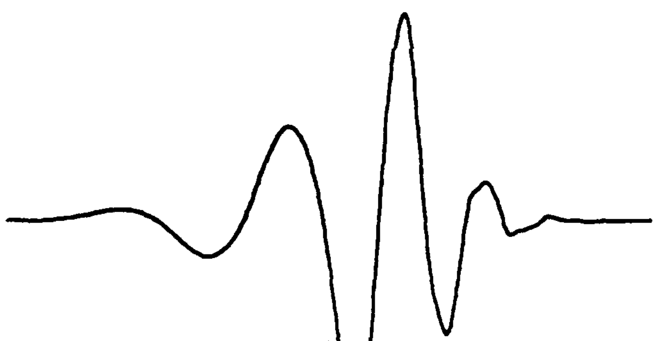
APPLICATIONS of WAVELETS to SIGNAL PROCESSING



20–22 March 1991

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Wright-Patterson Air Force Base, Ohio

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13. ABSTRACT *(Maximum 200 words)*

The Air Force Institute of Technology (AFIT) hosted the Symposium on Applications of Wavelets to Signal Processing during 20-22 March 1991 at the AFIT Science and Research Center, Wright-Patterson AFB, OH. This was the first major academic meeting of its kind held at AFIT. The symposium was sponsored by the Air Force Office of Scientific Research (AFOSR), Bolling AFB, DC. It focused on applications of wavelets to problems in image and speech processing, communications, and graphics and opened with a one-day short course, "A Tutorial on Wavelets." The symposium was attended by more than 220 researchers worldwide. They represented 15 foreign countries, 34 US universities, 28 private industrial firms, 15 DoD organizations, and six non-DoD US government organizations. Professor Alexander Grossmann was sponsored by the AFOSR Window on Science program. Volumes I and II contain, respectively, a synopsis of the symposium and the complete set of tutorial notes.

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AFIT/EN-TM-91-1

AFIT/AFOSR SYMPOSIUM ON
APPLICATIONS OF WAVELETS TO SIGNAL PROCESSING

FINAL REPORT

VOLUME II -- TUTORIAL NOTES

Final report for the period 3 October 1990 to 30 April 1991

AFOSR-616-91-0024

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Air Force Institute of Technology
Wright-Patterson AFB, OH 45433-6583

30 April 1991

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Errata

"A Tutorial on Wavelets"

p71. change v to u so that it reads

$$v = \langle u, e \rangle e.$$

p152. delete the comma following V_{-1}.

p199. change + to − between $\delta_{n,2l}$ and $\delta_{n,2l+1}$ and between $C_{m-1,2l}$ and $C_{m-1,2l+1}$ in the last two lines on the page.

p209. change the boldface R to [0,1] so that it reads "... between $G \in L^2([0,1])$ and ..."

AFIT/AFOSR Symposium
on
APPLICATIONS OF WAVELETS TO
SIGNAL PROCESSING

Final Report, Volume II
Tutorial Notes

G. Warhola, M. Oxley, S. Rogers,
M. Kabrisky, B. Suter

AFIT/EN-TM-91-1 30 April 1991

A Tutorial on Wavelets

presented by

AFIT Faculty

sponsored by

AFOSR

1

These notes were prepared by

G. Warhola

M. Oxley

S. Rogers

M. Kabrisky

B. Suter

SCHEDULE

0830-1000 Part I
Introduction to Wavelets in Signal Analysis

1000-1030 Break

1030-1200 Part II
Introduction to Wavelet Bases and Frames

1200-1330 Lunch

1330-1430 Part III
AFIT Applications of Wavelets

1430-1500 Break

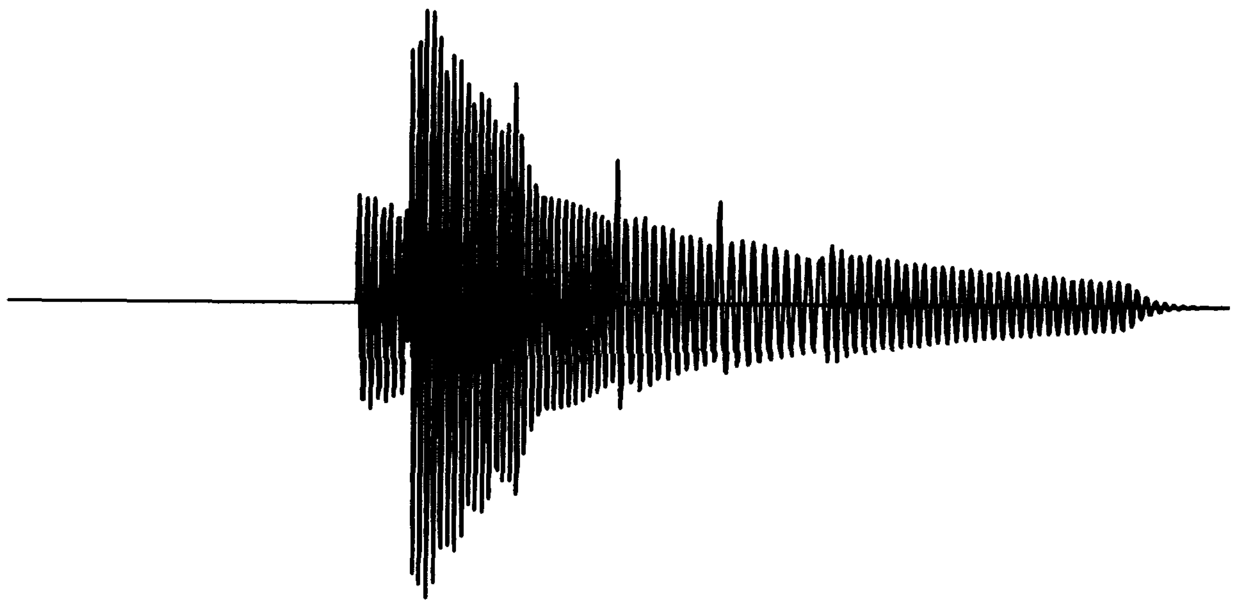
1500-1700 Part IV
Multiresolution Analysis and Orthonormal Bases

PART I

Introduction to Wavelets in Signal Analysis

* History
* Weyl-Heisenberg Wavelet Transforms
  ** Related Time-Frequency Distributions
* Affine Wavelet Transforms
  ** Basic Properties
  ** Examples
* Mathematical Necessities

TOY SIGNAL



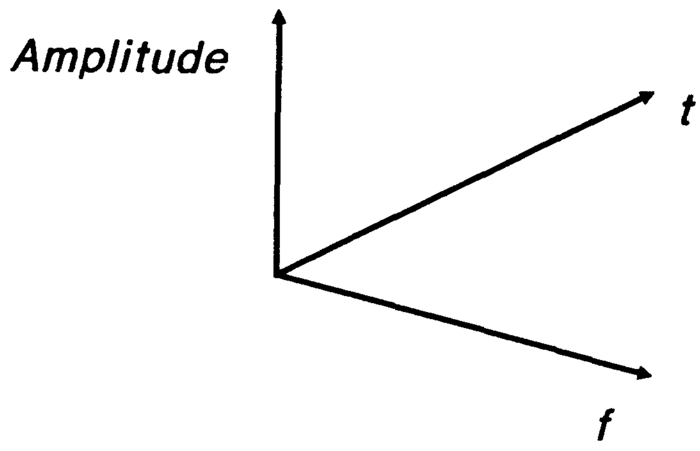
PROBLEMS IN SIGNAL PROCESSING

* Compression
* Recognition
  ** Is it there?
  ** Who's talking?
* Boundary Detection
  ** Transitions in Speech
  ** Edges in Images
* Inverse Problems
  ** Source Reconstruction
  ** Parameter ID

Fourier series and transforms are nice, but not good enough!

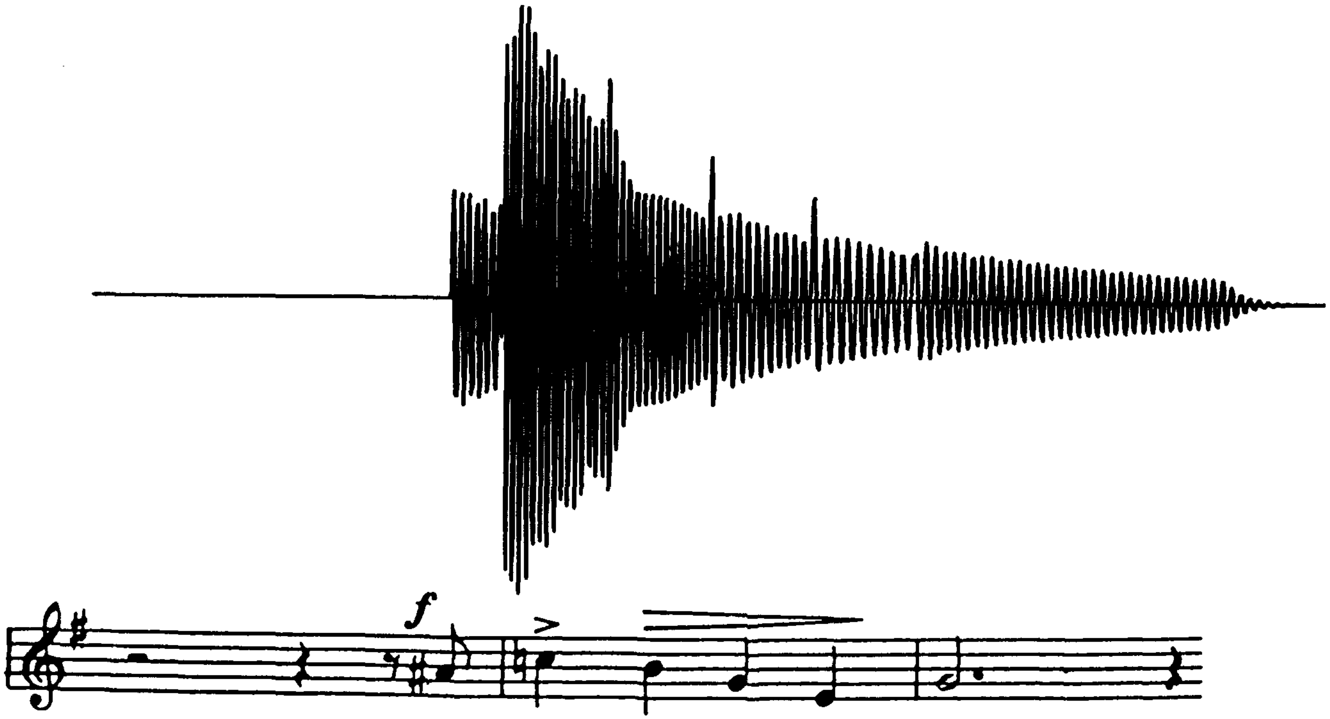
* Miraculous Cancellations at "Dead Spots"
* Data Dropouts $\Rightarrow$ Recompute all coefficients
* Timing of a frequency's predominant contribution not apparent
* Reconstruction dependent upon phase

TIME-FREQUENCY DISTRIBUTIONS



Track the time-evolution of "frequency"

TOY SIGNAL AND TIME-FREQUENCY REPRESENTATION



N. G. De Bruijn:

"What the composer really does, ..., is something entirely different from describing either f or $\mathcal{F}f$. Instead, he constructs a function of two variables. The variables are the time and the frequency, the function describes the intensity of the sound. He describes the function by a complicated set of dots on score paper. ...certainly vertical lines denote constant time, and horizontal lines denote constant frequency."

MATH HISTORY OF TIME-FREQUENCY DIST'S

1903	Runge	[R1]
1932	Wigner	[W1]
1946	Gabor	[G1]
1948	Ville	[V1]
1952	Duffin & Schaeffer	[DS]
1964	Calderon	[C1]
1966	Cohen	[C2]
1967	De Bruijn	[DB]
1980	Coifman & Rochberg	[CR]
	Bastiaans	[B2]
1981	Balian	[B1]
	Jannsen	[J1,2]
1982	Morlet, et al	[M1]
1984	Grossmann & Morlet	[GM]
	Jannsen	[J3]

HISTORY (cont'd)

1985	Meyer	[Me1]
	Low	[L1]
	Frazier & Jawerth	[FJ1]
1986	Daubechies, et al	[DGM]
	Lemarié & Meyer	[LM]
1987	Battle	[B5]
1988-89	Cohen	[C3]
	Daubechies	[Db1,2]
	Frazier & Jawerth	[FJ2]
	Heil & Walnut	[HW]
	Lemarié	[L2]
	Mallat	[Ma1-3]
	Meyer	[Me3]
1990	Meyer	[Me4]
	Resnikoff & Burrus	[RB]
	Daubechies	[Db3]

NOTATION

$\mathbf{R}$ - the real numbers
$\mathbf{C}$ - the complex numbers
$\mathbf{Z}$ - the integer numbers
i - the imaginary unit, $i = \sqrt{-1}$.

$$\begin{aligned} L^2(\mathbf{R}) &= \{\text{square-integrable complex-valued functions defined on } \mathbf{R}\ \} \\ &= \{x : \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty\} \end{aligned}$$

Similarly,

$$L^2([0,1]) = \{x : \int_0^1 |x(t)|^2 dt < \infty\}$$

"Finite-energy" continuous-time signals

$$
\begin{aligned}
l^2(\mathbf{Z}) &= \{\text{square-summable complex-valued functions defined on } \mathbf{Z}\ \} \\
&= \{\text{square-summable sequences}\} \\
&= \{\alpha = (\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots) : \sum_{n \in \mathbf{Z}} |\alpha_n|^2 < \infty\}
\end{aligned}
$$

"Finite-energy" discrete-time signals

Let "$\in$" denote membership in a set, family, class, etc.

e.g.

$$x \in L^2(\mathbf{R}) \iff \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

where

" $\iff$ " means "if and only if

WEYL-HEISENBERG WAVELET TRANSFORMS

Recall the Fourier Transform, $\hat{x}$, for $x \in L^2(\mathbf{R})$:

$$\mathcal{F}\{x\}(f) \doteq \hat{x}(f) = \int_{-\infty}^{\infty} x(t)\,\mathrm{e}^{-i2\pi f t} dt$$

Inversion Formula:

$$\mathcal{F}^{-1}\{\hat{x}\}(t) = x(t) = \int_{-\infty}^{\infty} \hat{x}(f)\,\mathrm{e}^{i2\pi f t} df$$

The "usual" inner product, $\langle \cdot \, , \, \cdot \rangle$, defined on $L^2(\mathbf{R})$ is given by

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t)\overline{y(t)}dt$$

where the"overbar" implies complex conjugation.

The "induced norm," $\| \cdot \|$, is given by

$$\begin{aligned} \|x\| &= \langle x, x \rangle^{1/2} \\ &= \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt \right)^{1/2} \end{aligned}$$

Then,

$$\hat{x}(f) = \langle x, E_f \rangle = \int_{-\infty}^{\infty} x(t)\, \mathrm{e}^{-i2\pi f t} dt$$

where $E_f(t) = \mathrm{e}^{i2\pi f t}$. Also,

$$x(t) = \langle \hat{x}, \mathcal{E}_t \rangle = \int_{-\infty}^{\infty} \hat{x}(f)\, \mathrm{e}^{i2\pi f t} df$$

where $\mathcal{E}_t(f) = \mathrm{e}^{-i2\pi f t}$. Note that

$$E_f(t) = \overline{\mathcal{E}_t(f)}$$

"Building blocks" for the Fourier Transform are complex sinusoids of continuous frequency, f, which satisfy

$$\langle E_f, E_{f_0} \rangle = \delta(f - f_0),$$

a Dirac δ - distribution.

Plancherel - "signal energy = spectrum energy"

$$\|x\|^2 = \|\hat{x}\|^2$$

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 \, df$$

Parserval - preservation of "geometry"

$$\langle x, y \rangle = \langle \hat{x}, \hat{y} \rangle$$

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)} dt = \int_{-\infty}^{\infty} \hat{x}(f)\overline{\hat{y}(f)} df$$

For $x \in L^2([0,1])$, a Fourier Series representation is

$$x(t) = \sum_{n \in \mathbf{Z}} c_n E_n(t) \ ,$$

a 1-periodic function of $t \in \mathbf{R}$, i.e. $x(t) = x(t+k) \ \forall\, k \in \mathbf{Z}$, where

$$\begin{aligned} c_n &= \int_0^1 x(t)\overline{E_n(t)}dt \\ &\doteq \langle x, E_n \rangle \end{aligned}$$

$\langle \cdot \, , \, \cdot \rangle$ in the context of the function space considered, $L^2([0,1])$ here.

"Building blocks" for Fourier Series are complex exponentials of discrete (integer values of) frequency, n , which satisfy

$$\langle E_n, E_m \rangle = \delta_{nm},$$

the Kronecker δ - function.

Fourier Coefficients

$$c_m = \int_0^1 x(t)\, \mathrm{e}^{-i2\pi mt} dt$$

as a Fourier Transform

$$c_m = \int_{-\infty}^{\infty} \chi_{[0,1)}(t) x(t)\, \mathrm{e}^{-i2\pi mt} dt$$

where

$$\chi_{[0,1)}(t) = \begin{cases} 1, & t \in [0,1) \\ 0, & \text{else}\,. \end{cases}$$

In a loose sense, c_m is a measure of the amount of frequency m present in x as viewed through the "window" $\chi_{[0,1)}$.

Move the window to the interval $[n, n+1)$ and define

$$c_{mn} \doteq \int_{-\infty}^{\infty} \chi_{[n,n+1)}(t)\, x(t)\, \mathrm{e}^{-i2\pi m t} dt \quad \forall\, m, n \in \mathbf{Z}$$

to measure the frequency m in the interval $[n, n+1)$.

Gabor - 1946 [G1] Gaussian Window, g ,

$$c_{mn} = \int_{-\infty}^{\infty} g(t - nt_0)\, x(t)\, \mathrm{e}^{-i2\pi m f_0 t}\, dt \quad \forall\, m, n \in \mathbf{Z}$$

with

$$\frac{1}{f_0 t_0} = 1$$

gives numerically unstable reconstruction!

Require $\frac{1}{f_0 t_0} > 1$ for stability (Nyquist, Balian-Low).

Generalize to a continuous frequency, p , and shift, q
$p, q \in \mathbf{R}$, where g is some "window".

Windowed ("Short Time") Fourier Transform

$$\mathrm{W}\mathcal{F}_g\{x\}(p,q) = \int_{-\infty}^{\infty} x(t)\,\overline{g(t-q)}\mathrm{e}^{-i2\pi pt}\,dt$$

"Weyl-Heisenberg Wavelet Transform"

Define

$$g^{(p,q)}(t) \doteq g(t-q)\,\mathrm{e}^{i2\pi pt}$$

then

$$\widehat{g^{(p,q)}}(f) = \hat{g}(f-p)\,\mathrm{e}^{-i2\pi qf} \quad .$$

(Note the windows, g and $\hat{g}$ in time and frequency domains, resp.)

$$\begin{aligned} \mathrm{W}\mathcal{F}_g\{x\}(p,q) &= \langle x, g^{(p,q)} \rangle \\ &= \int_{-\infty}^{\infty} x(t)\,\overline{g^{(p,q)}(t)}\,dt \end{aligned}$$

Suppose $||g||^2 = 1 = ||\hat{g}||^2$; then, $|g(\cdot)|^2$ and $|\hat{g}(\cdot)|^2$ are (probability) density functions. e.g.

$$g(t) = \pi^{-1/4} e^{-(t^2/2)}$$

If mean time and mean frequency are zero w.r.t. the densities, i.e.

$$\int_{-\infty}^{\infty} t\,|g(t)|^2\,dt = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f\,|\hat{g}(f)|^2\,df = 0$$

then

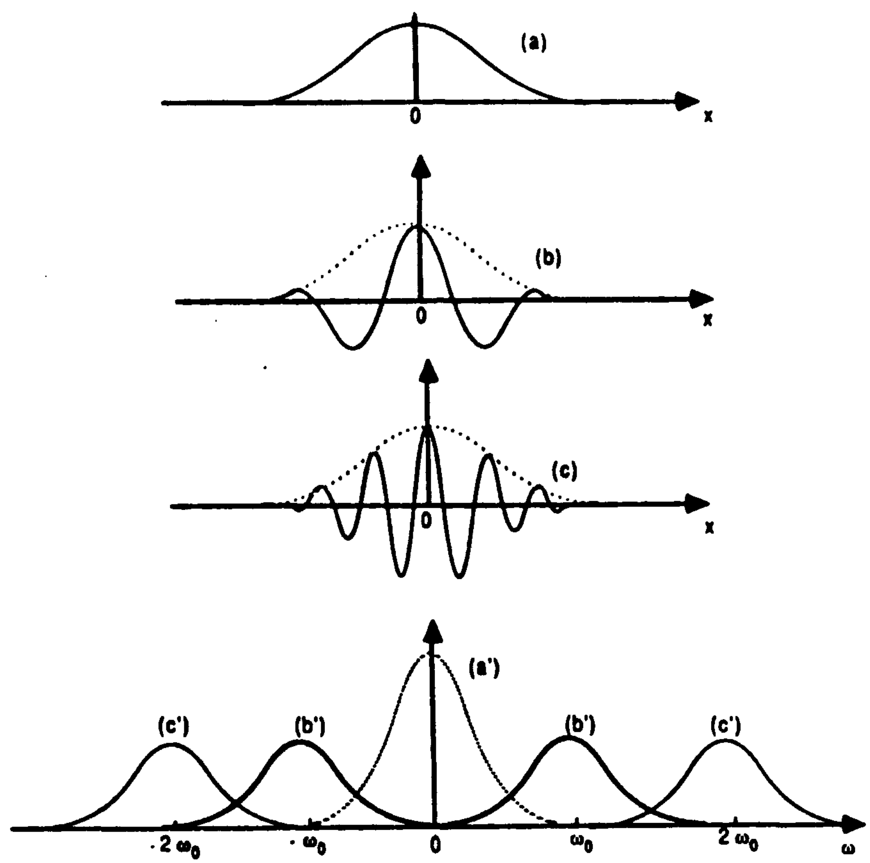
$g^{(p,q)}$ is "centered" around frequency p, time q.

$g^{(p,q)}$ is "centered" around frequency p, for each time q, i.e.

$$\int_{-\infty}^{\infty} f \left|\widehat{g^{(p,q)}}(f)\right|^2 df = p \quad \text{for all time } q$$

$$\int_{-\infty}^{\infty} t \left|g^{(p,q)}(t)\right|^2 dt = q \quad \text{for all frequencies } p\,.$$

$\mathrm{W}\mathcal{F}_g\{x\}(p,q)$ measures frequency p-content of x in window g centered at time q.



[Ma2]

Resolution of Identity

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathrm{W}\mathcal{F}_g\{x\}(p,q)|^2 \, dp \, dq \;=\; \|x\|^2 \|g\|^2$$

$$\|\mathrm{W}\mathcal{F}_g\{x\}\|^2 \;=\; \|x\|^2 \|g\|^2$$

Compare with Plancherel

$$\int_{-\infty}^{\infty} |\mathcal{F}\{x\}(f)|^2 \, df \;=\; \|x\|^2$$

$$\|\hat{x}\|^2 \;=\; \|x\|^2$$

Inversion Integral

$$\begin{aligned} x(t) &= \frac{1}{\|g\|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{W}\mathcal{F}_g\{x\}(p,q)\, g^{(p,q)}(t) dp\, dq \\ &= \frac{1}{\|g\|^2} \langle \mathrm{W}\mathcal{F}_g\{x\}, \overline{g^{(p,q)}} \rangle \end{aligned}$$

Weyl-Heisenberg Building Blocks"

$$g^{(p,q)}(t) = g(t-q)\,\mathrm{e}^{i2\pi pt}$$

Include the Window!

W-H $\Rightarrow$ Translate and Modulate

Folklore

Fourier Inversion of W-H

$$x(t)\overline{g(t-q)} = \int_{-\infty}^{\infty} \mathrm{W}\mathcal{F}_g\{x\}(p,q)\,\mathrm{e}^{i2\pi pt}dp$$

"Overlap and Add" to "recover" x .

Second Moments

$$\sigma_p^2 = \int_{-\infty}^{\infty} f^2 \left|\hat{g}(f)\right|^2 df$$
$$\sigma_q^2 = \int_{-\infty}^{\infty} t^2 \left|g(t)\right|^2 dt$$

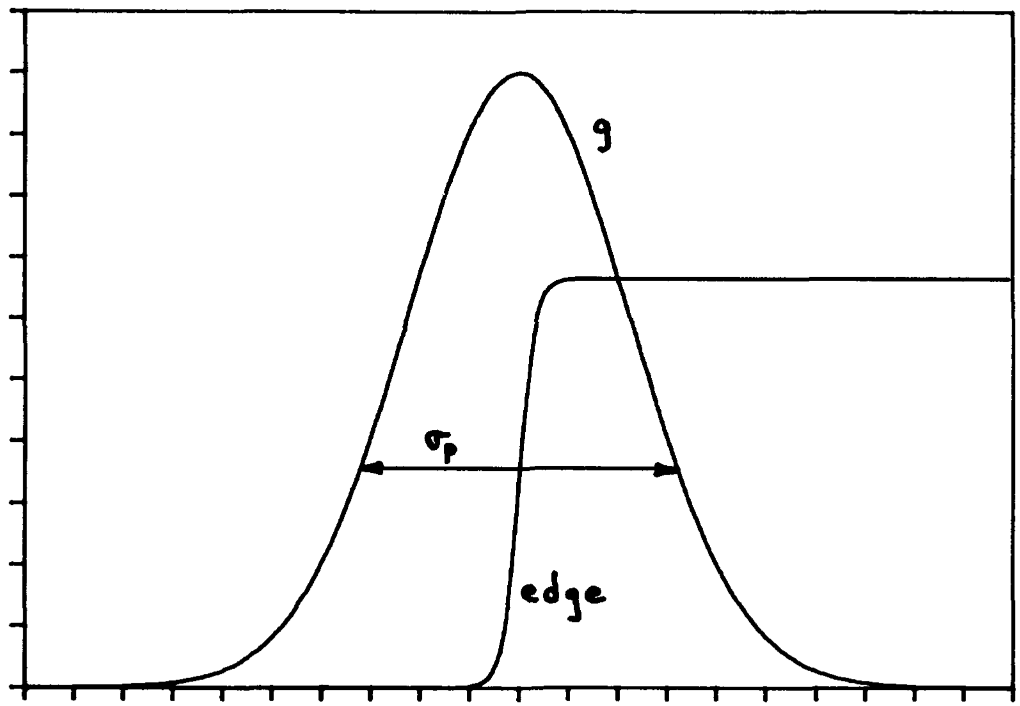
Standard Deviations of $g^{(p,q)}$ are resolution limits satisfying

$$\sigma_p \sigma_q \geq \frac{1}{4\pi}$$

"Uncertainty Principle"

Resolution Limitation of W-H

Fixed-width window, g, limits precision of event/frequency determination.



[Ma2]

Overcome poor time resolution of high frequencies with the

AFFINE WAVELET TRANSFORM.

$$\mathrm{W}_h\{x\}(a,b) = \int_{-\infty}^{\infty} x(t)\, \overline{h^{(a,b)}(t)}\, dt$$

where

$$h^{(a,b)}(t) \doteq \frac{1}{\sqrt{a}}\, h\!\left(\frac{t-b}{a}\right)$$

for some "Mother (Analyzing) Wavelet" h .

- $W_h\{x\}$ correlates x with Dilations and Translations of h .
- "Analyzing wavelets" $h^{(a,b)}$ all have same shape.
  - •• Compare with W-H.

$\mathrm{W}_h\{x\}$ as an Inverse Fourier Transform

Let

$$h_a(t) = \frac{1}{\sqrt{a}}\, h\left(\frac{t}{a}\right)$$

$$\widehat{h_a}(f) = \sqrt{a}\,\hat{h}(af)$$

then

$$\mathrm{W}_h\{x\}(a,b) = \int_{-\infty}^{\infty} x(t)\,\overline{h_a(t-b)}\,dt\ .$$

Let

$$\begin{aligned}\gamma_a(t) &= h_a(-t)\\ \widehat{\gamma_a}(f) &= \hat{h}_a(-f)\\ &= \sqrt{a}\,\hat{h}(-af)\quad.\end{aligned}$$

Then

$$\begin{aligned}\mathrm{W}_h\{x\}(a,b) &= \int_{-\infty}^{\infty} x(t)\,\overline{\gamma_a(b-t)}\,dt\\ &= (x * \overline{\gamma_a})(b)\quad.\end{aligned}$$

Have

$$\mathrm{W}_h\{x\}(a,b) = (x * \overline{\gamma_a})(b)$$

which implies

$$\begin{aligned}\mathrm{W}_h\{x\}(a,b) &= \mathcal{F}^{-1}\left\{\hat{x}\,(\cdot)\overline{\widehat{\gamma_a}(-(\cdot))}\right\}(b)\\ &= \mathcal{F}^{-1}\left\{\hat{x}(\cdot)\sqrt{a}\,\overline{\hat{h}(a(\cdot))}\right\}(b)\\ \mathrm{W}_h\{x\}(a,b) &= \sqrt{a}\int_{-\infty}^{\infty}\hat{x}(f)\,\overline{\hat{h}(af)}\,\mathrm{e}^{i2\pi fb}df\ .\end{aligned}$$

Fourier Transform:

$$\widehat{\mathrm{W}_h\{x\}}(a,f) = \sqrt{a}\,\hat{x}(f)\,\overline{\hat{h}(af)}$$

Mostly work with real mother wavelets, h , so that

$$\overline{\hat{h}(f)} = \hat{h}(-f)$$

$$\hat{h}(-f)\,\hat{h}(f) = \left|\hat{h}(f)\right|^2 \quad .$$

Easy to get wavelet inversion integral from the Inverse Fourier Transform

Inversion Integral

Given h real

$$\widehat{\mathrm{W}_h\{x\}}(a,f) = \sqrt{a}\,\hat{x}(f)\,\hat{h}(-af)$$

then

$$\begin{aligned}\int_0^\infty \widehat{\mathrm{W}_h\{x\}}(a,f)\,\hat{h}(af)a^{-3/2}\,da &= \hat{x}(f)\int_0^\infty \frac{|\hat{h}(af)|^2}{a}\,da \\ &= \hat{x}(f)\,C_h\end{aligned}$$

where

$$C_h \doteq \int_0^\infty \frac{|\hat{h}(\xi)|^2}{\xi}\,d\xi < \infty$$

is necessarily assumed (more on this, later).

Inverse Fourier Transform:

$$\begin{aligned} C_h\, x(t) &= \int_{-\infty}^{\infty}\int_0^{\infty} \widehat{\mathrm{W}_h\{x\}}(a,f)\sqrt{a}\,\hat{h}(af)\, a^{-2} da\, \mathrm{e}^{i2\pi f t} df \\ &= \int_0^{\infty} \mathcal{F}^{-1}\Big\{\widehat{\mathrm{W}_h\{x\}}(a,\cdot)\, h_a(\cdot)\Big\}(t) a^{-2} da \\ &= \int_0^{\infty} (\mathrm{W}_h\{x\} * h_a)(t) a^{-2} da \\ C_h\, x(t) &= \int_{-\infty}^{\infty}\int_0^{\infty} \mathrm{W}_h\{x\}(a,b)\, \frac{1}{\sqrt{a}}\, h\left(\frac{t-b}{a}\right) a^{-2} da\, db \end{aligned}$$

a weighted sum of wavelet "building blocks".

As an inner product

$$\begin{aligned} \mathrm{W}_h\{x\}(a,b) &= \int_{-\infty}^{\infty} x(t)\, \frac{1}{\sqrt{a}}\, \overline{h\left(\frac{t-b}{a}\right)}\, dt \\ &= \left\langle x, h^{(a,b)} \right\rangle . \end{aligned}$$

Admissibility Condition

$$C_h = \int_0^\infty \frac{|\hat{h}(f)|^2}{f}\, df < \infty$$

implies $\hat{h}(0) = 0$ "sufficiently fast" .

That is, for h to be a mother wavelet for an affine wavelet transform, h must satisfy the condition

$$\int_{-\infty}^{\infty} h(t)dt = 0 \; .$$

Observe, $\hat{h}$ is a bandpass filter.

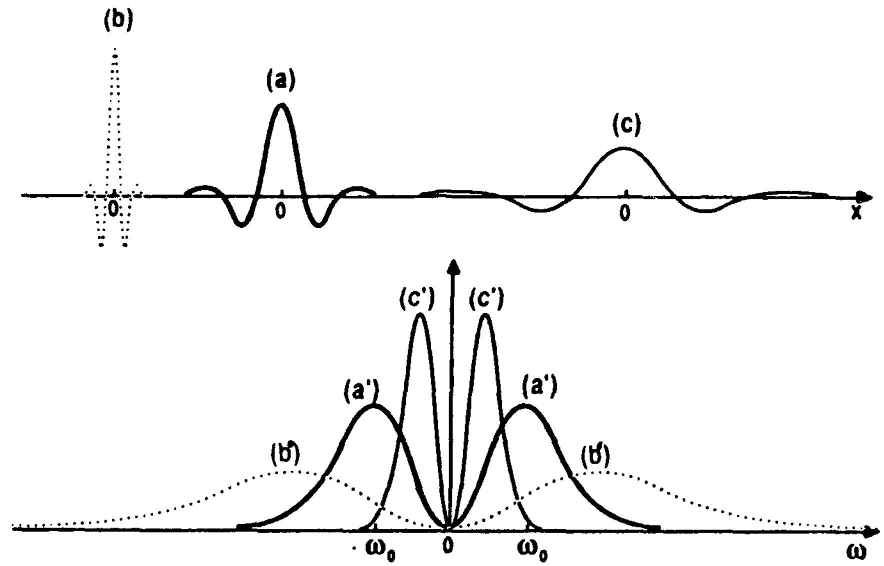


Fig. 10. (a) Graph of a wavelet $\psi(x)$. (b) Graph of $\psi_{s_1}(x)$ for $s_1 > 1$. (c) Graph of $\psi_{s_2}(x)$ for $s_2 < 1$. The curves (a'), (b'), and (c') are, respectively, the Fourier transform of the function shown in (a), (b), and (c). They have the same rms bandwidth on a logarithmic scale.

[Ma2]

Resolution Properties

Take mother wavelet, h , normalized

$$\|h\|^2 = 1 = \|\hat{h}\|^2$$

so that $|h|^2$, $|\hat{h}|^2$ are density functions.

Center frequency, f_0 of passband in $\hat{h}$ defined by

$$\int_0^\infty (f - f_0)\,|\hat{h}(f)|^2 df = 0$$

Second Moment (mean-squared bandwidth) about f_0 :

$$\sigma_f^2 = \int_0^\infty (f - f_0)^2\,|\hat{h}(f)|^2 df$$

Second Moment about time origin:

$$\sigma_t^2 = \int_0^\infty t^2\,|h(t)|^2\,dt$$

Wavelet $h_a(t-b) = \frac{1}{\sqrt{a}}\, h\left(\frac{t-b}{a}\right)$; $\hat{h}_a(f) = \sqrt{a}\,\hat{h}(af)$

Passband centered at f_0/a with mean-squared bandwidth $(\sigma_f/a)^2$

Energy in wavelet centered at $t = b$ with standard deviation $a\,\sigma_t$.

$a \downarrow 0 \;\Rightarrow$ "zoom in" on details

Elementary Example

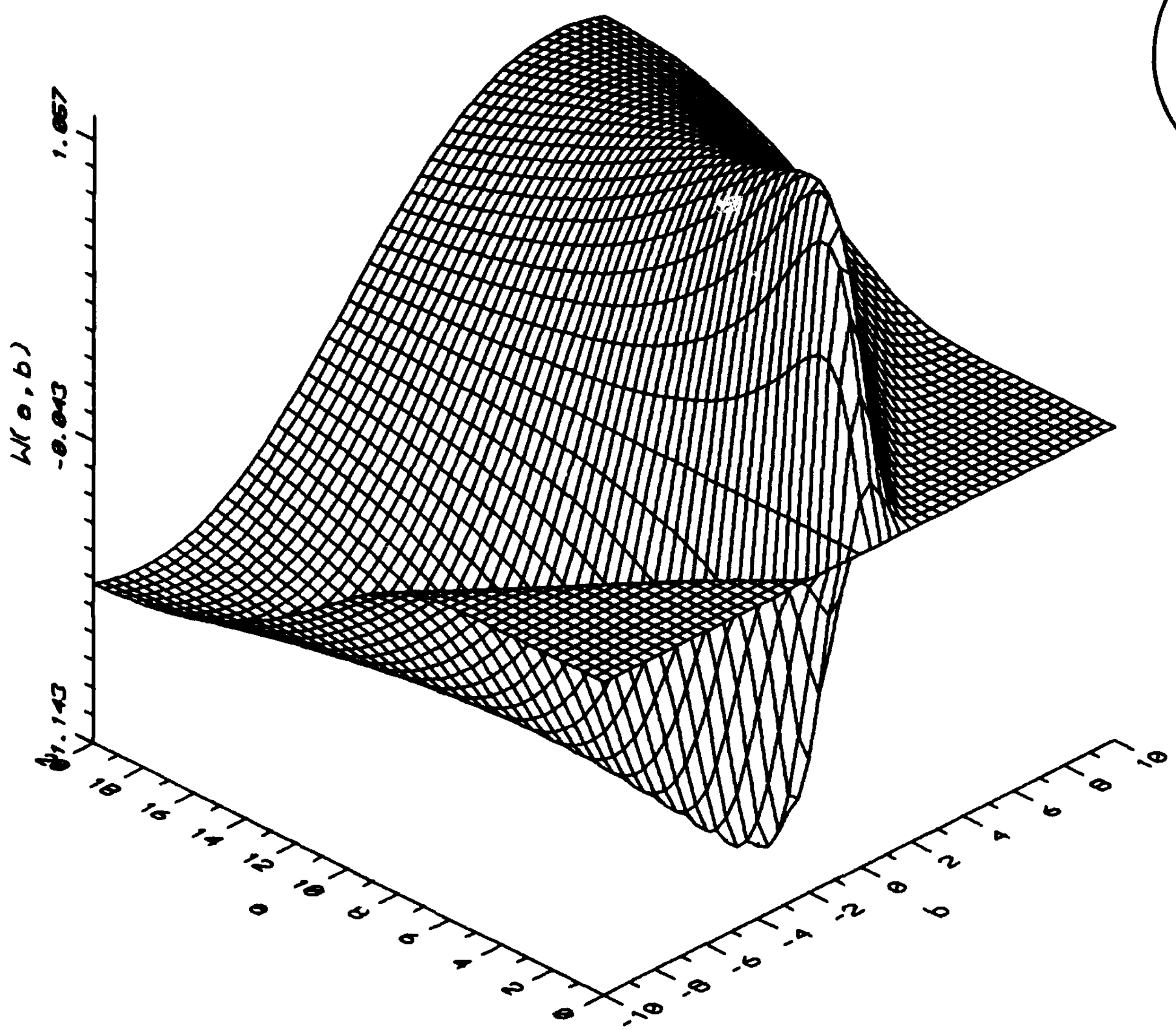
$x(t) = \mathrm{e}^{-kt^2}, k > 0$, analyzed w.r.t. mother wavelet

$$h(t) = \frac{d}{dt}\mathrm{e}^{-\pi t^2}$$

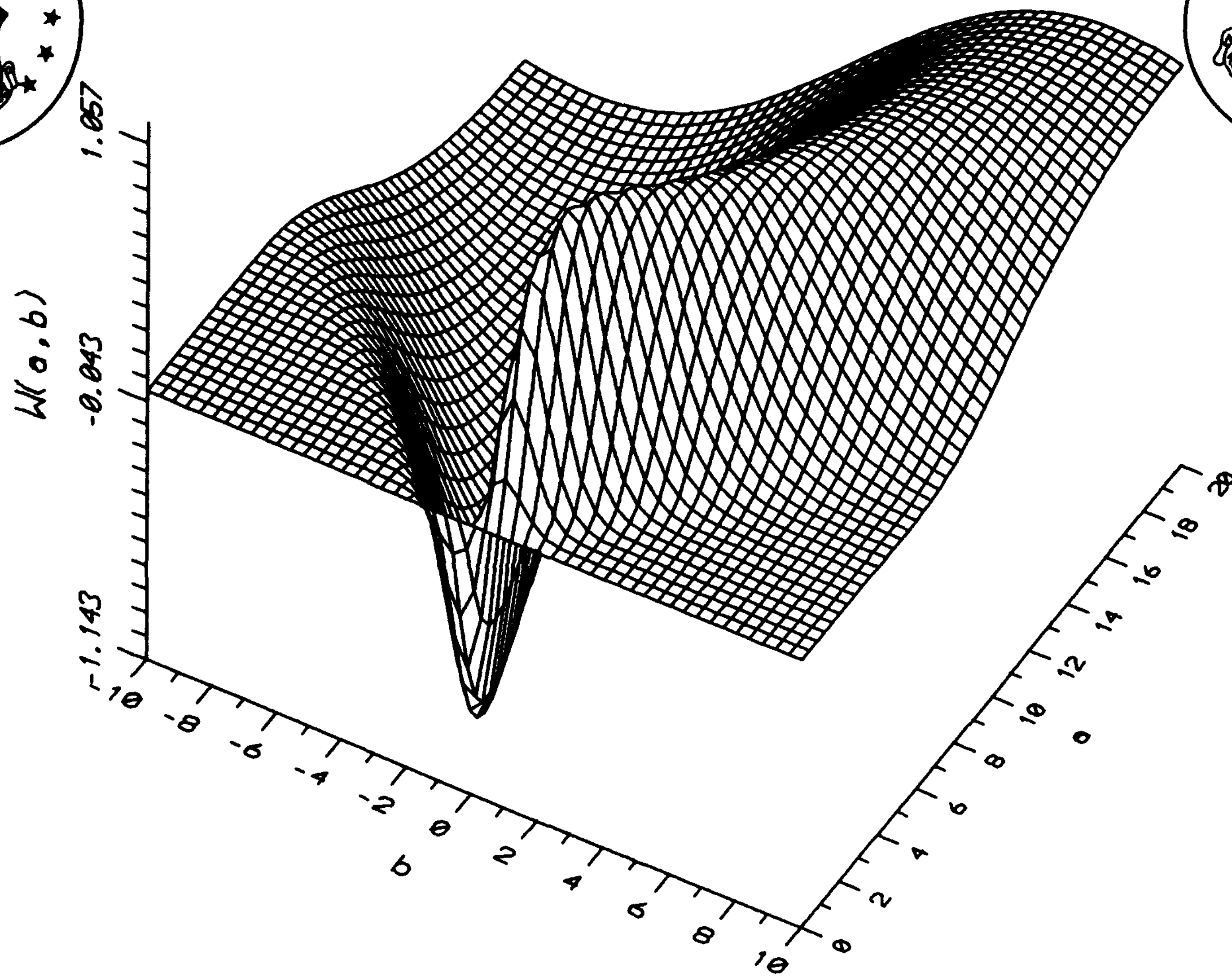
$$\begin{aligned} \mathrm{W}_h\{x\}(a,b) &= \mathcal{F}^{-1}\Big\{\hat{x}(\cdot)\sqrt{a}\,\hat{h}\,(-a(\cdot))\Big\}(b) \\ &= \frac{2ka^{3/2}b}{\left(1+\frac{ka^2}{\pi}\right)^{3/2}} \exp\left\{-\frac{kb^2}{1+\frac{ka^2}{\pi}}\right\} \end{aligned}$$

Note:

$$a^{-3/2}\,\mathrm{W}_h\{x\}(a,b) \longrightarrow -\frac{d}{dt}x(t)\,|_{t=b} \text{ as } a \downarrow 0.$$



1st-Deriv Gaussian Wavelet Tx of Gaussian



1st-Deriv Gaussian Wavelet Tx of Gaussian

Tuteur's Biomedical Application of Affine Wavelets [T1]

Nature of biomedical signals: short time duration cyclic signals whose exact shape is unknown. Moreover, shape and location of identifying characteristics change from cycle to cycle.

Averaging is ineffective.

Example. ECG waveform of child with cardiac defect – Ventricular Late Potential (VLP)

Pulse period of ECG waveform $\approx$ 1.05 seconds.

Duration of VLP signal $\approx$ 0.1 seconds.

$$\frac{\text{Peak VLP Amplitude}}{\text{Peak ECG Amplitude}} \approx 0.05.$$

Mother "wavelet" chosen

$$\begin{aligned} h(t) &= \mathrm{e}^{-t^2/2+imt} \\ \hat{h}(f) &= \sqrt{2\pi}\,\mathrm{e}^{-(2\pi f-m)^2/2} \end{aligned}$$

Not strictly a wavelet, since $\hat{h}(0) \neq 0$. But for m large enough, $\hat{h}(0) \approx 0$.

Tuteur chose $m = \pi\sqrt{2/\ln 2} \approx 5.336$, for which $\hat{h}(0) = 1.64\mathrm{X}10^{-6}$.

Fig. 1 "Normal" ECG

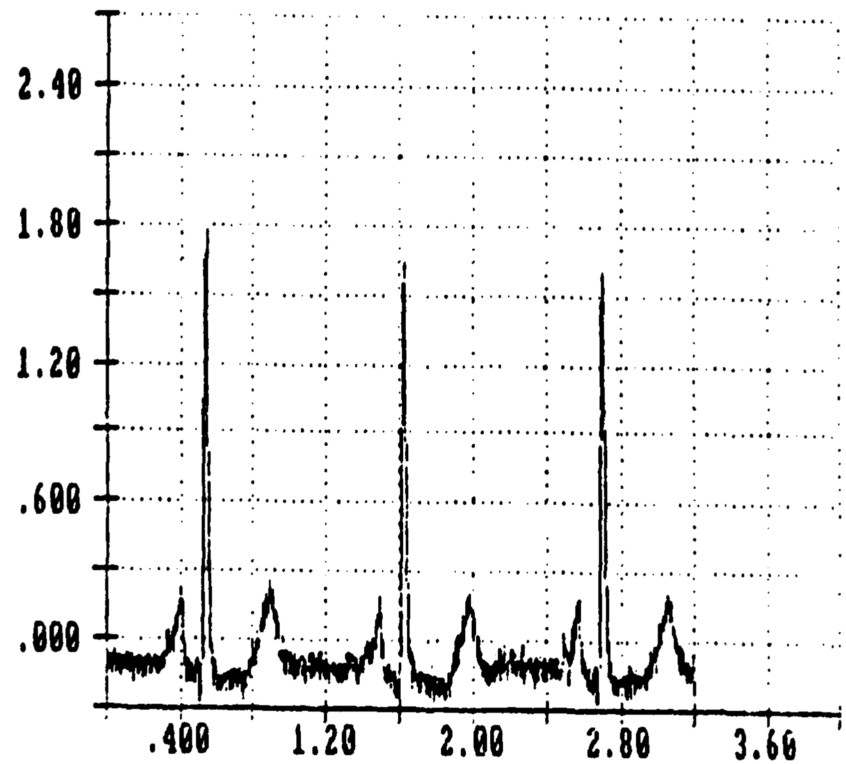


Fig. 2 ECG with artificially added VLP

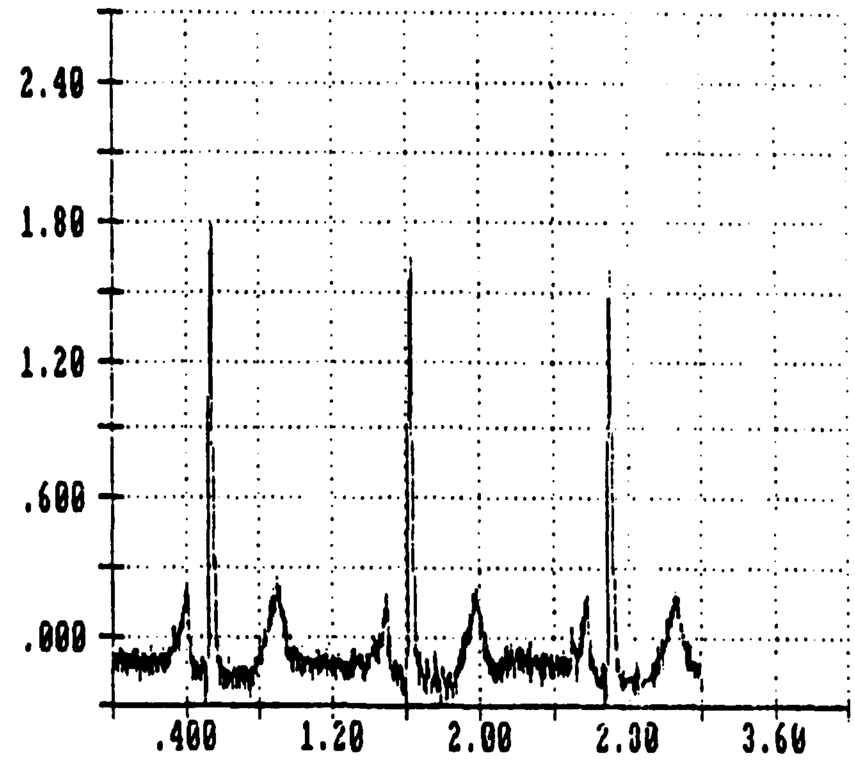
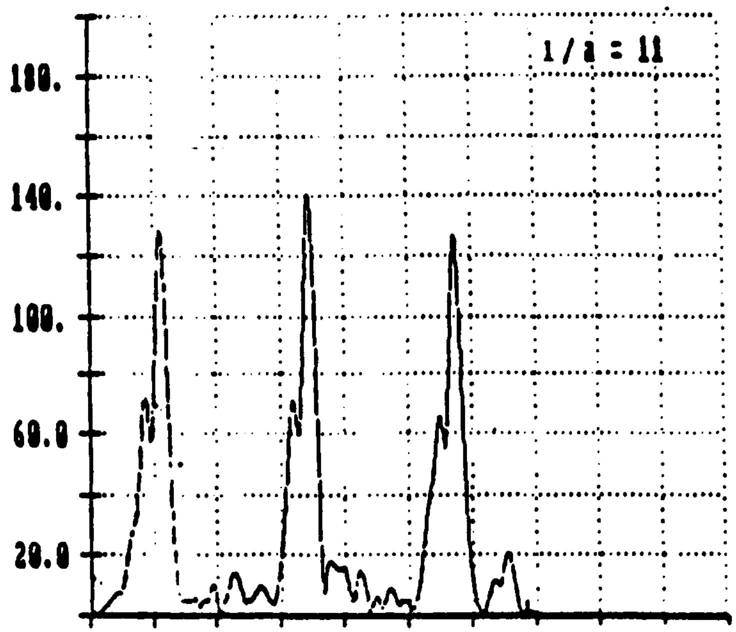
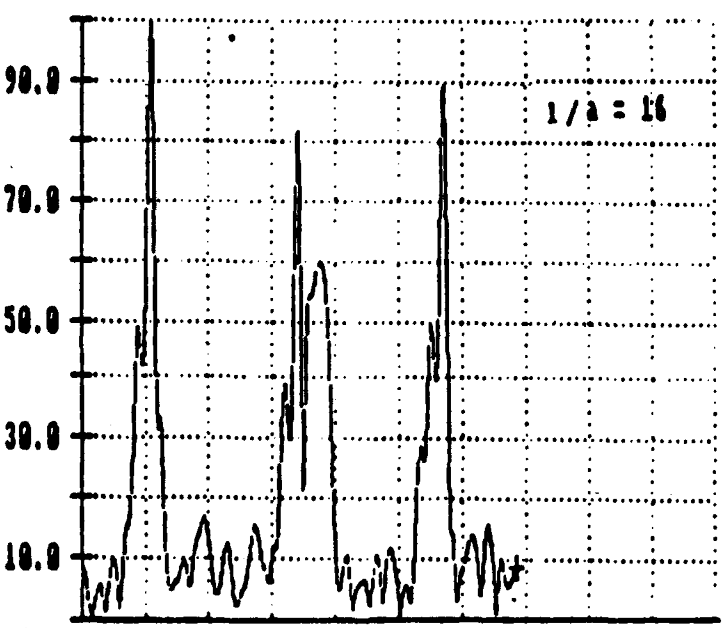
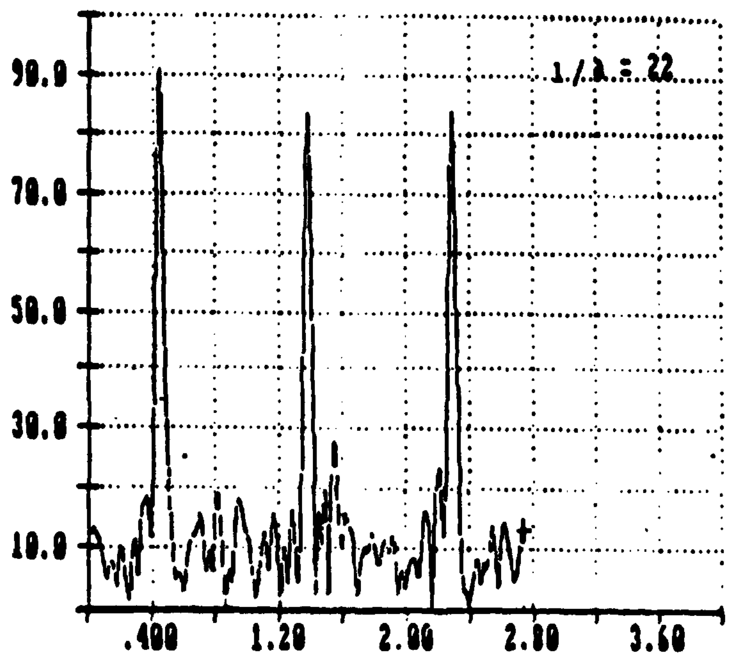


Fig. 3 Wavelet transforms of the clipped "abnormal" ECG for dilation parameter a = 1/11, 1/16, 1/22. Note the bulge to the right of the middle QRS peak indicating the presence of the VLP.





90.0
70.0
50.0
30.0
10.0
1/a = 16


1/λ = 22
90.0
70.0
50.0
30.0
10.0
.400
1.20
2.00
2.80
3.60

(Very) Elementary Properties of $\mathrm{W}_h\{x\}(a,b)$

Linearity:

$$W_h\{\alpha x + \beta y\} = \alpha \mathrm{W}_h\{x\} + \beta \mathrm{W}_h\{y\} \quad \forall\, \alpha, \beta \in \mathbf{C}$$

Scaling:

$$\mathrm{W}_h\{x([\cdot]/\lambda)\}(a,b) = \sqrt{|\lambda|}\,\mathrm{W}_h\{x(\cdot)\}(a/\lambda, b/\lambda) \quad \forall\, \lambda \in \mathbf{R} \setminus \{0\}$$

Time Shift:

$$\mathrm{W}_h\{x([\cdot] - t_0)\}(a,b) = \mathrm{W}_h\{x(\cdot)\}(a, b - t_0) \quad \forall\, t_0 \in \mathbf{R}$$

Time Differentiation: $x'(t) \doteq \frac{d}{dt}x(t)$

$$\mathrm{W}_h\{x'\}(a,b) = \frac{\partial}{\partial b}\mathrm{W}_h\{x\}(a,b)$$

Convolution:

$$\begin{aligned}\mathrm{W}_h\{x * y\}(a,b) &= [x * \mathrm{W}_h\{y\}(a,\cdot)](b) \\ &= [\mathrm{W}_h\{x\}(a,\cdot) * y](b)\end{aligned}$$

PART II

Introduction to Wavelet Bases and Frames

* Weyl-Heisenberg
* Affine

BASIC CONCEPTS of a HILBERT SPACE

Inner product Space – a linear space, X ,and a function $\langle \cdot, \cdot \rangle$ called an inner product defined on pairs of points (vectors) in X having the properties for each $u, v, w \in X$:

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$
2. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle \quad \forall$ scalars α
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle u, u \rangle > 0 \quad \forall\, u \neq 0$

Extends the notion of dot product for finite-dimensional vectors.

Norm, $\|\cdot\|$, induced by the inner product $\langle\cdot\,,\,\cdot\rangle$:

$$\|u\|^2 = \langle u, u\rangle \quad \forall\, u \in X$$

Extends the notion of length, or size, for finite-dimensional vectors. A sequence $\{u_n\}_{n=1}^{\infty}$ of vectors (points, functions, sequences, random variables, etc.) in an inner product space, X, is said to converge to the limit vector u, and we write $u_n \longrightarrow u$ if

$$\lim_{n\to\infty} \|u_n - u\| = 0\ ;$$

equivalently if,

$$\lim_{n\to\infty} \langle u_n - u, u_n - u\rangle = 0\ .$$

If the limit u is in the space, X, for all convergent sequences, $\{u_n\}_{n=1}^{\infty}$,
we say X is <u>complete</u>.

A complete inner product space is called a <u>Hilbert Space</u>.

Example

$$x(t) = \begin{cases} 0 & , \ t = 0 \\ 1 & , \ 0 < t \le 1/2 \\ 0 & , \ 1/2 < t \le 1 \end{cases}$$

$$\hat{x}(k) = \int_0^1 x(t)\, \mathrm{e}^{-i2\pi kt} dt$$

$$x_n(t) = \sum_{k=-n}^{n} \hat{x}(k)\, \mathrm{e}^{i2\pi kt}$$

$x_n(t) \not\longrightarrow x(t)$ at $t = 0, \frac{1}{2}, 1$; but $x_n \longrightarrow x$ in the Hilbert Space $L^2([0,1])$ since

$$\lim_{n\to\infty} \|x - x_n\|^2 = \lim_{n\to\infty} \int_0^1 |x(t) - x_n(t)|^2 dt = 0 \ .$$

Geometry

<u>Angle</u> $\theta \in (-\pi, \pi]$ between vectors

$$\langle u, v \rangle = \|u\| \, \|v\| \, \cos\theta$$

If $\langle u, v \rangle = 0$, we say u and v are <u>orthogonal</u> and write $u \perp v$.
If $\langle u, u \rangle = 1$, i.e., if $\|u\| = 1$, u is <u>normalized</u> (a unit vector).

A set $S = \{\ldots, u_{-1}, u_0, u_1, \ldots\}$ (which may be finite) is said to be <u>orthonormal</u> provided for all i, j

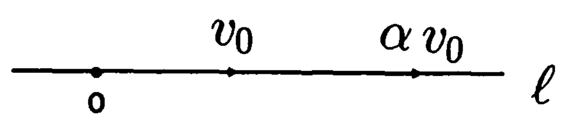
$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } \; i = j \\ 0, & \text{else} \; . \end{cases}$$

Ex. $\{ e^{i2\pi nt} \}_{n \in \mathbf{Z}}$ is an orthonormal set in $L^2([0,1])$.

A linear manifold, $\mathcal{M}$, in a Hilbert Space, $\mathcal{H}$, is a subset of $\mathcal{H}$ which is also a linear space. ($\mathcal{M}$ must contain the origin)

Simplest linear manifold: The line, ℓ generated by v_0 given by

$$\ell = \{\alpha\, v_0 : \ \forall \ \text{scalars} \ \alpha\}$$



The orthogonal projection of a vector $u \in \mathcal{H}$ onto the linear manifold, ℓ, (the line generated by v_0) is the vector $v \in \ell$ given by

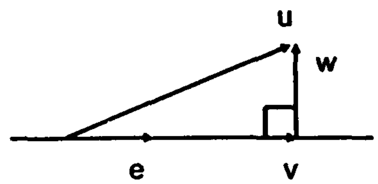
$$v = \langle v, e \rangle e$$

where

$$e = \frac{v_0}{\|v_0\|}$$

is a unit vector. Given v, can always write an orthogonal decomposition of u

$$u = v + w$$



where

$$w \perp v.$$

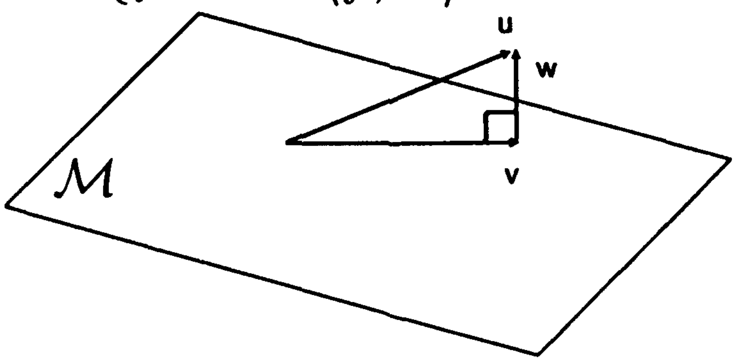
Hence the name "orthogonal projection".

For any manifold $\mathcal{M} \subset \mathcal{H}$, for which $\mathcal{M}$ is also a Hilbert Space, we may write any vector $u \in \mathcal{H}$ as

$$u = v + w$$

where $v \in \mathcal{M}$ and $w \in \mathcal{M}^{\perp}$. We call $\mathcal{M}^{\perp}$ the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$. It is defined by

$$\mathcal{M}^{\perp} \doteq \{f \in \mathcal{H} : \langle f, m \rangle = 0 \;\; \forall\, m \in \mathcal{M}\}$$



This is the "Classical Projection Theorem"

Orthonormal Bases

An orthonormal set $\{e_n\}_{n\in\mathbf{Z}} \subset \mathcal{H}$ having the property that

$$\langle e_n, v\rangle = 0 \quad \text{for all} \quad n \in \mathbf{Z} \quad \Longrightarrow v = 0$$

is an <u>orthonormal basis</u> for $\mathcal{H}$. Given such an orthonormal basis, any $x \in \mathcal{H}$ may be written as a <u>generalized Fourier series</u>

$$x = \sum_{n\in\mathbf{Z}} c_n\, e_n$$

where

$$c_n = \langle x, e_n\rangle$$

are <u>generalized Fourier coefficients</u> and the series converges in norm.

Orthonormal bases are linearly independent sets; i.e., for any integers M and N,

$$\sum_{i=M}^{N} \alpha_k e_k = 0 \implies \alpha_k = 0 \ \forall\, k = M, M+1, \ldots, N;$$

but the converse is not true.

Ex.1 Two non-orthogonal, non-collinear, coplanar vectors in $\mathbf{R}^2$ are linearly independent.

Ex.2 $\left\{ e^{i2\pi t}, \cos(2\pi t) \right\}$ is a linearly independent non-orthogonal set in $L^2([0,1])$.

With an orthonormal basis, we have Parseval's Identity:

$$\begin{aligned} \langle x, y \rangle &= \sum_{n \in \mathbf{Z}} \langle x, e_n \rangle \langle e_n, y \rangle \\ &= \sum_{n \in \mathbf{Z}} \langle x, e_n \rangle \overline{\langle y, e_n \rangle} \end{aligned}$$

Setting $y = x$ yields Plancherel's Identity

$$\|x\|^2 = \langle x, x \rangle = \sum_{n \in \mathbf{Z}} |\langle x, e_n \rangle|^2 \ .$$

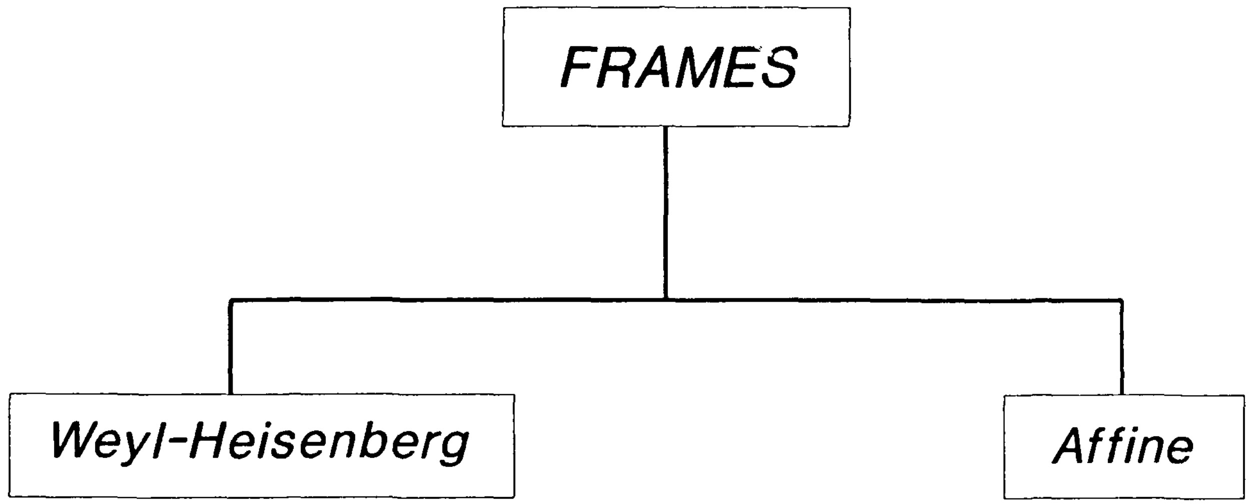
When $x \in L^2(\mathbf{R})$ (finite energy, continuous time signals), this says that the generalized Fourier coefficients form a finite energy sequence

$$(\cdots, \langle x, e_{-1} \rangle, \langle x, e_0 \rangle, \langle x, e_1 \rangle, \cdots) \in l^2(\mathbf{Z})$$

A Hilbert space which possesses a countable orthonormal basis is said to be separable.

All separable Hilbert spaces are essentially

(i.e.,equivalent to) $l^2(\mathbf{Z})$!!!


FRAMES
Weyl-Heisenberg
Affine

WAVELET BASES AND FRAMES

Recall the continuous W-H Wavelet (Windowed Fourier) Transform:

$$\begin{aligned} \mathrm{W}\mathcal{F}_g\{x\}(p,q) &= \int_{-\infty}^{\infty} x(t)\,\overline{g(t-q)}\,\mathrm{e}^{-i2\pi pt}dt\ ; \qquad p,q\in\mathbf{R} \\ &= \langle x, g^{(p,q)}\rangle \end{aligned}$$

where

$$g^{(p,q)}(t) = g(t-q)\,\mathrm{e}^{i2\pi pt}$$

For computations, we discretize p and q .

Choose $p_0 > 0,\ q_0 > 0$ and define

$$\begin{aligned} g_{mn}(t) &= g^{(m p_0, n q_0)}(t) \ ; \qquad m, n \in \mathbf{Z} \\ &= g(t - nq_0)\, \mathrm{e}^{i2\pi m p_0 t} \end{aligned}$$

Analogous to Fourier coefficients, $c_n = \langle x, E_n \rangle$, the Weyl-Heisenburg Wavelet coefficients are

$$\begin{aligned} c_{mn} &= \langle x, g_{mn} \rangle \\ &= \mathrm{W}\mathcal{F}_g\{x\}(mp_0, nq_0) \end{aligned}$$

Questions

1. Given the coefficients, c_{mn} , can we reconstruct the original signal x ?

2. (a) To what extent (if at all) does

$$\sum_{m,n\in\mathbf{Z}} \langle x, g_{mn}\rangle \, g_{mn}$$

represent x ;

2. (b) i.e., can we ever treat $\{g_{mn}\}_{m,n\in\mathbf{Z}}$ as an orthonormal basis of $L^2(\mathbf{R})$?

3. Are there restrictions on how finely we must sample the (p, q) plane?

Answers

1. Sometimes

2. (a) Sometimes, to within computable error.

2. (b) Yes

3. Yes

Answers are given by the notion of "Frames." [DS], [HW], [Db3].

A sequence, $\{\phi_n\}_{n\in\mathbf{Z}}$, in a Hilbert space, $\mathcal{H}$ is called a <u>frame</u> if we can find two positive numbers A and B $(B \geqslant A > 0)$ such that, for all vectors $x \in \mathcal{H}$ we have

$$A\|x\|^2 \leq \sum_{n\in\mathbf{Z}} |\langle x, \phi_n\rangle|^2 \leq B\|x\|^2$$

Constants A and B are called <u>frame bounds</u>.

If $A = B$, the frame is <u>tight</u>.

If B/A is "close to 1" the frame is <u>snug</u> with "snugness", S , given by

$$S = \left(\frac{B}{A} - 1\right)^{-1} .$$

If upon deletion of an arbitrary single element, the sequence ceases to be a frame, the frame is <u>exact</u>.

Analyze the frame definition when $\mathcal{H} = L^2(\mathbf{R})$.

$$A \|x\|^2 \leq \sum_{n \in \mathbf{Z}} |\langle x, \phi_n \rangle|^2 \leq B \|x\|^2$$

For each $x \in \mathcal{H}$, $\{\langle x, \phi_n \rangle\}_{n \in \mathbf{Z}}$ is a sequence with "energy" given by the sum above. The energy is bounded by the energy in x and non-zero for all non-zero vectors $x \in \mathcal{H}$. Compare with Plancherel for orthonormal bases.

Only $x = 0$ is orthogonal to every member of a frame since

$$\begin{aligned} x \perp \phi_n \Longrightarrow \quad \langle x, \phi_n \rangle &= 0 \\ \Longrightarrow \quad A\,\|x\| &\leq 0 \\ \Longrightarrow \quad \|x\| &= 0 \end{aligned}$$

since $A > 0$.

An orthonormal basis is a tight, exact, frame with frame constants

$$A = B = 1.$$

The frame definition reduces to Plancherel's Identity.

Examples of Frames [HW]

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$.

Ex.1 $\{e_1, e_1, e_2, e_2, e_3, e_3, \ldots\}$ is a tight inexact frame with $A = B = 2$, not an orthonormal basis, but contains one.

Ex.2 $\{e_1, e_2/2, e_3/3, \ldots\}$ is a complete orthogonal sequence, but not a frame. There is no non-zero lower bound, A ; for, if $x = e_M$, we have

$$A \|e_M\|^2 \leq \frac{1}{M^2} \sum_{n \in \mathbf{Z}} |\langle e_n, e_M \rangle|^2$$

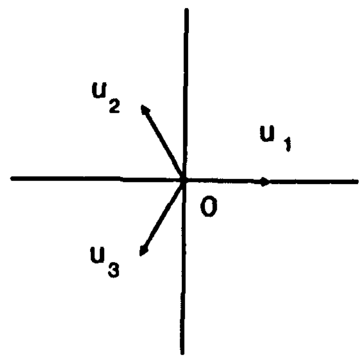
$$A \leq \frac{1}{M^2} \quad \text{for all integers } M.$$

<u>Ex.3</u> $\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, \ldots\}$ is a tight inexact frame with $A = B = 1$ and no non-redundant subsequence is a frame.

<u>Ex.4</u> $\{2e_1, e_2, e_3, \ldots\}$ is a non-tight exact frame for which $A = 1$, $B = 4$.

Example in $\mathbf{R}^2$ [Db3]

$$\begin{aligned} u_1 &= e_1 \\ u_2 &= -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2 \\ u_3 &= -\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2 \end{aligned}$$



where

$$e_1 = (0,1), \quad e_2 = (1,0)$$

Then any $v \in \mathbf{R}^2$ satisfies

$$\sum_{k=1}^{3} |\langle v, u_k \rangle|^2 = \frac{3}{2}\|v\|^2$$

and

$$v = \frac{2}{3}\sum_{k=1}^{3}\langle v, u_k \rangle\, u_k$$

The frame in Daubechies' example is tight with $A = B = \frac{3}{2}$, not exact.

In tight frames, the frame constants $A = B$ indicates rate of redundancy of the frame.

A tight frame of normalized vectors with $A = B = 1$ is an orthonormal basis.

Frames and Bases [HW]

A sequence $\{\phi_n\}_{n\in\mathbf{Z}}$ in a Hilbert space, $\mathcal{H}$, is a <u>basis</u> for $\mathcal{H}$ if for every $x \in \mathcal{H}$ there are *unique* scalars c_n so that x can be represented by

$$x = \sum_{n\in\mathbf{Z}} c_n \phi_n \quad .$$

The basis is <u>bounded</u> if $0 < \inf_n \|\phi_n\| \leq \sup_n \|\phi_n\| < \infty$.

It is an <u>unconditional basis</u> if every permutation of the above series converges.

In a Hilbert space, $\mathcal{H}$, each bounded unconditional basis is equivalent to an orthonormal basis; i.e., there is an isomorphism $T : \mathcal{H} \to \mathcal{H}$ such that $\phi_n = Te_n$ for all n.

A sequence in a Hilbert space, $\mathcal{H}$, is an exact frame for $\mathcal{H}$ if and only if it is a bounded unconditional basis for $\mathcal{H}$.

Hence, exact frames are equivalent to orthonormal bases.

Frame Operator on $L^2(\mathbf{R})$

Given a frame $\{\phi_n\}_{n\in\mathbf{Z}}$ for $L^2(\mathbf{R})$, define the frame operator $T : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ by

$$Tx = \sum_{k\in\mathbf{Z}} \langle x, \phi_k \rangle \, \phi_k$$

T is bounded and invertible.

Any "signal" $x \in L^2(\mathbf{R})$ may be written

$$x = \sum_{n \in \mathbf{Z}} \langle x, \phi_n \rangle \widetilde{\phi}_n$$

where

$$\widetilde{\phi}_n = T^{-1} \phi_n \quad .$$

The family $\{\widetilde{\phi}_n\}_{n \in \mathbf{Z}}$ is another frame called the <u>dual frame</u> having frame bounds B^{-1}, A^{-1} such that $0 < B^{-1} \leq A^{-1} < \infty$.

To construct x exactly from its frame coefficients, $\langle x, \phi_n \rangle$, one has to be able to compute $T^{-1}\phi_n$. To do so, write the identity

$$T = \left(\frac{2}{A+B}\right)^{-1} \left\{I - \left[I - \frac{2T}{A+B}\right]\right\}$$

where I is the Identity operator, i.e., $Ix = x, \ \forall\, x \in \mathcal{H}$. Then,

$$T^{-1} = \frac{2}{A+B} \left\{I - \mathcal{O}\right\}^{-1}$$

where

$$\mathcal{O} = I - \frac{2T}{A+B} \ .$$

Can show

$$\|\mathcal{O}\| \leq \frac{B/A - 1}{B/A + 1} < 1$$

so that a "geometric series" expansion applies:

$$T^{-1} = \frac{2}{A+B} \sum_{k=0}^{\infty} \mathcal{O}^k.$$

Hence,

$$\widetilde{\phi}_j = T^{-1}\phi_j = \frac{2}{A+B} \sum_{k=0}^{\infty} \left[I - \frac{2T}{A+B} \right]^k \phi_j$$

with faster convergence when B/A is closer to 1.

Keep only the first term $(k = 0)$ in the series,

$$\tilde{\phi}_j = \frac{2}{A+B}\,\phi_j + \mathrm{O}\left(\frac{B}{A} - 1\right)$$

so that

$$x \approx \frac{2}{A+B} \sum_{n \in \mathbf{Z}} \langle x, \phi_n \rangle\, \phi_n$$

with error that vanishes as snugness, $S = \left(\frac{B}{A} - 1\right)^{-1}$, increases.

For tight frames, have $A = B$, and the frame operator is $T = AI$, so that

$$x = \frac{1}{A} \sum_{n \in \mathbf{Z}} \langle x, \phi_n \rangle \, \phi_n \; .$$

If the elements $\{\phi_n\}_{n \in \mathbf{Z}}$ of a tight frame having $A = B = 1$ have unit norm, $\|\phi_n\| = 1, \; \forall \, n \in \mathbf{Z}$, the frame is an orthonormal basis for $L^2(\mathbf{R})$ so that

$$x = \sum_{n \in \mathbf{Z}} \langle x, \phi_n \rangle \, \phi_n \; .$$

Weyl-Heisenberg Frames

The frame operator is

$$Tx = \sum_{m,n \in \mathbf{Z}} \langle x, g_{mn} \rangle g_{mn}$$

where we have used the earlier definition

$$g_{mn}(t) = g(t - nq_0)\, e^{i2\pi m p_0 t}$$

Then,

$$x(t) = \sum_{m,n \in \mathbf{Z}} \langle x, g_{mn} \rangle\, \widetilde{g}_{mn}(t)$$

In the W-H case, don't have a double-infinity of dual basis elements, $\widetilde{g}_{mn}$, to compute, since

$$\widetilde{g}_{mn}(t) = \widetilde{g}_{00}(t - nq_0)\, \mathrm{e}^{-i2\pi m p_0 t}$$

where

$$\widetilde{g}_{00} = T^{-1} g \ .$$

Use the rapidly converging series to compute only $\widetilde{g}_{00}$.

Affine Frames

Recall the continuous Affine Wavelet Transform

$$\begin{aligned} \mathrm{W}_h\{x\}(a,b) &= \int_{-\infty}^{\infty} x(t)\overline{h^{(a,b)}(t)}dt \\ &= \langle x, h^{(a,b)} \rangle \end{aligned}$$

where

$$h^{(a,b)}(t) = \frac{1}{\sqrt{a}}\, h\left(\frac{t-b}{a}\right)$$

for a mother wavelet, h.

For computations, discretize a and b. Choose a dilation step $a_0 > 1$, and choose

$$a = a_0^m \quad \forall \; m \in \mathbf{Z} \; .$$

Adapt the translation b to the varying dilations a_0^m via

$$b = n \, a_0^m \, b_0$$

for a chosen translation step, $b_0 > 0$, and define

$$\begin{aligned} h_{m\,n}(t) &\doteq h^{(a_0^m, nb_0 a_0^m)}(t) \\ &= a_0^{-m/2} h\left(a_0^{-m} t - nb_0\right) \end{aligned}$$

The affine wavelet coefficients are then

$$\langle x, h_{mn} \rangle = \mathrm{W}_h\{x\}(a_0^m, nb_0 a_0^m) \ .$$

When $\{h_{mn}\}_{m,n\in\mathbf{Z}}$ constitutes a frame for $L^2(\mathbf{R})$, then

$$\begin{aligned} x &= \sum_{m,n\in\mathbf{Z}} \langle x, h_{mn} \rangle \widetilde{h}_{mn} \\ &\approx \frac{2}{A+B} \sum_{m,n\in\mathbf{Z}} \langle x, h_{mn} \rangle \, h_{mn} \end{aligned}$$

The affine wavelet frame operator is

$$Tx = \sum_{m,n} \langle x, h_{mn} \rangle \, h_{mn}$$

and

$$\widetilde{h}_{mn} = T^{-1} h_{mn} \,,$$

computable from the rapidly converging (for B/A close to 1) geometric series for T^{-1} .

Contrary to W-H frames, affine frames require computation of a single infinity of dual basis functions, $\widetilde{h}_{mn}$:

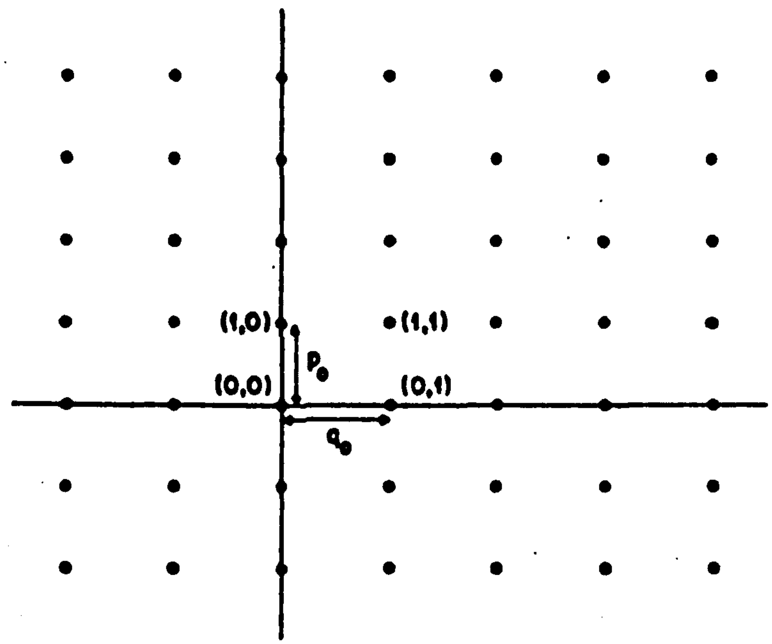
$$\widetilde{h}_{mn}(t) = a_0^{-m/2}\widetilde{h}_{0n}(a_0^{-m}t)$$

where

$$\widetilde{h}_{0n}(t) = T^{-1}h_{0n}.$$

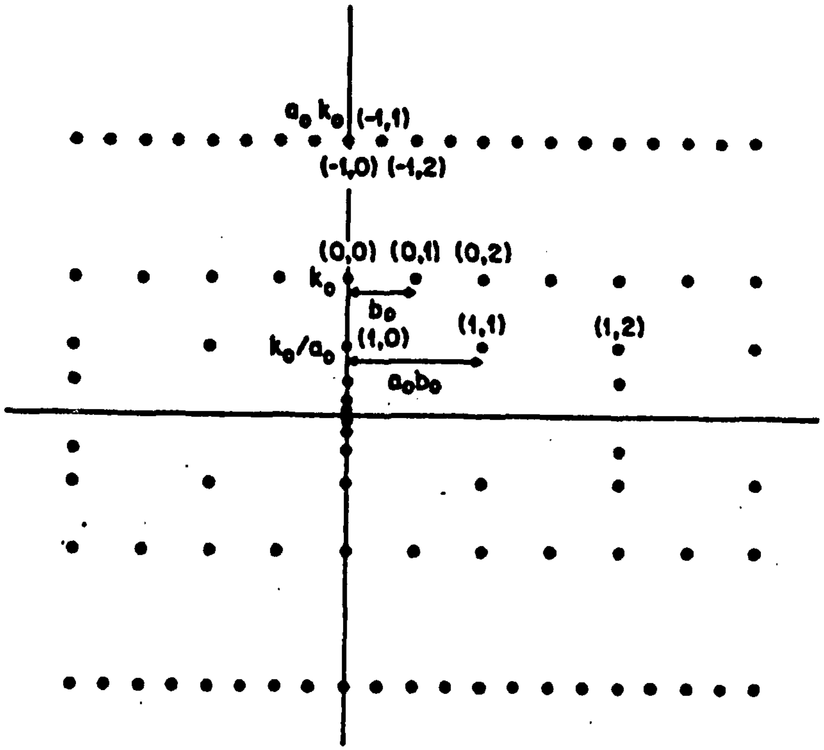
Again, the rapidly converging series for T^{-1} reduces the amount of computation required in practice for snug frames.

W-H Wavelet Lattice



[Db3]

Affine Wavelet Lattice



[Db3]

Lattice Spacings

W-H Critical Product, $p_0 q_0 = 1$, above which $\{g_{mn}\}_{m,n \in \mathbf{Z}}$ is not a frame; i.e.,

$$\text{must have } \frac{1}{p_0 q_0} \geq 1.$$

At the critical value, only slowly-decaying or non-smooth windows, g, can generate a frame according to the celebrated Balian-Low Theorem [B1,L1].

Balian-Low

<u>Theorem</u>. Given $g \in L^2(\mathbf{R})$, $p_0 > 0$. If the associated

$$g_{m\,n}(t) = g(t - nq_0)\, e^{i2\pi p_0 m t}$$

constitute a frame with $p_0 q_0 = 1$, then either

(a) $\breve{g} \notin L^2(\mathbf{R})$ where $\breve{g}(t) = t\, g(t)$

or

(b) $g' \notin L^2(\mathbf{R})$ where $g'(t) = \frac{d}{dt}\, g(t)$.

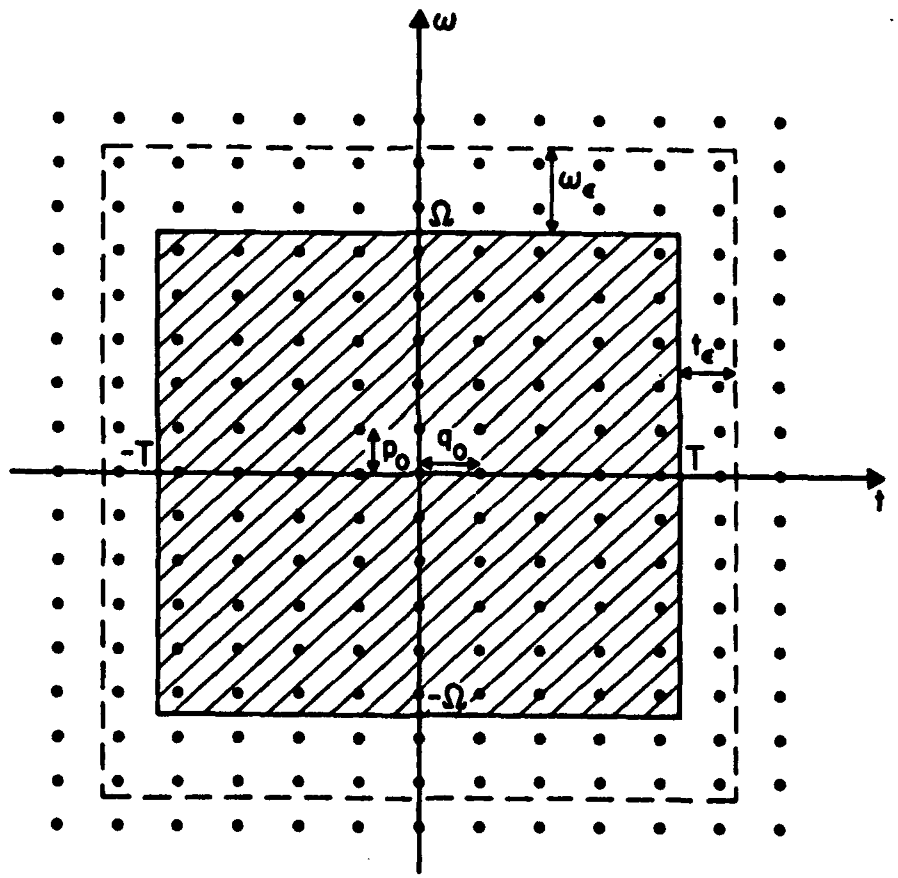
Affine wavelets lattices have no *a priori* limitations on lattice spacings, other than

$$a_0 > 1$$
$$b_0 > 0$$

Some choices lead to snugger frames than others, however.

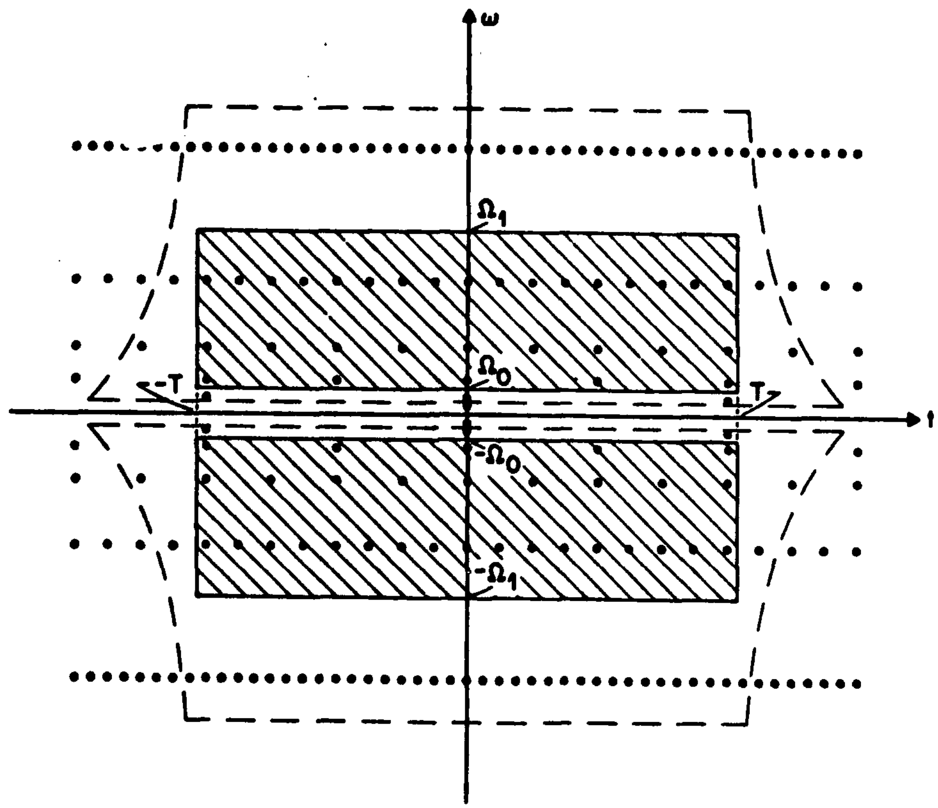
See Daubechies [Db3] for more on this and estimation (computation) of frame bounds.

Weyl-Heisenberg Localization



[Db3]

Affine Localization



[Db3]

Related Time-Frequency Distributions

- Cohen
- Cross Wigner
- Cross Ambiguity
- Gabor Representation

Cohen's Distribution [CM]

$$C_{f,g}(t, w; \Phi) =$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i2\pi(\xi t - \tau w - \xi w)} \Phi(\xi, \tau)\, f(u + \frac{\tau}{2})\, \overline{g(u - \frac{\tau}{2})} du\, d\tau\, d\xi$$

where $\Phi(\xi, \tau)$ is some kernel function.

Cross Wigner Distribution [CM]

$$W_{f,g}(a,b) = \int_{-\infty}^{\infty} \mathrm{e}^{-i2\pi bx} f(a + \frac{x}{2})\, \overline{g(a - \frac{x}{2})}\, dx$$

Cross Ambiguity Function

$$A_{f,g}(\nu, \tau) = \int_{-\infty}^{\infty} \mathrm{e}^{-i2\pi\nu t} f(t + \frac{\tau}{2}) g(t - \frac{\tau}{2})\, dt$$

Gabor Representation [J1],[DGM]

$$f(t) = \sum_{m,n \in \mathbf{Z}} c_{m,n}\, \mathrm{e}^{-i2\pi m\beta(t-n\alpha)} g(t-n\alpha)$$

Why bother with frames?

- Physical processes modelled by non-orthogonal "building blocks"
- Redundancy in frames $\Longrightarrow$ oversampling
  - •• Cancellation of roundoff errors

PART III

AFIT Applications of Wavelets

* Biological Connections
* Speech: One Dimensional Signal Processing
* Images: Two Dimensional Signal Processing

BIOLOGICAL CONNECTIONS

* Cochlear Data
* Cortical Data
* Occulometer Data
* Illusions

MECHANICAL COMPONENTS OF THE EAR

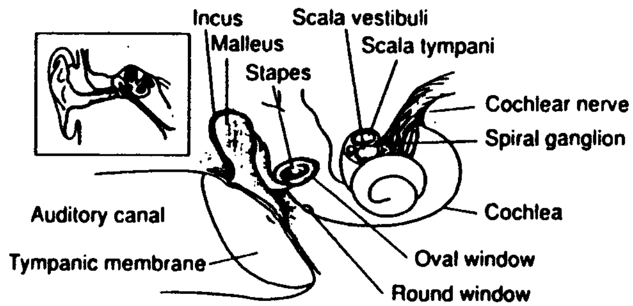


Figure 52-1. The tympanic membrane, the ossicular system of the middle ear, and the inner ear.

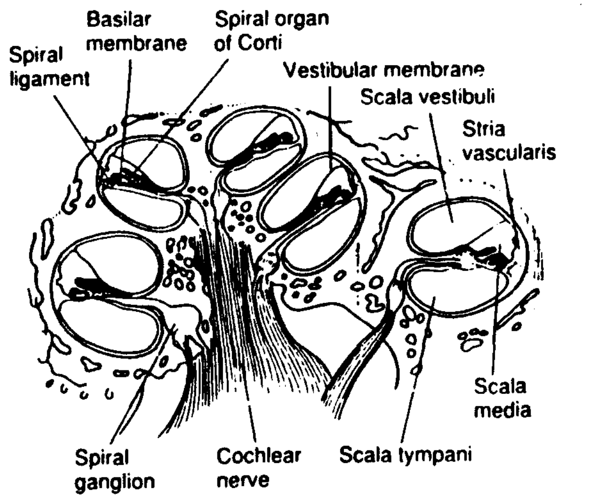


Figure 52-2. The cochlea. (From Goss, C. M. [ed.]: Gray's Anatomy of the Human Body. Philadelphia, Lea & Febiger.)

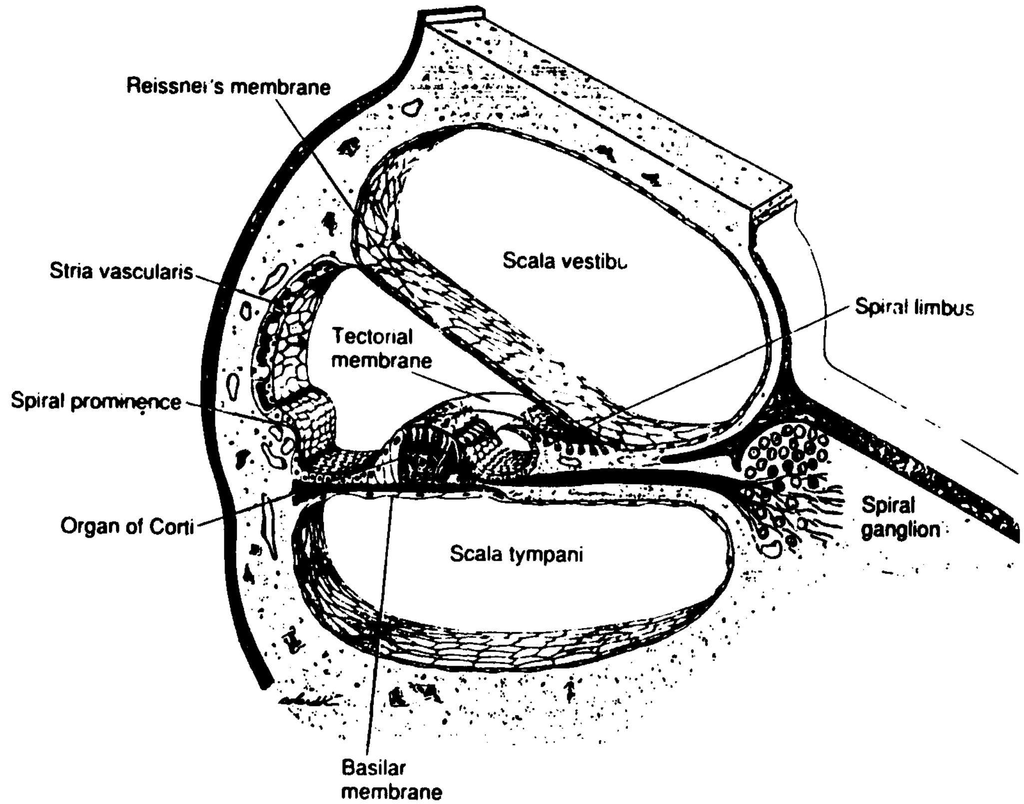


Figure 52-3. A section through one of the turns of the cochlea. (Drawn by Sylvia Colard Keene. From Fawcett: A Textbook of Histology, 11th Ed. Philadelphia, W. B. Saunders Company, 1986.)

Figures from Textbook of Medical Physiology, 8th Ed. by Guyton.

TUNING CHARACTERISTICS OF THE EAR

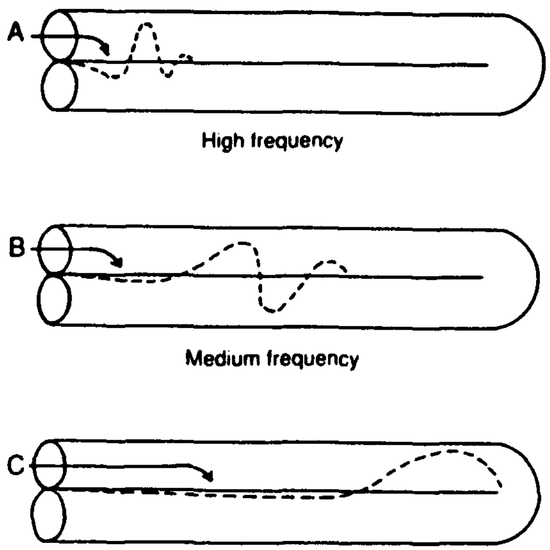


Figure 52-5. "Traveling waves" along the basilar membrane for high, medium, and low frequency sounds.

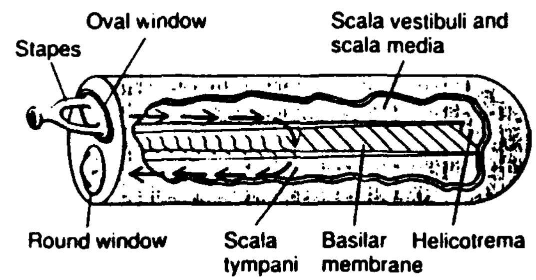


Figure 52-4. Movement of fluid in the cochlea following forward thrust of the stapes.

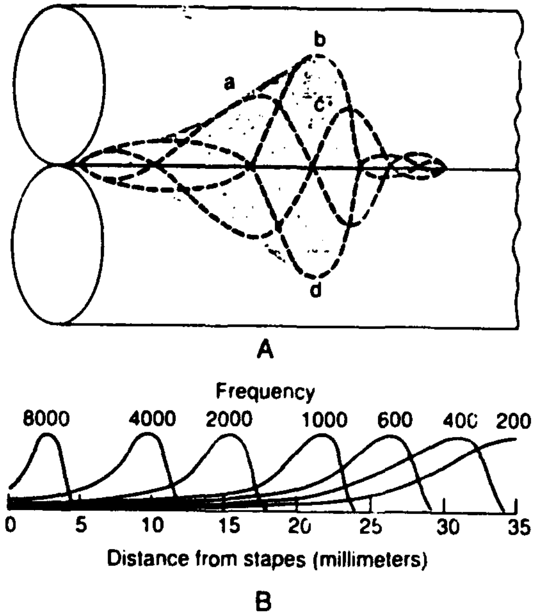


Figure 52-6. *A*, Amplitude pattern of vibration of the basilar membrane for a medium frequency sound. *B*, Amplitude patterns for sounds of all frequencies between 200 and 8000 per second, showing the points of maximum amplitude (the resonance points) on the basilar membrane for the different frequencies.

CORTICAL STRUCTURE IN BRAIN

CEREBELLAR CONNECTIONS

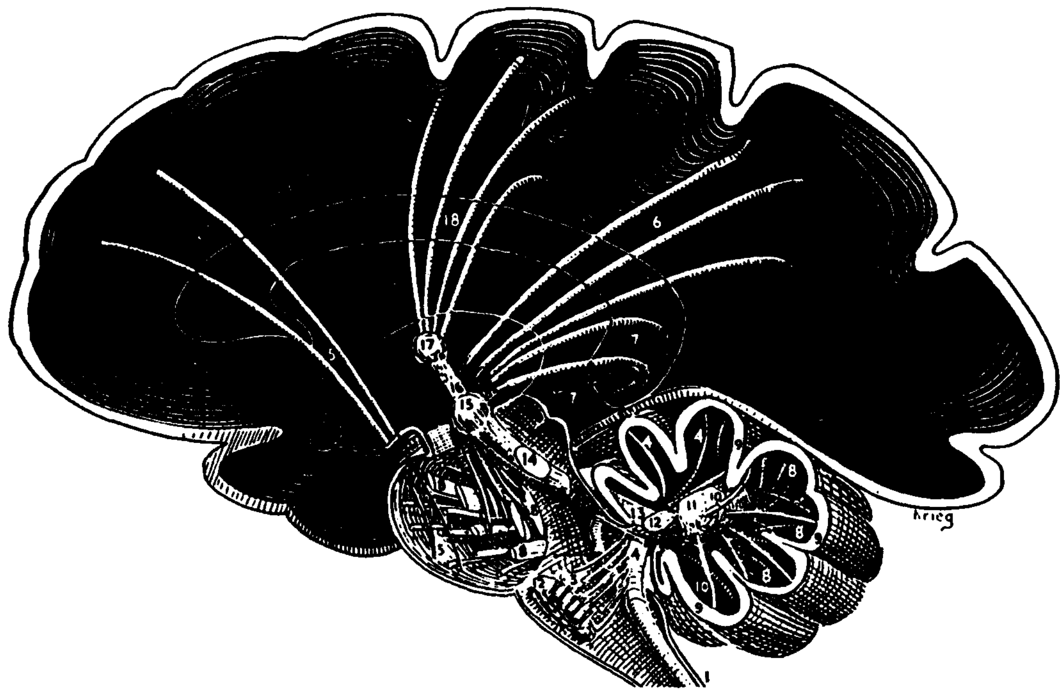
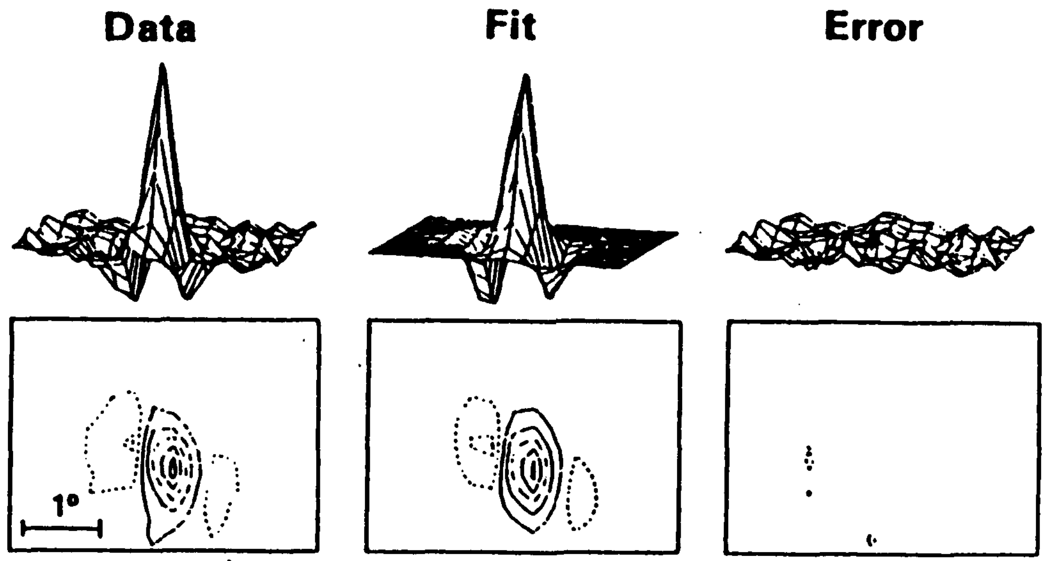


Figure from Krieg, W.J.S. Functional Neuroanatomy, Blakiston Company, 1942.

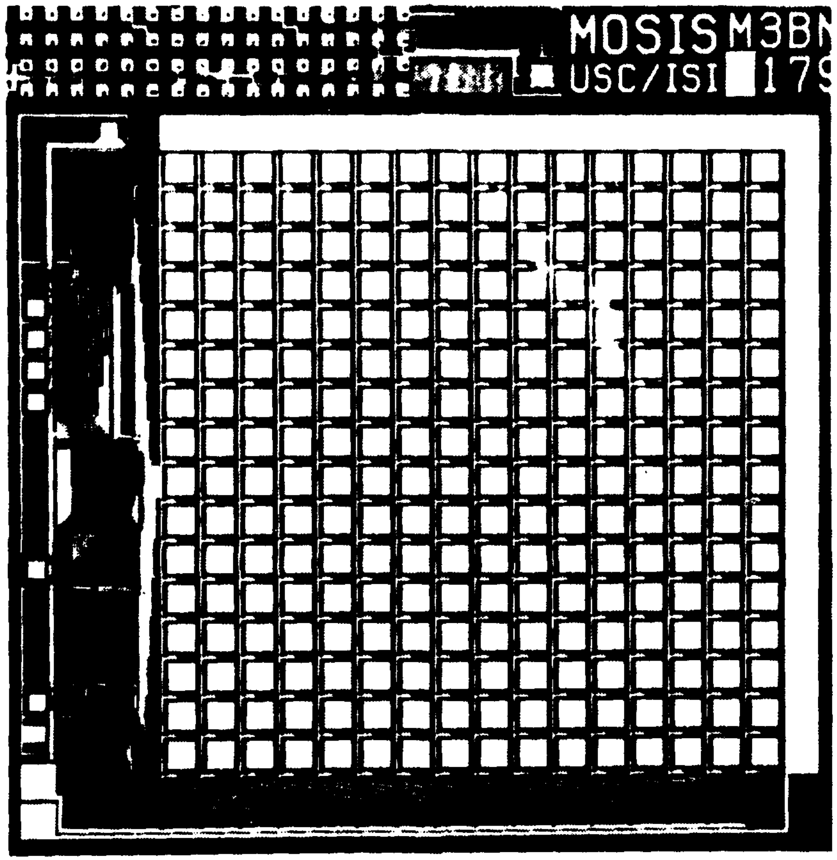
VISUAL IMPULSE RESPONSE OF MAMMALIAN VISION



Jones, J. and L. Palmer. "An Evaluation of the Two Dimensional Gabor Filter Model of Simple Receptive Fields in Cat Striate Cortex," *Journal of Neurophysiology*, *58*:1233–1258 (1987).

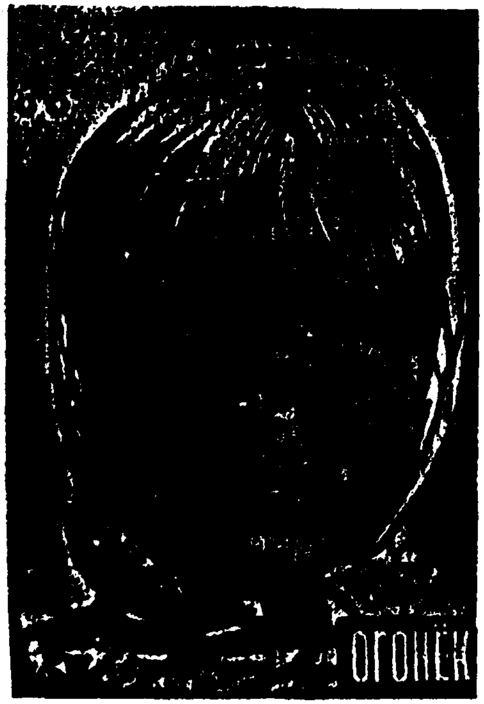
Ln Z Image Transform from the retina to the visual cortex

AFIT MULTIPLEXED MULTIELECTRODE CORTICAL ELECTRODE



The new 16 x 16 multiplexed electrode. The electrodes are 250 microns square. At the left is gate line driver; the on-board multiplexer for the 16 column output lineis in the lower left corner.

A



B

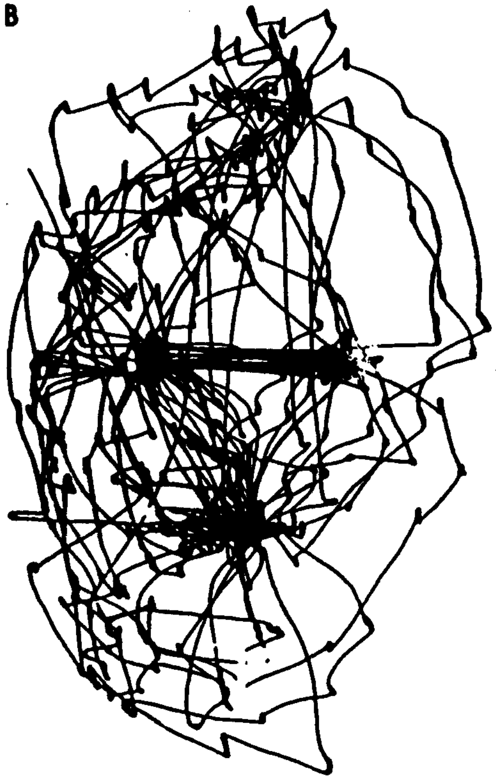


FIGURE 35 Eye movements during examination of a photograph by a normal subject. (*A*) Photograph given to subject for examination. (*B*) Recording of the eye movements during the examination of the photograph. (*After Yarbus, 1961.*)

Figures from A.L. Luria. <u>Higher Cortical Function</u>.

AFIT EYE SCAN DATA

RELATIONSHIP OF EYE SCAN PATTERN TO GABOR FILTERED IMAGE

MÜLLER-LYER ILLUSION SHOWN TO BE A CONSEQUENCE OF LOW PASS FOURIER FILTERING

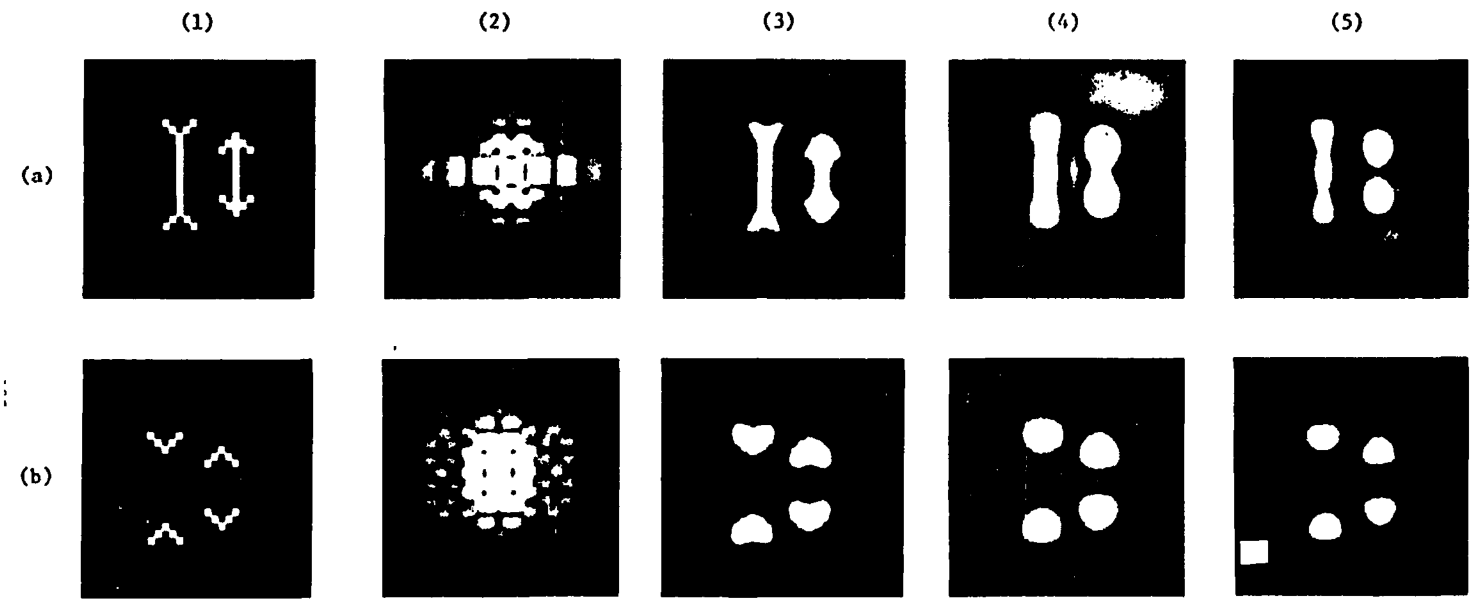
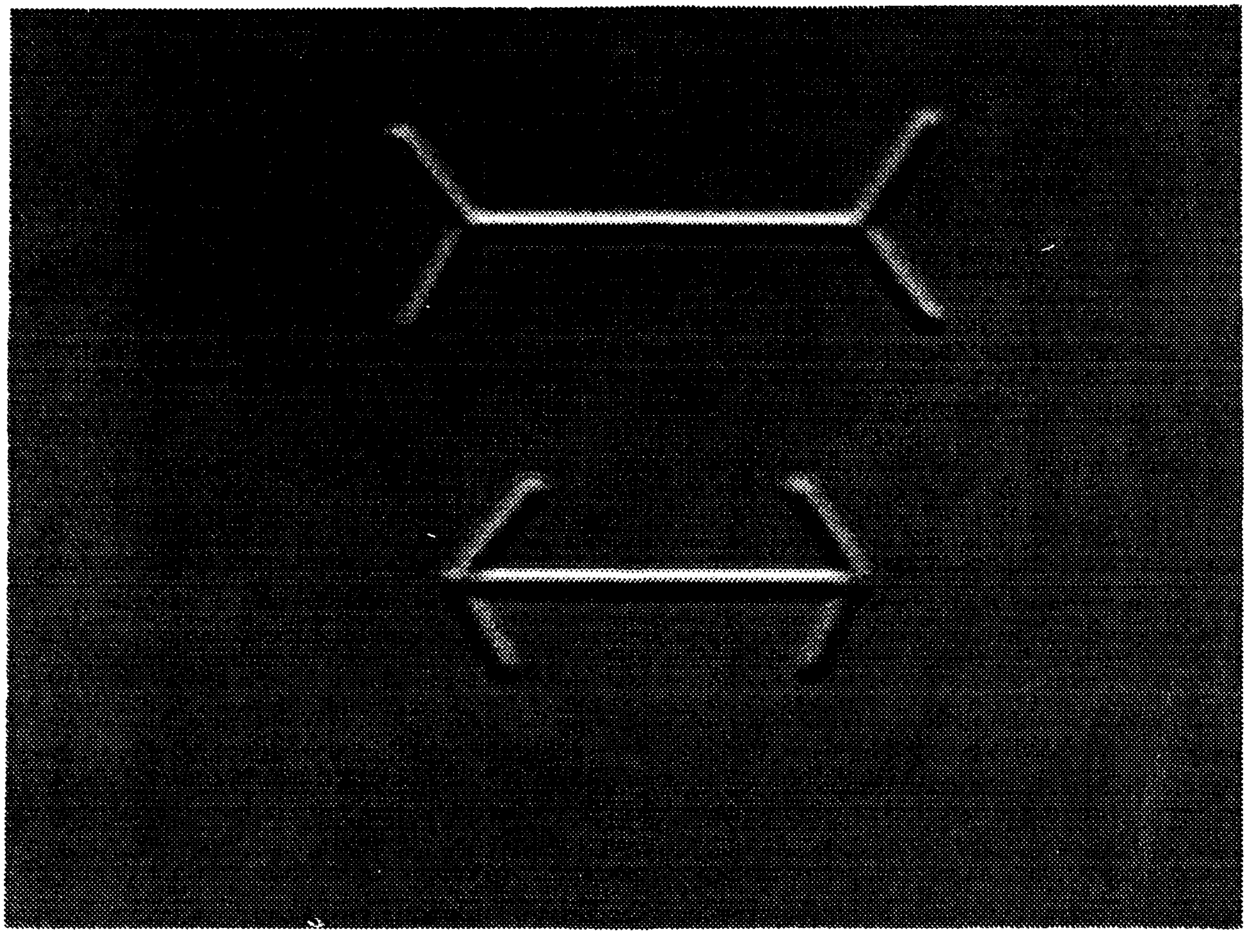


Figure 50. Müller-Lyer Illusions: (a) Standard Müller-Lyer Illusion; (b) Müller-Lyer Illusion with Fins Only; (c) Müller-Lyer Illusion with Unequal Size Fins Going in the Same Direction; (1) Original Illusions; (2) MTF Filtered Magnitude Spectra; (3) MTF Filtered Illusions; (4) MTF (5 by 5)L Filtered Illusions; (5) Low-Pass (5 by 5) Filtered Illusions. (Section 12.3.1)

Figures from Ginsburg, A.P. "Visual information processing based on Spatial Filters Constrained by Biological Data," AMRL-TR 78-129.

MÜLLER-LYER ILLUSION SHOWN TO BE A CONSEQUENCE OF GABOR FILTERING

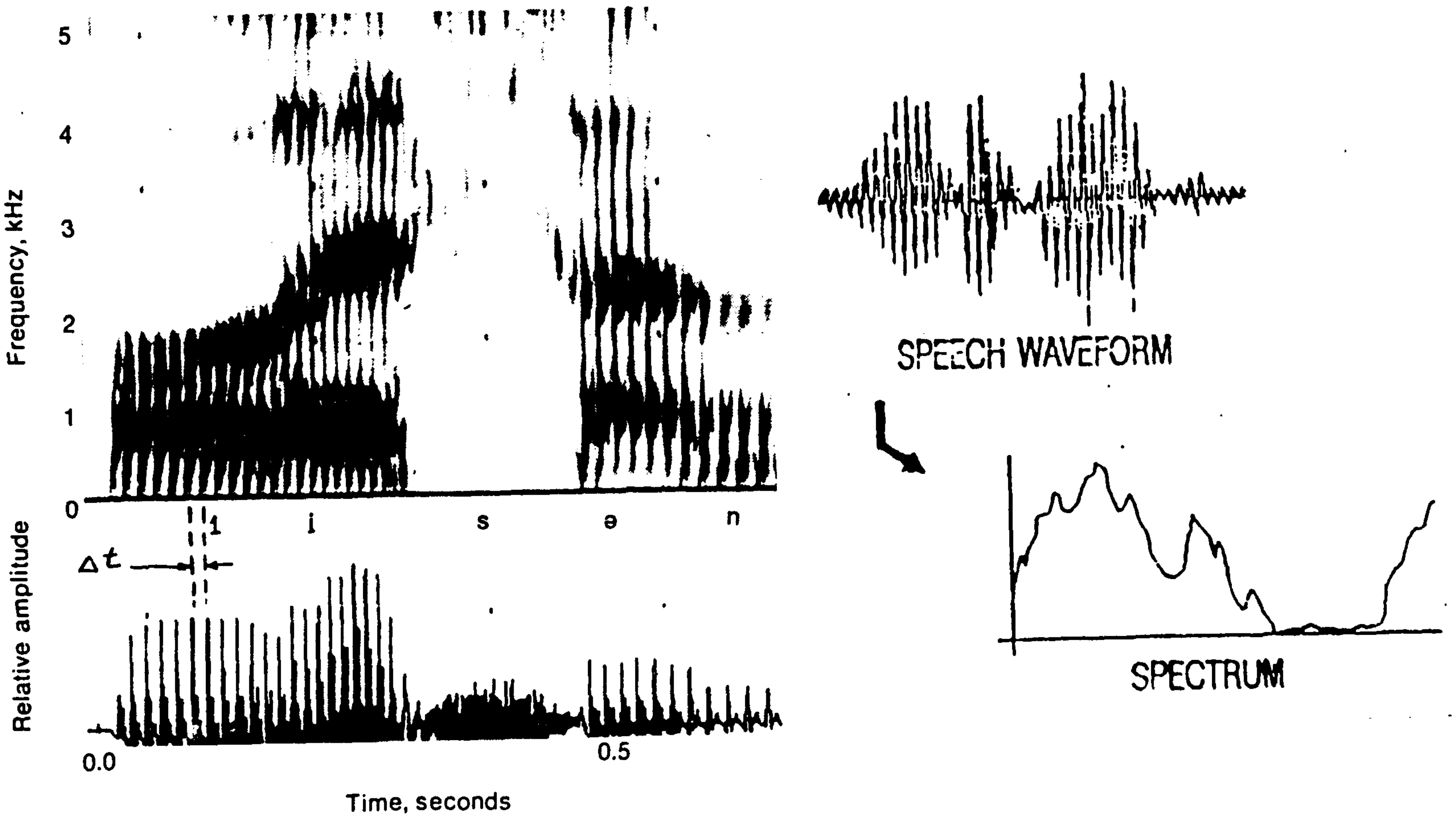


SPEECH: ONE DIMENSIONAL SIGNAL PROCESSING

* Problem Definition
* Speech Coding (Audio Tape)
* Speech Recognition

NATURE OF THE SPEECH SIGNAL

PROCESSED BY WINDOWED FOURIER TRANSFORM



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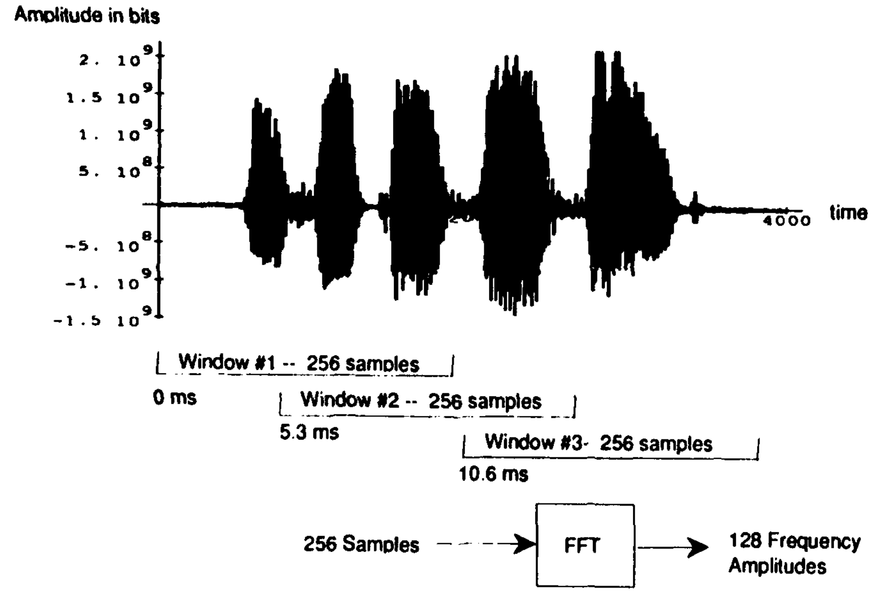
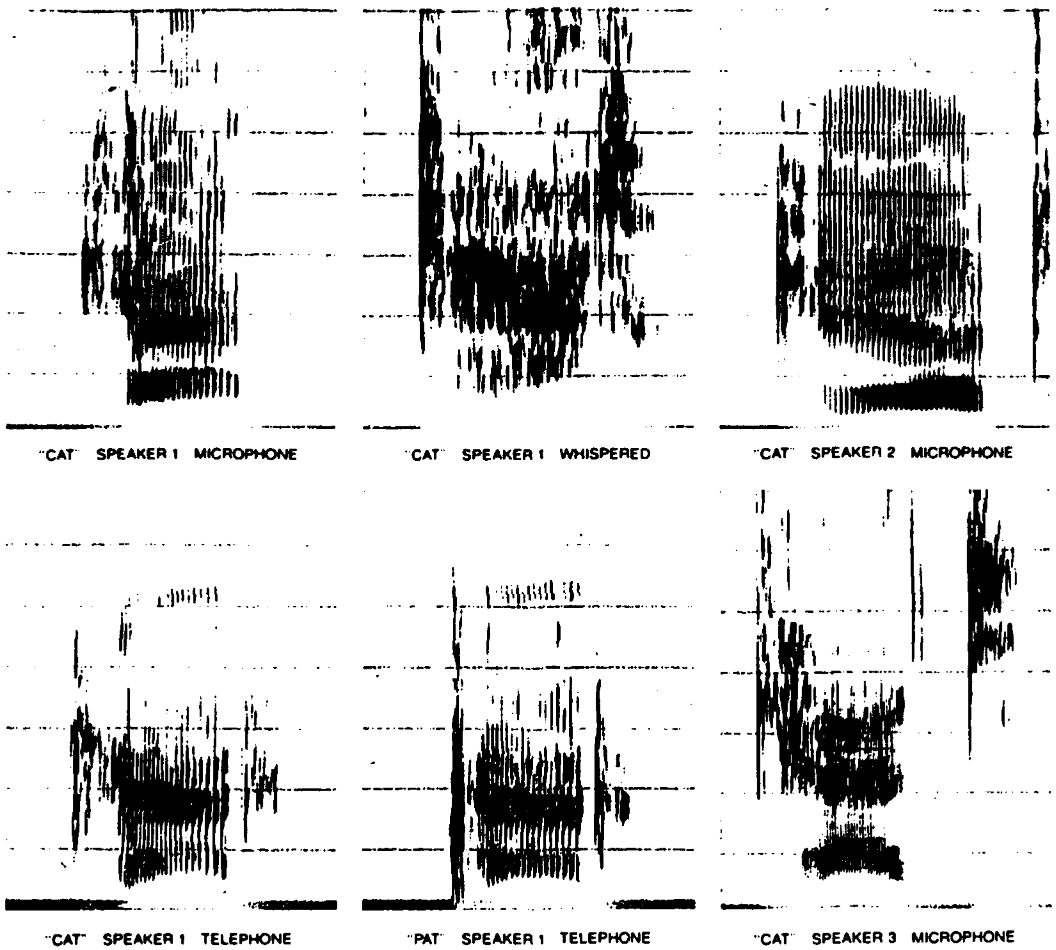
SPEECH RECOGNITION DETAILS

WINDOWING PARAMETERS



INTRINSIC VARIABILITY IN SPEECH SIGNAL



VARIABILITY OF HUMAN SPEECH, illustrated here by means of sound spectrograms, is one of the principal difficulties encountered in building an automated system for speech recognition. Spectrograms of distinct but acoustically similar words may be more alike than spectrograms of the same word pronounced under various conditions by different speakers. Automatic speech recognition must be able to attend only to relevant spectral differences (when they exist) and must disregard apparent differences that are linguistically irrelevant. The sound spectrogram represents a series of amplitude spectra over time. Time varies along the horizontal axis and frequency varies along the vertical axis. The darker the mark on the graph, the greater the amplitude of the waveform at that frequency and time.

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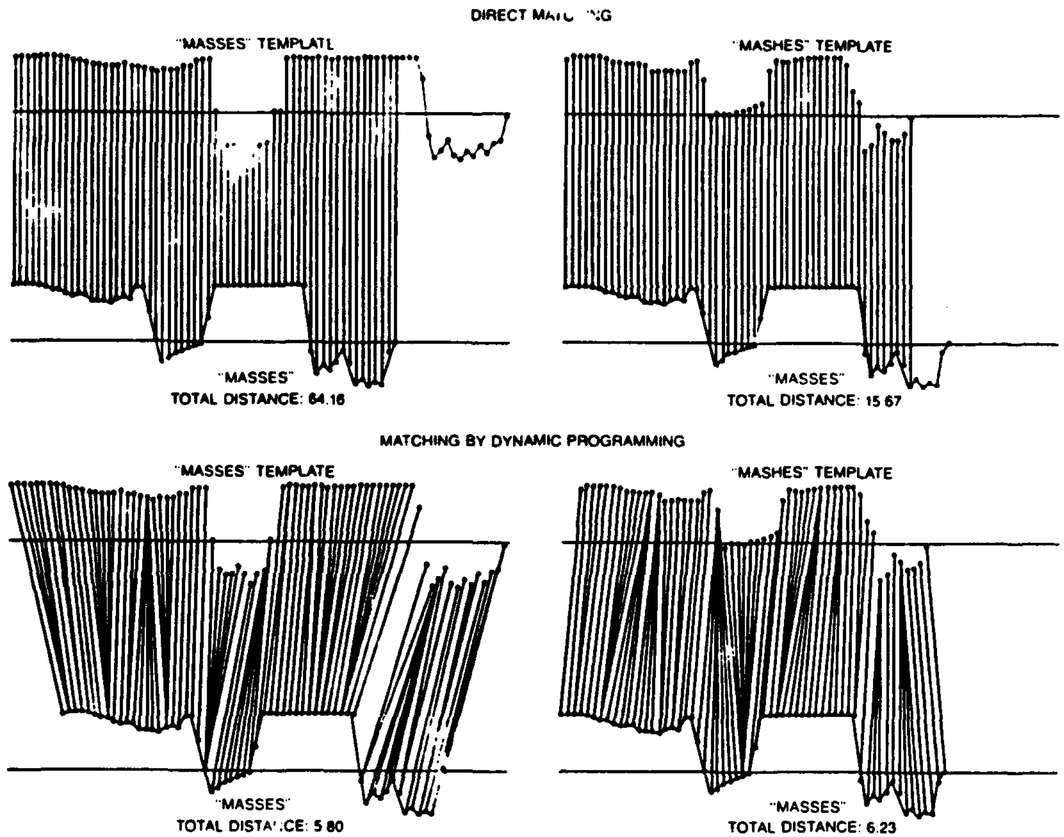
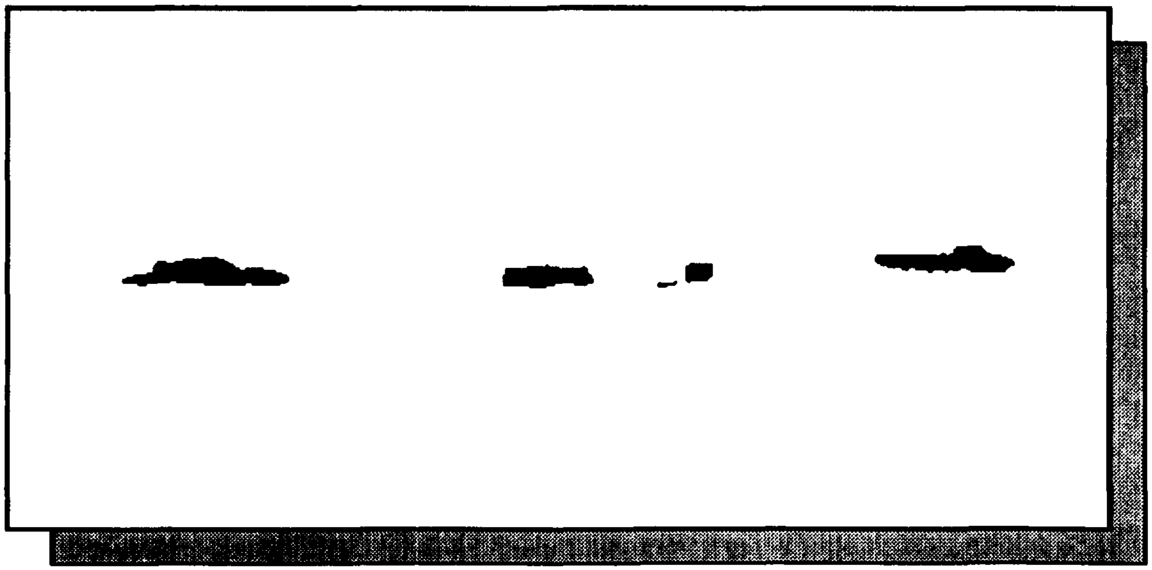
TIME WARPING OF SPEECH SIGNAL



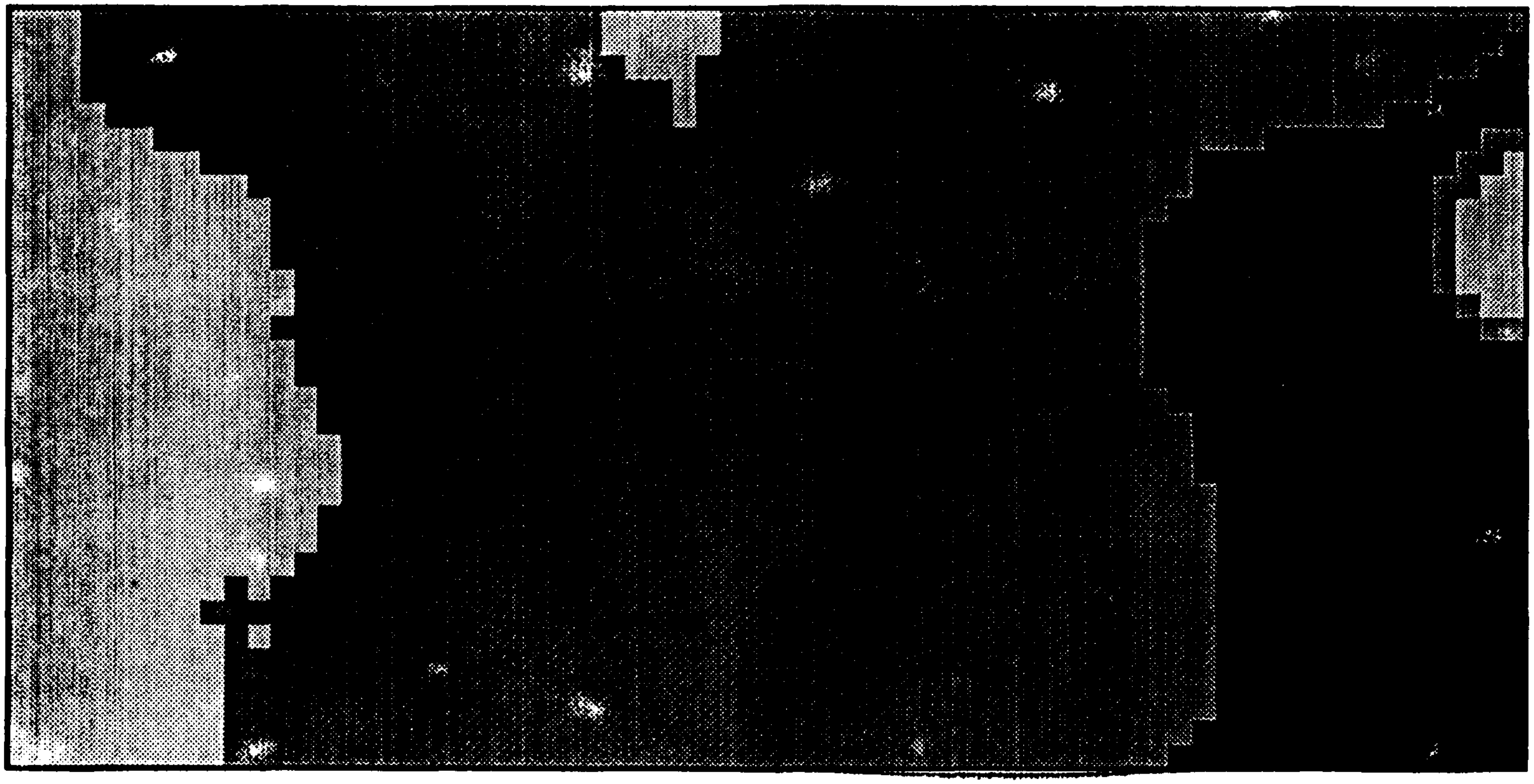
COMPARISON STAGE *of word recognition* **is carried out by compressing and stretching stored templates according to an optimization process called dynamic programming. For each stored template, dynamic programming seeks to associate every frame of the input word with some frame of the template in such a way that a distance measure of overall fit between the input and the template is minimized. The nonuniform time alignment of the stored template with the spoken word allows for** ***variations in the rate of speech and in*** **the relative lengths of the vowels and consonants in a word. Here matching the templates (*black*) to the input (*color*) without dynamic programming yields a misidentification, indicated by the distance scores, that is corrected when the compression and expansion procedure is applied. Dynamic programming is often done with the aid of a computer but should not be confused with computer programming.**

IMAGES: TWO DIMENSIONAL SIGNAL PROCESSING

* Image Segmentation
* Object Recognition

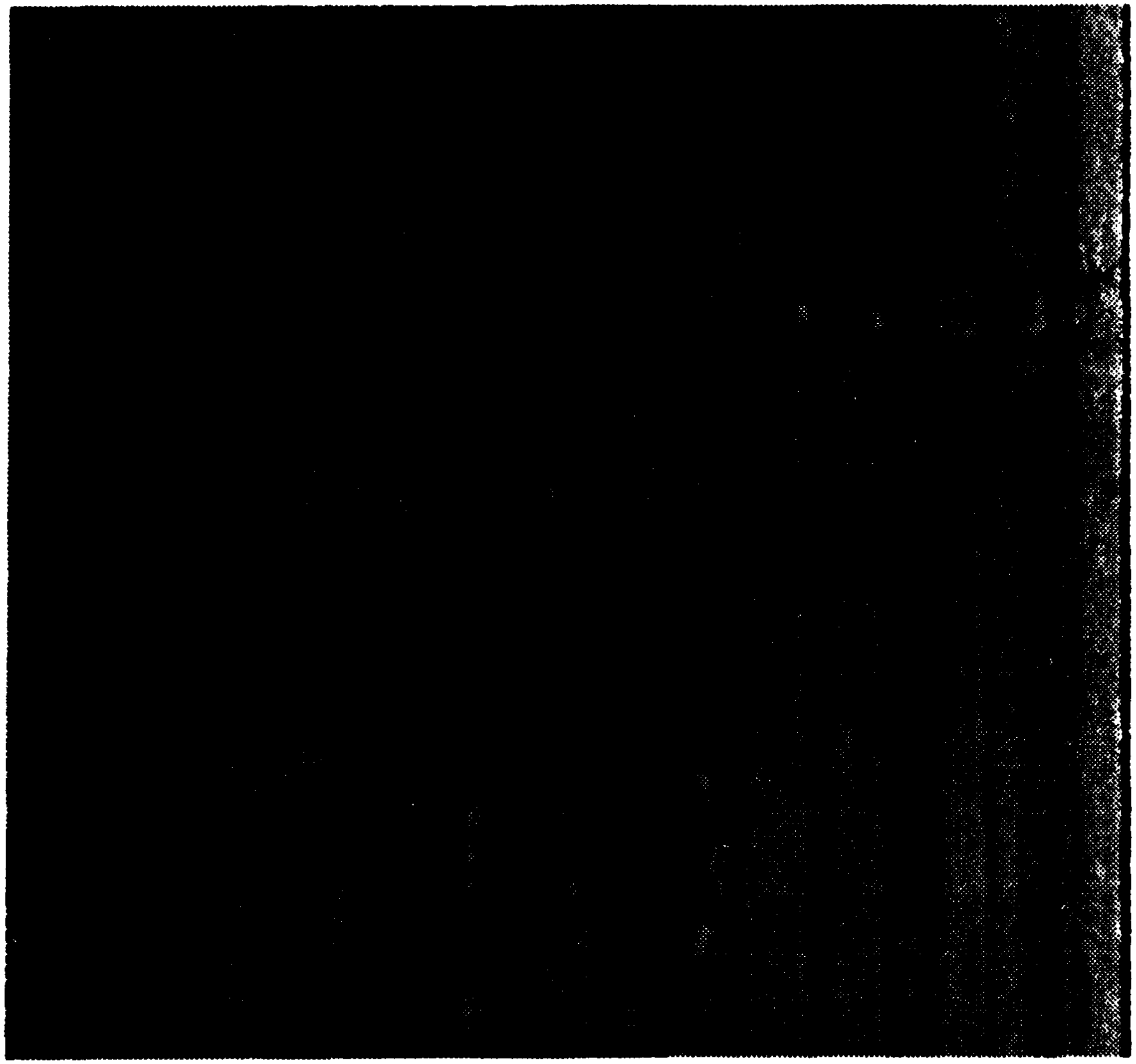


SAR Segmentation Example

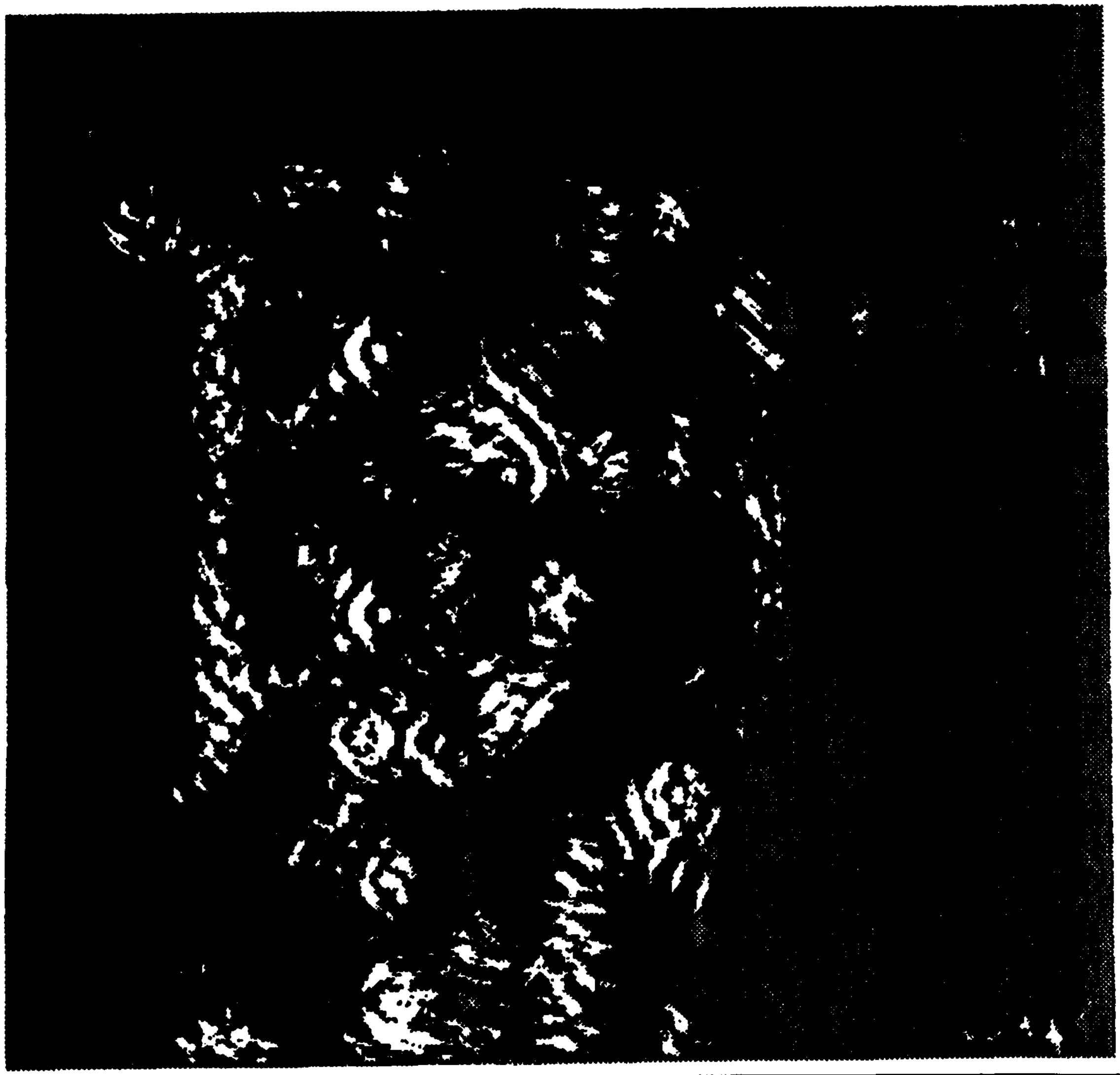


M85F28 RBF Output (4 Gabor 4 Fractal) 5 x 5 Median

VLSI SEGMENTATION EXAMPLE: ORIGINAL
Photo Micrograph of Silicon Chip



VLSI SEGMENTATION EXAMPLE: GABOR
Combination of Six Rotations



VLSI SEGMENTATION EXAMPLE:
Candidate Contact Locations

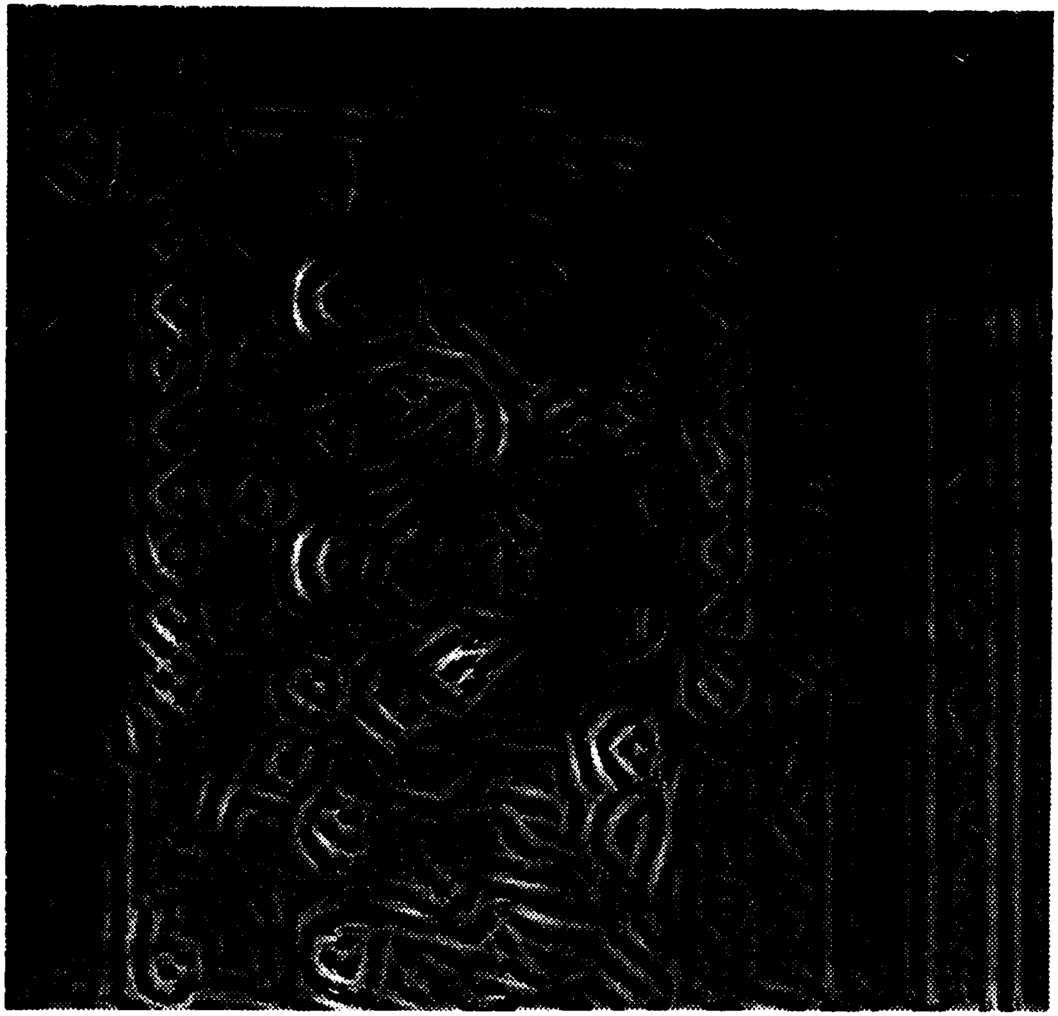
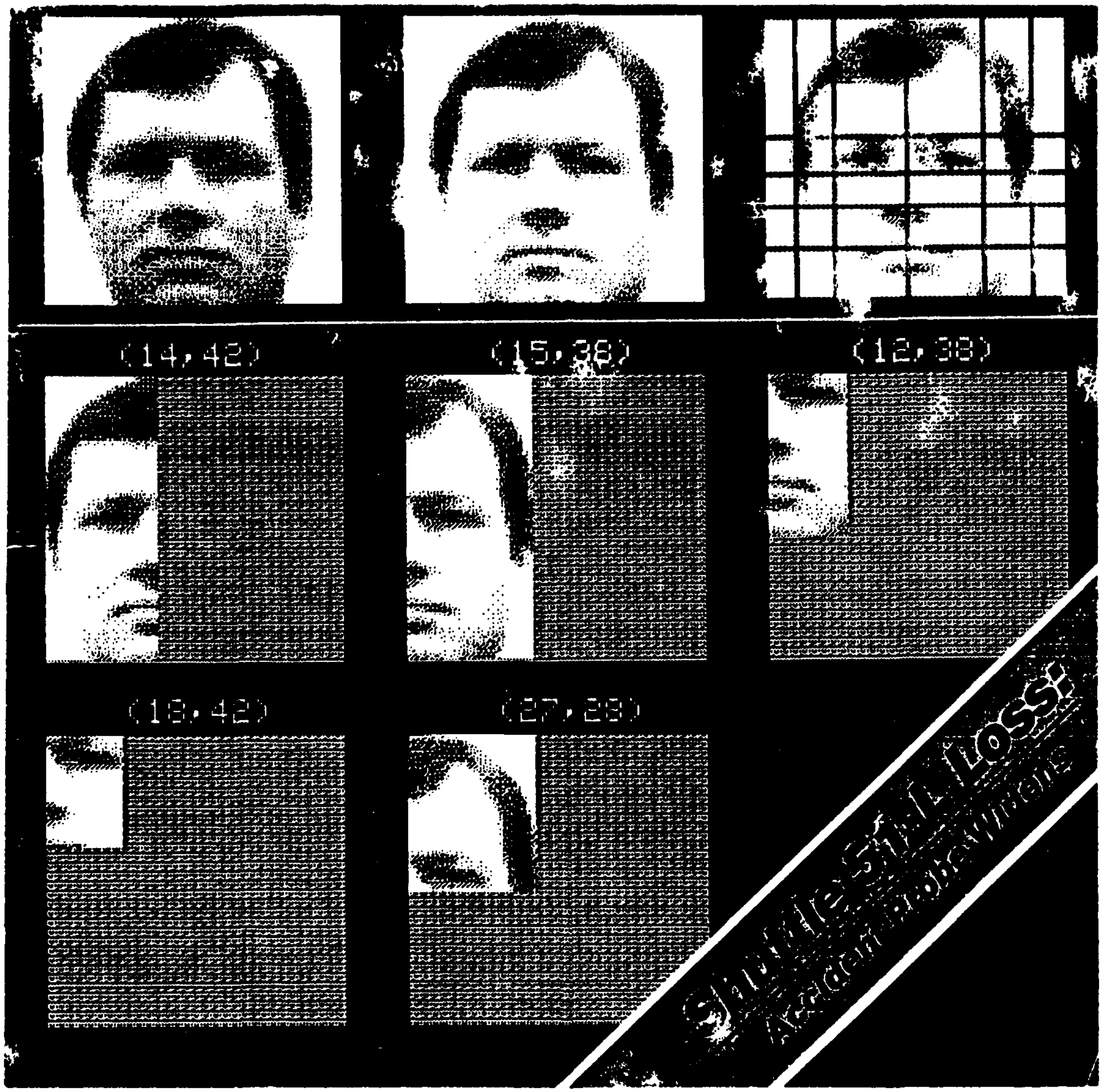


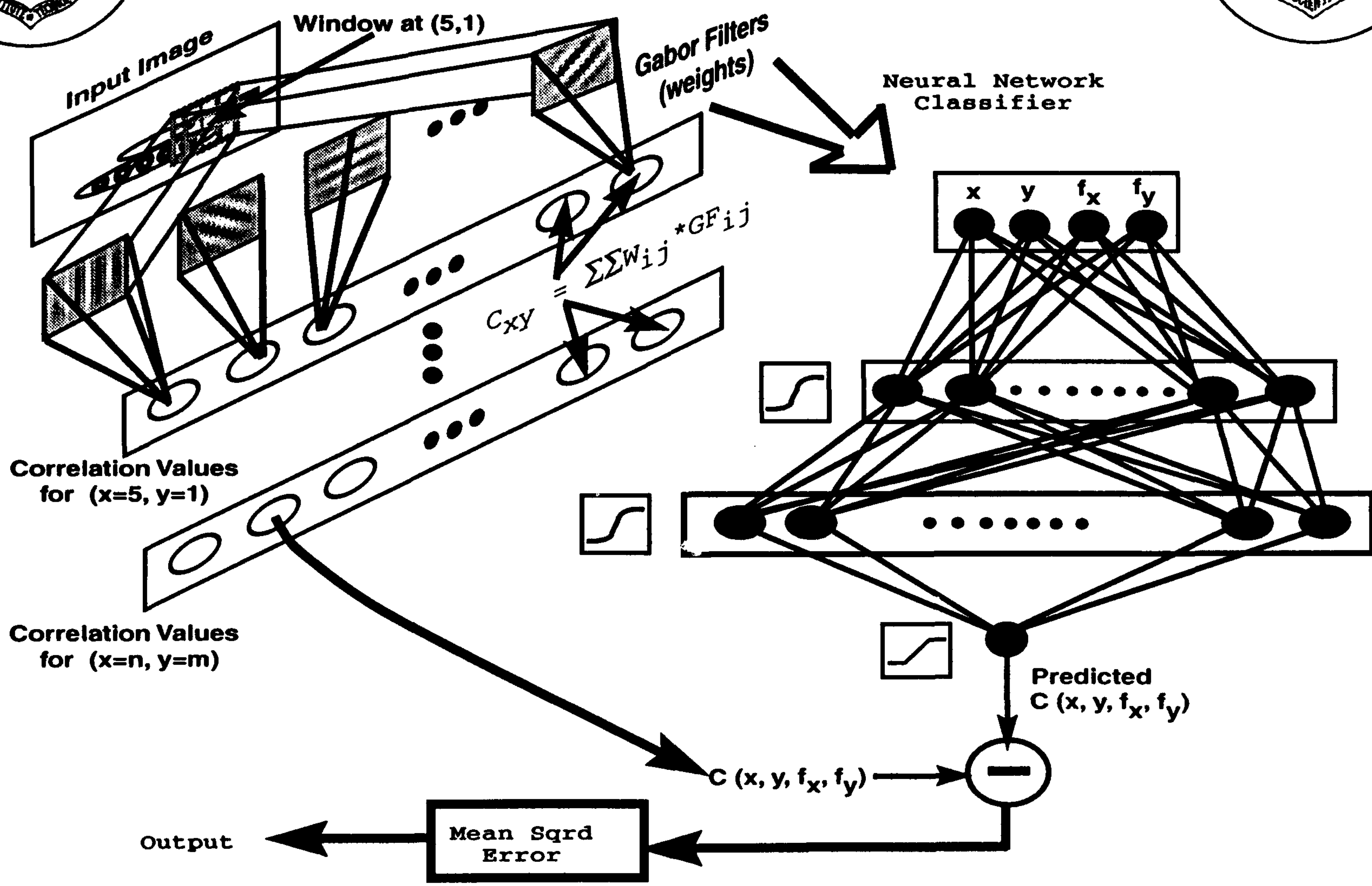
IMAGE RECOGNITION EXAMPLES

* Face Recognition
* Target Recognition
* Contact Recognition

FACE RECOGNITION EXAMPLE



Implementation of Wavelet Technology



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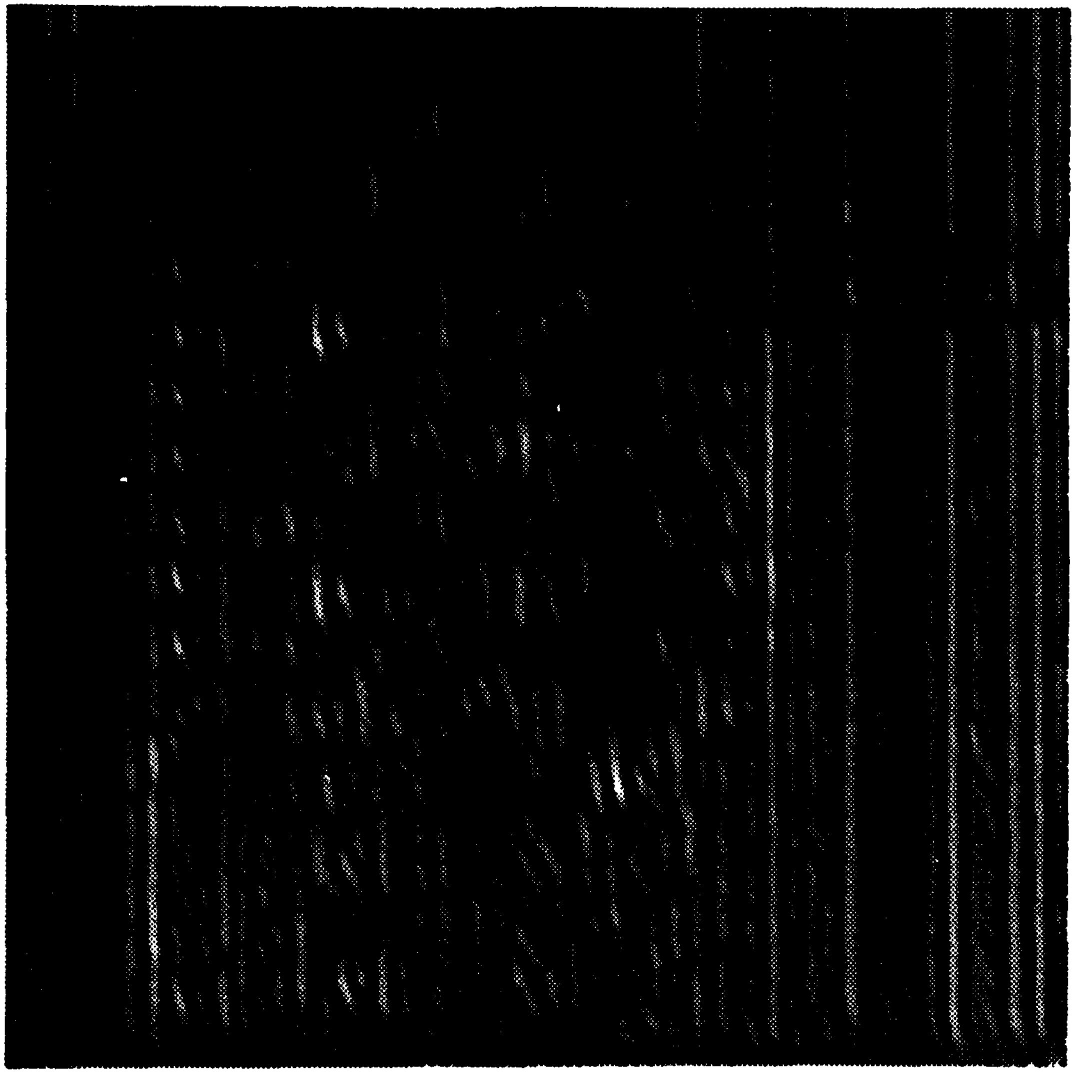
CONTACT RECOGNITION EXAMPLE
(without preprocessing via Gabor, couldn't find pads)



PART IV

Multiresolution Analysis and Orthonormal Bases

* Spline Bases
* Daubechies' Compactly Supported Bases

MULTIRESOLUTION ANALYSIS AND ORTHONORMAL BASES [Ma1-3; Db1,2]

Orthonormal Bases are nice!

- No redundancy
- No coupling $\Longrightarrow$ Diagonal Matrices
- "Nice" decomposition of spaces
- Energy conserved (Plancherel)
- Geometry preserved (Parseval)

. . . at the expense of

- wider support
- more computational precision required

Multiresolution Analysis

- Construct orthonormal (ON) bases
  - spline bases
  - compactly supported orthonormal (CSON) bases
- Use ON bases

Multiresolution Analysis on $L^2(\mathbf{R})$

The idea:

- Write $x \in L^2(\mathbf{R})$ as sequence of successive approximations
- Each approximation, a smoothing
- Successive approximations more concentrated
- "Zoom in" on details at different resolutions

A multiresolution analysis on $L^2(\mathbf{R})$ consists of

1. A family of embedded closed linear manifolds (subspaces) $V_m \subset L^2(\mathbf{R}), \quad m \in \mathbf{Z}$,

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1}, \subset V_{-2} \subset \cdots$$

satisfying,

2. $$\bigcap_{m \in \mathbf{Z}} V_m = \{0\}, \qquad \overline{\bigcup_{m \in \mathbf{Z}} V_m} = L^2(\mathbf{R})$$

and

3. $f(\cdot) \in V_m \iff f(2\cdot) \in V_{m-1}$, and for which

4. there is a function, $\phi \in V_0$, so that for each $m \in \mathbf{Z}$ the set $\{\phi_{m,n}\}_{n \in \mathbf{Z}}$, where,

$$\phi_{m,n}(t) = 2^{-m/2}\phi\left(2^{-m}t - n\right); \quad n \in \mathbf{Z}$$

constitutes an unconditional basis for V_m.

Property 4 means

(a) $V_m = \overline{\text{lin span}\,\{\phi_{m,n}\}_{n\in\mathbf{Z}}}$

and

(b) there are finite constants $B \geq A > 0$ so that for all sequences $(c_n)_{n\in\mathbf{Z}} \in l^2(\mathbf{Z})$

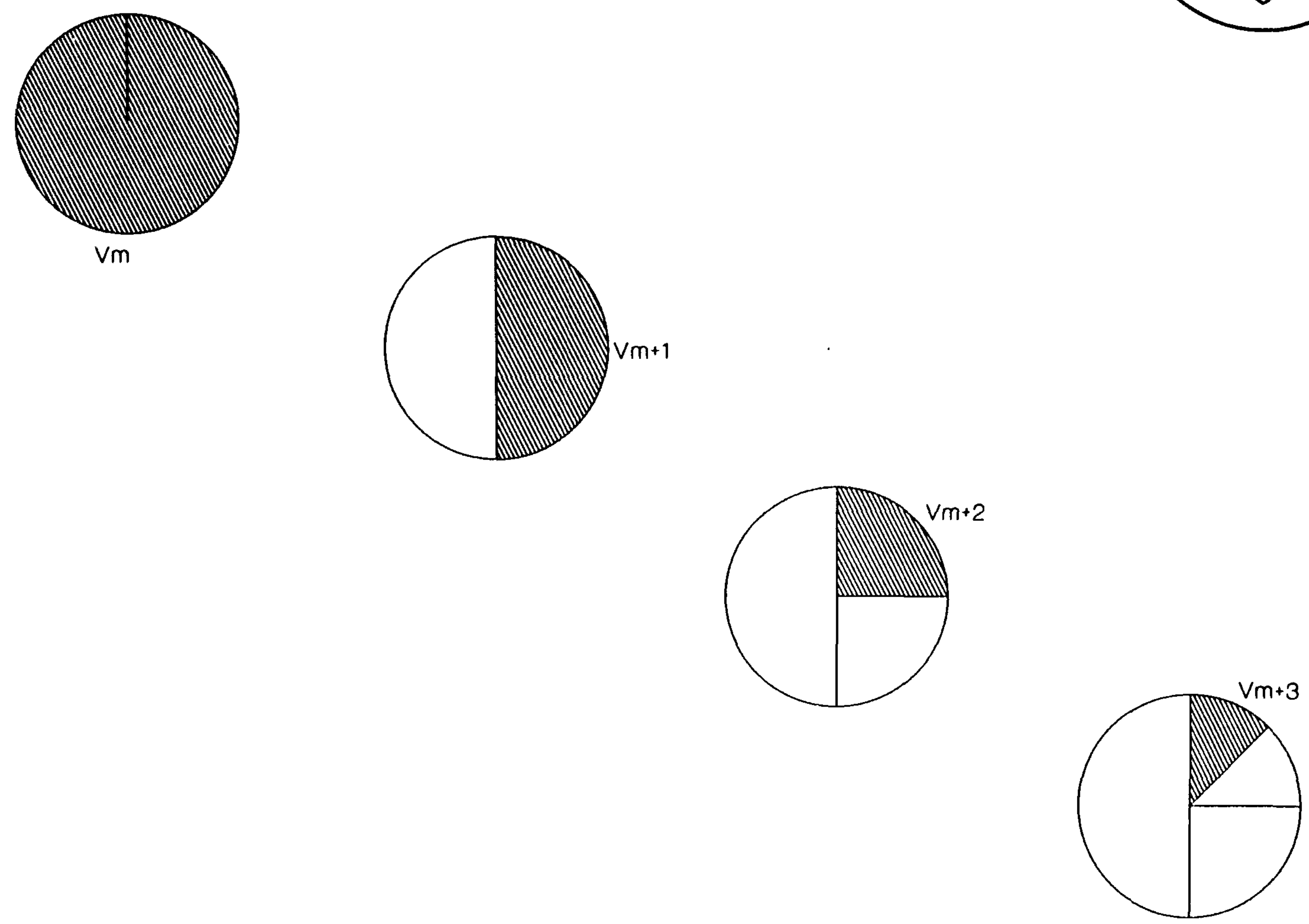
$$A \sum_n |c_n|^2 \leq \left\| \sum_n c_n\, \phi_{m,n} \right\|^2 \leq B \sum_n |c_n|^2 .$$

Property 1

Whatever can be "seen" at a coarse resolution can be seen at a finer resolution; can also see more at finer resolutions.

Property 2

In the limits, coarser resolutions see nothing and finer resolutions see it all.


Vm
Vm+1
Vm+2
Vm+3

Property 3

Scaling, self-similar, aspect of affine wavelets.

Property 4

Unconditional basis equivalent to orthonormal basis.
Translations of a fixed dilation are "building blocks" for the respective resolution levels.
As a consequence,

$$f(\cdot) \in V_m \Longrightarrow f(\cdot - 2^m n) \in V_m \quad \forall\, n \in \mathbf{Z}$$

Property 4 can be satisfied by requiring translates of ϕ to be an ON basis for V_0, meaning that any $x \in V_0$ may be written

$$x(t) = \sum_{n \in \mathbf{Z}} c_n \, \phi(t-n) = \sum_{n \in \mathbf{Z}} c_n \, \phi_{0,n}(t)$$

where

$$c_n = \langle x, \phi_{0,n} \rangle \,.$$

(We can make such a set $\{\phi_{0,n}\}_{n \in \mathbf{Z}}$).

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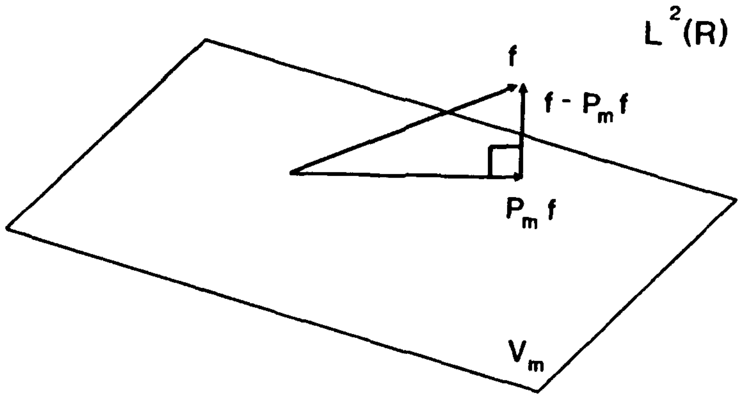
upon correction.

For $f \in L^2(\mathbf{R})$, let $P_m f$ be the orthogonal projection of f onto V_m.

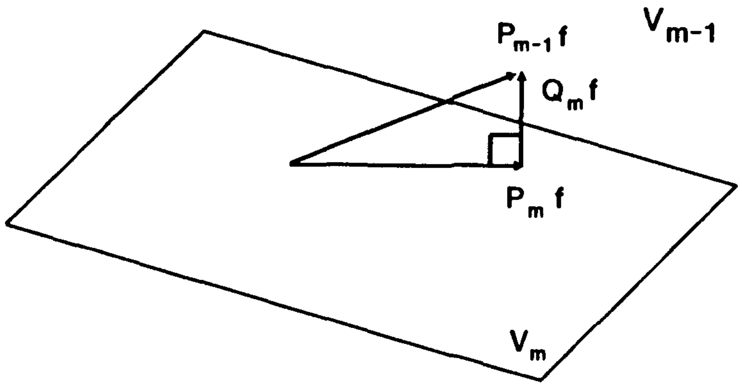
$$P_m f = \sum_{n \in \mathbf{Z}} \langle f, \phi_{m,n} \rangle \, \phi_{m,n}$$



Properties 1 and 2 $\Longrightarrow \lim_{m \to -\infty} P_m f = f \ \ \forall f \in L^2(\mathbf{R})$

$P_{m-1}f$: approximation to f based on $(m-1)^{\underline{st}}$ resolution level

Since $V_m \subset V_{m-1}$, we can consider $P_m P_{m-1} f$ and can easily show $P_m P_{m-1} f = P_m f$, approximation at next coarsest level.



Each embedded subspace V_k is also a Hilbert space.
Consideration of

$$Q_m f \doteq P_{m-1} f - P_m f$$

is the information difference between successive approximations.

By the Classical Projection Theorem (CPT),

$$Q_m f \perp V_m$$

i.e.,

$$Q_m f \in W_m$$

where W_m is the orthogonal complement of V_m in V_{m-1} (not $L^2(\mathbf{R})$!!!):

$$W_m = \{f \in V_{m-1} : \langle f, v \rangle = 0 \;\; \forall\, v \in V_m\}$$

By CPT, we have a *unique* representation of any $P_{m-1}f \in V_{m-1}$:

$$P_{m-1}f = P_m f + Q_m f \quad \forall\, f \in L^2(\mathbf{R}).$$

This implies that V_{m-1} is a direct sum:

$$V_{m-1} = V_m \oplus W_m$$

<u>Definition</u>. A linear space X is the <u>direct sum</u> of two linear manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ if any $x \in X$ is represented by $x = m_1 + m_2$ for unique vectors $m_1 \in \mathcal{M}_1$, $m_2 \in \mathcal{M}_2$.

Theorem: If $\mathcal{M}$ is a subspace in a Hilbert space, $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Furthermore, $\mathcal{M}^{\perp\perp} = \mathcal{M}$.

In our case, we have

$$\begin{aligned} V_m^{\perp} &= W_m \\ W_m^{\perp} &= V_m \end{aligned}$$

Furthermore, the orthogonal complements, $\{W_m\}_{m\in\mathbf{Z}}$, are mutually orthogonal

$$W_m \perp W_n \quad \text{if} \quad n \neq m$$

where (definition) two inner product spaces $\mathcal{A}$ and $\mathcal{B}$ are orthogonal if

$$\langle a, b\rangle = 0 \quad \forall\, a \in \mathcal{A}, b \in \mathcal{B}\ .$$

Having such a mutually orthogonal family of subspaces $\{W_m\}_{m \in \mathbf{Z}}$ we may write

$$\bigoplus_{m \in \mathbf{Z}} W_m = L^2(\mathbf{R})$$

meaning any $f \in L^2(\mathbf{R})$ may be written as

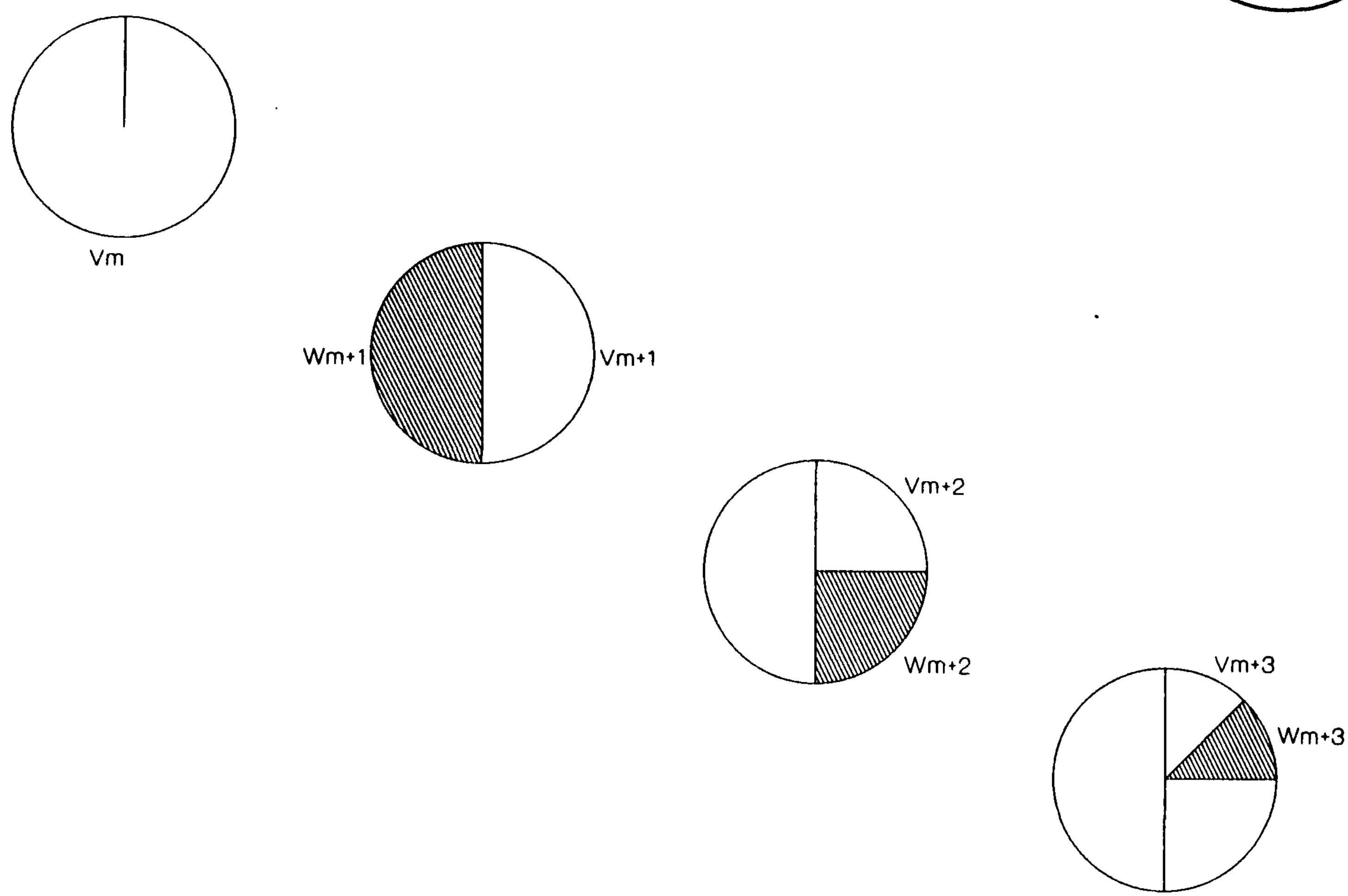
$$f = \sum_{m \in \mathbf{Z}} f_m$$

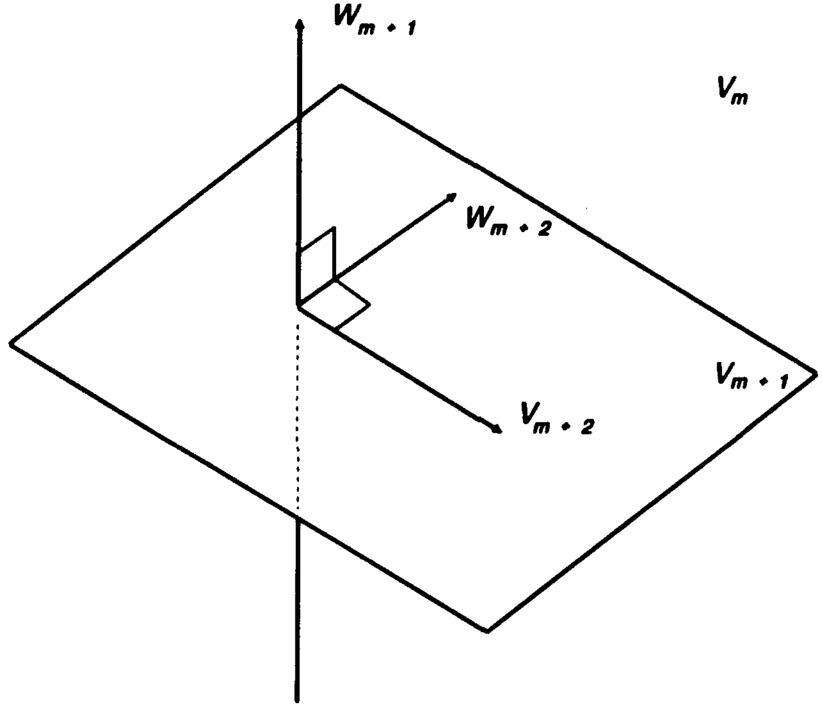
with

$$\|f\|^2 = \sum_{m \in \mathbf{Z}} \|f_m\|^2$$

where

$$f_m \in W_m \quad \text{for each } m \in \mathbf{Z} \,.$$


Vm
Wm+1
Vm+1
Vm+2
Wm+2
Vm+3
Wm+3


W_{m+1}
V_m
W_{m+2}
V_{m+1}
V_{m+2}

WANTED:

For each $m \in \mathbf{Z}$, an orthogonal basis $\{\psi_{m,n}\}_{n\in\mathbf{Z}}$ for W_m

The set $\{\psi_{m,n}\}_{m,n\in\mathbf{Z}}$ is then automatically an ON basis for $L^2(\mathbf{R})$

These are our wavelets!

Construction of ON Wavelets

Given a Multiresolution analysis of $L^2(\mathbf{R})$:

1. Construct a orthonormal basis for V_0 via filters M and H .
2. Construct wavelets via filter operations (G) on this basis.

Construction of ON basis for V_0

By Property 4, there is a function $g \in V_0$ ("generator") for which

$$\overline{\text{lin span}\,\{g_{0,n}\}_{n\in\mathbf{Z}}} = V_0 \,.$$

This implies any $v \in V_0$ may be written as

$$v(t) = \sum_{n\in\mathbf{Z}} \alpha_n\, g(t-n)$$

$$\hat{v}(f) = M(f)\,\hat{g}(f)$$

where the filter M is given by

$$M(f) = \sum_{n\in\mathbf{Z}} \alpha_n\, \mathrm{e}^{-i2\pi n f}$$

Observe, $M(f) = M(f+1)$ is "1-periodic". A function $\phi \in V_0$ yielding an ON basis $\{\phi_{0,n}\}_{n \in \mathbf{Z}}$ for V_0 follows from the Poisson summation formula

$$\sum_{n \in \mathbf{Z}} x(n) = \sum_{n \in \mathbf{Z}} \hat{x}(n) \quad .$$

This implies ϕ must satisfy

$$\sum_{k \in \mathbf{Z}} |\hat{\phi}(k+f)|^2 = 1, \quad \forall\, f \in \mathbf{R} \quad .$$

The result is

$$\hat{\phi}(f) = M(f)\, \hat{g}(f)$$

with

$$M(f) = \left(\sum_{k \in \mathbf{Z}} |\hat{g}(f+k)|^2 \right)^{-1/2} \quad .$$

Now,

$$\phi \in V_0 \Longleftrightarrow \phi(\cdot/2) \in V_1$$

and

$$V_1 \subset V_0 \quad .$$

Hence,

$$\frac{1}{2}\phi(t/2) \;=\; \sum_{k \in \mathbf{Z}} h_k \, \phi(t-k)$$

$$\hat{\phi}(2f) \;=\; H(f)\,\widehat{\phi}(f) \quad .$$

See that $\phi(\frac{t}{2})$ is a filtered version of $\phi(t)$, where ....

the filter, H, given by

$$H(f) = \sum_{k \in \mathbf{Z}} h_k \, \mathrm{e}^{-i2\pi k f}$$

is a 1-periodic function of f. Hence, $v(\cdot/2) \in V_m$ is a filtered version of $v(\cdot) \in V_{m-1}$.

<u>Necessarily</u> (can show that)

$$|\hat{\phi}(0)| = 1$$

i.e.,

$$\left| \int_{-\infty}^{\infty} \phi(t)dt \right| = 1$$

and

$$|H(f)|^2 + \left| H(f + \frac{1}{2}) \right|^2 = 1$$

subject to

$$|H(0)| = 1$$

Sufficient Conditions on H

Theorem. Let the filter, H, be defined by

$$H(f) = \sum_{k \in \mathbf{Z}} h_k \, e^{-i2\pi f k} \quad .$$

If

(*i*) $|h_k| < C(1 + h^2)^{-1}$, for some $C > 0$ ("regularity" condition)

(*ii*) $|H(0)| = 1$

(*iii*) $|H(f)|^2 + |H(f + 1/2)|^2 = 1 \quad \forall \, f \in \mathbf{R}$

(*iv*) $|H(f)| \neq 0 \quad \forall \, f \in [-1/4, 1/4]$

∘

∘

∘

then,

$$\hat{\phi}(f) = \Pi_{k=1}^{\infty} H(2^{-k} f)$$

is the Fourier transform of $\phi(t)$ such that $\{\phi_{0,n}\}_{n \in \mathbf{Z}}$ is an orthonormal basis of a subspace V_0 of $L^2(\mathbf{R})$. The corresponding subspaces, V_m, form a multiresolution approximation of $L^2(\mathbf{R})$.

2. Construction of ON basis for W_0

Want $\psi \in W_0$ such that

$$\{\psi_{0,n}\}_{n\in\mathbf{Z}} \text{ is an ON basis for } W_0 ,$$

so that

$$\{\psi_{m,n}\}_{n\in\mathbf{Z}} \text{ is an ON basis for each } W_m .$$

With $L^2(\mathbf{R}) = \bigoplus_{m\in\mathbf{Z}} W_m$ and $\{W_m\}_{m\in\mathbf{Z}}$ mutually orthogonal, the family $\{\psi_{m,n}\}_{m,n\in\mathbf{Z}}$ will constitute an (affine) wavelet ON basis for $L^2(\mathbf{R})$.

Recall, $W_1 \bigoplus V_1 = V_0$ so that $W_1 \subset V_0$.
We want $\psi \in W_0$ such that $\psi(\cdot/2) \in W_1 \subset V_0$. So write

$$\frac{1}{2}\psi\left(\frac{t}{2}\right) = \sum_{k\in\mathbf{Z}} g_k\,\phi(t-k)$$

$$\widehat{\psi}(2f) = G(f)\,\widehat{\phi}(f)$$

where

$$G(f) = \sum_{k\in\mathbf{Z}} g_k\,\mathrm{e}^{-i2\pi fk}$$

is a 1-periodic filter.

Sufficient conditions on G for the construction of ψ as a filtered version of ϕ are that the matrix

$$U = \begin{bmatrix} H(f) & G(f) \\ H\left(f + \frac{1}{2}\right) & G\left(f + \frac{1}{2}\right) \end{bmatrix}$$

be unitary; i.e., $\overline{U}^T U = I$. <u>One</u> possible choice for G is

$$G(f) = \mathrm{e}^{-i2\pi f}\,\overline{H(f + \frac{1}{2})} \ .$$

Then,

$$\frac{1}{2}\psi\left(\frac{t}{2}\right) = \mathcal{F}^{-1}\{G(\cdot)\,\hat{\phi}(\cdot)\}(t)$$

defines the mother wavelet.

Spline Wavelets

Let B_d be a basic spline of degree d defined by a d-fold convolution of characteristic functions

$$B_d(t) = \left[\left(\chi_{[0,1)} *\right)^d \chi_{[0,1)}\right](t)$$

where

$$\chi_{[0,1)}(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{else} . \end{cases}$$

$$
\begin{aligned}
\widehat{B_d}(f) &= \widehat{B_0}^{(d+1)}(f) \\
&= \mathrm{e}^{-i(d+1)\pi f}\,\mathrm{sinc}^{(d+1)}(f)
\end{aligned}
$$

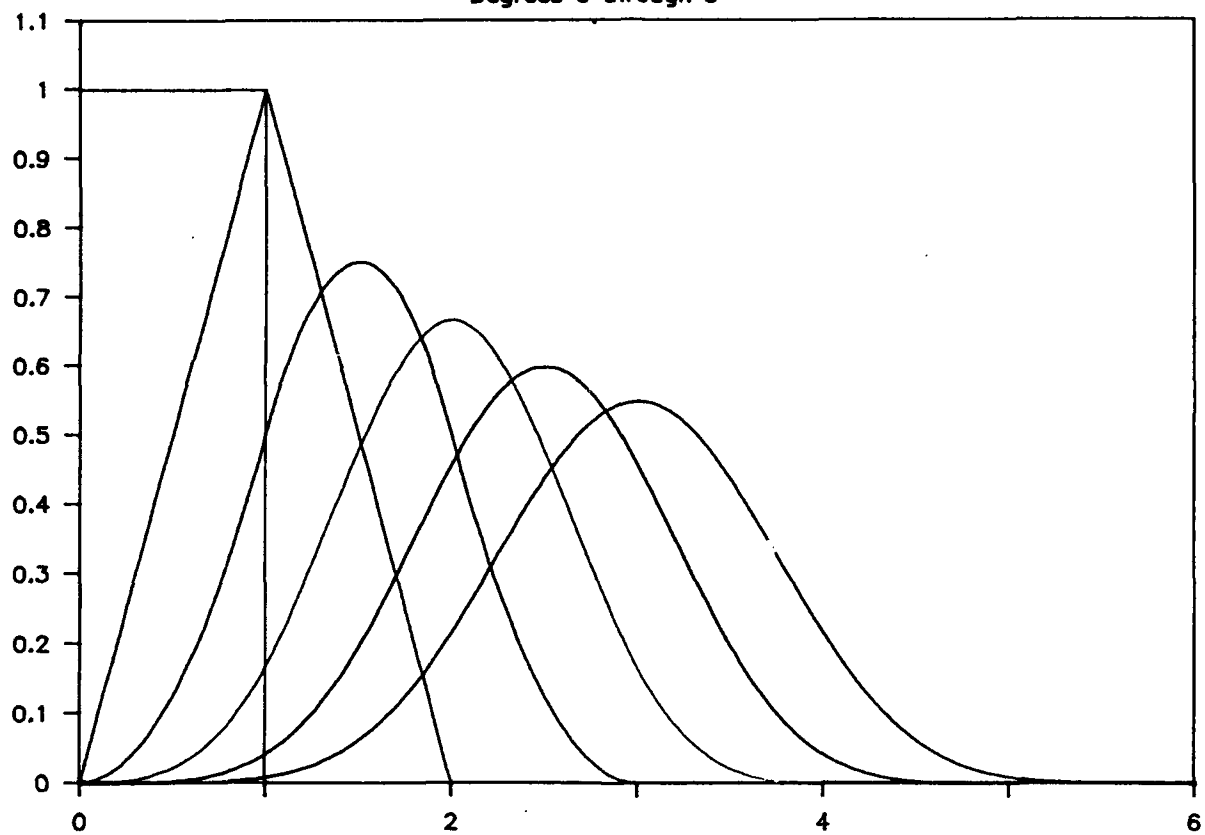
where

$$
\mathrm{sinc}(f) = \begin{cases} \dfrac{\sin(\pi f)}{\pi f}, & f \neq 0 \\ 1, & f = 0 \end{cases} .
$$

These splines are piecewise-polynomial functions on intervals $[n, n+1]$ for $n = 0, 1, 2, \ldots, d$ having $d-1$ continuous derivatives. Integer translates of B_d satisfy requirements for a multiresolution approximation; but they are not orthonormal.

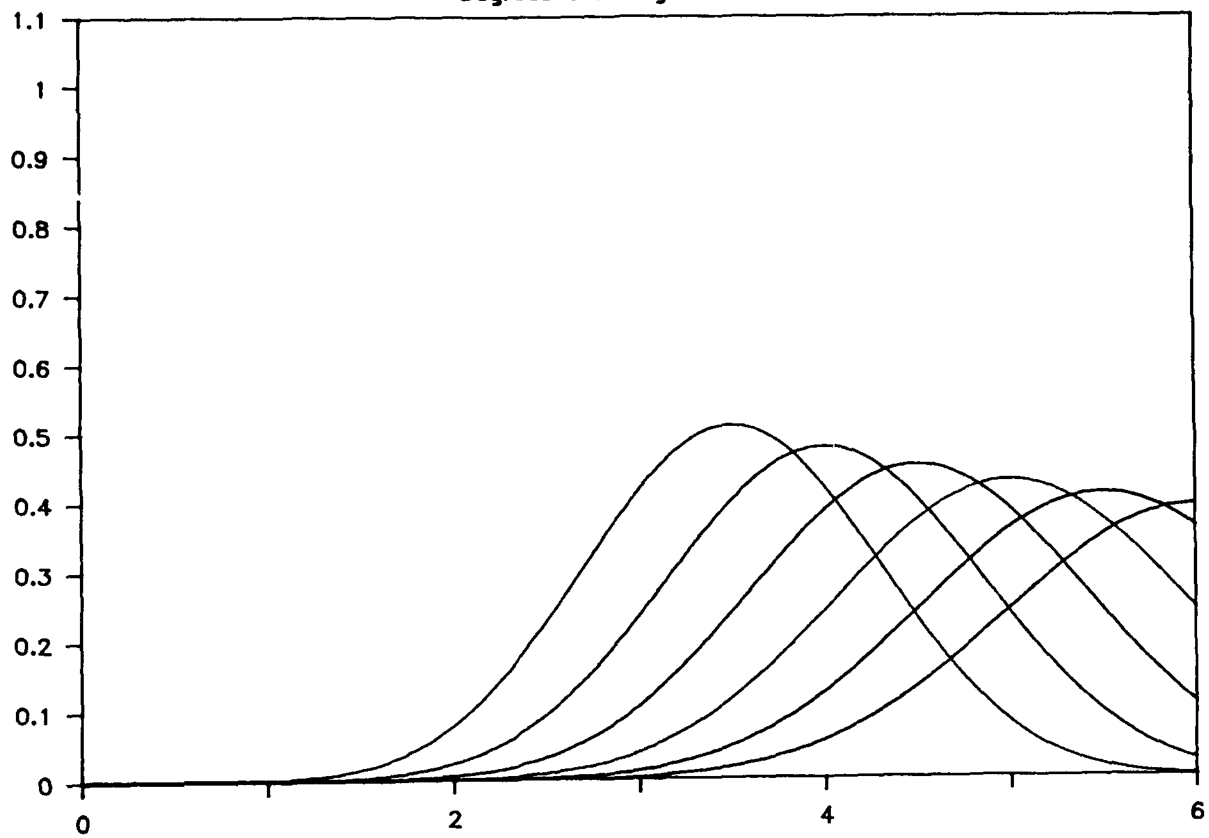
Basic Splines

Degrees 0 through 5



Basic Splines

Degrees 6 through 11



Take $g = B_d$ in the multiresolution algorithm to get an orthonormal basis for V_0 . Get

$$M^{-2}(f) = \sum_{k\in\mathbf{Z}} |\mathrm{sinc}(f+k)|^{2(d+1)}$$

$$\begin{aligned}\hat{\phi}(f) &= M(f)\,\hat{g}(f)\\ &= \mathrm{e}^{-i(d+1)\pi f}\,S_d(f)\end{aligned}$$

where

$$S_d^{-2}(f) = f^{2(d+1)} \sum_{k\in\mathbf{Z}} (k+f)^{-2(d+1)}$$

Notice that $\grave{\phi}$ has zeros of order $d{+}1$ at non-zero integers; $\hat{\phi}(0) = 1$.

$$
\begin{aligned}
H(f) &= \frac{\hat{\phi}(2f)}{\hat{\phi}(f)} \\
&= \frac{M(2f)}{M(f)} \left[\mathrm{e}^{-i\pi f} \cos(\pi f) \right]^{d+1} \quad .
\end{aligned}
$$

Take

$$
G(f) = \mathrm{e}^{-i2\pi f} \overline{H\left(f + \frac{1}{2}\right)}
$$

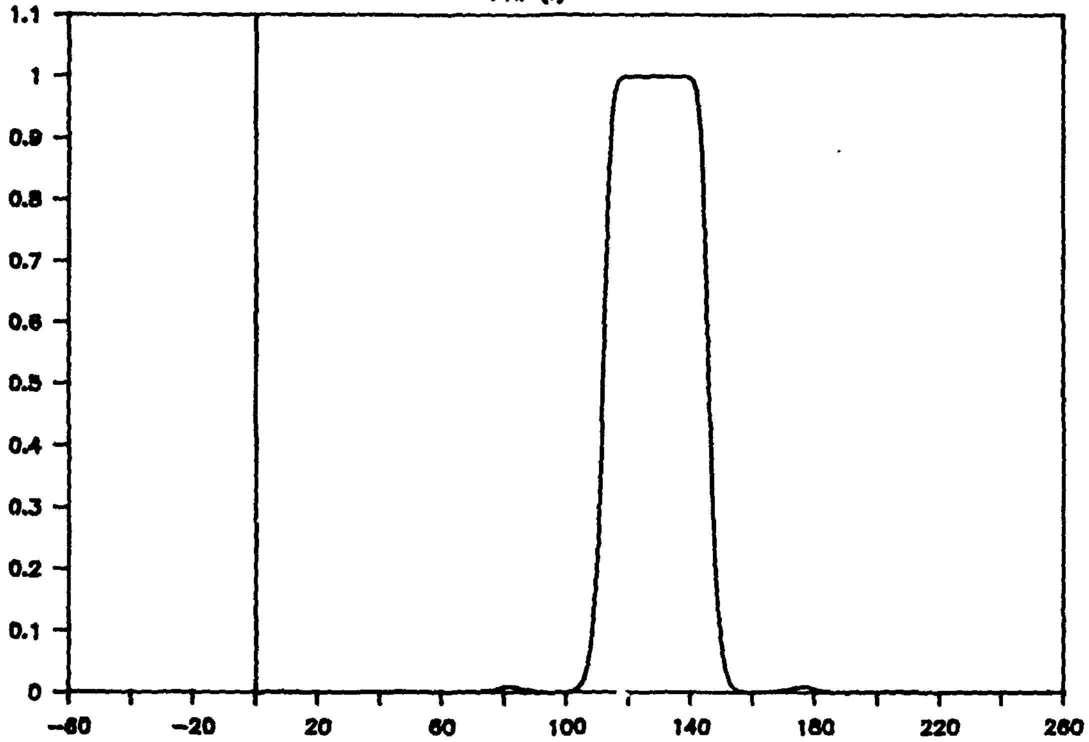
and get the Battle-Lemarié bases [B5,L6].

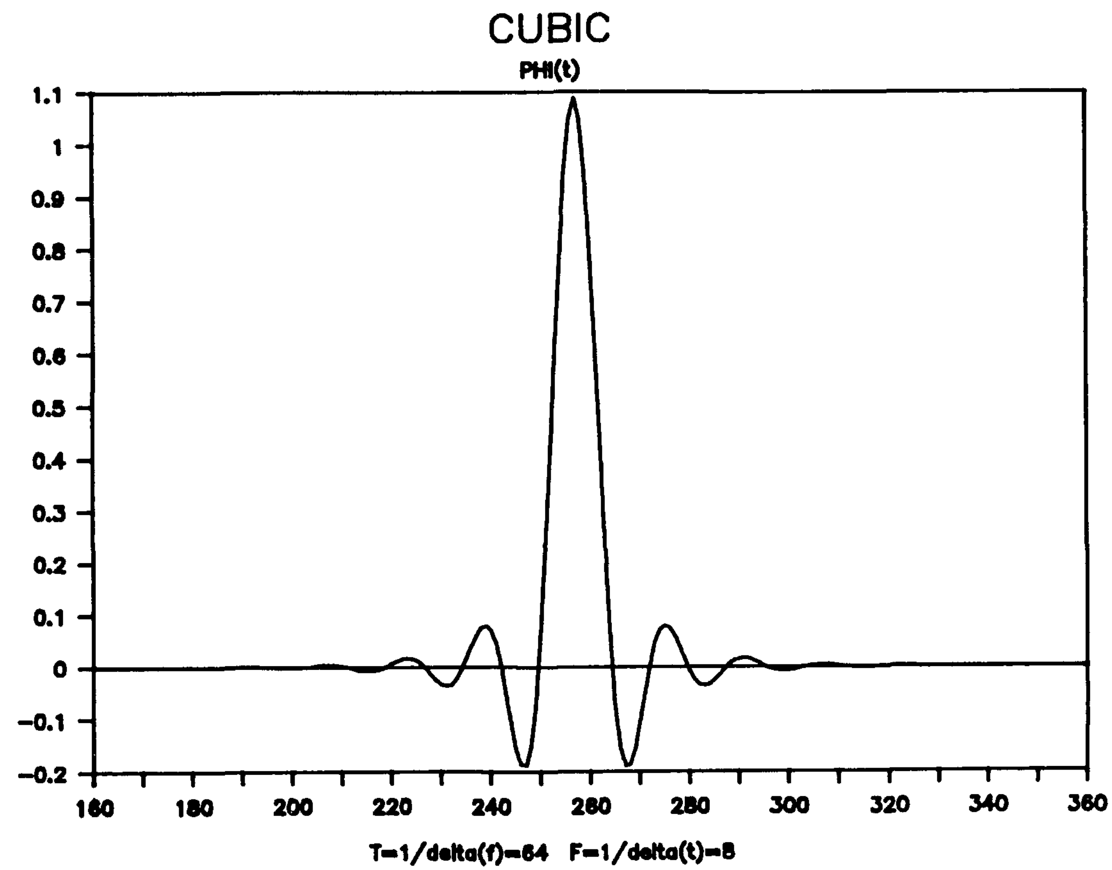
Battle-Lemarié Bases

- Orthonormal Basis for $L^2(\mathbf{R})$
- Piecewise Polynomials (Spline)
- Compact Representation
- Class C^k
- Non-compact Support
- Exponential decay
- Symmetric

CUBIC

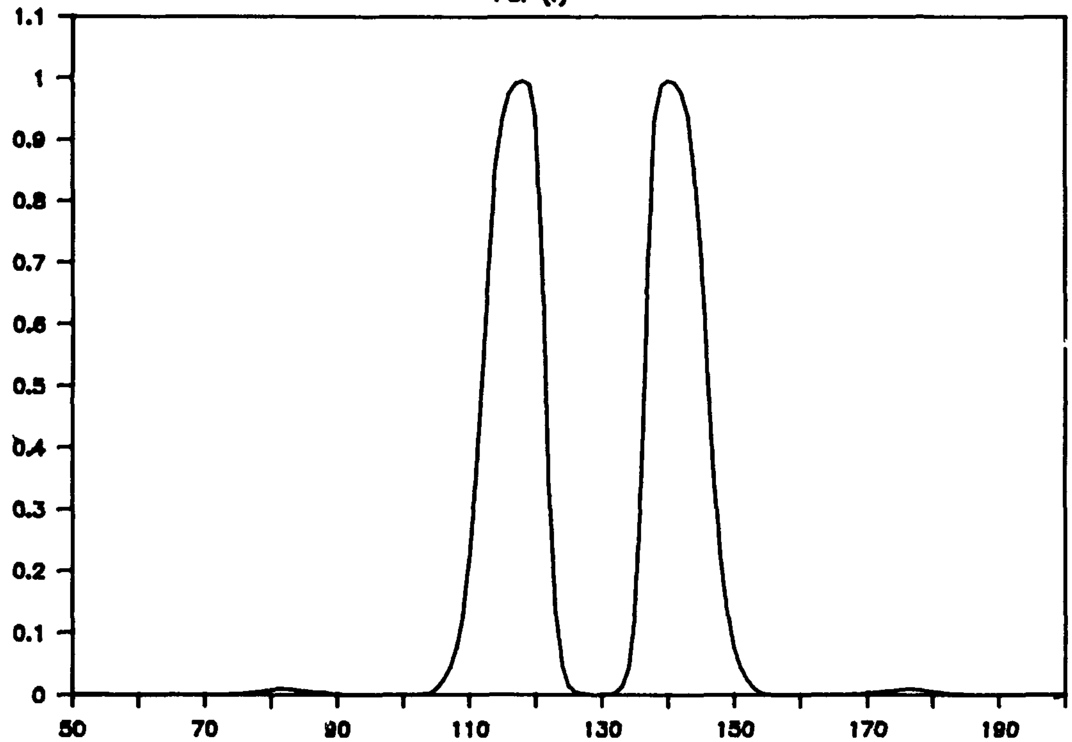
PHI^(f)



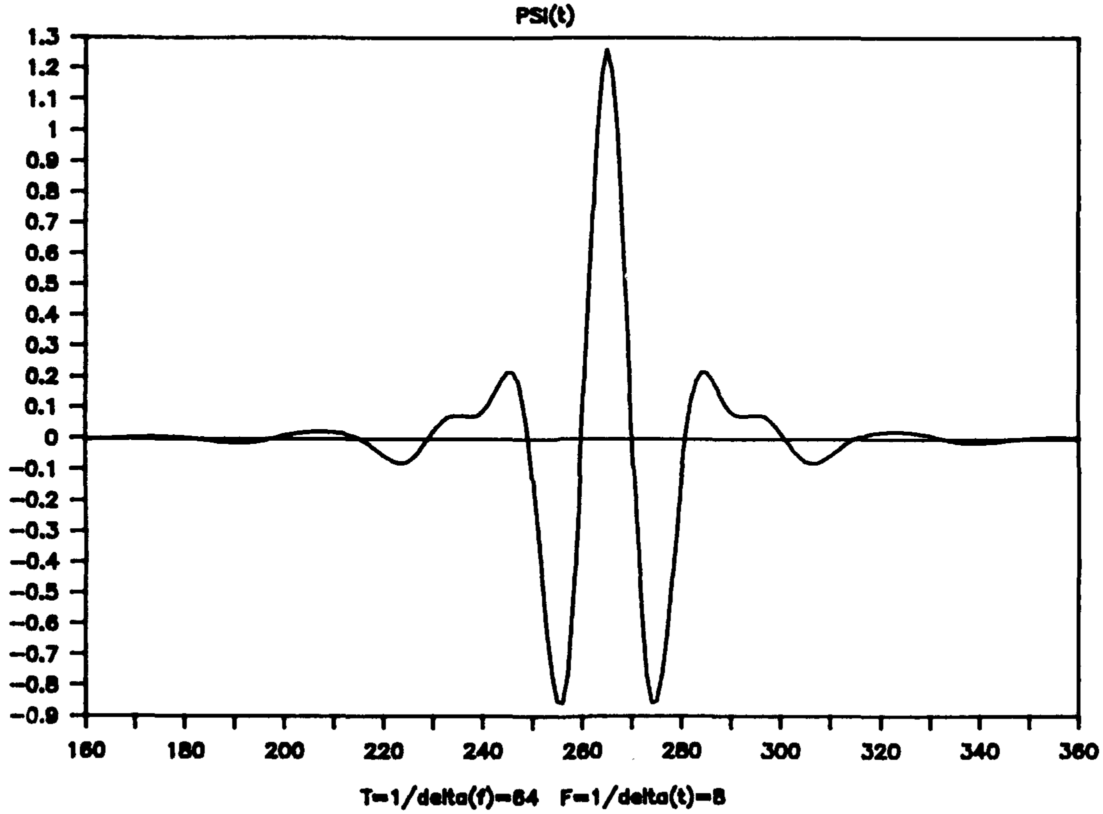

CUBIC
PHI(t)
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
160
180
200
220
240
260
280
300
320
340
360
T=1/delta(f)=64 F=1/delta(t)=8

CUBIC

PSI^(f)



CUBIC



PSI(t)
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
160
180
200
220
240
260
280
300
320
340
360
T=1/delta(f)=64 F=1/delta(t)=8

Multiresolution Analysis
Decomposition and Reconstruction

Successive $P_j f$: approximate, "blurred" versions of f .

Successive $Q_j f$: information difference between successive blurred versions.

$$Q_{m+1} f = P_m f - P_{m+1} f$$

where

$$[P_m f](x) = \sum_l c_{m,l}\, \phi_{m,l}(x)$$

and

$$[Q_m f](x) = \sum_l d_{m,l}\, \psi_{m,l}(x)$$

Get the coefficients:

$$c_{m,l} = \langle P_m f, \phi_{m,l} \rangle$$
$$c_{m,l} = \langle P_{m-1} f, \phi_{m,l} \rangle - \langle Q_m f, \phi_{m,l} \rangle$$

But

$$\phi_{m,l} \in V_m, \quad \text{and} \quad Q_m f \in W_m \perp V_m \ .$$

Therefore,

$$c_{m,l} = \sum_n c_{m-1,n} \langle \phi_{m-1,n}, \phi_{m,l} \rangle \ .$$

Similarly,

$$d_{m,l} = \sum_n c_{m-1,n} \langle \phi_{m-1,n}, \psi_{m,l} \rangle \ .$$

Discrete Filters

$$\langle \phi_{m-1,n}\,,\phi_{m,l}\rangle = 2^{-(m-1)/2}\,2^{-m/2}\int_{-\infty}^{\infty}\phi\left(2^{-(m-1)}t-n\right)\phi\left(2^{-m}t-l\right)dt$$

Let $\frac{u}{2} = 2^{-m}t - l$. Then,

$$\langle \phi_{m-1,n}\,,\phi_{m,l}\rangle = 2^{-1/2}\int_{-\infty}^{\infty}\phi\left(\frac{u}{2}\right)\phi\left(u-[n-2l]\right)du\,.$$

Define

$$h(n) \doteq 2^{-1/2}\int_{-\infty}^{\infty}\phi\left(\frac{t}{2}\right)\phi\left(t-n\right)dt$$

then

$$\langle \phi_{m-1,n}\,,\phi_{m,l}\rangle = h\left(n-2l\right).$$

independent of the level m !!!

Hence,

$$c_{m,l} = \sum_n c_{m-1,n}\, h\,(n - 2l)$$

gives the $m^{\underline{\text{th}}}$ level coefficients from a discrete filter operation on $(m-1)^{\underline{\text{st}}}$ level.

$\Longrightarrow$ Recursion

Similarly,

$$\langle \phi_{m-1,n} \, , \psi_{m,l} \rangle = g(n-2l) \quad \forall m \in \mathbf{Z}$$

where

$$g(n) = 2^{-1/2} \int_{-\infty}^{\infty} \psi\left(\frac{t}{2}\right) \phi\,(t-n)\, dt$$

so that

$$d_{m,l} = \sum_{n} c_{m-1,n}\, g\,(n-2l)$$

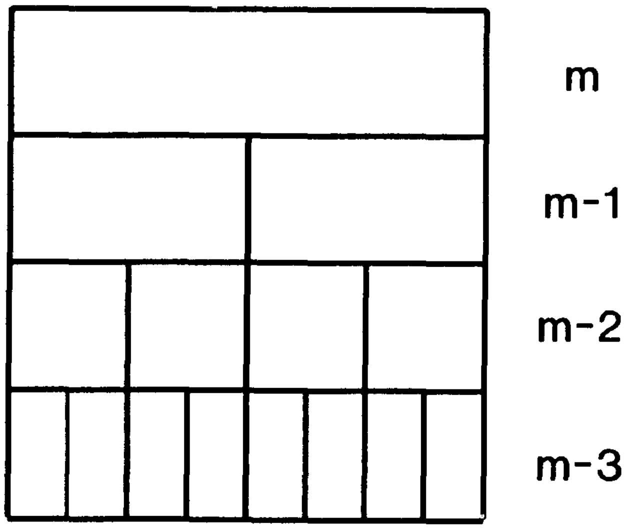
$\Longrightarrow$ Wavelet coefficients from discrete filter operation on scaling function coefficients at finer resolution level.

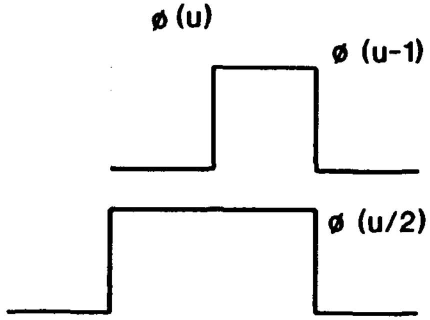
Conversely, if for a fixed $m \in \mathbf{Z}$, we know $\{c_{m,n}\}_{n\in\mathbf{Z}}$ and $\{d_{m,n}\}_{n\in\mathbf{Z}}$, we can reconstruct the finer, $(m-1)^{st}$ resolution level,

$$\begin{aligned} P_{m-1}f &= P_m f + Q_m f \\ &= \sum_n c_{m,n}\,\phi_{m,n} + \sum_n d_{m,n}\,\psi_{m,n} \quad . \end{aligned}$$

Since $c_{m-1,k} = \langle P_{m-1}f, \phi_{m-1,k}\rangle$, we get

$$\begin{aligned} c_{m-1,k} &= \sum_n c_{m,n}\,\langle \phi_{m,n}, \phi_{m-1,k}\rangle + \sum_n d_{m,n}\,\langle \psi_{m,n}, \phi_{m-1,k}\rangle \\ &= \sum_n c_{m,n}\,h(k-2n) + \sum_n d_{m,n}\,g(k-2n) \quad . \end{aligned}$$


m
m-1
m-2
m-3

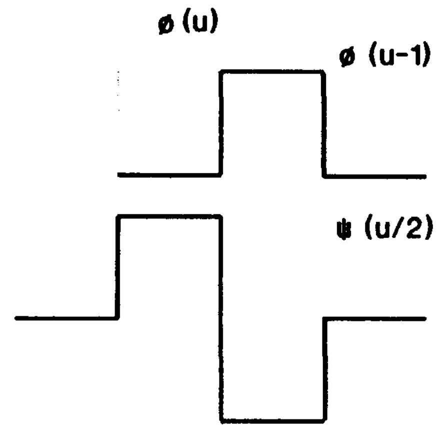


Haar System

$$\int_{-\infty}^{\infty} \phi\left(\frac{u}{2}\right) \phi\left(u - [n - 2l]\right) du = \begin{cases} 1, & \text{if } n = 2l \text{ or } n = 2l+1 \\ 0, & \text{else .} \end{cases}$$

$$= \delta_{n,2l} + \delta_{n,2l+1}$$

$$c_{m,l} = \sum_n c_{m-1,n} \left[2^{-1/2}\left(\delta_{n,2l} + \delta_{n,2l+1}\right)\right] = \frac{1}{\sqrt{2}}(c_{m-1,2l} + c_{m-1,2l+1})$$



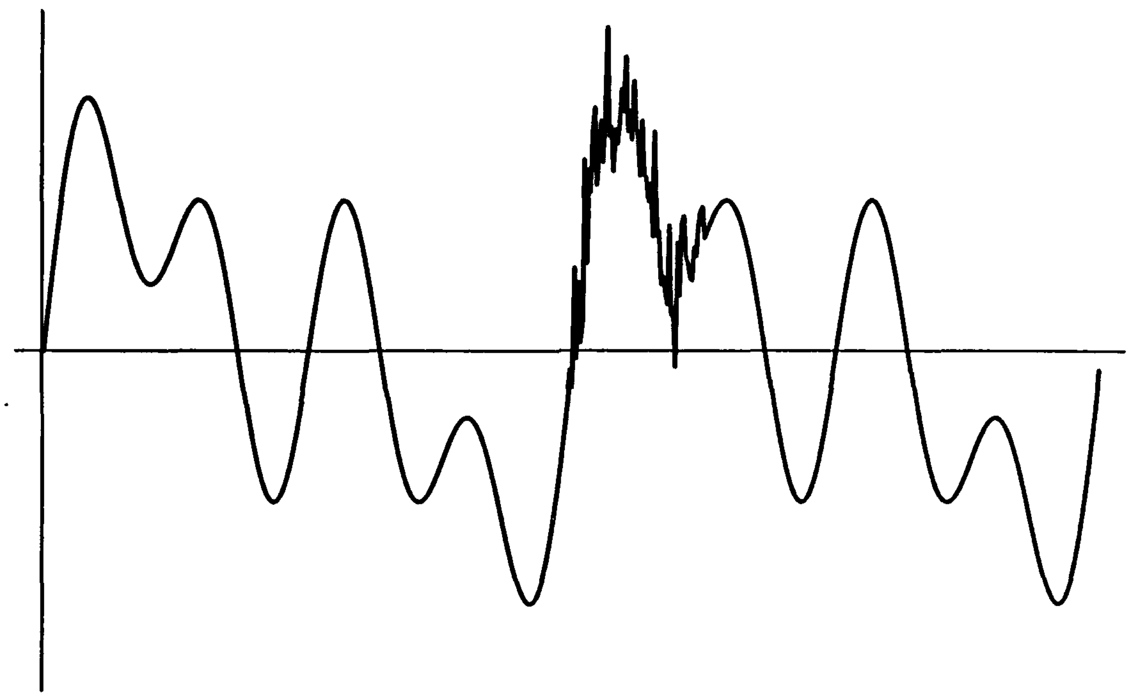
Haar System (cont'd)

$$\int_{-\infty}^{\infty} \psi\left(\frac{u}{2}\right) \phi\left(u - [n - 2l]\right) du = \begin{cases} 1, & \text{if } n = 2l \\ -1, & \text{if } n = 2l + 1 \\ 0, & \text{else .} \end{cases}$$

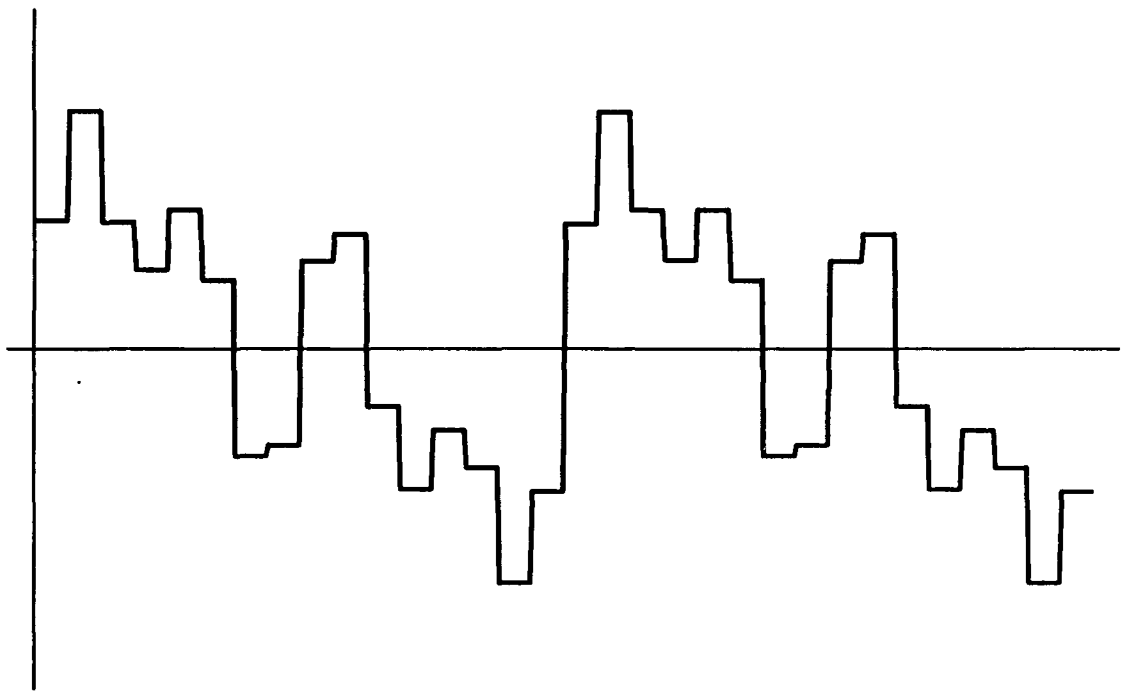
$$= \delta_{n,2l} + \delta_{n,2l+1}$$

$$d_{m,l} = \sum_n c_{m-1,n} \left[2^{-1/2} \left(\delta_{n,2l} + \delta_{n,2l+1}\right)\right] = \frac{1}{\sqrt{2}}(c_{m-1,2l} + c_{m-1,2l+1})$$

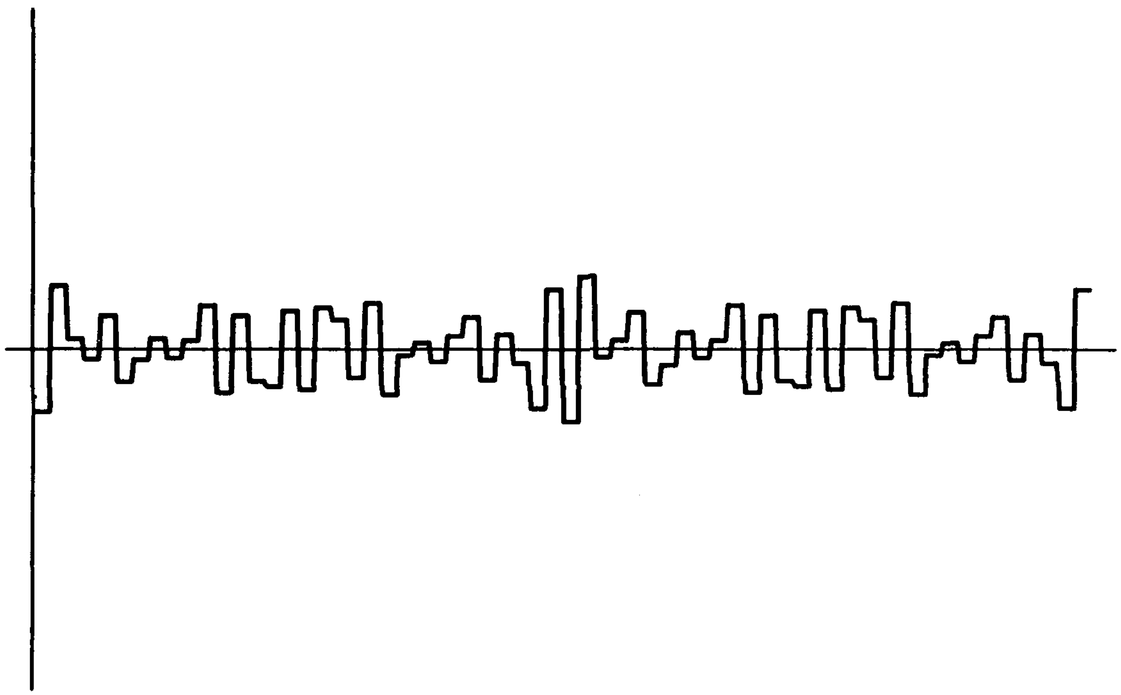
Original Signal f



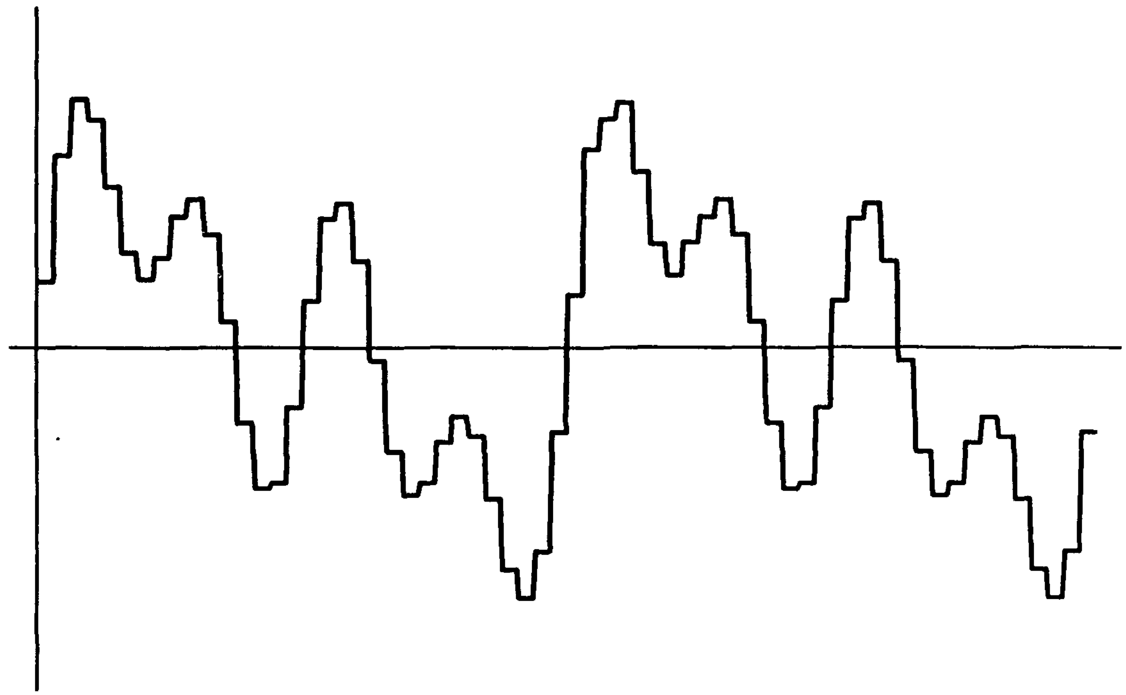
$P_{-5} f$



$Q_{-5}f$

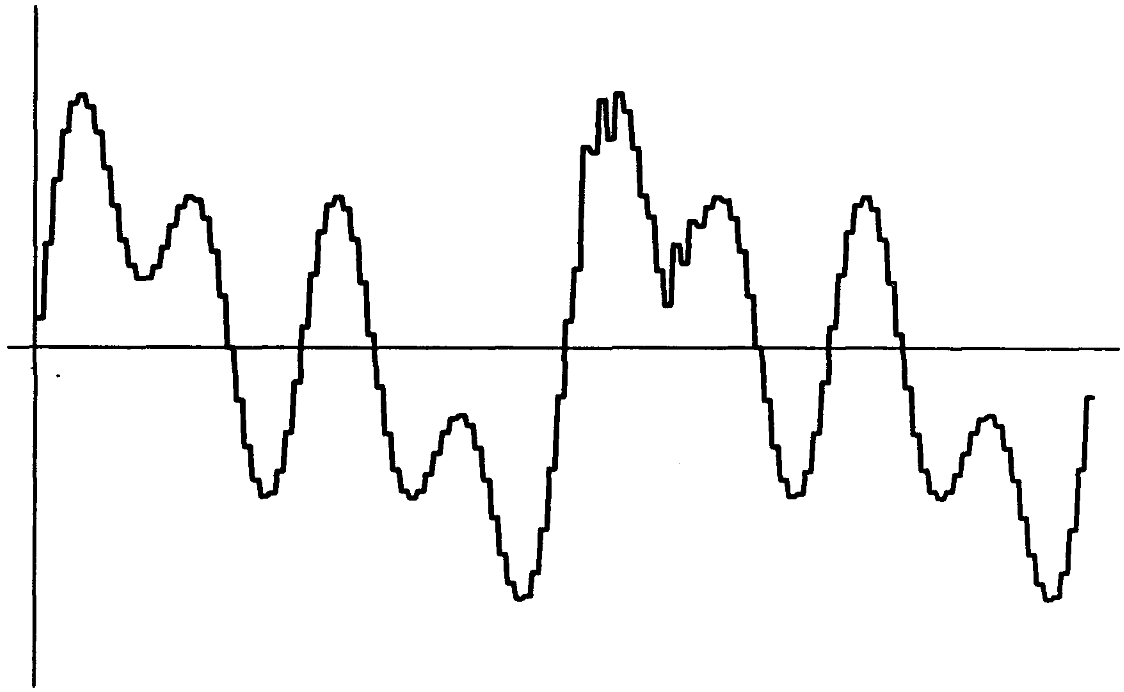


$P_{-6}f$

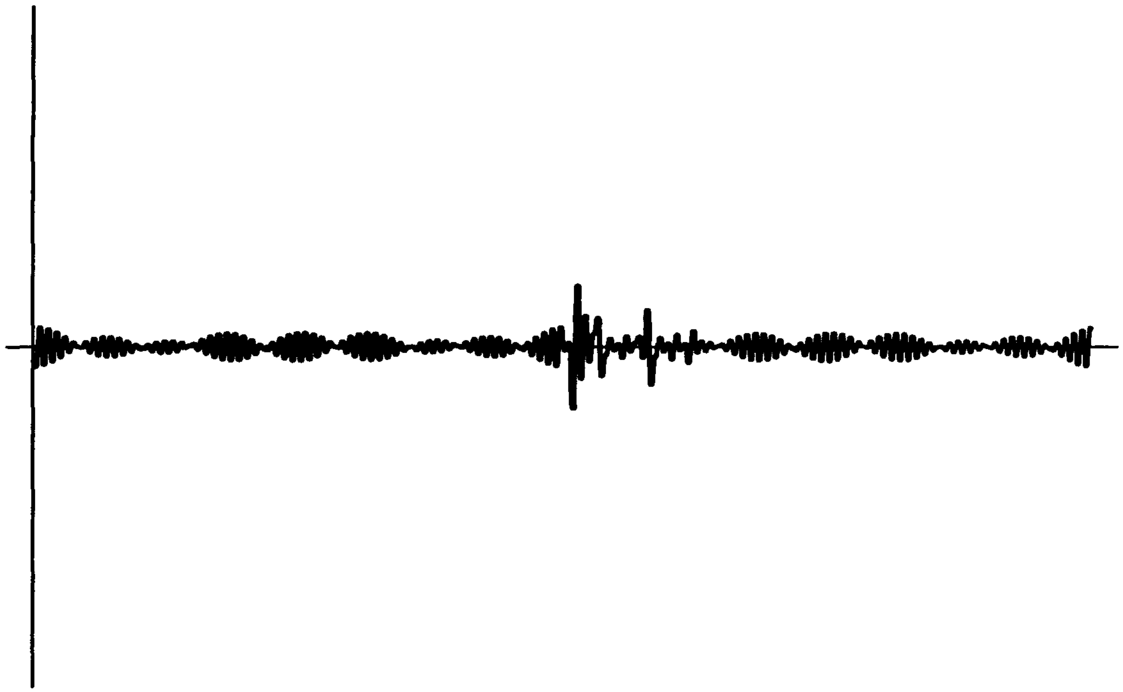


$Q_{-6}f$

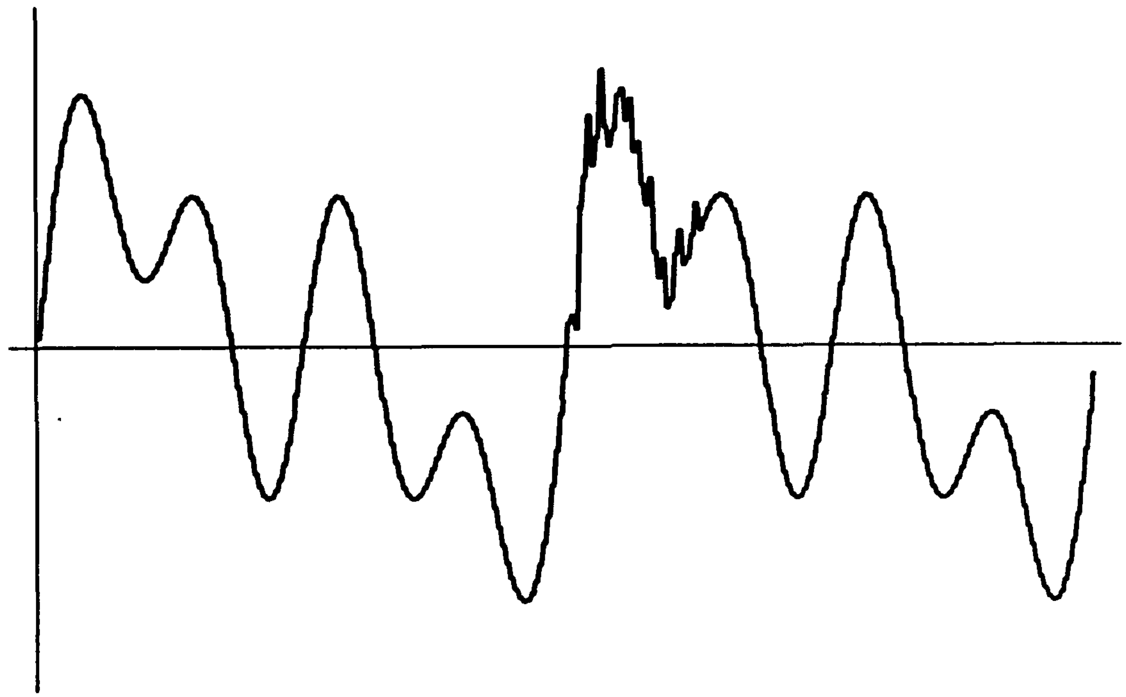
$P_{-7} f$



$Q_{-7}f$



$P_{-8}f$



Quadrature Mirror Filters

Choose

$$G(f) = \mathrm{e}^{-i2\pi f}\overline{H\left(f+\frac{1}{2}\right)}$$

then

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{-i2\pi f}\overline{H\left(f+\frac{1}{2}\right)}\,\mathrm{e}^{i2\pi f t}df \\ &= \int_{-\infty}^{\infty} \overline{H\left(f+\frac{1}{2}\right)\,\mathrm{e}^{-i2\pi f(t-1)}}df\ . \end{aligned}$$

Let $u = f + \frac{1}{2}$. Then

$$g(t) = e^{i\pi(1-t)} \int_{-\infty}^{\infty} \overline{H(u)\, e^{i2\pi u(1-t)}} du = e^{i\pi(1-t)} \overline{h(1-t)} \ .$$

Evaluate g at integers

$$g(n) = (-1)^{(1-n)} \overline{h(1-n)}, \quad \forall\, n \in \mathbf{Z} \ .$$

This defines a Quadrature Mirror Filter (QMF).

Note the correspondence between $G \in L^2(\mathbf{R})$ and $g \in l^2(\mathbf{Z})$ by

$$g(n) = \mathcal{F}^{-1}\{G\}(n) \ .$$

Daubechies' Compactly Supported Orthonormal (CSON) Wavelets [Db1]

Multiresolution Analysis computation of orthonormal wavelets having

- Compact support
- Arbitrarily high degree of regularity
- Linear increase of support width with regularity
- Non-compact representation
- Asymmetric

based on construction of a discrete filter (sequence); graphical construction of the wavelets.

Recall the filter

$$H(f) = \sum_{n \in \mathbf{Z}} h(n)\, \mathrm{e}^{-i2\pi f n}$$

such that

$$\hat{\phi}(2f) = H(f)\hat{\phi}(f)$$

to get $\phi(\frac{t}{2}) \in V_1$ from $\phi(t) \in V_0$.

Control $h(n)$ to build CSON wavelets.

Sufficient Conditions on h

Regularity

$$\sum_n |h(n)||n|^{\varepsilon} < \infty \text{ for some } \varepsilon > 0$$

Orthogonality

$$\sum_n h(n-2k)h(n-2l) = \delta_{kl}$$

Normalization

$$\sum_n h(n) = \sqrt{2}$$

Sufficient Conditions on H

H divisible by trigonometric polynomial:

$$H(f) = \left[\frac{1}{2}\left(1 + \mathrm{e}^{i2\pi f}\right)\right]^N \left[\sum_n z(n)\,\mathrm{e}^{i2\pi n f}\right]$$

where

$$\sum_n |z(n)||n|^\varepsilon < \infty, \quad \text{some} \quad \varepsilon > 0$$

$$\sup_{f \in \mathbf{R}} \left|\sum_n z(n)\,\mathrm{e}^{i2\pi n f}\right| < 2^{N-1}$$

Define

$$\begin{aligned} \hat{\phi}(f) &= \Pi_{k=1}^{\infty} H(2^{-k} f) \\ g(n) &= (-1)^n h(-n+1) \\ \psi(t) &= \sqrt{2} \sum_n g(n) \phi(2t-n) \end{aligned}$$

Then, $\{\phi_{m,n}\}_{m,n \in \mathbf{Z}}$ defines a multiresolution analysis in $L^2(\mathbf{R})$ and $\{\psi_{m,n}\}_{m,n \in \mathbf{Z}}$ is the associated orthonormal wavelet basis.

Rather than perform Fourier inversion of infinite product to obtain $\phi(t)$, use a graphical, recursive, construction

$$\phi(t) = \lim_{k\to\infty} \eta_k(t)$$

where

$$\eta_k(t) = \sqrt{2} \sum_{n\in\mathbf{Z}} h(n)\, \eta_{k-1}(2t-n)$$

$$\eta_0(t) = \chi_{[-1/2,1/2)}(t) \quad .$$

If h is a finite sequence (FIR) the iteration of characteristic functions yields ϕ with finite support!!!

With $h(n) = 0$ for $n < 0$ or $n > 2N - 1$, get compact supports, given by

$$\begin{aligned} \operatorname{supp}(\phi_N) &= [0, 2N-1] \\ \operatorname{supp}(\psi_N) &= [-(N-1), N] \ . \end{aligned}$$

Observe linear increase of support width in N. Coefficients $h(n)$ for $N = 2, 3, \ldots, 10$ are computed numerically and given in [Db1] ($N = 1$ corresponds to the Haar basis.)

Table 1. The coefficients $h_N(n)$ ($n = 0, \cdots, 2N-1$) for $N = 2, 3, \cdots, 10$.

	n	$h_N(n)$		n	$h_N(n)$
$N = 2$	0	.482962913145	$N = 8$	0	.054415842243
	1	.836516303738		1	.312871590914
	2	.224143868042		2	.675630736297
	3	−.129409522551		3	.585354683654
$N = 3$	0	.332670552950		4	−.015829105256
	1	.806891509311		5	−.284015542962
	2	.459877502118		6	.000472484574
	3	−.135011020010		7	.128747426620
	4	−.085441273882		8	−.017369301002
	5	.035226291882		9	−.044088253931
$N = 4$	0	.230377813309		10	.013981027917
	1	.714846570553		11	.008746094047
	2	.630880767930		12	−.004870352993
	3	−.027983769417		13	−.000391740373
	4	−.187034811719		14	.000675449406
	5	.030841381836		15	−.000117476784
	6	.032883011667			
	7	−.010597401785			

excerpt from [Db1]

Regularity

We say

$$x \in C^{\alpha} \iff \int_{-\infty}^{\infty} |\hat{x}(f)|(1+|f|)^{1+\alpha} df < \infty$$

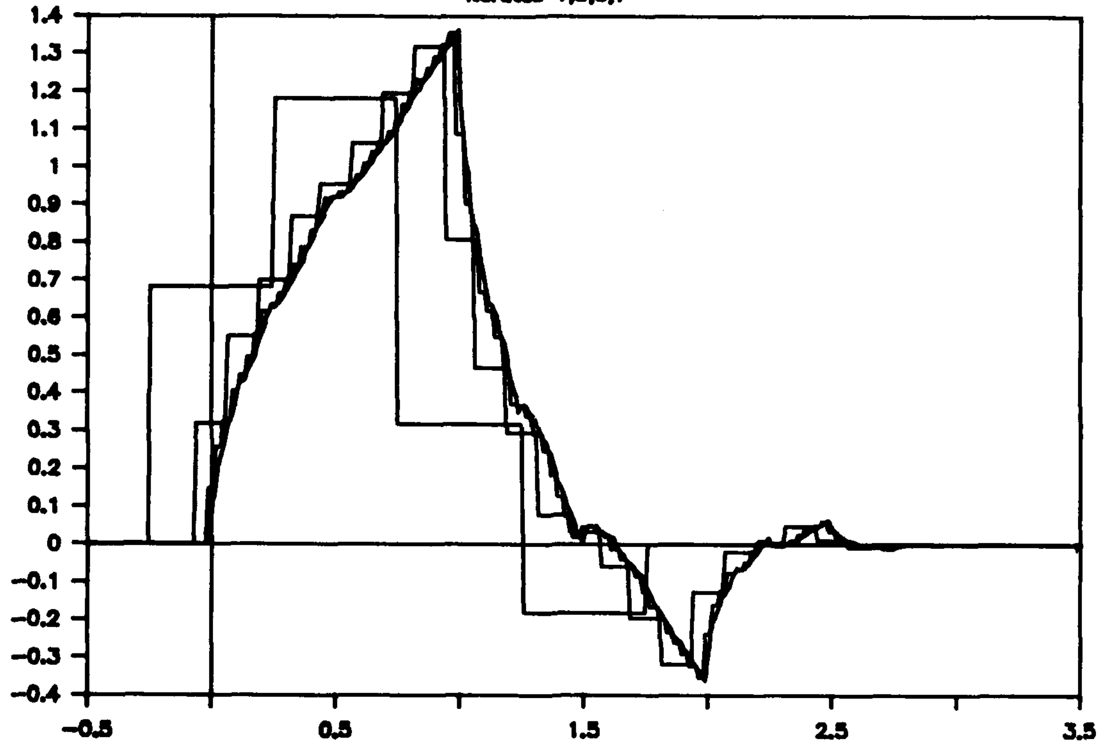
for $\alpha > 0,\ \alpha \neq 1, 2, 3, \ldots$

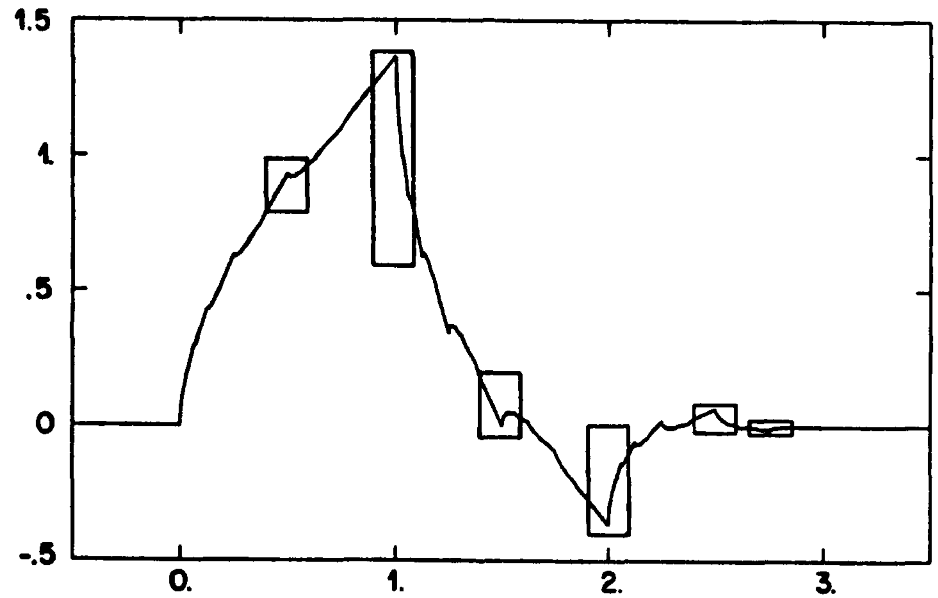
Note: if $\alpha = k = 1, 2, 3, \ldots,$ one gets the usual differentiability of x ; i.e., $x \in C^{k}$

- Regularity grows like λN , for some $\lambda > 0$.
- Regularity and support width are linearly related.

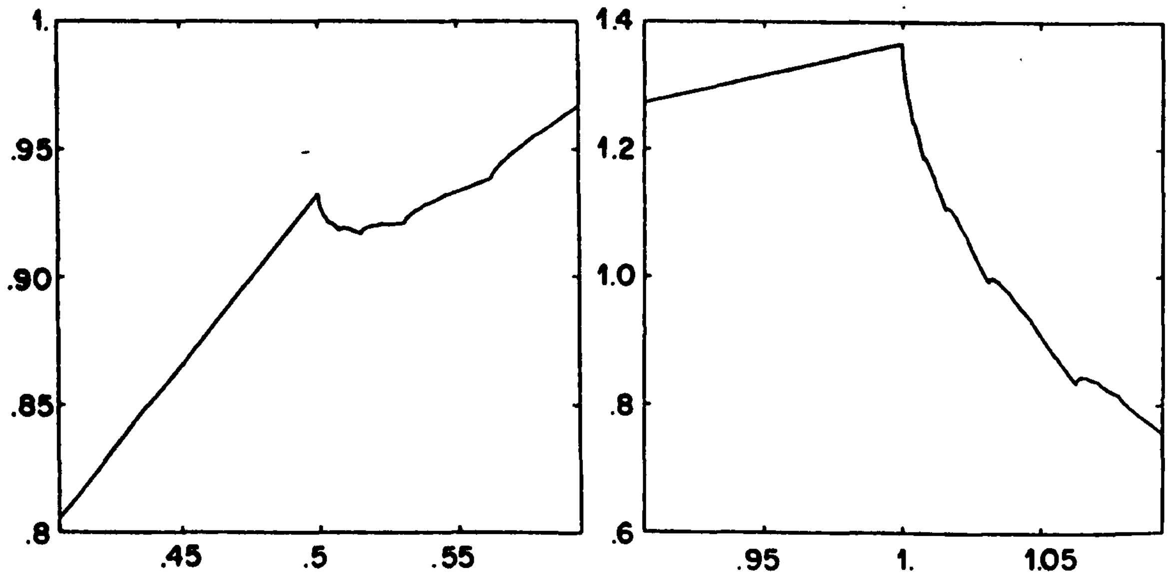
PHI−2

Iterates 1,3,5,7

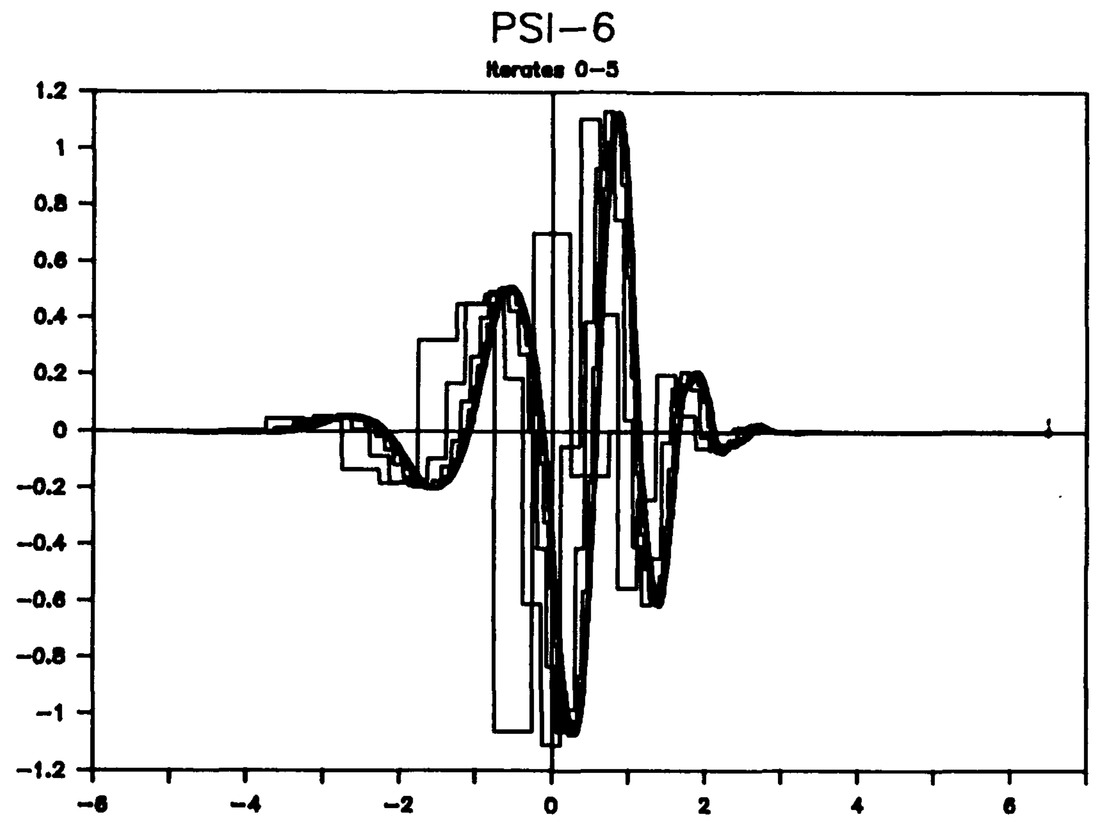


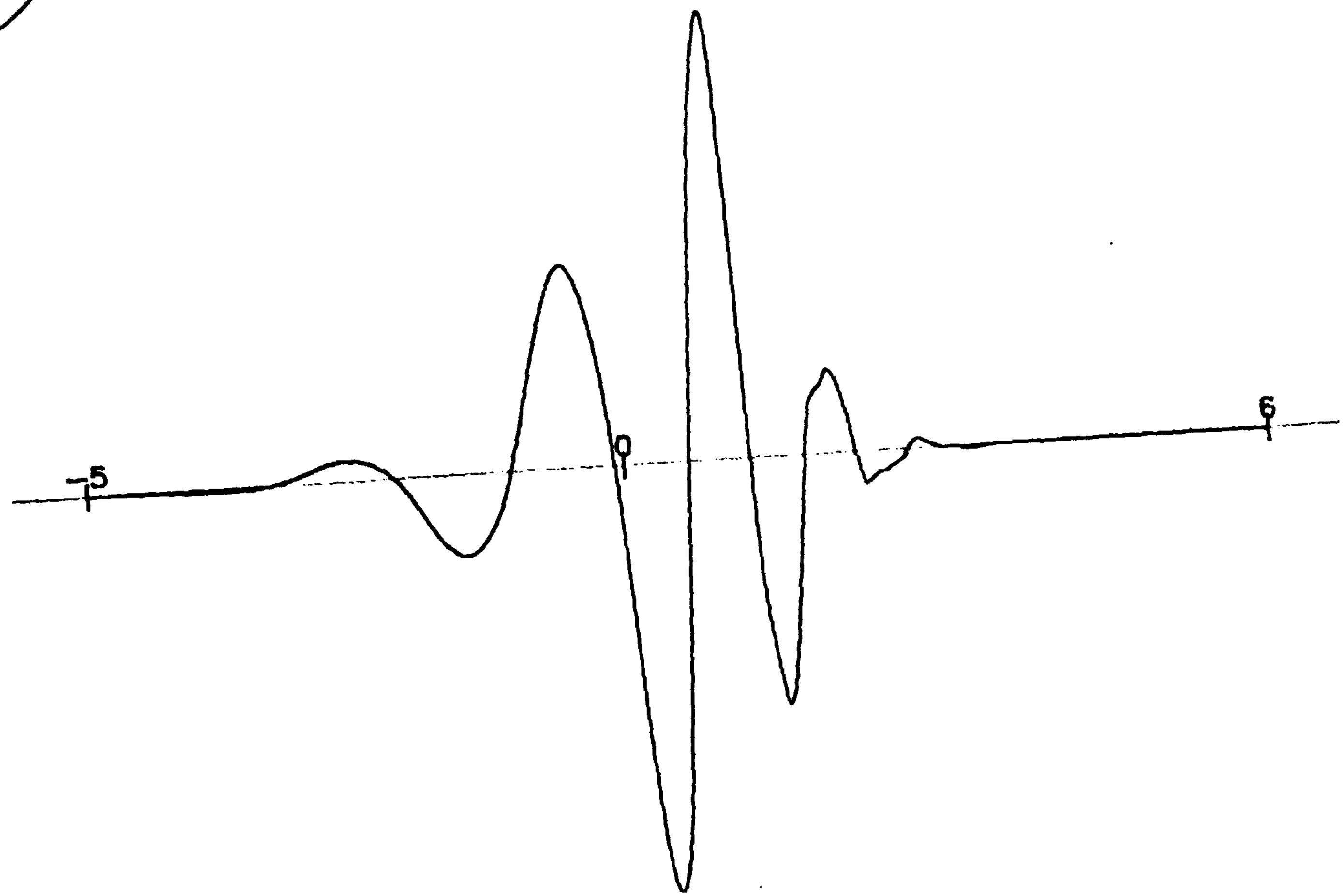


[Db1]



[Db1]


PSI-6
Iterates 0-5
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-6
-4
-2
0
2
4
6


-5
0
6

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