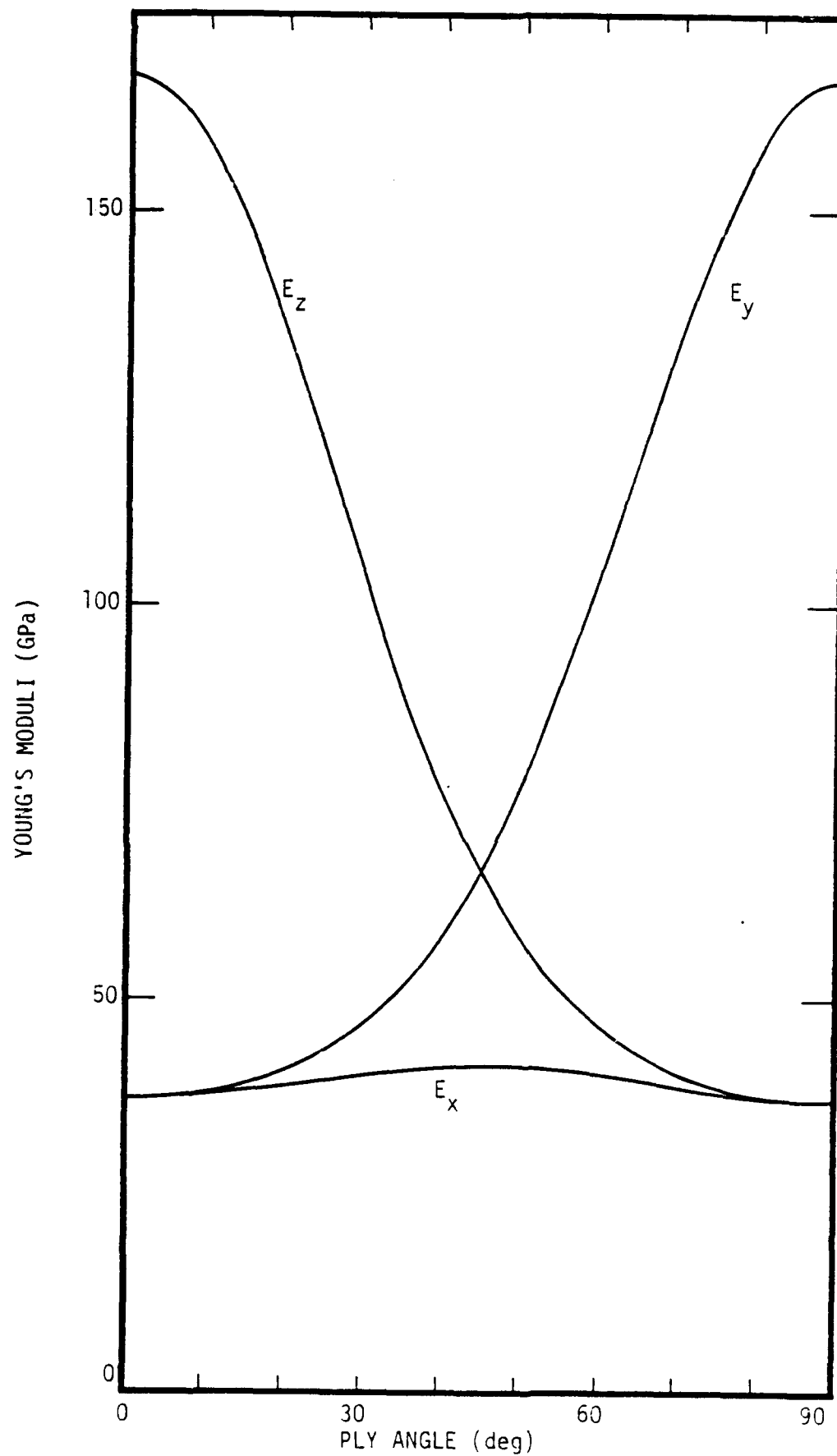


Figure 5.4. Calculated elastic (Young's) moduli for a metal-matrix laminate as a function of lay-up angle.

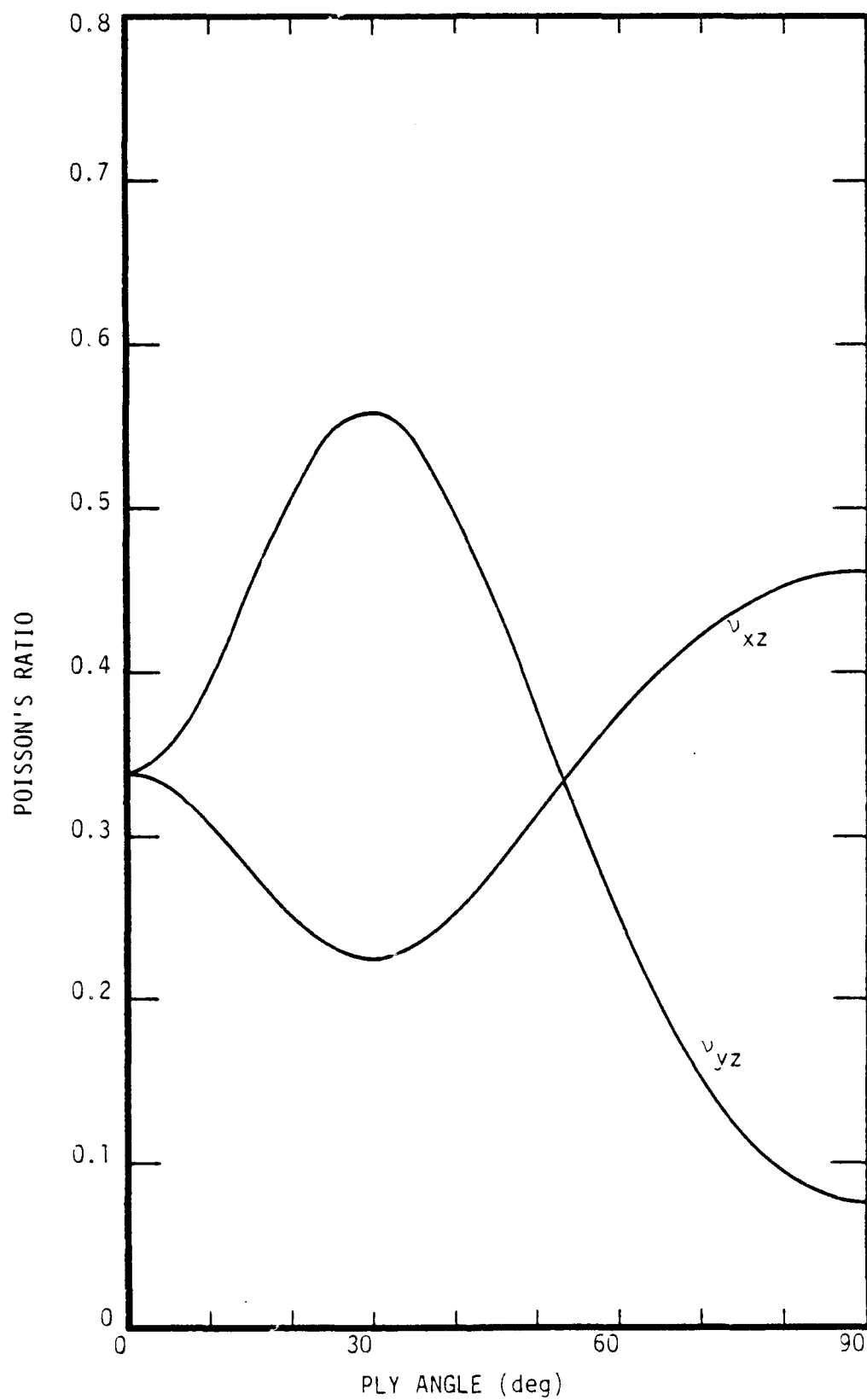


Figure 5.5. Two calculated Poisson's ratios for a metal-matrix laminate as a function of lay-up angle.

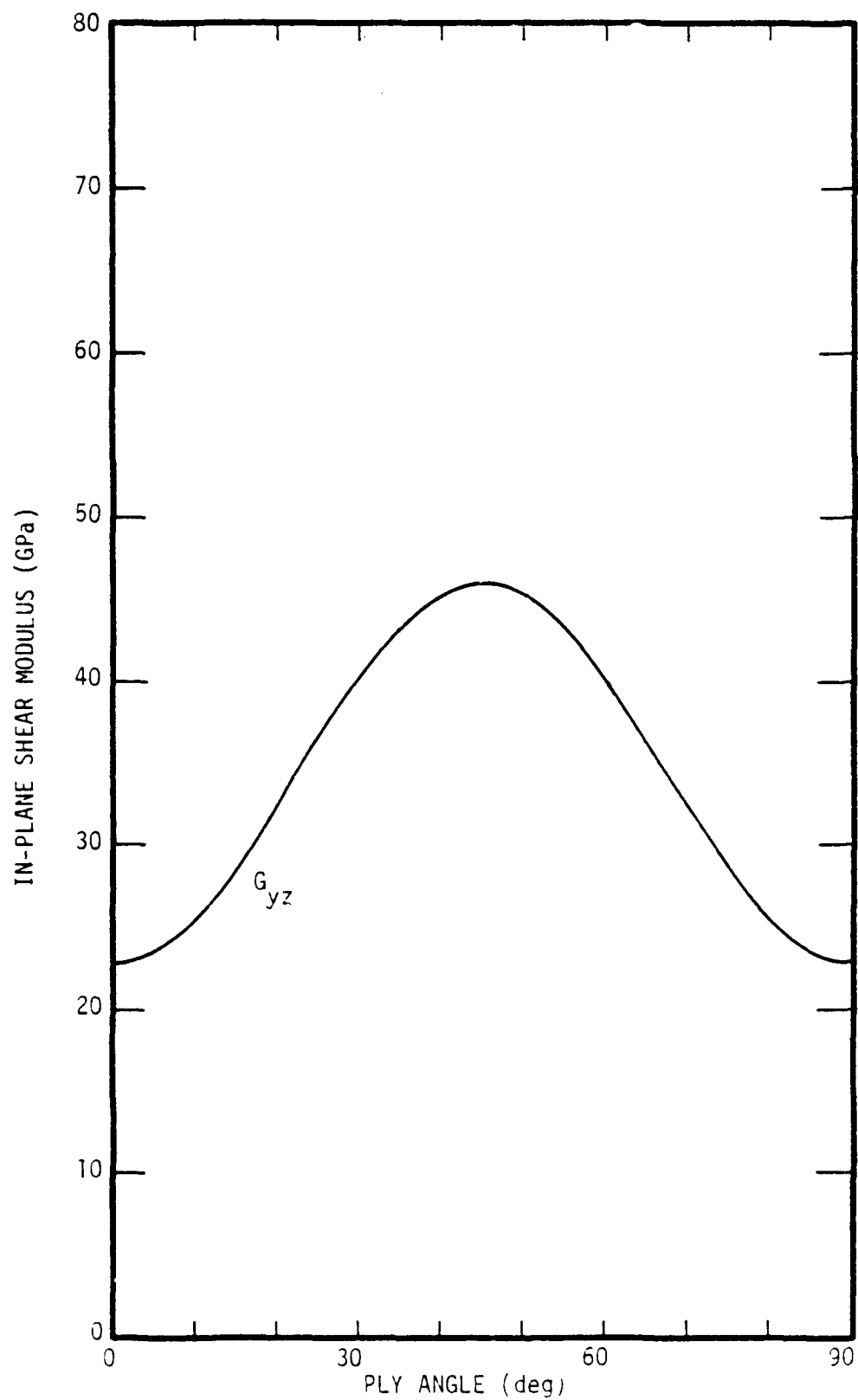


Figure 5.6. Calculated in-plane shear modulus for a metal-matrix laminate as a function of lay-up angle.

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6. TRANSVERSE ELASTIC PROPERTIES OF A UNIDIRECTIONALLY REINFORCED COMPOSITE AND THE PRUFC CODE

In a recent report,^[2] Hashin and Humphreys present numerical results for a typical graphite/aluminum composite as calculated using a finite-element code (ANSYS) to determine the stress-strain state in the unit cell of a model in which the fibers are arranged in a perfect hexagonal array. The advantage of their method is that it is expected to give a very accurate representation of the behavior of a composite in terms of the properties of the fiber and matrix components. The disadvantage is, as stated in the report, "The ANSYS computer code, ..., is very expensive to run for such cases." Consequently, it is of interest to compare these finite element results with those obtained with PRUFC^[3] and the S-CUBED plastic-flow code,^[1] which are inexpensive to run but use the concentric-cylinder approximation.

We present first a comparison of calculated elastic properties of a graphite/aluminum composite as obtained from the properties of the constituents. In the example given in Ref. 2, the properties of the fiber and aluminum matrix are as given in Table 6.1.

A comparison of the composite parameters as calculated with PRUFC and those calculated by Hashin and Humphreys for a 30 percent by volume of fibers is given in Table 6.2 below. Also given are values obtained from the "Composite Cylinder Assemblage" (CCA) model of a composite (Hashin, Ref. 8).

The only significant differences are in the transverse properties E_T , G_{TT} , and ν_{TT} (only two of which are independent, because of the transverse isotropy). Not all the transverse properties differ, however. If one considers the transverse bulk modulus k whose reciprocal is given by

$$\frac{1}{k} = \frac{2\epsilon_x}{\sigma_0} = 2 \left[\frac{1 - \nu_{TT}}{E_T} - \frac{2\nu_{AT}^2}{E_A} \right], \quad (6.1)$$

$$\sigma_x = \sigma_y = \sigma_0, \quad \epsilon_z = 0,$$

TABLE 6.1

CONSTITUENT PROPERTIES FOR A GRAPHITE/ALUMINUM COMPOSITE

Elastic Constant	T-50 Graphite Fiber	2024-T4 Al Matrix
E_A (GPa)	388.2	72.4
E_T (GPa)	7.6	72.4
G_{AT} (GPa)	14.9	27.2
G_{TT} (GPa)	2.6	27.2
ν_{AT}^*	0.41	0.33
ν_{TT}	0.45	0.33
α_A ($^{\circ}\text{C}^{-1}$)	-0.68×10^{-6}	22.5×10^{-6}
α_T ($^{\circ}\text{C}^{-1}$)	9.74×10^{-6}	22.5×10^{-6}

A = Axial (Longitudinal)

T = Transverse

*This Poisson's ratio is the transverse contraction for an axial stress.

TABLE 6.2

CALCULATED PROPERTIES FOR A GRAPHITE/ALUMINUM COMPOSITE
(30 Percent by Volume of Fibers)

Elastic Constant	PRUFC	Finite-Element Method	CCA
E_A (GPa)	167.18	167.14*	--
E_T (GPa)	33.06	42.26	41.78
G_{TT} (GPa)	10.86	15.13	14.99
G_{AT} (GPa)	22.82	23.20	22.87
ν_{TT}	0.522	0.396	0.394
ν_{AT}	0.338	0.340	0.338
α_A ($^{\circ}\text{C}^{-1}$)	6.29×10^{-6}	6.36×10^{-6}	6.36×10^{-6}
α_T ($^{\circ}\text{C}^{-1}$)	25.65×10^{-6}	25.69×10^{-6}	25.65×10^{-6}

the values as obtained by the PRUFC code and the finite element method are, respectively,

$$k = 38.19 \text{ GPa (PRUFC)}$$

$$k = 38.73 \text{ GPa (F.E.)} ,$$

a good agreement. The difficulty is that the concentric-cylinder approximation to a composite does not provide a unique definition of the transverse shear modulus.

In the PRUFC code as documented in Ref. 3, the transverse shear modulus is obtained from a consideration of the deformation of concentric cylinders subjected to a stress field at its outer surface the same as that acting on a fictitious cylindrical surface in a homogeneous material subjected to the stress field

$$\sigma_{xx} = -\sigma_{yy} = \sigma_0 = \text{const.}$$

$$\epsilon_z = 0 .$$

(see Appendix B of Reference 9). For the fictitious, homogeneous cylinder, of course, the resulting strain will be uniform throughout its volume and is related to the stress field by

$$\sigma_0 = 2G_{TT}\epsilon_{xx} , \tag{6.2}$$

involving only the transverse shear modulus, G_{TT} . For the concentric cylinder, however, for which an exact solution of the equations of elasticity is available,^[9] the effective strain $\delta x/x$ (see Figure 6.1) at the outer surface will, in general, no longer be constant but will vary with the angle θ . In fact, for some combinations of properties and fiber volume fractions, the effective strain $\delta x/x$ may go from positive to negative as $\theta \rightarrow 90$ degrees. Thus, if the relation (6.2) above is used to obtain the shear modulus from the concentric-cylinder model, the value of ϵ_{xx} used must be some sort of average. The same problem of nonuniqueness

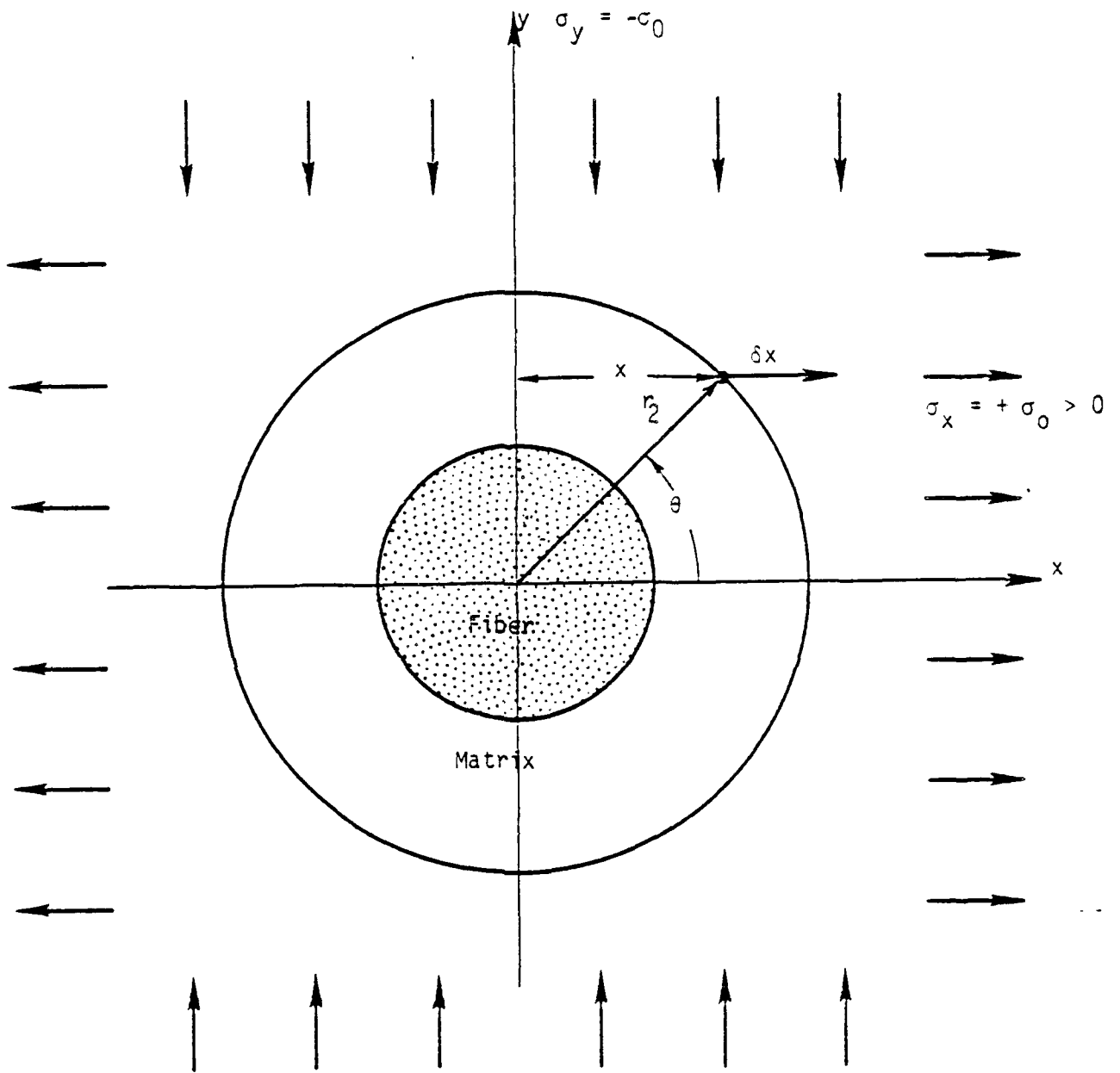


Figure 6.1 Concentric cylinder geometry.

occurs with Hashin's CCA model,^[8] the value for the transverse shear modulus quoted in Table 6.2 being the greater of calculated upper and lower bounds (see page 295 of Reference 8).

In the original version of PRUFC, a volume weighted average was used for ϵ_{xx} in Eq. (6.2) to obtain the transverse shear modulus, i.e.,

$$\bar{\epsilon}_{xx} = \frac{4}{\pi r_2^2} \int_0^{r_2} \left(\frac{\delta x}{x} \right) x dy , \quad (6.3)$$

δx being the displacement at the outer surface $r = r_2$. This procedure results in the values for the transverse properties given in Table 6.2. Since the transverse shear modulus obtained in this manner is significantly below that as calculated by the finite-element method, which is presumably the better value, several other averaging alternatives have been examined. Results are given in Figure 6.2, where the shear modulus is plotted versus fiber volume fraction.

The use of

$$\bar{\epsilon}_{xx} = \left(\frac{\delta x}{x} \right) \theta = 0 \quad (6.4)$$

results in a decrease in the calculated shear modulus relative to the volume weighted average (6.3), and a greater discrepancy with Hashin and Humphrey's result for the 30 percent composite. A closer agreement with the finite element result is obtained if one uses the linear average

$$\bar{\epsilon}_{xx} = \frac{1}{r_2} \int_0^{r_2} \frac{\delta x}{x} dy , \quad (6.5)$$

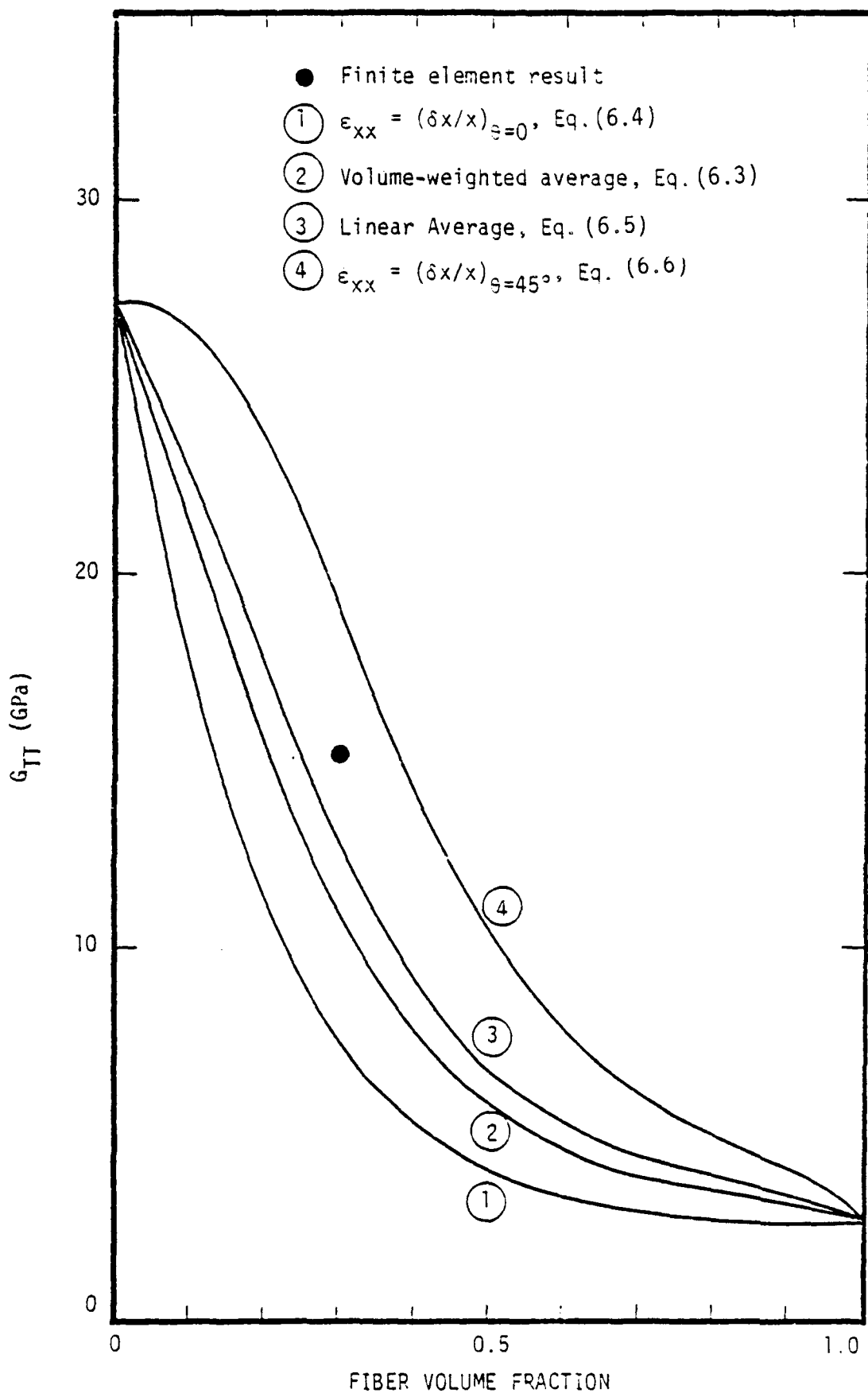


Figure 6.2. PRUFC results for the transverse shear modulus of a graphite/aluminum composite as a function of fiber volume fraction for various possible averaging procedures using stress boundary conditions. Also shown is the finite-element result of Hashin and Humphreys^[2] for 30 percent by volume of fibers.

but the PRUFC calculation is still low. Finally, the use of

$$\bar{\epsilon}_{xx} = \left(\frac{\delta x}{x} \right)_{\theta=45^\circ} \quad (6.6)$$

in Eq.(6.2) results in a PRUFC shear modulus greater than that obtained with the finite element method. The use of $(\delta x/x)$ at 45° suggests itself because this is also the value of

$$\frac{1}{2} \left(\frac{\delta x}{x} - \frac{\delta y}{y} \right) ,$$

which turns out to be independent of θ . Calculated results for v_{xy} and $\bar{\epsilon}_x$ using the various values of $\bar{\epsilon}_{xx}$ are given in Figures 6.3 and 6.4.

An alternative approach is to specify the displacement on the outer surface of the concentric cylinder model and to solve the equations of elasticity for the resulting stress field. As outlined in Ref. 9, the radial and hoop displacements are of the form

$$u^{(\alpha)}(r, \theta) = u^{(\alpha)}(r) \cos 2\theta \quad (6.7)$$

$$v^{(\alpha)}(r, \theta) = v^{(\alpha)}(r) \sin 2\theta , \quad (6.8)$$

and the condition now that the displacements on the outer surface,

$$\frac{\delta x}{x} = - \frac{\delta y}{y} = \epsilon_0 = \text{constant} \quad (6.9)$$

be the same as that on a cylindrical surface within a homogeneous material subjected to the uniform strain field $\epsilon_{xx} = -\epsilon_{yy} = \epsilon_0$, is satisfied by requiring that

$$v^{(2)}(r_2) = - u^{(2)}(r_2) . \quad (6.10)$$

The stress boundary conditions (B.9) and (B.10) of Ref. 9 are replaced by

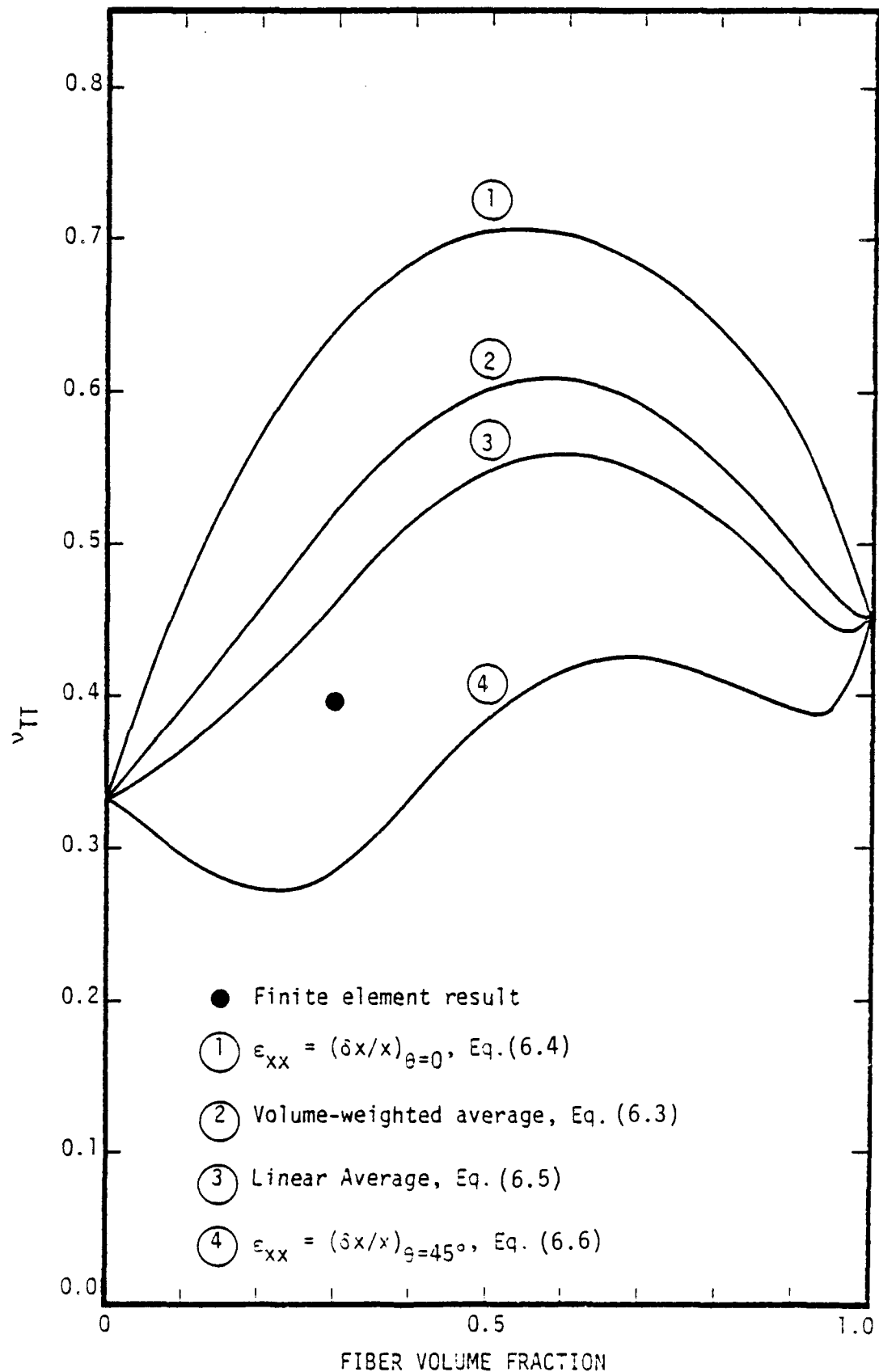


Figure 6.3. PRUFC results for the transverse Poisson's ratio of a graphite/aluminum composite as a function of fiber volume fraction for various possible averaging procedures using stress boundary conditions. Also shown is the finite-element result of Hashin and Humphreys[2] for 30 percent by volume of fibers.

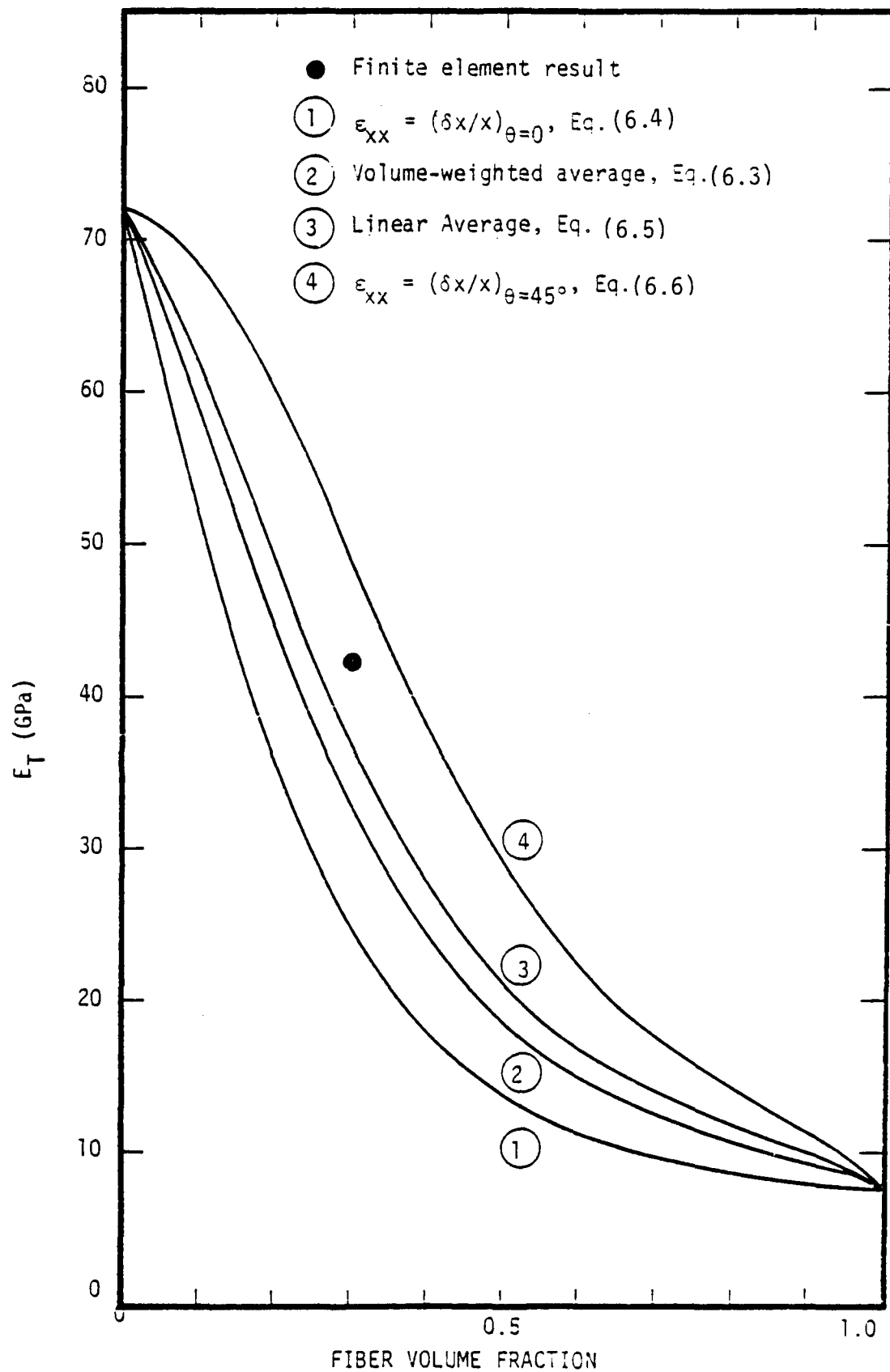


Figure 6.4. PRUFC results for the transverse elastic modulus of a graphite/aluminum composite as a function of fiber volume fraction for various possible averaging procedures using stress boundary conditions. Also shown is the finite-element result of Hashin and Humphreys[2] for 30 percent by volume of fibers.

$$u^{(2)}(r_2) = \epsilon_0 , \quad (6.11)$$

$$v^{(2)}(r_2) = - \epsilon_0 , \quad (6.12)$$

the other conditions, (B.5) - (B.8) remaining the same. Equations (6.11) and (6.12) follow from the relations for the cartesian components of displacement in terms of the radial and hoop components,

$$\delta x = u(r) \cos 2\theta \cos \theta - v(r) \sin 2\theta \sin \theta \quad (6.13)$$

$$\delta y = u(r) \cos 2\theta \sin \theta + v(r) \sin 2\theta \cos \theta . \quad (6.14)$$

As before, the boundary conditions lead to a set of six linear equations for the coefficients of the elastic solutions, Eqs.(B.27) of Ref. 9, the bottom two rows of the matrix K being replaced by

$$\begin{aligned} K_{51} &= 1, \quad K_{52} = 1, \quad K_{53} = 1, \\ K_{54} &= 1, \quad K_{55} = 0, \quad K_{56} = 0, \\ K_{61} &= -1, \quad K_{62} = -\frac{1}{2} \frac{(3 + \eta^{(2)})}{\eta^{(2)}} , \\ K_{63} &= -\frac{1}{2} (1 - \eta^{(2)}), \quad K_{64} = 1 , \\ K_{65} &= 0, \quad K_{66} = 0 , \end{aligned} \quad (6.15)$$

where the applied strain and outer radius have been normalized to $\epsilon = 1$, $r_2 = 1$. The coefficient K_{23} as given in Ref. 9 is in error. It should be

$$K_{23} = -\frac{1}{2} (1 - \eta^{(2)}) \frac{1}{r_1} .$$

From Eq. (B.13) of Ref. 9, the radial stress at the surface of the outer shell is given by

$$\begin{aligned}\sigma_{rr}^{(2)}(1, \theta) &= [C_{11}^{(2)}(A_2 + 3B_2 - C_2 - 3D_2) - C_{12}^{(2)}] \cos 2\theta \quad (6.16) \\ &\equiv \sigma_{rr0} \cos 2\theta\end{aligned}$$

and from (B.15) the shear stress is

$$\begin{aligned}\sigma_{r\theta}^{(2)}(1, \theta) &= C_{66}^{(2)} \left[-A_2 - \frac{3}{2} \frac{(3 + \eta^{(2)})}{\eta^{(2)}} B_2 \right. \\ &\quad \left. + \frac{1}{2} (1 - \eta^{(2)}) C_2 - 3D_2 - 1 \right] \sin 2\theta \\ &\equiv \sigma_{r\theta 0} \sin 2\theta\end{aligned} \quad (6.17)$$

(Actually, the above quantities are stresses per unit of applied strain, σ/ϵ_0). As in Eq. (6.2) above, the transverse shear modulus will be obtained from the relation

$$\bar{\sigma}_{xx} = 2G_{TT}\epsilon_0 \quad (6.18)$$

where $\bar{\sigma}_{xx}$ is some sort of average at the outer surface. In arriving at an average, we consider the x-component of the surface traction rather than $\sigma_{xx}(r, \theta)$ itself. In general, the force per unit area on a surface with normal unit vector $\hat{n}$ is given by

$$\vec{f} = \hat{n} \cdot \vec{\sigma}, \quad (6.19)$$

and for the cylindrical surface $\hat{n} = \hat{r}$ so that

$$\vec{f} = \hat{r} \sigma_{rr} + \hat{\theta} \sigma_{r\theta}, \quad (6.20)$$

and the x-component is

$$f_x = \sigma_{rr} \cos \theta - \sigma_{r\theta} \sin \theta \quad (6.21)$$

(Note that σ_{xx} itself also depends upon $\sigma_{\theta\theta}$). The stress per projected area perpendicular to the x-axis is

$$f_x^* = f_x r_2 \frac{d\theta}{dy} \quad (6.22)$$

and the linear average is then given by

$$\bar{\sigma}_{xx} = \bar{f}_x^* = \int_0^1 (\sigma_{rr} \cos \theta - \sigma_{r\theta} \sin \theta) \frac{d\theta}{dy} dy, \quad (6.23)$$

since $r_2 = 1$, and the result is

$$\bar{\sigma}_{xx} = \frac{1}{3} \sigma_{rr0} - \frac{2}{3} \sigma_{r\theta 0} \quad (6.24)$$

where σ_{rr0} and $\sigma_{r\theta 0}$ are defined by (6.16) and (6.17).

A comparison of the calculated transverse shear modulus using the linear-average, displacement boundary conditions with that from the linear-average, stress boundary conditions is given in Figure 6.5. Also plotted is the arithmetic mean of the two shear moduli, which although still high with respect to the finite element result appears to be the most plausible way to obtain the transverse properties using the concentric-cylinder approximation, and the PRUFC code has been revised accordingly. A comparison of results for the 30 percent composite using the revised PRUFC code with those obtained by the finite-element method is given in Table 6.3.

Hashin and Humphreys^[2] also present calculated results for a thermal process in which the yield strength of the aluminum matrix material in a 30 percent composite is exceeded. The yield-strength model they used is given as Figure 6.6, which is reproduced from Ref. 2. This isotropic hardening model was used in the S-CUBED residual stress code^[1] along with the fiber and matrix properties as given in Table 6.1, and two of the calculations were repeated. Figure 6.7 gives the axial strain as calculated with the S-CUBED model^[1] for a 30 percent composite, initially stress free at 371°C, as it is cooled to room temperature and reheated. Using the isotropic hardening model of Figure 6.6, the matrix first reaches

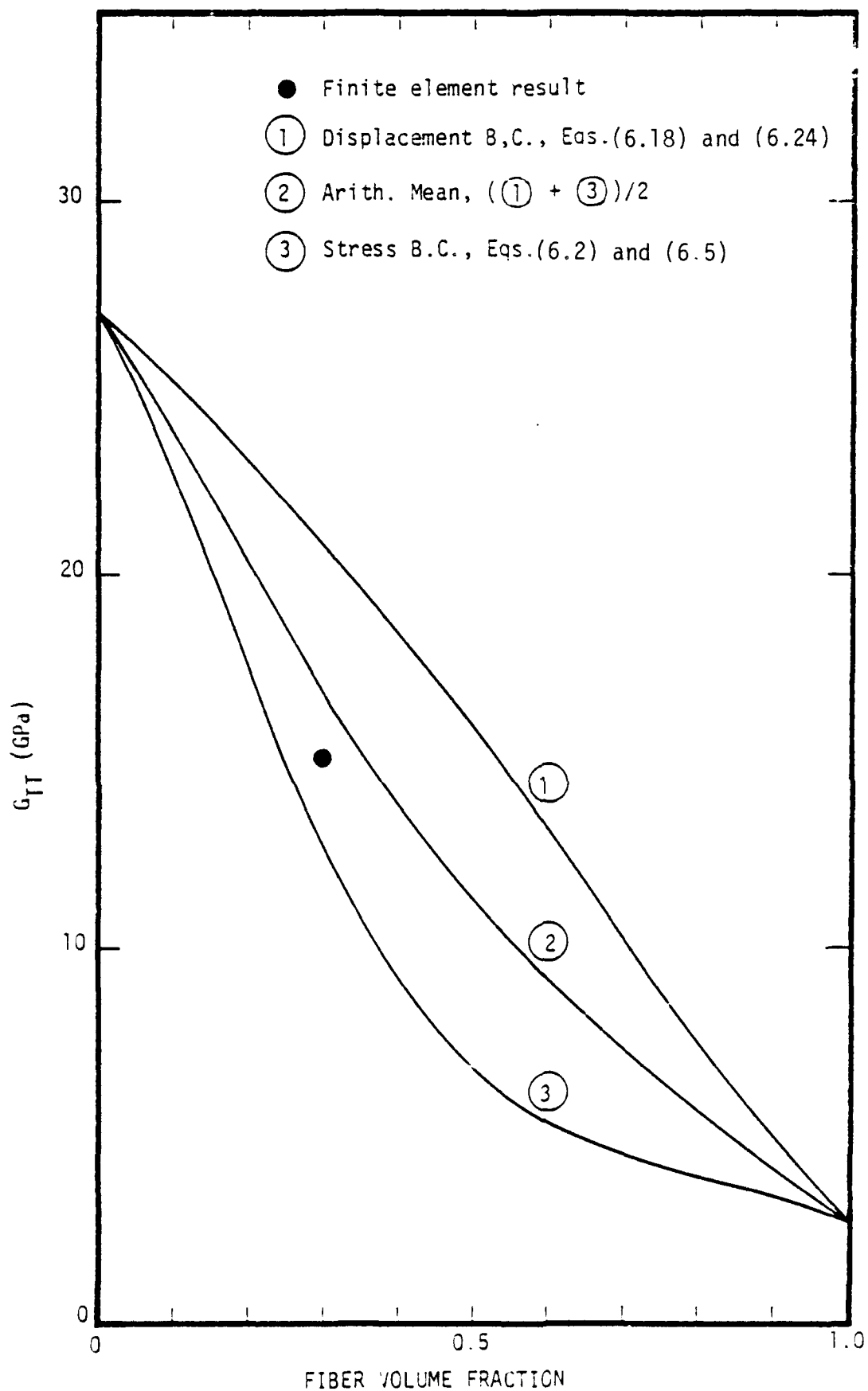


Figure 6.5. Comparison of the transverse shear modulus for a graphite/aluminum composite as calculated with displacement boundary conditions and stress boundary conditions. Also shown is the finite element result of Hashin and Humphreys.[2]

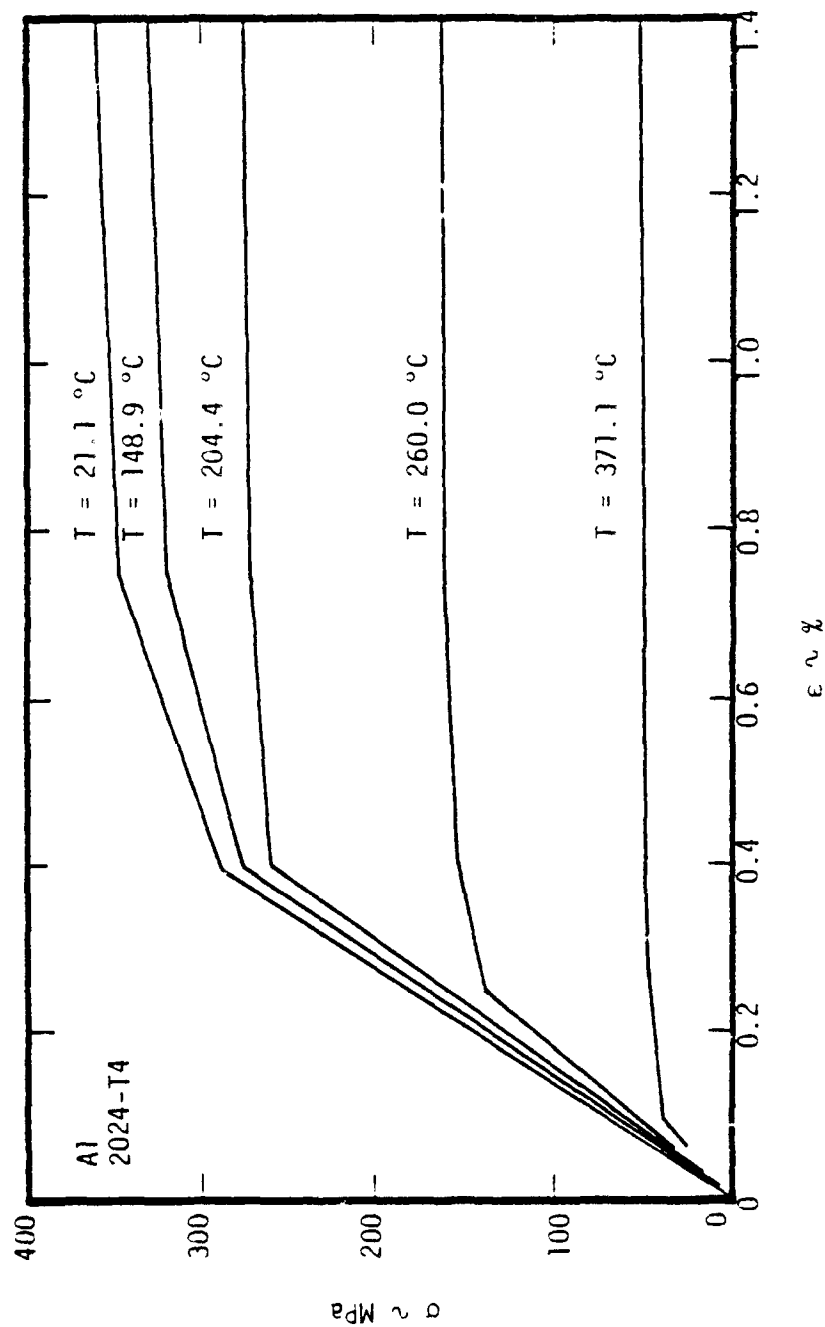


Figure 6.6. Isotropic hardening model used by Hashin and Humphreys, reproduced from Ref. 2.

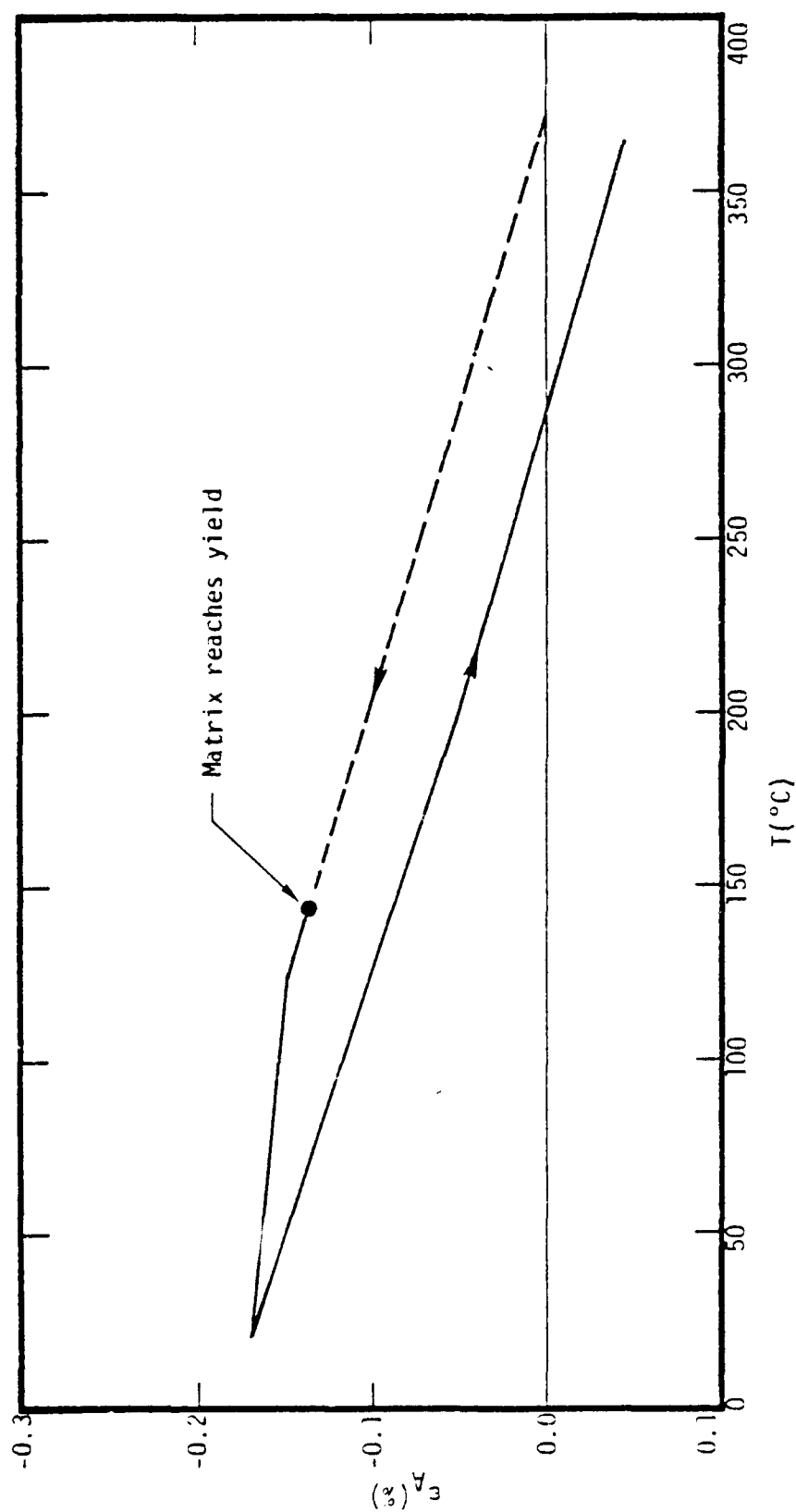


Figure 6.7. Axial strain as calculated with the S-CUBED residual stress model for a metal-matrix composite cooled from an initial stress-free state at 371°C. Results for the same process using a finite element method are given in Figure 6.8.

the yield surface at $\sim 142^\circ\text{C}$. Upon reheating from room temperature, the matrix immediately falls off the yield surface and does not

TABLE 6.3

CALCULATED PROPERTIES USING THE REVISED PRUFC* CODE
FOR A GRAPHITE/ALUMINUM COMPOSITE
(30 percent by volume of fibers).

Elastic Constant	PRUFC*	Finite-Element Method
E_A (GPa)	167.18	167.14
E_T (GPa)	45.22	42.26
G_{TT} (GPa)	16.80	15.13
G_{AT} (GPa)	22.82	23.20
ν_{TT}	0.346	0.396
ν_{AT}	0.338	0.340
α_A ($^\circ\text{C}^{-1}$)	6.29×10^{-6}	6.36×10^{-6}
α_T ($^\circ\text{C}^{-1}$)	25.65×10^{-6}	25.69×10^{-6}

*Revised as per above.

reach it again until about 350°C . The initial segment from 371°C to 142°C is shown as dashed because the present version of the S-CUBED code does not produce output for this interval. The results for the same process obtained by Hashin and Humphreys^[2] using the finite element method and ANSYS code are given in Figure 6.8. The agreement between the two types of calculation appears to be quite satisfactory.

Calculated results for the transverse strain for the same thermal cycle are given in Figures 6.9 and 6.10. Again the agreement is quite satisfactory. We conclude that, when their use is appropriate, the revised S-CUBED PRUFC code and concentric-cylinder residual-stress model⁽¹⁾ will give results for metal-matrix composites which are as satisfactory as those obtained with the more complex finite element method.

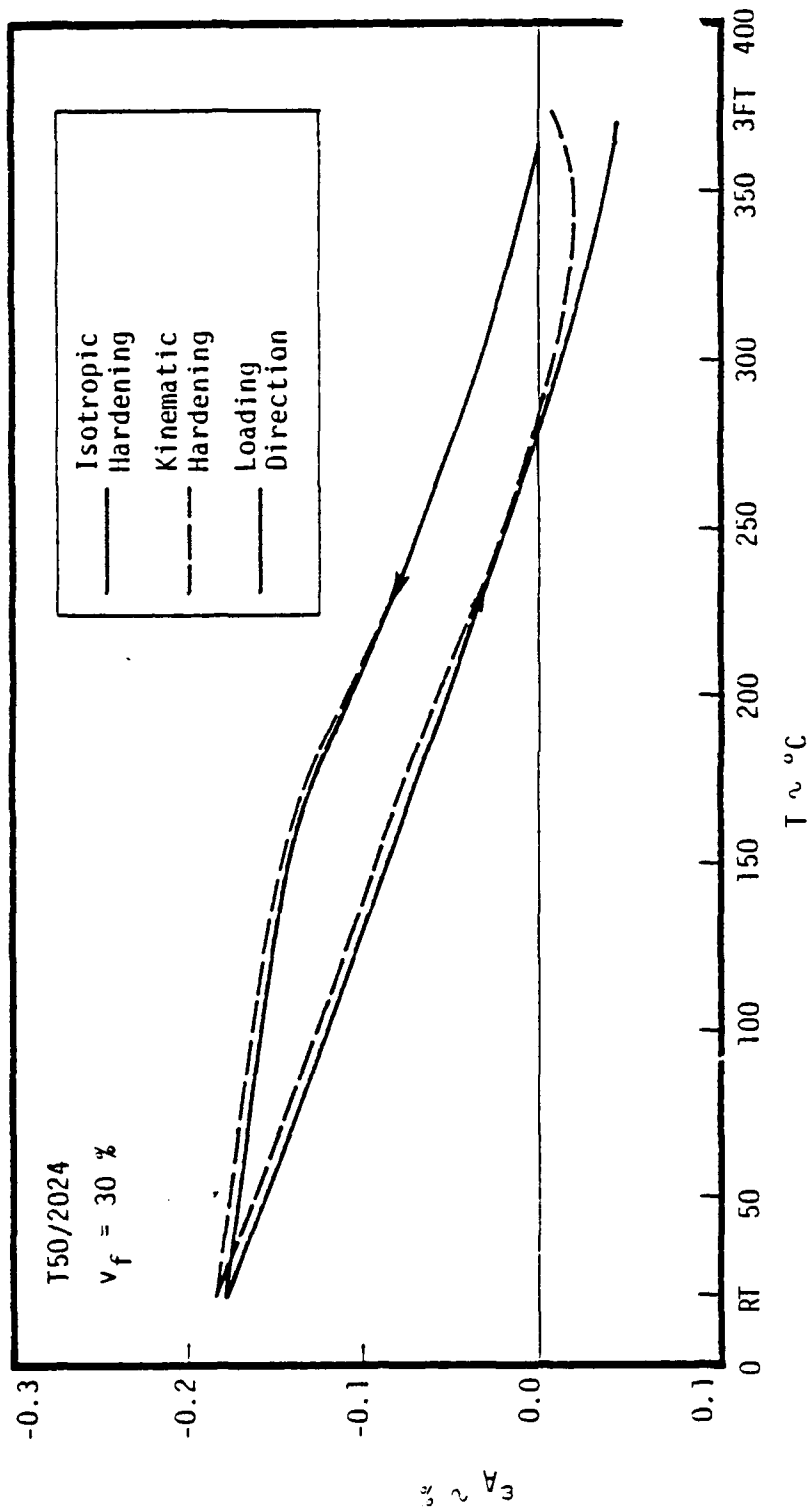


Figure 6.8. Hashin and Humphreys result for the axial strain of a metal-matrix composite, reproduced from Ref. 2. Compare with Figure 6.7.

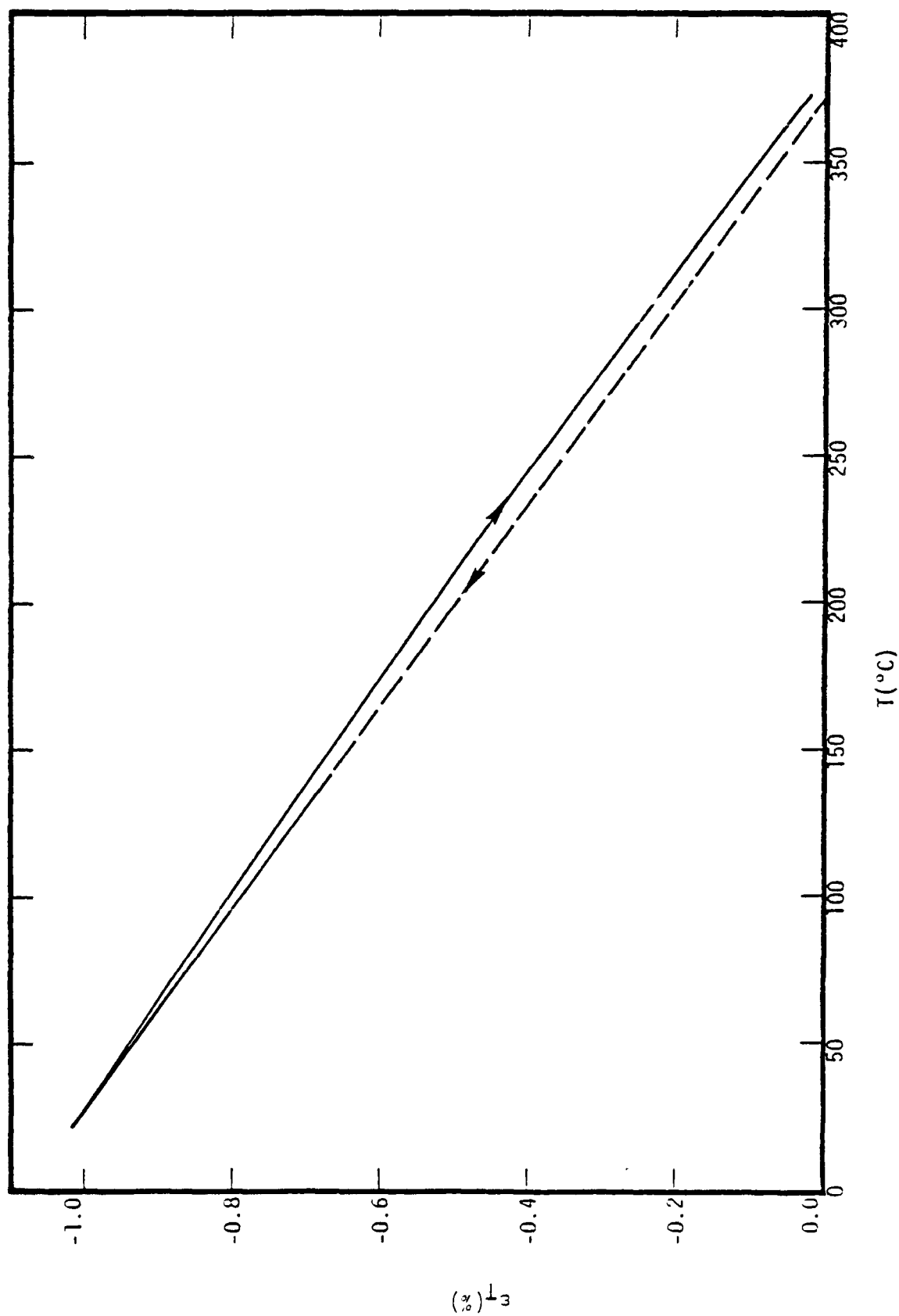


Figure 6.9. Transverse strain as calculated with the S-CUBED residual stress model for a metal-matrix composite cooled from an initial stress-free state at 371°C. Results for the same process using a finite element method are given in Figure 6.10.

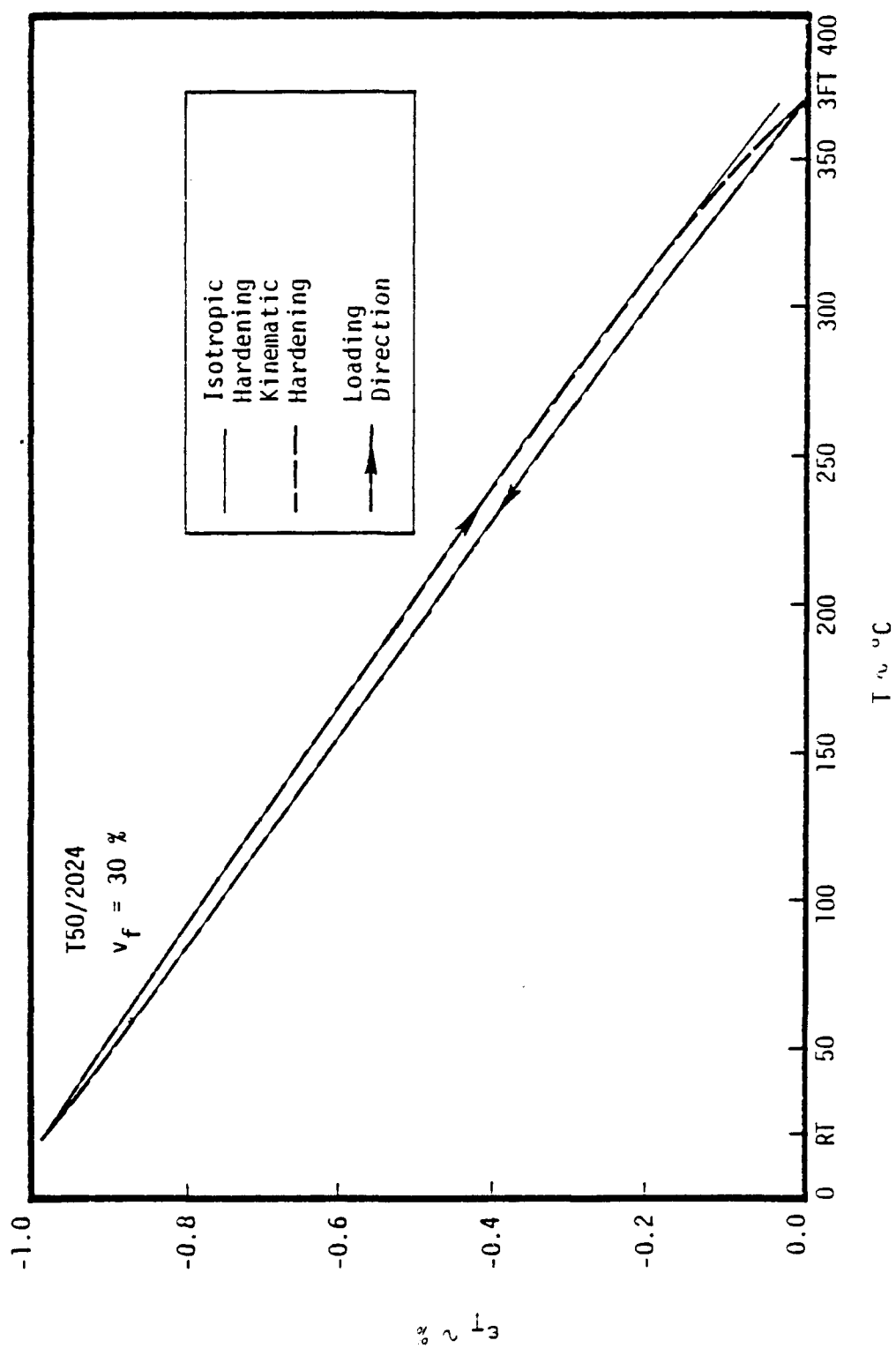


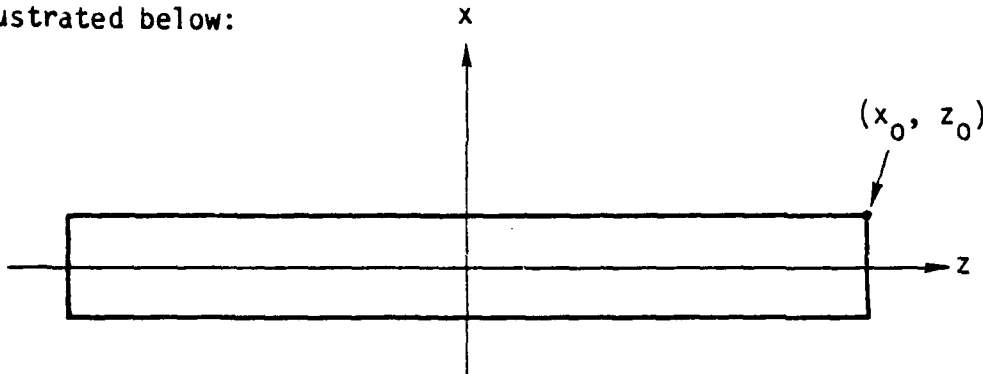
Figure 6.10. Hashin and Humphrey's result for the transverse strain of a metal-matrix composite, reproduced from Ref. 2. Compare with Figure 6.9.

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7. THERMAL DEFORMATIONS AND STRESSES IN AN ELASTIC PLATE UNDER AN ARBITRARY TEMPERATURE DISTRIBUTION

The material in this section has previously been submitted as a Progress Report to the Defense Supply Service (Ref. 10). It is reproduced here since it might prove useful in computing the permanent bending deformations of a metal-matrix composite under X-ray energy deposition.

The method of separation of variables is used to obtain the elastic displacements in a two-dimensional plate subject to an arbitrary transverse temperature distribution. The geometry is illustrated below:



The plate, of width $2x_0$ and length $2z_0$, is assumed to be anisotropic-transversely isotropic with the xy plane being the plane of isotropy. The stresses are then given by

$$\begin{aligned}
 \sigma_{xx} &= C_{11} \frac{\partial u_1}{\partial x} + C_{12} \frac{\partial u_2}{\partial y} + C_{13} \frac{\partial u_3}{\partial z} - \gamma_1 T(x) \\
 \sigma_{yy} &= C_{12} \frac{\partial u_1}{\partial x} + C_{11} \frac{\partial u_2}{\partial y} + C_{13} \frac{\partial u_3}{\partial z} - \gamma_1 T(x) \\
 \sigma_{zz} &= C_{13} \frac{\partial u_1}{\partial x} + C_{13} \frac{\partial u_2}{\partial y} + C_{33} \frac{\partial u_3}{\partial z} - \gamma_3 T(x) \\
 \sigma_{zx} &= C_{44} \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \right)
 \end{aligned}
 \tag{7.1}$$

where u_1 , u_2 , u_3 are the displacements in the x , y , and z directions, $T(x)$ is the applied temperature distribution, and γ_1 , γ_3 are the stress-temperature coupling coefficients, which are given in terms of the linear expansion coefficients β_1 , β_3 by

$$\begin{aligned}\gamma_1 &= (C_{11} + C_{12}) \beta_1 + C_{13} \beta_3 \\ \gamma_3 &= 2C_{13} \beta_1 + C_{33} \beta_3\end{aligned}\quad (7.2)$$

With appropriate changes, as indicated below, the method presented here could be adapted to an isotropic plate.

In the case of plane strain, $\epsilon_{yy} = 0$, the two-dimensional problem of interest here is

$$\begin{aligned}\sigma_{xx} &= C_{11} \frac{\partial u_1}{\partial x} + C_{13} \frac{\partial u_3}{\partial z} - \gamma_1 T(x) \\ \sigma_{zz} &= C_{13} \frac{\partial u_1}{\partial x} + C_{33} \frac{\partial u_3}{\partial z} - \gamma_3 T(x) \\ \sigma_{zx} &= C_{44} \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \right)\end{aligned}\quad (7.3)$$

The above relations are also valid for the case of plane stress, $\sigma_{yy} = 0$, provided that the actual elastic constants are replaced by

$$\begin{aligned}C_{11} &\rightarrow C_{11} - \frac{C_{12}^2}{C_{11}} \quad , \quad C_{13} \rightarrow C_{13} - \frac{C_{12}C_{13}}{C_{11}} \\ C_{33} &\rightarrow C_{33} - \frac{C_{13}^2}{C_{11}} \quad , \quad \gamma_1 \rightarrow \gamma_1 \left(1 - \frac{C_{12}}{C_{11}} \right) \\ \gamma_3 &\rightarrow \gamma_3 - \frac{C_{13}}{C_{11}} \gamma_1\end{aligned}\quad (7.4)$$

The equations for elastic equilibrium,

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} &= 0\end{aligned}\tag{7.5}$$

become with the use of (7.3)

$$\begin{aligned}C_{11} \frac{\partial^2 u_1}{\partial x^2} + C_{44} \frac{\partial^2 u_1}{\partial z^2} + (C_{13} + C_{44}) \frac{\partial^2 u_3}{\partial x \partial z} &= \gamma_1 \frac{\partial T}{\partial x} \\ C_{33} \frac{\partial^2 u_3}{\partial z^2} + C_{44} \frac{\partial^2 u_3}{\partial x^2} + (C_{13} + C_{44}) \frac{\partial^2 u_1}{\partial x \partial z} &= 0\end{aligned}\tag{7.6}$$

In the next two sections, we outline the solution of the set of equations (7.6) subject to the condition that the shear and normal stresses vanish at the surface of the plate. The formal solution itself is exact; the numerical evaluation, however, is approximate in the sense that the Fourier coefficients are obtained from the solution of a system of coupled equations of the form

$$\begin{aligned}C_{1m} &= a_m + \sum_n b_{mn} A_{1n} \\ A_{1n} &= f_n + \sum_m g_{nm} C_{1m},\end{aligned}\tag{7.7}$$

where A_{1n} and C_{1m} are the desired expansion coefficients, and a_m , f_n , b_{mn} , g_{nm} are constants. Thus, computational considerations limit one to the use of a few hundred terms.

In Section 7.1 we outline the solution for an applied temperature distribution which is an odd function of x , and in Section 7.2 the solution appropriate for an even function. The displacements corresponding to an arbitrary temperature function are obtained from a superposition of the two solutions.

Numerical results are presented in Section 7.3. In particular, we present calculated shear strains for a discontinuous applied temperature distribution of the form

$$T(x) = \begin{cases} 0 & , -x_0 < x < 0.8 x_0 \\ 1 & , 0.8x_0 \leq x \leq x_0 \end{cases}$$

This problem is mathematically equivalent to the uniform heating of two bonded plates which have the same elastic constants but different coefficients of thermal expansion.

7.1 ODD TEMPERATURE FUNCTION

This part of the solution has been reported in Reference 9, but in order that this report be self-contained, is repeated here. For an odd temperature distribution, the appropriate combination of functions to represent the elastic displacements is

$$\begin{aligned} u_1(x, z) &= \sum_n B_n(x) \cos kz + \sum_m D_m(z) \cos jx \\ u_3(x, z) &= \sum_n A_n(x) \sin kz + \sum_m C_m(z) \sin jx \end{aligned} \quad (7.9)$$

where

$$\begin{aligned} k &= (n + \frac{1}{2}) \frac{\pi}{z_0} , \quad n = 0, 1, 2, \dots \\ j &= (m + \frac{1}{2}) \frac{\pi}{x_0} , \quad m = 0, 1, 2, \dots \end{aligned} \quad (7.10)$$

and $A_n(x)$ and $C_m(z)$ are odd functions of their respective variables, and $B_n(x)$ and $D_m(z)$ are even functions. The functions $A_n(x)$, $B_n(x)$, $C_m(z)$ and $D_m(z)$ are subject to the following constraints as determined from the boundary conditions:

- 1) Zero shear stress along (x_0, z)

$$A'_n(x_0) - kB_n(x_0) = 0 \quad (7.11)$$

2) Zero shear stress along (x, z_0)

$$jC_m(z_0) + D'_m(z_0) = -\frac{2}{x_0} \sum_n (-1)^n \int_0^{x_0} [A'_n(x) - kB_n(x)] \cos jx dx \quad (7.12)$$

3) Zero normal stress along (x_0, z)

$$C_{13}kA_n(x_0) + C_{11}B'_n(x_0) = -\frac{2}{z_0} \sum_m (-1)^m \int_0^{z_0} [C_{13}C'_m(z) - jC_{11}D_m(z)] \cos kz dz + \frac{2}{z_0} \gamma_1 T(x_0) \frac{(-1)^n}{k} \quad (7.13)$$

4) Zero normal stress along (x, z_0)

$$C_{33}C'_m(z_0) - C_{13}jD_m(z_0) = \frac{2}{x_0} \gamma_3 \int_0^{x_0} T(x) \sin jx dx \quad (7.14)$$

The differential equations for $A_n(x)$, $B_n(x)$, $C_m(z)$, $D_m(z)$ are obtained from a substitution of equations (7.9) into equations (7.6). The resulting equations for $A_n(x)$ and $B_n(x)$ are

$$\begin{aligned} C_{44}A''_n(x) - C_{33}k^2A_n(x) - (C_{13} + C_{44})kB'_n(x) &= 0 \\ C_{11}B''_n(x) - C_{44}k^2B_n(x) + (C_{13} + C_{44})kA'_n(x) &= 0 \end{aligned} \quad (7.15)$$

and for $C_m(z)$ and $D_m(z)$ are

$$\begin{aligned} C_{33}C''_m(z) - C_{44}j^2C_m(z) - (C_{13} + C_{44})jD'_m(z) &= 0 \\ C_{44}D''_m(z) - C_{11}j^2D_m(z) + (C_{13} + C_{44})jC'_m(z) &= 0 \end{aligned} \quad (7.16)$$

$$= \frac{2}{x_0} \gamma_1 \int_0^{x_0} \frac{\partial T}{\partial x} \cos jx dx .$$

The appropriate solutions are

$$A_n(x) = A_{1n} \frac{\sinh(a_1 kx)}{\cosh(a_1 kx_0)} + A_{2n} \frac{\sinh(a_2 kx)}{\cosh(a_2 kx_0)} \quad (7.17)$$

$$B_n(x) = A_{1n} S_1 \frac{\cosh(a_1 kx)}{\cosh(a_1 kx_0)} + A_{2n} S_2 \frac{\cosh(a_2 kx)}{\cosh(a_2 kx_0)}$$

and

$$C_m(z) = C_{1m} \frac{\sinh(b_1 jz)}{\cosh(b_1 jz_0)} + C_{2m} \frac{\sinh(b_2 jz)}{\cosh(b_2 jz_0)} \quad (7.18)$$

$$D_m(z) = C_{1m} R_1 \frac{\cosh(b_1 jz)}{\cosh(b_1 jz_0)} + C_{2m} R_2 \frac{\cosh(b_2 jz)}{\cosh(b_2 jz_0)}$$

$$- \frac{2}{x_0} \frac{\gamma_1}{C_{11} j^2} \int_0^x \frac{\partial T}{\partial x} \cos jx dx ,$$

where A_{1n} , A_{2n} , C_{1m} , C_{2m} are constant expansion coefficients to be determined from the boundary conditions (7.11 - 7.14). The hyperbolic cosines in the denominators were factored out to facilitate the avoidance of overflow during numerical evaluation. The quantities a_1 , a_2 are the positive roots of the equation

$$C_{11}C_{44}a^4 - (C_{11}C_{33} - 2C_{13}C_{44} - C_{13}^2)a^2 + C_{33}C_{44} = 0 \quad (7.19)$$

with

$$S_i = \frac{(C_{44}a_i^2 - C_{33})}{(C_{13} + C_{44})a_i} , \quad i = 1, 2$$

and b_1 , b_2 are the positive roots of

$$C_{33}C_{44}b^4 - (C_{11}C_{33} - 2C_{13}C_{44} - C_{13}^2)b^2 + C_{11}C_{44} = 0 \quad (7.20)$$

with

$$R_i = \frac{(C_{33}b_i^2 - C_{44})}{(C_{13} + C_{44})b_i}, \quad i = 1, 2$$

Note that in the limit of isotropy, equation (7.19) has only one positive root, $a^2 = 1$, and the solution for $A_n(x)$ becomes a linear combination of $\sinh(kx)$ and $x \cosh(kx)$.

Substitution of (7.17) and (7.18) into the boundary conditions (7.11 - 7.14) yields the following equations for the expansion coefficients,

- 1) Zero shear stress along (x_0, z)

$$A_{1n}(a_1 - S_1) + A_{2n}(a_2 - S_2) = 0 \quad (7.21)$$

- 2) Zero shear stress along (x, z_0)

$$\begin{aligned} & C_{1m}(1 + R_1b_1) \tanh(b_1jz_0) + C_{2m}(1 + R_2b_2) \tanh(b_2jz_0) \\ & = -\frac{2}{x_0} (-1)^m \sum_n (-1)^n k \left[A_{1n} \frac{a_1 - S_1}{j^2 + a_1^2 k^2} + A_{2n} \frac{a_2 - S_2}{j^2 + a_2^2 k^2} \right] \end{aligned} \quad (7.22)$$

- 3) Zero normal stress along (x_0, z)

$$\begin{aligned} & A_{1n}(C_{13} + C_{11}S_1a_1) \tanh(a_1kx_0) \\ & + A_{2n}(C_{13} + C_{11}S_2a_2) \tanh(a_2kx_0) \\ & = -\frac{2}{z_0} (-1)^n \sum_m (-1)^m j \left[C_{1m} \frac{C_{13}b_1 - C_{11}R_1}{k^2 + b_1^2 j^2} + C_{2m} \frac{C_{13}b_2 - C_{11}R_2}{k^2 + b_2^2 j^2} \right] \end{aligned} \quad (7.23)$$

In arriving at (7.23) from (7.13), use has been made of the fact that $T(x)$ is an odd function and that

$$T(x_0) = \frac{2}{x_0} \sum_m (-1)^m \int_0^{x_0} T(x) \sin jx dx.$$

4) Zero normal stress along (x, z_0)

$$C_{1m} (C_{33}b_1 - C_{13}R_1) + C_{2m} (C_{33}b_2 - C_{13}R_2) \quad (7.24)$$

$$= \frac{2}{x_0} \left(\gamma_3 - \gamma_1 \frac{C_{13}}{C_{11}} \frac{1}{j} \right) \int_0^{x_0} T(x) \sin jx dx .$$

From (7.24),

$$C_{2m} = - Q_2 C_{1m} + Q_1 \frac{1}{j} \int_0^{x_0} T(x) \sin jx dx \quad (7.25)$$

where

$$Q_1 \equiv \frac{2}{x_0} \frac{\left(\gamma_3 - \gamma_1 \frac{C_{13}}{C_{11}} \right)}{\left(C_{33}b_2 - C_{13}R_2 \right)} \quad (7.26)$$

$$Q_2 \equiv \frac{C_{33}b_1 - C_{13}R_1}{C_{33}b_2 - C_{13}R_2} \quad (7.27)$$

Note that Q_1 above is defined differently than in Reference 9.
From (7.21)

$$A_{2n} = - Q_3 A_{1n} \quad (7.28)$$

$$Q_3 \equiv \frac{a_1 - S_1}{a_2 - S_2} \quad (7.29)$$

Substitution of (7.25) and (7.28) into (7.22) and (7.23) gives two sets of coupled equations for the coefficients A_{1n} and C_{1m} ,

$$\begin{aligned} C_{1m} Q_{5m} = & - Q_1 \frac{1}{j} \left[\int_0^{x_0} T(x) \sin jx dx \right] (1 + R_2 b_2) \tanh (b_2 j z_0) \\ & - \frac{2}{x_0} (-1)^m \sum_n (-1)^n k \left[\frac{a_1 - S_1}{j^2 + a_1^2 k^2} - Q_3 \frac{a_2 - S_2}{j^2 + a_2^2 k^2} \right] A_{1n} \end{aligned} \quad (7.30)$$

$$\begin{aligned}
A_{1n}Q_{4n} = & -\frac{2}{z_0} (-1)^n Q_1 (C_{13}b_2 - C_{11}R_2) \sum_m (-1)^m \frac{\int_0^{x_0} T(x) \sin jx dx}{k^2 + b_2^2 j^2} \\
& - \frac{2}{z_0} (-1)^n \sum_m (-1)^m j \left[\frac{C_{13}b_1 - C_{11}R_1}{k^2 + b_1^2 j^2} \right. \\
& \left. - Q_2 \frac{C_{13}b_2 - C_{11}R_2}{k^2 + b_2^2 j^2} \right] C_{1m}, \quad (7.31)
\end{aligned}$$

where

$$Q_{5m} \equiv (1 + R_1 b_1) \tanh(b_1 j z_0) - Q_2 (1 + R_2 b_2) \tanh(b_2 j z_0), \quad (7.32)$$

$$Q_{4n} \equiv (C_{13} + C_{11} S_1 a_1) \tanh(a_1 k x_0) - Q_3 (C_{13} + C_{11} S_2 a_2) \tanh(a_2 k x_0). \quad (7.33)$$

The coupled sets of equations (7.30) and (7.31) are of the form

$$\begin{aligned}
C_{1m} &= a_m + \sum_n b_{mn} A_{1n} \\
A_{1n} &= f_n + \sum_m g_{nm} C_{1m}
\end{aligned} \quad (7.34)$$

which can be solved with the following iteration procedure:

$$\begin{aligned}
C_{1m}^{(0)} &= a_m \\
A_{1n}^{(1)} &= f_n + \sum_m g_{nm} C_{1m}^{(0)} \\
C_{1m}^{(1)} &= a_m + \sum_n b_{mn} A_{1n}^{(1)} \\
A_{1n}^{(2)} &= f_n + \sum_m g_{nm} C_{1m}^{(1)}, \text{ etc.}
\end{aligned}$$

7.2 EVEN TEMPERATURE FUNCTION

For an even temperature distribution, the appropriate combination of functions for the displacements is

$$\begin{aligned} u_1(x, z) &= \sum_n B_n^*(x) \cos kz + \sum_m D_m^*(z) \sin jx \\ u_3(x, z) &= \sum_n A_n^*(x) \sin kz + \sum_m C_m^*(z) \cos jx, \end{aligned} \quad (7.36)$$

where as before,

$$k = (n + \frac{1}{2}) \frac{\pi}{z_0}, \quad n = 0, 1, 2, \dots$$

$$j = (m + \frac{1}{2}) \frac{\pi}{x_0}, \quad m = 0, 1, 2, \dots$$

and $B_n^*(x)$ and $C_m^*(z)$ are odd functions of their respective variables, and $A_n^*(x)$ and $D_m^*(z)$ are even functions. The boundary conditions impose the following constraints:

- 1) Zero shear stress along (x_0, z)

$$A_n^*(x_0) - kB_n^*(x_0) - \frac{2}{z_0} \sum_m (-1)^m \int_0^{z_0} [jC_m^*(z) - D_m^*(z)] \sin kz dz = 0 \quad (7.36)$$

- 2) Zero shear stress along (x, z_0)

$$jC_m^*(z_0) - D_m^*(z_0) = \frac{2}{x_0} \sum_n (-1)^n \int_0^{x_0} [A_n^*(x) - kB_n^*(x)] \sin jx dx \quad (7.37)$$

- 3) Zero normal stress along (x_0, z)

$$C_{13}kA_n^*(x_0) + C_{11}B_n^*(x_0) = \frac{2}{z_0} \gamma_1 T(x_0) \frac{(-1)^n}{k} \quad (7.38)$$

4) Zero normal stress along (x, z_0)

$$C_{33}C_m^{*'}(z_0) + C_{13}j D_m^{*'}(z_0) = \frac{2}{x_0} \gamma_3 \int_0^{x_0} T(x) \cos jx dx \quad (7.39)$$

The equations (7.6) for elastic equilibrium are unchanged, as are the equations (7.15) for $A_n^{*}(x)$ and $B_n^{*}(x)$. The equations for $C_m^{*}(z)$ and $D_m^{*}(z)$ become

$$\begin{aligned} C_{33}C_m^{*''}(z) - C_{44}j^2 C_m^{*}(z) + (C_{13} + C_{44}) j D_m^{*'}(z) &= 0 \\ C_{44}D_m^{*''}(z) - C_{11}j^2 D_m^{*}(z) - (C_{13} + C_{44}) j C_m^{*'}(z) \\ &= \frac{2}{x_0} \gamma_1 \int_0^{x_0} \frac{\partial T}{\partial x} \sin jx dx . \end{aligned}$$

The functions $A_n^{*}(x)$ and $B_n^{*}(x)$ are now

$$\begin{aligned} A_n^{*}(x) &= A_{1n}^{*} \frac{\cosh(a_1 kx)}{\cosh(a_1 kx_0)} + A_{2n}^{*} \frac{\cosh(a_2 kx)}{\cosh(a_2 kx_0)} \\ B_n^{*}(x) &= A_{1n}^{*} S_1 \frac{\sinh(a_1 kx)}{\cosh(a_1 kx_0)} + A_{2n}^{*} S_2 \frac{\sinh(a_2 kx)}{\cosh(a_2 kx_0)} \end{aligned} \quad (7.40)$$

and $C_m^{*}(z)$ and $D_m^{*}(z)$ become

$$\begin{aligned} C_m^{*}(z) &= C_{1m}^{*} \frac{\sinh(b_1 jz)}{\cosh(b_1 jz_0)} + C_{2m}^{*} \frac{\sinh(b_2 jz)}{\cosh(b_2 jz_0)} \\ D_m^{*}(z) &= C_{1m}^{*} R_1^{*} \frac{\cosh(b_1 jz)}{\cosh(b_1 jz_0)} + C_{2m}^{*} R_2^{*} \frac{\cosh(b_2 jz)}{\cosh(b_2 jz_0)} \\ &\quad - \frac{2}{x_0} \frac{\gamma_1}{C_{11}} \frac{1}{j^2} \int_0^{x_0} \frac{\partial T}{\partial x} \sin jx dx . \end{aligned} \quad (7.41)$$

The quantities a_1, a_2 are the positive roots of

$$C_{11}C_{33}a^4 - (C_{11}C_{33} - 2C_{13}C_{44} - C_{13}^2)a^2 + C_{33}C_{44} = 0$$

with

$$S_i = \frac{(C_{33}a_i^2 - C_{33})}{(C_{13} + C_{44})a_i}, \quad i = 1, 2 \quad (7.42)$$

the same as for the odd temperature function; and b_1, b_2 are the positive roots of

$$C_{33}C_{44}b^4 - (C_{11}C_{33} - 2C_{13}C_{44} - C_{13}^2)b^2 + C_{11}C_{44} = 0,$$

but now

$$R_i^* = - \frac{(C_{33}b_i^2 - C_{44})}{(C_{13} + C_{44})b_i}, \quad i = 1, 2. \quad (7.43)$$

The boundary conditions give the following relations for the expansion coefficients,

- 1) Zero shear stress along (x_0, z)

$$\begin{aligned} & A_{1n}^* (a_1 - S_1) \tanh(a_1 k x_0) + A_{2n}^* (a_2 - S_2) \tanh(a_2 k x_0) \\ &= \frac{2}{z_0} \frac{(-1)^n}{k} \sum_m (-1)^m j^2 \left[\frac{b_1 (1 - R_1^* b_1)}{k^2 + b_1^2 j^2} C_{1m}^* + \frac{b_2 (1 - R_2^* b_2)}{k^2 + b_2^2 j^2} C_{2m}^* \right] \end{aligned} \quad (7.44)$$

- 2) Zero shear stress along (x, z_0)

$$\begin{aligned} & C_{1m}^* (1 - R_1^* b_1) \tanh(b_1 j z_0) + C_{2m}^* (1 - R_2^* b_2) \tanh(b_2 j z_0) \\ &= \frac{2}{x_0} \frac{(-1)^m}{j} \sum_n (-1)^n k^2 \left[\frac{a_1 (a_1 - S_1)}{j^2 + a_1^2 k^2} A_{1n}^* + \frac{a_2 (a_2 - S_2)}{j^2 + a_2^2 k^2} A_{2n}^* \right] \end{aligned} \quad (7.45)$$

3) Zero normal stress along $(x_0 z)$

$$\begin{aligned} & A_{1n}^* (C_{13} + C_{11} S_1 a_1) + A_{2n}^* (C_{13} + C_{11} S_2 a_2) \\ &= \frac{2}{z_0} \gamma_1 T(x_0) \frac{(-1)^n}{k^2} \end{aligned} \quad (7.46)$$

4) Zero normal stress along (x, z_0)

$$\begin{aligned} & C_{1m}^* (C_{33} b_1 + C_{13} R_1^*) + C_{2m}^* (C_{33} b_2 + C_{13} R_2^*) \\ &= \frac{2}{x_0} \frac{\gamma_3}{j} \int_0^{x_0} T(x) \cos jx dx + \frac{2}{x_0} \frac{\gamma_1 C_{13}}{C_{11}} \frac{1}{j^2} \int_0^{x_0} \frac{\partial T}{\partial x} \sin jx dx \end{aligned} \quad (7.47)$$

From (7.46),

$$A_{2n}^* = -Q_1^* A_{1n}^* + Q_2^* T(x_0) \frac{(-1)^n}{k^2} \quad (7.48)$$

with

$$Q_1^* \equiv \frac{(C_{13} + C_{11} S_1 a_1)}{(C_{13} + C_{11} S_2 a_2)} \quad (7.49)$$

$$Q_2^* \equiv \frac{2\gamma_1}{z_0 (C_{13} + C_{11} S_2 a_2)} \quad (7.50)$$

and from (7.47),

$$\begin{aligned} C_{2m}^* &= -Q_3^* C_{1m}^* + Q_4^* \frac{1}{j} \int_0^{x_0} T(x) \cos jx dx \\ &+ Q_5^* \frac{1}{j^2} \int_0^{x_0} \frac{\partial T}{\partial x} \sin jx dx \end{aligned} \quad (7.51)$$

with

$$Q_3^* = \frac{(C_{33}b_1 + C_{13}R_1^*)}{(C_{33}b_2 + C_{13}R_2^*)} \quad (7.52)$$

$$Q_4^* = \frac{2\gamma_3}{x_0(C_{33}b_2 + C_{13}R_2^*)} \quad (7.53)$$

$$Q_5^* = \frac{2\gamma_1 C_{13}}{x_0 C_{11}(C_{33}b_2 + C_{13}R_2^*)} \quad (7.54)$$

Substitution into (7.44) and (7.45) yields the following coupled equations for A_{1n}^* and C_{1m}^* :

$$C_{1m}^* Q_{5m}^* = - (1-R_2^*b_2) \tanh(b_2 j z_0) \left[\frac{Q_4^*}{j} \int_0^{x_0} T(x) \cos j x dx + \frac{Q_5^*}{j^2} \int_0^{x_0} \frac{\partial T}{\partial x} \sin j x dx \right] \quad (7.55)$$

$$+ \frac{2}{x_0} \frac{(-1)^m}{j} a_2(a_2 - S_2) Q_2^* T(x_0) \sum_n \frac{1}{j^2 + a_2^2 k^2} + \frac{2}{x_0} \frac{(-1)^m}{j} \sum_n (-1)^n k^2 \left[\frac{a_1(a_1 - S_1)}{j^2 + a_2^2 k^2} - Q_1^* \frac{a_2(a_2 - S_2)}{j^2 + a_2^2 k^2} \right] A_{1n}^*$$

$$A_{1n}^* Q_{4n}^* = - Q_2^* T(x_0)(a_2 - S_2) \tanh(a_2 k x_0) \frac{(-1)^n}{k^2} + \frac{2b_2(1-R_2^*b_2)}{z_0} \frac{(-1)^n}{k} \sum_m (-1)^m \frac{1}{k^2 + b_2^2 j^2} \left[j Q_4^* \int_0^{x_0} T(x) \cos j x dx + Q_5^* \int_0^{x_0} \frac{\partial T}{\partial x} \sin j x dx \right] \quad (7.56)$$

$$+ \frac{2}{z_0} \frac{(-1)^n}{k} \sum_m (-1)^m j^2 \left[\frac{b_1(1-R_1^*b_1)}{k^2 + b_1^2 j^2} - Q_3 \frac{b_2(1-R_2^*b_2)}{k^2 + b_2^2 j^2} \right] C_{1m}^*$$

where

$$\begin{aligned} Q_{5m}^* &= (1-R_1^*b_1) \tanh(b_1jz_0) - Q_3^*(1-R_2^*b_2) \tanh(b_2jz_0) \\ Q_{4n}^* &= (a_1-S_1) \tanh(a_1kx_0) - Q_1^*(a_2-S_2) \tanh(a_2kx_0) \end{aligned} \quad (7.57)$$

Equations (7.55) and (7.56) are also of the form of (7.34) and can be solved with the same iteration procedure.

7.3 COMPUTATIONAL RESULTS

In this section we present calculated results for several discontinuous temperature distributions. The input, plane-stress elastic constants, the same as those used for the calculations reported in Reference 9, are appropriate to a 50 percent aluminum-graphite composite,

$$\begin{aligned} C_{11} &= 3.187 \text{ Msi} \\ C_{13} &= 1.118 \text{ Msi} \\ C_{33} &= 55.401 \text{ Msi} \\ C_{44} &= 3.298 \text{ Msi} \\ \gamma_1 &= 5.033 \times 10^{-5} \text{ Msi/}^\circ\text{F} \\ \gamma_3 &= 3.705 \times 10^{-5} \text{ Msi/}^\circ\text{F.} \end{aligned}$$

The thermal expansion coefficients are

$$\begin{aligned} \beta_1 &= 15.705 \times 10^{-6} \text{ }^\circ\text{F}^{-1} \\ \beta_3 &= 0.352 \times 10^{-6} \text{ }^\circ\text{F}^{-1} \end{aligned}$$

and the length to width ratio is 10,

$$\begin{aligned} x_0 &= 1 \\ z_0 &= 10 \end{aligned}$$

Figure 7.1 is a plot of the shear strain as a function of z along $x = 0$ for the odd temperature distribution

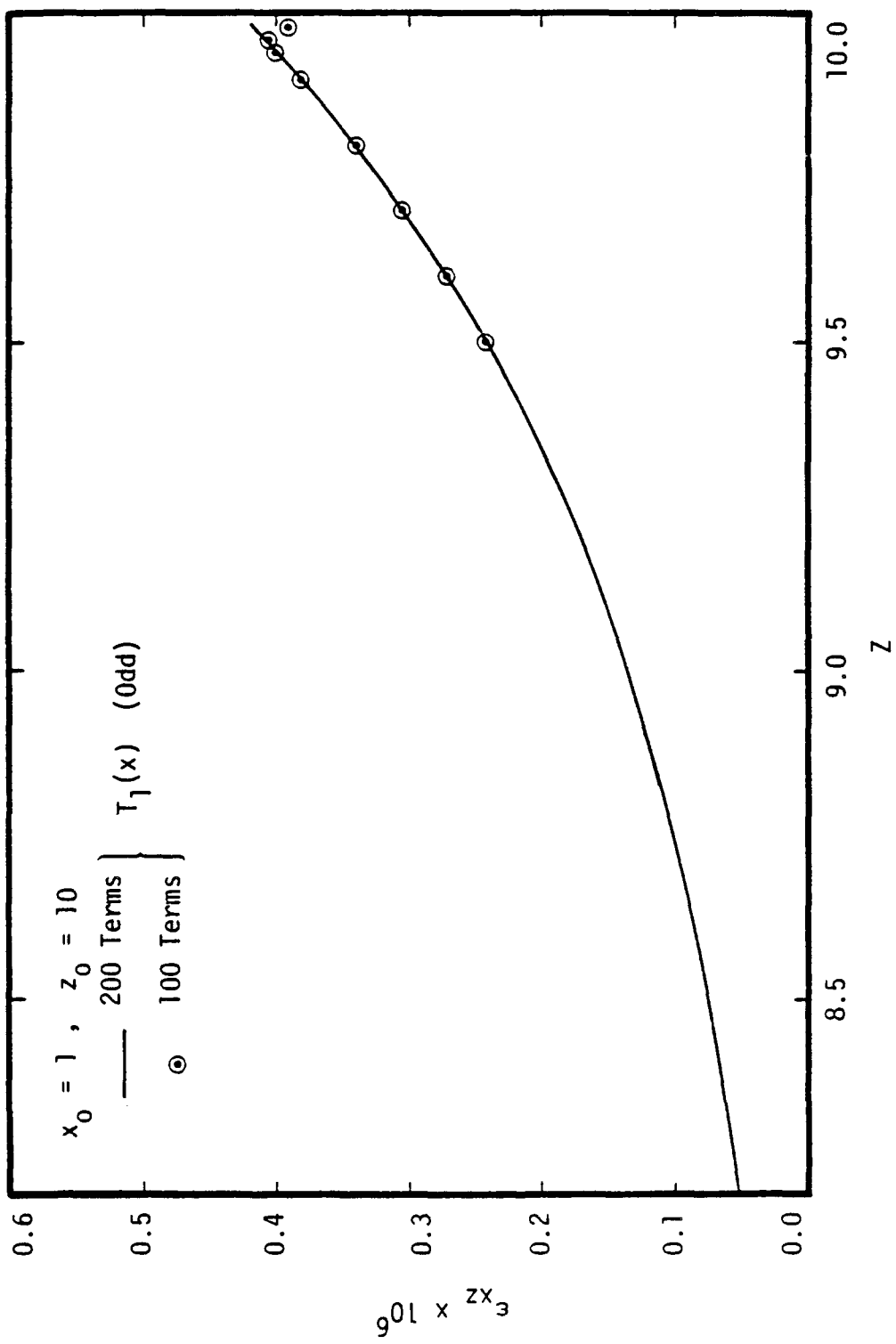


Figure 7.1. Shear strain along $z = 0$ for the odd temperature distribution $T_1(x)$.

$$T_1(x) = \begin{cases} -\frac{1}{2} & , \quad -x_0 \leq x < 0 \\ +\frac{1}{2} & , \quad 0 < x \leq x_0 \end{cases} \quad (7.58)$$

The solid line indicates the shear strain as calculated with 200 coefficients, and the dots indicate selected values as calculated with 100 coefficients. The numerical results give no indication of a singularity in the stress field; and, indeed, from Boggy's work on wedges (Reference 11), one would not expect one.

Figure 7.2 shows the results of two calculations for the shear strain as a function of z along $x = 0.8 x_0$ for the odd and even temperature distributions.

$$T_2(x) = \begin{cases} -\frac{1}{2} & , \quad -x_0 < x < -0.8 x_0 \\ 0 & , \quad -0.8 x_0 < x < +0.8 x_0 \\ -\frac{1}{2} & , \quad 0.8 x_0 < x < x_0 \end{cases} \quad (7.59)$$

and

$$T_3(x) = \begin{cases} +\frac{1}{2} & , \quad -x_0 < x < -0.8 x_0 \\ 0 & , \quad -0.8 x_0 < x < +0.8 x_0 \\ +\frac{1}{2} & , \quad 0.8 x_0 < x < x_0 \end{cases} \quad (7.60)$$

The shape of the curve corresponding to the even temperature distribution $T_3(x)$ was unexpected, but so far no algebraic or coding errors have been found. In any case, it has been verified that the longitudinal force at $z = 0$,

$$\int_{0.8 x_0}^{x_0} \sigma_{zz}(x, 0) dx \quad ,$$

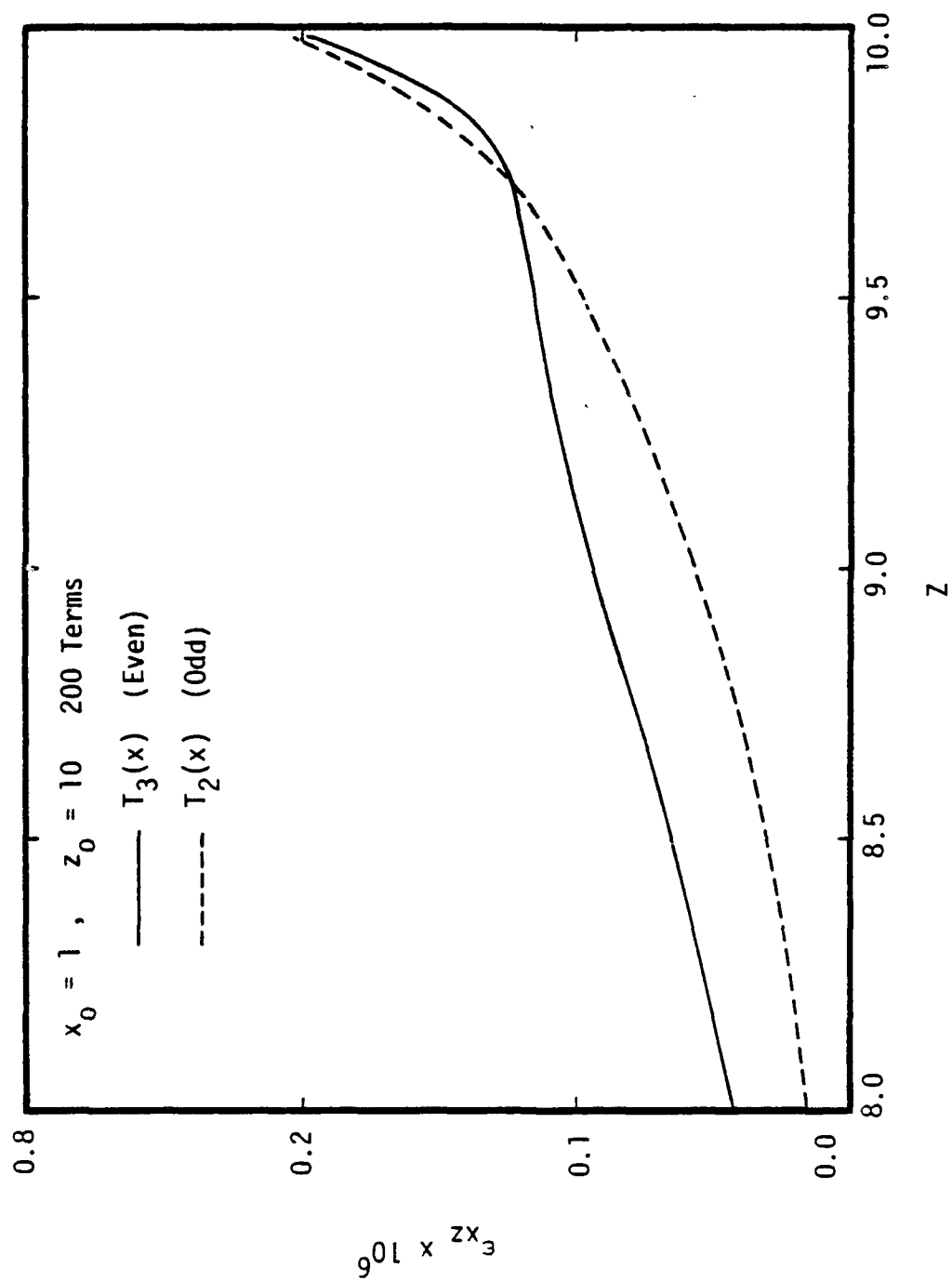


Figure 7.2. Shear strain along $x = 0.8x_0$ for the odd and even temperature distributions $T_2(x)$ and $T_3(x)$.

balances that of the shear stress,

$$\int_0^{z_0} \sigma_{xz} (0.8 x_0, z) dz .$$

A plot of the longitudinal strain ϵ_{zz} as a function of z along $x = 0.8 x_0$, corresponding to the temperature distributions $T_2(x)$ and $T_3(x)$, is given as Figure 7.3.

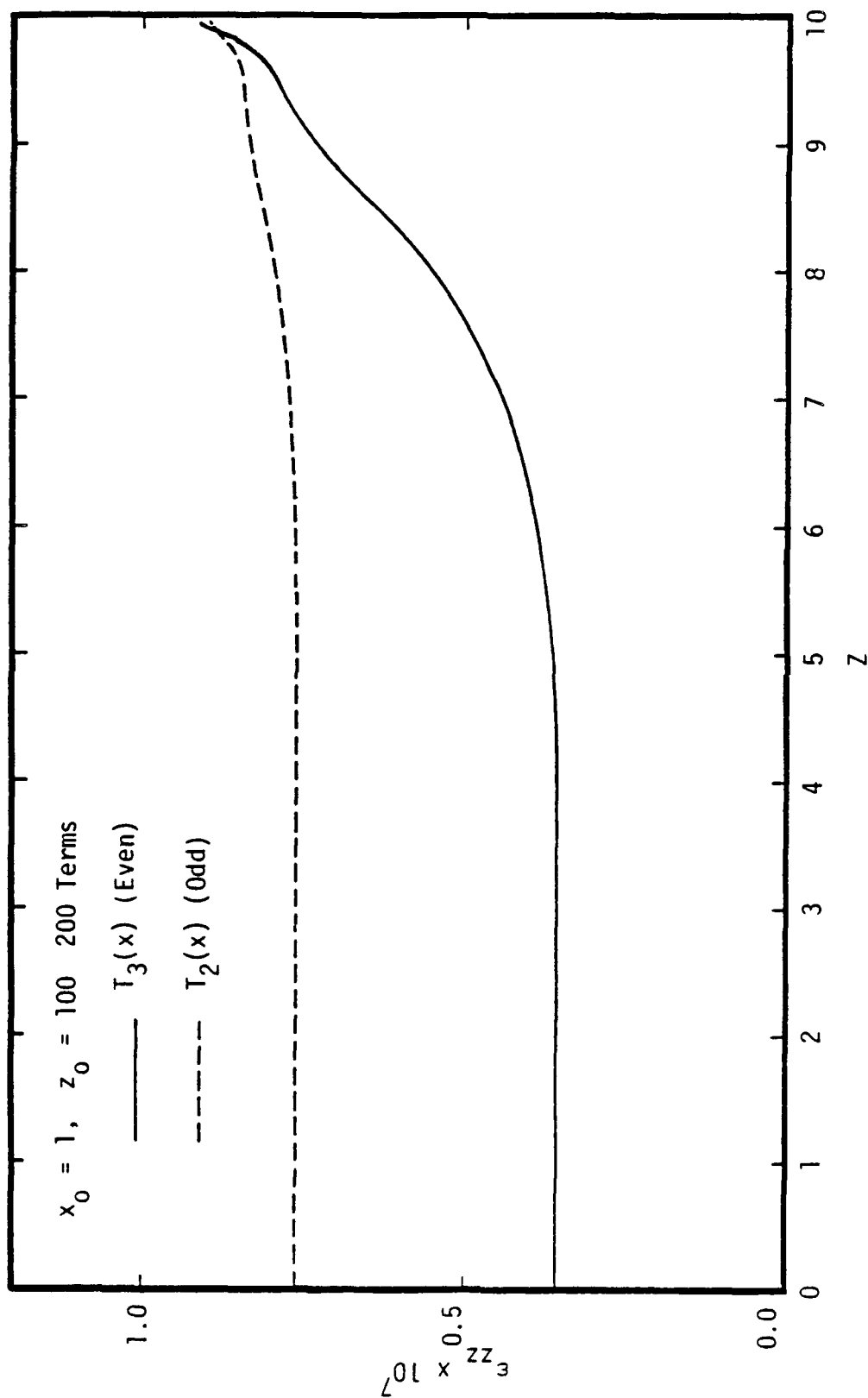


Figure 7.3. Longitudinal strain ϵ_{zz} along $x = 0.8x_0$ for the odd and even temperature distributions $T_2(x)$ and $T_3(x)$.

8. SUMMARY AND CONCLUSIONS

- a. For G/Al where the matrix is in a T4 temper (rapid quench to -240°F):

Axial stresses in the aluminum at room temperature are tensile and lie in the range 30 to 40 ksi.

Subsequent reheating will lower the axial stress, and the matrix will remain elastic up to a temperature of approximately 600°F , at which point the axial stress will have become compressive.

Residual hoop stresses in the matrix are tensile and range from approximately 10 to 20 ksi for all cases in which the fibers' transverse coefficient of thermal expansion remains less than or equal to that of the matrix.

The interfacial radial stress is compressive and ranges from 2 to 10 ksi so long as the fibers' transverse coefficient of thermal expansion does not exceed that of the matrix. This would appear to be a most desirable result in maintaining the integrity of these materials, since they are currently limited by their ability to bear significant radial tensile stresses across this interface.

- b. For G/Al where the matrix is in a T0 temper (slow quench to -240°F), the axial residual stresses are near zero at room temperature.
- c. For G/Al in the T4 temper, a reduction of approximately 5 to 10 ksi in the room temperature axial residual stress could be expected if the composite was quenched to -320°F .

- d. For the W/Al and SiC/Al composites, residual stress results are qualitatively similar to those of G/Al of equivalent temper.
- e. Calculated axial stress strain behavior of SiC/Al using a yielded matrix material in the T0 temper condition is in reasonable agreement with experimental data up to 200 ksi.
- f. For a three-layer laminate of G/Al, minimum longitudinal coefficient of thermal expansion occurs at a ply angle of approximately 20 degrees (i.e., 20/0/20).
- g. PRUFC yields G/Al thermoelastic properties in good agreement with those reported by Hashin and Humphries who used a finite element analysis (Ref.2).
- h. The S-C'BED residual stress code yields results for G/Al which are in good agreement with the ANSYS finite element results reported by Hashin and Humphries (Ref. 2).

REFERENCES

1. Rice, M. H. and G. A. Gurtman, "Residual Stresses in Fiber Reinforced Metal Matrix Composites," S-CUBED Report SSS-R-82-5447, March 1982 (Submitted to DARPA).
2. Hashin, Z. and E. A. Humphreys, "Elevated Temperature Behavior of Metal-Matrix Composites," Materials Sciences Corporation Report MSC TFR 1214/1502, November 1981 (AFOSR-TR-82-0212).
3. Rice, M. H. and G. A. Gurtman, "PRUFC: A Computer Code to Calculate the Properties of Unidirectional Fibrous Composites," S-CUBED Report SSS-R-81-4940, April 1981.
4. Rice, M. H. and G. A. Gurtman, "Residual Stresses and Mechanical Properties of SiC/Al composites," S-CUBED Report SSS-R-83-5934, December 1982.
5. Appendix II of Silicon Carbide -- 1973, eds., R. C. Marshall, J. W. Faust, Jr., and C. E. Ryan, University of S. Carolina Press (1973).
6. Hill, R., The Mathematical Theory of Plasticity, Oxford University Press (1950).
7. Private communication from D. J. A. Cornie (AVCO-SMC) to G. A. Gurtman (S-CUBED).
8. Hashin, Z., "Theory of Fiber Reinforced Materials," NASA Contractor Report NASA CR-1974, March 1972.
9. Gurtman, G. A., M. H. Rice and A. Maewal, "Thermomechanical Analysis of Graphite/Metal Matrix Composites," S-CUBED Report SSS-R-81-4862, February 1981.
10. Rice, M. H. and G. A. Gurtman, "Thermal Deformation of an Elastic Plate," S-CUBED Progress Report SSS-R-81-5075, July 1981.
11. Bogy, D. B., "Two Edge-Bonded Elastic Wedges of Different Materials and Wedge Angles Under Surface Traction," J. App. Mecn, 38, 377(1971).
12. Metals Handbook, Ninth Edition, Volume 2, "Properties and Selection: Nonferrous Alloys and Pure Metals," American Society for Metals (1979).
13. Military Handbook, MIL-HDBK-5A, "Metallic Materials and Elements for Aerospace Vehicle Structures," February 1966.

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APPENDIX A
ALUMINUM PROPERTIES

Thermoelastic properties and yield-strength models for 6061 and 2024 aluminum alloys are summarized below.

A.1 6061 ALUMINUM

For the 6061 aluminum matrix, the elastic modulus was input as a function of temperature, linear interpolation being used between the following tabular points in Table A1:

TABLE A1
ELASTIC MODULUS FOR 6061 ALUMINUM

T(°F)	E(Msi)
-459	11.8
75	10.0
240	9.8
400	9.0
600	7.0
1100	0.0

The Poisson's ratio was also allowed to vary with temperature according to

$$\nu = 0.5 - E/58.823 \quad (A.1)$$

where E is the elastic modulus in Msi as computed from the table above. The thermal expansion coefficient for the aluminum matrix was taken to be temperature dependent according to the tabulated points in Table A2:

TABLE A2
 LINEAR COEFFICIENT OF THERMAL EXPANSION FOR
 A1 6061 AND A1 2024

T (°F)	$\alpha (10^{-6} \text{ } ^\circ\text{F}^{-1})$
- 400	0
- 300	5
- 200	9
0	12
1100	18.5

Residual stress calculations were done with two different yield strength models. A model appropriate to 6061 A1 in the annealed state, or T0 temper, is depicted in Figure A1. The room-temperature yield stress at 0.2 percent offset is 8 ksi in accord with Ref. 12 (the elastic modulus at room temperature is taken as 10 Msi). A yield model appropriate to 6061 A1 in the T4 temper (rapid quenching) is shown in Figure A2. Here the yield stress at room temperature at the 0.2 percent offset point is 21 ksi, as given in Ref. 12. The shapes of the room-temperature stress-strain curves for both tempers were scaled from that for A1 201 in the 0 temper, for which experimental data were available (see p. 25 of Ref. 1). The variation of yield stress with temperature is scaled proportionately to that of other aluminums (see, for example, Ref. 13).

A.2 2024 ALUMINUM

For the 2024 aluminum matrix, the elastic modulus as a function of temperature was given by the tabulated points in Table A3:

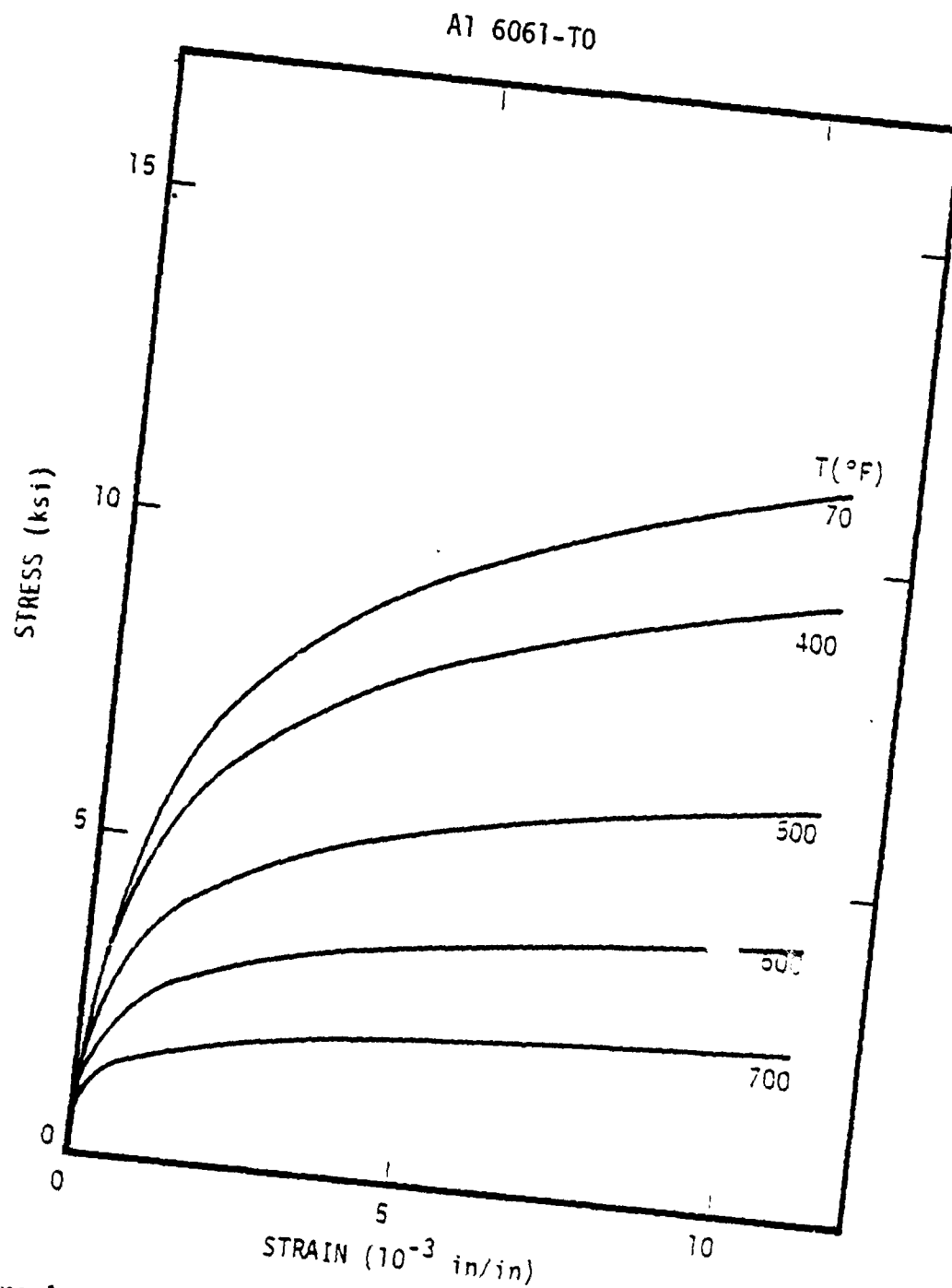


Figure A-1. Yield stress model for 6061 aluminum in the T0 temper (annealed).

Al 6061-T4

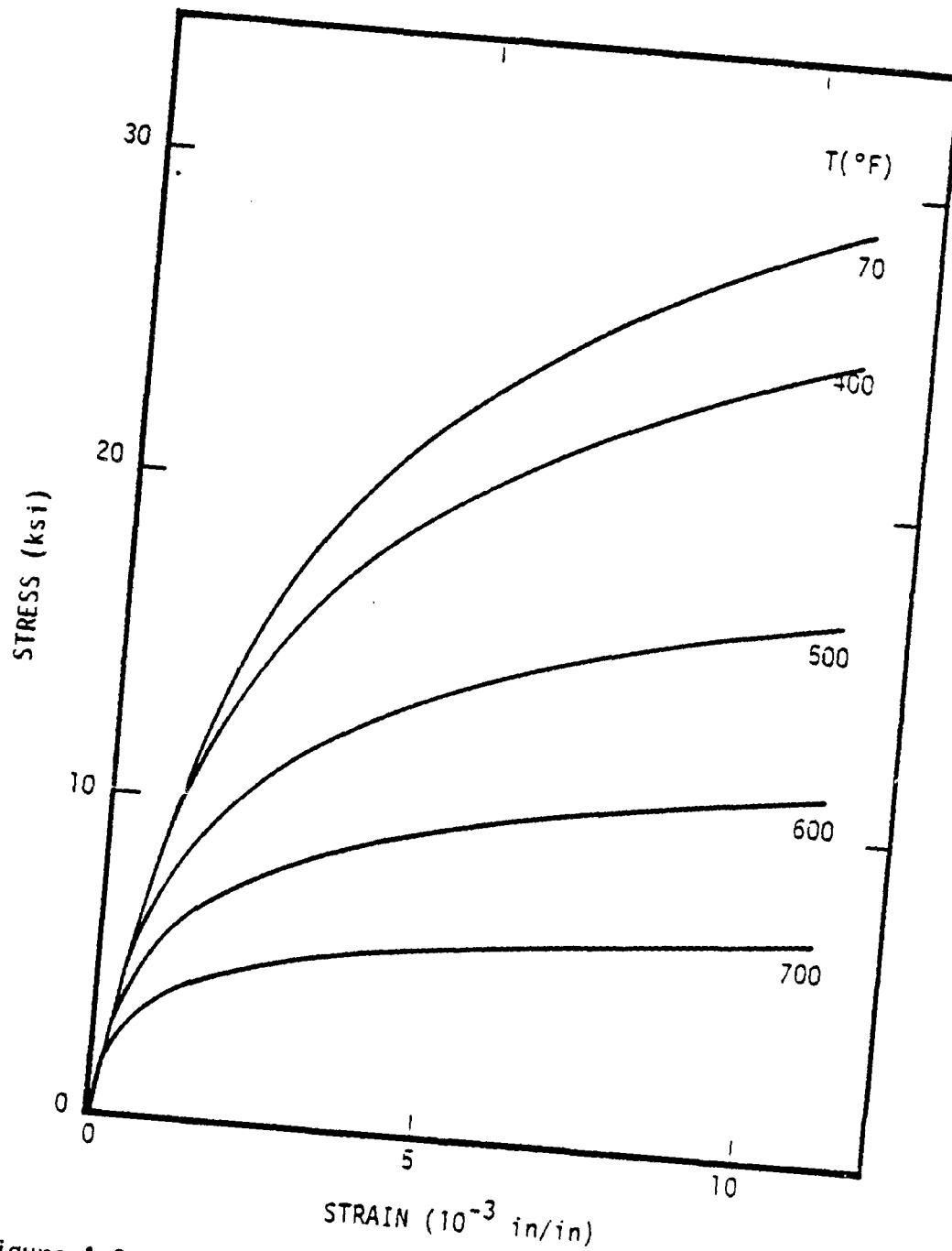


Figure A-2. Yield stress model for 6061 aluminum in the T4 temper.

TABLE A3
ELASTIC MODULUS FOR 2024 ALUMINUM

T(°F)	E(Msi)
- 459	12.39
75	10.5
240	10.29
400	9.45
600	7.35
1100	0

The Poisson's ratio was allowed to vary with temperature according to

$$\nu = 0.5 - E/61.765 \quad , \quad (A.2)$$

where E is the elastic modulus in Msi from Table A3. The thermal expansion coefficient was taken to be the same as that for Al 6061 as given in Table A2.

The yield stress model for the T0 (annealed) and T4 tempers are shown in Figures A3 and A4.

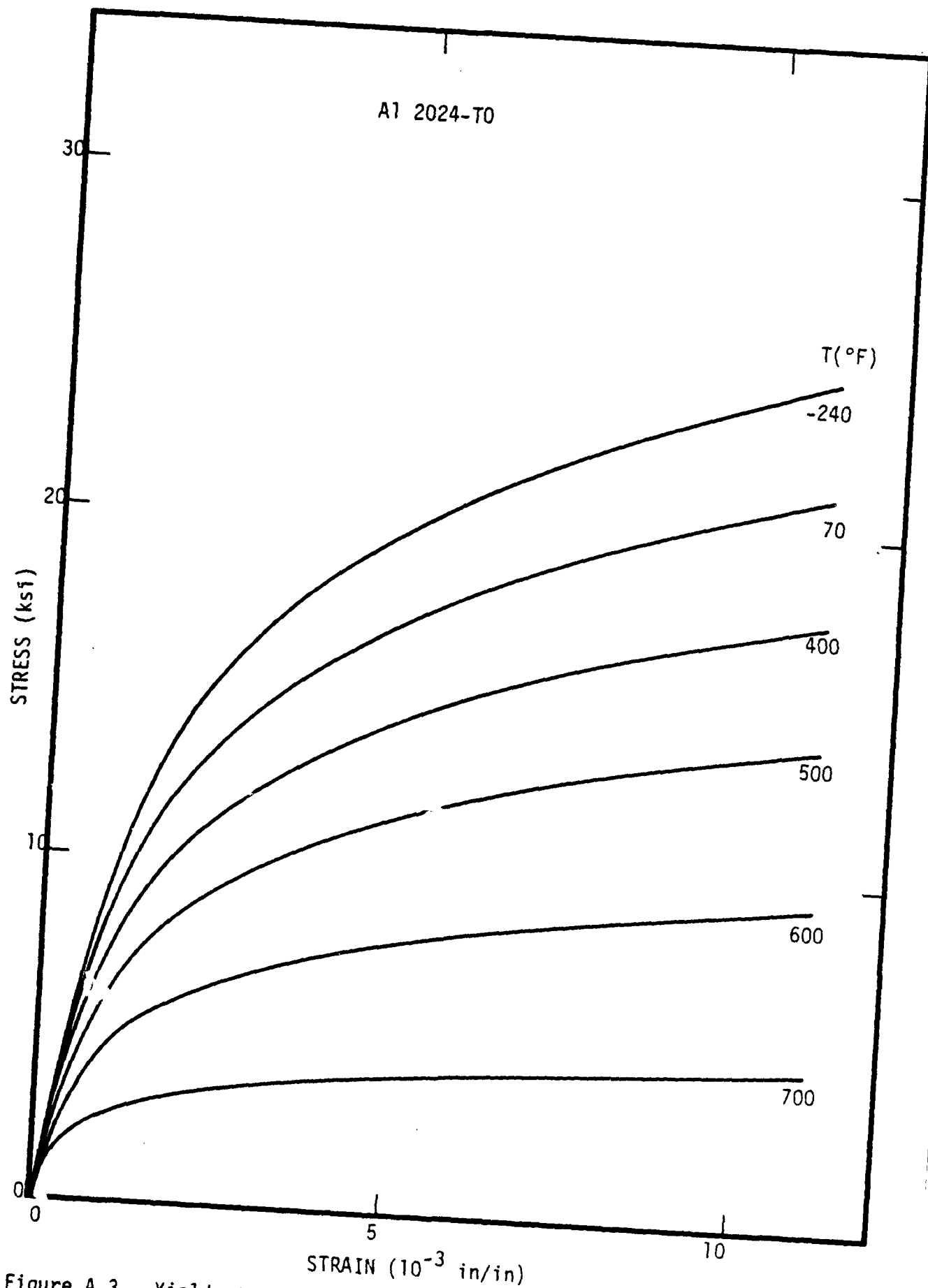


Figure A.3. Yield stress model for 2024 aluminum in the T0 temper (annealed).

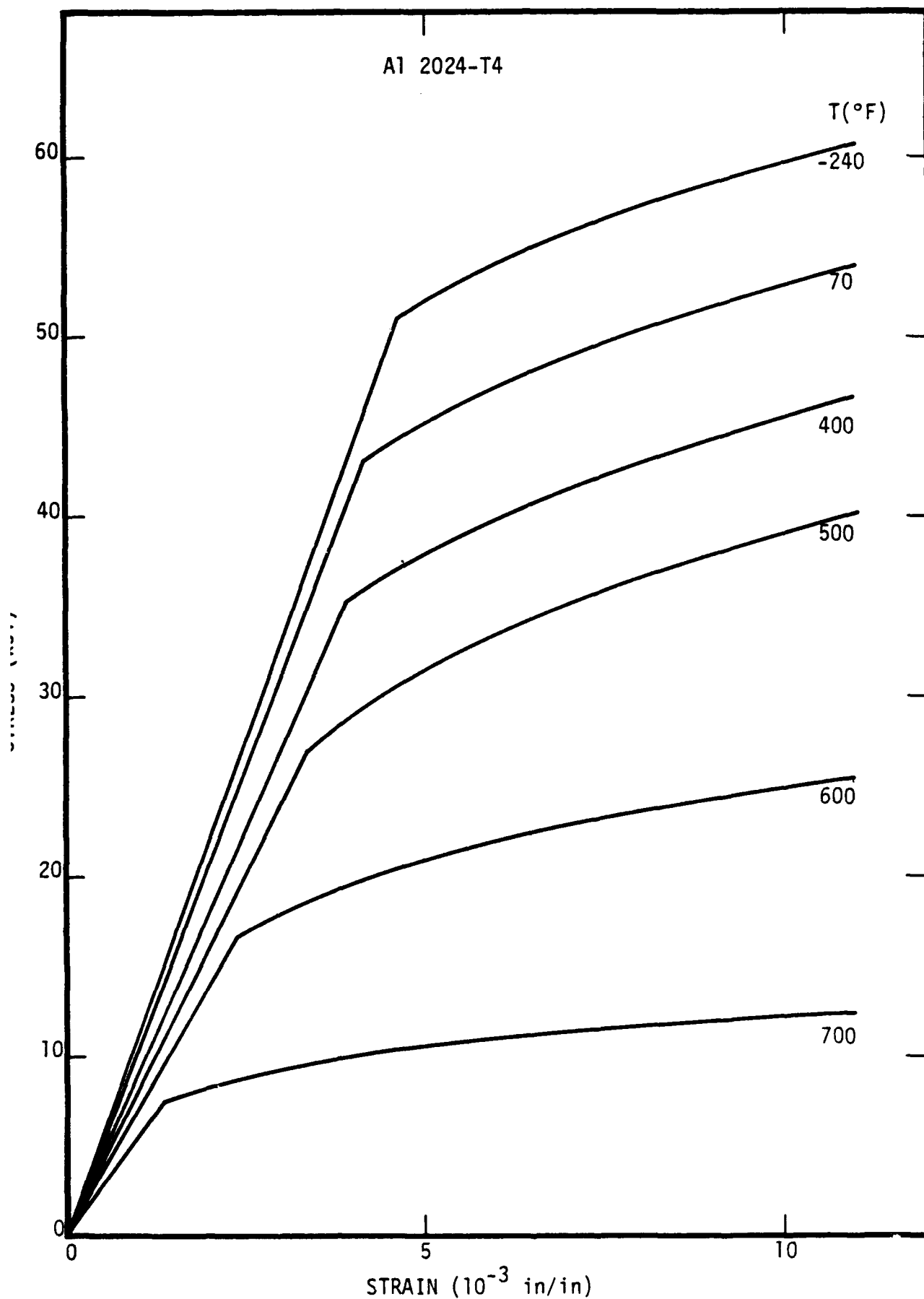


Figure A.4. Yield stress model for 2024 aluminum in the T4 temper.

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APPENDIX B

LAMINATE CODE

A description is given here of the code constructed at S-CUBED for calculating the thermoelastic properties of a cross-ply laminate whose individual plies may be anisotropic-transversely isotropic. The input properties may be either the overall elastic constants and thermal expansion coefficients of each type of ply (up to three types oriented in three different directions) or the corresponding quantities for the fiber and matrix constituents in each type of ply.

The required input deck is described in Section B1, a sample output is given in Section B2, and a FORTRAN listing of the code constitutes Section B3 below.

B.1 INPUT DATA DECK

Any consistent set of units may be used for the input parameters; the corresponding output will be in the same units.

<u>Card No.</u>	<u>Format</u>	<u>Columns</u>	<u>Input Parameter</u>	<u>Description</u>
1	I10	1-10	NPROB	Number of calculations
2	20A4	1-80	HEADER	User's problem description
3	8E10.0	1-10	VFRAC(J)	Volume fraction of first type of ply
		11-20	ANGLE(J)	Orientation of first type of ply with respect to laminate z-axis (degrees) (See discussion below).

<u>Card No.</u>	<u>Format</u>	<u>Columns</u>	<u>Input Parameter</u>	<u>Description</u>
		21-30	TYPE(J)	Type of input to follow. If TYPE>0 input for the individual fiber and matrix components is expected (2 additional cards) If TYPE<=0, input of overall ply properties is required (1 additional card).
3A	8E10.0		Ply properties	Card 3A is required if ply properties VFRAC>0 and TYPE<=0. Otherwise omit.
		1-10	ET(J)	Transverse elastic modulus for first type of ply
		11-20	EA(J)	Axial elastic modulus.
		21-30	NUTT(J)	Poisson's ratio: transverse contraction, transverse stress.
		31-40	NUA(J)	Poisson's ratio: transverse contraction, axial stress.
		41-50	GA(J)	Inplane shear modulus
		51-60	ALFT(J)	Transverse linear coefficient of thermal expansion
		61-70	ALFA(J)	Axial linear coefficient of thermal expansion.
3B	8E10.0		Fiber Properties	Cards 3B and 3C are required if VFRAC>0 and TYPE>0. Otherwise omit.

<u>Card No.</u>	<u>Format</u>	<u>Columns</u>	<u>Input Parameter</u>	<u>Description</u>
		1-10	ETF(J)	Transverse elastic modulus of fiber in first ply.
		11-20	EAF(J)	Axial elastic modulus of fiber.
		21-30	NUTTF(J)	Transverse-transverse Poisson's ratio.
		31-40	NUTAF(J)	Poisson's ratio: transverse contraction, axial stress.
		41-50	GAF(J)	Transverse-axial shear modulus for fiber.
		51-60	ALFTF(J)	Transverse linear coefficient of thermal expansion.
		61-70	ALFAF(J)	Axial linear coefficient of thermal expansion.
		71-80	VF(J)	Fiber volume fraction in first type of ply.
3C	8E10.0		Matrix	Identical to Card 3B, except that the properties correspond to the matrix material in ply J.

The sets of data cards 2, 3, ... 3C are repeated as many times as specified on Card 1. For each set of data, Card 3 must appear three times, once for each of three different orientations of plies. However, if VFRAC is set to zero on Card 3, no further data for that type of ply is required.

If the orientation ANGLE(J) is set to 0.0 on all three of Cards 3, the code will run a series of calculations for orientations

+ θ , - θ , 0.0 degrees of the three ply types, for increments in θ of 5 degrees from 0.0 degrees to 90 degrees. Otherwise the code will do one calculation of laminate properties for the ply orientations specified.

B.2 SAMPLE OUTPUT

A sample output is given below for the case of a laminate of SiC/Al plies, and in this case properties of the fiber and matrix constituents were input for each type of ply.

The first section of output is a printing of the user-supplied header followed by a reiteration of the input properties (fiber and matrix for each ply type in this case). The properties as calculated with the PRUFC code (ply coordinate system) are then given for the three ply types, and the stiffness coefficients and stress-temperature coupling coefficients are also printed.

The main output is then begun. In this example, the orientation angles $\text{ANGLE}(J)$, $J=1,2,3$, were set to zero, so a series of output results at + θ , - θ , 0.0 degrees are printed, only one of which is included here for the +35, -35, 0.0 degree case corresponding to input volume fractions of 0.25, 0.25, and 0.50 respectively. The stiffness and compliance coefficients are listed followed by the thermal expansion coefficients. Finally the intraply stress-temperature derivatives (M_{ij} per $^{\circ}\text{F}$ in this case) are given in both the laminate and ply coordinate systems.

The final bit of output is a summary for each of the orientations (5 degree increments) of the engineering elastic constants and thermal expansion coefficients for the laminate.

SIC/AL LAMINATE MSI DEG-F ELASTIC MATRIX

```

CONSTITUENT INPUT PROPERTIES OF PLY 1      VOL FRAC= .25000  AT  .00 DEG TO LAMINATE Z-AXIS
FIBER PROPERTIES      VOL FRAC= .45000
ELASTIC MODULUS TRANSVERSE= .58480*02      AXIAL= .63280*02      AXIAL SHEAR MODULUS= .23520*02
POISSONS RATIO NUTT .23700000      NUTA= .26000000      THERM EXP TRANSVERSE= .24200-05      AXIAL= .24200-05
MATRIX PROPERTIES      VOL FRAC= .55000
ELASTIC MODULUS TRANSVERSE= .10000*02      AXIAL= .10000*02      AXIAL SHEAR MODULUS= .37594*01
POISSONS RATIO NUTT .33000000      NUTA= .33000000      THERM EXP TRANSVERSE= .13100-04      AXIAL= .13100-04

CONSTITUENT INPUT PROPERTIES OF PLY 2      VOL FRAC= .25000  AT  .00 DEG TO LAMINATE Z-AXIS
FIBER PROPERTIES      VOL FRAC= .45000
ELASTIC MODULUS TRANSVERSE= .58480*02      AXIAL= .63280*02      AXIAL SHEAR MODULUS= .23520*02
POISSONS RATIO NUTT .23700000      NUTA= .26000000      THERM EXP TRANSVERSE= .24200-05      AXIAL= .24200-05
MATRIX PROPERTIES      VOL FRAC= .55000
ELASTIC MODULUS TRANSVERSE= .10000*02      AXIAL= .10000*02      AXIAL SHEAR MODULUS= .37594*01
POISSONS RATIO NUTT .33000000      NUTA= .33000000      THERM EXP TRANSVERSE= .13100-04      AXIAL= .13100-04

CONSTITUENT INPUT PROPERTIES OF PLY 3      VOL FRAC= .50000  AT  .00 DEG TO LAMINATE Z-AXIS
FIBER PROPERTIES      VOL FRAC= .45000
ELASTIC MODULUS TRANSVERSE= .58480*02      AXIAL= .63280*02      AXIAL SHEAR MODULUS= .23520*02
POISSONS RATIO NUTT .23700000      NUTA= .26000000      THERM EXP TRANSVERSE= .24200-05      AXIAL= .24200-05
MATRIX PROPERTIES      VOL FRAC= .55000
ELASTIC MODULUS TRANSVERSE= .10000*02      AXIAL= .10000*02      AXIAL SHEAR MODULUS= .37594*01
POISSONS RATIO NUTT .33000000      NUTA= .33000000      THERM EXP TRANSVERSE= .13100-04      AXIAL= .13100-04

```

PROPERTIES OF PLY 1 AS CALCULATED WITH PRUFC
 ET= .17104+02 EA= .33991+02 MUTT= .43445 NUTA= .29477
 GAT= .73956+01 GTT= .59619+01 ALFT= .89393-05 ALFA= .42390-05

PROPERTIES OF PLY 2 AS CALCULATED WITH PRUFC
 ET= .17104+02 EA= .33991+02 MUTT= .43445 NUTA= .29477
 GAT= .73956+01 GTT= .59619+01 ALFT= .89393-05 ALFA= .42390-05

PROPERTIES OF PLY 3 AS CALCULATED WITH PRUFC
 ET= .17104+02 EA= .33991+02 MUTT= .43445 NUTA= .29477
 GAT= .73956+01 GTT= .59619+01 ALFT= .89393-05 ALFA= .42390-05

STIFFNESS COEFFICIENTS OF PLY 1
 C11= .23849+02 C12= .11926+02 C13= .10545+02 C33= .40238+02
 C44= .73956+01 C66= .59619+01 GAMMA1= .36451-03 GAMMA3= .35898-03

STIFFNESS COEFFICIENTS OF PLY 2
 C11= .23849+02 C12= .11926+02 C13= .10545+02 C33= .40238+02
 C44= .73956+01 C66= .59619+01 GAMMA1= .36451-03 GAMMA3= .35898-03

STIFFNESS COEFFICIENTS OF PLY 3
 C11= .23849+02 C12= .11926+02 C13= .10545+02 C33= .40238+02
 C44= .73956+01 C66= .59619+01 GAMMA1= .36451-03 GAMMA3= .35898-03

SIC/AL LAMINATE MSI DEG-F ELASTIC MATRIX

PLIES AT +35.,-35., 0. DEG TO LAMINATE Z-AXIS

STIFFNESS COEFFICIENTS OF LAMINATE

C11= .23849+02 C12= .11699+02 C13= .10772+02 C14= .00000
 C22= .25061+02 C23= .12023+02 C24= .00000
 C33= .36038+02 C34= .00000
 C55= .71142+01 C56= .00000
 C66= .61593+01

COMPLIANCE COEFFICIENTS FOR LAMINATE

S11= .57113-01 S12= -.21990-01 S13= -.97349-02 S14= .00000
 S22= .55976-01 S23= -.12105-01 S24= .00000
 S33= .34698-01 S34= .00000
 S55= .35141-01 S56= .00000
 S66= .40576-01

THERMAL EXPANSION COEFFICIENTS FOR LAMINATE

ALF1= .93192-05 ALF2= .79810-05 ALF3= .45376-05 ALF4= .00000

PLY STRESS-TEMPERATURE DERIVATIVES (LAMINATE COORD SYSTEM)

PLY 1 DSIGXXDT= -.72760-11 DSIGYYDT= .15566-04 DSIGZZDT= -.55630-05 DSIGVZDT= -.40939-04
 PLY 2 DSIGXXDT= -.72760-11 DSIGYYDT= .15566-04 DSIGZZDT= -.55630-05 DSIGVZDT= -.40939-04
 PLY 3 DSIGXXDT= -.72760-11 DSIGYYDT= .15566-04 DSIGZZDT= .55630-05 DSIGVZDT= .00000

PLY STRESS-TEMPERATURE DERIVATIVES (PLY COORD SYSTEM)

PLY 1 DSIGXXDT= -.72760-11 DSIGYYDT= -.29855-04 DSIGZZDT= .39850-04 DSIGVZDT= .23930-04
 PLY 2 DSIGXXDT= -.72760-11 DSIGYYDT= -.29855-04 DSIGZZDT= .39850-04 DSIGVZDT= -.23930-04
 PLY 3 DSIGXXDT= -.72760-11 DSIGYYDT= .15566-04 DSIGZZDT= .55630-05 DSIGVZDT= .00000

SIC/AL LAMINATE MSI DEG-F ELASTIC MATRIX

LAMINATE ENGINEERING ELASTIC CONSTANTS AND COEFFS OF THERMAL EXPANSION

PLY ANGLES	EX	EY	EZ	NUXZ	MUYZ	GVZ	ALFX	ALFY	ALFZ
0., 0., 0.	.17104+02	.17104+02	.33991+02	.29477+00	.29477+00	.73956+01	.89393-05	.89393-05	.42390-05
5., -5., 0.	.17113+02	.17103+02	.33846+02	.29379+00	.29729+00	.74457+01	.89482-05	.89216-05	.42413-05
10., -10., 0.	.17141+02	.17106+02	.33420+02	.29108+00	.30443+00	.75901+01	.89745-05	.88675-05	.42496-05
15., -15., 0.	.17186+02	.17125+02	.32747+02	.28724+00	.31492+00	.78114+01	.90179-05	.87748-05	.42670-05
20., -20., 0.	.17249+02	.17182+02	.31877+02	.28321+00	.32687+00	.80827+01	.90770-05	.86405-05	.42986-05
25., -25., 0.	.17326+02	.17304+02	.31081+02	.28008+00	.33797+00	.83715+01	.91497-05	.84623-05	.43507-05
30., -30., 0.	.17414+02	.17522+02	.29835+02	.27892+00	.34594+00	.86429+01	.92321-05	.82408-05	.44291-05
35., -35., 0.	.17509+02	.17665+02	.28820+02	.28056+00	.34886+00	.88641+01	.93192-05	.79810-05	.45376-05
40., -40., 0.	.17603+02	.18356+02	.27905+02	.28543+00	.34562+00	.90085+01	.94031-05	.76931-05	.46764-05
45., -45., 0.	.17690+02	.19006+02	.27137+02	.29346+00	.33605+00	.90587+01	.94833-05	.73921-05	.48408-05
50., -50., 0.	.17765+02	.19806+02	.26541+02	.30412+00	.32992+00	.90085+01	.95503-05	.70950-05	.50219-05
55., -55., 0.	.17823+02	.20728+02	.26118+02	.31658+00	.30171+00	.88641+01	.96023-05	.68191-05	.52080-05
60., -60., 0.	.17865+02	.21725+02	.25849+02	.32980+00	.28024+00	.86429+01	.96393-05	.65755-05	.53873-05
65., -65., 0.	.17893+02	.22734+02	.25701+02	.34278+00	.25859+00	.83715+01	.96632-05	.63716-05	.55497-05
70., -70., 0.	.17908+02	.23683+02	.25641+02	.35459+00	.23842+00	.80827+01	.96768-05	.62096-05	.56881-05
75., -75., 0.	.17916+02	.24505+02	.25633+02	.36450+00	.22125+00	.78114+01	.96830-05	.60880-05	.57979-05
80., -80., 0.	.17919+02	.25139+02	.25649+02	.37195+00	.20422+00	.75901+01	.96859-05	.60049-05	.58770-05
85., -85., 0.	.17919+02	.25539+02	.25668+02	.37657+00	.20010+00	.74457+01	.96865-05	.59569-05	.59246-05
90., -90., 0.	.17919+02	.25675+02	.25675+02	.37813+00	.19735+00	.73956+01	.96865-05	.59404-05	.59404-05

PLY ANGLES	GXZ	GXY
0., 0., 0.	.73956+01	.59619+01
5., -5., 0.	.73888+01	.59663+01
10., -10., 0.	.73689+01	.59794+01
15., -15., 0.	.73365+01	.60009+01
20., -20., 0.	.72930+01	.60303+01
25., -25., 0.	.72401+01	.60669+01
30., -30., 0.	.71798+01	.61100+01
35., -35., 0.	.71142+01	.61583+01
40., -40., 0.	.70456+01	.62106+01
45., -45., 0.	.69762+01	.62656+01
50., -50., 0.	.69082+01	.63215+01
55., -55., 0.	.68435+01	.63766+01
60., -60., 0.	.67838+01	.64293+01
65., -65., 0.	.67308+01	.64776+01
70., -70., 0.	.66857+01	.65199+01
75., -75., 0.	.66496+01	.65547+01
80., -80., 0.	.66232+01	.65805+01
85., -85., 0.	.66072+01	.65964+01
90., -90., 0.	.66018+01	.66018+01

B.3 CODE LISTING

A FORTRAN listing of the code is given below.

APPENDIX B3

```

1*      C      LAMINATE MAIN PROGRAM
2*      REAL N1,N2,N3,NUXZ,NUYZ,MUTT,MUTA,MUTTF,MUTAF,MUTTH,MUTAM
3*      DIMENSION S(6,6),SINV(6,6),AX(19),AY(19),AZ(19),GAA(19)
4*      DIMENSION HEADER(20), A(4), SIGDOT(4,3), PHI(3), SIGPLY(4,3)
5*      DIMENSION CIN(8,3),CPRIME(6,6),E1(19),E2(19),E3(19),NUXZ(19),
6*      .      NUYZ(19),GXY(19),GXZ(19),TYPE(3),CDUM(8),GPRIME(4)
7*      COMMON/ONE/N1, N2, N3
8*      COMMON/TWO/C1(4,4), C2(4,4), C3(4,4)
9*      COMMON/THREE/GAMMA1(4), GAMMA2(4), GAMMA3(4)
10*     COMMON/FOUR/O1(2,2), O2(2,2), O3(2,2)
11*     COMMON/INPUT/VFRAC(3),ANGLE(3),ET(3),EA(3),MUTT(3),MUTA(3),GA(3),
12*     .ALFT(3),ALFA(3),ETF(3),EAF(3),MUTTF(3),MUTAF(3),GAF(3),ALFTF(3),
13*     .ALFAF(3),VF(3),ETH(3),EAM(3),MUTTH(3),MUTAM(3),GAM(3),ALFTM(3),
14*     .ALFAM(3),VM(3)
15*     DATA PI/3.1415927/
16*     READ(5,101) NPROB
17*     MNM=0
18*     1 CONTINUE
19*     READ(5,102) HEADER
20*     WRITE(6,100)
21*     WRITE(6,112) HEADER
22*     DO 11 J=1,3
23*     READ(5,103) VFRAC(J), ANGLE(J), TYPE(J)
24*     IF(VFRAC(J).LE.0.) GO TO 18
25*     IF(TYPE(J).GT.0.) GO TO 12

26*     READ(5,103) ET(J),EA(J),MUTT(J),MUTA(J),GA(J),ALFT(J),ALFA(J)
27*     WRITE(6,105) J, VFRAC(J), ANGLE(J)
28*     WRITE(6,106) ET(J), EA(J), GA(J)
29*     WRITE(6,107) MUTT(J), MUTA(J), ALFT(J), ALFA(J)
30*     GO TO 11
31*     18 DO 19 K=1,8
32*     19 CIN(K,J)=0.
33*     GO TO 11
34*     12 READ(5,103) ETF(J),EAF(J),MUTTF(J),MUTAF(J),GAF(J),ALFTF(J),ALFAF(
35*     .J),VF(J)
36*     READ(5,103) ETH(J),EAM(J),MUTTH(J),MUTAM(J),GAM(J),ALFTM(J),ALFAM(
37*     .J),VM(J)
38*     WRITE(6,108) J, VFRAC(J), ANGLE(J)
39*     WRITE(6,109) VF(J)
40*     WRITE(6,106) ETF(J), EAF(J), GAF(J)
41*     WRITE(6,107) MUTTF(J), MUTAF(J), ALFTF(J), ALFAF(J)
42*     WRITE(6,110) VM(J)
43*     WRITE(6,106) ETH(J), EAM(J), GAM(J)
44*     WRITE(6,107) MUTTH(J), MUTAM(J), ALFTM(J), ALFAM(J)
45*     11 CONTINUE
46*     N1=VFRAC(1)
47*     N2=VFRAC(2)
48*     N3=VFRAC(3)
49*     DO 13 J=1,3
50*     IF(VFRAC(J).LE.0.) GO TO 13
51*     IF(TYPE(J).LE.0.) GO TO 13
52*     CALL PRUFC(J,CIN)
53*     13 CONTINUE
54*     DO 14 J=1,3
55*     IF(VFRAC(J).LE.0.) GO TO 14
56*     IF(TYPE(J).GT.0.) GO TO 14
57*     CALL CCOEF(J,CIN)
58*     14 CONTINUE
59*     DO 22 J=1,3
60*     WRITE(6,115) J
61*     WRITE(6,116) (CIN(I,J),I=1,4)
62*     22 WRITE(6,117) (CIN(I,J),I=5,8)
63*     WRITE(6,100)
64*     WRITE(6,112) HEADER
65*     WRITE(6,132)
66*     ANGL=-5.
67*     KMAX=19
68*     IF((ABS(ANGLE(1))+ABS(ANGLE(2))+ABS(ANGLE(3)))>.0.) KMAX=1
69*     DO 10 KK=1,KMAX
70*     ANGL=ANGL + 5.
71*     IF(KMAX.EQ.1) ANGL=ANGLE(1)
72*     THETA=ANGL*PI/180.
73*     DO 15 K=1,8
74*     15 CDUM(K)=CIN(K,1)
75*     CALL CROT(CPRIME,GPRIME,CDUM,THETA)
76*     DO 20 J=1,4
77*     DO 20 I=1,4
78*     20 C1(I,J)=CPRIME(I,J)
79*     DO 21 J=1,4

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80*      21 GAMMA1(J)=GPRIME(J)
81*      DO 23 J=1,2
82*      DO 23 I=1,2
83*      23 D1(I,J)=CPRIME(I+4,J+4)
84*      THETA= - THETA
85*      IF(KMAX.EQ.1) THETA=ANGLE(2)*PI/180.
86*      DO 16 K=1,8
87*      16 CDUM(K)=CIN(K,2)
88*      CALL CROT(CPRIME,GPRIME,CDUM,THETA)
89*      DO 30 J=1,4
90*      DO 30 I=1,4
91*      30 C2(I,J)=CPRIME(I,J)
92*      DO 31 J=1,4
93*      31 GAMMA2(J)=GPRIME(J)
94*      DO 24 J=1,2
95*      DO 24 I=1,2
96*      24 D2(I,J)=CPRIME(I+4,J+4)
97*      THETA=0.
98*      IF(KMAX.EQ.1) THETA=ANGLE(3)*PI/180.
99*      DO 17 K=1,8
100*     17 CDUM(K)=CIN(K,3)
101*     CALL CROT(CPRIME,GPRIME,CDUM,THETA)
102*     DO 40 J=1,4
103*     DO 40 I=1,4
104*     40 C3(I,J)=CPRIME(I,J)
105*     DO 41 J=1,4
106*     41 GAMMA3(J)=GPRIME(J)
107*     DO 25 J=1,2
108*     DO 25 I=1,2
109*     25 D3(I,J)=CPRIME(I+4,J+4)
110*     CALL EXIS11,S21,S31,S41)
111*     CALL EYIS12,S22,S32,S42)
112*     CALL EZIS13,S23,S33,S43)
113*     CALL GXZIS14,S24,S34,S44)
114*     CALL GXZYIS55,S55,S56,S65,S66)
115*     DO 26 J=1,6
116*     DO 26 I=1,6
117*     26 S(I,J)=0.
118*     S(1,1)=S11
119*     S(1,2)=S12
120*     S(1,3)=S13
121*     S(1,4)=2.*S14
122*     S(2,1)=S21
123*     S(2,2)=S22
124*     S(2,3)=S23
125*     S(2,4)=2.*S24
126*     S(3,1)=S31
127*     S(3,2)=S32
128*     S(3,3)=S33
129*     S(3,4)=2.*S34
130*     S(4,1)=S41
131*     S(4,2)=S42
132*     S(4,3)=S43
133*     S(4,4)=2.*S44

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```

134*      S(5,5)=S55*2.
135*      S(5,6)=S56*2.
136*      S(6,5)=S65*2.
137*      S(6,6)=S66*2.
138*      CALL INVR5(6,S,SINV)
139*      CALL THERML(A,SIGDOT)
140*      AX(KK)=A(1)
141*      AY(KK)=A(2)
142*      AZ(KK)=A(3)
143*      E1(KK)=1./S11
144*      E2(KK)=1./S22
145*      E3(KK)=1./S33
146*      NUXZ(KK)= - S13*E3(KK)
147*      NUYZ(KK)= - S23*E3(KK)
148*      GAA(KK)=.5*SINV(4,4)
149*      GXZ(KK)=.5*SINV(5,5)
150*      GXY(KK)=.5*SINV(6,6)
151*      ANGL2= - ANGL
152*      ANGL3=0.
153*      IF(KMAX.NE.1) GO TO 51
154*      ANGL2=ANGLE(2)
155*      ANGL3=ANGLE(3)
156* 51 CONTINUE
157*      WRITE(6,111) ANGL, ANGL2, ANGL3
158*      WRITE(6,118)
159*      WRITE(6,119) SINV(1,1),SINV(1,2),SINV(1,3),SINV(4,1)
160*      WRITE(6,121) SINV(2,2),SINV(2,3),SINV(4,2)
161*      WRITE(6,122) SINV(3,3),SINV(4,3)
162*      XX=.5*SINV(4,4)
163*      XY=.5*SINV(5,5)
164*      XYY=.5*SINV(5,6)
165*      XXY=.5*SINV(6,6)
166*      WRITE(6,123) XX
167*      WRITE(6,137) XY, XYY, XXY
168*      WRITE(6,124)
169*      WRITE(6,125) S11,S12,S13,S14
170*      WRITE(6,126) S22,S23,S24
171*      WRITE(6,127) S33,S34
172*      WRITE(6,128) S44
173*      WRITE(6,138) S55,S56,S66
174*      WRITE(6,129)
175*      WRITE(6,131) A
176*      WRITE(6,135)
177*      DO 42 J=1,3
178*      IF(SIGDOT(4,J).NE.0.) GO TO 43
179*      GO TO 42
180* 43 DO 44 I=1,3
181* 44 IF(ABS(SIGDOT(I,J)/SIGDOT(4,J)).LT.1.E-07) SIGDOT(I,J)=0.
182* 42 CONTINUE
183*      PHI(1)=ANGL*PI/180.
184*      PHI(2)=ANGL2*PI/180.
185*      PHI(3)=ANGL3*PI/180.
186*      DO 62 J=1,3
187*      SI=SIN(PHI(J))

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188*      CO=COS(PHI(J))
189*      SI2=SI**2
190*      CO2=1. - SI2
191*      SIGPLY(1,J)=SIGDOT(1,J)
192*      SIGPLY(2,J)=SIGDOT(2,J)*CO2 - 2.*SIGDOT(4,J)*SI*CO + SIGDOT(3,J)
193*      .      *SI2
194*      SIGPLY(3,J)=SIGDOT(2,J)*SI2 - 2.*SIGDOT(4,J)*SI*CO + SIGDOT(3,J)
195*      .      *CO2
196*      SIGPLY(4,J)=SIGDOT(2,J)*SI*CO + SIGDOT(4,J)*(CO2 - SI2)
197*      .      - SIGDOT(3,J)*SI*CO
198*  62 CONTINUE
199*      DO 60 J=1,3
200*  60 WRITE(6,136) J, (SIGDOT(I,J),I=1,4)
201*      WRITE(6,141)
202*      DO 61 J=1,3
203*  61 WRITE(6,136) J, (SIGPLY(I,J),I=1,4)
204*      WRITE(6,132)
205*  10 CONTINUE
206*      WRITE(6,100)
207*      WRITE(6,112) HEADER
208*      WRITE(6,133)
209*      WRITE(6,134)
210*      ANGL=-5.
211*      DO 50 I=1,KMAX
212*      ANGL=ANGL + 5.
213*      ANGL2= - ANGL
214*      ANGL3=0.
215*      IF(KMAX.NE.1) GO TO 52
216*      ANGL=ANGLE(1)
217*      ANGL2=ANGLE(2)
218*      ANGL3=ANGLE(3)
219*  52 CONTINUE
220*  50 WRITE(6,130) ANGL,ANGL2,ANGL3,E1(I),E2(I),E3(I),NUXZ(I),NUYZ(I),
221*      .      GAA(I),AX(I),AY(I),AZ(I)
222*      WRITE(6,139)
223*      ANGL=-5.
224*      DO 27 I=1,KMAX
225*      ANGL=ANGL + 5.
226*      ANGL2= - ANGL
227*      ANGL3=0.
228*      IF(KMAX.NE.1) GO TO 53
229*      ANGL=ANGLE(1)
230*      ANGL2=ANGLE(2)
231*      ANGL3=ANGLE(3)
232*  53 CONTINUE
233*  27 WRITE(6,140) ANGL,ANGL2,ANGL3,GXZ(I),GXY(I)
234*      NNN=NNN + 1
235*      IF(NNN.LT.NPROB) GO TO 1
236*      STOP
237*  100 FORMAT(1H1)
238*  101 FORMAT(1I10)
239*  102 FORMAT(20A4)
240*  103 FORMAT(8E10.0)
241*  104 FORMAT(1H ,20A4,/)

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242* 105 FORMAT(///,' INPUT PROPERTIES OF PLY',I2,' VOL FRAC=',F7.5,'
243* . AT',F7.2,' DEG TO LAMINATE 2 - AXIS',/)
244* 106 FORMAT(' ELASTIC MODULUS TRANSVERSE=',E11.5,' AXIAL=',E11.5,'
245* . AXIAL SHEAR MODULUS=',E11.5)
246* 107 FORMAT(' POISSONS RATIO NUTT',F11.8,' NUTA=',F11.8,10X,' THERM
247* . EXP TRANSVERSE=',E11.5,5X,' AXIAL=',E11.5)
248* 108 FORMAT(///,' CONSTITUENT INPUT PROPERTIES OF PLY',I2,5X,' VOL FRAC=
249* .',F7.5,' AT',F7.2,' DEG TO LAMINATE 2-AXIS')
250* 109 FORMAT(' FIBER PROPERTIES',5X,' VOL FRAC=',F7.5,/)
251* 110 FORMAT(' MATRIX PROPERTIES',5X,' VOL FRAC=',F7.5,/)
252* 111 FORMAT(' PLIES AT +',F3.0,' ',F4.0,' ',F3.0,' DEG TO LAMINATE 2-AX
253* .IS',/)
254* 112 FORMAT(1H,20A)
255* 113 FORMAT(1H,4E16.8)
256* 114 FORMAT(1H,8E14.5)
257* 115 FORMAT(' STIFFNESS COEFFICIENTS OF PLY',I2)
258* 116 FORMAT(1H,5X,' C11=',E11.5,5X,' C12=',E11.5,5X,' C13=',E11.5,5X,' C33
259* .=',E11.5)
260* 117 FORMAT(1H,5X,' C44=',E11.5,5X,' C66=',E11.5,5X,' GAMMA1=',E11.5,5X,'
261* .GAMMA3=',E11.5)
262* 118 FORMAT(' STIFFNESS COEFFICIENTS OF LAMINATE')
263* 119 FORMAT(1H,5X,' C11=',E11.5,5X,' C12=',E11.5,5X,' C13=',E11.5,5X,' C14
264* .=',E11.5)
265* 121 FORMAT(1H,25X,' C22=',E11.5,5X,' C23=',E11.5,5X,' C24=',E11.5)
266* 122 FORMAT(1H,45X,' C33=',E11.5,5X,' C34=',E11.5)
267* 123 FORMAT(1H,65X,' C44=',E11.5)
268* 124 FORMAT(' COMPLIANCE COEFFICIENTS FOR LAMINATE')
269* 125 FORMAT(1H,5X,' S11=',E11.5,5X,' S12=',E11.5,5X,' S13=',E11.5,5X,' S14
270* .=',E11.5)
271* 126 FORMAT(1H,25X,' S22=',E11.5,5X,' S23=',E11.5,5X,' S24=',E11.5)
272* 127 FORMAT(1H,45X,' S33=',E11.5,5X,' S34=',E11.5)
273* 128 FORMAT(1H,65X,' S44=',E11.5)
274* 129 FORMAT(' THERMAL EXPANSION COEFFICIENTS FOR LAMINATE')
275* 131 FORMAT(1H,5X,' ALF1=',E11.5,5X,' ALF2=',E11.5,5X,' ALF3=',E11.5,5X,'
276* .ALF4=',E11.5)
277* 120 FORMAT(1H,4E16.8)
278* 130 FORMAT(' +',F3.0,' ',F4.0,' ',F3.0,9E12.5)
279* 132 FORMAT(' *****',/)
280* 133 FORMAT(///,' LAMINATE ENGINEERING ELASTIC CONSTANTS AND COEFFS OF T
281* .HERMAL EXPANSION')
282* 134 FORMAT(' PLY ANGLES ',5X,' EX',10X,' EY',10X,' EZ',9X,' NUXZ',8X,'
283* .NUYZ',9X,' GYZ',8X,' ALFX',8X,' ALFY',8X,' ALFZ',/)
284* 135 FORMAT(' PLY STRESS-TEMPERATURE DERIVATIVES (LAMINATE COORD SYST
285* .EM)',/)
286* 136 FORMAT(1H,5X,' PLY',I2,5X,' DSIGXXDT=',E11.5,5X,' DSIGYYDT=',E11.5,5
287* .X,' DSIGZZDT=',E11.5,5X,' DSIGYZDT=',E11.5)
288* 137 FORMAT(1H,25X,' C55=',E11.5,5X,' C56=',E11.5,5X,' C66=',E11.5)
289* 138 FORMAT(1H,25X,' S55=',E11.5,5X,' S56=',E11.5,5X,' S66=',E11.5)
290* 140 FORMAT(' +',F3.0,' ',F4.0,' ',F3.0,2E12.5)
291* 139 FORMAT(' PLY ANGLES ',5X,' GXZ',9X,' GXY',/)
292* 141 FORMAT(' PLY STRESS-TEMPERATURE DERIVATIVES (PLY COORD SYSTEM)',
293* ./)
294* END

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1*      SUBROUTINE CCOEF(IJ,CIN)
2*      REAL NUTT,NUTA,NUAT,NUTTF,NUTAF,NUTTM,NUTAM
3*      DIMENSION CIN(8,3)
4*      COMMON/INPUT/VFRAC(3),ANGLE(3),ET(3),EA(3),NUTT(3),NUTA(3),GA(3),
5*      .ALFT(3),ALFA(3),ETF(3),EAF(3),NUTTF(3),NUTAF(3),GAF(3),ALFTF(3),
6*      .ALFAF(3),VFM(3),ETM(3),EAM(3),NUTTM(3),NUTAM(3),GAM(3),ALFTM(3),
7*      .ALFAM(3),VM(3)
8*      NUAT=ET(IJ)*NUTA(IJ)/EA(IJ)
9*      DELTA=1. - NUTT(IJ)**2 - 2.*NUTA(IJ)*NUAT - 2.*NUTT(IJ)*NUTA(IJ)*NUAT
10*     CIN(1,IJ)=ET(IJ)*(1. - NUTA(IJ)*NUAT)/DELTA
11*     CIN(2,IJ)=ET(IJ)*(NUTT(IJ) + NUTA(IJ)*NUAT)/DELTA
12*     CIN(3,IJ)=ET(IJ)*(1. + NUTT(IJ)*NUTA(IJ)/DELTA
13*     CIN(4,IJ)=EA(IJ)*(1. - NUTT(IJ)**2)/DELTA
14*     CIN(5,IJ)=GA(IJ)
15*     CIN(6,IJ)=.5*(CIN(1,IJ) - CIN(2,IJ))
16*     CIN(7,IJ)=(CIN(1,IJ) + CIN(2,IJ))*ALFT(IJ) + CIN(3,IJ)*ALFA(IJ)
17*     CIN(8,IJ)=2.*CIN(3,IJ)*ALFT(IJ) + CIN(4,IJ)*ALFA(IJ)
18*     RETURN
19*     END

```

END OF COMPILATION: NO DIAGNOSTICS.

```

1*      SUBROUTINE CROTICPRIME,GPRIME,CIN,THETA)
2*      C      SEE MEMO 13 OCT 82 RICE TO BURTMAN
3*      C      COMPUTES CPRIME(I,J) ELASTIC CONSTANTS IN ROTATED COORD SYSTEM
4*      C      FROM THE CIN ELASTIC CONSTANTS IN THE FIBER SYSTEM
5*      C      CIN(1)=C11, CIN(2)=C12, CIN(3),C13, CIN(4)=C33, CIN(5)=C44,
6*      C      CIN(6)=C66 = .5*(C11 - C12) IF MATERIAL IS TRANSVERSELY ISOTROPIC
7*      C      COORD SYSTEM ROTATED BY ANGLE THETA ABOUT THE X(=1) AXIS, WHERE
8*      C      THE FIBER DIRECTION IS THE OLD Z(=3) AXIS
9*      C      CIN(7) = GAMAX, CIN(8) = GAMAZ
10*     DIMENSION CPRIME(6,6), CIN(8), GPRIME(4)
11*     X=SIN(THETA)
12*     Y=COS(THETA)
13*     X2=X**2
14*     Y2=Y**2
15*     X4=X2*X2
16*     Y4=Y2*Y2
17*     X3=X2*X
18*     Y3=Y2*Y
19*     CPRIME(1,1)=CIN(1)
20*     CPRIME(1,2)=Y2*CIN(2) + X2*CIN(3)
21*     CPRIME(1,3)=X2*CIN(2) + Y2*CIN(3)
22*     CPRIME(1,4)=X*Y*(CIN(3) - CIN(2))
23*     CPRIME(1,5)=0.
24*     CPRIME(1,6)=0.
25*     CPRIME(2,1)=CPRIME(1,2)
26*     CPRIME(2,2)=Y4*CIN(1) + X4*CIN(4) + 2.*X2*Y2*CIN(3)
27*     + 4.*X2*Y2*CIN(5)
28*     CPRIME(2,3)=X2*Y2*(CIN(1) + CIN(4)) + (X4 + Y4)*CIN(3)
29*     - 4.*X2*Y2*CIN(5)
30*     CPRIME(2,4)= - X*Y3*CIN(1) + (X*Y3 - X3*Y)*CIN(3) + X3*Y*CIN(4)
31*     + 2.*(X*Y3 - X3*Y)*CIN(5)
32*     CPRIME(2,5)=0.
33*     CPRIME(2,6)=0.
34*     CPRIME(3,1)=CPRIME(1,3)
35*     CPRIME(3,2)=CPRIME(2,3)
36*     CPRIME(3,3)=Y4*CIN(4) + X4*CIN(1) + 2.*X2*Y2*CIN(3) + 4.*X2*Y2
37*     *CIN(5)
38*     CPRIME(3,4)= - X3*Y*CIN(1) + X*Y3*CIN(4) + (X3*Y - X*Y3)*CIN(3)
39*     + 2.*(X3*Y - X*Y3)*CIN(5)
40*     CPRIME(3,5)=0.
41*     CPRIME(3,6)=0.
42*     CPRIME(4,1)=CPRIME(1,4)
43*     CPRIME(4,2)=CPRIME(2,4)
44*     CPRIME(4,3)=CPRIME(3,4)
45*     CPRIME(4,4)=X2*Y2*(CIN(1) + CIN(4)) - 2.*X2*Y2*CIN(3)
46*     + (Y2 - X2)**2)*CIN(5)
47*     CPRIME(4,5)=0.
48*     CPRIME(4,6)=0.
49*     CPRIME(5,1)=0.
50*     CPRIME(5,2)=0.
51*     CPRIME(5,3)=0.
52*     CPRIME(5,4)=0.
53*     CPRIME(5,5)=Y2*CIN(5) + X2*CIN(6)
54*     CPRIME(5,6)=X*Y*(CIN(5) - CIN(6))
55*     CPRIME(6,1)=0.
56*     CPRIME(6,2)=0.
57*     CPRIME(6,3)=0.
58*     CPRIME(6,4)=0.
59*     CPRIME(6,5)=CPRIME(5,6)
60*     CPRIME(6,6)=Y2*CIN(6) + X2*CIN(5)
61*     GPRIME(1)=CIN(7)
62*     GPRIME(2)=Y2*CIN(7) + X2*CIN(8)
63*     GPRIME(3)=X2*CIN(7) + Y2*CIN(8)
64*     GPRIME(4)=(CIN(8) - CIN(7))*X*Y
65*     RETURN
66*     END

```

END OF COMPILATION: NO DIAGNOSTICS.

```

1*      SUBROUTINE EX(S11,S21,S31,S41)
2*      DIMENSION X(6)
3*      REAL N1, N2, N3, K
4*      COMMON/ONE/N1, N2, N3
5*      COMMON/TWO/C1(4,4), C2(4,4), C3(4,4)
6*      COMMON/COEF/K(6,7)
7*      K(1,1)=C1(1,1)
8*      K(1,2)=0.
9*      K(1,3)=0.
10*     K(1,4)=C1(1,2)
11*     K(1,5)=C1(1,3)
12*     K(1,6)=2.*C1(1,4)
13*     K(1,7)=1.
14*     K(2,1)=0.
15*     K(2,2)=C2(1,1)
16*     K(2,3)=0.
17*     K(2,4)=C2(1,2)
18*     K(2,5)=C2(1,3)
19*     K(2,6)=2.*C2(1,4)
20*     K(2,7)=1.
21*     K(3,1)=0.
22*     K(3,2)=0.
23*     K(3,3)=C3(1,1)
24*     K(3,4)=C3(1,2)
25*     K(3,5)=C3(1,3)
26*     K(3,6)=2.*C3(1,4)
27*     K(3,7)=1.
28*     K(4,1)=N1*C1(1,2)
29*     K(4,2)=N2*C2(1,2)
30*     K(4,3)=N3*C3(1,2)
31*     K(4,4)=N1*C1(2,2) + N2*C2(2,2) + N3*C3(2,2)
32*     K(4,5)=N1*C1(2,3) + N2*C2(2,3) + N3*C3(2,3)
33*     K(4,6)=2.*(N1*C1(2,4) + N2*C2(2,4) + N3*C3(2,4))
34*     K(4,7)=0.
35*     K(5,1)=N1*C1(1,3)
36*     K(5,2)=N2*C2(1,3)
37*     K(5,3)=N3*C3(1,3)
38*     K(5,4)=N1*C1(2,3) + N2*C2(2,3) + N3*C3(2,3)
39*     K(5,5)=N1*C1(3,3) + N2*C2(3,3) + N3*C3(3,3)
40*     K(5,6)=2.*(N1*C1(3,4) + N2*C2(3,4) + N3*C3(3,4))
41*     K(5,7)=0.
42*     K(6,1)=N1*C1(1,4)
43*     K(6,2)=N2*C2(1,4)
44*     K(6,3)=N3*C3(1,4)
45*     K(6,4)=N1*C1(2,4) + N2*C2(2,4) + N3*C3(2,4)
46*     K(6,5)=N1*C1(3,4) + N2*C2(3,4) + N3*C3(3,4)
47*     K(6,6)=2.*(N1*C1(4,4) + N2*C2(4,4) + N3*C3(4,4))
48*     K(6,7)=0.
49*     CALL ROOT(6,X)
50*     EPSX=N1*X(1) + N2*X(2) + N3*X(3)
51*     S11=EPSX
52*     S21=X(4)
53*     S31=X(5)
54*     S41=X(6)
55*     RETURN
56*     END

```

END OF COMPILATION: NO DIAGNOSTICS.

```

1*      SUBROUTINE EY(S12,S22,S32,S42)
2*      DIMENSION X(6)
3*      REAL N1, N2, N3, K
4*      COMMON/ONE/N1, N2, N3
5*      COMMON/TWO/C1(4,4), C2(4,4), C3(4,4)
6*      COMMON/COEF/K(6,7)
7*      K(1,1)=C1(1,1)
8*      K(1,2)=0.
9*      K(1,3)=0.
10*     K(1,4)=C1(1,3)
11*     K(1,5)=2.*C1(1,4)
12*     K(1,6)= - C1(1,2)
13*     K(2,1)=0.
14*     K(2,2)=C2(1,1)
15*     K(2,3)=0.
16*     K(2,4)=C2(1,3)
17*     K(2,5)=2.*C2(1,4)
18*     K(2,6)= - C2(1,2)
19*     K(3,1)=0.
20*     K(3,2)=0.
21*     K(3,3)=C3(1,1)
22*     K(3,4)=C3(1,3)
23*     K(3,5)=2.*C3(1,4)
24*     K(3,6)= - C3(1,2)
25*     K(4,1)=N1*C1(1,3)
26*     K(4,2)=N2*C2(1,3)
27*     K(4,3)=N3*C3(1,3)
28*     K(4,4)=N1*C1(3,3) + N2*C2(3,3) + N3*C3(3,3)
29*     K(4,5)=2.*(N1*C1(3,4) + N2*C2(3,4) + N3*C3(3,4))
30*     K(4,6)= - (N1*C1(2,3) + N2*C2(2,3) + N3*C3(2,3))
31*     K(5,1)=N1*C1(1,4)
32*     K(5,2)=N2*C2(1,4)
33*     K(5,3)=N3*C3(1,4)
34*     K(5,4)=N1*C1(3,4) + N2*C2(3,4) + N3*C3(3,4)
35*     K(5,5)= 2.*(N1*C1(4,4) + N2*C2(4,4) + N3*C3(4,4))
36*     K(5,6)= - (N1*C1(2,4) + N2*C2(2,4) + N3*C3(2,4))
37*     CALL ROOT(S,X)
38*     SIGY1=C1(1,2)*X(1) + C1(2,2) + C1(2,3)*X(4) + 2.*C1(2,4)*X(5)
39*     SIGY2=C2(1,2)*X(2) + C2(2,2) + C2(2,3)*X(4) + 2.*C2(2,4)*X(5)
40*     SIGY3=C3(1,2)*X(3) + C3(2,2) + C3(2,3)*X(4) + 2.*C3(2,4)*X(5)
41*     SIGY=N1*SIGY1 + N2*SIGY2 + N3*SIGY3
42*     EPSX=N1*X(1) + N2*X(2) + N3*X(3)
43*     S12=EPSX/SIGY
44*     S22=1./SIGY
45*     S32=X(4)/SIGY
46*     S42=X(5)/SIGY
47*     RETURN
48*     END

```

END OF COMPILATION: NO DIAGNOSTICS.

```

1*      SUBROUTINE EZ(S13,S23,S33,S43)
2*      DIMENSION X(6)
3*      REAL N1, N2, N3, K
4*      COMMON/ONE/N1, N2, N3
5*      COMMON/TWO/C1(4,4), C2(4,4), C3(4,4)
6*      COMMON/COEF/K(6,7)
7*      K(1,1)=C1(1,1)
8*      K(1,2)=0.
9*      K(1,3)=0.
10*     K(1,4)=C1(1,2)
11*     K(1,5)=2.*C1(1,4)
12*     K(1,6)= - C1(1,3)
13*     K(2,1)=0.
14*     K(2,2)=C2(1,1)
15*     K(2,3)=0.
16*     K(2,4)=C2(1,2)
17*     K(2,5)=2.*C2(1,4)
18*     K(2,6)= - C2(1,3)
19*     K(3,1)=0.
20*     K(3,2)=0.
21*     K(3,3)=C3(1,1)
22*     K(3,4)=C3(1,2)
23*     K(3,5)=2.*C3(1,4)
24*     K(3,6)= - C3(1,3)
25*     K(4,1)=N1*C1(1,2)
26*     K(4,2)=N2*C2(1,2)
27*     K(4,3)=N3*C3(1,2)
28*     K(4,4)=N1*C1(2,2) + N2*C2(2,2) + N3*C3(2,2)
29*     K(4,5)=2.*(N1*C1(2,4) + N2*C2(2,4) + N3*C3(2,4))
30*     K(4,6)= - (N1*C1(2,3) + N2*C2(2,3) + N3*C3(2,3))
31*     K(5,1)=N1*C1(1,4)
32*     K(5,2)=N2*C2(1,4)
33*     K(5,3)=N3*C3(1,4)
34*     K(5,4)=N1*C1(2,4) + N2*C2(2,4) + N3*C3(2,4)
35*     K(5,5)=2.*(N1*C1(4,4) + N2*C2(4,4) + N3*C3(4,4))
36*     K(5,6)= - (N1*C1(3,4) + N2*C2(3,4) + N3*C3(3,4))
37*     CALL ROOT(5,X)
38*     SIGZ1=C1(1,3)*X(1) + C1(2,3)*X(4) + C1(3,3) + 2.*C1(3,4)*X(5)
39*     SIGZ2=C2(1,3)*X(2) + C2(2,3)*X(4) + C2(3,3) + 2.*C2(3,4)*X(5)
40*     SIGZ3=C3(1,3)*X(3) + C3(2,3)*X(4) + C3(3,3) + 2.*C3(3,4)*X(5)
41*     SIGZ=N1*SIGZ1 + N2*SIGZ2 + N3*SIGZ3
42*     EPSX=N1*X(1) + N2*X(2) + N3*X(3)
43*     S13=EPSX/SIGZ
44*     S23=X(4)/SIGZ
45*     S33=1./SIGZ
46*     S43=X(5)/SIGZ
47*     RETURN
48*     END

```

END OF COMPILATION: NO DIAGNOSTICS.

```

1*      SUBROUTINE GXZXY(S55,S56,S65,S66)
2*      DIMENSION X(6)
3*      REAL N1,N2,N3,K
4*      COMMON/ONE/N1, N2, N3
5*      COMMON/FOUR/D1(2,2), D2(2,2), D3(2,2)
6*      COMMON/COEF/K(6,7)
7*      DO 10 J=1,7
8*      DO 10 I=1,6
9*      10 K(I,J)=0.
10*      K(1,1)=D1(1,1)
11*      K(1,2)=D1(1,2)
12*      K(1,7)=.5
13*      K(2,3)=D2(1,1)
14*      K(2,4)=D2(1,2)
15*      K(2,7)=.5
16*      K(3,5)=D3(1,1)
17*      K(3,6)=D3(1,2)
18*      K(3,7)=.5
19*      K(4,1)=D1(1,2)
20*      K(4,2)=D1(2,2)
21*      K(5,3)=D2(1,2)
22*      K(5,4)=D2(2,2)
23*      K(6,5)=D3(1,2)
24*      K(6,6)=D3(2,2)
25*      CALL RQOT(6,X)
26*      EPS13=N1*X(1) + N2*X(3) + N3*X(5)
27*      EPS12=N1*X(2) + N2*X(4) + N3*X(6)
28*      S55=.5*EPS13
29*      S65=.5*EPS12
30*      K(1,7)=0.
31*      K(2,7)=0.
32*      K(3,7)=0.
33*      K(4,7)=.5
34*      K(5,7)=.5
35*      K(6,7)=.5
36*      CALL ROOT(6,X)
37*      EPS13=N1*X(1) + N2*X(3) + N3*X(5)
38*      EPS12=N1*X(2) + N2*X(4) + N3*X(6)
39*      S56=.5*EPS13
40*      S66=.5*EPS12
41*      RETURN
42*      END

```

END OF COMPILATION: NO DIAGNOSTICS.

```

1*      SUBROUTINE GYZ(S14,S24,S34,S44)
2*      DIMENSION X(6)
3*      REAL N1, N2, N3, K
4*      COMMON/ONE/N1, N2, N3
5*      COMMON/TWO/C1(4,4), C2(4,4), C3(4,4)
6*      COMMON/COEF/K(6,7)
7*      K(1,1)=C1(1,1)
8*      K(1,2)=0.
9*      K(1,3)=0.
10*     K(1,4)=C1(1,2)
11*     K(1,5)=C1(1,3)
12*     K(1,6)= - 2.*C1(1,4)
13*     K(2,1)=0.
14*     K(2,2)=C2(1,1)
15*     K(2,3)=0.
16*     K(2,4)=C2(1,2)
17*     K(2,5)=C2(1,3)
18*     K(2,6)= - 2.*C2(1,4)
19*     K(3,1)=0.
20*     K(3,2)=0.
21*     K(3,3)=C3(1,1)
22*     K(3,4)=C3(1,2)
23*     K(3,5)=C3(1,3)
24*     K(3,6)= - 2.*C3(1,4)
25*     K(4,1)=N1*C1(1,2)
26*     K(4,2)=N2*C2(1,2)
27*     K(4,3)=N3*C3(1,2)
28*     K(4,4)=N1*C1(2,2) + N2*C2(2,2) + N3*C3(2,2)
29*     K(4,5)=N1*C1(2,3) + N2*C2(2,3) + N3*C3(2,3)
30*     K(4,6)= - 2.*(N1*C1(2,4) + N2*C2(2,4) + N3*C3(2,4))
31*     K(5,1)=N1*C1(1,3)
32*     K(5,2)=N2*C2(1,3)
33*     K(5,3)=N3*C3(1,3)
34*     K(5,4)=N1*C1(2,3) + N2*C2(2,3) + N3*C3(2,3)
35*     K(5,5)=N1*C1(3,3) + N2*C2(3,3) + N3*C3(3,3)
36*     K(5,6)= - 2.*(N1*C1(3,4) + N2*C2(3,4) + N3*C3(3,4))
37*     CALL ROOT(S,X)
38*     SIGYZ1=C1(1,4)*X(1) + C1(2,4)*X(4) + C1(3,4)*X(5) + 2.*C1(4,4)
39*     SIGYZ2=C2(1,4)*X(2) + C2(2,4)*X(4) + C2(3,4)*X(5) + 2.*C2(4,4)
40*     SIGYZ3=C3(1,4)*X(3) + C3(2,4)*X(4) + C3(3,4)*X(5) + 2.*C3(4,4)
41*     SIGYZ=N1*SIGYZ1 + N2*SIGYZ2 + N3*SIGYZ3
42*     EPSX=N1*X(1) + N2*X(2) + N3*X(3)
43*     S14=.5*EPSX/SIGYZ
44*     S24=.5*X(4)/SIGYZ
45*     S34=.5*X(5)/SIGYZ
46*     S44=.5/SIGYZ
47*     RETURN
48*     END

```

END OF COMPILATION: NO DIAGNOSTICS.

1*		SUBROUTINE INVERT	INV00100
2*	C	INVERTS 3 X 3 MATRIX	INV00200
3*		COMMON/MIX2/C(3,3), Q(3), XX	INV00300
4*		COMMON/SOUT/S(3,3)	INV00400
5*		D=C(1,1)*C(2,2)*C(3,3) - C(3,2)*C(2,3) - C(1,2)*C(2,1)*C(3,3)	INV00500
6*		- C(3,1)*C(2,3) + C(1,3)*C(2,1)*C(3,2) - C(3,1)*C(2,2)	INV00600
7*		S(1,1)=(C(2,2)*C(3,3) - C(3,2)*C(2,3))/D	INV00700
8*		S(1,2)=-(C(1,2)*C(3,3) - C(3,2)*C(1,3))/D	INV00800
9*		S(1,3)=(C(1,2)*C(2,3) - C(2,2)*C(1,3))/D	INV00900
10*		S(2,1)=-(C(2,1)*C(3,3) - C(3,1)*C(2,3))/D	INV01000
11*		S(2,2)=(C(1,1)*C(3,3) - C(3,1)*C(1,3))/D	INV01100
12*		S(2,3)=-(C(1,1)*C(2,3) - C(2,1)*C(1,3))/D	INV01200
13*		S(3,1)=(C(2,1)*C(3,2) - C(3,1)*C(2,2))/D	INV01300
14*		S(3,2)=-(C(1,1)*C(3,2) - C(3,1)*C(1,2))/D	INV01400
15*		S(3,3)=(C(1,1)*C(2,2) - C(2,1)*C(1,2))/D	INV01500
16*		RETURN	INV01600
17*		END	INV01700

END OF COMPILATION: NO DIAGNOSTICS.

1*		SUBROUTINE INVRS(N,XIN,XOUT)
2*	C	COMPUTES INVERSE XOUT OF INPUT MATRIX XIN
3*		DIMENSION C(6,7), XIN(6,6), XOUT(6,6), X(6)
4*		COMMON/COEF/C
5*		DO 30 J=1,6
6*		DO 30 I=1,6
7*	30	C(I,J)=XIN(I,J)
8*		DO 10 J=1,N
9*		DO 20 K=1,N
10*		C(K,N+1)=0.
11*		IF(K.EQ.J) C(K,N+1)=1.
12*	20	CONTINUE
13*		CALL ROOT(N,X)
14*		DO 40 L=1,N
15*	40	XOUT(L,J)=X(L)
16*	10	CONTINUE
17*		RETURN
18*		END

END OF COMPILATION: NO DIAGNOSTICS.

```

1*      SUBROUTINE MIX                                MIX00100
2*      C      COMPUTE EFFECTIVE ELASTIC CONSTANTS FROM THOSE OF CONSTITUENTS, MIX00200
3*      C      C11 + C12, C13, C33, GAMMA1, GAMMA3 MIX00300
4*      REAL N1, N2, K(2), KE MIX00400
5*      COMMON/ELAST/N1, C11(2), C12(2) MIX00500
6*      COMMON/MIX1/C1(3,3), C2(3,3), GAMMA1(3), GAMMA2(3), BETA1(3), MIX00600
7*      .      BETA2(3) MIX00700
8*      COMMON/MIX2/CE(3,3), GAMMAE(3), KE MIX00800
9*      K(1)=C1(1,1) + C1(1,2) MIX00900
10*     K(2)=C2(1,1) + C2(1,2) MIX01000
11*     N2=1. - N1 MIX01100
12*     H=N2*K(1) + N1*K(2) + C2(1,1) - C2(1,2) MIX01200
13*     KE=N1*K(1) + N2*K(2) - (K(1) - K(2))*2*N1*V2/H MIX01300
14*     CE(1,3)=N1*C1(1,3) + N2*C2(1,3) - (K(1) - K(2))*(C1(1,3) - MIX01400
15*     .      C2(1,3))*N1*N2/H MIX01500
16*     CE(3,3)=N1*C1(3,3) + N2*C2(3,3) - 2.*(C1(1,3) - C2(1,3))*2*N1*N2/H MIX01600
17*     .      H MIX01700
18*     GAMMAE(1)=N1*GAMMA1(1) + N2*GAMMA2(1) - (GAMMA1(1) - GAMMA2(1)) MIX01800
19*     .      *(K(1) - K(2))*N1*N2/H MIX01900
20*     GAMMAE(3)=N1*GAMMA1(3) + N2*GAMMA2(3) - 2.*(C1(1,3) - C2(1,3))* MIX02000
21*     .      (GAMMA1(1) - GAMMA2(1))*N1*N2/H MIX02100
22*     CE(2,3)=CE(1,3) MIX02200
23*     CE(3,1)=CE(1,3) MIX02300
24*     CE(3,2)=CE(1,3) MIX02400
25*     GAMMAE(2)=GAMMAE(1) MIX02500
26*     RETURN MIX02600
27*     END MIX02700

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END OF COMPILATION: NO DIAGNOSTICS.

1*		SUBROUTINE MIXCC	MXC001
2*	C	CALCULATE COMPOSITE C11 AND C12 SEPARATELY	MXC002
3*		REAL N1, K(6,7), KE	MXC003
4*		DIMENSION X(6)	MXC004
5*		COMMON/ELAST/N1, C11(2), C12(2)	MXC005
6*		COMMON/MIX1/C1(3,3), C2(3,3), GAMMA1(3), GAMMA2(3), BETA1(3),	MXC006
7*		BETA2(3)	MXC007
8*		COMMON/MIX2/CE(3,3), GAMMAE(3), KE	MXC008
9*		COMMON/COEF/K	MXC009
10*		C11(1)=C1(1,1)	MXC010
11*		C11(2)=C2(1,1)	MXC011
12*		C12(1)=C1(1,2)	MXC012
13*		C12(2)=C2(1,2)	MXC013
14*		IF(N1.LE.0) N1=1.E-05	
15*		CALL SETUP(1)	
16*		CALL ROOT(6,X)	MXC015
17*		SUM=X(1) + X(2) + X(3) + X(4)	MXC016
18*		ETA=C12(2)/C11(2)	MXC017
19*		M=-X(1) - .5*(3. + ETA)*X(2)/ETA - .5*(1. - ETA)*X(3) + X(4)	MXC018
20*		SUM=SUM/3. - 2.*M/3.	
21*		Y1=1./SUM	
22*		IF(N1.LE.0) N1=1.E-05	
23*		CALL SETUP(2)	
24*		CALL ROOT(6,X)	
25*		SIGRO=C11(2)*(X(1) + 3.*X(2) - X(3) - 3.*X(4)) - C12(2)	
26*		C66= .5*(C11(2) - C12(2))	
27*		SIGRTO=C66*(-X(1) - 1.5*(3. + ETA)*X(2)/ETA + .5*(1. - ETA)*X(3)	
28*		- 3.*X(4) - 1.)	
29*		Y2=SIGRO/3. - 2.*SIGRTO/3.	
30*		Y=.5*(Y1 + Y2)	
31*		CE(1,1)=.5*(Y + KE)	MXC021
32*		CE(1,2)=.5*(KE - Y)	MXC022
33*		CE(2,2)=CE(1,1)	MXC023
34*		CE(2,1)=CE(1,2)	MXC024
35*		RETURN	MXC025
36*		END	MXC026

END OF COMPILATION: NO DIAGNOSTICS.

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1*      SUBROUTINE PRUFC(IJ,CIN)
2*      REAL NUXY(2), NUXZ(2), NUZX(2), N1
3*      REAL NUTT,NUTA,NUTTF,NUTAF,NUTTM,NUTAM
4*      REAL NUCXYC,NUCXZC,NUCZXC
5*      REAL KX(2), KZ(2), KXE, KZE, KE
6*      DIMENSION CIN(8,3)
7*      DIMENSION BETAE(3), EX(2), EZ(2), RHO(2), G(2)
8*      COMMON/MIX1/C1(3,3), C2(3,3), GAMMA1(3), GAMMA2(3), BETA1(3),
9*      .      BETA2(3)
10*     COMMON/MIX2/CE(3,3), GAMMAE(3), KE
11*     COMMON/ELAST/N1,C11(2), C12(2)
12*     COMMON/SOUT/S(3,3)
13*     COMMON/INPUT/VFRAC(3),ANGLE(3),ET(3),EA(3),NUTT(3),NJTA(3),GA(3),
14*     .ALFT(3),ALFA(3),ETF(3),EAF(3),NUTTF(3),NUTAF(3),GAF(3),ALFTF(3),
15*     .ALFAF(3),VFA(3),ETM(3),EAM(3),NUTTM(3),NUTAM(3),GAM(3),ALFTM(3),
16*     .ALFAM(3),VM(3)
17*     EX(1)=ETF(IJ)
18*     EZ(1)=EAF(IJ)
19*     NUXY(1)=NUTTF(IJ)
20*     NUXZ(1)=NUTAF(IJ)
21*     BETAE(1)=ALFTF(IJ)
22*     BETA1(3)=ALFAF(IJ)
23*     KX(1)=1.
24*     KZ(1)=1.
25*     RHO(1)=1.
26*     G(1)=GAF(IJ)
27*     EX(2)=ETM(IJ)
28*     EZ(2)=EAM(IJ)
29*     NUXY(2)=NUTTM(IJ)
30*     NUXZ(2)=NUTAM(IJ)
31*     BETA2(1)=ALFTM(IJ)
32*     BETA2(3)=ALFAM(IJ)
33*     KX(2)=1.
34*     KZ(2)=1.
35*     RHO(2)=1.
36*     G(2)=GAM(IJ)
37*     C      CALCULATE CSUBIJ'S FOR FIBER
38*     NUZX(1)=(EX(1)*NUXZ(1))/EZ(1)
39*     DELTA=1. - NUXY(1)**2 - 2.*NUXZ(1)*NUZX(1) - 2.*NUXY(1)*NUXZ(1)
40*     .      *NUZX(1)
41*     C1(1,1)=EX(1)*(1. - NUXZ(1)*NUZX(1))/DELTA
42*     C1(1,2)=EX(1)*(NUXY(1) + NUXZ(1)*NUZX(1))/DELTA
43*     C1(1,3)=EX(1)*(1. + NUXY(1))*NUXZ(1)/DELTA
44*     C1(3,3)=EZ(1)*(1. - NUXY(1)**2)/DELTA
45*     C1(2,1)=C1(1,2)
46*     C1(2,2)=C1(1,1)
47*     C1(2,3)=C1(1,3)
48*     C1(3,1)=C1(1,3)
49*     C1(3,2)=C1(1,3)
50*     C      CALCULATE CSUBIJ'S FOR MATRIX
51*     NUZX(2)=(EX(2)*NUXZ(2))/EZ(2)
52*     DELTA=1. - NUXY(2)**2 - 2.*NUXZ(2)*NUZX(2) - 2.*NUXY(2)*NUXZ(2)
53*     .      *NUZX(2)
54*     C2(1,1)=EX(2)*(1. - NUXZ(2)*NUZX(2))/DELTA
55*     C2(1,2)=EX(2)*(NUXY(2) + NUXZ(2)*NUZX(2))/DELTA
56*     C2(1,3)=EX(2)*(1. + NUXY(2))*NUXZ(2)/DELTA
57*     C2(3,3)=EZ(2)*(1. - NUXY(2)**2)/DELTA

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PRU00400

PRU00600

PRU01200

PRU01300

PRU01400

PRU01500

PRU01600

PRU03400

PRU03500

PRU03600

PRU03700

PRU03800

PRU03900

PRU04000

PRU04100

PRU04200

PRU04300

PRU04400

PRU04500

PRU04600

PRU04700

PRU04800

PRU04900

PRU05000

PRU05100

PRU05200

PRU05300

PRU05400

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58*      C2(2,1)=C2(1,2)                                PRU05500
59*      C2(2,2)=C2(1,1)                                PRU05600
60*      C2(2,3)=C2(1,3)                                PRU05700
61*      C2(3,1)=C2(1,3)                                PRU05800
62*      C2(3,2)=C2(1,3)                                PRU05900
63*      C      CALCULATE THERMAL STRESS COEFFS          PRU07900
64*      BETA1(2)=BETA1(1)                                PRU08000
65*      BETA2(2)=BETA2(1)                                PRU08100
66*      DO 30 I=1,3                                      PRU08200
67*      GAMMA1(I)=0.                                      PRU08300
68*      DO 30 JJ=1,3
69*      30 GAMMA1(I)=GAMMA1(I) + C1(I,JJ)*BETA1(JJ)
70*      DO 40 I=1,3
71*      GAMMA2(I)=0.                                      PRU08700
72*      DO 40 JJ=1,3
73*      40 GAMMA2(I)=GAMMA2(I) + C2(I,JJ)*BETA2(JJ)
74*      C      CALCULATE ELASTIC AND THERMAL STRESS COEFFS FOR COMPOSITE AS PRU09200
75*      C      FUNCTION OF FIBER VOLUME FRACTION, C11 + C12, C13, C33, GAMMA1, PRU09300
76*      C      GAMMA3 PRU09400
77*      N1=VF(IJ)
78*      CALL MIX PRU09800
79*      C      CALCULATE THERMAL EXPANSION COEFFS FOR COMPOSITE PRU09900
80*      BETA(1)=.5*(GAMMAE(1)*CE(3,3) - GAMMAE(3)*CE(1,3))/(1.5*KE PRU10000
81*      . *CE(3,3) - CE(1,3)**2) PRU10100
82*      BETA(3)=(1.5*KE*GAMMAE(3) - CE(1,3)*GAMMAE(1))/(1.5*KE PRU10200
83*      . *CE(3,3) - CE(1,3)**2) PRU10300
84*      BETA(2)=BETA(1) PRU10400
85*      SAVEX=BETA(1)
86*      SAVEZ=BETA(3)
87*      CIN(7,J)=GAMMAE(1)
88*      CIN(8,J)=GAMMAE(3)
89*      C      THERM COND FOR COMPOSITE PRU10700
90*      KZE=N1*KZ(1) + (1. - N1)*KZ(2) PRU10800
91*      Q=(KX(2) - KX(1))/(KX(1) + KX(2) + N1*(KX(2) - KX(1))) PRU10900
92*      KXE=N1*KX(1)*(1. + (1. - N1)*Q) + (1. - N1)*KX(2)*(1. - N1*Q) PRU11000
93*      C      COMPOSITE DENSITY PRU11100
94*      RHOC=N1*RHO(1) + (1. - N1)*RHO(2) PRU11200
95*      C      COMPUTE C11 AND C12 FOR COMPOSITE PRU11400
96*      CALL MIXCC PRU11500
97*      CALL INVERT PRU11600
98*      CIN(1,J)=CE(1,1)
99*      CIN(2,J)=CE(1,2)
100*      CIN(3,J)=CE(1,3)
101*      CIN(4,J)=CE(3,3)
102*      C      COMPOSITE INPLANE SHEAR MODULUS PRU12200
103*      CIN(5,J)=G(2)*(2.*N1*G(1) + (1. - N1)*(G(1) + G(2)))/(2.*N1*G(2)
104*      . + (1. - N1)*(G(1) + G(2)))
105*      CIN(6,J)=.5*(CE(1,1) - CE(1,2))
106*      C      CALCULATE COMPOSITE ENGINEERING MODULI AND SAVE FOR PRINTING PRU12600
107*      ECXC=1./S(1,1)
108*      ECZC=1./S(3,3)
109*      NUCXYC= - S(1,2)*ECXC
110*      NUCXZC= - S(1,3)*ECZC
111*      NUCZYC=NUCXZC*ECXC/ECZC
112*      CALL STRESS(SIGZ1,SIGZ2,SIGR,SIGH11,SIGH21,SIGH22,SAVEX,SAVEZ)
113*      WRITE(6,103)
114*      WRITE(6,100) J
115*      100 FORMAT(/,' PROPERTIES OF PLY',I2,' AS CALCULATED HIT4 PRUFC')
116*      WRITE(6,101) ECXC,ECZC,NUCXYC,NUCXZC
117*      101 FORMAT(1H ,5X,'ET=',E11.5,5X,'EA=',E11.5,5X,'NUTT=',F8.5,5X,'NUTA=
118*      .',F8.5)
119*      WRITE(6,102) CIN(5,J),CIN(6,J),BETA(1),BETA(3)
120*      102 FORMAT(1H ,5X,'GAT=',E11.5,5X,'GTT=',E11.5,5X,'ALFT=',E11.5,5X,'AL
121*      .FA=',E11.5)
122*      103 FORMAT(/)
123*      RETURN
124*      END PRU22400

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END OF COMPILATION: NO DIAGNOSTICS.

1*	SUBROUTINE ROOT(N,XOUT)	R0000100
2*	C SOLVES N LINEAR SIMULTANEOUS EQUATIONS, N.LE.6 UNLESS	R0000200
3*	C DIMENSIONS ARE CHANGED	R0000300
4*	IMPLICIT DOUBLE PRECISION(A-H,O-Z)	R0000400
5*	REAL XOUT(6), XKIN(6,7)	R0000500
6*	DIMENSION X(6), XK(6,7)	R0000600
7*	COMMON/COEF/XKIN	R0000700
8*	NP1=N + 1	R0000800
9*	NM1=N - 1	R0000900
10*	NI=0	R0001000
11*	DO 10 I=1,N	R0001100
12*	DO 10 J=1,NP1	R0001200
13*	10 XK(I,J)=XKIN(I,J)	R0001300
14*	20 NI=NI + 1	R0001400
15*	IF(NI.GT.NM1) GO TO 80	R0001500
16*	NIPI=NI + 1	R0001600
17*	A=XK(NI,NI)	R0001700
18*	IM=NI	R0001800
19*	DO 30 I=NIPI,N	R0001900
20*	IF(0ABS(XK(I,NI)).LE.0ABS(A)) GO TO 30	R0002000
21*	A=XK(I,NI)	R0002100
22*	IM=I	R0002200
23*	30 CONTINUE	R0002300
24*	IF(IM.LE.NI) GO TO 50	R0002400
25*	DO 40 J=NI,NP1	R0002500
26*	A=XK(IM,J)	R0002600
27*	XK(IM,J)=XK(NI,J)	R0002700
28*	40 XK(NI,J)=A	R0002800
29*	50 CONTINUE	R0002900
30*	A=XK(NI,NI)	R0003000
31*	DO 60 J=NI,NP1	R0003100
32*	60 XK(NI,J)=XK(NI,J)/A	R0003200
33*	DO 70 I=NIPI,N	R0003300
34*	DO 70 J=NIPI,NP1	R0003400
35*	70 XK(I,J)=XK(I,NI)*XK(NI,J) - XK(I,J)	R0003500
36*	GO TO 20	R0003600
37*	80 CONTINUE	R0003700
38*	X(N)=XK(N,NP1)/XK(N,N)	R0003800
39*	NX=N	R0003900
40*	90 NX=NX - 1	R0004000
41*	NXP1=NX + 1	R0004100
42*	IF(NX.LT.1) GO TO 110	R0004200
43*	X(NX)=XK(NX,NP1)	R0004300
44*	DO 100 K=NXP1,N	R0004400
45*	100 X(NX)=X(NX) - XK(NX,K)*X(K)	R0004500
46*	X(NX)=X(NX)/XK(NX,NX)	R0004600
47*	GO TO 90	R0004700
48*	110 CONTINUE	R0004800
49*	DO 120 I=1,N	R0004900
50*	120 XOUT(I)=X(I)	R0005000
51*	RETURN	R0005100
52*	END	R0005200

NO OF COMPILATION:

NO DIAGNOSTICS.

1*		SUBROUTINE SETUPIN)	
2*	C	SETS UP MATRIX OF COEFFS USED BY SUBROUTINE MIXCC TO OBTAIN	SET00200
3*	C	C11 AND C12	SET00300
4*		REAL N1, K(6,7)	SET00400
5*		DIMENSION ETA(2), C66(2)	SET00500
6*		COMMON/ELAST/N1, C11(2), C12(2)	SET00600
7*		COMMON/COEF/K	SET00700
8*		IF(N.EQ.2) GO TO 10	
9*		R1=SQRT(N1)	SET00800
10*		R12=R1**2	SET00900
11*		R14=R12**2	SET01000
12*		R16=R12**3	SET01100
13*		ETA(1)=C12(1)/C11(1)	SET01200
14*		ETA(2)=C12(2)/C11(2)	SET01300
15*		C66(1)=.5*(C11(1) - C12(1))	SET01400
16*		C66(2)=.5*(C11(2) - C12(2))	SET01500
17*		K(1,1)=R14	SET01600
18*		K(1,2)=R16	SET01700
19*		K(1,3)=R12	SET01800
20*		K(1,4)=1.	SET01900
21*		K(1,5)=-R14	SET02000
22*		K(1,6)=-R16	SET02100
23*		K(1,7)=0.	SET02200
24*		K(2,1)=-ETA(1)*ETA(2)*R14	SET02300
25*		K(2,2)=-.5*ETA(1)*(3. + ETA(2))*R16	SET02400
26*		K(2,3)=-.5*ETA(1)*ETA(2)*(1. - ETA(2))*R12	SET02500
27*		K(2,4)=ETA(1)*ETA(2)	SET02600
28*		K(2,5)=ETA(1)*ETA(2)*R14	SET02700
29*		K(2,6)=.5*ETA(2)*(3. + ETA(1))*R16	SET02800
30*		K(2,7)=0.	SET02900
31*		K(3,1)=R14*(C11(2) - C12(2))	SET03000
32*		K(3,2)=0.	SET03100
33*		K(3,3)=-R12*(C11(2) - ETA(2)*C12(2))	SET03200
34*		K(3,4)=-3.*(C11(2) - C12(2))	SET03300
35*		K(3,5)=-R14*(C11(1) - C12(1))	SET03400
36*		K(3,6)=0.	SET03500
37*		K(3,7)=0.	SET03600
38*		K(4,1)=-2.*ETA(1)*ETA(2)*R14*C66(2)	SET03700
39*		K(4,2)=-C66(2)*R16*ETA(1)*(3.*(1. + ETA(2))	SET03800
40*		K(4,3)=-C66(2)*ETA(1)*ETA(2)*(1. + ETA(2))*R12	SET03900
41*		K(4,4)=-6.*C66(2)*ETA(1)*ETA(2)	SET04000
42*		K(4,5)=2.*C66(1)*ETA(1)*ETA(2)*R14	SET04100
43*		K(4,6)=3.*C66(1)*ETA(2)*(1. + ETA(1))*R16	SET04200
44*		K(4,7)=0.	SET04300
45*		K(5,1)=C11(2) - C12(2)	SET04400
46*		K(5,2)=0.	SET04500
47*		K(5,3)=-C11(2) + ETA(2)*C12(2)	SET04600
48*		K(5,4)=-3.*(C11(2) - C12(2))	SET04700
49*		K(5,5)=0.	SET04800
50*		K(5,6)=0.	SET04900
51*		K(5,7)=1.	SET05000
52*		K(6,1)=-2.*ETA(2)*C66(2)	SET05100
53*		K(6,2)=-3.*C66(2)*(1. + ETA(2))	SET05200
54*		K(6,3)=-C66(2)*ETA(2)*(1. + ETA(2))	SET05300
55*		K(6,4)=-6.*ETA(2)*C66(2)	SET05400
56*		K(6,5)=0.	SET05500
57*		K(6,6)=0.	SET05600
58*		K(6,7)=-ETA(2)	SET05700
59*		RETURN	
60*	10	CONTINUE	
61*		ETA(2)=C12(2)/C11(2)	
62*		K(5,1)=1.	
63*		K(5,2)=1.	
64*		K(5,3)=1.	
65*		K(5,4)=1.	
66*		K(5,5)=0.	
67*		K(5,6)=0.	
68*		K(5,7)=1.	
69*		K(6,1)=-ETA(2)	
70*		K(6,2)=-.5*(3. + ETA(2))	
71*		K(6,3)=-.5*(1. - ETA(2))*ETA(2)	
72*		K(6,4)=ETA(2)	
73*		K(6,5)=0.	
74*		K(6,6)=0.	
75*		K(6,7)=-ETA(2)	
76*		RETURN	SET05800
77*		END	SET05900

1*	SUBROUTINE STRESS(SIGZ1,SIGZ2,SIGR,SIGH11,SIGH21,SIG422,BETAX,	STR00100
2*	BETAZ)	STR00200
3*	C CALCULATES INTERIOR FIBER AND MATRIX STRESSES	STR00300
4*	REAL N1	STR00400
5*	COMMON/MIX1/C1(3,3), C2(3,3), GAMMA1(3), GAMMA2(3), BETA1(3),	STR00500
6*	BETA2(3)	STR00600
7*	COMMON/ELAST/N1, C11(2), C12(2)	STR00700
8*	A2=(C2(1,3)*BETAZ + (C2(1,1) + C2(1,2))*BETAX - GAMMA2(1))/	STR00800
9*	(N1+2.*C2(1,1))	STR00900
10*	B2=BETAX - A2*N1	STR01000
11*	B1=A2 + B2	STR01100
12*	SIGZ1=2.*C1(1,3)*B1 + C1(3,3)*BETAZ - GAMMA1(3)	STR01200
13*	SIGZ2=2.*C2(1,3)*B2 + C2(3,3)*BETAZ - GAMMA2(3)	STR01300
14*	SIGR=(C1(1,1) + C1(1,2))*B1 + C1(1,3)*BETAZ - GAMMA1(1)	STR01400
15*	SIGH11=SIGR	STR01500
16*	SIGH21=C2(1,2)*(-A2 + B2) + C2(1,1)*(A2 + B2) + C2(1,3)*BETAZ	STR01600
17*	- GAMMA2(1)	STR01700
18*	SIGH22=C2(1,2)*(-A2*N1 + B2) + C2(1,1)*(A2*N1 + B2) + C2(1,3)	STR01800
19*	*BETAZ - GAMMA2(1)	STR01900
20*	RETURN	STR02000
21*	END	STR02100

1 OF COMPILATION: NO DIAGNOSTICS.

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1*      SUBROUTINE THERML(ALF,SIGDOT)
2*      DIMENSION X(6), ALF(4), SIGDOT(4,3)
3*      REAL N1, N2, N3, K
4*      COMMON/ONE/N1, N2, N3
5*      COMMON/TWO/C1(4,4), C2(4,4), C3(4,4)
6*      COMMON/COEF/K(6,7)
7*      COMMON/THREE/GAMMA1(4), GAMMA2(4), GAMMA3(4)
8*      K(1,1)=C1(1,1)
9*      K(1,2)=0.
10*     K(1,3)=0.
11*     K(1,4)=C1(1,2)
12*     K(1,5)=C1(1,3)
13*     K(1,6)=2.*C1(1,4)
14*     K(1,7)=GAMMA1(1)
15*     K(2,1)=0.
16*     K(2,2)=C2(1,1)
17*     K(2,3)=0.
18*     K(2,4)=C2(1,2)
19*     K(2,5)=C2(1,3)
20*     K(2,6)=2.*C2(1,4)
21*     K(2,7)=GAMMA2(1)
22*     K(3,1)=0.
23*     K(3,2)=0.
24*     K(3,3)=C3(1,1)
25*     K(3,4)=C3(1,2)
26*     K(3,5)=C3(1,3)
27*     K(3,6)=2.*C3(1,4)
28*     K(3,7)=GAMMA3(1)
29*     K(4,1)=N1*C1(1,2)
30*     K(4,2)=N2*C2(1,2)
31*     K(4,3)=N3*C3(1,2)
32*     K(4,4)=N1*C1(2,2) + N2*C2(2,2) + N3*C3(2,2)
33*     K(4,5)=N1*C1(2,3) + N2*C2(2,3) + N3*C3(2,3)
34*     K(4,6)=2.*(N1*C1(2,4) + N2*C2(2,4) + N3*C3(2,4))
35*     K(4,7)=N1*GAMMA1(2) + N2*GAMMA2(2) + N3*GAMMA3(2)
36*     K(5,1)=N1*C1(1,3)
37*     K(5,2)=N2*C2(1,3)
38*     K(5,3)=N3*C3(1,3)
39*     K(5,4)=N1*C1(2,3) + N2*C2(2,3) + N3*C3(2,3)
40*     K(5,5)=N1*C1(3,3) + N2*C2(3,3) + N3*C3(3,3)
41*     K(5,6)=2.*(N1*C1(3,4) + N2*C2(3,4) + N3*C3(3,4))
42*     K(5,7)=N1*GAMMA1(3) + N2*GAMMA2(3) + N3*GAMMA3(3)
43*     K(6,1)=N1*C1(1,4)
44*     K(6,2)=N2*C2(1,4)
45*     K(6,3)=N3*C3(1,4)
46*     K(6,4)=N1*C1(2,4) + N2*C2(2,4) + N3*C3(2,4)
47*     K(6,5)=N1*C1(3,4) + N2*C2(3,4) + N3*C3(3,4)
48*     K(6,6)=2.*(N1*C1(4,4) + N2*C2(4,4) + N3*C3(4,4))
49*     K(6,7)=N1*GAMMA1(4) + N2*GAMMA2(4) + N3*GAMMA3(4)
50*     CALL ROOT(6,X)
51*     EPSX=N1*X(1) + N2*X(2) + N3*X(3)
52*     ALF(1)=EPSX
53*     ALF(2)=X(4)
54*     ALF(3)=X(5)
55*     ALF(4)=X(6)
56*     SIGDOT(1,1)=C1(1,1)*X(1) + C1(1,2)*X(4) + C1(1,3)*X(5)
57*     + 2.*C1(1,4)*X(6) - GAMMA1(1)
58*     SIGDOT(2,1)=C1(1,2)*X(1) + C1(2,2)*X(4) + C1(2,3)*X(5)
59*     + 2.*C1(2,4)*X(6) - GAMMA1(2)
60*     SIGDOT(3,1)=C1(1,3)*X(1) + C1(2,3)*X(4) + C1(3,3)*X(5)
61*     + 2.*C1(3,4)*X(6) - GAMMA1(3)
62*     SIGDOT(4,1)=C1(1,4)*X(1) + C1(2,4)*X(4) + C1(3,4)*X(5)
63*     + 2.*C1(4,4)*X(6) - GAMMA1(4)
64*     SIGDOT(1,2)=C2(1,1)*X(2) + C2(1,2)*X(4) + C2(1,3)*X(5)
65*     + 2.*C2(1,4)*X(6) - GAMMA2(1)

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66*      SIGDOT(2,2)=C2(1,2)*X(2) + C2(2,2)*X(4) + C2(2,3)*X(5)
67*      .      + 2.*C2(2,4)*X(6) - GAMMA2(2)
68*      SIGDOT(3,2)=C2(1,3)*X(2) + C2(2,3)*X(4) + C2(3,3)*X(5)
69*      .      + 2.*C2(3,4)*X(6) - GAMMA2(3)
70*      SIGDOT(4,2)=C2(1,4)*X(2) + C2(2,4)*X(4) + C2(3,4)*X(5)
71*      .      + 2.*C2(4,4)*X(6) - GAMMA2(4)
72*      SIGDOT(1,3)=C3(1,1)*X(3) + C3(1,2)*X(4) + C3(1,3)*X(5)
73*      .      + 2.*C3(1,4)*X(6) - GAMMA3(1)
74*      SIGDOT(2,3)=C3(1,2)*X(3) + C3(2,2)*X(4) + C3(2,3)*X(5)
75*      .      + 2.*C3(2,4)*X(6) - GAMMA3(2)
76*      SIGDOT(3,3)=C3(1,3)*X(3) + C3(2,3)*X(4) + C3(3,3)*X(5)
77*      .      + 2.*C3(3,4)*X(6) - GAMMA3(3)
78*      SIGDOT(4,3)=C3(1,4)*X(3) + C3(2,4)*X(4) + C3(3,4)*X(5)
79*      .      + 2.*C3(4,4)*X(6) - GAMMA3(4)
80*      RETURN
81*      END

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END OF COMPILATION: NO DIAGNOSTICS.