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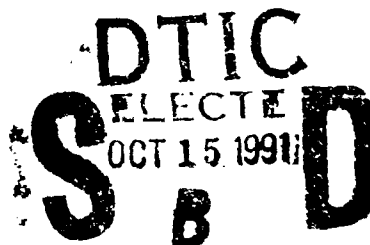
Propulsion and Auxiliary Systems Department

Research and Development Report

**Moving-Frame Time Dependence in
A Differential Turbomachinery
Equation with Viscous Correction**

by

Herman B. Urbach



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Turbomachinery Equation with Viscous Correction

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CONTENTS

	Page
Nomenclature	iv
Abstract	1
Administrative Information	1
Acknowledgment	1
Introduction	1
The Transformation between Absolute and Moving Frames	2
Derivation of the Differential and Integral Forms	4
Uncoupling the Substantial Total Enthalpy Derivative in the Moving and Absolute Frames	4
The Substantial Derivative of the Total Rothalpy	5
Crypto-Steady State Conditions	6
Choosing a Frame for Time Dependence	7
A Sinusoidal Velocity Moving Frame	7
Some Characteristics of the New Differential Turbomachinery Equation	8
Integration of the Total Enthalpy Rate	9
Derivation of the Integral Form from the Differential Form	9
The Substantial Total Enthalpy Rate in a Two-Dimensional Device	11
The Stream Function, Velocity and Relative Enthalpy in the Frame of the Blade	13
The Substantial Total Enthalpy Derivative	14
Enthalpy Transfer in Steady-State Frames	15
References	17

FIGURES

1. Configurational relationships between the absolute and moving
coordinate systems. 3
3. The single-blade linear turbine: (a) sailplane version, (b) sailboat version,
and (c) the rotating cylinder blade. 12

NOMENCLATURE

$\bar{e}$	A unit vector
f	Any continuous function
h	Enthalpy per unit mass of fluid
h_o	Total enthalpy per unit mass of fluid
h_{ow}	Relative total specific rothalpy
H_o	Total enthalpy of a fluid; in linear systems, the total enthalpy of a fluid per unit length of blade
L	Lift per unit length of blade
m	Mass flow rate
n	A constant
p	Local static pressure
p_o	Total pressure
q	A generalized curvilinear coordinate
$\dot{q}$	A specific heat rate
r	Radial coordinate of a cylindrical coordinate system
$\bar{R}$	Position vector for the point of interest in a fluid
s	Entropy per unit mass of fluid
t	Time
T	Absolute temperature
$\bar{U}$	Velocity of the blade and a function of the radius
$\bar{V}$	Velocity
(VIS)	The integrated viscous term of Equation 27
$\bar{W}$	Relative fluid velocity in a moving rotor frame
$\bar{W}'$	A time-dependent component of the velocity in the moving frame
z	Axial coordinate of a cylindrical coordinate system
δ_i^j	Kronecker's delta function

Γ	The circulation
ϕ	The dissipation
θ	Tangential angle coordinate in cylindrical coordinates
ν	The kinematic viscosity
$\bar{\pi}$	The stress tensor separated from the thermodynamic pressure
ρ	Local fluid density
ψ	A stream function
ω	Angular velocity

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ABSTRACT

A differential turbomachinery equation describing the energy transfer between a fluid and any body moving in that fluid was derived. The derivation is based upon the Coriolis form of the Navier-Stokes Equations. A differential equation for the total relative rothalpy is also obtained. Both equalities contain a rigorous viscous correction. Both may be evaluated in the absolute and moving frame.

On integration of the differential equations, a form of the Euler Turbomachinery Equation with viscous correction is derived. The resultant form contains two distinct work rate terms for the axial and radial components of the flow. The fact that integration yields a result which approximates the classic Euler Turbomachinery Equation constitutes confirmation of the derivation.

An application of the equation to an ideal infinite linear cylinder with bound vorticity was developed, yielding the expected known result.

ADMINISTRATIVE INFORMATION

This study is a further development of earlier work (1987) performed in partial fulfillment of thesis requirements of the Aerospace Department of the University of Maryland at College Park. Professor Everett Jones was the graduate advisor and director. Publication of this manuscript was supported by the David Taylor Research Center under the Independent Research and Development Program (IR&IED) of Dr. Bruce Douglas.

The work has significant implications to turbomachinery studies of efficiency and noise in the Power Systems Division, Code 272, of the Propulsion and Auxiliary Systems Department.

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INTRODUCTION

A monument of turbomachinery technology is the Turbomachinery Equation (ultimately due to Euler's analysis of torque in fluids) which is based upon thermodynamic definitions of work and Newton's Laws. Since the Navier-Stokes Equations and Crocco's Equation[1,2] in a rotating (moving) frame are also based upon thermodynamics and Newton's Laws, they must in principle contain the Turbomachinery Equation in differential form and, on integration, in integral form.

The Coriolis' transformation relates moving rotor frames and absolute or laboratory frames so that integration of the energy rate may be conveniently performed in a time-independent frame with a time-independent set of coordinates.

The transformation leads to simplified expressions for the substantial derivative of the total enthalpy uncoupled from the substantial derivative of the entropic energy. The uncoupling generates a differential Turbomachinery Equation which, on integration, yields a novel form of the Turbomachinery Equation corrected for viscous non-ideal flow.

The classical Turbomachinery Equation is an integrated expression arising from Euler's analysis of fluid torque and mechanical shaft work transferred between a fluid and a rotor moving in that fluid. Since heat transfer is negligible in most turbomachines, the shaft work has been equated to the specific total enthalpy transfer (see Horlock [3], page 77, Equation 4.3) as follows:

$$\Delta h_o = \Delta(UV_U) \quad (1)$$

where V_U is the component of absolute velocity, $\bar{V}$, in the direction $\bar{U}$ of the rotor (or energy-transferring device). An inconsistency arises in the consideration of a propeller windmilling on a frictionless shaft in a moving fluid. Since the shaft delivers no work a viscous correction is necessary whether the right member of (1) represents work or a change in the specific total enthalpy.

An application of the differential turbomachinery equation is described for a two-dimensional, ideal, linear turbine.

THE TRANSFORMATION BETWEEN ABSOLUTE AND MOVING FRAMES

In the following discussion the subscripts v and w represent the absolute and the moving frame coordinate and vector values. (See Figure 1 and the Nomenclature for definitions of quantities.) A frequently used relation connecting the absolute and moving frame velocities is

$$\bar{V} = \bar{W} + \bar{U}. \quad (2)$$

Following Spannhake [4], spatial derivatives in the moving frame and time derivatives have the following relationship:

$$\frac{1}{r} \left(\frac{\partial}{\partial \theta_w} \right)_{\theta_v} = -\frac{1}{r\omega} \left(\frac{\partial}{\partial t} \right)_{\theta_v} = -\frac{1}{U} \left(\frac{\partial}{\partial t} \right)_{\theta_v}. \quad (3)$$

Similar results are obtained with cartesian coordinates.

$$\left(\frac{\partial}{\partial y_w} \right)_{y_v} = -\frac{1}{U} \left(\frac{\partial}{\partial t} \right)_{y_v}. \quad (4)$$

Equations (3) and (4) define, in fact, the crypto-steady criterion [5], which if U is constant, indicates that a frame exists in which the flow regime may be truly steady state.

Since the vector operator ∇ is independent of time and since vector operators are independent of frame,

$$\nabla_v = \nabla_w = \nabla. \quad (5)$$

Now from Equation (3) for any static function f ,

$$\left(\frac{\partial f}{\partial t} \right)_{q_i} = -U \left(\frac{\partial f}{\partial q_{wu}} \right) = -\bar{U} \cdot \nabla_w f = -\bar{U} \cdot \nabla f. \quad (6)$$

where q_i represents all the position coordinates in the absolute frame and q_{wu} is the mov-

The Transformation

$$t_v = t_w = t$$

$$z_v = z_w + z_0 = z_w = z \text{ if } z_0 = 0$$

$$r_v = r_w = r$$

$$\theta_v = \theta_w + \omega t$$

$$\bar{R}_v = \bar{R}_w + r\omega t \bar{e}_\theta$$

$$\bar{V} = \bar{W} + \bar{U}$$

$$\left(\frac{\partial}{\partial \theta_w} \right)_{\theta_v} = - \frac{1}{\omega} \left(\frac{\partial}{\partial t} \right)_{\theta_v}$$

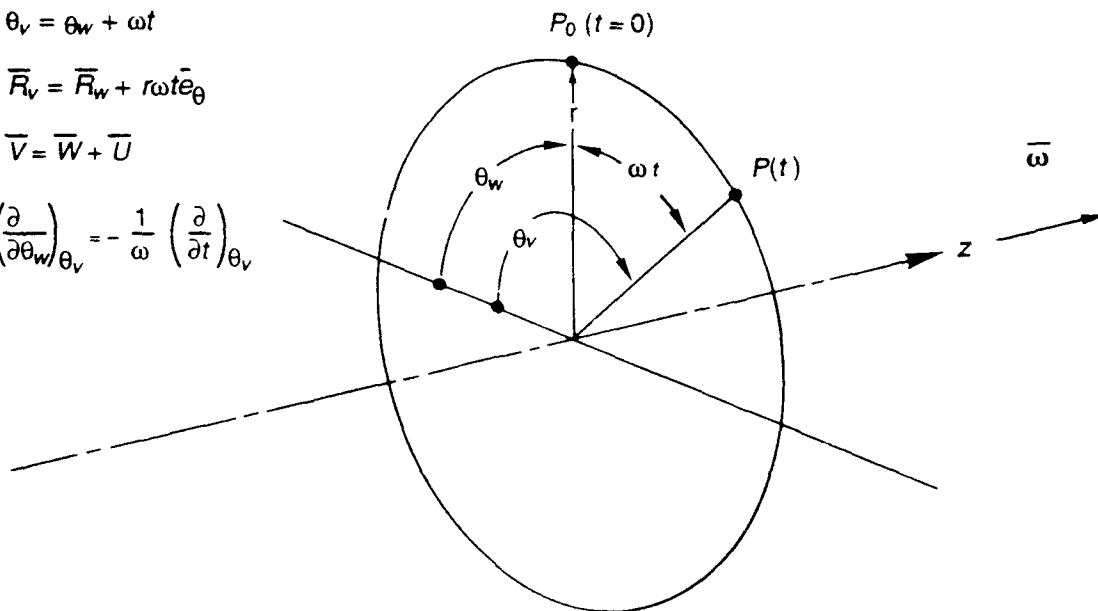


Fig. 1. Configurational relationships between the absolute and moving coordinate systems.

ing-frame coordinate in the direction of the lifting body. Equation (6) is an extension of the usual crypto-steady relation.

To appreciate the significance of the final equation of this section, consider an observer located on a blade of a rotating windmill in an infinite fluid which at infinity translates with uniform constant velocity. The observer cannot detect any time-dependent changes in the fluid at any given point on the blade. However, on moving to the absolute frame, the observer notes changes in velocity, pressure and temperature as each blade passes (See Dean [6]). Thus, the inequality

$$\left(\frac{\partial}{\partial t} \right)_{\theta_v} \neq \left(\frac{\partial}{\partial t} \right)_{\theta_w} \quad (7)$$

indicates that in the moving frame all partial time derivatives may vanish while the absolute partial time derivative may be finite.

Note, however, in contrast with the inequality (7) that the substantial derivative of any static quantity at a point is independent of the coordinate system or frame and is therefore an invariant with respect to frame.

DERIVATION OF THE DIFFERENTIAL AND INTEGRAL FORMS

UNCOUPLING THE SUBSTANTIAL TOTAL ENTHALPY DERIVATIVE IN THE MOVING AND ABSOLUTE FRAMES

Relationships between absolute and relative flow fields are given by the Coriolis form of the Navier-Stokes Equations [3,7-10].

$$\begin{aligned} \frac{\partial \bar{V}}{\partial t} + \nabla V^2/2 - \bar{V} \times (\nabla \times \bar{V}) \\ = \frac{\partial \bar{W}}{\partial t} + \nabla W^2/2 - \bar{W} \times (\nabla \times \bar{W}) + 2\bar{\omega} \times \bar{W} - \nabla U^2/2 \\ = \frac{-\nabla p}{\rho} + \frac{1}{\rho} \nabla \cdot \bar{\pi}' , \end{aligned} \quad (8)$$

where $\bar{\pi}'$ represents the stress tensor excluding the pressure tensor, $\rho \delta_{ij}$, and $\bar{U}$ is independent of time. The relative acceleration, $\partial \bar{W}/\partial t$, is defined in the relative frame, i.e., $(\partial \bar{W}/\partial t)_w$.

Using the absolute-frame equality of (8) and the gradient form of Gibb's equation of state, the substantial derivative of the total enthalpy is obtained in terms of the partial derivative of the pressure, the substantial derivative of the entropic energy, $T(Ds/Dt)$, and the stress tensor $\bar{\pi}'$. (See Wu et al. [7, 8].

$$\frac{Dh_o}{Dt} = \frac{1}{\rho} \left(\frac{\partial p}{\partial t} \right)_v + T \frac{Ds}{Dt} + \frac{\bar{V}}{\rho} \cdot (\nabla \cdot \bar{\pi}') . \quad (9)$$

At this point non-ideal analysis stops unless the partial derivative of the pressure with respect to time is resolved in terms of the pressure gradient [9]. Employing Equation (6) with the static pressure as the arbitrary function the result is

$$\frac{Dh_o}{Dt} = -\frac{\bar{U}}{\rho} \cdot \nabla p + T \frac{Ds}{Dt} + \frac{\bar{V}}{\rho} \cdot (\nabla \cdot \bar{\pi}') . \quad (10)$$

Now Equation (8) may be used to replace the pressure gradient in (10). From the dot product of $\bar{U}$ with the moving frame equality in (8) the pressure gradient term becomes

$$\begin{aligned} -\bar{U} \cdot \frac{\nabla p}{\rho} &= \bar{U} \cdot \frac{\partial \bar{W}}{\partial t} + \bar{U} \cdot (\bar{W} \cdot \nabla) \bar{W} + 2\bar{U} \cdot \bar{\omega} \times \bar{W} \\ &\quad - \bar{U} \cdot \nabla U^2/2 - \frac{1}{\rho} \bar{U} \cdot (\nabla \cdot \bar{\pi}') . \end{aligned} \quad (11)$$

The gradient of the static pressure is of course invariant in all frames, but the moving frame is convenient and preferable in (11). Combining (10) with (11), the substantial derivative of the total enthalpy coupled with the substantial derivative of the entropic energy is obtained in terms of the moving-frame flow field variables.

$$\frac{Dh_o}{Dt} - T \frac{Ds}{Dt} = \bar{U} \cdot \frac{\partial \bar{W}}{\partial t} + \bar{U} \cdot (\bar{W} \cdot \nabla) \bar{W} + 2\bar{U} \cdot \bar{\omega} \times \bar{W} + \frac{1}{\rho} \bar{W} \cdot (\nabla \cdot \bar{\pi}') . \quad (12)$$

Wu (reference [8], page 91) noted that in the moving frame following a particle of fluid

$$T \frac{Ds}{Dt} = \dot{q} + \frac{1}{\rho} (\bar{\pi}' \cdot \nabla) \cdot \bar{W} , \quad (13)$$

where $\dot{q}$ represents heat transfer, and the second term is the specific dissipation. If it is assumed that heat sources and sinks, and thermal conduction are negligible, then from (12) and (13)

$$\frac{Dh_o}{Dt} = \bar{U} \cdot \frac{\partial \bar{W}}{\partial t} + \bar{U} \cdot (\bar{W} \cdot \nabla) \bar{W} + 2\bar{U} \cdot \bar{\omega} \times \bar{W} + \frac{1}{\rho} \nabla \cdot (\bar{\pi}' \cdot \bar{W}) . \quad (14)$$

Equation (14), the essential development of this paper, is a differential turbomachinery equation expressed in terms of the velocity vector of the moving frame. It is universally applicable in any moving frame whether rotating or not.

THE SUBSTANTIAL DERIVATIVE OF THE TOTAL ROTHALPY

Using the moving frame equality of Equation (8) and Gibb's Equation of state for the pressure, a viscous moving-frame version of Crocco's Equation may be obtained.

$$\left(\frac{\partial \bar{W}}{\partial t} \right)_{*} + \nabla W^2/2 - \bar{W} \times (2\bar{\omega} + \nabla \times \bar{W}) - \nabla U^2/2 = -\nabla h + T \nabla s + \frac{1}{\rho} \nabla \cdot \bar{\pi}' . \quad (15)$$

In (15) the subscript w on the relative acceleration term implies that spatial variables in the relative frame, r, θ_w and z are fixed.

Wu's total relative rothalpy, h_{ow} , may be defined as follows:

$$h_{ow} \equiv h + W^2/2 - U^2/2 = h_o - UV_U = h + V^2/2 - UV'_U. \quad (16)$$

Terms may be regrouped in (15) to obtain an expression explicit in Δh_{ow} .

$$\nabla h_{ow} = - \left(\frac{\partial \bar{V}}{\partial t} \right)_w + \bar{W} \times (2\bar{\omega} + \nabla \times \bar{W}) + T \nabla s + \frac{1}{\rho} \nabla \cdot \bar{\pi}'. \quad (17)$$

Now the definition of the substantial derivative of the total relative rothalpy in the moving frame is as follows:

$$\left(\frac{Dh_{ow}}{Dt} \right)_w \equiv \left(\frac{\partial h}{\partial t} \right)_w + \left(\frac{\partial W^2/2}{\partial t} \right)_w + \bar{W} \cdot \nabla h_{ow}. \quad (18)$$

Taking the dot product of the relative velocity $\bar{W}$ on (17) and adding the dot product to (18) yields

$$\left(\frac{Dh_{ow}}{Dt} \right)_w = \left(\frac{\partial h}{\partial t} \right)_w + T \bar{W} \cdot \nabla s + \frac{\bar{W}}{\rho} \cdot (\nabla \cdot \bar{\pi}'). \quad (19)$$

Using Gibb's Equation to eliminate the enthalpy yields an equation which is analogous to (9) in the relative frame.

$$\left(\frac{Dh_{ow}}{Dt} \right)_w = \frac{1}{\rho} \left(\frac{\partial p}{\partial t} \right)_w + T \left(\frac{Ds}{Dt} \right)_w + \frac{\bar{W}}{\rho} \cdot (\nabla \cdot \bar{\pi}'). \quad (20)$$

Since, as noted above, the substantial derivative of the static quantities are invariant in all frames, using (13) and assuming that heat transfer is negligible, (20) may be rewritten thus:

$$\left(\frac{Dh_{ow}}{Dt} \right)_w = \frac{1}{\rho} \left(\frac{\partial p}{\partial t} \right)_w + \frac{1}{\rho} \nabla \cdot (\bar{W} \cdot \bar{\pi}'). \quad (21)$$

The quantity h_{ow} in Equation (21) exhibits an explicit time dependence in $\partial p/\partial t$ which was deliberately ignored in crypto-steady flow [9] but recently reconsidered [12]. Also, in (21) if the viscosity does not vanish, h_{ow} also depends upon the last term [9] which includes the viscous work and the viscous dissipation.

CRYPTO-STEADY STATE CONDITIONS

We now examine the case where the observer is fixed to an infinite circular cylinder with bound circulation translating uniformly in a infinite fluid. The observer fixed to the moving cylinder cannot sense any time dependence. Of course time-dependent velocity and pressure are immediately sensed when the observer moves to the absolute frame.

In another example the observer is fixed to a blade of an isolated rotating air screw or a marine screw in an infinite uniform fluid. No time-dependence can be sensed in the moving frame and a crypto-steady-state prevails.

Under crypto-steady-state conditions the partial derivative of the pressure with time must vanish. This poses a great simplification which justifies common practice [11] in the design of marine screws where time dependence is generally ignored.

Before departing from the crypto-steady-state, it is pertinent to ask what happens to rotor energy transfer if there is steady-state in the absolute frame so that $\partial p / \partial t_v$ vanishes. Clearly, the viscous terms in (9), (10), and (14) do not vanish. In (14) only the viscous term of the right member makes any contribution to bladeless, viscously coupled discs and concentric cylinders.

Now, if viscosity vanishes and the time-dependent terms vanish because we impose the crypto-steady condition, then (21) reduces to

$$\bar{W} \cdot \nabla h_{ow} = 0 \quad (22)$$

The non-trivial solution is

$$\nabla h_{ow} = \nabla \left(h + W^2/2 - U^2/2 \right) = \nabla (h_o - UV_U) = 0 \quad (23)$$

Equation (23) represents gradient forms of the classic Turbomachinery Equation in the moving and absolute frame. Integration of the right member of (23) over a stream tube in the absolute frame yields the classic Turbomachinery Equation (1). The argument demonstrates that Equation (1) is strictly true only for isentropic and steady-state flow.

CHOOSING A FRAME FOR TIME DEPENDENCE

Since the analyses are based upon two frames of reference, the hypothesis now proposed that time dependence may be associated with disturbances either in the bulk flow or with flow local to the moving frame. Therefore, formulation of a mathematical description of the flow entering into the calculational domain may be most conveniently described in bulk frame coordinates (the absolute frame, and subsequently transferred to the moving frame). On the other hand flow disturbances arising near the moving body are most conveniently described in terms of the moving frame coordinates. Since the discussion centers on rotors, it is natural to consider subsequent time fluctuations in the moving rotating frame.

A SINUSOIDAL VELOCITY IN THE MOVING-FRAME

Now in Equation (14) a small sinusoidal moving-frame oscillation is imposed on the relative velocity $\bar{W}$ leading to a new velocity, $\bar{W} + \bar{W}'$. The ratio of the magnitude of time-independent $\bar{W}'$ to time-independent $\bar{W}$ will be arbitrarily fixed at ≤ 0.05 . The

imposed frequency of the time-dependent $\bar{W}'$ is at least an order of magnitude greater than the blade velocity. If the oscillating frequency is too low, it will exhibit a period that is long relative to the blade period and is therefore not considered. Thus, prior to Reynolds time averaging (which should not employ too large an interval to avoid time smoothing) the velocity and other thermodynamic quantities will fluctuate at a point (on the blade, for example) during a blade period. With Reynolds averaging, the partial derivative of $\bar{W}'$ with respect to time will vanish.

However, the Reynolds average over non-linear terms will lead to non-vanishing Reynolds stress terms which will not be addressed here. As a result of the Reynolds time-averaging process, Equation (14) may be written

$$\frac{Dh_o}{Dt} = \bar{U} \cdot \{(\bar{W} \cdot \nabla) \bar{W}\} + 2\bar{U} \cdot \bar{\omega} \times \{\bar{W}\} + \{1/\rho \nabla \cdot \bar{\pi}' \cdot \bar{W}\} \quad (24)$$

where the braces represent the Reynolds time average and $\bar{W}$ and $\bar{\pi}'$ include the time-dependent $\bar{W}'$. This notation will not be used again.

Now, note that arbitrary samples of turbulent flow over short (relative to the blade) periods may be written as a Fourier summation of regular sinusoidal oscillations. For the limited picture developed here, Equation (24) is again applicable by the argument above.

SOME CHARACTERISTICS OF THE NEW DIFFERENTIAL TURBOMACHINERY EQUATION

Equation (14), an explicit statement of the substantial derivative of the total enthalpy, represents the power transfer at any point in a flow field between the fluid and a body moving at constant speed in that fluid.

With (13) it is now possible to uncouple the total-enthalpy, pressure relationship of (10) from the entropic energy.

$$\frac{Dh_o}{Dt} = -\frac{\bar{U}}{\rho} \cdot \nabla p + \frac{1}{\rho} \nabla \cdot (\bar{\pi}' \cdot \bar{W}) + \frac{1}{\rho} \bar{U} \cdot (\nabla \cdot \bar{\pi}') \quad (25)$$

An inviscid Equation (25) indicates that neither axial nor radial pressure gradients are germane to the calculation of specific total enthalpy transfer. (Mass flow rates are of course a function of axial or radial pressure gradients.) Note, in contrast to reference [3], p. 7), for the inviscid case, only transverse pressure gradients parallel to $\bar{U}$ contribute to total enthalpy transfer. The impulse stages of turbomachines prove that axial or radial pressure gradients need play no role in energy transfer.

For bladeless viscous coupling of discs or concentric cylinders, the gradient of the pressure parallel to $\bar{U}$ vanishes. This is equivalent to steady-state pressure in the absolute frame (see reference [6]).

A proper test of (14) would be whether, on integration over the rotor blade-to-blade flow, it would predict a total enthalpy transfer compatible with that of the classic Turbomachinery Equation (1). Therefore, the integration of (14) for crypto-steady conditions will be performed as a test in the three-dimensional domain. Then an application of the

new equation will be developed in a two-dimensional linear turbine. Finally, the steady-state viscous problem of two concentric cylinders will be examined.

INTEGRATION OF THE TOTAL ENTHALPY RATE

In the integration process it will be assumed that the flow may be divided into streams which pass between a given pair of blades. In the rotating frame the streamtube walls are fixed steady-state walls associated with a steady-state mass flow rate m which may consist of radial and axial mass flow components.

DERIVATION OF THE INTEGRAL FORM FROM THE DIFFERENTIAL FORM

The differential form of the turbomachinery equation (14) may be integrated to yield

$$\begin{aligned} \iiint \rho \frac{Dh_o}{Dt} d\tau \\ = \iiint \rho [U \cdot (\bar{W} \cdot \nabla) \bar{W} + 2U \cdot \bar{\omega} \times \bar{W}] r dr d\theta dz \\ + \iiint \nabla \cdot (\bar{\pi}' \cdot \bar{W}) r dr d\theta dz . \end{aligned} \quad (26)$$

The first term of the right member of (26) is the tangential component of the convective term obtained on dot multiplication with $\bar{U}$, i.e.:

$$\begin{aligned} \rho \bar{U} \cdot (\bar{W} \cdot \nabla) \bar{W} \\ = \rho U \left(W_r \frac{\partial W_\theta}{\partial r} + \frac{W_\theta}{r} \frac{\partial W_\theta}{\partial \theta} + W_z \frac{\partial W_\theta}{\partial z} + \frac{W_r W_\theta}{r} \right) . \end{aligned} \quad (27)$$

The first and last terms of (27) will be combined in an integral identified by $I_{1,4}$ thus:

$$I_{1,4} = \iiint \rho r W_r \frac{\partial W_\theta}{\partial r} r dr d\theta dz . \quad (28)$$

The factors in (28) may be rearranged to express the radial mass flow which must be constant in steady flow. Following arguments presented previously [9]

$$I_{1,4} = m_r \int d(U \bar{W}_\theta) = m_r [(U \bar{W}_\theta)_{r2} - (U \bar{W}_\theta)_{r1}] , \quad (29)$$

where

$$\bar{W}_\theta(r) = \frac{1}{\Delta\theta \Delta z} \int \int_{\theta_{\min}}^{\theta_{\max}} W_\theta d\theta dz . \quad (30)$$

The third term of the right member of (27) may be written to show the axial mass flow rate m_z explicitly.

$$I_3 = \int \left(\int \int \rho \bar{W}_z r dr d\theta \right) \frac{\partial \bar{U} \bar{W}_\theta}{\partial z} dz, \quad (31)$$

where the axial velocity $\bar{W}_z$, $\bar{W}_\theta$, and $\bar{U}$ have been averaged over r and θ . Substituting the axial mass rate for the parenthesis in (31),

$$I_3 = m_z \int d(\bar{U} \bar{W}_\theta) = m_z [(\bar{U} \bar{W}_\theta)_{z2} - (\bar{U} \bar{W}_\theta)_{z1}] . \quad (32)$$

The second term of the right member of (27) provides an integral which contains the tangential kinetic energy.

$$\begin{aligned} I_2 &= \int \int \frac{\rho \omega r}{2} \left(\int \frac{\partial W_\theta^2}{\partial \theta} d\theta \right) dr dz \\ &= \int \int \frac{\rho \omega r}{2} [W_\theta^2(\theta_2) - W_\theta^2(\theta_1)] dr dz = 0 . \end{aligned} \quad (33)$$

Since the tangential velocities at the blade walls are equal to the blade velocity, the integral vanishes.

Now identifying the second term of the right member of Equation (27) as I_5 , we may write

$$I_5 = \int \int \int 2\rho \omega^2 r W_r r dr d\theta dz . \quad (34)$$

Finally, following the arguments above,

$$I_5 \approx m_r [(U^2)_{r2} - (U^2)_{r1}] . \quad (35)$$

Summing the components of integration $I_{1,4}$ through I_5 ,

$$\begin{aligned} \int \int \int \rho \frac{Dh_o}{Dt} d\tau &= m_r [\Delta_r (U \bar{W}_\theta) + \Delta_r (U^2)] \\ &\quad + m_z \Delta_z (\bar{U} \bar{W}_\theta) + (VIS) \end{aligned} \quad (36)$$

where Δ_r and Δ_z represent the change along r and z respectively, and (VIS) is the integrated viscous term.

Now adding $m_z \Delta_z \bar{U}^2$, which is zero, to Equation (37),

$$\int \int \int \rho \frac{Dh_o}{Dt} d\tau = m_r \Delta_r \left[(\bar{U} + \bar{\bar{W}}_\theta) \bar{U} \right] + m_z \Delta_z \left[(U + \bar{\bar{W}}_\theta) U \right] + (VIS) . \quad (37)$$

In Equation (37) the terms $\bar{\bar{W}}_\theta$ are averaged over θ and z in the first term and over r and θ in the second term. If the total steady-state mass rate m between a pair of blades is

$$m = m_r + m_z , \quad (38)$$

then,

$$\Delta h_o = f_r \Delta_r (\bar{U} \bar{\bar{V}}_\theta) + f_z \Delta_z (U \bar{\bar{V}}_\theta) + (VIS)/m . \quad (39)$$

Equation (39) represents classic Turbomachinery Equation with mixed flows and the terms $\bar{\bar{V}}_\theta$ and $\bar{U}$ are averaged over the blade space where necessary. The coefficients f_r and f_z represent the radial and axial fractions of the mass flow.

The viscous term is a novel feature of the derivation which explains losses in rotors during windmilling. With the exception of the viscous energy term (VIS) , the integral expression equation (39) exhibits a formal similarity and compatibility with the classic Turbomachinery Equation. The differences arise because the integration in (39) has been performed over the entire blade-to-blade volume rather than a stream tube. It is easy to combine the axial and radial flow terms by letting Δ vary in both r and z . The derivation lends credence to the hypothesis that Equation (14) is indeed a differential turbomachinery equation.

A two-dimensional application and test of the differential form (14) on an ideal linear device where the solution is known precisely will now be examined.

THE SUBSTANTIAL TOTAL ENTHALPY RATE IN A TWO-DIMENSIONAL DEVICE

An infinite circular cylinder with bound circulation, as shown in Figure 2, is an elemental linear turbine. It may be considered as an infinite sail on a sailboat or an infinite wing on a sailplane. The device extracts energy from the ideal inviscid working fluid. Work is performed on the sailplane (fixed to a vertical rail) by raising its height at uniform speed U against gravity. Work on the sailboat is performed by moving the boat at uniform speed U which elevates a weight attached at minus infinity by an infinite tether. In the moving frame the apparent velocity of the ideal working fluid at infinite distance is W_o . The relationship between the absolute and moving coordinate system and the velocities is given by the transformation of Figure 1.

Since the flow field is ideal, the flow domain may be described by a potential function or its conjugate stream function. The lift is therefore the ideal lifting force, L , of the Kutta-Joukowski Equation given by

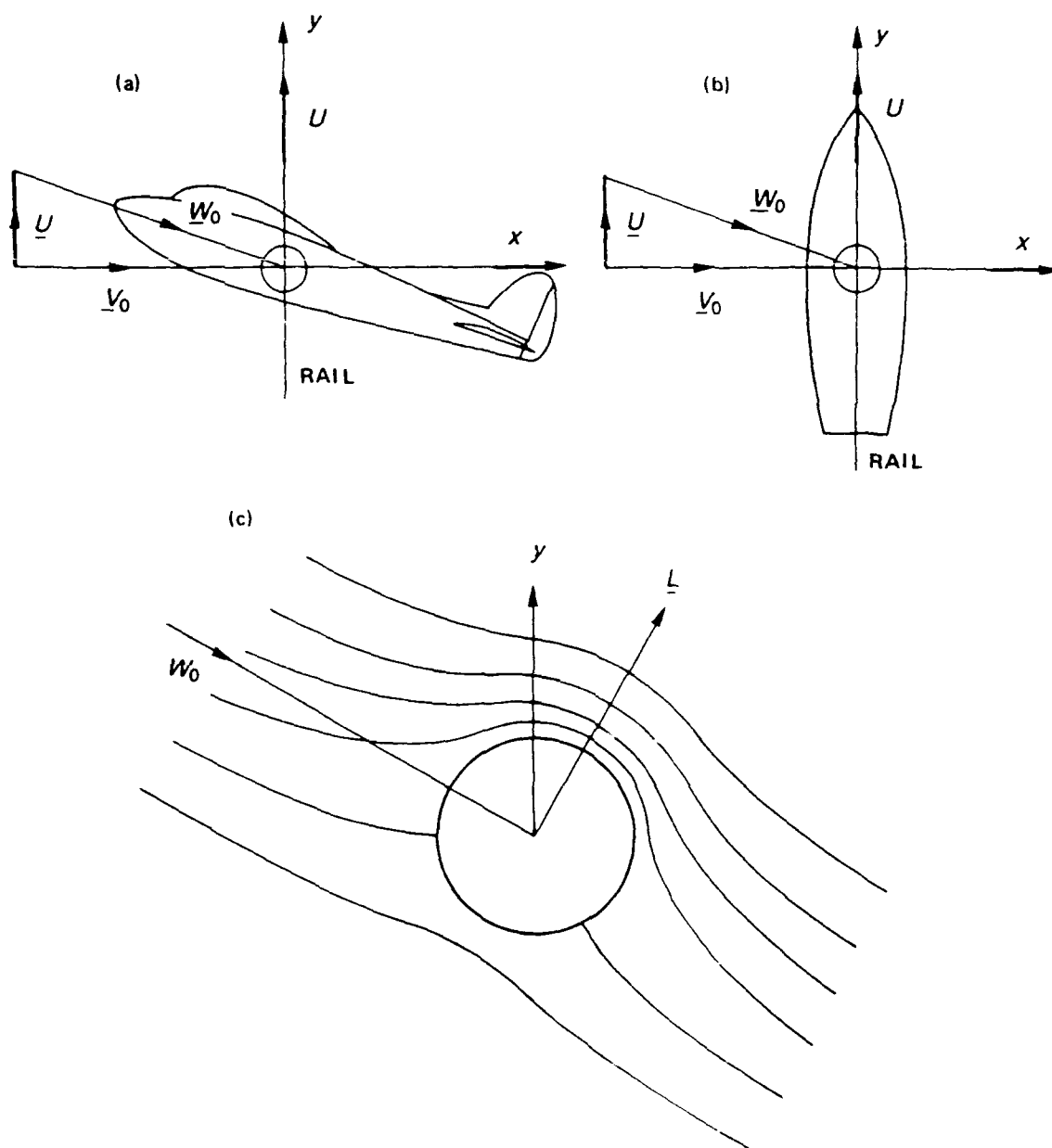


Fig. 2. The single-blade linear turbine: (a) sailplane version, (b) sailboat version, and (c) the rotating cylinder blade.

$$L = \rho W_o \Gamma , \quad (40)$$

where Γ is the scalar circulation. The units are force per unit length of cylinder. In the absolute and moving frame the lift component L_y directed parallel to the y axis of Figure 2 is given by

$$L_y = \rho W_{ox} \Gamma = \rho V_{ox} \Gamma , \quad (41)$$

where the subscript x represents the x component. Recalling that U is the velocity of motion of the device (sail or wing or rotating cylinder) as perceived in the absolute frame, the power is the product of U and L_y .

$$\text{Power/unit length} = \rho U W_{ox} \Gamma . \quad (42)$$

Since we assume that there is no heat rate,

$$\frac{DH_o}{Dt} = -\rho U W_{ox} \Gamma , \quad (43)$$

where H_o is the total enthalpy of the system per unit length. Equation (43) is the anticipated relationship which should ultimately be developed from the differential form (14).

THE STREAM FUNCTION, VELOCITY AND RELATIVE ENTHALPY IN THE FRAME OF THE BLADE

Since ideal flow has been assumed in the moving frame of the blade, the stream function, ψ , is the usual function modified for motion along the y axis.

$$\begin{aligned} \psi = & -W_{oy}(1 - a^2/r^2)r \cos \theta + W_{ox}(1 - a^2/r^2)r \sin \theta \\ & + (\Gamma/2\pi) \ln(r/a) . \end{aligned} \quad (44)$$

The constant a is the radius of the cylinder. The cartesian velocity components are obtained by the usual transformation [9] as follows:

$$\begin{aligned} W_x = & W_{ox} + \frac{a^2 W_{ox}(y^2 - x^2)}{(x^2 + y^2)^2} - \frac{2a^2 W_{oy}xy}{(x^2 + y^2)^2} \\ & + \frac{\Gamma}{2\pi} \frac{y}{(x^2 + y^2)} , \end{aligned} \quad (45)$$

$$\begin{aligned} W_y = & W_{oy} - \frac{2a^2 W_{ox}xy}{(x^2 + y^2)^2} + \frac{a^2 W_{oy}(x^2 - y^2)}{(x^2 + y^2)^2} \\ & - \frac{\Gamma}{2\pi} \frac{x}{(x^2 + y^2)} . \end{aligned} \quad (46)$$

Now the relative vorticity must vanish because potential flow cannot have vorticity. A check of the vorticity in the relative frame shows that indeed it vanishes. Also, the time-dependent term vanishes.

THE SUBSTANTIAL TOTAL ENTHALPY DERIVATIVE

In the linear two-dimensional system, the differential form of the turbomachinery Equation (14) is simplified because the rotation vanishes.

$$\frac{Dh_o}{Dt} = \bar{U} \cdot (\bar{W} \cdot \nabla) \bar{W} . \quad (47)$$

Since the vorticity vanishes

$$\bar{W} \cdot \nabla \bar{W} = \nabla W^2 / 2 , \quad (48)$$

and

$$\frac{Dh_o}{Dt} = \bar{U} \cdot \nabla W^2 / 2 = \frac{U}{2} \frac{\partial W^2}{\partial y_w} . \quad (49)$$

The integrated substantial total enthalpy rate per unit length [where the subscript on y in (49) has been dropped] is

$$\frac{DH_o}{Dt} = \int \int \frac{\rho U}{2} \frac{\partial (W_x^2 + W_y^2)}{\partial y} dy dx . \quad (50)$$

Integration of (50) will be performed over all space per unit length z of the blade. The choice of time is immaterial since the fluid dynamics are steady state in the moving frame and the thermodynamic rates over all space are invariant with time. It is understood that the integration applies only to the fluid domain and that boundaries at solid walls are observed.

Now without going into details [9] the integral of the total derivative in (50) is

$$\begin{aligned} \frac{DH_o}{Dt} = & -\rho U \int_{-a}^a \left(\frac{4UW_{ox} x(a^2 - x^2)^{1/2}(2x^2 - a^2)}{a^4} \right. \\ & + \frac{2W_{ox}\Gamma x(a^2 - x^2)^{1/2}}{\pi a^4} + \frac{8W_{ox}Ux(a^2 - x^2)^{1/2}}{a^2} \\ & + \frac{W_{ox}\Gamma(a^2 - x^2)^{1/2}}{\pi a^4} + \frac{4W_{ox}Ux(a^2 - x^2)^{1/2}(a^2 - 2x^2)}{a^4} \\ & \left. + \frac{W_{ox}\Gamma(a^2 - x^2)^{1/2}(a^2 - 2x^2)}{\pi a^4} \right) dx . \quad (51) \end{aligned}$$

Note that only odd terms in y make any contribution to (51). Since the first and fifth terms cancel, only four terms remain. The integration with respect to x is performed through a transformation employing

$$x = a \cos \theta , \quad (52)$$

with integration limits given by

$$\begin{aligned} \theta &= \pi \text{ when } x = -a , \\ \theta &= 0 \text{ when } x = a . \end{aligned} \quad (53)$$

Making the substitutions

$$\begin{aligned} \frac{DH_o}{Dt} = & -\rho U \left(-\frac{2W_{ox}\Gamma}{\pi} \int_{\pi}^0 \cos^2 \theta \sin^2 \theta d\theta \right. \\ & - 8W_{ox}Ua \int_{\pi}^0 \cos \theta \sin^2 \theta d\theta - \frac{W_{ox}\Gamma}{\pi} \int_{\pi}^0 \sin^2 \theta d\theta \\ & \left. - \frac{W_{ox}\Gamma}{\pi} \int_{\pi}^0 \sin^2 \theta (1 - 2 \cos^2 \theta) d\theta \right) . \end{aligned} \quad (54)$$

In (54) the second integral makes no contribution because it is antisymmetric. The first integral cancels the second term in the last integral to yield from the surviving terms

$$\frac{DH_o}{Dt} = -\rho W_{ox}\Gamma U = -\rho V_{ox}\Gamma U . \quad (55)$$

Equation (55) is identical with (43) and this result illustrates a useful application of the differential form and constitutes confirmation of the validity of the differential turbomachinery Equation (14). For the linear case, the energy transfer rate of the rotor is proportional to the component of the kinetic energy gradient parallel to the moving rotor.

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new equation will be developed in a two-dimensional linear turbine. Finally, the steady-state viscous problem of two concentric cylinders will be examined.

INTEGRATION OF THE TOTAL ENTHALPY RATE

In the integration process it will be assumed that the flow may be divided into streams which pass between a given pair of blades. In the rotating frame the streamtube walls are fixed steady-state walls associated with a steady-state mass flow rate m which may consist of radial and axial mass flow components.

DERIVATION OF THE INTEGRAL FORM FROM THE DIFFERENTIAL FORM

The differential form of the turbomachinery equation (14) may be integrated to yield

$$\begin{aligned} \iiint \rho \frac{Dh_o}{Dt} d\tau \\ = \iiint \rho [U \cdot (W \cdot \nabla)W + 2U \cdot \bar{\omega} \times W] r dr d\theta dz \\ + \iiint \nabla \cdot (\bar{\pi} \cdot W) r dr d\theta dz . \end{aligned} \quad (26)$$

The first term of the right member of (26) is the tangential component of the convective term obtained on dot multiplication with U , i.e.:

$$\begin{aligned} \rho U \cdot (W \cdot \nabla)W \\ = \rho U \left(W_r \frac{\partial W_\theta}{\partial r} + \frac{W_\theta}{r} \frac{\partial W_\theta}{\partial \theta} + W_z \frac{\partial W_\theta}{\partial z} + \frac{W_r W_\theta}{r} \right) . \end{aligned} \quad (27)$$

The first and last terms of (27) will be combined in an integral identified by $I_{1,4}$ thus:

$$\begin{aligned} I_{1,4} &= \iiint \rho r w \frac{W_r}{r} \frac{\partial W_\theta}{\partial r} r dr d\theta dz \\ &= \int \left(\iint \rho W_r r d\theta dz \right) \frac{\partial r w \bar{W}_\theta}{\partial r} dr \end{aligned} \quad (28)$$

where we have used the mean value theorem to take $\bar{W}_\theta(r)$ outside the double integral. Thus,

$$\bar{W}_\theta(r) = \frac{1}{\Delta\theta \Delta z} \int_{\theta_{\min}}^{\theta_{\max}} \int_{z_{\min}}^{z_{\max}} W_\theta d\theta dz . \quad (29)$$

The factor in parentheses in the right member of (28) is the radial mass flow at any point r .

$$m_r(r) = \int \int \rho W_r r d\theta dz = m f_r(r) , \quad (30)$$

where $f_r(r)$ is the radial fraction of the mass flow rate m . Then

$$I_{1,4} = m \int f_r(r) \frac{\partial r \omega \bar{\bar{W}}_\theta}{\partial r} dr = m \int f_r d(U \bar{\bar{W}}_\theta) . \quad (31)$$

Now in the steady state the mass value of $f_r(r)$, $\bar{f}_r$, is a constant given by

$$\bar{f}_r = \int_{\text{entrance}}^{\text{exit}} f_r (U \bar{\bar{W}}_\theta) d(U \bar{\bar{W}}_\theta) / \Delta(U \bar{\bar{W}}_\theta) , \quad (32)$$

and,

$$I_{1,4} = m \bar{f}_r \Delta_r (U \bar{\bar{W}}_\theta) = \dot{m}_r [(U \bar{\bar{W}}_\theta)_{r_2} - (U \bar{\bar{W}}_\theta)_{r_1}] . \quad (33)$$

Using the same arguments as used for the radial mass flow, the third term of the right member of (27) may be written to show the axial mass flow rate $m_z(z)$ explicitly.

(See argument above for $m_r(r)$.) With the velocities $\bar{\bar{W}}_z$, $\bar{\bar{W}}_\theta$, and $\bar{U}$ averaged over r and θ , one obtains

$$I_3 = \dot{m}_z [(\bar{U} \bar{\bar{W}}_\theta)_{z_2} - (\bar{U} \bar{\bar{W}}_\theta)_{z_1}] . \quad (34)$$

The second term of the right member of (27) provides an integral, I_2 , which contains the tangential kinetic energy.

$$\begin{aligned} I_2 &= \int \int \frac{\rho \omega r}{2} \left(\int \frac{\partial W_\theta^2}{\partial \theta} d\theta \right) dr dz \\ &= \int \int \frac{\rho \omega r}{2} [W_\theta^2(\theta_2) - W_\theta^2(\theta_1)] dr dz = 0 . \end{aligned} \quad (35)$$

Since the tangential velocities at the blade walls are equal to the blade velocity, the integral vanishes.

Now identifying the second term of the right member of Equation (27) as I_5 , we may write

$$I_5 = \int \int \int 2 \rho \omega^2 r W_r r dr d\theta dz = \dot{m}_r [(U^2)_{r_2} - (U^2)_{r_1}] . \quad (36)$$

Summing the components of integration (33) through (36)

$$\begin{aligned} \int \int \int \rho \frac{Dh_\theta}{Dt} d\tau &= \dot{m}_r [\Delta_r (U \bar{\bar{W}}_\theta) + \Delta_r (U^2)] \\ &\quad + \dot{m}_z \Delta_z (\bar{U} \bar{\bar{W}}_\theta) + (VIS) . \end{aligned} \quad (37)$$

where Δ_r and Δ_z represent the change along r and z respectively, and (VIS) is the integrated viscous term.

Now adding $\bar{m}_z \Delta_z \bar{U}^2$, which is zero, to Equation (37),

$$\int \int \int \rho \frac{Dh_o}{Dt} d\tau = \bar{m}_r \Delta_r \left[(\bar{U} + \bar{W}_\theta) \bar{U} \right] + \bar{m}_z \Delta_z \left[(U + \bar{W}_\theta) U \right] + (VIS) = m \Delta h_o . \quad (38)$$

In Equation (38) the terms $\bar{W}_\theta$ are averaged over θ and z in the first term and over r and θ in the second term. Finally,

$$\Delta h_o = \bar{f}_r \Delta_r (\bar{U} \bar{V}_\theta) + \bar{f}_z \Delta_z (U \bar{V}_\theta) + (VIS)/m = \Delta (\bar{U} \bar{V}_\theta) + (VIS)/m . \quad (39)$$

Equation (39) represents the classic Turbomachinery Equation with mixed flows and the terms $\bar{V}_\theta$ and $\bar{U}$ are averaged over the blade space where necessary.

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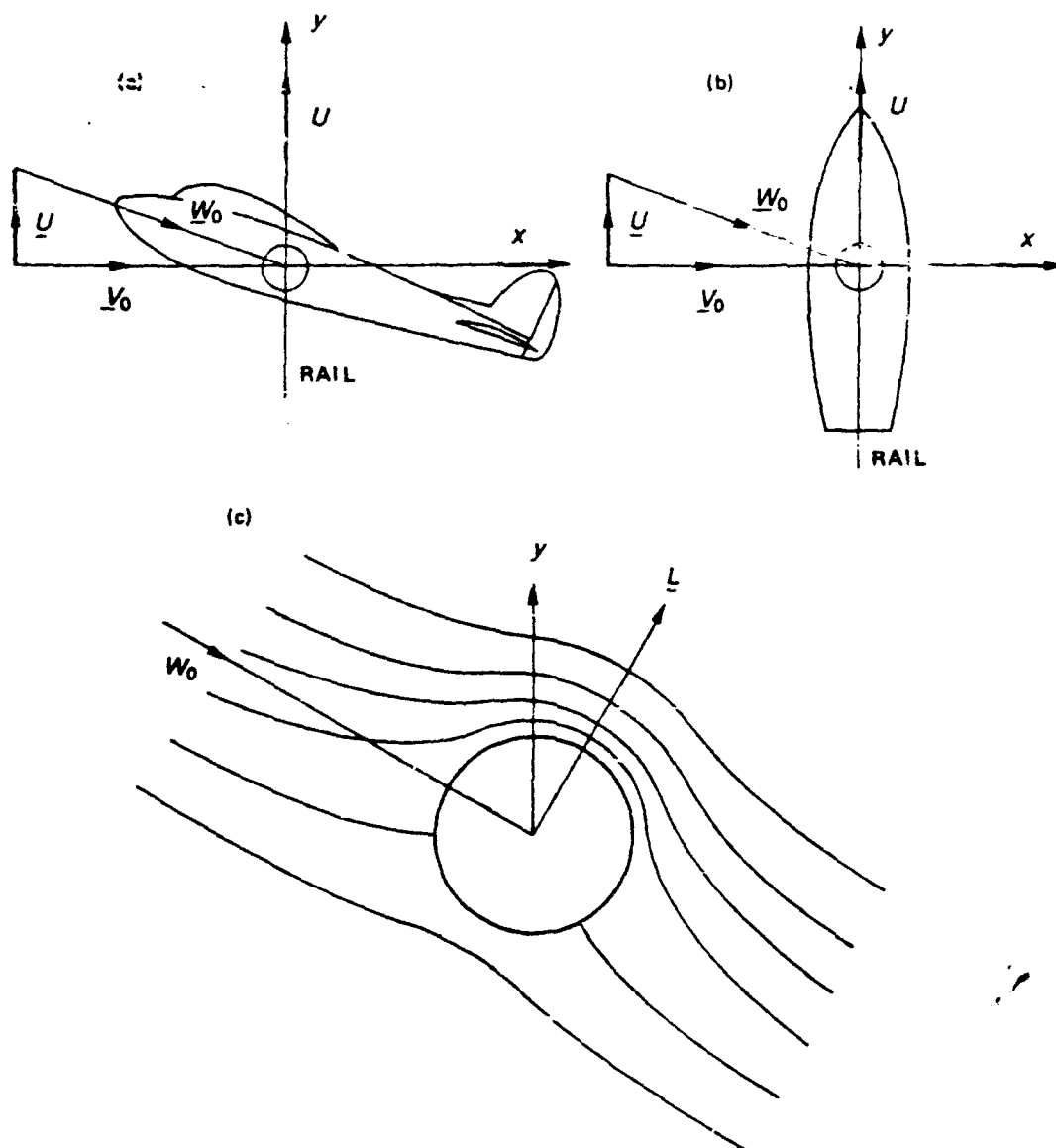


Fig. 2. The single-blade linear turbine: (a) sailplane version, (b) sailboat version, and (c) the rotating cylinder blade.