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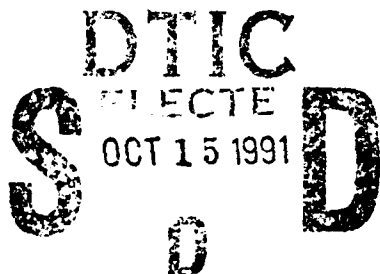


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TECHNICAL REPORT
NCSC TR 426-90

SEPTEMBER 1991

**ELECTROMAGNETIC FIELDS OF A UNIFORM
SPHERE IN A UNIFORM CONDUCTING
MEDIUM WITH APPLICATION
TO DIPOLE SOURCES**



W. M. WYNN

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13. ABSTRACT (Maximum 200 words) Vector spherical harmonic expansions are developed for the electromagnetic field of a uniform sphere in a uniform conducting medium in the presence of an arbitrary localized distribution of current. The source is either interior to or exterior to the sphere. Explicit expressions then are developed for all cases for magnetic and current dipole sources, and a complete numerical treatment is given for the case of exterior dipole and field point. The general treatment includes the transverse electric-transverse magnetic representation of the electromagnetic field in a source-free region, and demonstration of the formation of the electric and magnetic field vectors from $E \cdot r$ and $B \cdot r$ alone. General expressions are given relating the scattered field expansion coefficients to the source expansion coefficients, including the special case of a perfectly conducting sphere. The specialized treatment of dipole sources includes explicit expressions for the electric and magnetic field components both interior to and exterior to the sphere, for either internal or external dipoles, for the general ac case and for the dc limit. The detailed treatment of the dipole sources includes various series and asymptotic representations of the Bessel functions J_ν and $H_\nu^{(1)}$, which are incorporated into Hewlett-Packard Basic 3.0 codes for the calculation of the dipole field components for the ac and dc cases for exterior dipole and field point.				
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INTRODUCTION

The analysis of electromagnetic fields in uniform media reduces to the solution of the vector Helmholtz equation. This equation is a vector partial differential equation of the form $\nabla \times \nabla \times \mathbf{A} - k^2 \mathbf{A} = 0$, and generally leads to coupled equations for the field components, but it can be separated in the spherical and cylindrical coordinate systems.¹ For cylindrical coordinates, detailed treatments can be found for localized sources in the presence of a layered cylinder,² and for localized sources outside a perfectly conducting wedge,³ as well as less detailed treatments of the conducting wedge.^{4,5} In the present work, a detailed analysis is given for the electromagnetic fields inside and outside a uniform sphere immersed in a uniform medium, in the presence of a localized source either interior or exterior to the sphere.

The solution is given in terms of a vector spherical harmonic expansion, with expansion coefficients for the scattered fields expressed in terms of the coefficients appropriate to the source distribution. The source is treated using a clever technique due to Jackson.⁶ A separate treatment of the case of a perfectly conducting sphere also is included.

For explicit results, the source is taken to be a magnetic or current dipole. Without loss of generality, the dipoles are placed on the z-axis, and oriented radially or tangentially to the sphere, which is centered on the origin. Detailed expressions are given for the electric and magnetic field components, both interior and exterior to the sphere, for both interior and exterior dipoles. These expressions are relatively simple, and do not involve any recursion formulas for the coefficients, contrary to what has been reported elsewhere.⁷ For completeness, the dc limits of the expressions also are given. This is useful for direct applications, to provide more transparent forms to check for physical consistency of the field expressions, and to provide a numerical check against the codes for the fields in the low-frequency limit.

Numerical codes written in Hewlett-Packard BASIC, Version 3.0, are given for both the ac and dc field expressions for the single case of exterior dipole and field point. The bulk of the code given, in particular the geometrical transformations and the Bessel function and Legendre polynomial routines, is applicable to the other three cases. The numerical treatment of the spherical Bessel functions incorporates several representations found in Abramowitz and Stegun.⁸

BACKGROUND

For a time-harmonic source $e^{-i\omega t}$, Maxwell's equations for a uniform medium take the form

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} \quad (1)$$

$$\nabla \times \mathbf{B} = \mu((\sigma - i\omega\epsilon)\mathbf{E} + \nabla \times \mathbf{M}_s + \mathbf{J}_s) = \mu \mathbf{J} \quad (2)$$

$$\nabla \cdot (\epsilon \mathbf{E}) = \rho \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (4)$$

where σ , ϵ , and μ are the medium conductivity, permittivity, and permeability, respectively, and $\mathbf{M}_s$ and $\mathbf{J}_s$ are source magnetization and current density. The sources are included in a general way, but will be specialized to point magnetic and current dipoles for explicit calculations.

The equation of continuity is

$$\nabla \cdot \mathbf{J} - i\omega\rho = 0 \quad (5)$$

that, upon substitution of Equations (2) and (3), can be written

$$\nabla \cdot \left(\mathbf{E} + \frac{\mathbf{J}_s}{\sigma - i\omega\epsilon} \right) = 0. \quad (6)$$

With the introduction of the divergenceless vector

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{J}_s}{\sigma - i\omega\epsilon} \quad (7)$$

that is identical to $\mathbf{E}$ in a source-free region, Equations (1) and (2) can be written

$$\nabla \times \mathbf{E}' = i\omega\mathbf{B} + \frac{\nabla \times \mathbf{J}_s}{\sigma - i\omega\epsilon} \quad (8)$$

and

$$\nabla \times \mathbf{B} = \mu((\sigma - i\omega\epsilon)\mathbf{E}' + \nabla \times \mathbf{M}_s). \quad (9)$$

With the introduction of the complex wave number

$$k^2 = \mu(i\omega\sigma - \omega^2\epsilon) \quad (10)$$

Equations (8) and (9) can be written

$$\nabla \times \mathbf{E}' = i\omega \left(\mathbf{B} + \frac{\nabla \times \mathbf{J}_s}{k^2} \right) \quad (11)$$

and

$$\nabla \times \mathbf{B} = \frac{k^2}{i\omega} \mathbf{E}' + \mu \nabla \times \mathbf{M}_s. \quad (12)$$

Applying the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ to Equations (11) and (12) then results in the vector equations with sources given by

$$(\nabla^2 + k^2)\mathbf{E}' = -i\omega\mu \left(\frac{\nabla \times \nabla \times \mathbf{J}_s}{k^2} + \nabla \times \mathbf{M}_s \right) \quad (13)$$

and

$$(\nabla^2 + k^2)\mathbf{B} = -\mu(\nabla \times \nabla \times \mathbf{M}_s + \nabla \times \mathbf{J}_s). \quad (14)$$

In a source free region, the fields $\mathbf{E}'$ and $\mathbf{B}$, satisfying the associated homogeneous forms of Equations (13) and (14), can be expanded in terms of transverse electric and transverse magnetic parts in the form

$$\mathbf{E}' = \nabla \times (\mathbf{r}\Psi_E) + \alpha \nabla \times \nabla \times (\mathbf{r}\Psi_M) \quad (15)$$

and

$$\mathbf{B} = \nabla \times (\mathbf{r}\Psi_M) + \beta \nabla \times \nabla \times (\mathbf{r}\Psi_E) \quad (16)$$

where $\mathbf{r}$ is the radius vector, provided that Ψ_E and Ψ_M satisfy the scalar Helmholtz equation

$$\nabla^2 \Psi_E + k^2 \Psi_E = 0 \quad (17)$$

and

$$\nabla^2 \Psi_M + k^2 \Psi_M = 0. \quad (18)$$

This may be verified by applying the operator $\nabla \times \nabla \times (\dots)$ to Equations (15) and (16) in source free regions, and applying the appropriate vector identities.

The coefficients α and β can be determined by inserting Equations (15) and (16) in Equation (11) in a source free region and using the identity

$$\nabla \times \nabla \times (\mathbf{r}\Psi) = k^2 \mathbf{r}\Psi + \nabla \left(\Psi + r \frac{\partial \Psi}{\partial r} \right) \quad (19)$$

to give the result

$$\alpha = \frac{i\omega}{k^2} \quad \text{and} \quad \beta = -\frac{i}{\omega}. \quad (20)$$

Equation (19) is a nontrivial identity, and is developed in detail below Equation (69) and preceding discussion).

The vectors $\mathbf{E}'$ and $\mathbf{B}$ can be determined from the projections $\mathbf{r} \cdot \mathbf{E}'$ and $\mathbf{r} \cdot \mathbf{B}$, as can be seen by constructing the projections using Equations (15) and (16), applying Equation (19) and the identity $\mathbf{r} \cdot (\nabla \Psi \times \mathbf{r}) = 0$. This results in

$$\mathbf{r} \cdot \mathbf{E}' = \frac{i\omega}{k^2} \left[k^2 r^2 \Psi_M + r \frac{\partial}{\partial r} \left(\Psi_M + r \frac{\partial \Psi_M}{\partial r} \right) \right] \quad (21)$$

and

$$\mathbf{r} \cdot \mathbf{B} = -i\omega \left[k^2 r^2 \Psi_E + r \frac{\partial}{\partial r} \left(\Psi_E + r \frac{\partial \Psi_E}{\partial r} \right) \right]. \quad (22)$$

This shows that the projections $\mathbf{r} \cdot \mathbf{E}'$ and $\mathbf{r} \cdot \mathbf{B}$ determine Ψ_M , and Ψ_E , respectively, and consequently determine $\mathbf{E}'$ and $\mathbf{B}$ via Equations (15) and (16).

The projections $\mathbf{r} \cdot \mathbf{E}'$ and $\mathbf{r} \cdot \mathbf{B}$ can be determined everywhere by means of the identities

$$\mathbf{r} \cdot (\nabla^2 + k^2) \mathbf{E}' = (\nabla^2 + k^2) (\mathbf{r} \cdot \mathbf{E}') \quad (23)$$

and

$$\mathbf{r} \cdot (\nabla^2 + k^2) \mathbf{B} = (\nabla^2 + k^2) (\mathbf{r} \cdot \mathbf{B}) \quad (24)$$

together with Equations (13) and (14). These together give the inhomogeneous scalar Helmholtz equations

$$(\nabla^2 + k^2) (\mathbf{r} \cdot \mathbf{E}') = \mathbf{r} \cdot \mathbf{S}_E = \rho_E \quad (25)$$

and

$$(\nabla^2 + k^2) (\mathbf{r} \cdot \mathbf{B}) = \mathbf{r} \cdot \mathbf{S}_B = \rho_B \quad (26)$$

where ρ_E and ρ_B are given by

$$\rho_E = -i\omega\mu\mathbf{r} \cdot \left(\frac{\nabla \times \nabla \times \mathbf{J}_S}{k^2} + \nabla \times \mathbf{M}_S \right) \quad (27)$$

and

$$\rho_B = -\mu\mathbf{r} \cdot (\nabla \times \nabla \times \mathbf{M}_S + \nabla \times \mathbf{J}_S). \quad (28)$$

ANGULAR MOMENTUM OPERATOR AND VECTOR SPHERICAL HARMONICS

The subsequent introduction of spherical boundaries near a localized source distribution is greatly facilitated by the introduction of the angular momentum operator

$$\mathbf{L} = -i\mathbf{r} \times \nabla \quad (29)$$

and the normalized vector spherical harmonics

$$\mathbf{X}_{l,m}(\theta, \phi) = \frac{1}{\sqrt{l(l+1)}} \mathbf{L} Y_{l,m}(\theta, \phi). \quad (30)$$

By simple manipulation using the identity $\nabla \times (\mathbf{r}\Psi) = -\mathbf{r} \times \nabla\Psi + \Psi\nabla \times \mathbf{r} = -\mathbf{r} \times \nabla\Psi$, Equations (15) and (16) can be written

$$\mathbf{E}' = -i\mathbf{L}\Psi_E + \frac{\omega}{k^2} \nabla \times \mathbf{L}\Psi_M \quad (31)$$

and

$$\mathbf{B} = -i\mathbf{L}\Psi_M - \frac{1}{\omega} \nabla \times \mathbf{L}\Psi_E. \quad (32)$$

Since Ψ_M and Ψ_E are solutions to the scalar Helmholtz equation, they can be expanded in spherical coordinate eigenfunctions of the form

$$\Psi_{l,m} = f_l(kr)Y_{l,m}(\theta, \phi) \quad (33)$$

where f_l is a spherical Bessel function satisfying the differential equation

$$\frac{d^2 f_l}{dr^2} + \frac{2}{r} \frac{df_l}{dr} + \left(k^2 - \frac{l(l+1)}{r^2} \right) f_l = 0 \quad (34)$$

and the $Y_{l,m}$ are spherical harmonics satisfying, in particular,

$$\mathbf{L}^2 Y_{l,m} = l(l+1)Y_{l,m}. \quad (35)$$

Now note the following series of substitutions resulting in a very useful identity:

$$\begin{aligned} \mathbf{L}[f_l(kr)Y_{l,m}(\theta, \phi)] &= -i\mathbf{r} \times \nabla[f_l(kr)Y_{l,m}(\theta, \phi)] \\ &= -i\mathbf{r} \times [Y_{l,m}(\theta, \phi)\nabla_r f_l(kr) + f_l(kr)\nabla_{\theta, \phi} Y_{l,m}(\theta, \phi)] \\ &= -i\mathbf{r} \times [f_l(kr)\nabla_{\theta, \phi} Y_{l,m}(\theta, \phi)] = f_l(kr)[-i\mathbf{r} \times \nabla Y_{l,m}(\theta, \phi)] \\ &= f_l \mathbf{L}Y_{l,m}(\theta, \phi) = \sqrt{l(l+1)} f_l(kr) \mathbf{X}_{l,m}(\theta, \phi). \end{aligned} \quad (36)$$

If the scalar functions are written as the expansions

$$\Psi_E = i \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{a_{l,m}^E}{\sqrt{l(l+1)}} = f_l(kr)Y_{l,m}(\theta, \phi) \quad (37)$$

and

$$\Psi_M = i \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{a_{l,m}^M}{\sqrt{l(l+1)}} g_l(kr)Y_{l,m}(\theta, \phi) \quad (38)$$

then these may be substituted into Equations (31), (32), and (36) to give

$$\mathbf{E}' = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^E f_l \mathbf{X}_{l,m} + \frac{i\omega}{k^2} a_{l,m}^M \nabla \times [g_l \mathbf{X}_{l,m}] \right\} \quad (39)$$

and

$$\mathbf{B} = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^M g_l \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^E \nabla \times [f_l \mathbf{X}_{l,m}] \right\}. \quad (40)$$

These forms will be used outside of the source region for both the primary fields due to the source, and the scattered fields due to spherical surfaces. The primary fields will be developed first. Then, in a subsequent section, the scattered fields will be developed by applying the appropriate boundary conditions, and utilizing various orthogonality properties of the vector spherical harmonics.

GREEN FUNCTION FOR THE PRIMARY FIELDS

For a localized source and no boundaries, the solutions to Equations (25) and (26) can be expressed in terms of a Green function by means of Green's theorem:

$$\int d^3 \mathbf{r}' \{ G(\mathbf{r} - \mathbf{r}') (\nabla'^2 + k^2) F(\mathbf{r}') - F(\mathbf{r}') (\nabla'^2 + k^2) G(\mathbf{r} - \mathbf{r}') \} = 0. \quad (41)$$

With G and F satisfying

$$(\nabla'^2 + k^2) G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \quad (42)$$

and

$$(\nabla'^2 + k^2) F(\mathbf{r}') = \rho(\mathbf{r}') \quad (43)$$

Equation (41) reduces to

$$F(\mathbf{r}) = \int d^3 \mathbf{r}' G(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}'). \quad (44)$$

As shown in a number of texts, such as Jackson,⁷ the free field Green function is given by

$$G(\mathbf{r} - \mathbf{r}') = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \quad (45)$$

and has the expansion in spherical coordinate eigenfunctions given by

$$G(\mathbf{r} - \mathbf{r}') = ik \sum_{l=0}^{\infty} j_l(kr^<) h_l^{(1)}(kr^>) \sum_{m=-l}^{m=l} Y_{l,m}^*(\theta', \phi') Y_{l,m}(\theta, \phi) \quad (46)$$

where j_l and $h_l^{(1)}$ are spherical Bessel functions of the first and third kind, respectively, and

$$r^{</>} = r, r^{</>} < / > r', r' \text{ otherwise.} \quad (47)$$

EXPANSION COEFFICIENTS FOR THE PRIMARY FIELDS

The primary fields outside the source region have the forms given in Equations (39) and (40). The expansion coefficients can be determined by evaluating $\mathbf{r} \cdot \mathbf{E}'$ and $\mathbf{r} \cdot \mathbf{B}$. First note that

$$\mathbf{r} \cdot \mathbf{X}_{l,m} = 0 \quad (48)$$

and

$$\mathbf{r} \cdot \nabla f(r) \times \mathbf{X}_{l,m} = \mathbf{X}_{l,m} \cdot \mathbf{r} \times \nabla f(r) = 0. \quad (49)$$

This leads to the identity

$$\begin{aligned} \mathbf{r} \cdot \nabla \times [f_l(kr) \mathbf{X}_{l,m}] &= f_l(kr) \mathbf{r} \cdot \nabla \times \mathbf{X}_{l,m} = f_l(kr) \mathbf{r} \times \nabla \cdot \mathbf{X}_{l,m} \\ &= i f_l(kr) \mathbf{L} \cdot \mathbf{X}_{l,m} = \frac{i f_l(kr)}{\sqrt{l(l+1)}} \mathbf{L}^2 Y_{l,m} = i f_l(kr) \sqrt{l(l+1)} Y_{l,m}. \end{aligned} \quad (50)$$

Thus the primary field expansion coefficients are given simply by

$$\mathbf{r} \cdot \mathbf{E}'^P = -\frac{\omega}{k^2} \sum_{l=0}^{\infty} \sqrt{l(l+1)} g_l(kr) \sum_{m=-l}^{m=l} a_{l,m}^{M,P} Y_{l,m} \quad (51)$$

and

$$\mathbf{r} \cdot \mathbf{B}^P = \frac{1}{\omega} \sum_{l=0}^{\infty} \sqrt{l(l+1)} f_l(kr) \sum_{m=-l}^{m=l} a_{l,m}^{E,P} Y_{l,m}. \quad (52)$$

When Equations (43) and (44) are applied to Equations (25) and (26), and the expansion Equation (46) is used, expansions are obtained for $\mathbf{r} \cdot \mathbf{E}'^P$ and $\mathbf{r} \cdot \mathbf{B}^P$. Then, Equation (51) becomes

$$\begin{aligned} & -\frac{\omega}{k^2} \sum_{l=0}^{\infty} \sqrt{l(l+1)} g_l(kr) \sum_{m=-l}^{m=l} a_{l,m}^{M,P} Y_{l,m}(\theta, \phi) \\ &= i k \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} Y_{l,m}(\theta, \phi) \int d^3 \mathbf{r}' \rho_E(\mathbf{r}') j_l(kr^{</>}) h_l^{(1)}(kr^{>}) Y_{l,m}^*(\theta', \phi'). \end{aligned} \quad (53)$$

For $r < r'$ for any source point, $f_l(kr) = g_l(kr) = j_l(kr)$ and this can be used together with the orthonormality of the spherical harmonics $Y_{l,m}$ to convert Equation (53) to

$$a_{l,m}^{M,P} = -\frac{ik^3}{\omega \sqrt{l(l+1)}} \int d^3 \mathbf{r}' \rho_E(\mathbf{r}') h_l^{(1)}(kr^{</>}) Y_{l,m}^*(\theta', \phi'). \quad (54)$$

A similiar treatment of $\mathbf{r} \cdot \mathbf{B}^P$ gives

$$a_{l,m}^{E,P} = \frac{ik\omega}{\sqrt{l(l+1)}} \int d^3\mathbf{r}' \rho_B(\mathbf{r}') h_l^{(1)}(kr') Y_{l,m}^*(\theta', \phi'). \quad (55)$$

For points $r > r'$ for any source point, $h_l^{(1)}(kr')$ is replaced by $j_l(kr')$ in Equations (54) and (55).

APPLICATION TO A SPHERE WITH AN EXTERNAL SOURCE

For points interior to the sphere and exterior points outside the source, the fields have expansions of the forms given in Equations (39) and (40). In particular, these forms hold for all points on the surface of the sphere, and will be useful in imposing boundary conditions. The interior fields will be designated with a 1 superscript, and the exterior scattered fields have a 2 superscript. To be regular at the origin, the interior fields have the form

$$\mathbf{E}^1 = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ a_{l,m}^{E,1} j_l(k_1 r) \mathbf{X}_{l,m} + \frac{i\omega}{k_1^2} a_{l,m}^{M,1} \nabla \times [j_l(k_1 r) \mathbf{X}_{l,m}] \right\} \quad (56)$$

and

$$\mathbf{B}^1 = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ a_{l,m}^{M,1} j_l(k_1 r) \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E,1} \nabla \times [j_l(k_1 r) \mathbf{X}_{l,m}] \right\} \quad (57)$$

while the exterior scattered fields are expressed in terms of outgoing waves and have the form

$$\mathbf{E}^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ a_{l,m}^{E,2} h_l^{(1)}(k_2 r) \mathbf{X}_{l,m} + \frac{i\omega}{k_2^2} a_{l,m}^{M,2} \nabla \times [h_l^{(1)}(k_2 r) \mathbf{X}_{l,m}] \right\} \quad (58)$$

and

$$\mathbf{B}^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ a_{l,m}^{M,2} h_l^{(1)}(k_2 r) \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E,2} \nabla \times [h_l^{(1)}(k_2 r) \mathbf{X}_{l,m}] \right\} \quad (59)$$

and, just outside the sphere surface, the primary fields have the form

$$\mathbf{E}^P = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ a_{l,m}^{E,P} j_l(k_2 r) \mathbf{X}_{l,m} + \frac{i\omega}{k_2^2} a_{l,m}^{M,P} \nabla \times [j_l(k_2 r) \mathbf{X}_{l,m}] \right\} \quad (60)$$

and

$$\mathbf{B}^P = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ a_{l,m}^{M,P} j_l(k_2 r) \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E,P} \nabla \times [j_l(k_2 r) \mathbf{X}_{l,m}] \right\}. \quad (61)$$

BOUNDARY CONDITIONS

The boundary conditions are continuity of the tangential $\mathbf{E}$ and tangential $\mathbf{H}$ fields at the surface of the sphere. If R is the radius of the sphere, then the boundary conditions can be expressed as

$$\mathbf{R} \times \mathbf{E}^1 = \mathbf{R} \times (\mathbf{E}^2 + \mathbf{E}^p) \quad (62)$$

and

$$\frac{1}{\mu_1} \mathbf{R} \times \mathbf{B}^1 = \frac{1}{\mu_2} \mathbf{R} \times (\mathbf{B}^2 + \mathbf{B}^p). \quad (63)$$

To apply the boundary conditions in Equations (62) and (63) it is necessary to develop a rather complex identity. First, note that

$$\mathbf{r} \times \nabla \times [f(r)\mathbf{X}_{l,m}] = \mathbf{r} \times \left[\frac{\dot{f}(r)}{r} \mathbf{r} \times \mathbf{X}_{l,m} + f(r) \nabla \times \mathbf{X}_{l,m} \right]. \quad (64)$$

The first term on the right-hand side of Equation (64) has the form

$$\mathbf{r} \cdot \mathbf{X}_{l,m} \frac{\dot{f}(r)}{r} \mathbf{r} - r \dot{f}(r) \mathbf{X}_{l,m} = -r \dot{f}(r) \mathbf{X}_{l,m} \quad (65)$$

that is, it is proportional to $\mathbf{X}_{l,m}$. Next, it will be demonstrated that the second term on the right-hand side of Equation (64) also is proportional to $\mathbf{X}_{l,m}$.

First, note that by a standard identity

$$\nabla \times (\mathbf{r} \times \nabla)F = r \nabla^2 F - (\nabla \cdot \mathbf{r}) \nabla F + (\nabla F \cdot \nabla) \mathbf{r} - (\mathbf{r} \cdot \nabla) \nabla F. \quad (66)$$

The second term on the right-hand side of Equation (66) is just $-3 \nabla F$. For the third term on the right-hand side of Equation (66)

$$(\nabla F \cdot \nabla) \mathbf{r} = \partial_i F \partial_i r_j = \partial_i F \delta_{i,j} = \partial_j F = \nabla F \quad (67)$$

and, the fourth term on the right-hand side of Equation (66) is

$$\begin{aligned} -(\mathbf{r} \cdot \nabla) \nabla F &= -r_i \partial_i \partial_j F = -r_i \partial_j \partial_i F \\ &= -(\partial_j r_i - \delta_{i,j}) \partial_i F = \nabla F - \nabla(\mathbf{r} \cdot \nabla F). \end{aligned} \quad (68)$$

Applying these results to Equation (66) produces the identity

$$\nabla \times (\mathbf{r} \times \nabla)F = r \nabla^2 F - \nabla \left(F + r \frac{\partial F}{\partial r} \right). \quad (69)$$

This result can be applied directly to $\nabla \times \mathbf{X}_{l,m}$:

$$\begin{aligned}\nabla \times \mathbf{X}_{l,m} &= \frac{1}{\sqrt{l(l+1)}} \nabla \times \mathbf{L} Y_{l,m} = -\frac{i}{\sqrt{l(l+1)}} \nabla \times (\mathbf{r} \times \nabla) Y_{l,m} \\ &= -\frac{i}{\sqrt{l(l+1)}} \left[\mathbf{r} \nabla^2 Y_{l,m} - \nabla \left(1 + r \frac{\partial}{\partial r} \right) Y_{l,m} \right].\end{aligned}\quad (70)$$

but,

$$\nabla^2 Y_{l,m} = -\frac{L^2}{r^2} Y_{l,m} = -\frac{l(l+1)}{r^2} Y_{l,m} \quad (71)$$

so,

$$\nabla \times \mathbf{X}_{l,m} = \frac{il(l+1)}{r^2} \mathbf{r} Y_{l,m} + \frac{i}{\sqrt{l(l+1)}} \nabla Y_{l,m} \quad (72)$$

and

$$\mathbf{r} \times \nabla \times \mathbf{X}_{l,m} = \frac{i}{\sqrt{l(l+1)}} \mathbf{r} \times \nabla Y_{l,m} = -\mathbf{X}_{l,m}. \quad (73)$$

With all the foregoing, Equation (64) reduces (for spherical Bessel functions $f_l(kr)$) to

$$\mathbf{r} \times \nabla \times [f_l(kr) \mathbf{X}_{l,m}] = -[f_l(kr) + kr \dot{f}_l(kr)] \mathbf{X}_{l,m}. \quad (74)$$

Application of the boundary conditions Equations (62) and (63), using the expansions Equations (56) through (61), and the identity Equation (74) gives

$$\begin{aligned}& \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ \frac{a_{l,m}^{M,1} j_l(k_1 R)}{\mu_1} - \frac{a_{l,m}^{M,2} h_l^{(1)}(k_2 R)}{\mu_2} - \frac{a_{l,m}^{M,P} j_l(k_2 R)}{\mu_2} \right\} \mathbf{r} \times \mathbf{X}_{l,m} \\ & + i \left[\frac{a_{l,m}^{E,1} \{j_l(k_1 R) + k_1 R \dot{j}_l(k_1 R)\}}{\omega \mu_1} - \frac{a_{l,m}^{E,2} \{h_l^{(1)}(k_2 R) + k_2 R \dot{h}_l^{(1)}(k_2 R)\}}{\omega \mu_2} \right. \\ & \quad \left. - \frac{a_{l,m}^{E,P} \{j_l(k_2 R) + k_2 R \dot{j}_l(k_2 R)\}}{\omega \mu_2} \right] \mathbf{X}_{l,m} \} = 0\end{aligned}\quad (75)$$

and

$$\begin{aligned}
& \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \{ [a_{l,m}^{E,1} j_l(k_1 R) - a_{l,m}^{E,2} h_l^{(1)}(k_2 R) - a_{l,m}^{E,P} j_l(k_2 R)] \mathbf{R} \times \mathbf{X}_{l,m} \\
& - i\omega \left[\frac{a_{l,m}^{M,1} \{j_l(k_1 R) + k_1 R \dot{j}_l(k_1 R)\}}{k_1^2} - \frac{a_{l,m}^{M,2} \{h_l^{(1)}(k_2 R) + k_2 R \dot{h}_l^{(1)}(k_2 R)\}}{k_2^2} \right. \\
& \left. - \frac{a_{l,m}^{M,P} \{j_l(k_2 R) + k_2 R \dot{j}_l(k_2 R)\}}{k_2^2} \right] \mathbf{X}_{l,m} \} = 0.
\end{aligned} \tag{76}$$

The vector spherical harmonics satisfy several orthogonality and normalization conditions which are useful in determining the expansion coefficients⁶

$$\mathbf{X}_{l',m'}^* \cdot \mathbf{r} \times \mathbf{X}_{l,m} = 0 \tag{77}$$

$$\int d\Omega \mathbf{X}_{l',m'}^* \cdot \mathbf{X}_{l,m} = \delta_{l',l} \delta_{m',m} \tag{78}$$

and

$$\begin{aligned}
& \int d\Omega (\mathbf{r} \times \mathbf{X}_{l',m'}^*) \cdot (\mathbf{r} \times \mathbf{X}_{l,m}) \\
& = \int d\Omega [(\mathbf{r} \cdot \mathbf{r}) (\mathbf{X}_{l',m'}^* \cdot \mathbf{X}_{l,m}) - (\mathbf{r} \cdot \mathbf{X}_{l',m'}^*) (\mathbf{r} \cdot \mathbf{X}_{l,m})] \\
& = r^2 \delta_{l',l} \delta_{m',m}
\end{aligned} \tag{79}$$

where the integration is over all solid angles, and $\delta_{i,j}$ is the Kronecker delta. If Equations (75) and (76) are scalar multiplied by $\mathbf{X}_{l',m'}^*$ or $\mathbf{R} \times \mathbf{X}_{l',m'}^*$ and the solid angle integral performed, Equation (77) through (79) may be applied to give

$$a_{l,m}^{M,1} j_l(k_1 R) - \tau a_{l,m}^{M,2} h_l^{(1)}(k_2 R) = \tau a_{l,m}^{M,P} j_l(k_2 R) \tag{80}$$

$$a_{l,m}^{M,1} [j_l(k_1 R) + k_1 R \dot{j}_l(k_1 R)] - \gamma a_{l,m}^{M,2} [h_l^{(1)}(k_2 R) + k_2 R \dot{h}_l^{(1)}(k_2 R)] \tag{81}$$

$$= \gamma a_{l,m}^{M,P} [j_l(k_2 R) + k_2 R \dot{j}_l(k_2 R)]$$

$$a_{l,m}^{E,1} j_l(k_1 R) - a_{l,m}^{E,2} h_l^{(1)}(k_2 R) = a_{l,m}^{E,P} j_l(k_2 R) \tag{82}$$

and

$$a_{l,m}^{E,1} [j_l(k_1 R) + k_1 R \dot{j}_l(k_1 R)] - \tau a_{l,m}^{E,2} [h_l^{(1)}(k_2 R) + k_2 R \dot{h}_l^{(1)}(k_2 R)] \tag{83}$$

$$= \tau a_{l,m}^{E,P} [j_l(k_2 R) + k_2 R \dot{j}_l(k_2 R)]$$

where $\tau = \mu_1/\mu_2$ and $\gamma = k_1^2/k_2^2$.

Using the Wronskian $\{j_l(u), h_l^{(1)}(u)\} = i/u^2$, introduce the determinants of coefficients

$$D_{E,l} = h_l^{(1)}(u_2) [j_l(u_1) + u_1 \dot{j}_l(u_1)] - \tau j_l(u_1) [h_l^{(1)}(u_2) + u_2 \dot{h}_l^{(1)}(u_2)] \quad (84)$$

and

$$D_{M,l} = \tau h_l^{(1)}(u_2) [j_l(u_1) + u_1 \dot{j}_l(u_1)] - \gamma j_l(u_1) [h_l^{(1)}(u_2) + u_2 \dot{h}_l^{(1)}(u_2)] \quad (85)$$

and the expressions

$$N_{E,l} = -j_l(u_2) [j_l(u_1) + u_1 \dot{j}_l(u_1)] + \tau j_l(u_1) [j_l(u_2) + u_2 \dot{j}_l(u_2)] \quad (86)$$

and

$$N_{M,l} = -\tau j_l(u_2) [j_l(u_1) + u_1 \dot{j}_l(u_1)] + \gamma j_l(u_1) [j_l(u_2) + u_2 \dot{j}_l(u_2)]. \quad (87)$$

where $u_1 = k_1 R$ and $u_2 = k_2 R$. Then, the solutions for the expansion coefficients, *valid for any localized exterior source distribution that does not include the sphere surface*, are given by

$$a_{l,m}^{E,1} = -\frac{i\tau}{u_2 D_{E,l}} a_{l,m}^{E,P} \quad (88)$$

$$a_{l,m}^{E,2} = \frac{N_{E,l}}{D_{E,l}} a_{l,m}^{E,P} \quad (89)$$

$$a_{l,m}^{M,1} = -\frac{i\tau\gamma}{u_2 D_{M,l}} a_{l,m}^{M,P} \quad (90)$$

and

$$a_{l,m}^{M,2} = \frac{N_{M,l}}{D_{M,l}} a_{l,m}^{M,P}. \quad (91)$$

ELECTRIC AND MAGNETIC FIELDS FOR A PERFECTLY CONDUCTING SPHERE

For a highly conducting sphere, the numerical results using the coefficient representations in Equations (84) through (91) may be unreliable. In this case it is best to treat the sphere as a perfect conductor. Then, the coefficients may be determined by means of the single boundary condition Equation (62)

$$\mathbf{R} \times (\mathbf{E} + \mathbf{E}^P) = 0 \quad (92)$$

at the surface of the sphere. Applying the expansions Equations (58) and (60), and the identity Equation (74), and dropping the 2 subscript/superscript for the exterior region, Equation (92) becomes

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l \left[-\frac{i\omega}{k^2} \{ a_{l,m}^M [h_l^{(1)}(kR) + kR \dot{h}_l^{(1)}(kR)] + a_{l,m}^{M,P} [j_l(kR) + kR \dot{j}_l(kR)] \} \mathbf{X}_{l,m} \right] \quad (93)$$

$$\{ a_{l,m}^E h_l^{(1)}(kR) + a_{l,m}^{E,P} j_l(kR) \} \mathbf{R} \times \mathbf{X}_{l,m} = 0.$$

Again, the properties of the vector spherical harmonics are used in Equations (77) through (79), and the resulting equations solved to give

$$a_{l,m}^M = \frac{N_{M,l}}{D_{M,l}} a_{l,m}^{M,P} \quad (94)$$

and

$$a_{l,m}^E = \frac{N_{E,l}}{D_{E,l}} a_{l,m}^{E,P} \quad (95)$$

again valid for any localized exterior source distribution not containing the sphere surface, where now,

$$N_{M,l} = -[j_l(kR) + kR \dot{j}_l(kR)] \quad (96)$$

$$D_{M,l} = h_l^{(1)}(kR) + kR \dot{h}_l^{(1)}(kR) \quad (97)$$

$$N_{E,l} = -j_l(kR) \quad (98)$$

and

$$D_{E,l} = h_l^{(1)}(kR). \quad (99)$$

SPECIALIZATION TO DIPOLE SOURCES

Explicit solutions will be constructed for current and magnetic dipole sources. For these, the current density and magnetization take the form

$$\mathbf{J}_s(\mathbf{r}) = \mathbf{p} \delta(\mathbf{r} - \mathbf{R}_0) \quad (100)$$

and

$$\mathbf{M}_s(\mathbf{r}) = \mathbf{m} \delta(\mathbf{r} - \mathbf{R}_0). \quad (101)$$

The current dipole moment $\mathbf{p}$ can be specified directly for a dipole source shorted to the medium, and has a non-zero value at dc. To represent an insulated dipole that couples only through the displacement current, $\mathbf{p}$ should be written $\mathbf{p} = i\omega \mathbf{p}'$, where $\mathbf{p}'$ is the electric dipole moment.

For $\mathbf{V}(\mathbf{r}') = \mathbf{v} \delta(\mathbf{r}' - \mathbf{R}_0)$, with $\mathbf{v}$ a constant vector, the following identities are valid:

$$\nabla' \times \mathbf{V} = -\mathbf{v} \times \nabla' \delta(\mathbf{r}' - \mathbf{R}_0) \quad (102)$$

$$\nabla' \times \nabla' \times \mathbf{V} = -\nabla'^2 \delta(\mathbf{r}' - \mathbf{R}_0) + (\mathbf{v} \cdot \nabla') \nabla' \delta(\mathbf{r}' - \mathbf{R}_0) \quad (103)$$

$$\int d\mathbf{r}'^3 F(\mathbf{r}, \mathbf{r}') \nabla' \delta(\mathbf{r}' - \mathbf{R}_0) = -\nabla_0 F(\mathbf{r}, \mathbf{R}_0) \quad (104)$$

$$\int d\mathbf{r}'^3 F(\mathbf{r}, \mathbf{r}') \nabla'^2 \delta(\mathbf{r}' - \mathbf{R}_0) = \nabla_0^2 F(\mathbf{r}, \mathbf{R}_0) \quad (105)$$

and

$$\int d\mathbf{r}'^3 F(\mathbf{r}, \mathbf{r}') \nabla' \otimes \nabla' \delta(\mathbf{r}' - \mathbf{R}_0) = \nabla_0 \otimes \nabla_0 F(\mathbf{r}, \mathbf{R}_0). \quad (106)$$

If Equations (100) and (101) are substituted in Equations (54) and (55) and Equations (102) through (106) applied, the resulting source expansion coefficients, separated according to source type, are (again, for an exterior source distribution, $r < r'$ for any source point near enough to the sphere surface)

$$a_{l,m}^{E,CD} = \frac{k_2^3 \mu_2}{\omega \epsilon_2 \sqrt{l(l+1)}} (\mathbf{R}_0 \times \mathbf{p}) \cdot \nabla_0 [h_l^{(1)}(k_2 R_0) Y_{l,m}^*(\theta_0, \phi_0)] \quad (107)$$

$$a_{l,m}^{M,CD} = -\frac{k_2^3 \mu_2}{\omega^2 \epsilon_2 \sqrt{l(l+1)}} \{ k_2^2 (\mathbf{R}_0 \cdot \mathbf{p}) h_l^{(1)}(k_2 R_0) Y_{l,m}^*(\theta_0, \phi_0) \quad (108)$$

$$+ \mathbf{p} \cdot \nabla_0 \left[Y_{l,m}^*(\theta_0, \phi_0) \frac{d}{dR_0} \{ R_0 h_l^{(1)}(k_2 R_0) \} \right] \}$$

$$a_{l,m}^{E,MD} = \frac{k_2 \omega \mu_2}{\sqrt{l(l+1)}} \{ k_2^2 (\mathbf{R}_0 \cdot \mathbf{m}) h_l^{(1)}(k_2 R_0) Y_{l,m}^*(\theta_0, \phi_0) \quad (109)$$

$$+ \mathbf{m} \cdot \nabla_0 \left[Y_{l,m}^*(\theta_0, \phi_0) \frac{d}{dR_0} \{ R_0 h_l^{(1)}(k_2 R_0) \} \right] \}$$

and

$$a_{l,m}^{M,MD} = -\frac{k_2^3 \mu_2}{\sqrt{l(l+1)}} (\mathbf{R}_0 \times \mathbf{m}) \cdot \nabla_0 [h_l^{(1)}(k_2 R_0) Y_{l,m}^*(\theta_0, \phi_0)]. \quad (110)$$

There are two cases for both types of dipole, radial and tangential. For the radial case

$$a_{l,m}^{E,CD,R} = 0 \quad (111)$$

$$a_{l,m}^{M,CD,R} = -\frac{pR_0k_2^3\mu_2}{\omega^2\epsilon_2\sqrt{l(l+1)}} \left[k_2^2 h_l^{(1)}(k_2 R_0) + \frac{1}{R_0} \frac{d^2}{dR_0^2} \{R_0 h_l^{(1)}(k_2 R_0)\} \right] Y_{l,m}^*(\theta_0, \phi_0) \quad (112)$$

$$a_{l,m}^{E,MD,R} = \frac{m\omega R_0 k_2 \mu_2}{\sqrt{l(l+1)}} \left[k_2^2 h_l^{(1)}(k_2 R_0) + \frac{1}{R_0} \frac{d^2}{dR_0^2} \{R_0 h_l^{(1)}(k_2 R_0)\} \right] Y_{l,m}^*(\theta_0, \phi_0) \quad (113)$$

and

$$a_{l,m}^{M,MD,R} = 0. \quad (114)$$

Without loss of generality, the tangential dipole can be taken to have only a θ component. Then Equations (107) through (110) become

$$a_{l,m}^{E,CD,T} = \frac{p k_2^3 \mu_2}{\omega \epsilon_2 \sqrt{l(l+1)}} h_l^{(1)}(k_2 R_0) \frac{1}{\sin \theta_0} \frac{\partial}{\partial \phi_0} Y_{l,m}^*(\theta_0, \phi_0) \quad (115)$$

$$a_{l,m}^{M,CD,T} = -\frac{p k_2^3 \mu_2}{\omega^2 \epsilon_2 \sqrt{l(l+1)} R_0} \frac{d}{dR_0} [R_0 h_l^{(1)}(k_2 R_0)] \frac{\partial}{\partial \theta_0} Y_{l,m}^*(\theta_0, \phi_0) \quad (116)$$

$$a_{l,m}^{E,MD,T} = \frac{m k_2 \omega \mu_2}{\sqrt{l(l+1)} R_0} \frac{d}{dR_0} [R_0 h_l^{(1)}(k_2 R_0)] \frac{\partial}{\partial \theta_0} Y_{l,m}^*(\theta_0, \phi_0) \quad (117)$$

and

$$a_{l,m}^{M,MD,T} = -\frac{m k_2^3 \mu_2}{\sqrt{l(l+1)}} h_l^{(1)}(k_2 R_0) \frac{1}{\sin \theta_0} \frac{\partial}{\partial \phi_0} Y_{l,m}^*(\theta_0, \phi_0). \quad (118)$$

To proceed further, it is useful to list some explicit results for the spherical harmonics and related functions:

$$Y_{l,m}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{-im\phi}. \quad (119)$$

$$P_l^m(\cos \theta) \propto \sin^m \theta, \quad \theta \rightarrow 0, \quad |m| \geq 1 \quad (120)$$

$$P_l^{-m}(\cos \theta) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(\cos \theta) \quad (121)$$

and

$$\frac{d}{d\theta} P_l^m(\cos \theta) = -(l+m)(l-m+1) P_l^{m-1}(\cos \theta) + m \cot \theta P_l^m(\cos \theta). \quad (122)$$

Again without loss of generality, the dipole may be taken to be on the z-axis: $\theta_0 \rightarrow 0^+$. Then, application of Equations (119) and (120) to Equations (112) and (113) shows that the source coefficients for the radial dipole vanish unless $m = 0$, while application of Equations (119) through (122) to Equations (115) through (118) shows that the source coefficients for the tangential dipole vanish unless $m = \pm 1$.

Before writing down the final results for the source coefficients, it is useful to list some special limits:

$$P_l^0(1) = 1 \quad (123)$$

$$\dot{P}_l^0(1) = \frac{l(l+1)}{2} \quad (124)$$

$$\sin \theta \dot{P}_l^1(\cos \theta)|_{\theta \rightarrow 0^+} = \frac{l(l+1)}{2} \quad (125)$$

and

$$\sin \theta \dot{P}_l^{-1}(\cos \theta)|_{\theta \rightarrow 0^+} = -\frac{1}{2}. \quad (126)$$

These can be used to simplify the dipole source coefficients. For points $r < R_0$ the explicit forms are

$$a_{l,0}^{E,CD,R} = 0 \quad (127)$$

$$a_{l,0}^{M,CD,R} = \frac{p k_2 \mu_2}{R_0} \sqrt{\frac{(2l+1)l(l+1)}{4\pi}} h_l^{(1)}(k_2 R_0) \quad (128)$$

$$a_{l,0}^{E,MD,R} = \frac{i \omega m k_2 \mu_2}{R_0} \sqrt{\frac{(2l+1)l(l+1)}{4\pi}} h_l^{(1)}(k_2 R_0) \quad (129)$$

$$a_{l,0}^{M,MD,R} = 0 \quad (130)$$

$$a_{l,-1}^{E,CD,T} = a_{l,1}^{E,CD,T} = -\frac{\omega p k_2 \mu_2}{2} \sqrt{\frac{2l+1}{4\pi}} h_l^{(1)}(k_2 R_0) \quad (131)$$

$$a_{l,\pm 1}^{M,CD,T} = \mp \frac{p k_2 \mu_2}{2R_0} \sqrt{\frac{2l+1}{4\pi}} [h_l^{(1)}(k_2 R_0) + k_2 R_0 \dot{h}_l^{(1)}(k_2 R_0)] \quad (132)$$

$$a_{l,\pm 1}^{E,MD,T} = \mp \frac{i \omega m k_2 \mu_2}{2R_0} \sqrt{\frac{2l+1}{4\pi}} [h_l^{(1)}(k_2 R_0) + k_2 R_0 \dot{h}_l^{(1)}(k_2 R_0)] \quad (133)$$

and

$$a_{l,-1}^{M,MD,T} = a_{l,1}^{M,MD,T} = \frac{imk_2^3 \mu_2}{2} \sqrt{\frac{2l+1}{4\pi}} h_l^{(1)}(k_2 R_0). \quad (134)$$

For $r > R_0$, $h_l^{(1)}(k_2 R_0)$ is replaced by $j_l(k_2 R_0)$ in Equations (127) through (134).

EXPRESSIONS FOR THE SCATTERED FIELDS FOR AN EXTERNAL DIPOLE

The scattered fields are determined by combining Equations (127) through (134), Equations (84) through (91) and Equations (56) through (59) with m suitably restricted. In the manipulation of $X_{l,0}$ and $X_{l,\pm 1}$ some useful relationships are

$$2 \cos \theta \dot{P}_l^0 - \sin^2 \theta \ddot{P}_l^0 = l(l+1)P_l^0 \quad (135)$$

and

$$\frac{P_l^1}{\sin^2 \theta} + 2 \cos \theta \dot{P}_l^1 - \sin^2 \theta \ddot{P}_l^1 = l(l+1)P_l^1. \quad (136)$$

These appear in in the construction of the radial components of the fields, and are a consequence of Equation (50).

In the final presentation of the field expressions, only the Legendre polynomials $P_l \equiv P_l^0$ and their first derivatives will appear by virtue of the relations

$$\frac{P_l^1}{\sin \theta} = -\dot{P}_l \quad \text{and} \quad \dot{P}_l^1 \sin \theta = l(l+1)P_l - \cos \theta \dot{P}_l \quad (137)$$

and the Bessel function derivative will be eliminated via the relation

$$f_l(u) + u \dot{f}_l(u) = (l+1)f_l(u) - u f_{l+1}(u). \quad (138)$$

MAGNETIC DIPOLE

Radial

(Interior)

$$E_r^1 = 0 \quad (139)$$

$$E_\theta^1 = 0 \quad (140)$$

$$E_{\phi}^1 = \frac{i\omega m\mu_1 \sin \theta}{4\pi R_0 R} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{E,l}} j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (141)$$

$$B_r^1 = \frac{m\mu_1}{4\pi R_0 R r} \sum_{l=0}^{\infty} \frac{1}{D_{E,l}} (2l+1) l(l+1) j_l(k_1 r) h_l^{(1)}(k_2 R_0) P_l \quad (142)$$

$$B_{\theta}^1 = -\frac{m\mu_1 \sin \theta}{4\pi R_0 R r} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{E,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (143)$$

$$B_{\phi}^1 = 0 \quad (144)$$

(Exterior)

$$E_r^2 = 0 \quad (145)$$

$$E_{\theta}^2 = 0 \quad (146)$$

$$E_{\phi}^2 = -\frac{\omega m\mu_2 k_2 \sin \theta}{4\pi R_0} \sum_{l=0}^{\infty} (2l+1) \frac{N_{E,l}}{D_{E,l}} h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (147)$$

$$B_r^2 = \frac{i m\mu_2 k_2}{4\pi R_0 r} \sum_{l=0}^{\infty} (2l+1) l(l+1) \frac{N_{E,l}}{D_{E,l}} h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) P_l \quad (148)$$

$$B_{\theta}^2 = -\frac{i m\mu_2 k_2 \sin \theta}{4\pi R_0 r} \sum_{l=0}^{\infty} (2l+1) \frac{N_{E,l}}{D_{E,l}} [(l+1) h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (149)$$

$$B_{\phi}^2 = 0 \quad (150)$$

Tangential

(Interior)

$$E_r^1 = \frac{i\omega m\mu_1 \sin \theta \sin \phi}{4\pi R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{M,l}} j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (151)$$

$$E_{\theta}^1 = \frac{i\omega m\mu_1 \sin \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{-1}{D_{E,l}} r j_l(k_1 r) [(l+1) h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \right. \\ \left. + \frac{1}{D_{M,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] R_0 h_l^{(1)}(k_2 R_0) [l(l+1) P_l - \cos \theta \dot{P}_l] \right\} \quad (152)$$

$$E_{\phi}^1 = -\frac{i\omega m\mu_1 \cos \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{-1}{D_{M,l}} R_0 h_l^{(1)}(k_2 R_0) [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] \dot{P}_l \right. \\ \left. + \frac{1}{D_{E,l}} r j_l(k_1 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (153)$$

$$B_r^1 = \frac{m\mu_1 \sin \theta \cos \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{E,l}} [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] j_l(k_1 r) \dot{P}_l \quad (154)$$

$$B_{\theta}^1 = \frac{m\mu_1 \cos \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{D_{M,l}} r R_0 k_1^2 j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \right. \\ \left. + \frac{1}{D_{E,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (155)$$

$$B_{\phi}^1 = -\frac{m\mu_1 \sin \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{D_{M,l}} r R_0 k_1^2 j_l(k_1 r) h_l^{(1)}(k_2 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right. \\ \left. + \frac{1}{D_{E,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \right\} \quad (156)$$

(Exterior)

$$E_r^2 = -\frac{\omega m\mu_2 k_2 \sin \theta \sin \phi}{4\pi r} \sum_{l=1}^{\infty} (2l+1) \frac{N_{M,l}}{D_{M,l}} h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (157)$$

$$E_{\theta}^2 = -\frac{\omega m\mu_2 k_2 \sin \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{N_{E,l}}{D_{E,l}} r h_l^{(1)}(k_2 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \right. \\ \left. + \frac{N_{M,l}}{D_{M,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] R_0 h_l^{(1)}(k_2 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (158)$$

$$E_{\phi}^2 = \frac{\omega m\mu_2 k_2 \cos \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{N_{M,l}}{D_{M,l}} R_0 h_l^{(1)}(k_2 R_0) [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] \dot{P}_l \right. \\ \left. + \frac{N_{E,l}}{D_{E,l}} r h_l^{(1)}(k_2 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (159)$$

$$B_r^2 = \frac{i m\mu_2 k_2 \sin \theta \cos \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} (2l+1) \frac{N_{E,l}}{D_{E,l}} h_l^{(1)}(k_2 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \quad (160)$$

$$B_{\theta}^2 = \frac{i\mu\mu_2 k_2 \cos \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{N_{M,l}}{D_{M,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \right. \\ \left. + \frac{N_{E,l}}{D_{E,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (161)$$

$$B_{\phi}^2 = -\frac{i\mu\mu_2 k_2 \sin \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{N_{M,l}}{D_{M,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right. \\ \left. + \frac{N_{E,l}}{D_{E,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \right\} \quad (162)$$

CURRENT DIPOLE

Radial

(Interior)

$$E_r^1 = -\frac{i\omega p \mu_1}{2\pi k_2^2 R_0 R r} \sum_{l=0}^{\infty} \frac{1}{D_{M,l}} (2l+1)l(l+1)j_l(k_1 r)h_l^{(1)}(k_2 R_0)P_l \quad (163)$$

$$E_{\theta}^1 = \frac{i\omega p \mu_1 \sin \theta}{2\pi k_2^2 R_0 R r} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{M,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (164)$$

$$E_{\phi}^1 = 0 \quad (165)$$

$$B_r^1 = 0 \quad (166)$$

$$B_{\theta}^1 = 0 \quad (167)$$

$$B_{\phi}^1 = -\frac{p \mu_1 \gamma \sin \theta}{2\pi R_0 R} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{M,l}} j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (168)$$

(Exterior)

$$E_r^2 = -\frac{\omega p \mu_2}{4\pi k_2 R_0 r} \sum_{l=0}^{\infty} \frac{N_{M,l}}{D_{M,l}} (2l+1)l(l+1)h_l^{(1)}(k_2 r)h_l^{(1)}(k_2 R_0)P_l \quad (169)$$

$$E_{\theta}^2 = \frac{\omega p \mu_2 \sin \theta}{4\pi k_2 R_0 r} \sum_{l=0}^{\infty} (2l+1) \frac{N_{M,l}}{D_{M,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (170)$$

$$E_{\phi}^2 = 0 \quad (171)$$

$$B_r^2 = 0 \quad (172)$$

$$B_\theta^2 = 0 \quad (173)$$

$$B_\phi^2 = \frac{ip\mu_2 k_2 \sin \theta}{4\pi R_0} \sum_{l=0}^{\infty} (2l+1) \frac{N_{M,l}}{D_{M,l}} h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (174)$$

Tangential

(Interior)

$$E_r^1 = -\frac{i\omega p \mu_1 \sin \theta \cos \phi}{2\pi k_2^2 R_0 R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{M,l}} j_l(k_1 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \quad (175)$$

$$E_\theta^1 = \frac{i\omega p \mu_1 \cos \phi}{4\pi k_2^2 R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{D_{E,l}} r R_0 k_2^2 j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \right. \quad (176)$$

$$\left. - \frac{2}{D_{M,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\}$$

$$E_\phi^1 = -\frac{i\omega p \mu_1 \sin \phi}{4\pi k_2^2 R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{D_{E,l}} r R_0 k_2^2 j_l(k_1 r) h_l^{(1)}(k_2 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right. \quad (177)$$

$$\left. - \frac{2}{D_{M,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \right\}$$

$$B_r^1 = \frac{p \mu_1 \sin \theta \sin \phi}{4\pi R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{E,l}} j_l(k_1 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (178)$$

$$B_\theta^1 = \frac{p \mu_1 \sin \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{2\gamma}{D_{M,l}} r j_l(k_1 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \right. \quad (179)$$

$$\left. + \frac{1}{D_{E,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] R_0 h_l^{(1)}(k_2 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right\}$$

$$B_\phi^1 = \frac{p \mu_1 \cos \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{D_{E,l}} R_0 h_l^{(1)}(k_2 R_0) [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] \dot{P}_l \right. \quad (180)$$

$$\left. + \frac{2\gamma}{D_{M,l}} r j_l(k_1 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\}$$

(Exterior)

$$E_r^2 = -\frac{\omega p \mu_2 \sin \theta \cos \phi}{4\pi k_2 R_0 r} \sum_{l=1}^{\infty} (2l+1) \frac{N_{M,l}}{D_{M,l}} h_l^{(1)}(k_2 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \quad (181)$$

$$E_\theta^2 = -\frac{\omega p \mu_2 \cos \phi}{4\pi k_2 R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{N_{E,l}}{D_{E,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \right. \quad (182)$$

$$\left. + \frac{N_{M,l}}{D_{M,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\}$$

$$E_\phi^2 = \frac{\omega p \mu_2 \sin \phi}{4\pi k_2 R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{N_{E,l}}{D_{E,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right. \quad (183)$$

$$\left. + \frac{N_{M,l}}{D_{M,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \right\}$$

$$B_r^2 = \frac{i p \mu_2 k_2 \sin \theta \sin \phi}{4\pi r} \sum_{l=1}^{\infty} (2l+1) \frac{N_{E,l}}{D_{E,l}} h_l^{(1)}(k_2 r) h_l^{(1)}(k_2 R_0) \dot{P}_l \quad (184)$$

$$B_\theta^2 = \frac{i p \mu_2 k_2 \sin \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{N_{M,l}}{D_{M,l}} r h_l^{(1)}(k_2 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] \dot{P}_l \right. \quad (185)$$

$$\left. + \frac{N_{E,l}}{D_{E,l}} R_0 h_l^{(1)}(k_2 R_0) [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\}$$

$$B_\phi^2 = -\frac{i p \mu_2 k_2 \cos \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{N_{E,l}}{D_{E,l}} R_0 h_l^{(1)}(k_2 R_0) [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] \dot{P}_l \right. \quad (186)$$

$$\left. + \frac{N_{M,l}}{D_{M,l}} r h_l^{(1)}(k_2 r) [(l+1)h_l^{(1)}(k_2 R_0) - k_2 R_0 h_{l+1}^{(1)}(k_2 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\}$$

DC LIMIT OF THE SCATTERED FIELDS FOR AN EXTERNAL DIPOLE

The dc limits of the field expressions given in Equations (139) through (186) are useful both to provide a more transparent form of expression for a consistency check of the expressions, and to provide explicit expressions for dc applications. These limiting forms are given below. Note that, in the dc limit, $\gamma \rightarrow \tau\delta$ where $\delta = \sigma_1/\sigma_2$.

DC MAGNETIC DIPOLE**Radial**

(Interior)

$$E_r^1 = 0 \quad (187)$$

$$E_\theta^1 = 0 \quad (188)$$

$$E_\phi^1 = 0 \quad (189)$$

$$B_r^1 = \frac{m\mu_1}{4\pi R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)l(l+1)}{(\tau+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} P_l \quad (190)$$

$$B_\theta^1 = -\frac{m\mu_1 \sin \theta}{4\pi R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)l(l+1)}{(\tau+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} \dot{P}_l \quad (191)$$

$$B_\phi^1 = 0 \quad (192)$$

(Exterior)

$$E_r^2 = 0 \quad (193)$$

$$E_\theta^2 = 0 \quad (194)$$

$$E_\phi^2 = 0 \quad (195)$$

$$B_r^2 = \frac{m\mu_2(\tau-1)}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l(l+1)^2}{(\tau+1)l+1} \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{R_0}\right)^{l+1} P_l \quad (196)$$

$$B_\theta^2 = \frac{m\mu_2(\tau-1) \sin \theta}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\tau+1)l+1} \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{R_0}\right)^{l+1} \dot{P}_l \quad (197)$$

$$B_\phi^2 = 0 \quad (198)$$

Tangential

(Interior)

$$E_r^1 = 0 \quad (199)$$

$$E_{\theta}^1 = 0 \quad (200)$$

$$E_{\phi}^1 = 0 \quad (201)$$

$$B_r^1 = -\frac{m\mu_1 \sin \theta \cos \phi}{4\pi R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)l}{(\tau+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} \dot{P}_l \quad (202)$$

$$B_{\theta}^1 = -\frac{m\mu_1 \cos \phi}{4\pi R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\tau+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} [l(l+1)P_l - \cos \theta \dot{P}_l] \quad (203)$$

$$B_{\phi}^1 = \frac{m\mu_1 \sin \phi}{4\pi R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\tau+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} \dot{P}_l \quad (204)$$

(Exterior)

$$E_r^2 = 0 \quad (205)$$

$$E_{\theta}^2 = 0 \quad (206)$$

$$E_{\phi}^2 = 0 \quad (207)$$

$$B_r^2 = -\frac{m\mu_2(\tau-1) \sin \theta \cos \phi}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\tau+1)l+1} \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{R_0}\right)^{l+1} \dot{P}_l \quad (208)$$

$$B_{\theta}^2 = \frac{m\mu_2(\tau-1) \cos \phi}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l}{(\tau+1)l+1} \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{R_0}\right)^{l+1} [l(l+1)P_l - \cos \theta \dot{P}_l] \quad (209)$$

$$B_{\phi}^2 = -\frac{m\mu_2(\tau-1) \sin \phi}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l}{(\tau+1)l+1} \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{R_0}\right)^{l+1} \dot{P}_l \quad (210)$$

DC CURRENT DIPOLE

Radial

(Interior)

$$E_r^1 = -\frac{p\tau}{2\pi\sigma_2 R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)l(l+1)}{(\delta+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} P_l \quad (211)$$

$$E_{\theta}^1 = \frac{p\tau \sin \theta}{2\pi\sigma_2 R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)(l+1)}{(\delta+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} \dot{P}_l \quad (212)$$

$$E_{\phi}^1 = 0 \quad (213)$$

$$B_r^1 = 0 \quad (214)$$

$$B_{\theta}^1 = 0 \quad (215)$$

$$B_{\phi}^1 = -\frac{p\mu_1\tau\delta r \sin\theta}{2\pi R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\delta+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} \dot{P}_l \quad (216)$$

(Exterior)

$$E_r^2 = \frac{p(\delta-1)}{4\pi\sigma_2 R_0 r R} \sum_{l=1}^{\infty} \frac{l(l+1)^2}{(\delta+1)l+1} \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{R_0}\right)^{l+1} P_l \quad (217)$$

$$E_{\theta}^2 = \frac{p(\delta-1)\sin\theta}{4\pi\sigma_2 R_0 r R} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\delta+1)l+1} \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{R_0}\right)^{l+1} \dot{P}_l \quad (218)$$

$$E_{\phi}^2 = 0 \quad (219)$$

$$B_r^2 = 0 \quad (220)$$

$$B_{\theta}^2 = 0 \quad (221)$$

$$B_{\phi}^2 = \frac{p(\delta-1)\mu_2 \sin\theta}{4\pi R_0 R} \sum_{l=1}^{\infty} \frac{l+1}{(\delta+1)l+1} \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{R_0}\right)^{l+1} \dot{P}_l \quad (222)$$

Tangential

(Interior)

$$E_r^1 = \frac{p\tau \sin\theta \cos\phi}{2\pi\sigma_2 R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)l}{(\delta+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} \dot{P}_l \quad (223)$$

$$E_{\theta}^1 = \frac{p\tau \cos\phi}{2\pi\sigma_2 R_0^3} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\delta+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} [l(l+1)P_l - \cos\theta \dot{P}_l] \quad (224)$$

$$E_{\phi}^1 = -\frac{p\tau \sin\phi}{2\pi\sigma_2 R_0^3} \sum_{l=1}^{\infty} \frac{2l+1}{(\delta+1)l+1} \left(\frac{r}{R_0}\right)^{l-1} \dot{P}_l \quad (225)$$

$$B_r^1 = \frac{p\mu_1 \sin \theta \sin \phi}{4\pi R_0^2} \sum_{l=1}^{\infty} \frac{2l+1}{(\tau+1)l+1} \left(\frac{r}{R_0} \right)^{l-1} \dot{P}_l \quad (226)$$

$$B_\theta^1 = \frac{p\mu_1 \sin \phi}{4\pi R_0^2} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \left\{ \frac{l+1}{(\tau+1)l+1} [l(l+1)P_l - \cos \theta \dot{P}_l] - \frac{2\tau\delta r l}{(\delta+1)l+1} \dot{P}_l \right\} \left(\frac{r}{R_0} \right)^{l-1} \quad (227)$$

$$B_\phi^1 = \frac{p\mu_1 \cos \phi}{4\pi R_0^2} \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \left\{ \frac{l+1}{(\tau+1)l+1} \dot{P}_l - \frac{2\tau\delta r l}{(\delta+1)l+1} [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \left(\frac{r}{R_0} \right)^{l-1} \quad (228)$$

(Exterior)

$$E_r^2 = -\frac{p(\delta-1) \sin \theta \cos \phi}{4\pi\sigma_2 R_0 R r} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\delta+1)l+1} \left(\frac{R}{R_0} \right)^{l+1} \left(\frac{R}{r} \right)^{l+1} \dot{P}_l \quad (229)$$

$$E_\theta^2 = \frac{p(\delta-1) \cos \phi}{4\pi\sigma_2 R_0 R r} \sum_{l=1}^{\infty} \frac{l}{(\delta+1)l+1} \left(\frac{R}{R_0} \right)^{l+1} \left(\frac{R}{r} \right)^{l+1} [l(l+1)P_l - \cos \theta \dot{P}_l] \quad (230)$$

$$E_\phi^2 = -\frac{p(\delta-1) \sin \phi}{4\pi\sigma_2 R_0 R r} \sum_{l=1}^{\infty} \frac{l}{(\delta+1)l+1} \left(\frac{R}{R_0} \right)^{l+1} \left(\frac{R}{r} \right)^{l+1} \dot{P}_l \quad (231)$$

$$B_r^2 = \frac{p\mu_2(\tau-1) \sin \theta \sin \phi}{4\pi R r} \sum_{l=1}^{\infty} \frac{l+1}{(\tau+1)l+1} \left(\frac{R}{R_0} \right)^{l+1} \left(\frac{R}{r} \right)^{l+1} \dot{P}_l \quad (232)$$

$$B_\theta^2 = -\frac{p\mu_2 \sin \phi}{4\pi R R_0 r} \sum_{l=1}^{\infty} \left\{ \frac{R_0(\tau-1)}{(\tau+1)l+1} [l(l+1)P_l - \cos \theta \dot{P}_l] - \frac{r(\delta-1)}{(\delta+1)l+1} \dot{P}_l \right\} \left(\frac{R}{R_0} \right)^{l+1} \left(\frac{R}{r} \right)^{l+1} \quad (233)$$

$$B_\phi^2 = \frac{p\mu_2 \cos \phi}{4\pi R R_0 r} \sum_{l=1}^{\infty} \left\{ -\frac{R_0(\tau-1)}{(\tau+1)l+1} \dot{P}_l + \frac{r(\delta-1)}{(\delta+1)l+1} [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \left(\frac{R}{R_0} \right)^{l+1} \left(\frac{R}{r} \right)^{l+1} \quad (234)$$

APPLICATION TO A SPHERE WITH AN INTERNAL SOURCE

The scattered fields still have the forms given in Equations (56) through (59). The primary field expansion now has the form

$$\mathbf{E}^P = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{E,P} h_l^{(1)}(k_1 r) \mathbf{X}_{l,m} + \frac{i\omega}{k_1^2} a_{l,m}^{M,P} \nabla \times [h_l^{(1)}(k_1 r) \mathbf{X}_{l,m}] \right\} \quad (235)$$

and

$$\mathbf{B}^P = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ a_{l,m}^{M,P} h_l^{(1)}(k_1 r) \mathbf{X}_{l,m} - \frac{i}{\omega} a_{l,m}^{E,P} \nabla \times [h_l^{(1)}(k_1 r) \mathbf{X}_{l,m}] \right\}. \quad (236)$$

The boundary conditions now are given by

$$\mathbf{R} \times (\mathbf{E}^1 + \mathbf{E}^P) = \mathbf{R} \times \mathbf{E}^2 \quad (237)$$

and

$$\frac{1}{\mu_1} \mathbf{R} \times (\mathbf{B}^1 + \mathbf{B}^P) = \frac{1}{\mu_2} \mathbf{R} \times \mathbf{B}^2. \quad (238)$$

In expanded form Equations (237) and (238) are

$$\begin{aligned} \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ \left[\frac{a_{l,m}^{M,1} j_l(k_1 R)}{\mu_1} + \frac{a_{l,m}^{M,P} h_l^{(1)}(k_1 R)}{\mu_1} - \frac{a_{l,m}^{M,2} h_l^{(1)}(k_2 R)}{\mu_2} \right] \mathbf{R} \times \mathbf{X}_{l,m} \right. \\ \left. + i \left[\frac{a_{l,m}^{E,1} \{j_l(k_1 R) + k_1 R \dot{j}_l(k_1 R)\}}{\omega \mu_1} + \frac{a_{l,m}^{E,P} \{h_l^{(1)}(k_1 R) + k_1 R \dot{h}_l^{(1)}(k_1 R)\}}{\omega \mu_1} \right. \right. \\ \left. \left. - \frac{a_{l,m}^{E,2} \{h_l^{(1)}(k_2 R) + k_2 R \dot{h}_l^{(1)}(k_2 R)\}}{\omega \mu_2} \right] \mathbf{X}_{l,m} \right\} = 0 \end{aligned} \quad (239)$$

and

$$\begin{aligned} \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left\{ [a_{l,m}^{E,1} j_l(k_1 R) + a_{l,m}^{E,P} h_l^{(1)}(k_1 R) - a_{l,m}^{E,2} h_l^{(1)}(k_2 R)] \mathbf{R} \times \mathbf{X}_{l,m} \right. \\ \left. - i \omega \left[\frac{a_{l,m}^{M,1} \{j_l(k_1 R) + k_1 R \dot{j}_l(k_1 R)\}}{k_1^2} + \frac{a_{l,m}^{M,P} \{h_l^{(1)}(k_1 R) + k_1 R \dot{h}_l^{(1)}(k_1 R)\}}{k_1^2} \right. \right. \\ \left. \left. - \frac{a_{l,m}^{M,2} \{h_l^{(1)}(k_2 R) + k_2 R \dot{h}_l^{(1)}(k_2 R)\}}{k_2^2} \right] \mathbf{X}_{l,m} \right\} = 0. \end{aligned} \quad (240)$$

These may be reduced using Equations (77) through (79) to give

$$\bar{\tau} a_{l,m}^{M,1} j_l(k_1 R) - a_{l,m}^{M,2} h_l^{(1)}(k_2 R) = -\bar{\tau} a_{l,m}^{M,P} h_l^{(1)}(k_1 R) \quad (241)$$

$$\bar{\gamma} a_{l,m}^{M,1} [j_l(k_1 R) + k_1 R \dot{j}_l(k_1 R)] - a_{l,m}^{M,2} [h_l^{(1)}(k_2 R) + k_2 R \dot{h}_l^{(1)}(k_2 R)] \quad (242)$$

$$= -\bar{\gamma} a_{l,m}^{M,P} [h_l^{(1)}(k_1 R) + k_1 R \dot{h}_l^{(1)}(k_1 R)]$$

$$a_{l,m}^{E,1} j_l(k_1 R) - a_{l,m}^{E,2} h_l^{(1)}(k_2 R) = -a_{l,m}^{E,P} h_l^{(1)}(k_1 R) \quad (243)$$

and

$$\begin{aligned} & \bar{\tau} a_{l,m}^{E,1} [j_l(k_1 R) + k_1 R j_l'(k_1 R)] - a_{l,m}^{E,2} [h_l^{(1)}(k_2 R) + k_2 R h_l^{(1)'}(k_2 R)] \\ & = -\bar{\tau} a_{l,m}^{E,P} [h_l^{(1)}(k_1 R) + k_1 R h_l^{(1)'}(k_1 R)] \end{aligned} \quad (244)$$

where now $\bar{\tau} = \mu_2/\mu_1$ and $\bar{\gamma} = k_2^2/k_1^2$ are the reciprocals of τ and γ . Proceeding as before, introduce

$$\bar{D}_{E,l} = \bar{\tau} h_l^{(1)}(u_2) [(l+1)j_l(u_1) - u_1 j_{l+1}(u_1)] - j_l(u_1) [(l+1)h_l^{(1)}(u_2) - u_2 h_{l+1}^{(1)}(u_2)] \quad (245)$$

and

$$\bar{D}_{M,l} = \bar{\gamma} h_l^{(1)}(u_2) [(l+1)j_l(u_1) - u_1 j_{l+1}(u_1)] - \bar{\tau} j_l(u_1) [(l+1)h_{l+1}^{(1)}(u_2) - u_2 h_{l+1}^{(1)}(u_2)] \quad (246)$$

and the expressions

$$\bar{N}_{E,l} = h_l^{(1)}(u_1) [(l+1)h_l^{(1)}(u_2) - u_2 h_{l+1}^{(1)}(u_2)] - \bar{\tau} h_l^{(1)}(u_2) [(l+1)h_l^{(1)}(u_1) - u_1 h_{l+1}^{(1)}(u_1)] \quad (247)$$

and

$$\bar{N}_{M,l} = \bar{\tau} h_l^{(1)}(u_1) [(l+1)h_l^{(1)}(u_2) - u_2 h_{l+1}^{(1)}(u_2)] - \bar{\gamma} h_l^{(1)}(u_2) [(l+1)h_l^{(1)}(u_1) - u_1 h_{l+1}^{(1)}(u_1)]. \quad (248)$$

where again, $u_1 = k_1 R$ and $u_2 = k_2 R$. Then, the solutions for the expansion coefficients, *valid for any localized interior source distribution that does not include the sphere surface*, are given by

$$a_{l,m}^{E,1} = \frac{\bar{N}_{E,l}}{\bar{D}_{E,l}} a_{l,m}^{E,P} \quad (249)$$

$$a_{l,m}^{E,2} = -\frac{i\bar{\tau}}{u_1 \bar{D}_{E,l}} a_{l,m}^{E,P} \quad (250)$$

$$a_{l,m}^{M,1} = \frac{\bar{N}_{M,l}}{\bar{D}_{M,l}} a_{l,m}^{M,P} \quad (251)$$

and

$$a_{l,m}^{M,2} = \frac{i\bar{\tau}\bar{\gamma}}{u_1 \bar{D}_{M,l}} a_{l,m}^{M,P} \quad (252)$$

For dipole sources, the expressions in Equations (127) through (134) may be used, with $h_l^{(1)}(k_2 R_0)$ replaced by $j_l(k_1 R_0)$ and then the 2 subscripts replaced by 1. All the angular information contained in $X_{l,0}$ and $X_{l,\pm 1}$ remains the same.

The end result is that the field expressions for the source interior to the sphere can be obtained directly from Equations (139) through (186) for the exterior source case, by means of simple exchanges and substitutions.

To obtain the new field components for region 1, replace $h_l^{(1)}()$ by $j_l()$ everywhere in the old expressions for region 2, replace γ, N , and D by $\bar{\gamma}, \bar{N}$, and $\bar{D}$ in these expressions, and then replace 2 by 1 everywhere it appears explicitly.

To obtain the new field components for region 2, first define $[f_l(u)] = (l+1)f_l(u) - uf_{l+1}(u)$, then replace $j_l(k_1r)$ and $[j_l(k_1r)]$ by $h_l^{(1)}(k_2r)$ and $[h_l^{(1)}(k_2r)]$, $h_l^{(1)}(k_2R_0)$ and $[h_l^{(1)}(k_2R_0)]$ by $j_l(k_1R_0)$ and $[j_l(k_1R_0)]$, all unbarred quantities by barred quantities, change the remaining subscripts from 1 to 2, and reverse the sign of $\bar{D}_{M,l}$. The results of these operations are listed in the following section.

EXPRESSIONS FOR THE SCATTERED FIELDS FOR AN INTERNAL DIPOLE

MAGNETIC DIPOLE

Radial

(Interior)

$$E_r^1 = 0 \quad (253)$$

$$E_\theta^1 = 0 \quad (254)$$

$$E_\phi^1 = -\frac{\omega m \mu_1 k_1 \sin \theta}{4\pi R_0} \sum_{l=0}^{\infty} (2l+1) \frac{\bar{N}_{E,l}}{\bar{D}_{E,l}} j_l(k_1r) j_l(k_1R_0) \dot{P}_l \quad (255)$$

$$B_r^1 = \frac{im \mu_1 k_1}{4\pi R_0 r} \sum_{l=0}^{\infty} (2l+1) l(l+1) \frac{\bar{N}_{E,l}}{\bar{D}_{E,l}} j_l(k_1r) j_l(k_1R_0) P_l \quad (256)$$

$$B_\theta^1 = -\frac{im \mu_1 k_1 \sin \theta}{4\pi R_0 r} \sum_{l=0}^{\infty} (2l+1) \frac{\bar{N}_{E,l}}{\bar{D}_{E,l}} [(l+1)j_l(k_1r) - k_1r j_{l+1}(k_1r)] j_l(k_1R_0) \dot{P}_l \quad (257)$$

$$B_\phi^1 = 0 \quad (258)$$

(Exterior)

$$E_r^2 = 0 \quad (259)$$

$$E_\theta^2 = 0 \quad (260)$$

$$E_{\phi}^2 = \frac{i\omega\mu_1\mu_2 \sin \theta}{4\pi R_0 R} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{E,l}} h_l^{(1)}(k_2 r) j_l(k_1 R_0) \dot{P}_l \quad (261)$$

$$B_r^2 = \frac{m\mu_2}{4\pi R_0 R r} \sum_{l=0}^{\infty} \frac{1}{D_{E,l}} (2l+1) l(l+1) h_l^{(1)}(k_2 r) j_l(k_1 R_0) P_l \quad (262)$$

$$B_{\theta}^2 = -\frac{m\mu_2 \sin \theta}{4\pi R_0 R r} \sum_{l=0}^{\infty} (2l+1) \frac{1}{D_{E,l}} [(l+1) h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] j_l(k_1 R_0) \dot{P}_l \quad (263)$$

$$B_{\phi}^2 = 0 \quad (264)$$

Tangential

(Interior)

$$E_r^1 = -\frac{\omega m \mu_1 k_1 \sin \theta \sin \phi}{4\pi r} \sum_{l=1}^{\infty} (2l+1) \frac{\bar{N}_{M,l}}{D_{M,l}} j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l \quad (265)$$

$$E_{\theta}^1 = -\frac{\omega m \mu_1 k_1 \sin \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{\bar{N}_{E,l}}{D_{E,l}} r j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \right. \\ \left. + \frac{\bar{N}_{M,l}}{D_{M,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] R_0 j_l(k_1 R_0) [l(l+1) P_l - \cos \theta \dot{P}_l] \right\} \quad (266)$$

$$E_{\phi}^1 = \frac{\omega m \mu_1 k_1 \cos \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{\bar{N}_{M,l}}{D_{M,l}} R_0 j_l(k_1 R_0) [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] \dot{P}_l \right. \\ \left. + \frac{\bar{N}_{E,l}}{D_{E,l}} r j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1) P_l - \cos \theta \dot{P}_l] \right\} \quad (267)$$

$$B_r^1 = \frac{i m \mu_1 k_1 \sin \theta \cos \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} (2l+1) \frac{\bar{N}_{E,l}}{D_{E,l}} j_l(k_1 r) [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \quad (268)$$

$$B_{\theta}^1 = \frac{i m \mu_1 k_1 \cos \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{\bar{N}_{M,l}}{D_{M,l}} r R_0 k_1^2 j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l \right. \\ \left. + \frac{\bar{N}_{E,l}}{D_{E,l}} [(l+1) j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1) j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1) P_l - \cos \theta \dot{P}_l] \right\} \quad (269)$$

$$B_{\phi}^1 = -\frac{i m \mu_1 k_1 \sin \phi}{4 \pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{\bar{N}_{M,l}}{\bar{D}_{M,l}} r R_0 k_1^2 j_l(k_1 r) j_l(k_1 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right. \\ \left. + \frac{\bar{N}_{E,l}}{\bar{D}_{E,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \right\} \quad (270)$$

(Exterior)

$$E_r^2 = -\frac{i \omega m \mu_2 \sin \theta \sin \phi}{4 \pi R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{\bar{D}_{M,l}} h_l^{(1)}(k_2 r) j_l(k_1 R_0) \dot{P}_l \quad (271)$$

$$E_{\theta}^2 = -\frac{i \omega m \mu_2 \sin \phi}{4 \pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{\bar{D}_{E,l}} r h_l^{(1)}(k_2 r) [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \right. \\ \left. + \frac{1}{\bar{D}_{M,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] R_0 j_l(k_1 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (272)$$

$$E_{\phi}^2 = -\frac{i \omega m \mu_2 \cos \phi}{4 \pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{\bar{D}_{M,l}} R_0 j_l(k_1 R_0) [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] \dot{P}_l \right. \\ \left. + \frac{1}{\bar{D}_{E,l}} r h_l^{(1)}(k_2 r) [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (273)$$

$$B_r^2 = \frac{m \mu_2 \sin \theta \cos \phi}{4 \pi R_0 R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{\bar{D}_{E,l}} [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] h_l^{(1)}(k_2 r) \dot{P}_l \quad (274)$$

$$B_{\theta}^2 = \frac{m \mu_2 \cos \phi}{4 \pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{-1}{\bar{D}_{M,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) j_l(k_1 R_0) \dot{P}_l \right. \\ \left. + \frac{1}{\bar{D}_{E,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (275)$$

$$B_{\phi}^2 = -\frac{m \mu_2 \sin \phi}{4 \pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{-1}{\bar{D}_{M,l}} r R_0 k_2^2 h_l^{(1)}(k_2 r) j_l(k_1 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right. \\ \left. + \frac{1}{\bar{D}_{E,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \right\} \quad (276)$$

CURRENT DIPOLE

Radial

(Interior)

$$E_r^1 = -\frac{\omega p \mu_1}{4\pi k_1 R_0 r} \sum_{l=0}^{\infty} \frac{\bar{N}_{M,l}}{\bar{D}_{M,l}} (2l+1)l(l+1)j_l(k_1 r)j_l(k_1 R_0)P_l \quad (277)$$

$$E_\theta^1 = \frac{\omega p \mu_1 \sin \theta}{4\pi k_1 R_0 r} \sum_{l=0}^{\infty} (2l+1) \frac{\bar{N}_{M,l}}{\bar{D}_{M,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] j_l(k_1 R_0) \dot{P}_l \quad (278)$$

$$E_\phi^1 = 0 \quad (279)$$

$$B_r^1 = 0 \quad (280)$$

$$B_\theta^1 = 0 \quad (281)$$

$$B_\phi^1 = \frac{i p \mu_1 k_1 \sin \theta}{4\pi R_0} \sum_{l=0}^{\infty} (2l+1) \frac{\bar{N}_{M,l}}{\bar{D}_{M,l}} j_l(k_1 r)j_l(k_1 R_0) \dot{P}_l \quad (282)$$

(Exterior)

$$E_r^2 = \frac{i \omega p \mu_2}{2\pi k_1^2 R_0 R r} \sum_{l=0}^{\infty} \frac{1}{\bar{D}_{M,l}} (2l+1)l(l+1)h_l^{(1)}(k_2 r)j_l(k_1 R_0)P_l \quad (283)$$

$$E_\theta^2 = -\frac{i \omega p \mu_2 \sin \theta}{2\pi k_1^2 R_0 R r} \sum_{l=0}^{\infty} (2l+1) \frac{1}{\bar{D}_{M,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] j_l(k_1 R_0) \dot{P}_l \quad (284)$$

$$E_\phi^2 = 0 \quad (285)$$

$$B_r^2 = 0 \quad (286)$$

$$B_\theta^2 = 0 \quad (287)$$

$$B_\phi^2 = \frac{p \mu_2 \bar{\gamma} \sin \theta}{2\pi R_0 R} \sum_{l=0}^{\infty} (2l+1) \frac{1}{\bar{D}_{M,l}} h_l^{(1)}(k_2 r)j_l(k_1 R_0) \dot{P}_l \quad (288)$$

Tangential

(Interior)

$$E_r^1 = -\frac{\omega p \mu_1 \sin \theta \cos \phi}{4\pi k_1 R_0 r} \sum_{l=1}^{\infty} (2l+1) \frac{\bar{N}_{M,l}}{D_{M,l}} j_l(k_1 r) [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \quad (289)$$

$$E_\theta^1 = -\frac{\omega p \mu_1 \cos \phi}{4\pi k_1 R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{\bar{N}_{E,l}}{D_{E,l}} r R_0 k_1^2 j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l \right. \\ \left. + \frac{\bar{N}_{M,l}}{D_{M,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (290)$$

$$E_\phi^1 = \frac{\omega p \mu_1 \sin \phi}{4\pi k_1 R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{\bar{N}_{E,l}}{D_{E,l}} r R_0 k_1^2 j_l(k_1 r) j_l(k_1 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right. \\ \left. + \frac{\bar{N}_{M,l}}{D_{M,l}} [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \right\} \quad (291)$$

$$B_r^1 = \frac{i p \mu_1 k_1 \sin \theta \sin \phi}{4\pi r} \sum_{l=1}^{\infty} (2l+1) \frac{\bar{N}_{E,l}}{D_{E,l}} j_l(k_1 r) j_l(k_1 R_0) \dot{P}_l \quad (292)$$

$$B_\theta^1 = \frac{i p \mu_1 k_1 \sin \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{\bar{N}_{M,l}}{D_{M,l}} r j_l(k_1 r) [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \right. \\ \left. + \frac{\bar{N}_{E,l}}{D_{E,l}} R_0 j_l(k_1 R_0) [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (293)$$

$$B_\phi^1 = -\frac{i p \mu_1 k_1 \cos \phi}{4\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{\bar{N}_{E,l}}{D_{E,l}} R_0 j_l(k_1 R_0) [(l+1)j_l(k_1 r) - k_1 r j_{l+1}(k_1 r)] \dot{P}_l \right. \\ \left. + \frac{\bar{N}_{M,l}}{D_{M,l}} r j_l(k_1 r) [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (294)$$

(Exterior)

$$E_r^2 = \frac{i \omega p \mu_2 \sin \theta \cos \phi}{2\pi k_1^2 R_0 R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{M,l}} h_l^{(1)}(k_2 r) [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \quad (295)$$

$$E_{\theta}^2 = \frac{i\omega p \mu_2 \cos \phi}{4\pi k_1^2 R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{D_{E,l}} r R_0 k_1^2 h_l^{(1)}(k_2 r) j_l(k_1 R_0) \dot{P}_l \right. \\ \left. + \frac{2}{D_{M,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (296)$$

$$E_{\phi}^2 = -\frac{i\omega p \mu_2 \sin \phi}{4\pi k_1^2 R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{D_{E,l}} r R_0 k_1^2 h_l^{(1)}(k_2 r) j_l(k_1 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right. \\ \left. + \frac{2}{D_{M,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \right\} \quad (297)$$

$$B_r^2 = \frac{p \mu_2 \sin \theta \sin \phi}{4\pi R r} \sum_{l=1}^{\infty} (2l+1) \frac{1}{D_{E,l}} h_l^{(1)}(k_2 r) j_l(k_1 R_0) \dot{P}_l \quad (298)$$

$$B_{\theta}^2 = \frac{p \mu_2 \sin \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ -\frac{2\bar{\gamma}}{D_{M,l}} r h_l^{(1)}(k_2 r) [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] \dot{P}_l \right. \\ \left. + \frac{1}{D_{E,l}} [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] R_0 j_l(k_1 R_0) [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (299)$$

$$B_{\phi}^2 = \frac{p \mu_2 \cos \phi}{4\pi R_0 R r} \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{1}{D_{E,l}} R_0 j_l(k_1 R_0) [(l+1)h_l^{(1)}(k_2 r) - k_2 r h_{l+1}^{(1)}(k_2 r)] \dot{P}_l \right. \\ \left. - \frac{2\bar{\gamma}}{D_{M,l}} r h_l^{(1)}(k_2 r) [(l+1)j_l(k_1 R_0) - k_1 R_0 j_{l+1}(k_1 R_0)] [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \quad (300)$$

DC LIMIT OF THE SCATTERED FIELDS FOR AN INTERNAL DIPOLE

The dc limits of the field expressions given in Equations (253) through (300) are not simply derived from the dc expressions for the external dipole, given in Equations (187) through (234). The limiting forms are derived directly and are given below. In writing the expressions, the results are given in terms of $\tau = \mu_1/\mu_2$ and $\delta = \sigma_1/\sigma_2$.

DC MAGNETIC DIPOLE

Radial

(Interior)

$$E_r^1 = 0 \quad (301)$$

$$E_{\theta}^1 = 0 \quad (302)$$

$$E_{\phi}^1 = 0 \quad (303)$$

$$B_r^1 = -\frac{m\mu_1(\tau-1)}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l^2(l+1)}{(\tau+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l P_l \quad (304)$$

$$B_{\theta}^1 = \frac{m\mu_1(\tau-1)\sin\theta}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\tau+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \dot{P}_l \quad (305)$$

$$B_{\phi}^1 = 0 \quad (306)$$

(Exterior)

$$E_r^2 = 0 \quad (307)$$

$$E_{\theta}^2 = 0 \quad (308)$$

$$E_{\phi}^2 = 0 \quad (309)$$

$$B_r^2 = \frac{m\mu_2\tau}{4\pi R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)l(l+1)}{(\tau+1)l+1} \left(\frac{R_0}{r}\right)^l P_l \quad (310)$$

$$B_{\theta}^2 = \frac{m\mu_2\tau\sin\theta}{4\pi R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)l}{(\tau+1)l+1} \left(\frac{R_0}{r}\right)^l \dot{P}_l \quad (311)$$

$$B_{\phi}^2 = 0 \quad (312)$$

Tangential

(Interior)

$$E_r^1 = 0 \quad (313)$$

$$E_{\theta}^1 = 0 \quad (314)$$

$$E_{\phi}^1 = 0 \quad (315)$$

$$B_r^1 = -\frac{m\mu_1(\tau-1)\sin\theta\cos\phi}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\tau+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \dot{P}_l \quad (316)$$

$$B_{\theta}^1 = -\frac{m\mu_1(\tau-1)\cos\phi}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{(l+1)}{(\tau+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l [l(l+1)P_l - \cos\theta \dot{P}_l] \quad (317)$$

$$B_{\phi}^1 = \frac{m\mu_1(\tau-1)\sin\phi}{4\pi R_0 r R} \sum_{l=1}^{\infty} \frac{l+1}{(\tau+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \dot{P}_l \quad (318)$$

(Exterior)

$$E_r^2 = 0 \quad (319)$$

$$E_{\theta}^2 = 0 \quad (320)$$

$$E_{\phi}^2 = 0 \quad (321)$$

$$B_r^2 = \frac{m\mu_2\tau\sin\theta\cos\phi}{4\pi R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)(l+1)}{(\tau+1)l+1} \left(\frac{R_0}{r}\right)^l \dot{P}_l \quad (322)$$

$$B_{\theta}^2 = -\frac{m\mu_2\tau\cos\phi}{4\pi R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\tau+1)l+1} \left(\frac{R_0}{r}\right)^l [l(l+1)P_l - \cos\theta \dot{P}_l] \quad (323)$$

$$B_{\phi}^2 = \frac{m\mu_2\tau\sin\phi}{4\pi R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\tau+1)l+1} \left(\frac{R_0}{r}\right)^l \dot{P}_l \quad (324)$$

DC CURRENT DIPOLE

Radial

(Interior)

$$E_r^1 = -\frac{p(\delta-1)}{4\pi\sigma_1 R_0 r R} \sum_{l=1}^{\infty} \frac{l^2(l+1)}{(\delta+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l P_l \quad (325)$$

$$E_{\theta}^1 = \frac{p(\delta-1)\sin\theta}{4\pi\sigma_1 R_0 r R} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\delta+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \dot{P}_l \quad (326)$$

$$E_{\phi}^1 = 0 \quad (327)$$

$$B_r^1 = 0 \quad (328)$$

$$B_{\theta}^1 = 0 \quad (329)$$

$$B_{\Phi}^1 = -\frac{p(\delta-1)\mu_1 \sin \theta}{4\pi R_0 R} \sum_{l=1}^{\infty} \frac{l}{(\delta+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \dot{P}_l \quad (330)$$

(Exterior)

$$E_r^2 = \frac{p}{2\pi\sigma_2 R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)l(l+1)}{(\delta+1)l+1} \left(\frac{R_0}{r}\right)^l P_l \quad (331)$$

$$E_{\theta}^2 = \frac{p \sin \theta}{2\pi\sigma_2 R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)l}{(\delta+1)l+1} \left(\frac{R_0}{r}\right)^l \dot{P}_l \quad (332)$$

$$E_{\Phi}^2 = 0 \quad (333)$$

$$B_r^2 = 0 \quad (334)$$

$$B_{\theta}^2 = 0 \quad (335)$$

$$B_{\Phi}^2 = \frac{p\mu_2 \sin \theta}{2\pi R_0 r} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\delta+1)l+1} \left(\frac{R_0}{r}\right)^l \dot{P}_l \quad (336)$$

Tangential

(Interior)

$$E_r^1 = -\frac{p(\delta-1) \sin \theta \cos \phi}{4\pi\sigma_1 R_0 R r} \sum_{l=1}^{\infty} \frac{l(l+1)}{(\delta+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \dot{P}_l \quad (337)$$

$$E_{\theta}^1 = -\frac{p(\delta-1) \cos \phi}{4\pi\sigma_1 R_0 R r} \sum_{l=1}^{\infty} \frac{l+1}{(\delta+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l [l(l+1)P_l - \cos \theta \dot{P}_l] \quad (338)$$

$$E_{\Phi}^1 = \frac{p(\delta-1) \sin \phi}{4\pi\sigma_1 R_0 R r} \sum_{l=1}^{\infty} \frac{l+1}{(\delta+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \dot{P}_l \quad (339)$$

$$B_r^1 = -\frac{p\mu_1(\tau-1) \sin \theta \sin \phi}{4\pi R r} \sum_{l=1}^{\infty} \frac{l}{(\tau+1)l+1} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \dot{P}_l \quad (340)$$

$$B_{\theta}^1 = -\frac{p\mu_1 \sin \phi}{4\pi R R_0 r} \sum_{l=1}^{\infty} \left\{ \frac{R_0(\tau-1)}{(\tau+1)l+1} [l(l+1)P_l - \cos \theta \dot{P}_l] - \frac{r(\delta-1)}{(\delta+1)l+1} \dot{P}_l \right\} \left(\frac{r}{R}\right)^l \left(\frac{R_0}{R}\right)^l \quad (341)$$

$$B_{\Phi}^1 = -\frac{p\mu_1 \cos \phi}{4\pi R R_0 r} \sum_{l=1}^{\infty} \left\{ \frac{R_0(\tau-1)}{(\tau+1)l+1} \dot{P}_l - \frac{r(\delta-1)}{(\delta+1)l+1} [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \left(\frac{r}{R} \right)^l \left(\frac{R_0}{R} \right)^l \quad (342)$$

(Exterior)

$$E_r^2 = \frac{p \sin \theta \cos \phi}{2\pi \sigma_2 R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)(l+1)}{(\delta+1)l+1} \left(\frac{R_0}{r} \right)^l \dot{P}_l \quad (343)$$

$$E_{\theta}^2 = -\frac{p \cos \phi}{2\pi \sigma_2 R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\delta+1)l+1} \left(\frac{R_0}{r} \right)^l [l(l+1)P_l - \cos \theta \dot{P}_l] \quad (344)$$

$$E_{\Phi}^2 = \frac{p \sin \phi}{2\pi \sigma_2 R_0 r^2} \sum_{l=1}^{\infty} \frac{(2l+1)}{(\delta+1)l+1} \left(\frac{R_0}{r} \right)^l \dot{P}_l \quad (345)$$

$$B_r^2 = \frac{p\tau\mu_2 \sin \theta \sin \phi}{4\pi r^2} \sum_{l=1}^{\infty} \frac{2l+1}{(\tau+1)l+1} \left(\frac{R_0}{r} \right)^l \dot{P}_l \quad (346)$$

$$B_{\theta}^2 = \frac{p\mu_2 \sin \phi}{4\pi R_0 r^2} \sum_{l=1}^{\infty} (2l+1) \left\{ \frac{\tau R_0}{(l+1)[(\tau+1)l+1]} [l(l+1)P_l - \cos \theta \dot{P}_l] - \frac{2r}{l[(\delta+1)l+1]} \dot{P}_l \right\} \left(\frac{R_0}{r} \right)^l \quad (347)$$

$$B_{\Phi}^2 = \frac{p\mu_2 \cos \phi}{4\pi R_0 r^2} \sum_{l=1}^{\infty} (2l+1) \left\{ \frac{\tau R_0}{(l+1)[(\tau+1)l+1]} \dot{P}_l - \frac{2r}{l[(\delta+1)l+1]} [l(l+1)P_l - \cos \theta \dot{P}_l] \right\} \left(\frac{R_0}{r} \right)^l \quad (348)$$

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APPENDIX A

**NUMERICAL METHODS AND HEWLETT-PACKARD BASIC 3.0 COMPUTER
CODES FOR CURRENT AND MAGNETIC DIPOLE FIELDS**

APPENDIX A

NUMERICAL METHODS AND HEWLETT-PACKARD BASIC 3.0 COMPUTER
CODES FOR CURRENT AND MAGNETIC DIPOLE FIELDS

In this Appendix, codes are given for computing the electric and magnetic fields external to a sphere, for dipole sources external to the sphere. Codes are given for ac sources (MAGNDIPSPH and CURRDIPSPH) and for dc sources (MGDIPSPHDC and CRDIPSPHDC). The Legendre polynomials and their first derivatives are generated in the subroutine PI_pldot based on the recursion formulas.^{A1} The spherical Bessel functions can be generated from closed-form expressions,^{A1} but these have large roundoff errors for any given argument as the order increases. Instead, the Bessel functions are generated from various representations, depending on the values of argument and order. These routines were taken from Abramowitz and Stegun,^{A1} and are discussed in detail below.

The structure of the remainder of this Appendix is as follows:

MAGNDIPSPH
CURRDIPSPH
MGDIPSPHDC
CRDIPSPHDC

The subroutines:

Geomdipsph
Geomfldpos
Geomfldb_e
Jcomb
Hcomb
Spherejnz
Spherehnz
Jnuevrywhr
Hnuevrywhr
PI_pldot
Ndm
Nde
Ndmicon
Ndeicon

Discussion of series representations of Bessel functions.

The subroutines:

Jn
H1n
Jnu
H1nu
Prinlogz
Atn2
Gamma

Discussion of asymptotic representations of Bessel functions for large argument.

The subroutines:

Jnasy
H1nasy
Pnz
Qnz

Discussion of uniform asymptotic representation of Bessel functions for large order.

The subroutines:

Jfiuniasym
Hfkuniasym
Jh1produni
Uk
Ukcoefs

PROGRAM MAGNDIPSPH

```

10  OPTION BASE 1
20  DIM Xf(3),Xd(3),M(3),Rds(3,3),Bp(3,2),Ep(3,2),B(3,2),E(3,2)
30  DIM PI(100),Pld(100),Ndma(100,2),Ndea(100,2),Et(3,2),Bt(3,2)
40      INPUT "FREQUENCY AND MEDIUM AND SPHERE(Ss<0  =>
INF)CONDUCTIVITIES(MHO/M)?" ,F,Sm,Ss
50  INPUT "RELATIVE PERMEABILITY OF MEDIUM AND SPHERE?" ,Mm,Ms
60  INPUT "RELATIVE PERMITTIVITY OF MEDIUM AND SPHERE?" ,Em,Es
70  INPUT "POSITION OF FIELD POINT?(M)" ,Xf(*)
80  INPUT "POSITION OF SOURCE POINT?(M)" ,Xd(*)
90  INPUT "MAGNETIC DIPOLE MOMENT VECTOR?(AMP-M^2)" ,M(*)
100 INPUT "SPHERE RADIUS(M) AND MAXIMUM POLAR ANGLE INDEX?" ,A,Ell
110 INPUT "RELATIVE ERROR FOR TRUNCATION?" ,Err
120 Lmax=Ell+1
130 W=2*PI*F
140 M0=4*PI*1.E-7
150 E0=8.85415E-12
160 Ut1=W*M0
170 Ut2=W*E0*Ut1
180 K12r=Ut2*Es*Ms
190 K12i=Ut1*Ms*Ss
200 K22r=Ut2*Em*Mm
210 K22i=Ut1*Mm*Sm
220 Md=SQR(K12r*K12r+K12i*K12i)
230 K1r=SQR((Md+K12r)/2)
240 K1i=SQR(ABS(Md-K12r)/2)
250 Md=SQR(K22r*K22r+K22i*K22i)
260 K2r=SQR((Md+K22r)/2)
270 K2i=SQR(ABS(Md-K22r)/2)
280 Mefac=W*M0*Mm/4/PI      ! Electric fields in volt/meter
290 Mefacr=Mefac*K2r
300 Mefaci=Mefac*K2i
310 Mmfac=M0*Mm/4/PI      ! Magnetic fields in tesla

```

```

320 Mmfacr=-Mmfac*K2i
330 Mmfaci=Mmfac*K2r
340 Mur=Ms/Mm
350 !Only frequency and medium constants to here
360 REDIM Ndma(Lmax,2),Ndea(Lmax,2)
370 IF Ss>0 THEN GOTO 410
380 CALL Ndmicon(Lmax,K2r,K2i,A,Ndma(*))
390 CALL Ndeicon(Lmax,K2r,K2i,A,Ndea(*))
400 GOTO 430
410 CALL Ndm(Lmax,K1r,K1i,K2r,K2i,Mur,A,Ndma(*))
420 CALL Nde(Lmax,K1r,K1i,K2r,K2i,Mur,A,Ndea(*))
430 !Added only sphere radius to here
440 CALL Geomdipsph(Xd(*),M(*),kds(*),R0,Mr,Mt)
450 CALL Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
460 Zr=K2r*R
470 Zi=K2i*R
480 Z0r=K2r*R0
490 Z0i=K2i*R0
500 REDIM Pl(Lmax),Pld(Lmax)
510 CALL Pl_pldot(Lmax-1,Ct,Pl(*),Pld(*))
520 MAT Ep= (0)
530 MAT Bp= (0)
540 L=2
550 MAT Et= (0)
560 MAT Bt= (0)
570 CALL Hcomb(L-1,Zr,Zi,R,Hr,Hzi,Chzr,Chzi)
580 CALL Hcomb(L-1,Z0r,Z0i,R0,H0r,H0i,Ch0r,Ch0i)
590 Te1r=Hr*H0r-Hzi*H0i
600 Te1i=Hr*H0i+Hzi*H0r
610 Te2r=Te1r*Ndma(L,1)-Te1i*Ndma(L,2)
620 Te2i=Te1r*Ndma(L,2)+Te1i*Ndma(L,1)
630 Tb2r=Te1r*Ndea(L,1)-Te1i*Ndea(L,2)
640 Tb2i=Te1r*Ndea(L,2)+Te1i*Ndea(L,1)
650 Et(1,1)=Mt*St*Sp*Te2r*Pld(L)/R
660 Et(1,2)=Mt*St*Sp*Te2i*Pld(L)/R
670 Te3r=H0r*Chzr-H0i*Chzi
680 Te3i=H0r*Chzi+H0i*Chzr
690 Te4r=Te3r*Ndma(L,1)-Te3i*Ndma(L,2)
700 Te4i=Te3r*Ndma(L,2)+Te3i*Ndma(L,1)
710 Te5r=Hr*Ch0r-Hzi*Ch0i
720 Te5i=Hr*Ch0i+Hzi*Ch0r
730 Te6r=Te5r*Ndea(L,1)-Te5i*Ndea(L,2)
740 Te6i=Te5r*Ndea(L,2)+Te5i*Ndea(L,1)
750 Bt(1,1)=Mr*Tb2r*L*(L-1)*Pl(L)/R/R0
760 Bt(1,2)=Mr*Tb2i*L*(L-1)*Pl(L)/R/R0
770 Bt(1,1)=Bt(1,1)+Mt*Cp*St*Te6r*Pld(L)/R
780 Bt(1,2)=Bt(1,2)+Mt*Cp*St*Te6i*Pld(L)/R
790 Tb3r=Chzr*Ch0r-Chzi*Ch0i
800 Tb3i=Chzr*Ch0i+Chzi*Ch0r
810 Tb4r=Tb3r*Ndea(L,1)-Tb3i*Ndea(L,2)
820 Tb4i=Tb3r*Ndea(L,2)+Tb3i*Ndea(L,1)
830 Tb5r=K22r*Te2r-K22i*Te2i

```

```

840 Tb5i=K22r*Te2i+K22i*Te2r
850 Tb6r=Te3r*Ndea(L,1)-Te3i*Ndea(L,2)
860 Tb6i=Te3r*Ndea(L,2)+Te3i*Ndea(L,1)
870 Et(2,1)=Et(2,1)+Mt*Sp*Te4r*(Pl(L)-Ct*Pld(L)/L/(L-1))
880 Et(2,2)=Et(2,2)+Mt*Sp*Te4i*(Pl(L)-Ct*Pld(L)/L/(L-1))
890 Et(2,1)=Et(2,1)-Mt*Sp*Te6r*Pld(L)/L/(L-1)
900 Et(2,2)=Et(2,2)-Mt*Sp*Te6i*Pld(L)/L/(L-1)
910 Bt(2,1)=Bt(2,1)-Mr*St*Tb6r*Pld(L)/R0
920 Bt(2,2)=Bt(2,2)-Mr*St*Tb6i*Pld(L)/R0
930 Bt(2,1)=Bt(2,1)+Mt*Cp*Tb5r*Pld(L)/L/(L-1)
940 Bt(2,2)=Bt(2,2)+Mt*Cp*Tb5i*Pld(L)/L/(L-1)
950 Bt(2,1)=Bt(2,1)+Mt*Cp*Tb4r*(Pl(L)-Ct*Pld(L)/L/(L-1))
960 Bt(2,2)=Bt(2,2)+Mt*Cp*Tb4i*(Pl(L)-Ct*Pld(L)/L/(L-1))
970 Te7r=Te1r*Ndea(L,1)-Te1i*Ndea(L,2)
980 Te7i=Te1r*Ndea(L,2)+Te1i*Ndea(L,1)
990 Te8r=Te3r*Ndma(L,1)-Te3i*Ndma(L,2)
1000 Te8i=Te3r*Ndma(L,2)+Te3i*Ndma(L,1)
1010 Et(3,1)=Et(3,1)+Mt*Cp*Te6r*(Pl(L)-Ct*Pld(L)/L/(L-1))
1020 Et(3,2)=Et(3,2)+Mt*Cp*Te6i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1030 Et(3,1)=Et(3,1)-Mt*Cp*Te8r*Pld(L)/L/(L-1)
1040 Et(3,2)=Et(3,2)-Mt*Cp*Te8i*Pld(L)/L/(L-1)
1050 Et(3,1)=Et(3,1)-Mr*Te7r*St*Pld(L)/R0
1060 Et(3,2)=Et(3,2)-Mr*Te7i*St*Pld(L)/R0
1070 Bt(3,1)=Bt(3,1)+Mt*Sp*Tb5r*(Pl(L)-Ct*Pld(L)/L/(L-1))
1080 Bt(3,2)=Bt(3,2)+Mt*Sp*Tb5i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1090 Bt(3,1)=Bt(3,1)+Mt*Sp*Tb4r*Pld(L)/L/(L-1)
1100 Bt(3,2)=Bt(3,2)+Mt*Sp*Tb4i*Pld(L)/L/(L-1)
1110 Ep(1,1)=Ep(1,1)+(2*L-1)*Et(1,1)
1120 Ep(1,2)=Ep(1,2)+(2*L-1)*Et(1,2)
1130 Ep(2,1)=Ep(2,1)+(2*L-1)*Et(2,1)
1140 Ep(2,2)=Ep(2,2)+(2*L-1)*Et(2,2)
1150 Ep(3,1)=Ep(3,1)+(2*L-1)*Et(3,1)
1160 Ep(3,2)=Ep(3,2)+(2*L-1)*Et(3,2)
1170 Bp(1,1)=Bp(1,1)+(2*L-1)*Bt(1,1)
1180 Bp(1,2)=Bp(1,2)+(2*L-1)*Bt(1,2)
1190 Bp(2,1)=Bp(2,1)+(2*L-1)*Bt(2,1)
1200 Bp(2,2)=Bp(2,2)+(2*L-1)*Bt(2,2)
1210 Bp(3,1)=Bp(3,1)+(2*L-1)*Bt(3,1)
1220 Bp(3,2)=Bp(3,2)+(2*L-1)*Bt(3,2)
1230 Ner1=Et(1,1)*Et(1,1)+Et(1,2)*Et(1,2)
1240 Ner2=Et(2,1)*Et(2,1)+Et(2,2)*Et(2,2)
1250 Ner3=Et(3,1)*Et(3,1)+Et(3,2)*Et(3,2)
1260 Dne1=Ep(1,1)*Ep(1,1)+Ep(1,2)*Ep(1,2)
1270 Dne2=Ep(2,1)*Ep(2,1)+Ep(2,2)*Ep(2,2)
1280 Dne3=Ep(3,1)*Ep(3,1)+Ep(3,2)*Ep(3,2)
1290 Nbr1=Bt(1,1)*Bt(1,1)+Bt(1,2)*Bt(1,2)
1300 Nbr2=Bt(2,1)*Bt(2,1)+Bt(2,2)*Bt(2,2)
1310 Nbr3=Bt(3,1)*Bt(3,1)+Bt(3,2)*Bt(3,2)
1320 Dnb1=Bp(1,1)*Bp(1,1)+Bp(1,2)*Bp(1,2)
1330 Dnb2=Bp(2,1)*Bp(2,1)+Bp(2,2)*Bp(2,2)
1340 Dnb3=Bp(3,1)*Bp(3,1)+Bp(3,2)*Bp(3,2)
1350 IF L=Lmax THEN 1570

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1360 IF Dnb1=0 THEN 1380
1370 IF Nbr1/Dnb1<Err*Err THEN
1380   IF Dnb2=0 THEN 1400
1390   IF Nbr2/Dnb2<Err*Err THEN
1400     IF Dnb3=0 THEN 1420
1410     IF Nbr3/Dnb3<Err*Err THEN
1420       IF Dne1=0 THEN 1440
1430       IF Ner1/Dne1<Err*Err THEN
1440         IF Dne2=0 THEN 1460
1450         IF Ner2/Dne2<Err*Err THEN
1460           IF Dne3=0 THEN 1480
1470           IF Ner3/Dne3<Err*Err THEN
1480             GOTO 1570
1490           END IF
1500         END IF
1510       END IF
1520     END IF
1530   END IF
1540 END IF
1550 L=L+1
1560 GOTO 550
1570 Tmpr=Mefacr*Ep(1,1)-Mefaci*Ep(1,2)
1580 Tmpi=Mefacr*Ep(1,2)+Mefaci*Ep(1,1)
1590 Ep(1,1)=-Tmpr
1600 Ep(1,2)=-Tmpi
1610 Tmpr=Mmfacr*Bp(1,1)-Mmfaci*Bp(1,2)
1620 Tmpi=Mmfacr*Bp(1,2)+Mmfaci*Bp(1,1)
1630 Bp(1,1)=Tmpr
1640 Bp(1,2)=Tmpi
1650 Tmpr=Mefacr*Ep(2,1)-Mefaci*Ep(2,2)
1660 Tmpi=Mefacr*Ep(2,2)+Mefaci*Ep(2,1)
1670 Ep(2,1)=-Tmpr
1680 Ep(2,2)=-Tmpi
1690 Tmpr=Mmfacr*Bp(2,1)-Mmfaci*Bp(2,2)
1700 Tmpi=Mmfacr*Bp(2,2)+Mmfaci*Bp(2,1)
1710 Bp(2,1)=Tmpr
1720 Bp(2,2)=Tmpi
1730 Tmpr=Mefacr*Ep(3,1)-Mefaci*Ep(3,2)
1740 Tmpi=Mefacr*Ep(3,2)+Mefaci*Ep(3,1)
1750 Ep(3,1)=Tmpr
1760 Ep(3,2)=Tmpi
1770 Tmpr=Mmfacr*Bp(3,1)-Mmfaci*Bp(3,2)
1780 Tmpi=Mmfacr*Bp(3,2)+Mmfaci*Bp(3,1)
1790 Bp(3,1)=-Tmpr
1800 Bp(3,2)=-Tmpi
1810 CALL Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
1820 PRINT E(*) !Electric field in original frame
1830 PRINT
1840 PRINT B(*) !Magnetic field in original frame
1850 END

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PROGRAM CURRDIPSPH

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10  OPTION BASE 1
20  DIM Xf(3),Xd(3),P(3),Rds(3,3),Bp(3,2),Ep(3,2),B(3,2),E(3,2)
30  DIM Pl(100),Pld(100),Ndma(100,2),Ndea(100,2),Et(3,2),Bt(3,2)
40  INPUT "FREQUENCY AND MEDIUM AND SPHERE(Ss<0 => INF)
CONDUCTIVITIES(MHO/M)?" ,F,Sm,Ss
50  INPUT "RELATIVE PERMEABILITY OF MEDIUM AND SPHERE?" ,Mm,Ms
60  INPUT "RELATIVE PERMITTIVITY OF MEDIUM AND SPHERE?" ,Em,Es
70  INPUT "POSITION OF FIELD POINT?(M)" ,Xf(*)
80  INPUT "POSITION OF SOURCE POINT?(M)" ,Xd(*)
90  INPUT "CURRENT DIPOLE MOMENT VECTOR?(AMP-M)" ,P(*)
100 INPUT " SPHERE RADIUS(M) AND MAXIMUM POLAR ANGLE INDEX?" ,A,Ell
110 Err=1.E-6
120 Lmax=Ell+1
130 W=2*PI*F
140 M0=4*PI*1.E-7
150 E0=8.85415E-12
160 Ut1=W*M0
170 Ut2=W*E0*Ut1
180 K12r=Ut2*Es*Ms
190 K12i=Ut1*Ms*Ss
200 K22r=Ut2*Em*Mm
210 K22i=Ut1*Mm*Sm
220 Md=SQR(K12r*K12r+K12i*K12i)
230 K1r=SQR((Md+K12r)/2)
240 K1i=SQR(ABS(Md-K12r)/2)
250 Md=SQR(K22r*K22r+K22i*K22i)
260 K2r=SQR((Md+K22r)/2)
270 K2i=SQR(ABS(Md-K22r)/2)
280 Mk2=K2r*K2r+K2i*K2i
290 Ik2r=K2r/Mk2
300 Ik2i=-K2i/Mk2
310 Eefac=M0*Mm*W/4/PI           !Electric fields in volt/meter
320 Eefacr=Eefac*Ik2r
330 Eefaci=Eefac*Ik2i
340 Emfac=M0*Mm/4/PI           !Magnetic fields in tesla
350 Emfacr=-Emfac*K2i
360 Emfaci=Emfac*K2r
370 Mur=Ms/Mm
380 !Only frequency and medium constants to here
390 REDIM Ndma(Lmax,2),Ndea(Lmax,2)
400 IF Ss>0 THEN 440
410 CALL Ndmicon(Lmax,K2r,K2i,A,Ndma(*))
420 CALL Ndeicon(Lmax,K2r,K2i,A,Ndea(*))
430 GOTO 460
440 CALL Ndm(Lmax,K1r,K1i,K2r,K2i,Mur,A,Ndma(*))
450 CALL Nde(Lmax,K1r,K1i,K2r,K2i,Mur,A,Ndea(*))
460 !Added only sphere radius to here
470 CALL Geomdipsph(Xd(*),P(*),Rds(*),R0,Pr,Pt)
480 CALL Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
490 Zr=K2r*R

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500 Zi=K2i*R
510 Z0r=K2r*R0
520 Z0i=K2i*R0
530 REDIM Pl(Lmax),Pld(Lmax)
540 CALL Pl_pldot(Lmax-1,Ct,Pl(*),Pld(*))
550 MAT Ep= (0)
560 MAT Bp= (0)
570 L=2
580 MAT Et= (0)
590 MAT Bt= (0)
600 CALL Hcomb(L-1,Zr,Zi,R,HZR,Hzi,Chzr,Chzi)
610 CALL Hcomb(L-1,Z0r,Z0i,R0,HZ0r,HZ0i,Chz0r,Chz0i)
620 Tb1r=HZr*HZ0r-Hzi*HZ0i
630 Tb1i=HZr*HZ0i+Hzi*HZ0r
640 Tb2r=Tb1r*Ndea(L,1)-Tb1i*Ndea(L,2)
650 Tb2i=Tb1r*Ndea(L,2)+Tb1i*Ndea(L,1)
660 Te2r=Tb1r*Ndma(L,1)-Tb1i*Ndma(L,2)
670 Te2i=Tb1r*Ndma(L,2)+Tb1i*Ndma(L,1)
680 Bt(1,1)=Pt*St*Sp*Tb2r*Pld(L)/R
690 Bt(1,2)=Pt*St*Sp*Tb2i*Pld(L)/R
700 Tb3r=HZ0r*Chzr-HZ0i*Chzi
710 Tb3i=HZ0r*Chzi+HZ0i*Chzr
720 Tb4r=Tb3r*Ndea(L,1)-Tb3i*Ndea(L,2)
730 Tb4i=Tb3r*Ndea(L,2)+Tb3i*Ndea(L,1)
740 Tb5r=HZr*Chz0r-Hzi*Chz0i
750 Tb5i=HZr*Chz0i+Hzi*Chz0r
760 Tb6r=Tb5r*Ndma(L,1)-Tb5i*Ndma(L,2)
770 Tb6i=Tb5r*Ndma(L,2)+Tb5i*Ndma(L,1)
780 Et(1,1)=Pr*Te2r*L*(L-1)*Pl(L)/R/R0
790 Et(1,2)=Pr*Te2i*L*(L-1)*Pl(L)/R/R0
800 Et(1,1)=Et(1,1)+Pt*Cp*St*Tb6r*Pld(L)/R
810 Et(1,2)=Et(1,2)+Pt*Cp*St*Tb6i*Pld(L)/R
820 Te3r=Chzr*Chz0r-Chzi*Chz0i
830 Te3i=Chzr*Chz0i+Chzi*Chz0r
840 Te4r=Te3r*Ndma(L,1)-Te3i*Ndma(L,2)
850 Te4i=Te3r*Ndma(L,2)+Te3i*Ndma(L,1)
860 Te5r=-K22r*Tb2r+K22i*Tb2i
870 Te5i=-K22r*Tb2i-K22i*Tb2r
880 Te6r=Tb3r*Ndma(L,1)-Tb3i*Ndma(L,2)
890 Te6i=Tb3r*Ndma(L,2)+Tb3i*Ndma(L,1)
900 Bt(2,1)=Bt(2,1)+Pt*Sp*Tb4r*(Pl(L)-Ct*Pld(L)/L/(L-1))
910 Bt(2,2)=Bt(2,2)+Pt*Sp*Tb4i*(Pl(L)-Ct*Pld(L)/L/(L-1))
920 Bt(2,1)=Bt(2,1)-Pt*Sp*Tb6r*Pld(L)/L/(L-1)
930 Bt(2,2)=Bt(2,2)-Pt*Sp*Tb6i*Pld(L)/L/(L-1)
940 Et(2,1)=Et(2,1)+Pr*St*Te6r*Pld(L)/R0
950 Et(2,2)=Et(2,2)+Pr*St*Te6i*Pld(L)/R0
960 Et(2,1)=Et(2,1)+Pt*Cp*Te5r*Pld(L)/L/(L-1)
970 Et(2,2)=Et(2,2)+Pt*Cp*Te5i*Pld(L)/L/(L-1)
980 Et(2,1)=Et(2,1)-Pt*Cp*Te4r*(Pl(L)-Ct*Pld(L)/L/(L-1))
990 Et(2,2)=Et(2,2)-Pt*Cp*Te4i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1000 Tb7r=Tb1r*Ndma(L,1)-Tb1i*Ndma(L,2)
1010 Tb7i=Tb1r*Ndma(L,2)+Tb1i*Ndma(L,1)

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1020 Tb8r=Tb3r*Ndea(L,1)-Tb3i*Ndea(L,2)
1030 Tb8i=Tb3r*Ndea(L,2)+Tb3i*Ndea(L,1)
1040 Bt(3,1)=Bt(3,1)-Pt*Cp*Tb6r*(Pl(L)-Ct*Pld(L)/L/(L-1))
1050 Bt(3,2)=Bt(3,2)-Pt*Cp*Tb6i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1060 Bt(3,1)=Bt(3,1)+Pt*Cp*Tb8r*Pld(L)/L/(L-1)
1070 Bt(3,2)=Bt(3,2)+Pt*Cp*Tb8i*Pld(L)/L/(L-1)
1080 Bt(3,1)=Bt(3,1)+Pr*Tb7r*St*Pld(L)/R0
1090 Bt(3,2)=Bt(3,2)+Pr*Tb7i*St*Pld(L)/R0
1100 Et(3,1)=Et(3,1)-Pt*Sp*Te5r*(Pl(L)-Ct*Pld(L)/L/(L-1))
1110 Et(3,2)=Et(3,2)-Pt*Sp*Te5i*(Pl(L)-Ct*Pld(L)/L/(L-1))
1120 Et(3,1)=Et(3,1)+Pt*Sp*Te4r*Pld(L)/L/(L-1)
1130 Et(3,2)=Et(3,2)+Pt*Sp*Te4i*Pld(L)/L/(L-1)
1140 Bp(1,1)=Bp(1,1)+(2*L-1)*Bt(1,1)
1150 Bp(1,2)=Bp(1,2)+(2*L-1)*Bt(1,2)
1160 Bp(2,1)=Bp(2,1)+(2*L-1)*Bt(2,1)
1170 Bp(2,2)=Bp(2,2)+(2*L-1)*Bt(2,2)
1180 Bp(3,1)=Bp(3,1)+(2*L-1)*Bt(3,1)
1190 Bp(3,2)=Bp(3,2)+(2*L-1)*Bt(3,2)
1200 Ep(1,1)=Ep(1,1)-(2*L-1)*Et(1,1)
1210 Ep(1,2)=Ep(1,2)-(2*L-1)*Et(1,2)
1220 Ep(2,1)=Ep(2,1)+(2*L-1)*Et(2,1)
1230 Ep(2,2)=Ep(2,2)+(2*L-1)*Et(2,2)
1240 Ep(3,1)=Ep(3,1)+(2*L-1)*Et(3,1)
1250 Ep(3,2)=Ep(3,2)+(2*L-1)*Et(3,2)
1260 Ner1=Et(1,1)*Et(1,1)+Et(1,2)*Et(1,2)
1270 Ner2=Et(2,1)*Et(2,1)+Et(2,2)*Et(2,2)
1280 Ner3=Et(3,1)*Et(3,1)+Et(3,2)*Et(3,2)
1290 Dne1=Ep(1,1)*Ep(1,1)+Ep(1,2)*Ep(1,2)
1300 Dne2=Ep(2,1)*Ep(2,1)+Ep(2,2)*Ep(2,2)
1310 Dne3=Ep(3,1)*Ep(3,1)+Ep(3,2)*Ep(3,2)
1320 Nbr1=Bt(1,1)*Bt(1,1)+Bt(1,2)*Bt(1,2)
1330 Nbr2=Bt(2,1)*Bt(2,1)+Bt(2,2)*Bt(2,2)
1340 Nbr3=Bt(3,1)*Bt(3,1)+Bt(3,2)*Bt(3,2)
1350 Dnb1=Bp(1,1)*Bp(1,1)+Bp(1,2)*Bp(1,2)
1360 Dnb2=Bp(2,1)*Bp(2,1)+Bp(2,2)*Bp(2,2)
1370 Dnb3=Bp(3,1)*Bp(3,1)+Bp(3,2)*Bp(3,2)
1380 IF L=Lmax THEN 1600
1390 IF Dnb1=0 THEN 1410
1400 IF Nbr1/Dnb1<Err*Err THEN
1410   IF Dnb2=0 THEN 1430
1420   IF Nbr2/Dnb2<Err*Err THEN
1430     IF Dnb3=0 THEN 1450
1440     IF Nbr3/Dnb3<Err*Err THEN
1450       IF Dne1=0 THEN 1470
1460       IF Ner1/Dne1<Err*Err THEN
1470         IF Dne2=0 THEN 1490
1480         IF Ner2/Dne2<Err*Err THEN
1490           IF Dne3=0 THEN 1510
1500           IF Ner3/Dne3<Err*Err THEN
1510             GOTO 1600
1520           END IF
1530         END IF

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1540   END IF
1550   END IF
1560   END IF
1570 END IF
1580 L=L+1
1590 GOTO 580
1600 Tmpr=Emfacr*Bp(1,1)-Emfaci*Bp(1,2)
1610 Tmpi=Emfacr*Bp(1,2)+Emfaci*Bp(1,1)
1620 Bp(1,1)=Tmpr
1630 Bp(1,2)=Tmpi
1640 Tmpr=Eefacr*Ep(1,1)-Eefaci*Ep(1,2)
1650 Tmpi=Eefacr*Ep(1,2)+Eefaci*Ep(1,1)
1660 Ep(1,1)=Tmpr
1670 Ep(1,2)=Tmpi
1680 Tmpr=Emfacr*Bp(2,1)-Emfaci*Bp(2,2)
1690 Tmpi=Emfacr*Bp(2,2)+Emfaci*Bp(2,1)
1700 Bp(2,1)=Tmpr
1710 Bp(2,2)=Tmpi
1720 Tmpr=Eefacr*Ep(2,1)-Eefaci*Ep(2,2)
1730 Tmpi=Eefacr*Ep(2,2)+Eefaci*Ep(2,1)
1740 Ep(2,1)=Tmpr
1750 Ep(2,2)=Tmpi
1760 Tmpr=Emfacr*Bp(3,1)-Emfaci*Bp(3,2)
1770 Tmpi=Emfacr*Bp(3,2)+Emfaci*Bp(3,1)
1780 Bp(3,1)=Tmpr
1790 Bp(3,2)=Tmpi
1800 Tmpr=Eefacr*Ep(3,1)-Eefaci*Ep(3,2)
1810 Tmpi=Eefacr*Ep(3,2)+Eefaci*Ep(3,1)
1820 Ep(3,1)=Tmpr
1830 Ep(3,2)=Tmpi
1840 CALL Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
1850 PRINT E(*) !Electric field in original frame
1860 PRINT
1870 PRINT B(*) !Magnetic field in original frame
1880 END

```

PROGRAM MGDIPSPHDC

```

10  OPTION BASE 1
20  DIM Xf(3),Xd(3),M(3),Rds(3,3),Bp(3),Ep(3),B(3),E(3)
30  DIM Pl(100),Pld(100),Et(3),Bt(3)
40  INPUT "MEDIUM AND SPHERE CONDUCTIVITIES(MHO/M)?",Sm,Ss
50  INPUT "RELATIVE PERMEABILITY OF MEDIUM AND SPHERE?",Mm,Ms
60  INPUT "POSITION OF FIELD POINT?(M)",Xf(*)
70  INPUT "POSITION OF SOURCE POINT?(M)",Xd(*)
80  INPUT "MAGNETIC DIPOLE MOMENT VECTOR?(AMP-M^2)",M(*)
90  INPUT " SPHERE RADIUS(M) AND MAXIMUM POLAR ANGLE INDEX?",A,Ell
100 Err=1.E-6
110 Lmax=Ell+1
120 T=Ms/Mm
130 M0=4*PI*1.E-7

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140 !Only medium constants to here
150 CALL Geomdipsph(Xd(*),M(*),Rds(*),R0,Mr,Mt)
160 CALL Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
170 Facm=M0*Mm*(T-1)/4/PI/A/R/R0
180 REDIM Pl(Lmax),Pld(Lmax)
190 CALL Pl_pldot(Lmax-1,Ct,Pl(*),Pld(*))
200 MAT Ep= (0)
210 MAT Bp= (0)
220 Rat=A*A/R/R0
230 Ratl=Rat
240 L=2
250 MAT Bt= (0)
260 Ratl=Ratl*Rat
270 Dt=1/((T+1)*(L-1)+1)
280 Bt(1)=-Mt*St*Cp*L*(L-1)*Ratl*Pld(L)*Dt
290 Bt(1)=Bt(1)+Mr*L*L*(L-1)*Ratl*Pl(L)*Dt
300 Bt(2)=Mt*Cp*(L-1)*Ratl*(L*(L-1)*Pl(L)-Ct*Pld(L))*Dt
310 Bt(2)=Bt(2)+Mr*St*L*(L-1)*Pld(L)*Ratl*Dt
320 Bt(3)=-Mt*Sp*(L-1)*Ratl*Pld(L)*Dt
330 MAT Bp= Bp+Bt
340 Nbr1=Bt(1)*Bt(1)
350 Nbr2=Bt(2)*Bt(2)
360 Nbr3=Bt(3)*Bt(3)
370 Dnb1=Bp(1)*Bp(1)
380 Dnb2=Bp(2)*Bp(2)
390 Dnb3=Bp(3)*Bp(3)
400 IF L=Lmax THEN 530
410 IF Dnb1=0 THEN 430
420 IF Nbr1/Dnb1<Err*Err THEN
430   IF Dnb2=0 THEN 450
440   IF Nbr2/Dnb2<Err*Err THEN
450     IF Dnb3=0 THEN 470
460     IF Nbr3/Dnb3<Err*Err THEN
470       GOTO 530
480     END IF
490   END IF
500 END IF
510 L=L+1
520 GOTO 250
530 Bp(1)=Facm*Bp(1)
540 Bp(2)=Facm*Bp(2)
550 Bp(3)=Facm*Bp(3)
560 CALL Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
570 PRINT E(*) !DC electric field in original frame(volt/meter)
580 PRINT
590 PRINT B(*) !DC magnetic field in original frame(tesla)
600 END

```

PROGRAM CRDIPSPHDC

```

10  OPTION BASE 1
20  DIM Xf(3),Xd(3),P(3),Rds(3,3),Bp(3),Ep(3),B(3),E(3)
30  DIM Pl(100),Pld(100),Et(3),Bt(3)
40  INPUT "MEDIUM AND SPHERE CONDUCTIVITIES(MHO/M)?",Sm,Ss
50  INPUT "RELATIVE PERMEABILITY OF MEDIUM AND SPHERE?",Mm,Ms
60  INPUT "POSITION OF FIELD POINT?(M)",Xf(*)
70  INPUT "POSITION OF SOURCE POINT?(M)",Xd(*)
80  INPUT "CURRENT DIPOLE MOMENT VECTOR?(AMP-M)",P(*)
90  INPUT " SPHERE RADIUS(M) AND MAXIMUM POLAR ANGLE INDEX?",A,Ell
100 Err=1.E-6
110 Lmax=Ell+1
120 M0=4*PI*1.E-7
130 Facm=M0*Mm/4/PI/A
140 Face=1/4/PI/A/Sm
141 T=Ms/Mm
150 Ee=Ss/Sm
170 !Only medium constants to here
180 CALL Geomdipsph(Xd(*),P(*),Rds(*),R0,Pr,Pt)
190 CALL Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
200 REDIM Pl(Lmax),Pld(Lmax)
210 CALL Pl_pldot(Lmax-1,Ct,Pl(*),Pld(*))
220 MAT Ep= (0)
230 MAT Bp= (0)
240 Rat=A*A/R/R0
250 Ratl=Rat
260 L=2
270 MAT Et= (0)
280 MAT Bt= (0)
290 Ratl=Ratl*Rat
300 Dt=1/((T+1)*(L-1)+1)
310 De=1/((Ee+1)*(L-1)+1)
320 Bt(1)=(T-1)*Pt*St*Sp*L*Ratl*Pld(L)*Dt
330 Et(1)=(Ee-1)*Pr*L*(L-1)*Pl(L)*Ratl*De
340 Et(1)=Et(1)-(Ee-1)*Pt*Cp*St*L*(L-1)*Ratl*Pld(L)*De
350 Bt(2)=Pt*Sp*(Ee-1)*Ratl*Pld(L)/R0*De
360 Bt(2)=Bt(2)-Pt*Sp*(T-1)*(L*(L-1)*Pl(L)-Ct*Pld(L))*Ratl/R*Dt
370 Et(2)=(Ee-1)*Pr*St*L*(L-1)*Ratl*Pld(L)*De
380 Et(2)=Et(2)+(Ee-1)*Pt*Cp*(L-1)*Ratl*(L*(L-1)*Pl(L)-Ct*Pld(L))*De
390 Bt(3)=(Ee-1)*Pr*St*L*Ratl*Pld(L)*De/R0
400 Bt(3)=Bt(3)+(Ee-1)*Pt*Cp*(L*(L-1)*Pl(L)-Ct*Pld(L))*Ratl/R0*De
410 Bt(3)=Bt(3)-(T-1)*Pt*Cp*Ratl*Pld(L)/R*Dt
420 Et(3)=-Pt*Sp*(Ee-1)*(L-1)*Ratl*Pld(L)*De
430 MAT Bp= Bp+Bt
440 MAT Ep= Ep+Et
450 Ner1=Et(1)*Et(1)
460 Ner2=Et(2)*Et(2)
470 Ner3=Et(3)*Et(3)
480 Dne1=Ep(1)*Ep(1)
490 Dne2=Ep(2)*Ep(2)
500 Dne3=Ep(3)*Ep(3)

```

```

510 Nbr1=Bt(1)*Bt(1)
520 Nbr2=Bt(2)*Bt(2)
530 Nbr3=Bt(3)*Bt(3)
540 Dnb1=Bp(1)*Bp(1)
550 Dnb2=Bp(2)*Bp(2)
560 Dnb3=Bp(3)*Bp(3)
570 IF L=Lmax THEN 790
580 IF Dnb1=0 THEN 600
590 IF Nbr1/Dnb1<Err*Err THEN
600   IF Dnb2=0 THEN 620
610   IF Nbr2/Dnb2<Err*Err THEN
620     IF Dnb3=0 THEN 640
630     IF Nbr3/Dnb3<Err*Err THEN
640       IF Dne1=0 THEN 660
650       IF Ner1/Dne1<Err*Err THEN
660         IF Dne2=0 THEN 680
670         IF Ner2/Dne2<Err*Err THEN
680           IF Dne3=0 THEN 700
690           IF Ner3/Dne3<Err*Err THEN
700             GOTO 790
710           END IF
720         END IF
730       END IF
740     END IF
750   END IF
760 END IF
770 L=L+1
780 GOTO 270
790 Bp(1)=Facm*Bp(1)/R
800 Ep(1)=Face*Ep(1)/R/R0
810 Bp(2)=Facm*Bp(2)
820 Ep(2)=Face*Ep(2)/R/R0
830 Bp(3)=Facm*Bp(3)
840 Ep(3)=Face*Ep(3)/R0/R
850 CALL Geomfldb_e(Ct,St,C_r,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
860 PRINT E(*) !DC electric field in original frame(volt/meter)
870 PRINT
880 PRINT B(*) !DC magnetic field in original frame(tesla)
890 END

```

```

10 SUB Geomdipsph(D(*),V(*),R(*),Dm,Vr,Vt)
20 !
30 !Produces transformation R(*) from general frame with source com-
40 !ponents D(*) and moment components V(*) to the standard frame
50 !with source on the z-axis, radial component Vr, and tangential
60 !component Vt oriented along the positive x-axis.
70 !
80 OPTION BASE 1
90 DIM R1(3,3),R2(3,3),R3(3,3),Rt(3,3),Vtm(3)
100 MAT R1= (0)
110 MAT R2= (0)

```

```

120  MAT R3= (0)
130  Dh=SQR(D(1)*D(1)+D(2)*D(2))
140  Dm=SQR(Dh*Dh+D(3)*D(3))
150  Std=Dh/Dm
160  Ctd=D(3)/Dm
170  IF Std>1.E-12 THEN
180    Cpd=D(1)/Dh
190    Spd=D(2)/Dh
200    R1(1,1)=Cpd
210    R1(1,2)=Spd
220    R1(2,1)=-Spd
230    R1(2,2)=Cpd
240    R1(3,3)=1
250    R2(1,1)=Ctd
260    R2(1,3)=-Std
270    R2(2,2)=1
280    R2(3,1)=Std
290    R2(3,3)=Ctd
300    MAT Rt= R2*R1
310  ELSE
320    MAT Rt= IDN
330  END IF
340  MAT Vtm= Rt*V
350  Vh=SQR(Vtm(1)*Vtm(1)+Vtm(2)*Vtm(2))
360  Vm=SQR(Vh*Vh+Vtm(3)*Vtm(3))
370  IF Vh/Vm>1.E-12 THEN
380    Cpv=Vtm(1)/Vh
390    Spv=Vtm(2)/Vh
400    R3(1,1)=Cpv
410    R3(1,2)=Spv
420    R3(3,1)=0
430    R3(2,1)=-Spv
440    R3(2,2)=Cpv
450    R3(3,3)=1
460  ELSE
470    MAT R3= IDN
480  END IF
490  MAT R= R3*Rt
500  Vt=Vh
510  Vr=Vtm(3)
520  SUBEND

10  SUB Geomfldpos(Xf(*),Rds(*),R,Ct,St,Cp,Sp)
20    !
30    !Transforms field point in general frame to special dipole-
40    !Sphere frame and converts it to polar coordinates in that
50    !Frame.
60    !
70  OPTION BASE 1
80  DIM X(3)
90  MAT X= Rds*Xf
100  Rh=SQR(X(1)*X(1)+X(2)*X(2))

```

```

110 R=SQR(Rh*Rh+X(3)*X(3))
120 St=Rh/R
130 Ct=X(3)/R
140 IF St>1.E-12 THEN
150   Cp=X(1)/Rh
160   Sp=X(2)/Rh
170 ELSE
180   Cp=1
190   Sp=0
200 END IF
210 SUBEND

10 SUB Geomfldb_e(Ct,St,Cp,Sp,Bp(*),Ep(*),Rds(*),B(*),E(*))
20   !
30   !Transforms complex polar coordinate fields in the special
40   !Dipole-sphere frame to cartesian coordinates in that frame
50   !And then rotates the fields back to the original general
60   !Frame.
70   !
80   OPTION BASE 1
90   DIM Irds(3,3),Pc(3,3),Et(3,2),Bt(3,2)
100  MAT Irds= INV(Rds)
110  Pc(1,1)=St*Cp
120  Pc(1,2)=Ct*Cp
130  Pc(1,3)=-Sp
140  Pc(2,1)=St*Sp
150  Pc(2,2)=Ct*Sp
160  Pc(2,3)=Cp
170  Pc(3,1)=Ct
180  Pc(3,2)=-St
190  Pc(3,3)=0
200  MAT Bt= Pc*Bp
210  MAT Et= Pc*Ep
220  MAT B= Irds*Bt
230  MAT E= Irds*Et
240 SUBEND

10 SUB Jcomb(L,Ur,Ui,R,Jr, Ji, Jcr, Jci)
20  CALL Spherejnz(L,Ur,Ui,Jr, Ji)
30  CALL Spherejnz(L+1,Ur,Ui,J1r,J1i)
40  Jcr=((L+1)*Jr-(Ur*J1r-Ui*J1i))/R
50  Jci=((L+1)*Ji-(Ur*J1i+Ui*J1r))/R
60 SUBEND

10 SUB Hcomb(L,Ur,Ui,R,Hr,Hi,Hcr,Hci)
20  CALL Spherehznz(L,Ur,Ui,Hr,Hi)
30  CALL Spherehznz(L+1,Ur,Ui,H1r,H1i)
40  Hcr=((L+1)*Hr-(Ur*H1r-Ui*H1i))/R
50  Hci=((L+1)*Hi-(Ur*H1i+Ui*H1r))/R
60 SUBEND

```

```

10 SUB Spherejnz(N,Zr,Zi,Sjr,Sji)
20 CALL Jnuevrywhr(N+.5,Zr,Zi,Jr, Ji)
30 Dm=Zr*Zr+Zi*Zi
40 Ur=Zr/Dm
50 Ui=-Zi/Dm
60 Nm=SQR(Ur*Ur+Ui*Ui)
70 Sur=SQR((Nm+Ur)/2)
80 Sui=SGN(Ui)*SQR(ABS(Nm-Ur)/2)
90 Sjr=SQR(PI/2)*(Sur*Jr-Sui*Ji)
100 Sji=SQR(PI/2)*(Sur*Ji+Sui*Jr)
110 SUBEND

```

```

10 SUB Spherehznz(N,Zr,Zi,Shr,Shi)
20 CALL Hnuevrywhr(N+.5,Zr,Zi,Hr,Hi)
30 Dm=Zr*Zr+Zi*Zi
40 Ur=Zr/Dm
50 Ui=-Zi/Dm
60 Nm=SQR(Ur*Ur+Ui*Ui)
70 Sur=SQR((Nm+Ur)/2)
80 Sui=SGN(Ui)*SQR(ABS(Nm-Ur)/2)
90 Shr=SQR(PI/2)*(Sur*Hr-Sui*Hi)
100 Shi=SQR(PI/2)*(Sur*Hi+Sui*Hr)
110 SUBEND

```

```

10 SUB Jnuevrywhr(Nu,Zr,Zi,Jr, Ji)
20 Zmag=SQR(Zr*Zr+Zi*Zi)
30 IF Nu<10 THEN
40   IF Zmag<10 THEN
50     CALL Jnu(Nu,Zr,Zi,Jr, Ji, 1.E-28)
60   ELSE
70     IF Nu=0 THEN
80       CALL Jnasy(Nu,Zr,Zi,Jr, Ji, 5)
90     ELSE
100      CALL Jfiuniasym(Nu,Zr,Zi,Jr, Ji, 5)
110    END IF
120  END IF
130 ELSE
140   CALL Jfiuniasym(Nu,Zr,Zi,Jr, Ji, 5)
150 END IF
160 SUBEND

```

```

10 SUB Hnuevrywhr(Nu,Zr,Zi,Hr,Hi)
20 Zmag=SQR(Zr*Zr+Zi*Zi)
30 IF Nu<10 THEN
40   IF Zmag<10 THEN
50     CALL H1nu(Nu,Zr,Zi,Hr,Hi, 1.E-28)
60   ELSE
70     IF Nu=0 THEN
80       CALL H1nasy(Nu,Zr,Zi,Hr,Hi, 5)
90     ELSE
100      CALL Hfkuniasym(Nu,Zr,Zi,Hr,Hi, 5)
110    END IF

```

```

120  END IF
130  ELSE
140  CALL Hfkuniasym(Nu,Zr,Zi,Hr,Hi,5)
150  END IF
160  SUBEND

10  SUB Pl_pldot(N,X,Pl(*),Pldot(*))
20  Pl(1)=1
30  IF N=0 THEN 60
40  Pl(2)=X
50  IF N=1 THEN 80
60  Pldot(1)=0
70  IF N=0 THEN 170
80  Pldot(2)=1
90  IF N=1 THEN 170
100  FOR L=3 TO N+1
110  C1=1/(L-1)
120  C2=1-C1
130  C3=2-C1
140  Pl(L)=C3*X*Pl(L-1)-C2*Pl(L-2)
150  Pldot(L)=(C3*X*Pldot(L-1)-Pldot(L-2))/C2
160  NEXT L
170  SUBEND

10  SUB Nde(Lmax,K1r,K1i,K2r,K2i,Mur,R,Ndea(*))
20  U1r=K1r*R
30  U1i=K1i*R
40  U2r=K2r*R
50  U2i=K2i*R
60  FOR L=1 TO Lmax
70  ON ERROR GOTO 110
80  CALL Spherejnz(L-1,U1r,U1i,J1r,J1i)
90  CALL Spherejnz(L,U1r,U1i,J11r,J11i)
100  GOTO 140
110  J1ratr=0
120  J1rati=1
130  GOTO 190
140  Dm=J1r*J1r+J1i*J1i
150  Rpr=J1r/Dm
160  Rpi=-J1i/Dm
170  J1ratr=Rpr*J11r-Rpi*J11i
180  J1rati=Rpr*J11i+Rpi*J11r
190  OFF ERROR
200  CALL Spherejnz(L-1,U2r,U2i,J2r,J2i)
210  CALL Spherejnz(L,U2r,U2i,J21r,J21i)
220  CALL Spherehnz(L-1,U2r,U2i,H2r,H2i)
230  CALL Spherehnz(L,U2r,U2i,H21r,H21i)
240  Nb1r=L*J2r-(U2r*J21r-U2i*J21i)
250  Nb1i=L*J2i-(U2r*J21i+U2i*J21r)
260  Nb2r=L-(U1r*J1ratr-U1i*J1rati)
270  Nb2i=-(U1i*J1rati+U1i*J1ratr)
280  Topr=Mur*Nb1r-(J2r*Nb2r-J2i*Nb2i)

```

```

290  Topi=Mur*Nb1i-(J2r*Nb2i+J2i*Nb2r)
300  Db1r=L*H2r-(U2r*H21r-U2i*H21i)
310  Db1i=L*H2i-(U2r*H21i+U2i*H21r)
320  Btmr=Mur*Db1r-(H2r*Nb2r-H2i*Nb2i)
330  Btmi=Mur*Db1i-(H2r*Nb2i+H2i*Nb2r)
340  Bm=Btmr*Btmr+Btmi*Btmi
350  Rbmr=Btmr/Bm
360  Rbmi=-Btmi/Bm
370  Ndea(L,1)=- (Topr*Rbmr-Topi*Rbmi)
380  Ndea(L,2)=- (Topr*Rbmi+Topi*Rbmr)
390  NEXT L
400  SUBEND

10  SUB Ndm(Lmax,K1r,K1i,K2r,K2i,Mur,R,Ndma(*))
20  U1r=K1r*R
30  U1i=K1i*R
40  U2r=K2r*R
50  U2i=K2i*R
60  K1sqr=K1r*K1r-K1i*K1i
70  K1sqi=2*K1r*K1i
80  K2sqr=K2r*K2r-K2i*K2i
90  K2sqi=2*K2r*K2i
100 Dm=K1r*K1r+K1i*K1i
110 Rpr=K1r/Dm
120 Rpi=-K1i/Dm
130 Ep2r=Rpr*K2r-Rpi*K2i
140 Ep2i=Rpr*K2i+Rpi*K2r
150 FOR L=1 TO Lmax
160   ON ERROR GOTO 200
170   CALL Spherejnz(L-1,U1r,U1i,J1r,J1i)
180   CALL Spherejnz(L,U1r,U1i,J11r,J11i)
190   GOTO 230
200   J1ratr=0
210   J1rati=1
220   GOTO 280
230   Dm=J1r*J1r+J1i*J1i
240   Rpr=J1r/Dm
250   Rpi=-J1i/Dm
260   J1ratr=Rpr*J11r-Rpi*J11i
270   J1rati=Rpr*J11i+Rpi*J11r
280   OFF ERROR
290   CALL Spherejnz(L-1,U2r,U2i,J2r,J2i)
300   CALL Spherejnz(L,U2r,U2i,J21r,J21i)
310   CALL Spherehznz(L-1,U2r,U2i,H2r,H2i)
320   CALL Spherehznz(L,U2r,U2i,H21r,H21i)
330   Nb1r=L*J2r-(U2r*J21r-U2i*J21i)
340   Nb1i=L*J2i-(U2r*J21i+U2i*J21r)
350   Nb2r=L-(U1r*J1ratr-U1i*J1rati)
360   Nb2i=-(U1r*J1rati+U1i*J1ratr)
370   Nfacr=Mur*(Ep2r*J2r-Ep2i*J2i)
380   Nfaci=Mur*(Ep2r*J2i+Ep2i*J2r)
390   Topr=Nb1r-(Nfacr*Nb2r-Nfaci*Nb2i)

```

```

400  Topi=Nb1i-(Nfacr*Nb2i+Nfaci*Nb2r)
410  Db1r=L*H2r-(U2r*H21r-U2i*H21i)
420  Db1i=L*H2i-(U2r*H21i+U2i*H21r)
430  Dfacr=Mur*(Ep2r*H2r-Ep2i*H2i)
440  Dfaci=Mur*(Ep2r*H2i+Ep2i*H2r)
450  Btmr=Db1r-(Dfacr*Nb2r-Dfaci*Nb2i)
460  Btmi=Db1i-(Dfacr*Nb2i+Dfaci*Nb2r)
470  Bm=Btmr*Btmr+Btmi*Btmi
480  Rbmr=Btmr/Bm
490  Rbmi=-Btmi/Bm
500  Ndma(L,1)=- (Topr*Rbmr-Topi*Rbmi)
510  Ndma(L,2)=- (Topr*Rbmi+Topi*Rbmr)
520  NEXT L
530  SUBEND

```

```

10  SUB Ndeicon(Lmax,K2r,K2i,R,Ndea(*))
20  U2r=K2r*R
30  U2i=K2i*R
40  FOR L=1 TO Lmax
50  CALL Spherejnz(L-1,U2r,U2i,J2r,J2i)
60  CALL Spherehnz(L-1,U2r,U2i,H2r,H2i)
70  Den=H2r*H2r+H2i*H2i
80  Ih2r=H2r/Den
90  Ih2i=-H2i/Den
100  Ndea(L,1)=- (J2r*Ih2r-J2i*Ih2i)
110  Ndea(L,2)=- (J2r*Ih2i+J2i*Ih2r)
120  NEXT L
130  SUBEND

```

```

140  SUB Ndmicon(Lmax,K2r,K2i,R,Ndma(*))
150  U2r=K2r*R
160  U2i=K2i*R
170  FOR L=1 TO Lmax
180  CALL Spherejnz(L-1,U2r,U2i,J2r,J2i)
190  CALL Spherejnz(L,U2r,U2i,J21r,J21i)
200  CALL Spherehnz(L-1,U2r,U2i,H2r,H2i)
210  CALL Spherehnz(L,U2r,U2i,H21r,H21i)
220  A=U2r*H21r-U2i*H21i
230  B=U2r*H21i+U2i*H21r
240  C=H2r*L-A
250  D=H2i*L-B
260  Den=C*C+D*D
270  Ihr=C/Den
280  Ihi=-D/Den
290  A=U2r*J21r-U2i*J21i
300  B=U2r*J21i+U2i*J21r
310  C=J2r*L-A
320  D=J2i*L-B
330  Ndma(L,1)=- (C*Ihr-D*Ihi)
340  Ndma(L,2)=- (C*Ihi+D*Ihr)
350  NEXT L
360  SUBEND

```

SERIES REPRESENTATIONS($\nu < 10, |z| < 10$)

The Bessel functions of the first kind $J_\nu(z)$ can be constructed from the series representation

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^k}{k! \Gamma(\nu + k + 1)}. \quad (\text{A1})$$

The Hankel function, or Bessel function of the third kind for "outgoing waves", $H_\nu^{(1)}(z)$ is related to $J_\nu(z)$ and the Bessel functions of the second kind, $Y_\nu(z)$, by

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z). \quad (\text{A2})$$

For non-integral values of ν , $Y_\nu(z)$ can be defined by

$$Y_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \quad (\text{A3})$$

but for integer values of ν , it is necessary to use the more complicated expression

$$Y_n(z) = -\frac{\left(\frac{z}{2}\right)^n}{\pi} \sum_{k=0}^{n-1} (n-k-1) \frac{1}{k!} \left(\frac{z^2}{4}\right)^k + \frac{2}{\pi} \ln\left(\frac{z}{2}\right) J_n(z) - \frac{\left(\frac{z}{2}\right)^n}{\pi} \sum_{k=0}^{\infty} \{\psi(k+1) + \psi(n+k+1)\} \frac{\left(-\frac{z^2}{4}\right)^k}{k!(n+k)!} \quad (\text{A4})$$

where $\psi(n)$ is given by

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \quad (n \geq 2), \quad (\text{A5})$$

and γ is the Euler constant 0.577215664901532860606512.

In the following seven subroutines, the functions $J_\nu(z)$ and $H_\nu^{(1)}(z)$ are generated. Use of these routines has been restricted to cases where the magnitudes of ν and z do not exceed 10. For larger values of ν and z , various asymptotic techniques are used. The discussion of these begins below, after the series expansion routines.

```

10 SUB Jn(N,Zzr,Zzi,Jnr,Jni,Err)
20   !N is the (Integer) order.
30   !Zzr and Zzi are the real and imaginary parts of the argument.
40   !Jnr and Jni are the real and imaginary parts of the Bessel function.
50   !Err is the relative truncation error in the truncation of the series
60   !expansion of the Bessel function.
70   M=0
80   Zr=Zzr/2
90   Zi=Zzi/2
100  U2=Zi*Zi-Zr*Zr
110  V2=-2*Zr*Zi
120  IF N=0 THEN
130    Tr=1
140    Ti=0
150    Dr=Tr
160    Di=Ti
170  ELSE
180    U=1
190    V=0
200    FOR L=1 TO N
210      Ut=U*Zr-V*Zi
220      Vt=U*Zi+V*Zr
230      U=Ut/L
240      V=Vt/L
250    NEXT L
260    Tr=U
270    Ti=V
280    Dr=Tr
290    Di=Ti
300  END IF
310  M=M+1
320  Ttr=(Tr*U2-Ti*V2)/M/(M+N)
330  Tti=(Tr*V2+Ti*U2)/M/(M+N)
340  Tr=Ttr
350  Ti=Tti
360  Dr=Dr+Tr
370  Di=Di+Ti
380  IF (Tr*Tr+Ti*Ti)/(Dr*Dr+Di*Di)<Err*Err THEN 400
390  GOTO 310
400  Jnr=Dr
410  Jni=Di
420 SUBEND

```

```

10 SUB H1n(N,Zzr,Zzi,H1nr,H1ni,Err)
20   !N is the (Integer) order
30   !Zzr and Zzi are the real and imaginary parts of the argument.
40   !H1nr and H1ni are the real and imaginary parts of the Bessel
50   !function. Err is the relative truncation error in the truncation
60   !of the series expansion of the Bessel function.
70   M=0

```

```

80  Zr=Zzr/2
90  Zi=Zzi/2
100 P1=0
110 P2=0
120 CALL Prinlogz(Zr,Zi,Lr,Li)
130 Gam=.577215664901533
140 Cr=PI-2*Li
150 Cii=2*(Gam+Lr)
160 U2=Zi*Zi-Zr*Zr
170 V2=-2*Zr*Zi
180 IF N=0 THEN
190   Sr=0
200   Si=0
210   Pwr=1
220   Pwi=0
230   Dr=Cr
240   Di=Cii
250 ELSE
260   U=1
270   V=0
280   Nm=1
290   FOR L=1 TO N
300     Ut=U*Zr-V*Zi
310     Vt=U*Zi+V*Zr
320     U=Ut/L
330     V=Vt/L
340     P2=P2+1/L
350     Nm=Nm*L
360   NEXT L
370   Dn=U*U+V*V
380   Uu=-V/Dn/Nm
390   Vv=-U/Dn/Nm
400   Nm=Nm/N
410   Pwr=U
420   Pwi=V
430   Sr=Nm*Uu
440   Si=Nm*Vv
450   Dr=Cr*Pwr-(Cii-P2)*Pwi+Sr
460   Di=Cr*Pwi+(Cii-P2)*Pwr+Si
470 END IF
480 M=M+1
490 P1=P1+1/M
500 P2=P2+1/(N+M)
510 Pwrt=(Pwr*U2-Pwi*V2)/M/(M+N)
520 Pwit=(Pwr*V2+Pwi*U2)/M/(M+N)
530 Pwr=Pwrt
540 Pwi=Pwit
550 Ci=Cii-P1-P2
560 Tr=Pwr*Cr-Pwi*Ci
570 Ti=Pwi*Cr+Pwr*Ci
580 Dr=Dr+Tr
590 Di=Di+Ti

```

```

600 IF M<N THEN
610   Ssr=-(Sr*U2-Si*V2)/M/(N-M)
620   Ssi=-(Sr*V2+Si*U2)/M/(N-M)
630   Sr=Ssr
640   Si=Ssi
650 ELSE
660   Sr=0
670   Si=0
680 END IF
690 Dr=Dr+Sr
700 Di=Di+Si
710 IF (Tr*Tr+Ti*Ti)/(Dr*Dr+Di*Di)<Err*Err THEN 730
720 GOTO 480
730 H1nr=Dr/PI
740 H1ni=Di/PI
750 SUBEND

```

```

10 SUB Jnu(Nu,Zxr,Zzi,Jnur,Jnui,Err)
20   !Nu is the order
30   !Zxr and Zzi are the real and imaginary parts of the argument.
40   !Jnur and Jnui are the real and imaginary parts of the Bessel
50   !function. Err is the relative truncation error in the truncation
60   !of the series expansion of the Bessel function.
70   M=0
80   Zr=Zxr/2
90   Zi=Zzi/2
100  U2=Zi*Zi-Zr*Zr
110  V2=-2*Zr*Zi
120  U=1
130  V=0
140  Dn=1
150  Dr=1
160  Di=0
170  M=M+1
180  Ut=U*U2-V*V2
190  Vt=U*V2+V*U2
200  U=Ut
210  V=Vt
220  Dn=Dn*(Nu+M)*M
230  Tr=U/Dn
240  Ti=V/Dn
250  Dr=Dr+Tr
260  Di=Di+Ti
270  IF (ABS(Tr)+ABS(Ti))/(ABS(Dr)+ABS(Di))<Err THEN 290
280  GOTO 170
290  Jnur=Dr
300  Jnui=Di
310  N=INT(Nu)
320  Eps=Nu-N
330  CALL Gamma(Nu+1,G0)

```

```

340 CALL Prinlogz(Zr,Zi,Lzr,Lzi)
350 Mo=EXP(Eps*Lzr)
360 Ph=Eps*Lzi
370 Cr=Mo*COS(Ph)
380 Ci=Mo*SIN(Ph)
390 M=0
400 Jr=(Cr*Jnur-Ci*Jnui)/G0
410 Ji=(Cr*Jnui+Ci*Jnur)/G0
420 U=1
430 V=0
440 IF N<0 THEN 570
450 IF N=0 THEN 670
460 U=Zr
470 V=Zi
480 M=M+1
490 IF M=N THEN 670
500 Ut=U*Zr-V*Zi
510 Vt=U*Zi+V*Zr
520 U=Ut
530 V=Vt
540 M=M+1
550 IF M=N THEN 670
560 GOTO 500
570 Ut=U*Zr-V*Zi
580 Vt=U*Zi+V*Zr
590 U=Ut
600 V=Vt
610 M=M-1
620 IF M=N THEN 640
630 GOTO 570
640 Dnm=U*U+V*V
650 U=U/Dnm
660 V=-V/Dnm
670 Jnur=U*Jr-V*Ji
680 Jnui=U*Ji+V*Jr
690 SUBEND

```

```

10 SUB H1nu(Nu,Zr,Zi,H1nur,H1nui,Err)
20 !See Jnu.
30 CALL Jnu(Nu,Zr,Zi,Jr,Ji,Err)
40 CALL Jnu(-Nu,Zr,Zi,Jmr,Jmi,Err)
50 C=COS(Nu*PI)
60 S=SIN(Nu*PI)
70 H1nur=Jr-(C*Ji-Jmi)/S
80 H1nui=Ji+(C*Jr-Jmr)/S
90 SUBEND

```

```

10 SUB Prinlogz(Zr,Zi,Lzr,Lzi)
20 Lzr=LOG(Zr*Zr+Zi*Zi)/2
30 CALL Atn2(Zr,Zi,Lzi)
40 SUBEND

```

```

10 SUB Atn2(X,Y,Theta)
20 IF X=0 THEN
30 IF Y<0 THEN
40 Theta=-PI/2
50 ELSE
60 Theta=PI/2
70 END IF
80 GOTO 200
90 END IF
100 Theta=ATN(Y/X)
110 IF X>=0 THEN
120 Theta=Theta
130 ELSE
140 IF Y>=0 THEN
150 Theta=Theta+PI
160 ELSE
170 Theta=Theta-PI
180 END IF
190 END IF
200 SUBEND

```

```

10 SUB Gamma(X,G)
20 OPTION BASE 1
30 !
40 !Computes Gamma(x) for any real x(where defined).
50 !Precision is 1E-12
60 !
70 DIM A(20),Z(20)
80 IF X=1 THEN
90 G=1
100 GOTO 540
110 END IF
120 IF X>=1 THEN
130 Xx=X
140 Y=1
150 Xx=Xx-1
160 IF Xx>=0 AND Xx<=1 THEN GOTO 280
170 Y=Xx*Y
180 Xx=Xx-1
190 GOTO 160
200 ELSE
210 Xx=X-1
220 Y=1

```

```

230  Xx=Xx+1
240  Y=Y/Xx
250  IF Xx>=0 AND Xx<=1 THEN 280
260  GOTO 230
270  END IF
280  A(1)=1
290  A(2)=.57721566490
300  A(3)=-.65587807152
310  A(4)=-.04200263503
320  A(5)=.16653861138
330  A(6)=-.04219773456
340  A(7)=-.00962197153
350  A(8)=.00721894325
360  A(9)=-.00116516759
370  A(10)=-.00021524167
380  A(11)=.00012805028
390  A(12)=-.00002013485
400  A(13)=-.00000125049
410  A(14)=.00000113303
420  A(15)=-.00000020563
430  A(16)=.00000000612
440  A(17)=.00000000500
450  A(18)=-.00000000118
460  A(19)=.00000000010
470  A(20)=.00000000001
480  Z(1)=Xx
490  FOR I=2 TO 20
500    Z(I)=Xx*Z(I-1)
510  NEXT I
520  Gi=DOT(A,Z)
530  G=Y*Xx/Gi
540  SUBEND

```

ASYMPTOTIC REPRESENTATIONS ($v < 10, |z| > 10$)

For $v < 10$ and large values of z , the following asymptotic expansions are useful

$$J_v(z) = \sqrt{\frac{2}{\pi z}} \{P(v, z) \cos \chi - Q(v, z) \sin \chi\} \quad (\text{A6})$$

and

$$H_v^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \{P(v, z) + iQ(v, z)\} e^{i\chi} \quad (\text{A7})$$

where $\chi = z - \left(\frac{1}{2}v + \frac{1}{2}\right)\pi$ and, with $\mu = 4v^2$, the functions $P(v, z)$ and $Q(v, z)$ are given by

$$P(v, z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{(v, 2k)}{(2z)^{2k}} \quad (A8)$$

$$Q(v, z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{(v, 2k+1)}{(2z)^{2k+1}} \quad (A9)$$

and

$$(v, 0) = 1, \quad (v, 2) = (\mu - 1) \frac{(\mu - 9)}{2!}, \quad (v, 4) = (\mu - 1)(\mu - 9)(\mu - 25) \frac{(\mu - 49)}{4!} \dots \quad (A10)$$

and

$$(v, 1) = \mu - 1, \quad (v, 3) = (\mu - 1)(\mu - 9) \frac{(\mu - 25)}{3!} \dots \quad (A11)$$

The next four subroutines apply these expressions to the calculation of the Bessel functions.

```

10 SUB Jnasy(N,Zr,Zi,Jnr,Jni,Trms)
20   !Note: Valid for arbitrary real N
30   !Zr and Zi real and imaginary parts of argument
40   !Jnr and Jni real and imaginary parts of the Bessel function.
50   !Trms is the number of terms retained in the asymptotic expansion.
60   CALL Pnz(N,Zr,Zi,Pnr,Pni,Trms)
70   CALL Qnz(N,Zr,Zi,Qnr,Qni,Trms)
80   Chir=Zr-(2*N+1)*PI/4
90   Chii=Zi
100  E=EXP(Chii)/2
110  Ei=1/4/E
120  C=COS(Chir)
130  S=SIN(Chir)
140  Cor=C*(E+Ei)
150  Coi=-S*(E-Ei)
160  Sir=S*(E+Ei)
170  Sii=C*(E-Ei)
180  Utlr=Pnr*Cor-Pni*Coi-Qnr*Sir+Qni*Sii
190  Utli=Pnr*Coi+Pni*Cor-Qnr*Sii-Qni*Sir
200  Zm=SQR(Zr*Zr+Zi*Zi)
210  Aa=SQR((Zm+Zr)/2)
220  Bb=SQR((Zm-Zr)/2)
230  Cc=Aa*Aa+Bb*Bb
240  Dd=SQR(2/PI)/Cc
250  Jnr=Dd*(Utlr*Aa+Utli*Bb)
260  Jni=Dd*(-Utlr*Bb+Utli*Aa)
270 SUBEND

```

```

10 SUB H1nasy(N,Zr,Zi,H1nr,H1ni,Trms)
20   !Note: Valid for arbitrary real N
30   !Zr and Zi real and imaginary parts of argument
40   !H1nr and H1ni real and imaginary parts of the Bessel function.
50   !Trms is the number of terms retained in the asymptotic expansion.
60 CALL Pnz(N,Zr,Zi,Pnr,Pni,Trms)
70 CALL Qnz(N,Zr,Zi,Qnr,Qni,Trms)
80 Chir=Zr-(2*N+1)*PI/4
90 Chii=Zi
100 E=EXP(-Chii)
110 C=COS(Chir)
120 S=SIN(Chir)
130 Utlr=(Pnr-Qni)*C-(Pni+Qnr)*S
140 Utli=(Pnr-Qni)*S+(Pni+Qnr)*C
150 Zm=SQR(Zr*Zr+Zi*Zi)
160 Aa=SQR((Zm+Zr)/2)
170 Bb=SGN(Zi)*SQR((Zm-Zr)/2)
180 Cc=Aa*Aa+Bb*Bb
190 Dd=E*SQR(2/PI)/Cc
200 H1nr=Dd*(Utlr*Aa+Utli*Bb)
210 H1ni=Dd*(-Utlr*Bb+Utli*Aa)
220 SUBEND

```

```

10 SUB Pnz(N,Zr,Zi,Pnr,Pni,Trms)
20   !Used with Jnasy and H1nasy.
30   M=0
40   Mu=4*N*N
50   Tr=1
60   Ti=0
70   Dr=Tr
80   Di=Ti
90   U2=64*(Zr*Zr-Zi*Zi)
100  V2=128*Zr*Zi
110  Dn=U2*U2+V2*V2
120  U=-U2/Dn
130  V=V2/Dn
140  M=M+2
150  Ttr=Tr*U-Ti*V
160  Tti=Tr*V+Ti*U
170  Tr=Ttr/M/(M-1)
180  Ti=Tti/M/(M-1)
190  Fac=1
200  FOR I=1 TO 2*M-1 STEP 2
210    Fac=Fac*(Mu-I*I)
220  NEXT I
230  Dr=Dr+Tr*Fac
240  Di=Di+Ti*Fac
250  IF M/2+1<Trms THEN 140

```

```

260 Pnr=Dr
270 Pni=Di
280 SUBEND

```

```

10 SUB Qnz(N,Zr,Zi,Qnr,Qni,Trms)
20 !Used with Jnasy and H1nasy.
30 M=1
40 Mu=4*N*N
50 Dnm=8*(Zr*Zr+Zi*Zi)
60 Tr=Zr/Dnm
70 Ti=-Zi/Dnm
80 Dr=Tr*(Mu-1)
90 Di=Ti*(Mu-1)
100 U2=64*(Zr*Zr-Zi*Zi)
110 V2=128*Zr*Zi
120 Dn=U2*U2+V2*V2
130 U=-U2/Dn
140 V=V2/Dn
150 M=M+2
160 Ttr=Tr*U-Ti*V
170 Tti=Tr*V+Ti*U
180 Tr=Ttr/M/(M-1)
190 Ti=Tti/M/(M-1)
200 Fac=1
210 FOR I=1 TO 2*M-1 STEP 2
220 Fac=Fac*(Mu-I*I)
230 NEXT I
240 Dr=Dr+Tr*Fac
250 Di=Di+Ti*Fac
260 IF (M+1)/2<Trms THEN 150
270 Qnr=Dr
280 Qni=Di
290 SUBEND

```

UNIFORM ASYMPTOTIC REPRESENTATION ($v > 10, |z| > 10, v > |z|$)

Abramowitz and Stegun^{A1} give uniform asymptotic expansions for the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$. These functions can be used to generate the functions $J_\nu(z)$ and $H_\nu^{(1)}(z)$ by means of the identities

$$J_\nu(z) = e^{\frac{i\pi}{2}} I_\nu\left(ze^{\frac{-i\pi}{2}}\right) \quad (\text{A12})$$

and

$$H_\nu^{(1)}(z) = \frac{-2i}{\pi} e^{\frac{i\pi}{2}} K_\nu\left(ze^{\frac{-i\pi}{2}}\right) \quad (\text{A13})$$

where the argument of z is restricted to $-\pi/2 < \arg z < \pi$. This restriction causes no problem, since the argument of z in electromagnetic applications is limited to $0 \leq \arg z < \pi/2$.

The uniform asymptotic expansions for $I_\nu(z)$ and $K_\nu(z)$ are given by ($w = z/\nu$)

$$I_\nu(z) \sim \frac{e^{\nu\eta}}{\sqrt{2\pi\nu\sqrt{1+w^2}}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right\} \quad (\text{A14})$$

and

$$K_\nu(z) \sim \frac{\pi e^{-\nu\eta}}{\sqrt{2\pi\nu\sqrt{1+w^2}}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k u_k(t)}{\nu^k} \right\} \quad (\text{A15})$$

where

$$t = \frac{1}{\sqrt{1+w^2}} \quad \text{and} \quad \eta = \frac{1}{t} + \ln \left(\frac{w}{1+\frac{1}{t}} \right). \quad (\text{A16})$$

The asymptotic representation of the product of J_ν and $H_\nu^{(1)}$ can be constructed from the product of their asymptotic representations, and this will help avoid overflow and underflow problems in computations. The product is

$$J_\nu(z) H_\nu^{(1)}(z') = \frac{-2i}{\pi} e^{\frac{i(\nu-\nu')\pi}{2}} I_\nu \left(z e^{\frac{-i\pi}{2}} \right) K_{\nu'} \left(z' e^{\frac{-i\pi}{2}} \right) \quad (\text{A17})$$

where

$$I_\nu(z) K_{\nu'}(z') \sim \frac{e^{(\nu\eta - \nu'\eta')}}{2\sqrt{\nu\nu'}\sqrt{(1+w^2)(1+w'^2)}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right\} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k u_k(t')}{\nu'^k} \right\}. \quad (\text{A18})$$

The functions $u_k(t)$ are defined by

$$u_0(t) = 1 \quad (\text{A19})$$

and

$$u_{k+1}(t) = \frac{1}{2} t^2 (1-t^2) u_k(t) + \frac{1}{8} \int_0^t d\tau (1-\tau^2) u_k(\tau). \quad (\text{A20})$$

Abramowitz and Stegun^{A1} give the functions u_k for values of $k = 0$ thru 4 and give references for $k = 5$ and 6. The following is a general method for generating u_k for any k . Use the representation

$$u_k(t) = a_{k,0}t^k + a_{k,1}t^{k+2} + \dots + a_{k,i}t^{k+2i} + \dots a_{k,k}t^{k+2k}. \quad (\text{A21})$$

Then the coefficients are given by the following recursion formulas

$$a_{k+1,0} = a_{k,0} \left[\frac{k}{2} + \frac{1}{8(k+1)} \right] \quad (\text{A22})$$

$$a_{k+1,i} = a_{k,i} \left[\frac{k+2i}{2} + \frac{1}{8(k+1+2i)} \right] - a_{k,i-1} \left[\frac{k+2(i-1)}{2} + \frac{5}{8(k+1+2i)} \right] \quad (\text{A23})$$

and

$$a_{k+1,k+1} = -a_{k,k} \left[\frac{k+2k}{2} + \frac{5}{8(k+1+2\{k+1\})} \right] \quad (\text{A24})$$

where $i = 1, 2, \dots, k$.

```

10 SUB Jfiuniasym(Nu,Zr,Zi,Jr,Ji,Trms)
20 !Nu is the order.
30 !Zr and Zi real and imaginary parts of argument
40 !Jr and Ji real and imaginary parts of the Bessel function.
50 !Trms is the number of terms retained in the asymptotic expansion.
60 DIM A(10,10),U(10,2),T(30,2)
70 REDIM A(Trms,Trms),U(Trms,2),T(3*Trms,2)
80 Wr=Zi/Nu
90 Wi=-Zr/Nu
100 Ur=1+Wr*Wr-Wi*Wi
110 Ui=2*Wr*Wi
120 Um=SQR(Ur*Ur+Ui*Ui)
130 Radr=SQR((Um+Ur)/2)
140 Radi=SGN(Ui)*SQR((Um-Ur)/2)
150 Radm=Radi*Radi+Radr*Radr
160 Sradm=SQR(Radm)
170 Radrtr=SQR((Sradm+Radr)/2)
180 Radrti=SGN(Radi)*SQR((Sradm-Radr)/2)
190 Radrtm=Radrtr*Radrtr+Radrti*Radrti
200 Iradrtr=Radrtr/Radrtm
210 Iradrti=-Radrti/Radrtm
220 Tr=Radr/Radm
230 Ti=-Radi/Radm
240 Dnr=1+Radr
250 Dni=Radi
260 Dnm=Dnr*Dnr+Dni*Dni
270 Nmr=Dnr/Dnm
280 Nmi=-Dni/Dnm

```

```

250  Dni=Radi
260  Dnm=Dnr*Dnr+Dni*Dni
270  Nmr=Dnr/Dnm
280  Nmi=-Dni/Dnm
290  Argr=Wr*Nmr-Wi*Nmi
300  Argi=Wr*Nmi+Wi*Nmr
310  CALL Prinlogz(Argr,Argi,Lgr,Lgi)
320  Etar=Radr+Lgr
330  Etai=Radi+Lgi
340  CALL Ukcoefs(Trms,A(*))
350  CALL Uk(Tr,Ti,Trms,A(*),T(*),U(*))
360  Dumr=0
370  Dumi=0
380  Nufac=Nu
390  FOR I=1 TO Trms
400    Nufac=Nufac/Nu
410    Dumr=Dumr+U(I,1)*Nufac
420    Dumi=Dumi+U(I,2)*Nufac
430  NEXT I
440  E=EXP(Nu*Etar)
450  Co=COS(Nu*Etai)
460  Si=SIN(Nu*Etai)
470  Fac=1/SQR(2*PI*Nu)
480  Facr=Fac*E*(Co*Iradrtr-Si*Iradrtr)
490  Faci=Fac*E*(Co*Iradrtr+Si*Iradrtr)
500  Inur=Facr*Dumr-Faci*Dumi
510  Inui=Facr*Dumi+Faci*Dumr
520  Cn=COS(Nu*PI/2)
530  Sn=SIN(Nu*PI/2)
540  Jr=Cn*Inur-Sn*Inui
550  Ji=Cn*Inui+Sn*Inur
560  SUBEND

```

```

10  SUB Hfkuniasym(Nu,Zr,Zi,H1r,H1i,Trms)
20    !Nu is the order.
30    !Zr and Zi real and imaginary parts of argument
40    !H1r and H1i real and imaginary parts of the Bessel function.
50    !Trms is the number of terms retained in the asymptotic expansion.
60    DIM A(10,10),U(10,2),T(30,2)
70    REDIM A(Trms,Trms),U(Trms,2),T(3*Trms,2)
80    Wr=Zi/Nu
90    Wi=-Zr/Nu
100   Ur=1+Wr*Wr-Wi*Wi
110   Ui=2*Wr*Wi
120   Um=SQR(Ur*Ur+Ui*Ui)
130   Radr=SQR((Um+Ur)/2)
140   Radi=SGN(Ui)*SQR((Um-Ur)/2)
150   Radm=Radi*Radi+Radr*Radr
160   Sradm=SQR(Radm)
170   Radrtr=SQR((Sradm+Radr)/2)

```

```

180 Radrti=SGN(Radi)*SQR((Sradm-Radr)/2)
190 Radrtm=Radtr*Radtr+Radrti*Radrti
200 Iradtr=Radtr/Radrtm
210 Iradrti=-Radrti/Radrtm
220 Tr=Radr/Radm
230 Ti=-Radi/Radm
240 Dnr=1+Radr
250 Dni=Radi
260 Dnm=Dnr*Dnr+Dni*Dni
270 Nmr=Dnr/Dnm
280 Nmi=-Dni/Dnm
290 Argr=Wr*Nmr-Wi*Nmi
300 Argi=Wr*Nmi+Wi*Nmr
310 CALL Prinlogz(Argr,Argi,Lgr,Lgi)
320 Etar=Radr+Lgr
330 Etai=Radi+Lgi
340 CALL Ukcoefs(Trms,A(*))
350 CALL Uk(Tr,Ti,Trms,A(*),T(*),U(*))
360 Dumr=0
370 Dumi=0
380 Nufac=-Nu
390 FOR I=1 TO Trms
400   Nufac=-Nufac/Nu
410   Dumr=Dumr+U(I,1)*Nufac
420   Dumi=Dumi+U(I,2)*Nufac
430 NEXT I
440 E=EXP(-Nu*Etar)
450 Co=COS(Nu*Etai)
460 Si=-SIN(Nu*Etai)
470 Fac=SQR(PI/2/Nu)
480 Facr=Fac*E*(Co*Iradrtr-Si*Iradrtr)
490 Faci=Fac*E*(Co*Iradrtr+Si*Iradrtr)
500 Knur=Facr*Dumr-Faci*Dumi
510 Knui=Facr*Dumi+Faci*Dumr
520 Cn=COS(Nu*PI/2)
530 Sn=-SIN(Nu*PI/2)
540 Jr=Cn*Knur-Sn*Knui
550 Ji=Cn*Knui+Sn*Knur
560 H1r=2*Ji/PI
570 H1i=-2*Jr/PI
580 SUBEND

```

```

10 SUB Jh1produni(Nu,Nup,Zr,Zi,Zpr,Zpi,Prodr,Prodi,Trms)

```

```

20   !Nu and Nup are the orders of the two Bessel functions in the
product.

```

```

30   !Zr and Zi are the real and imaginary parts of the argument of one

```

```

40   !factor and Zpr and Zpi are those for the other factor.

```

```

50   !Prodr and Prodi are the real and imaginary parts of the Bessel

```

```

60   !function product.

```

```

70   !Trms is the number of terms retained in the asymptotic expansion.

```

```

80  DIM A(10,10),U(10,2),T(30,2)
90  REDIM A(Trms,Trms),U(Trms,2),T(3*Trms,2)
100  Wr=Zi/Nu
110  Wi=-Zr/Nu
120  Ur=1+Wr*Wr-Wi*Wi
130  Ui=2*Wr*Wi
140  Um=SQR(Ur*Ur+Ui*Ui)
150  Radr=SQR((Um+Ur)/2)
160  Radi=SGN(Ui)*SQR((Um-Ur)/2)
170  Radm=Radi*Radi+Radr*Radr
180  Sradm=SQR(Radm)
190  Radrtr=SQR((Sradm+Radr)/2)
200  Radrti=SGN(Radi)*SQR((Sradm-Radr)/2)
210  Radrtm=Radrtr*Radrtr+Radrti*Radrti
220  Iradrtr=Radrtr/Radrtm
230  Iradrti=-Radrti/Radrtm
240  Tr=Radr/Radm
250  Ti=-Radi/Radm
260  Dnr=1+Radr
270  Dni=Radi
280  Dnm=Dnr*Dnr+Dni*Dni
290  Nmr=Dnr/Dnm
300  Nmi=-Dni/Dnm
310  Argr=Wr*Nmr-Wi*Nmi
320  Argi=Wr*Nmi+Wi*Nmr
330  CALL Prinlogz(Argr,Argi,Lgr,Lgi)
340  Etar=Radr+Lgr
350  Etai=Radi+Lgi
360  CALL Ukcoefs(Trms,A(*))
370  CALL Uk(Tr,Ti,Trms,A(*),T(*),U(*))
380  Dumr=0
390  Dumi=0
400  Nufac=Nu
410  FOR I=1 TO Trms
420    Nufac=Nufac/Nu
430    Dumr=Dumr+U(I,1)*Nufac
440    Dumi=Dumi+U(I,2)*Nufac
450  NEXT I
460  MAT U= (0)
470  Wr=Zpi/Nup
480  Wi=-Zpr/Nup
490  Ur=1+Wr*Wr-Wi*Wi
500  Ui=2*Wr*Wi
510  Um=SQR(Ur*Ur+Ui*Ui)
520  Radr=SQR((Um+Ur)/2)
530  Radi=SGN(Ui)*SQR((Um-Ur)/2)
540  Radm=Radi*Radi+Radr*Radr
550  Sradm=SQR(Radm)
560  Radrtr=SQR((Sradm+Radr)/2)
570  Radrti=SGN(Radi)*SQR((Sradm-Radr)/2)
580  Radrtm=Radrtr*Radrtr+Radrti*Radrti
590  Iradrtr=Radrtr/Radrtm

```

```

600  Iradrtpi=-Radrti/Radrtm
610  Tr=Radrt/Radm
620  Ti=-Radi/Radm
630  Dnr=1+Radrt
640  Dni=Radi
650  Dnm=Dnr*Dnr+Dni*Dni
660  Nmr=Dnr/Dnm
670  Nmi=-Dni/Dnm
680  Argr=Wr*Nmr-Wi*Nmi
690  Argi=Wr*Nmi+Wi*Nmr
700  CALL Prinlogz(Argr,Argi,Lgr,Lgi)
710  Etapr=Radrt+Lgr
720  Etapi=Radi+Lgi
730  CALL Uk(Tr,Ti,Trms,A(*),T(*),U(*))
740  Dumpr=0
750  Dumpi=0
760  Nufac=-Nup
770  FOR I=1 TO Trms
780    Nufac=-Nufac/Nup
790    Dumpr=Dumpr+U(I,1)*Nufac
800    Dumpi=Dumpi+U(I,2)*Nufac
810  NEXT I
820  Sumr=Dumpr*Dumpr-Dumi*Dumpi
830  Sumi=Dumpr*Dumpi+Dumi*Dumpr
840  Radsr=Iradrt*Iradrtpr-Iradrti*Iradrtpi
850  Radsi=Iradrt*Iradrtpi+Iradrti*Iradrtpr
860  Dumprodr=(Sumr*Radsr-Sumi*Radsi)/(2*SQR(Nu*Nup))
870  Dumprodi=(Sumr*Radsi+Sumi*Radsr)/(2*SQR(Nu*Nup))
880  Etargr=Nu*Etar-Nup*Etapr
890  Etargi=Nu*Etai-Nup*Etapi
900  E=EXP(Etargr)
910  Co=COS(Etargi)
920  Si=SIN(Etargi)
930  Ikprodr=E*(Dumprodr*Co-Dumprodi*Si)
940  Ikprodi=E*(Dumprodr*Si+Dumprodi*Co)
950  Cfr=2*SIN((Nu-Nup)*PI/2)/PI
960  Cfi=-2*COS((Nu-Nup)*PI/2)/PI
970  Prodr=Ikprodr*Cfr-Ikprodi*Cfi
980  Prodi=Ikprodr*Cfi+Ikprodi*Cfr
990  SUBEND

```

```

10  SUB Uk(Tr,Ti,K,A(*),T(*),Uk(*))
20    !Functions used in the uniform asymptotic expansion of the Bessel
30    !functions for large orders. A(*) is a matrix generated by the
40    !program Ukcoefs. T(*) and Uk(*) are output to the uniform asymp-
50    !totic Bessel function routines.
60    T(1,1)=Tr
70    T(1,2)=Ti
80    FOR I=2 TO 3*K
90      T(I,1)=T(I-1,1)*Tr-T(I-1,2)*Ti

```

```

100  T(I,2)=T(I-1,1)*Ti+T(I-1,2)*Tr
110  NEXT I
120  Uk(1,1)=1
130  Uk(1,2)=0
140  FOR I=2 TO K
150    FOR J=1 TO I
160      Uk(I,1)=Uk(I,1)+A(I,J)*T(2*J+I-3,1)
170      Uk(I,2)=Uk(I,2)+A(I,J)*T(2*J+I-3,2)
180    NEXT J
190  NEXT I
200  SUBEND

```

```

10  SUB Ukcoefs(K,A(*))
20    !K is the maximum index value. A(*) is output to Uk.
30    A(1,1)=1
40    FOR L=0 TO K-2
50      A(L+2,1)=A(L+1,1)*(L/2+1/8/(L+1))
60      A(L+2,L+2)=-A(L+1,L+1)*(3*L/2+5/24/(L+1))
70      IF L>=1 THEN
80        FOR M=1 TO L
90          A(L+2,M+1)=A(L+1,M+1)*((L+2*M)/2+1/(8*(L+1+2*M)))
100         A(L+2,M+1)=A(L+2,M+1)-A(L+1,M)*((L+2*(M-1))/2+5/(8*(L+1+2*M)))
110        NEXT M
120      END IF
130    NEXT L
140  SUBEND

```

REFERENCES

- A1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (Applied Mathematics Series 55, 1965), National Bureau of Standards.

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