Naval Coastal Systems Center

Panama City, Florida 32407-5000

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PERMEABLE PROLATE SPHEROID IN AN EXTERNAL FIELD

DEMAGNETIZATION AND GRADIOMETER PERFORMANCE

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A permeable prolate spheroid is used as a model for the core of a search coil gradiometer. The performance of the gradiometer is determined by flux concentration and demagnetization due to the core geometry, just as in the case of a search coil magnetometer. Performance curves are given for both magnetometer and gradiometer.

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INTRODUCTION

In an investigation of active magnetic detection, a basic issue is the decoupling of the source field from the detector. One concept for accomplishing this is a search coil gradiometer. The source coil is wrapped around the center of a permeable core, and two detection coils are wrapped around the core, one near each end, and connected in series opposition. The detection coils and/or the source coil can be made movable along the axis of the core, in order to achieve optimum nulling of the source-detector coupling. In addition to this, the device can be used in a discriminator configuration, in which the source frequency is rejected in the detector by means of a feedback circuit with some time constant appropriate to the application. Of course, this mode of operation requires relative motion between the detector and the scattering object, but in many applications this is acceptable.

At practical distances from a scattering object, the scattered field can be approximated by a linear combination of a uniform field and a uniform gradient. The uniform gradient is the quantity to be measured by the search coil gradiometer. It is well known that the presence of a permeable core will cause magnetic flux concentration, and will enhance the measurement of the magnetic field. The question to be addressed here is: what does the permeable core do to the field gradient?

THE MODEL

To investigate the effects of a permeable core, the core will be approximated by a permeable prolate spheroid. The spheroid is oriented with its symmetry axis along the z-axis, and a uniform field plus a uniform gradient field is applied. The magnetic potential for the external field is then given by

$$\Phi^e = -H_1^e x - H_2^e y - H_3^e z - \frac{G_{11}^e}{2}(x^2 - z^2) - \frac{G_{22}^e}{2}(y^2 - z^2) - G_{12}^e xy - G_{13}^e xz - G_{23}^e yz + C \tag{1}$$

where C is an arbitrary constant. The form given here produces a unform gradient tensor which is symmetric and traceless, consistent with the requirements $\nabla \cdot \mathbf{H} = 0$ and $\nabla \times \mathbf{H} = 0$.

To solve the boundary value problem, it is necessary to change to prolate spheroidal coordinates (ξ, η, ϕ) where $\xi (1 \le \xi < \infty)$ labels a family of confocal prolate spheroids, $\eta(-1 < \eta < 1)$ labels an orthogonal family of confocal hyperboids, and $\phi(0 \le \phi \le 2\pi)$ labels angular position about the symmetry axis.[1] With the foci located at $\pm a$ on the z-axis, the connection between the coordinate systems is as follows

$$\xi = \frac{r_1 + r_2}{2a} \tag{2}$$

1 "*Static and Dynamic Electricity*", by William R. Smythe, Third Edition, McGraw-Hill, 1968

$$\eta = \frac{r_1 - r_2}{2a} \tag{3}$$

and

$$\phi = \Gamma \mathrm{an}^{-1}(x, y). \tag{4}$$

where

$$r_1 = \sqrt{x^2 + y^2 + (z + a)^2} \tag{5}$$

and

$$r_2 = \sqrt{x^2 + y^2 + (z - a)^2}. \tag{6}$$

The inverse relationships are

$$x = a\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos\phi \tag{7}$$

$$y = a\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin\phi \tag{8}$$

and

$$z = a\xi\eta. \tag{9}$$

The general solution to Laplace's equation in prolate spheroidal coordinates is[1]

$$\Phi = \sum_{l=0}^{\infty}\sum_{m=0}^{l}[A_{l,m}\cos m\phi + B_{l,m}\sin m\phi]\,[C_{l,m}P_l^m(\xi) + D_{l,m}Q_l^m(\xi)]\,[E_{l,m}P_l^m(\eta) + F_{l,m}Q_l^m(\eta)] \tag{10}$$

where P_l^m and Q_l^m are the associated Legendre functions of the first and second kind, respectively. Note that $Q_l^m(\xi)$ is regular for $\xi \to \infty$ while $P_l^m(\xi)$ is regular for $\xi \to 1$. The first few functions have the explicit forms

$$P_0^0(\xi) = 1 \tag{11}$$

$$P_1^0(\xi) = \xi \tag{12}$$

$$P_1^1(\xi) = \sqrt{\xi^2 - 1} \tag{13}$$

$$P_2^0(\xi) = \frac{1}{2}(3\xi^2 - 1) \tag{14}$$

$$P_2^1(\xi) = 3\xi\sqrt{(\xi^2 - 1)} \tag{15}$$

$$P_2^2(\xi) = 3(\xi^2 - 1) \tag{16}$$

$$Q_0^0(\xi) = \frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\right) \tag{17}$$

$$Q_1^0(\xi) = \frac{1}{2}\xi\ln\left(\frac{\xi+1}{\xi-1}\right) - 1 \tag{18}$$

$$Q_1^1(\xi) = \sqrt{\xi^2 - 1}\left[\frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\right) - \frac{\xi}{\xi^2 - 1}\right] \tag{19}$$

$$Q_2^0(\xi) = \frac{1}{4}(3\xi^2 - 1)\ln\left(\frac{\xi+1}{\xi-1}\right) - \frac{3\xi}{2} \tag{20}$$

$$Q_2^1(\xi) = \sqrt{\xi^2 - 1}\left[\frac{3\xi}{2}\ln\left(\frac{\xi+1}{\xi-1}\right) - \frac{3\xi^2 - 2}{\xi^2 - 1}\right] \tag{21}$$

and

$$Q_2^2(\xi) = (\xi^2 - 1)\left[\frac{3}{2}\ln\left(\frac{\xi+1}{\xi-1}\right) + \frac{5\xi - 3\xi^3}{(\xi^2 - 1)^2}\right] \tag{22}$$

To put the external applied potential into prolate spheroidal coordinates it is necessary to use (Eq. 7-9), using (Eq. 11-16) for guidance. This gives

$$x = aP_1^1(\xi)P_1^1(\eta)\cos\phi \tag{23}$$

$$y = aP_1^1(\xi)P_1^1(\eta)\sin\phi \tag{24}$$

$$z = aP_1^0(\xi)P_1^0(\eta) \tag{25}$$

$$xz = \frac{a^2}{9}P_2^1(\xi)P_2^1(\eta)\cos\phi \tag{26}$$

$$yz = \frac{a^2}{9} P_2^1(\xi) P_2^1(\eta) \sin\phi \tag{27}$$

and

$$xy = \frac{a^2}{18} P_2^2(\xi) P_2^2(\eta) \sin 2\phi \tag{28}$$

for the obvious cases. To handle the terms involving squares of the coordinate values it is more convienient to use linear combinations. First

$$(x^2 - z^2) - (y^2 - z^2) = x^2 - y^2 = \frac{a^2}{9} P_2^2(\xi) P_2^2(\eta) \cos 2\phi. \tag{29}$$

The other combination is

$$\begin{aligned} (x^2 - z^2) + (y^2 - z^2) &= x^2 + y^2 - 2z^2 \\ &= a^2(\xi^2 - 1)(1 - \eta^2) - 2a^2\xi^2\eta^2 \\ &= -\frac{a^2}{3}(3\xi^2 - 1)(3\eta^2 - 1) - \frac{2a^2}{3} \\ &= -\frac{a^2}{12} P_2^0(\xi) P_2^0(\eta) + C'. \end{aligned} \tag{30}$$

The additive constant in (Eq. 30) can be absorbed into the constant in (Eq. 1), so it can be dropped without loss of generality. With the results given in (Eq. 29-30), the following expressions result

$$x^2 - z^2 = \frac{a^2}{18} P_2^2(\xi) P_2^2(\eta) \cos 2\phi - \frac{a^2}{24} P_2^0(\xi) P_2^0(\eta) \tag{31}$$

and

$$y^2 - z^2 = -\frac{a^2}{18} P_2^2(\xi) P_2^2(\eta) \cos 2\phi - \frac{a^2}{24} P_2^0(\xi) P_2^0(\eta). \tag{32}$$

The external potential can now be expressed in prolate spheroidal coordinates as

$$\Phi^e = \sum_{l=1}^{l=2} \sum_{m=0}^{m=l} [A^e_{l,m} \cos m\phi + B^e_{l,m} \sin m\phi] P^m_l(\xi) P^m_l(\eta) \tag{33}$$

where

$$B^e_{1,0} = B^e_{2,0} = 0 \tag{34}$$

$$A^e_{1,0} = -aH^e_3 \quad , \quad A^e_{2,0} = \frac{a^2}{48}(G^e_{11} + G^e_{22}) \tag{35}$$

$$A^e_{1,1} = -aH^e_1 \quad , \quad B^e_{1,1} = -aH^e_2 \tag{36}$$

$$A^e_{2,1} = -\frac{a^2}{9} G^e_{13} \quad , \quad B^e_{2,1} = -\frac{a^2}{9} G^e_{23} \tag{37}$$

$$A^e_{2,2} = \frac{a^2}{36}(G^e_{22} - G^e_{11}) \quad , \quad B^e_{2,2} = \frac{a^2}{18} G^e_{12}. \tag{38}$$

The total external potential has the form

$$\Phi = \Phi^e + \Phi^o \tag{39}$$

where the spheroid-induced part Φ^o has the form

$$\Phi^o = \sum_{l=1}^{l=2} \sum_{m=0}^{m=l} [A^o_{l,m} \cos m\phi + B^o_{l,m} \sin m\phi] Q^m_l(\xi) P^m_l(\eta). \tag{40}$$

The interior potential Φ^i is given by

$$\Phi^i = \sum_{l=1}^{l=2} \sum_{m=0}^{m=l} [A^i_{l,m} \cos m\phi + B^i_{l,m} \sin m\phi] P^m_l(\xi) P^m_l(\eta). \tag{41}$$

The coefficients are determined by applying the boundary conditions of continuity of the tangential **H** field and normal **B** field at the surface of the spheroid where $\xi = \bar{\xi}$. The **H** field is given by $\mathbf{H} = -\nabla\Phi$, and in prolate spheroidal coordinates, this gives

$$H_\xi = -\frac{1}{h_1}\frac{\partial\Phi}{\partial\xi} \tag{42}$$

$$H_\eta = -\frac{1}{h_2}\frac{\partial\Phi}{\partial\eta} \tag{42}$$

and

$$H_\phi = -\frac{1}{h_3}\sin\phi\frac{\partial\Phi}{\partial\phi} \tag{43}$$

where the scale functions are

$$h_1 = \sqrt{\frac{\xi^2 - 1}{\xi^2 - \eta^2}} \tag{44}$$

$$h_2 = \sqrt{\frac{1 - \eta^2}{\xi^2 - \eta^2}} \tag{45}$$

and

$$h_3 = \frac{1}{\sqrt{(\xi^2 - 1)(1 - \eta^2)}}. \tag{46}$$

The boundary conditions are

$$H_\eta^e + H_\eta^o = H_\eta^i|_{\xi=\xi} \tag{47}$$

and

$$H_\xi^e + H_\xi^o = \tau H_\xi^i|_{\xi=\xi} \tag{48}$$

where $\tau = \mu_i/\mu_o$.

Upon application of (Eq. 47) after deleting common factors and using orthogonality of the trigonometric functions, there results

$$A^o_{l,m} Q^m_l(\xi) - A^i_{l,m} P^m_l(\xi) = -A^e_{l,m} P^m_l(\xi) \tag{49}$$

and

$$B^o_{l,m} Q^m_l(\xi) - B^i_{l,m} P^m_l(\xi) = -B^e_{l,m} P^m_l(\xi). \tag{50}$$

The remaining equations are obtained from (Eq. 48), using orthogonality of the trigonometric functions:

$$A^o_{l,m} \dot{Q}^m_l(\xi) - \tau A^i_{l,m} \dot{P}^m_l(\xi) = -A^e_{l,m} \dot{P}^m_l(\xi) \tag{51}$$

and

$$B^o_{l,m} \dot{Q}^m_l(\xi) - \tau B^i_{l,m} \dot{P}^m_l(\xi) = -B^e_{l,m} \dot{P}^m_l(\xi). \tag{52}$$

Define the determinant of coefficients $D_{l,m}$ and the Wronskian $W_{l,m}$, respectively, as

$$D_{l,m} = P^m_l(\xi) \dot{Q}^m_l(\xi) - \tau \dot{P}^m_l(\xi) Q^m_l(\xi) \tag{53}$$

and

$$W_{l,m} = P^m_l(\xi) \dot{Q}^m_l(\xi) - \dot{P}^m_l(\xi) Q^m_l(\xi). \tag{54}$$

Then the solutions for the coefficients are given by

$$A^o_{l,m} = \frac{(\tau - 1) P^m_l(\xi) \dot{P}^m_l(\xi)}{D_{l,m}} A^e_{l,m} \tag{55}$$

$$B^o_{l,m} = \frac{(\tau - 1) P^m_l(\xi) \dot{P}^m_l(\xi)}{D_{l,m}} B^e_{l,m} \tag{56}$$

$$A^i_{l,m} = \frac{W_{l,m}}{D_{l,m}} A^e_{l,m} \tag{57}$$

and

$$B^i_{l,m} = \frac{W_{l,m}}{D_{l,m}} B^e_{l,m}. \tag{58}$$

AXIAL FIELD AND GRADIENT AT THE CENTER OF THE SPHEROID

The performance of the permeable spheroid as a magnetic sensor core is determined by the combined effects of the permeability and the demagnetizing effect of the body. These can be examined, for both the field and gradient, by evaluating the two expressions at the origin. Since the expansion for Φ^i is identical in form to that for Φ^e, (Eq. 41) can be written in terms of cartesian coordinates with the aid of (Eq. 23-30) used in reverse. This gives

$$\Phi^i = \frac{A^i_{1,1}}{a}x + \frac{B^i_{1,1}}{a}y + \frac{A^i_{1,0}}{a}z + \frac{9A^i_{2,1}}{a^2}xz + \frac{9B^i_{2,1}}{a^2}yz + \frac{18B^i_{2,2}}{a^2}xy \tag{59}$$

$$-\frac{12A_{2,0}}{a^2}(x^2+y^2-2z^2) + \frac{9A^i_{2,2}}{a^2}(x^2-y^2).$$

Then

$$H^i_j = -\frac{\partial \Phi^i}{\partial x_j} \tag{60}$$

and

$$G^i_{jk} = -\frac{\partial^2 \Phi^i}{\partial x_j \partial x_k}. \tag{61}$$

Thus, the axial field is

$$H^i_3 = -\frac{A^i_{1,0}}{a} - \frac{9A^i_{2,1}}{a^2}x - \frac{9B^i_{2,1}}{a^2}y - \frac{48A^i_{2,0}}{a^2}z \tag{62}$$

and the axial gradient is

$$G^i_{33} = \frac{48A^i_{2,0}}{a^2}. \tag{63}$$

Applying (Eq. 35) and (Eq. 57), the ratio of the interior axial flux density at the origin to the applied flux density is given by

$$\frac{B_3^i(0,0,0)}{B_3^e} = \tau\frac{W_{1,0}}{D_{1,0}} = f\tau \tag{64}$$

and the ratio of the axial induction gradients everywhere in the interior is given by

$$\frac{\mu G_{33}^i}{\mu_0 G_{33}^e} = \tau\frac{W_{2,0}}{D_{2,0}} = g\tau. \tag{65}$$

The Wronskian $W_{l,m}(\bar{\xi})$ has the explicit form

$$W_{l,m}(\bar{\xi}) = \frac{(-1)^{m+1}4^m}{(\bar{\xi}^2-1)}\frac{\Gamma\left(\frac{l+m+2}{2}\right)\Gamma\left(\frac{l+m+1}{2}\right)}{\Gamma\left(\frac{l-m+2}{2}\right)\Gamma\left(\frac{l-m+1}{2}\right)} \tag{66}$$

which, for $m = 0$ reduces to

$$W_{l,0}(\bar{\xi}) = -\frac{1}{\bar{\xi}^2-1}. \tag{67}$$

With these results, the quantities f and g are given by

$$f = \frac{1}{(\tau-1)\left[\bar{\xi}(\bar{\xi}^2-1)\ln\sqrt{\frac{\bar{\xi}+1}{\bar{\xi}-1}}-\bar{\xi}^2\right]+\tau} \tag{68}$$

and

$$g = \frac{4}{4+3(\tau-1)\bar{\xi}\left[(3\bar{\xi}^2-1)\ln\sqrt{\frac{\bar{\xi}+1}{\bar{\xi}-1}}-6\bar{\xi}\right]}. \tag{69}$$

If D and L are the spheroid diameter and length, respectively, then, with $\varepsilon = D/L$, $\bar{\xi} = 1/\sqrt{1-\varepsilon^2}$. The functions f and g are plotted in Figure 1., for a value of $\tau = 100$. It is clear from these curves that demagnetization is more significant for the gradient than for the field, and an L/D of 27 is required to get 50% of the benefit of the higher permeability.

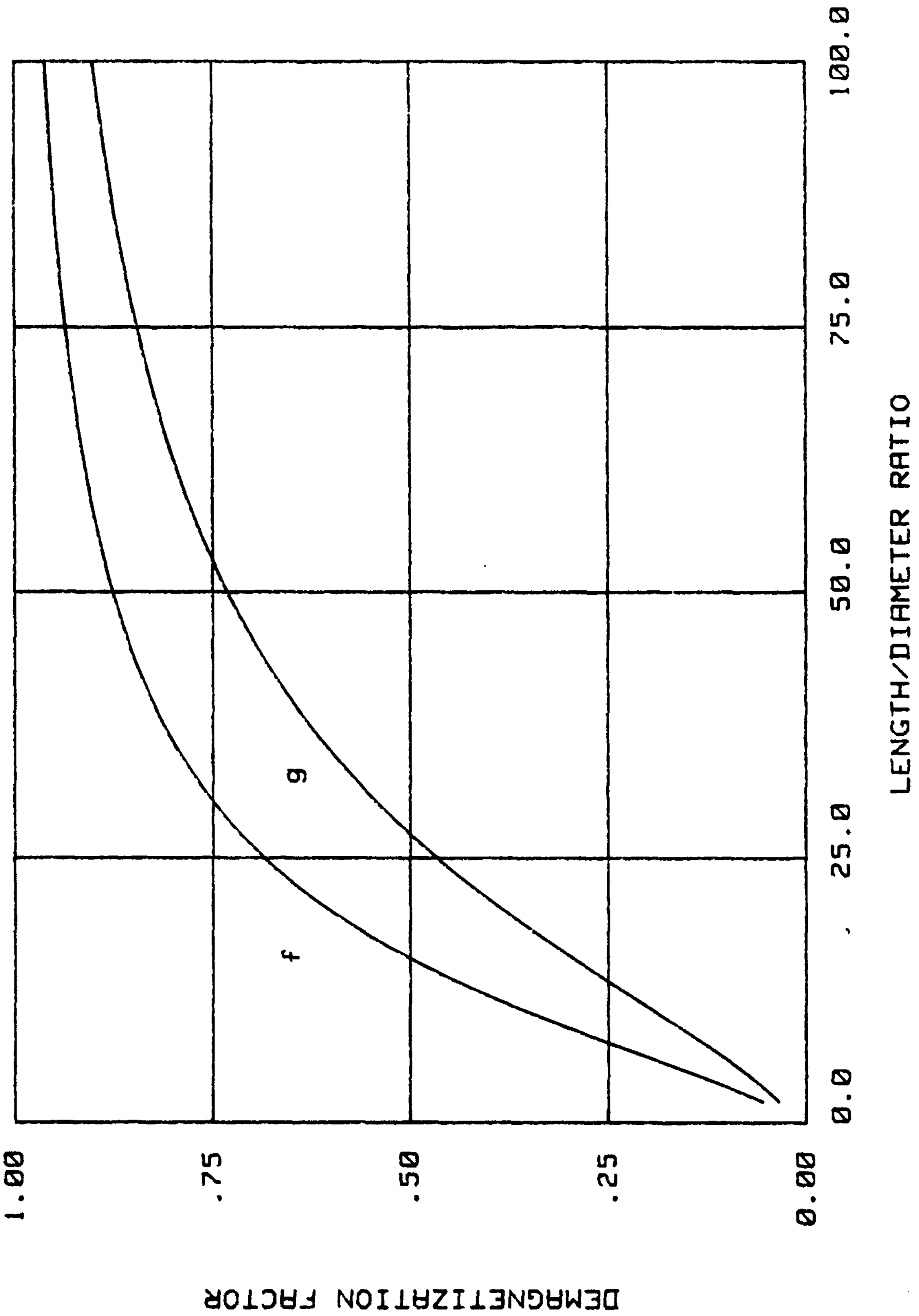

FIGURE 1. FIELD AND GRADIENT DEMAGNETIZATION FACTORS AS A FUNCTION OF LENGTH TO DIAMETER RATIO

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