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SOLUTION STRATEGIES FOR SECOND ORDER, NONLINEAR, ONE DIMENSIONAL, TWO POINT BOUNDARY VALUE PROBLEMS BY FEM ANALYSIS

by

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Solution Strategies for Second Order,
Nonlinear, One Dimensional, Two Point Boundary Value Problems
by FEM Analysis

by

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ABSTRACT

This research demonstrates the Galerkin FEM's ability to provide approximate solutions of second order, nonlinear, one dimensional, two point boundary value problems. The research concentrates on the development of linearization, iteration, and interpolation strategies for the solution of differential equations containing the nonlinear u^2 term. Additionally, various numerical considerations are explored. Over 2000 cases were analyzed using various strategies and results detailing the efficacy of strategy combinations are presented. A linearization strategy known as quasilinearization consistently yielded excellent approximate solutions of the nonlinear differential equations investigated. It converged in a minimum number of iterations and was capable of solving equations which have large function order and activity over their specified domain.

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LIST OF SYMBOLS

a - FEM 2x2 Elemental Differential Operator Matrix
A - FEM NxN System Differential Operator Matrix
B - FEM Nx1 Natural Boundary Condition Vector
CPU - Actual Time Required for Iteration Process
CPU* - Measure of accuracy and computational effort
D - Length of Domain
f - Internal System Excitation
f - FEM 2x1 Element Excitation Vector
F - FEM Nx1 System Excitation Vector
g - Element 2x1 Linear Shape Function Vector
G - Global Nx1 Linear Shape Function Vector
h- Arbitrary function utilized in the linearized Galerkin integral
I - FEM 2x2 Element Linearization Matrix
L - FEM NxN System Linearization Matrix
m - Arbitrary function utilized in classical linearization
N - Number of system degrees of freedon
q - Arbitrary function utilized in quasilinearization
R - Residual
u - Dependent Variable
u - FEM Solution Nx1 Vector
v - Arbitrary function utilized in constant linearization
w - Weighting factor
x - Independent Variable
α_i - Distance from origin of x to origin of element i
η - Nondimensionalized coordinate system
ξ - Element coordinate system
$\mathscr{L}$ - Linear Differential Operator
$\mathscr{L}(\ )$ - Nonlinear Differential Operator

Subscripts

i - Iteration number

j - Global nodal point number

m - Modified

Superscripts

* - Variable which changes during the iteration process

~ - Derived from FEM approximation; approximate value

T - Vector transpose

I. INTRODUCTION

Differential equations which describe natural phenomena are developed through the application of the conservation principles of mass, momentum, and/or energy to a continuous media. After application of these principles to a small or differential element of the media, the resulting equation provides a relationship between one or more derivatives of an unknown function whose behavior is desired. The differential equation is linear when it contains only derivatives of the function with either constant or independent variable function coefficients. The solutions of linear differential equations are unique based on their boundary and/or initial conditions. Various analytical and numerical strategies have been developed to obtain these solutions. The finite element method (FEM), which is used in this research, is one of the more popular and accurate techniques that has been developed to solve differential equations.

When the coefficients of the derivatives are functions of the desired unknown, (that is, the dependent variable), or when the dependent variable and/or its derivatives do not appear linearly in the differential equation, then the differential equation becomes nonlinear and conventional analytical techniques usually do not work. Nonlinear operators occur in a number of differential equations that describe the behavior of natural phenomenon, e.g., the Navier-Stokes equations for fluid flow, a beam on an inelastic foundation, the Falkner Skan equation, etc. This research investigates various strategies for solving nonlinear, second order, one dimensional, two point boundary value problems using FEM analysis. The Galerkin method of weighted residuals (MWR) with discrete basis functions is the FEM technique used to compute approximate solutions of these equations.

A. LINEAR DIFFERENTIAL EQUATIONS

Under most circumstances, this particular FEM provides excellent approximations to linear differential equations of the form

$$\mathcal{L}\,u - f = 0, \quad 0 < x < D \tag{1.1}$$

with appropriate essential and natural boundary conditions where

- $\mathcal{L}$ is the sum of arbitrary linear operators such as $\frac{d}{dx}(\,)$, $\frac{d^2}{dx^2}(\,)$, etc.
- x is the one-dimensional independent variable

- u is the dependent variable of x
- f is an internal excitation to the system
- D is the domain of the differential equation

If the dominant operator in equation (1.1) is of odd order (non self-adjoint form), then the Galerkin method doesn't work and the Petrov-Galerkin FEM must be utilized. The Galerkin FEM transforms the differential equation into a system of linear algebraic equations of the form

$$\mathbf{Au} - \mathbf{F} = \mathbf{0} \tag{1.2.a}$$

where

- **A** is an NxN coefficient matrix characterizing the operator(s) $\mathcal{L}$, and N is the number of system degrees of freedom (DOF) in the approximation.
- **u** is the Nx1 FEM approximate solution vector
- **F** is an Nx1 FEM vector representation of the excitation function f

This system of equations is readily solved by matrix algebra for the response variable,

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{F} \tag{1.2.b}$$

where equation (1.2.b) implies the solution of equation (1.2.a) and does not mean that A inverse is actually obtained.

Chapter II of the research demonstrates the Galerkin Method's ability to accurately model linear differential equations, regardless of the magnitude of the solution, i.e., $u = x^2$, x^3, etc. Additionally, various methods of modeling the excitation function, f, are examined.

B. NONLINEAR DIFFERENTIAL EQUATIONS

The remaining research examines various nonlinear, second order, differential equations of the form

$$\mathcal{L}u + \mathcal{L}(u) - f = 0 \quad 0 < x < D \tag{1.3}$$

with appropriate boundary conditions where all terms are as previously defined and $\mathcal{L}(u)$ is the sum of arbitrary nonlinear terms such as $u\frac{du}{dx}$, $\sin(u)$, u^2, etc. Though this research concentrates on the u^2 nonlinear term, the principles presented should allow for the adequate analysis of any nonlinear term that might be encountered.

The first step in the solution of equation (1.3) is linearization of the nonlinear term. Once the equation is linearized with respect to the dependent variable, the Galerkin FEM is used in an iterative fashion to solve a linear system of algebraic equations in the form

$$\mathbf{Au} + \mathbf{L}^{*}\mathbf{u} - \mathbf{F} = \mathbf{0} \tag{1.4}$$

where the coefficients of $\mathbf{L}^{*}$ are functions of u evaluated at the previous iteration, denoted $\mathbf{u}_{i-1}$, and where subscripts i and i-1 refer to the present and past iterations. To begin the iteration process, a value is initially assumed for $\mathbf{u}_{i-1}$ and the system of equations is solved for $\mathbf{u}_i$. The new values of $\mathbf{u}_i$ and the input values of $\mathbf{u}_{i-1}$ at each node are compared and tested against a convergence criteria. If convergence is not obtained, the new value of $\mathbf{u}_i$ is substituted back into the system of equations for $\mathbf{u}_{i-1}$ and the iteration process continues until convergence is obtained. The final iteration yields the FEM approximate solution of the nonlinear differential equation.

Chapter III provides the general principles and considerations which are involved in solving nonlinear differential equations. Chapter IV utilizes these principles in the solution of two different equations containing the nonlinear term u^2. Final conclusions are made based on the solution results as to which problem solving strategy yields the most accurate approximation while using the least amount of computer time.

II. LINEAR APPLICATIONS OF THE GALERKIN FINITE ELEMENT METHOD (FEM)

This section examines the Galerkin method of weighted residuals (MWR) using discrete linear shape functions, that is the Galerkin FEM, as applied to linear second order, one dimensional, differential equations, that is, two point boundary value problems. In section A, a general FEM procedure for the differential equation

$$u'' - f(x) = 0 \quad a < x < b \tag{2.1}$$

with two boundary conditions, one at each end point of the domain, is presented. In particular, several strategies for the FEM representation of the excitation function $f(x)$ are developed.

In section B, the various strategies for FEM representation of the excitation function are implemented on equation (2.1) for three different excitation functions. The three $f(x)$ functions were selected to provide the solutions of x^2, x^3, x^4 to equation (2.1).

A. GENERAL FEM PROCEDURE

Here, a general procedure for the FEM formulation of equation (2.1) is presented. The Galerkin FEM process transforms a differential equation into a system of linear algebraic whose solution is an approximation to the exact solution of the differential equation. The transformation requires the following three steps:

- Step 1: Form an N degree of freedom approximation, say $\tilde{u} = \mathbf{G}^T\mathbf{u}$, where:
  $\tilde{u}$ is the approximate solution
  $\mathbf{G}^T$ is the 1xN transpose vector of linear shape functions
  $\mathbf{u}$ is the Nx1 vector of the FEM solution, $\tilde{u}$, at each node
- Step 2: Form the Residual $R = \tilde{u}'' - f(x)$, where:
  $f(x)$ is the excitation function in the differential equation
- Step 3 Form the Galerkin integral equations $\int_a^b \mathbf{G}R\,dx = \mathbf{0}$

The evaluation of these integral equations gives a system of N linear algebraic equations whose solution is the approximation, $\tilde{u}$. The details of the FEM formulation for equation (2.1) follows. Substitution of the residual R into the Galerkin integral equation and separating terms yields

$$\int_a^b \mathbf{G}\tilde{u}''dx - \int_a^b \mathbf{G}f(x)dx = 0 \tag{2.2.a}$$

From step 1, $\tilde{u}'' = (\mathbf{G}^T\mathbf{u})'' = (\mathbf{G}^T)''\mathbf{u}$. Substitution of this into the first term of equation (2.2.a) and moving the second term to the right hand side yields

$$\int_a^b \mathbf{G}(\mathbf{G}^T)''dx\,\mathbf{u} = \int_a^b \mathbf{G}f(x)dx \tag{2.2.b}$$

Note that the term on the left hand side is the FEM representation of $\mathcal{L}u = u''$ and the term on the right hand side is the FEM representation of $f(x)$. The result of an integration by parts of the left hand side leaves

$$\mathbf{G}(\mathbf{G}^T)'\mathbf{u}\Big|_a^b - \int_a^b \mathbf{G}'(\mathbf{G}^T)'dx\,\mathbf{u} = \int_a^b \mathbf{G}f(x)dx \tag{2.2.c}$$

Each term is now evaluated individually.

Boundary Term , $\mathbf{G}(\mathbf{G}^T)'\mathbf{u}\Big|_a^b$

Differentiating the equation in step 1 of the formulation process yields $(\mathbf{G}^T)'\mathbf{u} = \tilde{u}'$. Substitution of $\tilde{u}'$ into the above equation and evaluating it at the upper and lower limits gives

$$\mathbf{G}\tilde{u}'\big|_b - \mathbf{G}\tilde{u}'\big|_a = \begin{bmatrix} G_1(b)\tilde{u}'(b) \\ G_2(b)\tilde{u}'(b) \\ \cdots \\ \cdots \\ G_N(b)\tilde{u}'(b) \end{bmatrix} - \begin{bmatrix} G_1(a)\tilde{u}'(a) \\ G_2(a)\tilde{u}'(a) \\ \cdots \\ \cdots \\ G_N(a)\tilde{u}'(a) \end{bmatrix} \tag{2.3.a}$$

In equation (2.3.a), G_i denotes that it is the linear shape function associated with the i^{th} system degree of freedom (SDOF) at the i^{th} system nodal point. The ' a ' and ' b ' arguments of G and u' are the endpoint values of the domain where these functions are evaluated. These endpoints could be denoted by their system node identities, which are 1 and N, at the left and right end points respectively. With this notation, equation (2.3.a) becomes:

$$\mathbf{G}\tilde{u}'\big|_b - \mathbf{G}\tilde{u}'\big|_a = \begin{bmatrix} G_1(N)\tilde{u}'(N) \\ G_2(N)\tilde{u}'(N) \\ \cdots \\ \cdots \\ G_N(N)\tilde{u}'(N) \end{bmatrix} - \begin{bmatrix} G_1(1)\tilde{u}'(1) \\ G_2(1)\tilde{u}'(1) \\ \cdots \\ \cdots \\ G_N(1)\tilde{u}'(1) \end{bmatrix} \qquad (2.3.b)$$

Due to the Kroenecker delta property[1] of the linear shape functions, all terms are equal to zero except $G_N(N)\tilde{u}'(N)$ and $G_1(1)\tilde{u}'(1)$. This yields a single vector comprised of the natural boundary conditions at each end of the domain and is designated **B** where

$$\mathbf{G}\tilde{u}'\big|_b - \mathbf{G}\tilde{u}'\big|_a = \mathbf{B} = \begin{bmatrix} -\tilde{u}'(a) \\ 0 \\ \cdots \\ \cdots \\ 0 \\ \tilde{u}'(b) \end{bmatrix} \qquad (2.3.c)$$

When natural boundary conditions, u', are present, they are used for the corresponding $\tilde{u}'$ in equation (2.3.c).

Differential Operator , $\int_a^b \mathbf{G}'(\mathbf{G}^T)'dx\ \mathbf{u}$

The integral $\int_a^b \mathbf{G}'(\mathbf{G}^T)'dx$, associated with the u'' operator, is reduced to the element coordinate level for evaluation and becomes the 2x2 element **a** matrix

$$\mathbf{a} = \int_0^{l_e} \mathbf{g}'(\mathbf{g}^T)'d\xi \qquad (2.4.a)$$

where ξ is the local coordinate axis, l_e is the length of the element, and **g** is the 2x1 vector of linear shape functions, given by

---

[1] $G_i(j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$\mathbf{g} = \begin{bmatrix} 1 - \dfrac{\xi}{l_e} \\ \dfrac{\xi}{l_e} \end{bmatrix} \qquad (2.4.b)$$

as shown in Figure 1. Differentiation of the g functions gives the 2x1 g′ vector,

$$\mathbf{g}' = \begin{bmatrix} -\dfrac{1}{l_e} \\ \dfrac{1}{l_e} \end{bmatrix} \qquad (2.4.c)$$

Substituting g′ and $(\mathbf{g}^T)'$ into the integral of equation (2.4.a) gives

$$\mathbf{a} = \int_0^{l_e} \begin{bmatrix} -\dfrac{1}{l_e} \\ +\dfrac{1}{l_e} \end{bmatrix} \begin{bmatrix} -\dfrac{1}{l_e} & \dfrac{1}{l_e} \end{bmatrix} d\xi = \int_0^{l_e} \begin{bmatrix} +\dfrac{1}{l_e^2} & -\dfrac{1}{l_e^2} \\ -\dfrac{1}{l_e^2} & +\dfrac{1}{l_e^2} \end{bmatrix} d\xi \qquad (2.4.d)$$

$$= \frac{1}{l_e} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

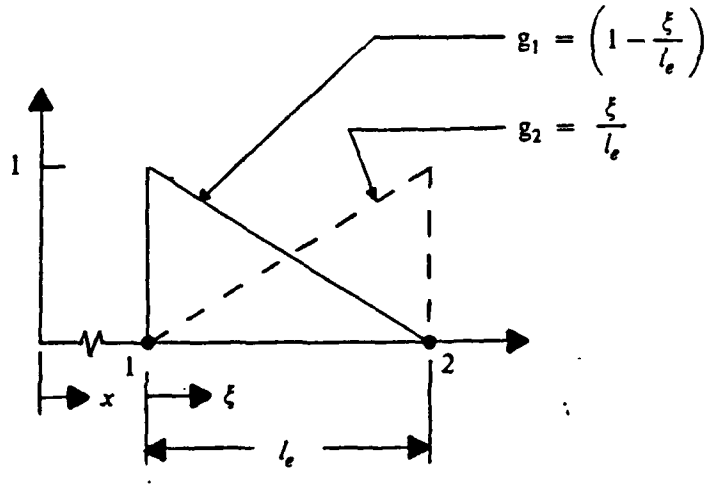


Figure 1. One Dimensional Linear Shape Functions

After construction in an element DO loop, the 2x2 **a** matrix for each element is then distributed into the NxN system **A** matrix in accordance with a correspondence table,

which relates local DOF to system DOF, and where N is the number of system degrees of freedom. Each element has two local degrees of freedom, LDOF 1 at the left end of an element, and LDOF 2 at the right end of the element. The correspondence between LDOF 1 and LDOF 2 of the i^{th} element and the j^{th} system degree of freedom (SDOF J) is $J = (i - 1) + k$ where k is 1 or 2. Upon assembly of all the element matrices, we obtain the system **Au** term.

Excitation , $\int_a^b \mathbf{G} f(x)\, dx$

This term determines the forcing function (or excitation) vector **F** , that is

$$\mathbf{F} = \int_a^b \mathbf{G} f(x)\, dx \tag{2.5}$$

and is obtained by assembling the 2x1 element excitation vectors **f**. The **f** vectors can be either modeled as a lumped approximation term in several different ways or integrated exactly to yield a consistent forcing function. In this study, two lumped approximations and the exact integration are developed and thereafter compared to determine which yields the more accurate solution. A third approximation method is also described. Although this third method is not used in the evaluation of the excitation function in this chapter on linear systems, it is used in the next chapter on the nonlinear systems portion of the research. It should be noted that as the number of elements approaches infinity and the element length approaches zero, each approximation technique described below yields the exact value of the excitation integral.

- Midpoint Lumped Approximation of $f(x)$

  The midpoint lumped approximation method for evaluation of the excitation vector is the simplest and crudest approximation. This approximation involves evaluating the function $f(x)$ at the midpoint of the element and multiplying this value by the element length. Half of this area $(f(l_e/2)l_e)$ is then placed at the left element node and the other half at the right element node as illustrated in Figure 2 on page 9 for three different arbitrary $f(x)$. For the monotonically increasing function in Figure 2.a, this method places too much area at the left local nodal point (LNP) and not enough at the right. Conversely, when $f(x)$ is monotonically decreasing, too little area is placed at the left node while the right node gets too much. When $f(x)$ is concave over the element, too little area is placed at each node (Figure 2.b), while for the convex case (Figure 2.c), each LNP receives too much area.

- 1/4 - 3/4 Lumped Approximation of $f(x)$

  This technique, which is a refinement of the previous one, evaluates $f(x)$ at the quarter point and three quarter point of an element. Each value is multiplied by half of the element length and the resulting areas are placed at the left and right

element nodal points respectively as shown in Figure 3 on page 10 for the same three arbitrary functions. This method provides a better approximation than the lumped midpoint technique, especially for the monotonically increasing function in Figure 3.a, as it uses two points to capture the behavior of the curve instead of one.

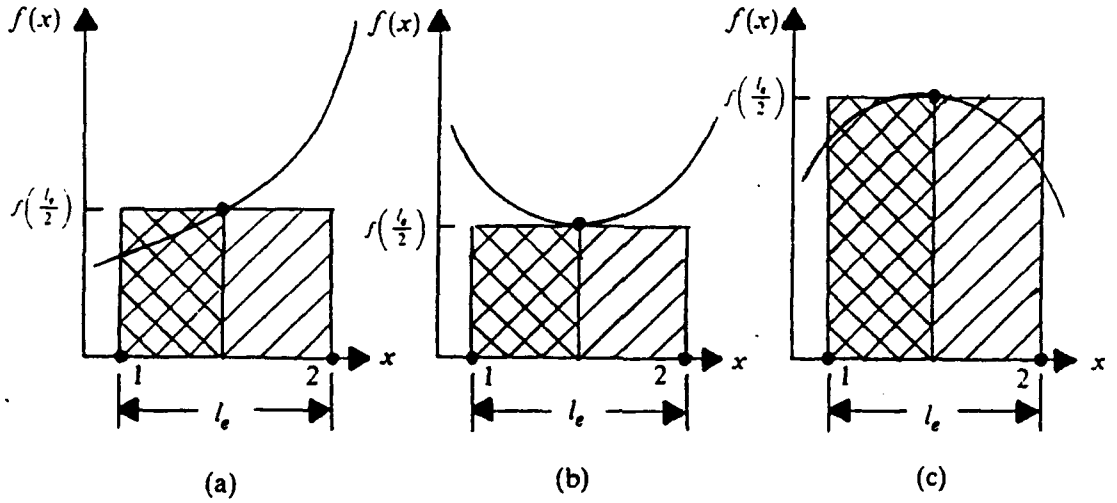


Figure 2. Illustration of Midpoint Lumped Approximation

- Linear Approximation

  This method approximates $f(x)$ over the element in terms of the linear shape functions where

$$f(x) \approx \bar{f}(x) = f(x_1)\left(1 - \frac{\xi}{l_e}\right) + f(x_2)\left(\frac{\xi}{l_e}\right) \tag{2.6}$$

  as shown in Figure 4 on page 10. It overestimates the area for concave curves (Figure 4.a) and underestimates for convex curves as shown in Figure 4.b. Note that $f(x_1)$ and $f(x_2)$ can be generalized to coefficients f_1 and f_2 to give a better linear fit of the curve. Also, the linear shape functions can be replaced by higher order shape functions which provides an n^{th} order approximation of $f(x)$ as opposed to a linear one. This approximation method is not utilized for the evaluation of forcing functions, but is used later in the nonlinear portion of the research to approximate other types of functions.

- Consistent

  The consistent solution requires transforming $f(x)$ into the element coordinate system, $f(\xi)$, by substituting $x = \alpha_i + \xi$ into $f(x)$ and performing the integration over the lengths of the elements. The coordinate transformation is illustrated in Figure 5 on page 11.

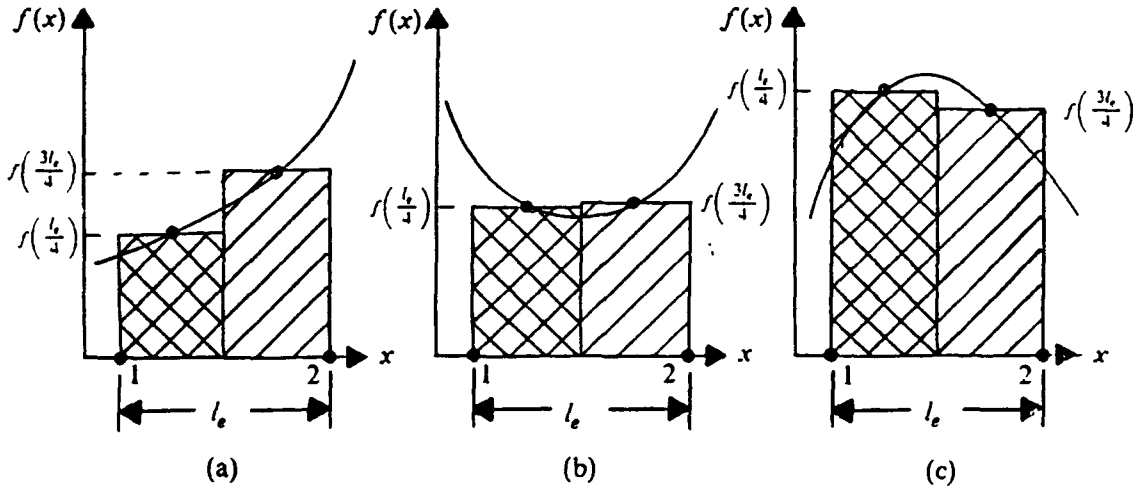


Figure 3. Illustration of 1/4 - 3/4 Lumped Approximation

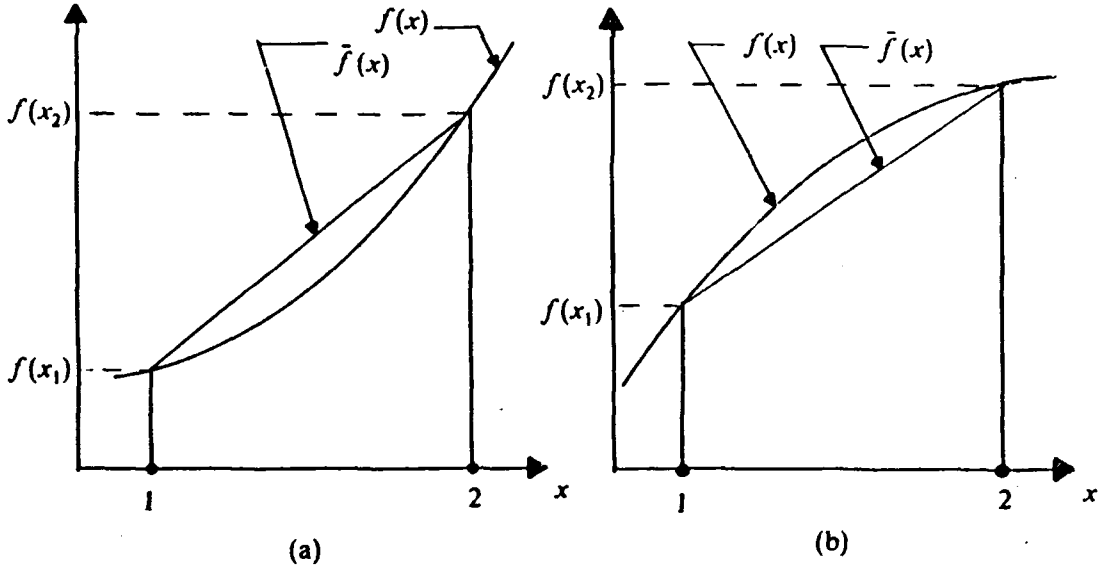


Figure 4. Illustration of Linear Approximation

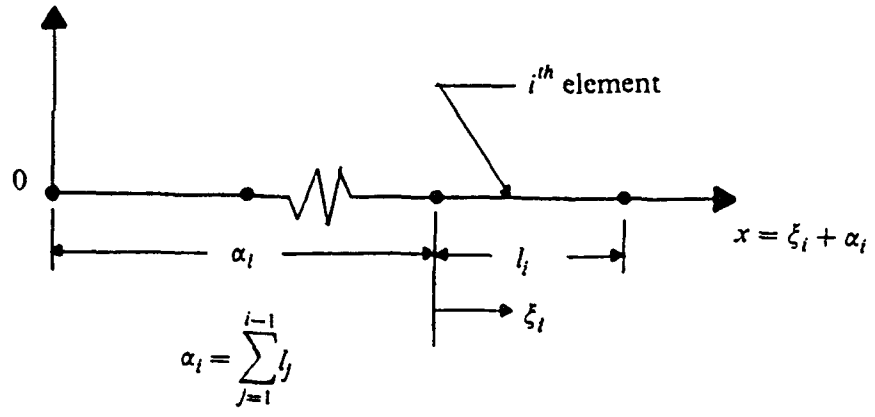


Figure 5. Global to Local Coordinate Transformation

The integration for each of the above methods is performed on the element level, producing the 2x1 **f** vector. The **f** vector for each element is then distributed into the Nx1 system force vector **F** in accordance with the local to global nodal point correspondence.

Substitution of **A**, **F**, and **B** into equation (2.2.c) yields a linear system of algebraic equations in the form

$$-\mathbf{Au} = \mathbf{F} - \mathbf{B} \tag{2.7.a}$$

The $(\mathbf{F} - \mathbf{B})$ term can be combined into a single vector designated $\mathbf{F}_m$ leaving

$$-\mathbf{Au} = \mathbf{F}_m \tag{2.7.b}$$

which can be solved for **u**, the FEM approximate solution at each nodal point.

B. RESULTS

In order to obtain specific results, the following equations are analyzed over the domain $0 < x < 2$:

$$u'' = 2; \quad u(0) = 0, \; u'(2) = 4 \qquad u_{exact} = x^2 \tag{2.8}$$

$$u'' = 6x; \quad u(0) = 0, \; u'(2) = 12 \qquad u_{exact} = x^3 \tag{2.9}$$

$$u'' = 12x^2; \quad u(0) = 0, \; u'(2) = 32 \qquad u_{exact} = x^4 \tag{2.10}$$

The excitation function in each of the above equations is evaluated using the three methods previously described in II.A. The detailed formulation of these different f vectors is shown in Appendix A for the forcing function of equation (2.9) where $f(x) = 6x$. The other two $f(x)$ are evaluated in a similar manner and the results for all three $f(x)$ functions are shown in Table 1.

Table 1. FORCING FUNCTIONS FOR VARIOUS F(X)

$f(x)$	Midpoint Approximation	1/4 - 3/4 Approximation	Consistent
2	$\begin{bmatrix} l_e \\ l_e \end{bmatrix}$	$\begin{bmatrix} l_e \\ l_e \end{bmatrix}$	$\begin{bmatrix} l_e \\ l_e \end{bmatrix}$
$6x$	$3l_e \begin{bmatrix} \alpha + \frac{l_e}{2} \\ \alpha + \frac{l_e}{2} \end{bmatrix}$	$3l_e \begin{bmatrix} \alpha + \frac{l_e}{4} \\ \alpha + \frac{3l_e}{4} \end{bmatrix}$	$\begin{bmatrix} 3\alpha l_e + l_e^2 \\ 3\alpha l_e + 2l_e^2 \end{bmatrix}$
$12x^2$	$6l_e \begin{bmatrix} \left(\alpha + \frac{l_e}{2}\right)^2 \\ \left(\alpha + \frac{l_e}{2}\right)^2 \end{bmatrix}$	$6l_e \begin{bmatrix} \left(\alpha + \frac{l_e}{4}\right)^2 \\ \left(\alpha + \frac{3l_e}{4}\right)^2 \end{bmatrix}$	$\begin{bmatrix} 6\alpha^2 l_e + 4\alpha l_e^2 + l_e^3 \\ 6\alpha^2 l_e + 8\alpha l_e^2 + 3l_e^3 \end{bmatrix}$

The FORTRAN programs and results for a ten element analysis of each equation with the various forcing functions are provided in Appendix B. The first problem considered is that presented by equation (2.8). Due to the nature of the forcing function in this equation, i.e., a constant, all three formulation options provide the same result. The FEM approximation, shown in Figure 6, provided the exact solution at each nodal point for a 10 element analysis.

The solutions of equations (2.9) and (2.10) were then considered. The different forcing function formulations in Table 1 were used in solving these differential equations. For clarity purposes, only a portion of the plots comparing each solution process to the exact solution in an area of rapidly changing gradient are shown in Figure 7 on page 14 (for $u'' = 6x$) and Figure 8 on page 15 (for $u'' = 12x^2$).

The midpoint approximation method for F provides a solution which is larger than the exact at each nodal point for both equations (2.9) and (2.10). The approximation is worse for equation (2.10) in Figure 8 as the midpoint method provides an overesti-

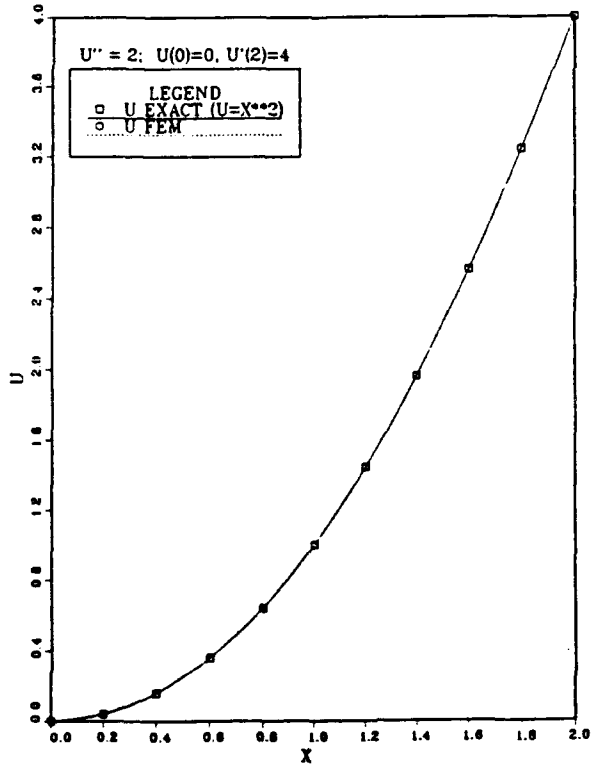


Figure 6. Comparison of Solutions for Equation 2.8 Using 10 Elements.

nation of the actual value of the excitation integral at each node due to its inability to account for the quadratic nature of the forcing function.

The quarter/three-quarter point approximation of F provides solutions for equations (2.9) and (2.10) that are quite close to the exact solution at each nodal point. This technique provides a much better approximation of the area under the forcing function curve because it discretizes the area into two independent sections, where as the mid-point technique did not. Thus, this technique is quite accurate in approximating excitation functions such as x^2 which are monotonic and quadratic in nature.

The consistent formulation method provides the exact solution for both equations (2.9) and (2.10) at each nodal point, even when the solution curve has a rapidly changing gradient such as $u = x^4$ as shown in Figure 8. It was expected that this technique would provide the most accurate solution for all cases as it yields the exact area given by the Galerkin excitation integral.

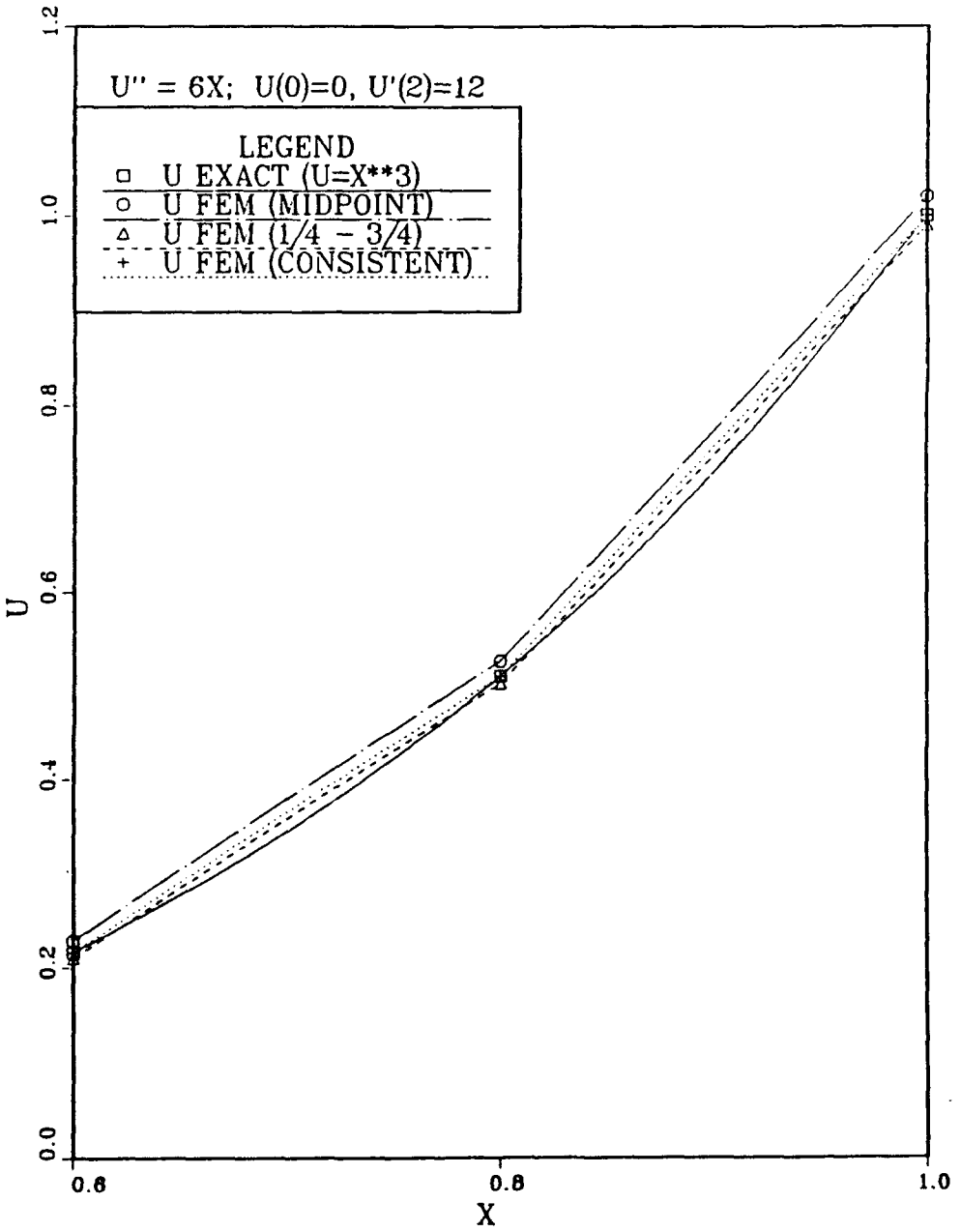


Figure 7. Comparison of Solutions for Equation 2.9 Using 10 Elements.

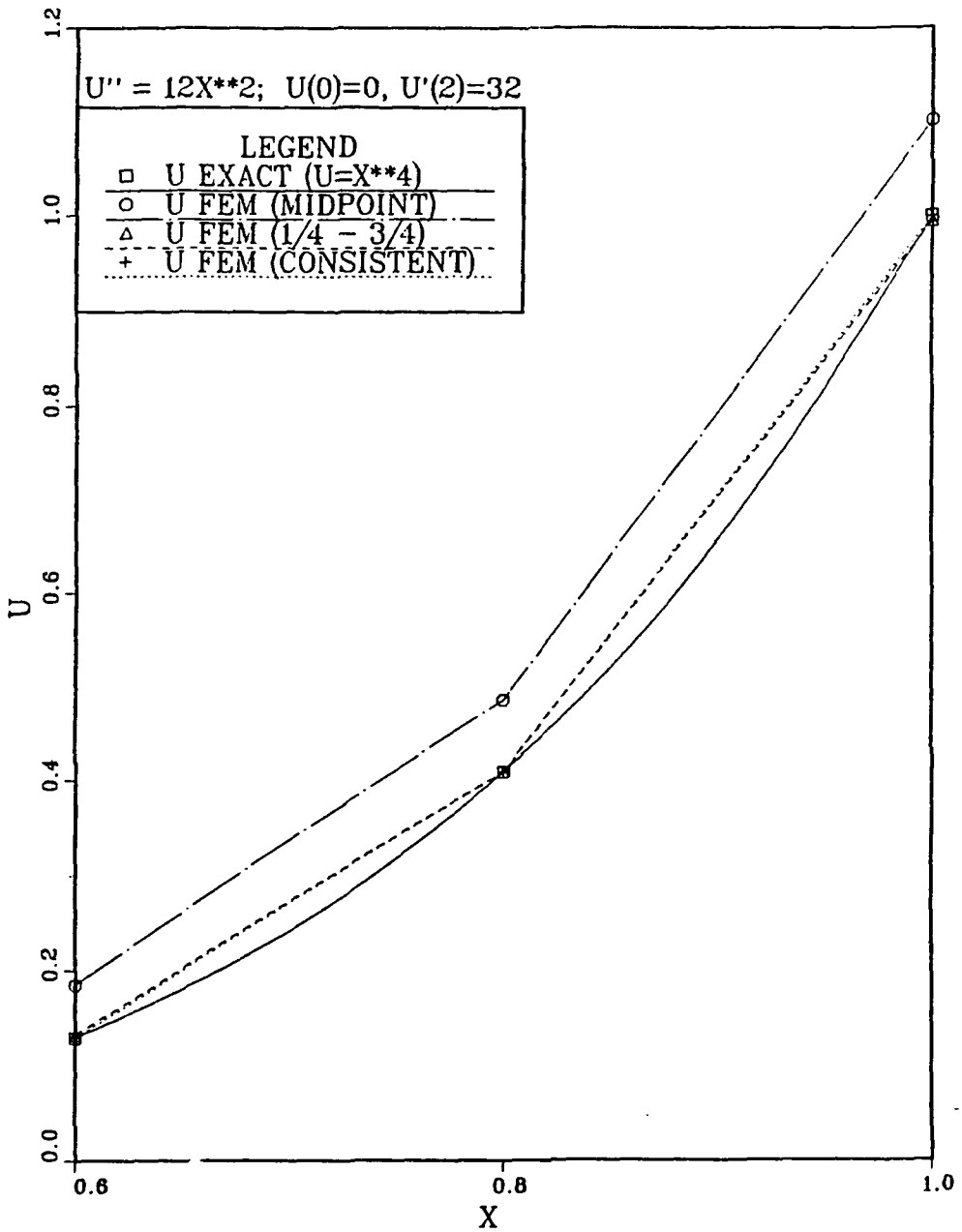


Figure 8. Comparison of Solutions for Equation 2.10 Using 10 Elements.

C. CONCLUSIONS

Linear shape functions provide an efficient and accurate interpolation for approximating second order, linear, one dimensional, differential equations using the Galerkin FEM, regardless of the magnitude of the solution. Therefore, they are utilized throughout the remainder of the research when approximating linear and nonlinear operators of second order or less. An open question which remains is whether higher order elements will work for nonlinear two point boundary value problems when linear elements do not.

Additionally, two important observations regarding the use of a consistent forcing function analysis are made.

- The use of this technique in evaluating the excitation integral provides for a very accurate solution process. Thus, this method is employed in the remainder of the research whenever possible. In those cases where the integration cannot be performed analytically, a Simpson's Rule approximation to the integral is used so that the resulting error is kept to a minimum.
- For linear problems, this method provides very accurate approximations over large domains using a minimum number of elements. To illustrate this point, equation (2.10) was solved over the domain $0 < x < 10$ using only two elements. The FEM approximation provided the "exact" solution at $x = 5$ and $x = 10$ as shown in Figure 9 on page 17.

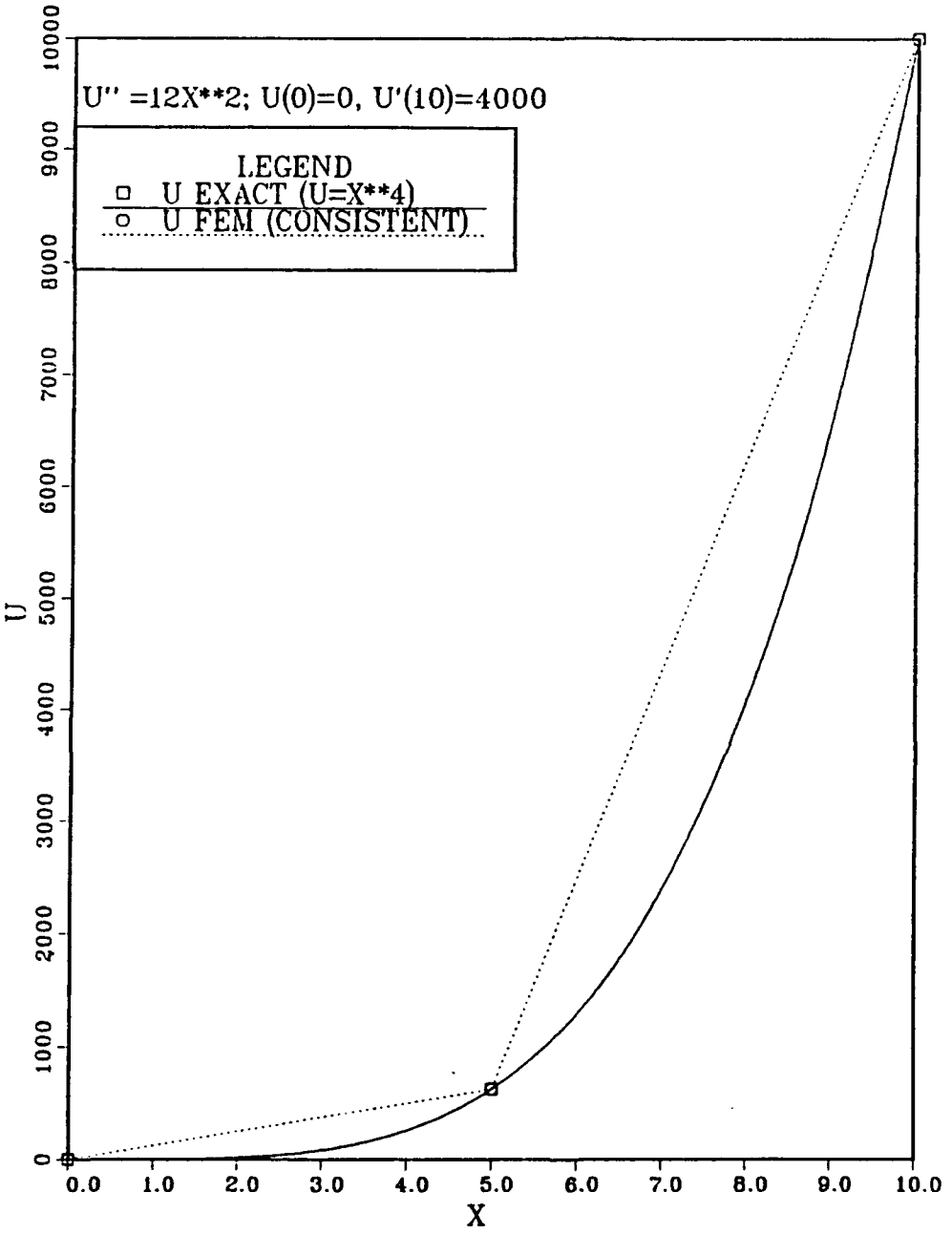


Figure 9. Solution of Equation 2.10 Over a Large Domain Using Two Elements

III. ONE DIMENSIONAL SECOND ORDER NONLINEAR SYSTEMS

This chapter outlines various solution strategies which are later analyzed on their ability to solve second order nonlinear differential equations of the form

$$\mathcal{L}u + \mathcal{L}(u) - f(x) = 0 \quad x \in D \tag{3.1.a}$$

where $\mathcal{L}$ denotes linear operators and $\mathcal{L}(\,)$ denotes nonlinear operators. Because equation (3.1.a) is nonlinear, a closed form analytical solution is, in general, not possible. Therefore numerical solutions are obtained by a variety of approximation techniques. This chapter sets c : a general procedure, consisting of three steps, for obtaining numerical solutions of equation (3.1.a). The three steps, when used with the Galerkin FEM, have the effect of transforming the original nonlinear differential equation into a system of linear algebraic equations.

The first step in the procedure consists of a 'linearization' of the nonlinear $\mathcal{L}(u)$ term(s). Thus, $\mathcal{L}(u)$ is transformed to $\mathcal{L}^{*}u$ where $\mathcal{L}^{*}$ can be obtained by a number of different strategies, three of which are described in section III.A.

Once the nonlinear differential equation, equation (3.1.a), has been transformed to a linear differential equation of the form

$$\mathcal{L}u + \mathcal{L}^{*}u = f(x) \tag{3.1.b}$$

the second step is associated with how the $\mathcal{L}^{*}u$ term in equation (3.1.b) is evaluated in the FEM Galerkin integral equations. A number of interpolation procedures are described for this step in Section III.B.

The third step in the solution procedure is defining the iterative process by which a solution of the linear algebraic equations developed by the Galerkin FEM is obtained. In particular, the efficacy of the iterative method involves two considerations.

- the selection of an initial estimate to begin the iteration
- the methodology for subsequent iterations

Section III.C describes a number of iteration strategies.

Thus sections III.A, III.B, and III.C cover the three basic steps in the solution procedure; that is, linearization, interpolation, and iteration. The selection of a particular strategy within each of these steps defines a specific solution procedure which can

be utilized in approximating the solution of equation (3.1.a). In addition to these sections, section III.D discusses several numerical aspects which either affect, or are used in evaluating, the efficacy of a solution procedure.

A. LINEARIZATION STRATEGIES

The first step in analyzing nonlinear equations using the Galerkin FEM is linearization of the nonlinear term(s). Three different linearization strategies are investigated in this research. There is no implication that these are the only strategies that exist.

1. Constant Linearization

The constant linearization method transforms the nonlinear term into a function of u^* where

$$\mathcal{L}(u) \approx \mathcal{L}^* u = v(u^*) \tag{3.2.a}$$

As discussed in Chapter 1, the solution process for nonlinear equations using the Galerkin FEM is iterative in nature. On the first iteration, u^* is set equal to an assumed value at each node and for subsequent iterations, each nodal value of u^* is based on the FEM approximation. Thus, $v(u^*)$ is a known function evaluated at each node and is taken to the right hand side, leaving a linear equation of the form

$$\mathcal{L}u = f(x) - v(u^*) \tag{3.2.b}$$

As an example, consider the nonlinear term $u'u$. The constant linearization technique linearizes this term as $v(u^*) = (u^*)'u^*$, where $(u^*)'$ is evaluated using finite difference techniques.

2. Classical Linearization

The classical linearization strategy linearizes the nonlinear term as a known function coefficient multiplying the dependent variable where

$$\mathcal{L}(u) \approx \mathcal{L}^* u = \left(m(u^*)\right) u \tag{3.3.a}$$

which in a sense maintains the functional nature of the dependent variable by allowing it to be kept on the left hand side of the differential equation in a linear fashion. As in the constant linearization strategy, $m(u^*)$ is a known function coefficient where the values of u^* are assumed for the first iteration and are based on the FEM approximation for

subsequent iterations. Substituting the results of equation (3.3.a) into equation (3.1.a), the linearized differential equation takes the form

$$\mathcal{L}u + \left(m(u^*)\right) u = f(x) \tag{3.3.b}$$

Using the nonlinear term given as an example in the previous section, that is $u'u$, the classical linearization technique provides two alternatives. The first is $(m(u^*))\, u = (u^*)'u$ which keeps the full effect of the dependent variable, u. The second is $(m(u^*))\, u = u^*u'$ which maintains the full effect of the derivative term, u'.

3. Quasilinearization

Quasilinearization is covered extensively by Bellman and Kalaba in [Ref. 1: p. 36] where the nonlinear term is set equal to $q(u)$ and linearized as

$$\begin{aligned} \mathcal{L}(u) \approx \mathcal{L}^* u &= q(u^*) + (u - u^*)\, q'(u^*) \\ &= \left(q'(u^*)\right) u + \left(q(u^*) - u^* q'(u^*)\right) \end{aligned} \tag{3.4.a}$$

Comparing equation (3.4.a) with equations (3.2.a) and (3.3.a), it can be seen that quasilinearization is a combination of constant and classical linearization with the coefficient functions determined in a different manner. Substituting the results of equation (3.4.a) into equation (3.1.a), the linearized differential equation becomes

$$\mathcal{L}u + \left(q'(u^*)\right) u = f(x) - \left(q(u^*) - u^* q'(u^*)\right) \tag{3.4.b}$$

As an example, consider the u^2 nonlinear term. The quasilinearization technique defines $q(u) = u^2$, from which $q(u^*) = (u^*)^2$ and $q'(u^*) = 2u^*$. Substitution of these functions into equation (3.4.a) yields a linearization of the form $\mathcal{L}^* u = 2u^* u - (u^*)^2$. As u^* and u begin to approach the same value at each node during the iteration process, $\mathcal{L}^* u$ begins to approach the original nonlinear term, namely u^2.

B. INTERPOLATION STRATEGIES

The second step of the solution process is the evaluation of the Galerkin integrals. Based on the type of linearization strategy utilized, one or both of the following integrals are obtained.

$$\text{Linearization Vector: } \int_D \mathbf{G}\, h(u^*) dx \tag{3.5.a}$$

$$\text{Linearization Matrix: } \int_D \mathbf{G}\mathbf{G}^{\mathrm{T}}\, h(u^*)dx\, \mathbf{u} \qquad (3.5.b)$$

The constant linearization strategy results in a linearization vector where $h(u^*)$ in equation (3.5.a) is replaced by $v(u^*)$ from equation (3.2.a). The classical linearization technique yields a linearization matrix where $h(u^*)$ in equation (3.5.b) is replaced by $m(u^*)$ from equation (3.3.a). The quasilinearization method utilizes both integrals where $h(u^*)$ becomes $(q(u^*) - u^* q'(u^*))$ in equation (3.5.a) and $q'(u^*)$ in equation (3.5.b). Since u^* is derived from an FEM approximation utilizing linear shape functions, it varies linearly between nodes over each element. Thus, when equations (3.5.a) and (3.5.b) are reduced to the element level for evaluation, there are numerous interpolation strategies available to approximate $h(u^*)$. Here, a few strategies are discussed for each of the above integrals.

1. Linearization Vector

Reducing the integral in equation (3.5.a) to the element level yields

$$\int_0^{l_e} \mathbf{g}\, h(u^*)d\xi \qquad (3.6)$$

which is quite similar to the forcing function integral discussed in Chapter II. Three interpolation techniques similar to those used for the forcing function are examined in this research. The integral in equation (3.6) yields a 2x1 element vector denoted as $\mathbf{f}^*$; the '*' superscript meaning that the vector changes with each iteration as the values of u^* are updated. The $\mathbf{f}^*$ are then distributed into a system linearization vector denoted as $\mathbf{F}^*$ in accordance with the local to global nodal point correspondence.

a. Midpoint Lumped Approximation

This method evaluates $h(u^*)$ at the midpoint of the element and brings this value outside the integral as a constant. Since u^* varies linearly over the element, its value at the midpoint of the element is simply the average of the values of u^* at the two nodes of the element, $(u_i^*)_i$ and $(u_{i+1}^*)_i$. Substituting this into equation (3.6) leaves

$$\mathbf{f}^* = h\left(\frac{(u_j^*)_i + (u_{j+1}^*)_i}{2}\right)\int_0^{l_e}\begin{bmatrix} 1-\frac{\xi}{l_e} \\ \frac{\xi}{l_e}\end{bmatrix} d\xi$$
$$= h\left(\frac{(u_j^*)_i + (u_{j+1}^*)_i}{2}\right)\begin{bmatrix}\frac{l_e}{2} \\ \frac{l_e}{2}\end{bmatrix} \tag{3.7}$$

where $h(\,)$ denotes that the function h is evaluated at the argument of ( ).

b. 1/4 - 3/4 Lumped Approximation

This method takes the $h(u^*)$ term inside the shape function vector yielding

$$\mathbf{f}^* = \int_0^{l_e}\begin{bmatrix} h(u^*)\left(1-\frac{\xi}{l_e}\right) \\ h(u^*)\frac{\xi}{l_e}\end{bmatrix} d\xi \tag{3.8.a}$$

In the first term, u^* is evaluated at $l_e/4$ while in the second term it is evaluated at $3l_e/4$. These values are again easily determined due to the linear variation of u^* over the element and are given by equations (3.8.b) and (3.8.c).

$$u^*\left(\frac{l_e}{4}\right) = (u_j^*)_i + \frac{1}{4}\left((u_{j+1}^*)_i - (u_j^*)_i\right)$$
$$= \frac{3}{4}(u_j^*)_i + \frac{1}{4}(u_{j+1}^*)_i \tag{3.8.b}$$

$$u^*\left(\frac{3l_e}{4}\right) = (u_j^*)_i + \frac{3}{4}\left((u_{j+1}^*)_i - (u_j^*)_i\right)$$
$$= \frac{1}{4}(u_j^*)_i + \frac{3}{4}(u_{j+1}^*)_i \tag{3.8.c}$$

Substitution of these expressions for u^* into equation (3.8.a) gives

$$\mathbf{f}^* = \int_0^{l_e} \begin{bmatrix} h\left(\frac{3}{4}(u_j^*)_i + \frac{1}{4}(u_{j+1}^*)_i\right)\left(1 - \frac{\xi}{l_e}\right) \\ h\left(\frac{1}{4}(u_j^*)_i + \frac{3}{4}(u_{j+1}^*)_i\right)\left(\frac{\xi}{l_e}\right) \end{bmatrix} d\xi$$
$$= \frac{l_e}{2} \begin{bmatrix} h\left(\frac{3}{4}(u_j^*)_i + \frac{1}{4}(u_{j+1}^*)_i\right) \\ h\left(\frac{1}{4}(u_j^*)_i + \frac{3}{4}(u_{j+1}^*)_i\right) \end{bmatrix} \qquad (3.8.d)$$

where again, $h(\,)$ denotes that the function h is to be evaluated at ( ).

c. Linear Approximation

This method evaluates u^* in a linear manner over the entire element as a function of the element coordinate ξ where

$$u^* = (u_j^*)_i\left(1 - \frac{\xi}{l_e}\right) + (u_{j+1}^*)_i\left(\frac{\xi}{l_e}\right) \qquad (3.9.a)$$

Substitution of this expression into equation (3.6) yields

$$\mathbf{f}^* = \int_0^{l_e} \begin{bmatrix} 1 - \frac{\xi}{l_e} \\ \frac{\xi}{l_e} \end{bmatrix} h\left((u_j^*)_i\left(1 - \frac{\xi}{l_e}\right) + (u_{j+1}^*)_i\left(\frac{\xi}{l_e}\right)\right) d\xi \qquad (3.9.b)$$

Since $h(u^*)$ is no longer a constant but a function of ξ, this integral must be evaluated for each specific $h(u^*)$. Examples of each of these approximations is provided in Appendix C.

2. Linearization Matrix

At the element level, the integral in equation (3.5.b) becomes

$$\int_0^{l_e} \mathbf{g}\mathbf{g}^T h(u^*) d\xi \qquad (3.10)$$

which is similar to the Galerkin FEM differential operator integral discussed in Chapter II, with the $\mathbf{g}\mathbf{g}^T$ term producing a 2x2 element matrix. In order to preserve the 2x2 nature of this matrix, $h(u^*)$ must remain as a single term multiplying each term in the matrix. Two techniques for evaluating this integral are examined in this research. The

resulting 2x2 element matrix is denoted as $\mathbf{l}^*$ where the '*' superscript again indicates that the matrix is changing with each iteration. These are then distributed into the system linearization matrix $\mathbf{L}^*$ in accordance with the local to global nodal point correspondence.

a. Midpoint Lumped Approximation

This method evaluates u^* at the midpoint of the element using the same process as in the linearization vector analysis and brings $h(u^*)$ outside the integral as a constant. Substitution of this expression into equation (3.10) gives

$$\begin{aligned} \mathbf{l}^* &= h\left(\frac{(u_j^*)_i + (u_{j+1}^*)_i}{2}\right)\int_0^{l_e}\begin{bmatrix} 1-\frac{\xi}{l_e} \\ \frac{\xi}{l_e}\end{bmatrix}\begin{bmatrix} 1-\frac{\xi}{l_e} & \frac{\xi}{l_e}\end{bmatrix}d\xi \\ &= h\left(\frac{(u_j^*)_i + (u_{j+1}^*)_i}{2}\right)\begin{bmatrix} \frac{l_e}{3} & \frac{l_e}{6} \\ \frac{l_e}{6} & \frac{l_e}{3}\end{bmatrix} \end{aligned} \tag{3.11}$$

where $h(\ )$ denotes that the function h is to be evaluated at ( ).

b. Linear Approximation

This method transforms u^* into a linear function of the element coordinate ξ as given by equation (3.9.a). This expression is substituted into equation (3.10) giving

$$\mathbf{l}^* = \int_0^{l_e}\begin{bmatrix} 1-\frac{\xi}{l_e} \\ \frac{\xi}{l_e}\end{bmatrix}\begin{bmatrix} 1-\frac{\xi}{l_e} & \frac{\xi}{l_e}\end{bmatrix} h\left((u_j^*)_i\left(1-\frac{\xi}{l_e}\right) + (u_{j+1}^*)_i\left(\frac{\xi}{l_e}\right)\right)d\xi \tag{3.12}$$

As for linear evaluation of the linearization vector, equation (3.12) must be evaluated for each $h(u^*)$. An example of each of these approximations is provided in Appendix D.

C. ITERATION STRATEGIES

Having evaluated the Galerkin integrals and developed a set of linear algebraic equations, the last step in the solution process is the determination of an iteration strategy. The main goal of an iteration strategy is to obtain a convergent approximation in a minimum number of steps while at the same time, keeping the computational effort of the iteration process to a minimum. Two of the factors which control the rate of convergence within a specific linearization strategy are

- how close the initial assumed values of u^* are to the actual solution
- the nature in which the FEM solution , $\tilde{u}$, obtained at the end of each iteration, is utilized to obtain a value of u^* to begin each of the subsequent iterations

1. Initial Iteration

In order to begin any iteration process, an initial estimate, or guess, must be made as to the value of the variable which is to be determined. The relative accuracy of this guess with respect to the true solution of the differential equation greatly affects the ability of the solution process to converge as well as its rate of convergence. If the starting point for the iteration process is too far away from the actual solution, the likelihood for divergence or convergence to a nonsolution of the differential equation is very high.

The first step in formulating an initial iteration strategy is developing an idea as to what the activity range of the solution of the differential equation might be, i.e., how much and how quickly is the dependent variable changing over the prescribed domain. Since the solution is not known, this information must be obtained from the boundary conditions, the domain length, and insight as to the physics of the system.

Two possible combinations of boundary conditions exist for two point boundary value problems.

- Essential-Essential (E-E), where the magnitude of the dependent variable is specified at each end of the domain
- Essential-Natural (E-N), where the magnitude of the dependent variable is specified at one end of the domain, and the slope or rate of change of the dependent variable is specified at the other end.

The essential-essential combination provides information as to the probable activity range of the solution over the system domain, which is referred to as function order in this research and is defined below.

Function Order provides a relative magnitude of the range of values in the solution function as indicated by two essential boundary conditions. This relative magnitude is determined by writing the values of the essential boundary conditions in power ten exponent form and then taking the quotient of the maximum value over the minimum value. When the minimum valued boundary condition is 0.0, it should be written as 1×10^0 in order that the quotient does not become undefined. The magnitude of the exponent in this quotient defines the function order of the solution while the decimal value provides a ranking of how different function orders of the same magnitude compare. For example, take a differential equation which has boundary conditions of $u(0) = 0$ and $u(2) = 25$. These are written in power ten exponent form as $u(0) = 1.00 \times 10^0$ and $u(2) = 0.25 \times 10^2$. The quotient of these

two values is 0.25×10^2. Thus, the solution function is said to have a function order of two.

When an E-N boundary condition combination is specified, determination of the dependent variable activity range is not as straight forward as in the E-E situation. Instead, the activity range must be determined from a knowledge of the physics of the system as well as the actual values of the specified boundary conditions. The importance of properly estimating the activity range of the dependent variable cannot be overemphasized as this estimation is utilized in determining an initial iteration strategy, which is the most critical step in the solution procedure.

When the function order is zero or one, or the activity range is determined to be small based on the physics of the system, the magnitude of the dependent variable does not change appreciably over the prescribed domain. Thus, a reasonable estimate of the dependent variable for the first iteration would involve utilizing the essential boundary condition value(s). To examine the validity of this line of reasoning, the following initial iteration strategies are utilized in the present research when the solution has a function order of one or less.

- u^* set equal to the value of the left essential boundary condition.
- u^* set equal to the value of the right essential boundary condition
- u^* set equal to the average value of the two essential boundary conditions

When the function order is greater than one, there is a chance that utilization of any of the above criteria for the initial iteration values could result in divergence or convergence to a nonsolution of the differential equation. Any number of guesses could be made for the initial iteration values, but the chances of a random guess providing a convergent solution of the differential equation are quite low. Instead, a systematic approach given by the following four steps is suggested.

- Examine the physical system to which the differential equation applies and determine which terms on the left hand side of the equation tend to dominate.
- Neglect the nondominant term(s) and solve the equation for the dependent variable by any means possible, i.e., separation of variables, undetermined coefficients, successive integration, etc.
- Determine the value of the dependent variable at each node and let these be the values used in the initial iteration.
- If divergence results, reexamine the physical system and reevaluate the dominance of each of the terms. Then repeat steps two and three.

The process of determining initial iteration values that lead to a convergent solution can be very time consuming and frustrating. But, in most instances when the function order is greater than one, the selection of initial iteration values that are 'reasonably close' to the actual solution is crucial if a convergent solution is to be obtained.

To amplify the above guidelines, consider a one dimensional constant cross-sectional area heat fin with nonlinear conduction given by

$$k(T)T'' - c_1(T - T_\infty) = 0 \tag{3.13}$$

where c_1 is a constant based on the fin geometry and the convection coefficient, and T_∞ is the temperature of the convective fluid. When the fin is short or the value of $k(T)$ for the given range of operating temperatures is much greater than c_1, conduction tends to dominate and the convection term, $c_1(T - T_\infty)$, is negligible. A value can then be assumed for $k(T)$ based on the the physical system and boundary condition temperature(s). The equation thus becomes linear and can be integrated twice to yield $T^*(x)$. T^* is then determined at each node which provides the values used for the initial iteration.

When the constant c_1 is much larger than $k(T)$, the convection term tends to dominate and conduction can be neglected. Thus, T^* is set equal to T_∞ at each node for the initial iteration. In those cases where conduction dominates over one half of the domain and convection over the other, T^* can be determined by using a combination of both these initial iteration strategies. The process of determining an initial iteration strategy is further developed in IV.A.5.

2. Subsequent Iterations

After the initial values of u^* are determined, the next step is to develop an iteration procedure for subsequent u^* values which will result in convergence with the least amount of computational effort. To aid in this development and subsequent discussions, the following notation is utilized.

- $(u_j^*)_i$ is the value of u used in the iteration process where j denotes the node number and i is the iteration number.
- $(\tilde{u}_j)_i$ is the value of u returned by the FEM approximation where j and i are defined as above.

Two different strategies for determining u^* are investigated in the present research.

a. Previous Value Strategy

This method takes the value of $\tilde{u}$ from the previous iteration and sets it equal to u^* for the next iteration process where

$$(u_j^*)_i = (\tilde{u}_j)_{i-1} \tag{3.14.a}$$

This is the simplest iteration scheme, but does not take into account how $\tilde{u}$ is changing during the iteration process.

b. Average Value Strategy

This method uses the average value of $\tilde{u}$ from the last two iterations yielding

$$(u_j^*)_i = \frac{(\tilde{u}_j)_{i-1} + (\tilde{u}_j)_{i-2}}{2} \tag{3.14.b}$$

This method should enhance convergence when $\tilde{u}$ is oscillating about the final convergent solution while at the same time not adversely affect those situations where $\tilde{u}$ is converging monotonically.

c. Additional Strategies

Numerous other strategies can be utilized in an attempt to increase the rate of convergence. The following are not investigated in this research but are presented as topics requiring further research.

- K-step Strategy - This method takes into account k previous iterations where

$$(u_j^*)_i = \frac{(\tilde{u}_j)_{i-1} + (\tilde{u}_j)_{i-2} + \cdots + (\tilde{u}_j)_{i-k}}{k} \tag{3.14.c}$$

- Weighted Average Strategy - A method which assigns a weighting factor to each iterative value of $\tilde{u}$ where

$$(u_j^*)_i = \frac{w_1(\tilde{u}_j)_{i-1} + w_2(\tilde{u}_j)_{i-2}}{w_1 + w_2} \tag{3.14.d}$$

  The objective is to find the optimum combination of weighting factors, w_1 and w_2, that yields the minimum number of iterations.

- Weighted K-step Strategy - A combination of the previous two strategies where

$$(u_j^*)_i = \frac{w_1(\tilde{u}_j)_{i-1} + w_2(\tilde{u}_j)_{i-2} + \cdots + w_k(\tilde{u}_j)_{i-k}}{\sum_{n=1}^{k} w_n} \tag{3.14.e}$$

- Rate of Change Strategy - This method would require using some from of a Taylor series expansion to model how $\tilde{u}$ changes from iteration to iteration.

D. NUMERICAL CONSIDERATIONS

The following numerical aspects are considered in evaluating the ability of each solution procedure to provide a convergent solution of the nonlinear differential

equations that are investigated. Again, a solution procedure is a specific combination of linearization, interpolation, and iteration strategies.

1. Convergence/Divergence

For a convergent solution, the absolute value of the difference between $(\tilde{u}_j)_i$ and $(\tilde{u}_j)_{i-1}$ at each node decreases as the number of iterations, i, increases. In order to achieve the best approximation for a given number of elements, it is desirable to let the maximum value of $|(\tilde{u}_j)_i - (\tilde{u}_j)_{i-1}|$ at all nodes reach some minimum value before exiting the iteration process. Thus, the following percent difference convergence criterion is utilized: convergence is reached when the absolute value of the maximum difference between $(\tilde{u}_j)_i$ and $(\tilde{u}_j)_{i-1}$ at any node divided by $(\tilde{u}_j)_i$ at the same node it less than .0001 (.01%). This is shown mathematically in equation (3.15) and is but one of many convergence criteria that could be utilized.

$$\text{Convergence Criterion: } \left| \frac{\text{MAX } |(\tilde{u}_j)_i - (\tilde{u}_j)_{i-1}|}{(u_j)_i} \right| < .0001 \qquad (3.15)$$

The solution procedure is considered a failure if any of the following situations occur.

- convergence does not occur within 200 iterations
- divergence occurs, i.e., $\tilde{u}$ increases without bound as the number of iterations increases
- convergence to a nonsolution of the differential equation

2. Critical Number of Degrees of Freedom (DOF)

In some instances, especially when the function order is greater than two or the activity range is large, there may be a specific or critical number of DOF below which the solution procedure will not converge. To examine this phenomenon, each solution strategy is initially evaluated using three DOF i.e., two elements. The number of DOF is then increased until either a grid independent solution is obtained or the number of DOF exceeds 100; and a critical number of DOF, if it exists, is determined.

3. Stability

Stability in numerical analysis applications is based on the approximation method's ability to converge as it relates to the time and displacement discretization that are utilized. For example, the finite difference explicit method is stable for $r < \frac{1}{2}$ where $r = \frac{\Delta t}{(\Delta x)^2}$, while the Crank-Nicholsen implicit method is stable for any value of r, i.e., unconditionally stable. For this research, three types of stability are defined.

- Unconditional Stability - There is no critical number of DOF required to guarantee a convergent solution.
- Conditional Stability - There is a critical number of DOF below which a convergent solution cannot be obtained.
- Unstable - The solution process diverges regardless of the number of DOF utilized.

4. Multiplicity of Solutions

Nonlinear differential equations do not necessarily have a unique solution. The equations utilized in this research were developed based on known solutions. When a solution different from them is obtained, that solution is checked at two different points in the domain by passing a parabola through the point in question and two adjacent points. If the equations developed from both of these parabolas satisfy the differential equation, the solution procedure is considered to have provided a valid approximation.

5. Boundary Condition Effects

Two point boundary value problems require a boundary condition at each end. Two combinations are valid; either an essential (Dirichlet) boundary condition at each end or an essential at one end and a natural (Cauchy) at the other. The effect that each of these combinations has on the performance of each solution procedure is investigated.

6. Computational Efficiency

This research defines the most computationally efficient solution procedure as the one which provides the most accurate results for the least amount of computer time. CPU time by itself though is not an effective measure of efficiency. For example, a solution process that uses two elements will take much less time to run than one with 40, but the 40 element solution is likely to be much more accurate. There should be some critical number of elements for a specific solution procedure beyond which the increase in CPU time for the additional calculations is not matched by a proportional increase in solution accuracy.

The different solution strategies in this research are compared using a factor defined as

$$CPU^{*} = CPU \times \text{average \% error} \qquad (3.16.a)$$

CPU is the amount of computer time (in seconds) required to complete the iteration process. A timing subroutine installed in the FORTRAN solution program starts when

the iteration process is entered and stops when it is exited. Average % error is the average percent difference between the FEM approximation and the exact solution given by

$$\text{avg \% error} = \frac{\sum_{i=1}^{N} |\,\%\ \text{error}\,|_i}{N} \qquad N = \begin{matrix}\text{number of system} \\ \text{degrees of freedom}\end{matrix} \tag{3.16.b}$$

As the CPU time increases due to an increase in the number of elements, the percent error should decrease and approach zero. Therefore, the solution strategy yielding the minimum CPU* is defined as the most computationally efficient for a given equation and nonlinear operator.

IV. APPLICATIONS

A. PRELIMINARIES

1. Equations, Domains, and Boundary Conditions

This chapter evaluates the performance of various combinations of the linearization, interpolation, and iteration strategies previously developed in solving second order, nonlinear, one dimensional, differential equations containing the u^2 nonlinear term. In order to investigate the effect of the many numerical considerations described in the preceding chapter, two nonlinear differential equations over three domains with appropriate essential and natural boundary conditions are solved. The differential equations considered were

$$u'' - u^2 = 6 - 9x^4 \qquad u_{exact} = 3x^2 \tag{4.1}$$

$$u'' + u^2 = 60x + 100x^6 \qquad u_{exact} = 10x^3 \tag{4.2}$$

Domains and Boundary Conditions

- Domain 1: $0 < x < 1$
  Eqn. (4.1) $u(0) = 0,\ u(1) = 3$ or $u'(1) = 6$
  Eqn. (4.2) $u(0) = 0,\ u(1) = 10$ or $u'(1) = 30$
- Domain 2: $0 < x < 2$
  Eqn. (4.1) $u(0) = 0,\ u(2) = 12$ or $u'(2) = 12$
  Eqn. (4.2) $u(0) = 0,\ u(2) = 80$ or $u'(2) = 120$
- Domain 3: $0 < x < 5$
  Eqn. (4.2) $u(0) = 0,\ u(5) = 75$ or $u'(5) = 30$
  Eqn. (4.2) $u(0) = 0,\ u(5) = 1250$ or $u'(5) = 750$

In all cases the left end point of the domain has an essential boundary condition. Either an essential or natural boundary condition is provided at the right end of the domain in order to investigate the effect of the two different boundary condition combinations, namely essential-essential (E-E) and essential-natural (E-N) as discussed in III.D.5.

Equations (4.1) and (4.2) were developed by starting with a known solution, u_{exact}, and working backwards to form a second order nonlinear differential equation. The equations were kept simple, i.e., only one linear and one nonlinear term, due to the number of solution procedures that required evaluation, as well as the many numerical considerations involved. Still, equations (4.1) and (4.2) are viable representations of two

engineering phenomena that are described by second order differential equations, such as

- An axially loaded bar embedded in a nonlinear elastic medium as shown in Figure 10 on page 34.
- A constant cross-sectional area heat fin with internal heat generation and nonlinear convection as shown in Figure 11 on page 35.

2. Related Engineering Phenomena

a. Bar Problem

Consider the bar problem of Figure 10 subjected to distributed force $p(x)$, embedded in a nonlinearly elastic media which exerts an opposing distributed force proportional to the square of the displacement, u. A free body analysis of a differential element yields

$$F(x) + dF(x) + p(x)dx - F(x) - u^2(x)dx = 0 \qquad (4.3.a)$$

Cancelling the $F(x)$ terms, taking the applied excitation $p(x)$ to the right hand side of the equation and dividing by dx gives

$$\frac{dF(x)}{dx} - u^2(x) = -p(x) \qquad (4.3.b)$$

From solid mechanics, the following relations are known,

$$F = \sigma A \qquad (4.4.a)$$

$$\sigma = \varepsilon E \qquad (4.4.b)$$

$$\varepsilon = \frac{du}{dx} \qquad (4.4.c)$$

where F is the axial force, σ is the axial stress, A is the cross-sectional area, ε is the strain, and E is Young's Modulus. Substitution of equations (4.4.c) and (4.4.b) into equation (4.4.a) provides a relation for the force as

$$F = EAu' \qquad (4.4.d)$$

Differentiating equation (4.4.d) with respect to x yields

$$\frac{dF}{dx} = EAu'' \qquad (4.4.e)$$

Substitution of equation (4.4.c) into equation (4.3.b) leaves

$$EAu'' - u^2 = -p(x) \tag{4.5}$$

which is similar in form to equations (4.1) and (4.2), if the stiffness, EA, is set equal to unity.

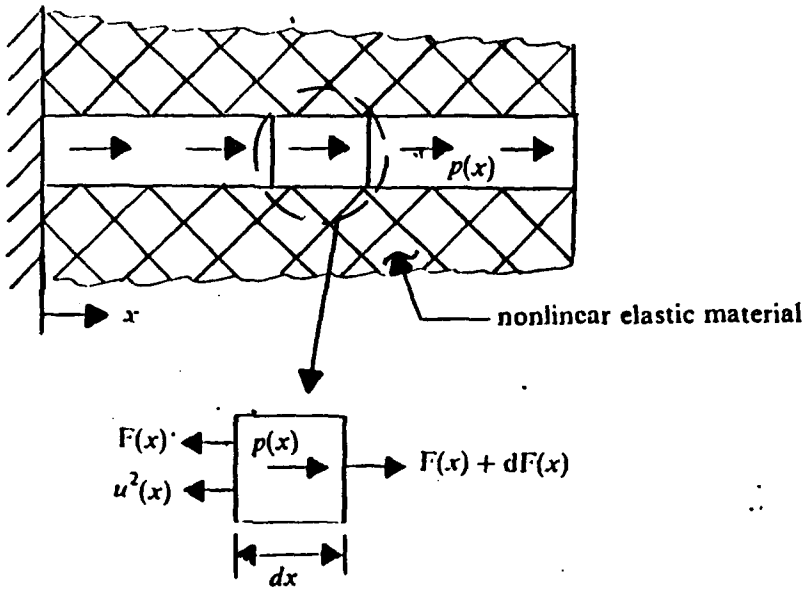


Figure 10. Axially Loaded Bar Embedded in a Nonlinearly Elastic Material

b. Heat Fin Problem

Now consider the heat fin problem of Figure 11 on page 35 where $\dot{q}(x)$ is the volumetric heat generation/unit length and the heat transfer coefficient, h, is a linear function of the temperature, T, that is $h = aT$. An energy balance on the fin differential element yields

$$q_{cond} - \dot{q}(x)dx - (q_{cond} + dq_{cond}) - dq_{conv} = 0 \tag{4.6.a}$$

Cancelling the q_{cond} terms, dividing by dx, and rearranging terms yields

$$-\frac{dq_{cond}}{dx} - \frac{dq_{conv}}{dx} = \dot{q}(x) \tag{4.6.b}$$

The following relations are known from heat transfer principles,

$$q_{cond} = -k\Lambda_c \frac{dT}{dx} \tag{4.7.a}$$

$$q_{conv} = h\Lambda_s(T - T_\infty) \tag{4.7.b}$$

where Λ_c is the cross-sectional area of the fin and is constant, Λ_s is the surface area of the fin and is written as Px where P is the perimeter of the heat fin. T_∞ is the ambient temperature of the convective media, and k is the thermal conductivity coefficient. Substitution of equations (4.7.a) and (4.7.b) into equation (4.6.b) and performing the appropriate differentiation yields

$$kA_cT'' - hP(T - T_\infty) = \dot{q}(x)$$

Letting $h = aT$, $T_\infty = 0$, and dividing through by $k\Lambda_c$ yields

$$T'' - \frac{aP}{k\Lambda_c} T^2 = \frac{1}{k\Lambda_c} \dot{q}(x) \tag{4.8.b}$$

which is similar to equations (4.1) and (4.2) if coefficient $(aP/k\Lambda_c)$ is set equal to unity.

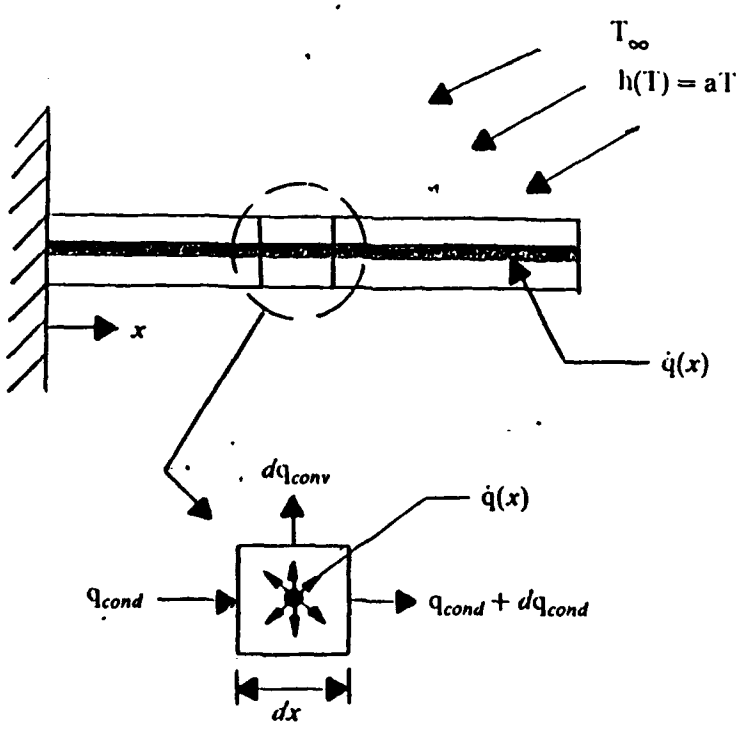


Figure 11. Heat Fin With Nonlinear Convection

3. Function Order and Domain Nondimensionalization

The exact solutions on which equations (4.1) and (4.2) are based, and the three domains of these equations previously described, were judiciously chosen to provide information as to the effect of function order on the performance of the solution procedures investigated in this analysis. The function order of each equation was determined over each domain based on the specified E-E boundary conditions and is provided in Table 2.

In order to determine whether an advantage is gained by the nondimensionalization of a domain, the following investigation was undertaken for both equations over domains two and three with equation (4.1) over domain three provided as an example. In this case, the dimensional variable x was replaced by the nondimensional independent variable $\eta = x/5$. Since $d\eta = dx/5$ and $x = 5\eta$, equation (4.1) was transformed to

$$u'' - 25u^2 = 150 - 140625u^4 \qquad 0 < \eta < 1$$

where differentiation now is with respect to η. Analysis of each nondimensionalized equation established that the transfer of 'domain activity' to 'differential equation activity' did not result in any computational gain.

Table 2. FUNCTION ORDER OF EQUATIONS (4.1) AND (4.2)

Equation	Domain	Function Order
(4.1)	One	1
	Two	2
	Three	2
(4.2)	One	2
	Two	2
	Three	4

4. General Solution Procedure

Subsequent analyses using the solution procedures discussed next, show that the efficacy of any particular solution procedure depends strongly on the function order of the problem being solved. The general solution procedure consists of the three steps;

linearization, interpolation, and iteration; shown in Figure 12 on page 38. First, equations (4.1) and (4.2) are linearized using the linearization strategies developed in III.A. Secondly, an interpolation strategy developed in III.B is used with the Galerkin FEM and the linearized differential equation is transformed into a set of linear algebraic equations. Finally, these algebraic equations are solved iteratively using various combinations of iteration strategies developed in III.C. The results of each solution procedure, where a solution procedure is a particular set of selected strategies, are then compared and evaluated based on their performance in approximating the solutions of equations (4.1) and (4.2).

5. Initial Iteration Strategy

The initial iteration strategy is the weak link in the overall solution procedure. For function order one problems, the initial iteration strategy is simply to utilize the essential boundary conditions as described in III.C.1. This is the strategy adopted for equation (4.1) over domain one, as well as equation (4.2) over the same domain because the latter problem becomes function order two only at the right boundary point of the domain.

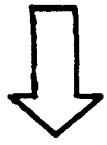
Another initial iteration strategy is utilized for equations (4.1) and (4.2) over domains two and three because their function order is greater than one. The physical systems to which equations (4.1) and (4.2) apply are generic in nature and do not provide the necessary information for determining which of the terms on the left hand side of the equation dominate the behavior of the system. However, since each of these equations is based on a known solution, the terms on the left side can be written as specific functions of x. This information is utilized to devise an initial iteration strategy for each equation over domains two and three based on the indicated dominance of the linear and nonlinear terms. Normally, the solution of the nonlinear differential equation is unknown. In these cases, the physical system must be scrutinized and a determination made as to which processes and their corresponding operators dominate over each part of the domain. Having once assessed the dominance of each operator over various parts of the domain, the following strategies can be utilized to generate the initial iteration values.

- Where the u'' dominates, the initial iteration vector is obtained by solving $u'' = f(x)$.
- Where the u^2 dominates, the initial iteration vector is obtained by solving $u^2 = f(x)$

SOLUTION PROCEDURE

NONLINEAR DIFFERENTIAL EQUATION

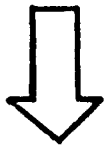
$$\mathcal{L}u + \mathcal{L}(u) - f = 0$$



LINEARIZATION STRATEGY

LINEAR DIFFERENTIAL EQUATION

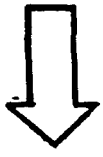
$$\mathcal{L}^*u - f = 0$$



FEM WITH INTERPOLATION STRATEGY

RECURSIVE LINEAR ALGEBRAIC EQUATIONS

$$(\mathbf{A} + \mathbf{A}^*_{i-1})u_i - \mathbf{F}^*_{i-1} - \mathbf{F} = 0$$



INIT. ITER. STRAT.
SUBS. ITER. STRAT.
(i = ITER. C'NTER)

$\mathbf{u}_n$

CONVERGED SOLN AFTER n ITER'S

Figure 12. Solution Procedure for Nonlinear Differential Equations

In equation (4.1), $u_{exact} = 3x^2$ which yields $u'' = 6$ and $u^2 = 9x^4$. These functions are plotted in Figure 13 on page 40 for the domain $0 < x < 2$ and in Figure 14 on page 41 for the domain $0 < x < 5$. From Figure 13, the u'' term is slightly more dominant than the u^2 term over the first half of the domain. Over the second half of the domain, the u^2 term is clearly dominant. Therefore, an appropriate initial iteration strategy is to utilize a procedure which neglects the u^2 term in determining the initial values of $(u_j^*)_0$ over $0 < x < 1$ and neglects the u'' term over $1 \le x < 2$. The first part of the initial value procedure involves integrating equation (4.1) twice without the imposition of any boundary conditions. The second part requires taking the square root of the right hand side of equation (4.1). Thus, the initial iteration values used in solving equation (4.1) over domain two are determined using equation (4.9).

$$(u_j^*)_0 = \begin{cases} 3x^2 - \frac{3}{10}x^6 & 0 < x < 1 \\ \sqrt{9x^4 - 6} & 1 \le x < 2 \end{cases} \tag{4.9}$$

From Figure 14 it is apparent that the u^2 term dominates over a majority of the domain. Thus, the u'' term is completely neglected and the initial iteration values are determined using equation (4.10).

$$(u_j^*)_0 = \sqrt{|9x^4 - 6|} \tag{4.10}$$

Nonlinear differential equation (4.2) was developed using $u_{exact} = 10x^3$ which yields $u'' = 60x$ and $u^2 = 100x^6$. These functions are plotted in Figure 15 on page 42 for a domain of $0 < x < 2$ and in Figure 16 on page 43 over domain $0 < x < 5$. Examination of Figure 15 shows the u'' term slightly dominating the u^2 term over the first half of the domain and the u^2 term clearly dominating over the second half of the domain. Thus, the same procedure for determining initial iteration values as was utilized for equation (4.1) over domain two is employed and the results are given by equation (4.11).

$$(u_j^*)_0 = \begin{cases} 10x^3 + \frac{100}{56}x^8 & 0 < x < 1 \\ \sqrt{60x + 100x^6} & 1 \le x < 2 \end{cases} \tag{4.11}$$

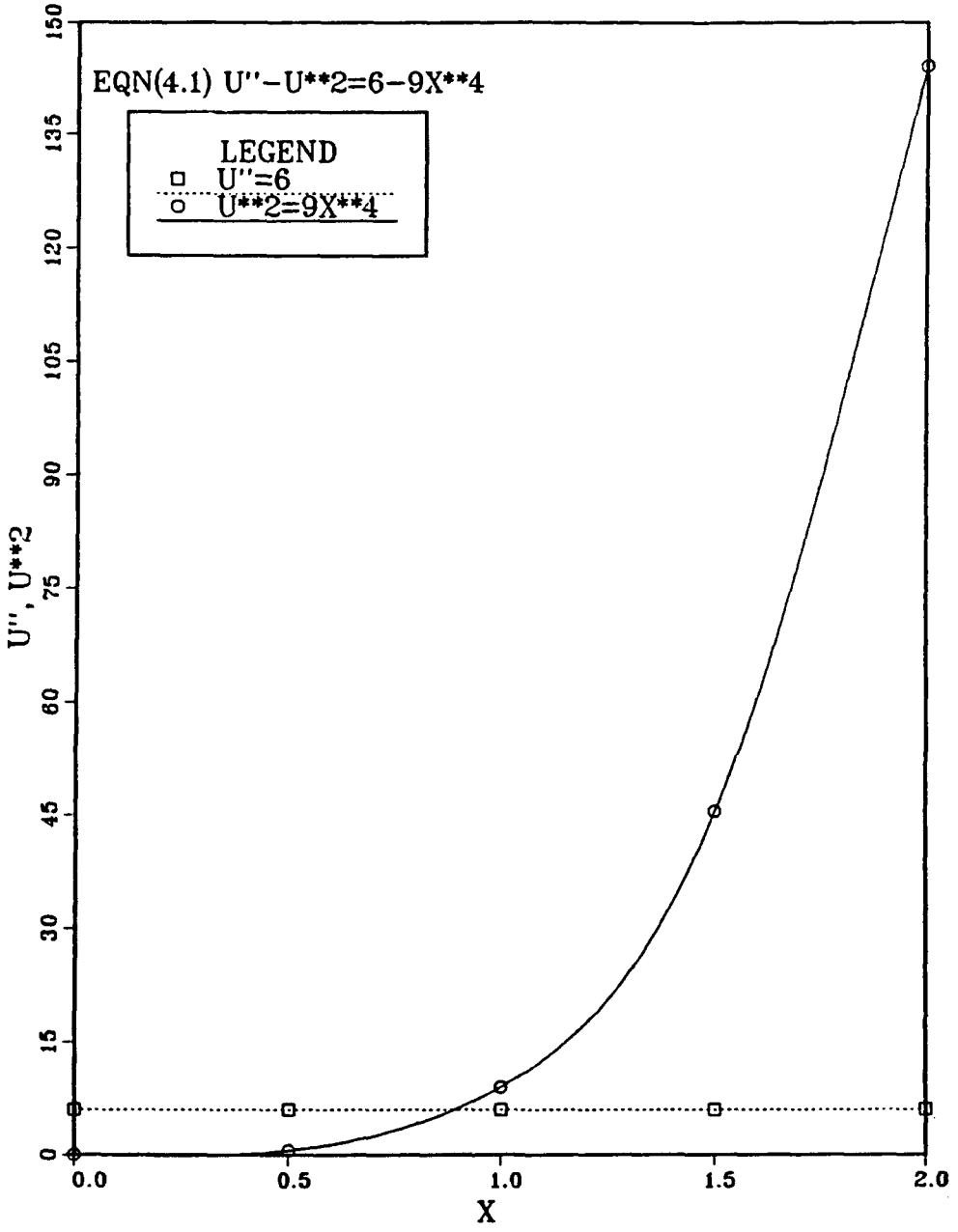


Figure 13. Dominance of Terms in Equation (4.1) over Domain Two

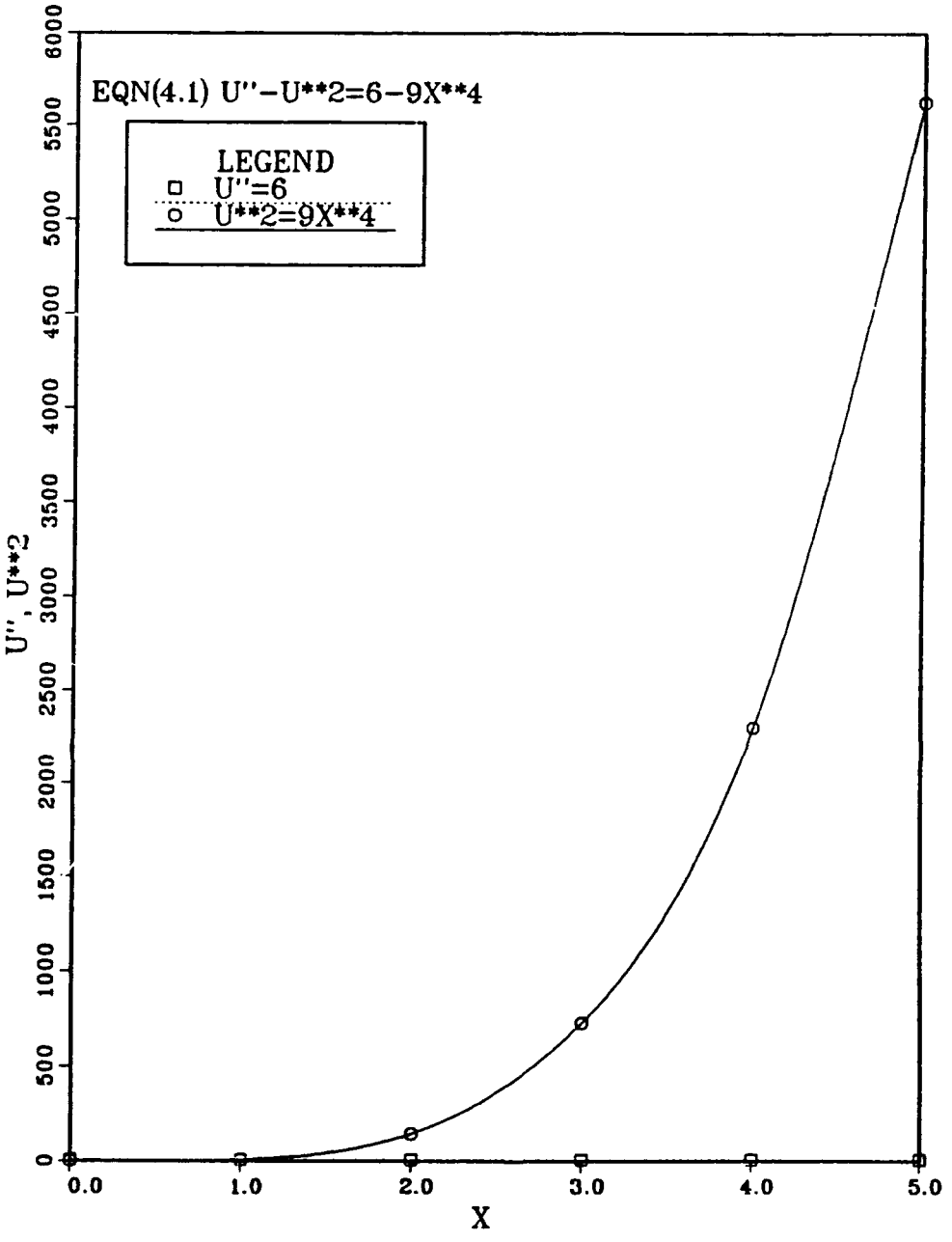


Figure 14. Dominance of Terms in Equation (4.1) over Domain Three

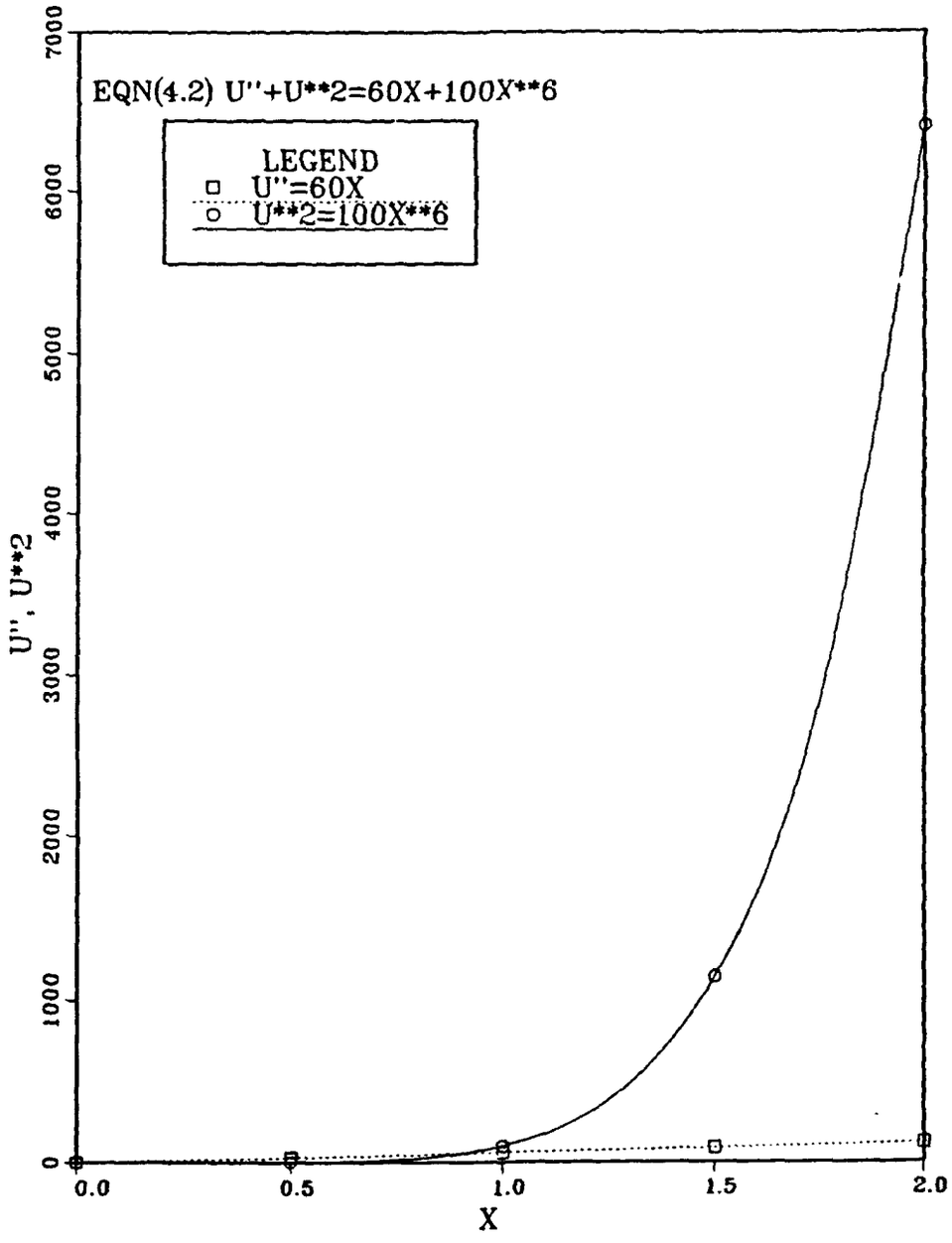


Figure 15. Dominance of Terms in Equation (4.2) over Domain Two

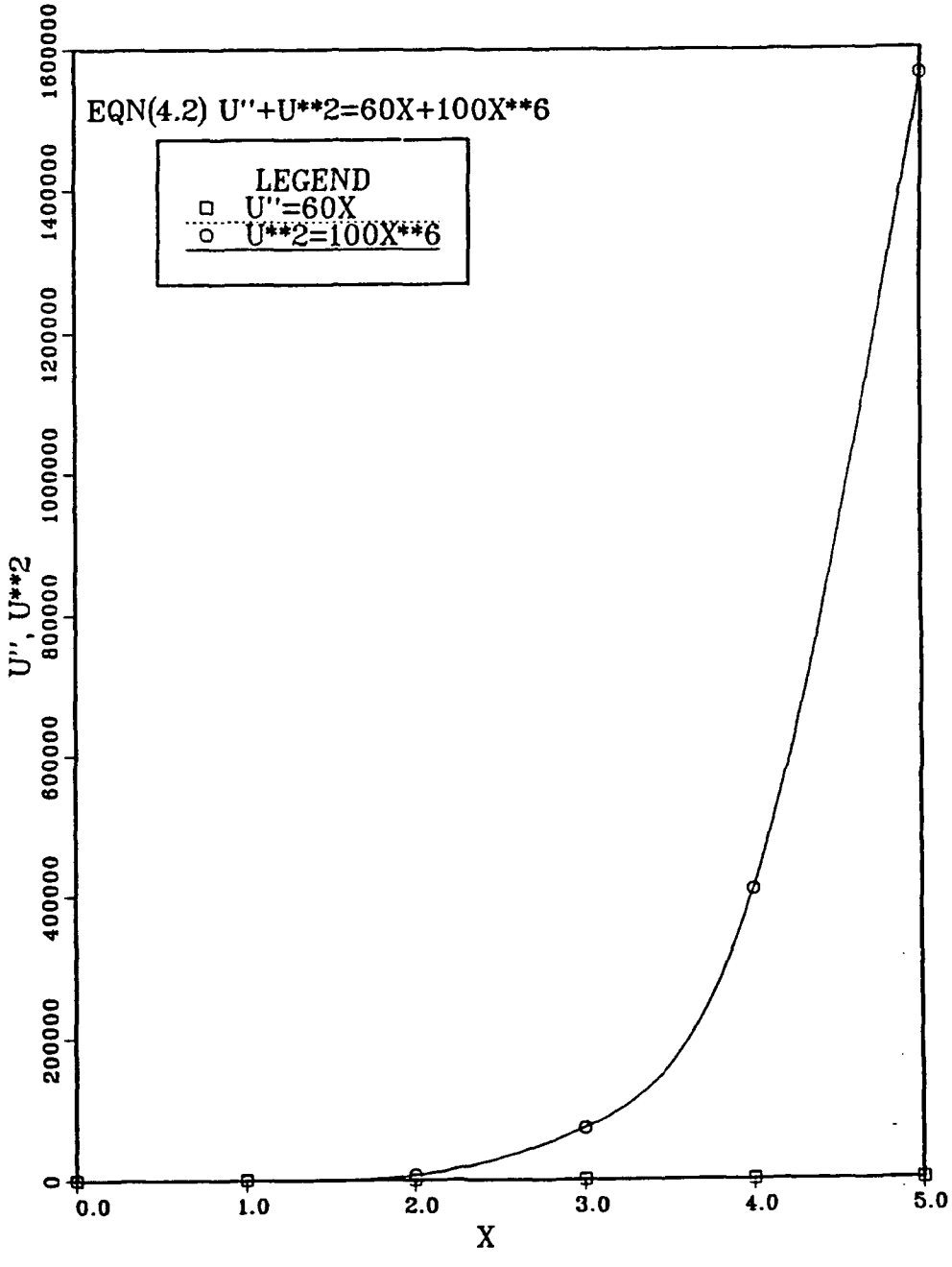


Figure 16. Dominance of Terms in Equation (4.2) over Domain Three

Over the domain $0 < x < 5$, Figure 16 clearly shows the domination of the u^2 term over a majority of the domain. Therefore the u'' term is neglected and the initial iteration values are determined using equation (4.12).

$$(u_j^*)_0 = \sqrt{60x + 100x^6} \tag{4.12}$$

The efficacy of the above initial iteration strategies was confirmed by the analyses which were undertaken.

B. CONSTANT LINEARIZATION

1. Problem Formulation

The constant linearization strategy, described in III.A.1, transforms the nonlinear term, u^2, into a constant linear term as shown in equation (4.13)

$$u^2 \approx \mathcal{L}^* u = (u^*)^2 \tag{4.13}$$

where u^* is determined as outlined in III.C. This process results in a linear differential equation of the form

$$u'' = \pm (u^*)^2 + f(x) \qquad x \in D \tag{4.14}$$

where '+' is for equation (4.1), '−' is for equation (4.2) and $f(x)$ represents the excitation function in each equation. The Galerkin FEM formulation process outlined in Chapter II transforms equation (4.14) into

$$\mathbf{G}(\mathbf{G}^T)'\mathbf{u}\Big|_a^b - \int_D \mathbf{G}'(\mathbf{G}^T)' dx\, \mathbf{u} = \pm \int_D \mathbf{G}(u^*)^2 dx + \int_D \mathbf{G} f(x) dx \tag{4.15}$$

where a and b in the first term represent the value of x at the left and right boundary of the domain, respectively. The left hand side of equation (4.15) is similar to that of equation (2.2.c) and upon evaluation yields $\mathbf{B} - \mathbf{Au}$, where the vector $\mathbf{B}$ is only present when a natural boundary condition is specified. The integrals on the right hand side are now evaluated.

Linearization Vector , $\int_D \mathbf{G}(u^*)^2 dx$

This integral is evaluated as outlined in III.B.1 where $h(u^*)$ in equation (3.6) is set equal to $(u^*)^2$. The detailed formulation of the 2x1 $\mathbf{f}^*$ element vectors for the three different interpolation strategies is given in Appendix C with the final results shown in

Table 3 on page 45. The 2x1 $\mathbf{f}^*$ vectors are then distributed into the system linearization vector, $\mathbf{F}^*$, in accordance with the local to global nodal point correspondence. The $\mathbf{F}^*$ vector continuously changes during the iteration process as the values of u^* are revised.

Table 3. CONSTANT LINEARIZATION ELEMENT VECTORS

Interpolation Strategy	$\mathbf{f}^*$
Midpoint Approximation	$\left(\frac{(u_j^*)_i + (u_{j-1}^*)_i}{2}\right)^2 \begin{bmatrix} \frac{l_e}{2} \\ \frac{l_e}{2} \end{bmatrix}$
1/4 - 3/4 Approximation	$\frac{l_e}{2} \begin{bmatrix} \left(\frac{3}{4}(u_j^*)_i + \frac{1}{4}(u_{j-1}^*)_i\right)^2 \\ \left(\frac{1}{4}(u_j^*)_i + \frac{3}{4}(u_{j-1}^*)_i\right)^2 \end{bmatrix}$
Linear Approximation	$l_e \begin{bmatrix} \frac{(u_j^*)_i^2}{4} + \frac{(u_j^*)_i(u_{j-1}^*)_i}{6} + \frac{(u_{j-1}^*)_i^2}{12} \\ \frac{(u_j^*)_i^2}{12} + \frac{(u_j^*)_i(u_{j-1}^*)_i}{6} + \frac{(u_{j-1}^*)_i^2}{4} \end{bmatrix}$

Excitation , $\int_D \mathbf{G} f(x) dx$

Transforming x to the element coordinate system, ξ, the element integral for $\mathbf{f}$ becomes

$$\mathbf{f} = \int_0^{l_e} \mathbf{g} f(\alpha_l + \xi) d\xi \tag{4.16}$$

where α_l is the distance from the origin of x to the origin of ξ of the element for which $\mathbf{f}$ is being evaluated. Equation (4.16) is evaluated using the consistent technique described in II.A, which is detailed in Appendix A, where $f(\,)$ is replaced by the appropriate excitation function in equations (4.1) and (4.2). The resulting 2x1 element excitation vectors are presented in Table 4 on page 46. The $\mathbf{f}$ vectors remain steady (constant) during the iteration process and are distributed into an Nx1 system force vector, $\mathbf{F}$, in accordance with the local to global DOF correspondence.

Table 4. ELEMENT FORCE VECTORS FOR EQUATIONS (4.1) AND (4.2)

$f(x)$	f
$6-9x^4$	$l_e\begin{bmatrix}3\\3\end{bmatrix}-\frac{9}{2}\alpha^4 l_e\begin{bmatrix}1\\1\end{bmatrix}-6\alpha^3 l_e^2\begin{bmatrix}1\\2\end{bmatrix}-\frac{9}{2}\alpha^3 l_e^3\begin{bmatrix}1\\3\end{bmatrix}-\frac{9}{5}\alpha l_e^4\begin{bmatrix}1\\4\end{bmatrix}-\frac{1}{30}l_e^5\begin{bmatrix}1\\5\end{bmatrix}$
$60x+100x^6$	$30\alpha l_e\begin{bmatrix}1\\1\end{bmatrix}+10l_e^2\begin{bmatrix}1\\2\end{bmatrix}+50\alpha^6 l_e\begin{bmatrix}1\\1\end{bmatrix}+100\alpha^5 l_e^2\begin{bmatrix}1\\2\end{bmatrix}+125\alpha^4 l_e^3\begin{bmatrix}1\\3\end{bmatrix}+$ $100\alpha^3 l_e^4\begin{bmatrix}1\\4\end{bmatrix}+50\alpha^2 l_e^5\begin{bmatrix}1\\5\end{bmatrix}+\frac{100}{7}\alpha l_e^6\begin{bmatrix}1\\6\end{bmatrix}+\frac{25}{16}l_e^7\begin{bmatrix}1\\7\end{bmatrix}$

Final FEM Equations

Substitution of the matrix and vector equivalents for each integral into equation (4.15) yields a system of equations given by

$$\mathbf{B}-\mathbf{Au}=\mathbf{F}^*+\mathbf{F} \qquad (4.17.a)$$

where the $\pm$ sign in equation (4.15) is incorporated into $\mathbf{F}^*$. The natural boundary condition vector **B**, when present for an E-N problem does not change during the iteration process and is taken to the right side and subtracted from **F** yielding $\mathbf{F}_m$, where the m subscript indicates that the excitation vector is modified for the given natural boundary condition. The system of equations then takes the final form of

$$-\mathbf{Au}_i=\mathbf{F}^*_{i-1}+\mathbf{F}_m \qquad (4.17.b)$$

where subscripts i and $i-1$ refer to the iteration counter. Equation 4.17.b is solved iteratively for **u**, with $\mathbf{F}^*$ changing after each iteration, until convergence is obtained.

2. Results

a. General

Equations (4.1) and (4.2) were each solved over domain one using 24 different solution procedures while 12 different procedures were utilized for both domains two and three. The FORTRAN programs utilized for the constant linearization strategy are contained in Appendix E. A summary of the strategies utilized in each solution procedure is shown in Table 6 on page 49 and Table 7 on page 50 for equation (4.1), and Table 8 on page 51 and Table 9 on page 52 for equation (4.2), with the following performance information provided in the results portion of each table.

- Convergence (III.D.1)
- Stability (III.D.3)
- Number of iterations required to obtain convergence
- Average percent error (III.D.6)
- CPU* (III.D.6)

The results for each solution procedure over each of the domains were obtained using the number of elements shown in Table 5. Though these number of elements are not necessarily the number required to obtain a grid independent solution for each solution procedure, they do provide a common baseline upon which the performance of individual solution procedures for a particular problem may be compared. This is also the number of elements utilized in the table of results for each of the solution procedures developed by the other two linearization strategies.

Table 5. NUMBER OF ELEMENTS UPON WHICH SOLUTION PROCEDURE RESULTS ARE BASED

Equation	Domain	Number of Elements
4.1	1	10
	2	20
	3	25
4.2	1	20
	2	40
	3	50

Two general observations can be made based on the results of Table 6 through Table 9. One is that the constant linearization technique begins to fail as the function order over the given domain begins to approach two. This is clearly shown in Table 8 as only half of the solution procedures provided convergent solutions of equation (4.2), whose function order is just barely two over the domain $0 < x < 1$. As soon as the domain length was increased to 1.1, these remaining methods diverged as shown in Table 9. This is most likely due to equation (4.2) becoming fully order two as the domain is increased past $x = 1.0$. In order to determine where the solutions begin to breakdown for equation (4.1), the domain was extended in 0.1 increments until all solution procedures diverged. The results of this investigation showed that some solution procedures for equation (4.1) were able to provide convergent solutions over a

domain of $0 < x < 1.7$, which approaches the region where equation (4.1) shifts in function order from one to two. This failure to provide convergent solutions of equations (4.1) and (4.2) is not due to a lack of elements used in the approximation, but is a characteristic of this particular linearization strategy.

The second observation is that the number of iterations required to obtain convergence is strictly a function of the iteration strategies utilized, both initial and subsequent, and is independent of the interpolation strategy. Also, it was found that the number of iterations is independent of the number of degrees of freedom utilized. That is, the same number of iterations were required to reach convergence whether two or 20 elements were utilized. The accuracy of the approximation, on the other hand, seems to be independent of the iteration strategy, and only a function of the interpolation strategy and the number of degrees of freedom. The effect of each problem parameter in the different solution procedures for both equations (4.1) and (4.2) over domain one is now examined, as no convergent solutions were obtained for domains two and three.

b. Boundary Conditions

The use of an essential boundary condition at both ends of the domain increased the rate of convergence by a factor of two to five over those strategies which utilized an essential and natural boundary condition combination. In fact, only three of the twelve strategies which used an E-N boundary condition combination provided convergent approximations of the differential equation. Suffice it to say, the essential-natural boundary condition combination does not produce very efficient results when utilized with the constant linearization strategy and therefore this strategy should not be used for E-N problems. That being the case, the remaining comments are directed at those strategies which utilized an essential-essential boundary condition combination.

c. Initial Iteration Strategy

Each of the initial iteration strategies, described in III.C.1, provided nearly the same results within a specific subsequent iteration and interpolation strategy combination when the function order was small, as in equation (4.1). But, when the function order begins to approach two, as in equation (4.2), the use of the right essential boundary condition for the initial iteration values led to divergence, while the other two strategies provided similar convergent results.

d. Subsequent Iteration Strategy

The use of the previous value strategy (III.C.2.a) generally led to convergence using less iterations than the average value strategy (III.C.2.b). The main reason for this is most likely a result of the constant linearization strategy providing for

Table 6. SOLUTION PROCEDURES AND RESULTS USING CONSTANT LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAIN ONE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i')_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 1$	Prev. Value	Midpt.	E	E	L	C	S	7	0.48	0.0112
			E	E	M	C	S	7	0.48	0.0081
			E	E	R	C	S	8	0.48	0.0129
			E	N	L	C[a]	----	200+	----	-----
$0 < x < 1$	Prev. Value	1/4-3/4	E	E	L	C	S	7	0.12	0.0028
			E	E	M	C	S	7	0.12	0.0028
			E	E	R	C	S	8	0.12	0.0032
			E	N	L	C[a]	----	200+	----	-----
$0 < x < 1$	Prev. Value	Linear	E	E	L	C	S	7	0.16	0.0027
			E	E	M	C	S	7	0.16	0.0043
			E	E	R	C	S	8	0.16	0.0043
			E	N	L	C[a]	----	200+	----	-----
$0 < x < 1$	Avg. Value	Midpt.	E	E	L	C	S	9	0.48	0.0127
			E	E	M	C	S	7	0.50	0.0079
			E	E	R	C	S	9	0.50	0.0132
			E	N	L	C	S	38	0.27	0.0270
$0 < x < 1$	Avg. Value	1/4-3/4	E	E	L	C	S	9	0.12	0.0025
			E	E	M	C	S	7	0.12	0.0022
			E	E	R	C	S	9	0.11	0.0037
			E	N	L	C	S	38	0.31	0.0286
$0 < x < 1$	Avg. Value	Linear	E	E	L	C	S	9	0.16	0.0041
			E	E	M	C	S	7	0.15	0.0035
			E	E	R	C	S	9	0.17	0.0046
			E	N	L	C	S	38	0.47	0.0471

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Oscillates between two nonsolutions of the differential equation

Table 7. SOLUTION PROCEDURES AND RESULTS USING CONSTANT LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAINS TWO AND THREE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 2$	Prev. Value	Midpt.	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Prev. Value	1/4-3/4	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Prev. Value	Linear	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Avg. Value	Midpt.	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Avg. Value	1/4-3/4	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Avg. Value	Linear	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	Midpt.	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	1/4-3/4	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	Linear	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	Midpt.	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	1/4-3/4	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	Linear	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: (1) - See equation (4.9) (2) - See equation (4.10)

Table 8. SOLUTION PROCEDURES AND RESULTS USING CONSTANT LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAIN ONE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 1$	Prev. Value	Midpt.	E	E	L	C	S	12	12.9	1.4595
			E	E	M	C	S	13	13.0	2.0773
			E	E	R	D	U	-----	----	-----
			E	N	L	D	U	-----	----	-----
$0 < x < 1$	Prev. Value	1/4-3/4	E	E	L	C	S	12	2.12	0.2198
			E	E	M	C	S	13	1.99	0.3380
			E	E	R	D	U	-----	----	-----
			E	N	L	D	U	-----	----	-----
$0 < x < 1$	Prev. Value	Linear	E	E	L	C	S	12	4.62	0.4922
			E	E	M	C	S	13	4.77	0.6664
			E	E	R	D	U	-----	----	-----
			E	N	L	D	U	-----	----	-----
$0 < x < 1$	Avg. Value	Midpt.	E	E	L	C	S	17	12.8	1.7412
			E	E	M	C	S	18	13.1	2.8829
			E	E	R	D	U	-----	----	-----
			E	N	L	C[a]	S	32	----	-----
$0 < x < 1$	Avg. Value	1/4-3/4	E	E	L	C	S	17	2.22	0.3619
			E	E	M	C	S	18	1.97	0.3799
			E	E	R	D	U	-----	----	-----
			E	N	L	C[a]	S	32	----	-----
$0 < x < 1$	Avg. Value	Linear	E	E	L	C	S	17	4.58	0.7091
			E	E	M	C	S	18	4.89	1.0410
			E	E	R	D	U	-----	----	-----
			E	N	L	C[a]	S	32	----	-----

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Converges to a nonsolution of the differential equation

Table 9. SOLUTION PROCEDURES ND RESULTS USING CONSTANT LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAINS TWO AND THREE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i')_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 2$	Prev. Value	Midpt.	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Prev. Value	1/4-3/4	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Prev. Value	Linear	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Avg. Value	Midpt.	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Avg. Value	1/4-3/4	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 2$	Avg. Value	Linear	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	Midpt.	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	1/4-3/4	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	Linear	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	Midpt.	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	1/4-3/4	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	Linear	E	E	(2)	D	U	-----	----	-----
			E	N	(2)	D	U	-----	----	-----

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: (1) - See equation (4.11) (2) - See equation (4.12)

monotonic convergence when used with an E-E combination. The essential boundary conditions provide a range that the values of u should be between and thus provide strong guidance during the iteration process. The one exception to this involves those cases where a natural boundary condition is given at the right end of the domain. In these instances, only the average value iteration strategy resulted in convergent solutions. This E-N combination does not fix the value of the dependent variable at the right end, so the approximate solution most likely oscillates about the exact solution during the iteration process. The average value strategy tends to enhance the convergence of this type of approximation, which might explain why this strategy resulted in convergence of the E-N problem while the previous value strategy did not.

e. Interpolation Strategy

The 1/4-3/4 interpolation method consistently provided the most accurate approximations as indicated by the low average percent error values. This is because the solution function for both equations (4.1) and (4.2) is a polynomial function of x and the 1/4-3/4 method was shown in Chapter II to provide better approximations of the integrals of these functions than the midpoint or linear techniques.

f. Overall Performance

The overall performance, which factors both computational effort (i.e., CPU time) and solution accuracy, of a particular solution procedure is indicated by its respective value of CPU*; where the lower the value, the more efficient the solution procedure. In Table 6, the CPU times upon which the CPU* values are based were all at or below the clock subroutine accuracy of $\pm$ .03 seconds. Thus, the CPU* values in this table should be used with caution. A comparison of the solution procedures using average percent error values (% Dif in Table 6) indicates the previous value iteration strategy combined with the 1/4-3/4 interpolation strategy provides the most accurate solution of equation (4.1) over domain one.

Due to the increase in the number of elements and iterations required for convergence of equation (4.2), more CPU time was required by the solution procedures in Table 8. Hence, the CPU* times in Table 8 are all based on CPU times much greater than the clock subroutine accuracy and therefore are all valid. Again, the combination of previous value iteration and 1/4-3/4 interpolation strategies provided for the most efficient procedure.

3. Conclusions

The constant linearization strategy can generally provide approximations of nonlinear differential equations when the function activity is less than order two over the

given domain. The relative 'crudeness' of this linearization technique requires that as much information as possil concerning the actual value of the dependent variable over the given domain be known. Thus, the knowledge of an essential boundary condition at each end of the domain is almost a prerequisite to obtain a convergent solution of a nonlinear differential equation when using this linearization method. If these conditions are met, an iteration strategy utilizing the magnitude of the smallest valued boundary condition for the initial iteration and the previous value strategy for subsequent iterations should result in convergence with a minimum number of iterations and expend the least amount of CPU time for a given number of degrees of freedom. The 1/4-3/4 interpolation strategy should yield the most accurate approximation as most solutions of engineering problems are monotonically increasing or decreasing functions, or at worst, convex or concave over the given domain.

C. CLASSICAL LINEARIZATION

1. Problem Formulation

The classical linearization strategy transforms the u^2 nonlinear term into a linear term as described in III.A.2 and shown in equation (4.18)

$$u^2 \approx \mathscr{L}^* u = u^* u \tag{4.18}$$

where u^* is determined as outlined in III.C. Substitution of equation (4.18) into equations (4.1) and (4.2) yields a linear differential equation of the form

$$u'' \pm u^* u = f(x) \qquad x \in D \tag{4.19}$$

where the '$-$' is for equation (4.1), the '$+$' is for equation (4.2) and $f(x)$ is again the respective excitation function in each equation. The Galerkin FEM formulation process transforms equation (4.19) into

$$\mathbf{G}(\mathbf{G}^{\mathrm{T}})'\mathbf{u}\Big|_a^b - \int_D \mathbf{G}'(\mathbf{G}^{\mathrm{T}})' dx\, \mathbf{u} \pm \int_D \mathbf{G}\mathbf{G}^{\mathrm{T}} u^* dx\, \mathbf{u} = \int_D \mathbf{G} f(x) dx \tag{4.20}$$

The first two terms on the left side of equation (4.20) again provide $\mathbf{B} - \mathbf{A}\mathbf{u}$ where the **B** vector is present only when a natural boundary condition is provided. The integral on the right side of equation (4.20), which was evaluated in IV.B.1, gives the system excitation vector, **F**. The only term remaining to be evaluated is the third term on the left side of equation (4.20), the linearization matrix integral.

Linearization Matrix , $\int_D \mathbf{G}\mathbf{G}^T u^* dx\, \mathbf{u}$

This integral is evaluated as outlined in III.B.2 where $h(u^*)$ in equation (3.10) is replaced by u^*. The detailed formulation of the element linearization matrices, $\mathbf{l}^*$, for the two different interpolation strategies is given in Appendix D, with the final results provided in Table 10. The $\mathbf{l}^*$ are then distributed into the system linearization matrix, $\mathbf{L}^*$, based on the local to global nodal point correspondence. The $\mathbf{L}^*$ matrix is updated during each iteration as the values of u^* are revised.

Table 10. CLASSICAL LINEARIZATION ELEMENT MATRICES

Interpolation Strategy	$\mathbf{l}^*$
Midpoint Approximation	$\left(\frac{(u_i^*)_i + (u_{i+1}^*)_i}{2}\right)\begin{bmatrix} \frac{l_e}{3} & \frac{l_e}{6} \\ \frac{l_e}{6} & \frac{l_e}{3} \end{bmatrix}$
Linear Approximation	$\frac{l_e}{12}\begin{bmatrix} 3(u_i^*)_i + (u_{i+1}^*)_i & (u_i^*)_i + (u_{i+1}^*)_i \\ (u_i^*)_i + (u_{i+1}^*)_i & (u_i^*)_i + 3(u_{i+1}^*)_i \end{bmatrix}$

Substitution of the matrix and vector equivalents for each integral into equation (4.20) yields a system of equations given by

$$\mathbf{B} - \mathbf{A}\mathbf{u} + \mathbf{L}^*\mathbf{u} = \mathbf{F} \tag{4.21.a}$$

where the $\pm$ sign in equation (4.20) is incorporated into $\mathbf{L}^*$. Again, $\mathbf{F}$ and $\mathbf{B}$ are combined to yield $\mathbf{F}_m$ and the system of linear algebraic equations takes the final form

$$\left(-\mathbf{A} + \mathbf{L}_{i-1}^*\right)\mathbf{u}_i = \mathbf{F}_m \tag{4.21.b}$$

Equation (4.21.b) is solved iteratively for $\mathbf{u}_i$, with $\mathbf{L}_{i-1}^*$ being calculated using u_{i-1}^* values , until convergence is obtained.

2. Results

a. General

Sixteen different solution procedures were utilized to solve equations (4.1) and (4.2) over domain one and eight procedures were utilized for each equation over both domains two and three. The FORTRAN programs for constant linearization are contained in Appendix F. The different strategies utilized in each procedure and the

corresponding results are provided in Table 11 on page 56, and Table 12 on page 57 for equation (4.1); and Table 13 on page 58, and Table 14 on page 59 for equation (4.2). The number of elements utilized in each solution procedure is shown in Table 5 on page 47.

Table 11. SOLUTION PROCEDURES AND RESULTS USING CLASSICAL LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAIN ONE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 1$	Prev. Value	Midpt.	E	E	L	C	S	6	0.24	0.0048
			E	E	M	C	S	4	0.24	0.0040
			E	E	R	C	S	5	0.24	0.0032
			E	N	L	C	S	12	0.10	0.0030
$0 < x < 1$	Prev. Value	Linear	E	E	L	C	S	6	0.16	0.0032
			E	E	M	C	S	4	0.16	0.0016
			E	E	R	C	S	5	0.16	0.0038
			E	N	L	C	S	12	0.38	0.0115
$0 < x < 1$	Avg. Value	Midpt.	E	E	L	C	S	8	0.24	0.0048
			E	E	M	C	S	5	0.24	0.0048
			E	E	R	C	S	8	0.24	0.0064
			E	N	L	C	S	12	0.08	0.0019
$0 < x < 1$	Avg. Value	Linear	E	E	L	C	S	8	0.16	0.0027
			E	E	M	C	S	5	0.16	0.0043
			E	E	R	C	S	8	0.16	0.0043
			E	N	L	C	S	12	0.36	0.0107

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

Table 12. SOLUTION PROCEDURES AND RESULTS USING CLASSICAL LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAINS TWO AND THREE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i')_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 2$	Prev. Value	Midpt.	E	E	(1)	C	S	17	0.38	0.0678
			E	N	(1)	C	S	77	0.34	0.2262
$0 < x < 2$	Prev. Value	Linear	E	E	(1)	C	S	17	0.25	0.0458
			E	N	(1)	C	S	76	0.29	0.2239
$0 < x < 2$	Avg. Value	Midpt.	E	E	(1)	C	S	12	0.36	0.0423
			E	N	(1)	C	S	18	0.34	0.0572
$0 < x < 2$	Avg. Value	Linear	E	E	(1)	C	S	12	0.23	0.0298
			E	N	(1)	C	S	18	0.25	0.0465
$0 < x < 5$	Prev. Value	Midpt.	E	E	(1)	C[a]	S	200+	0.60	2.0412
			E	N	(1)	C[a]	S	200+	0.96	3.2696
$0 < x < 5$	Prev. Value	Linear	E	E	(1)	C[a]	S	200+	0.43	1.4574
			E	N	(1)	C[a]	S	200+	0.65	2.2315
$0 < x < 5$	Avg. Value	Midpt.	E	E	(1)	C	S	13	0.60	0.1365
			E	N	(1)	C	S	10	0.65	0.0986
$0 < x < 5$	Avg. Value	Linear	E	E	(1)	C	S	13	0.42	0.0777
			E	N	(1)	C	S	13	0.42	0.0863

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Convergence was imminent and obtained within another 50 iterations
(1) - See equation (4.10)

Table 13. SOLUTION PROCEDURES AND RESULTS USING CLASSICAL LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAIN ONE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_j^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 1$	Prev. Value	Midpt.	E	E	L	C	S	9	6.94	0.7851
			E	E	M	C	S	8	6.98	0.6507
			E	E	R	C	S	10	6.91	0.8740
			E	N	L	C[a]	S	17	----	-----
$0 < x < 1$	Prev. Value	Linear	E	E	L	C	S	9	4.61	0.4300
			E	E	M	C	S	8	4.66	0.5275
			E	E	R	C	S	10	4.58	0.5785
			E	N	L	C[b]	S	200+	----	-----
$0 < x < 1$	Avg. Value	Midpt.	E	E	L	C	S	13	6.91	1.0120
			E	E	M	C	S	11	7.04	0.9376
			E	E	R	C	S	15	6.90	1.1933
			E	N	L	C[a]	S	17	----	-----
$0 < x < 1$	Avg. Value	Linear	E	E	L	C	S	12	4.57	0.5988
			E	E	M	C	S	11	4.71	0.6376
			E	E	R	C	S	15	4.57	0.8058
			E	N	L	C[a]	S	17	----	-----

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Converges to another solution of the differential equation
b - Convergence to another solution of the differential equation was imminent

Table 14. SOLUTION PROCEDURES AND RESULTS USING CLASSICAL LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAINS TWO AND THREE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 2$	Prev. Value	Midpt.	E	E	(1)	C[a]	S	87	----	-----
			E	N	(1)	(2)	----	200+	----	-----
$0 < x < 2$	Prev. Value	Linear	E	E	(1)	C[a]	S	95	----	-----
			E	N	(1)	(2)	----	200+	----	-----
$0 < x < 2$	Avg. Value	Midpt.	E	E	(1)	C[a]	S	40	----	-----
			E	N	(1)	C[a]	S	42	----	-----
$0 < x < 2$	Avg. Value	Linear	E	E	(1)	C[a]	S	35	----	-----
			E	N	(1)	C[a]	S	52	----	-----
$0 < x < 5$	Prev. Value	Midpt.	E	E	(1)	(2)	----	200+	----	-----
			E	N	(1)	(2)	----	200+	----	-----
$0 < x < 5$	Prev. Value	Linear	E	E	(1)	(2)	----	200+	----	-----
			E	N	(1)	(2)	----	200+	----	-----
$0 < x < 5$	Avg. Value	Midpt.	E	E	(1)	C[a]	S	117	----	-----
			E	N	(1)	C[a]	S	80	----	-----
$0 < x < 5$	Avg. Value	Linear	E	E	(1)	C[a]	S	43	----	-----
			E	N	(1)	C[a]	S	105	----	-----

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Converges to a nonsolution of the differential equation
(1) - See equation (4.12)
(2) - Nonconvergent solution

Three general observations can be made from the results shown in these tables. The first is that, for solution procedures which yielded valid converged approximations, the number of iterations to convergence was strictly a function of the initial and subsequent iteration strategies and independent of the interpolation strategy. Additionally, the number of iterations to convergence was independent of the number of elements utilized. The accuracy of the solution, on the other hand, was strictly a function

of the interpolation strategy and the number of degrees of freedom used in the approximation.

The second observation is that the classical linearization technique provided valid approximations of equation (4.1) over domain three, but was not able to solve equation (4.2) over domain two, despite the fact both of these equations over these respective domains have the same function order. The main reason for this lies in the fact that the solution of equation (4.2) over domain two is changing more rapidly than that of equation (4.1) over domain three. Thus, it appears that the classical linearization technique can provide valid approximations of nonlinear differential equations that have a function order of two, but whose rate of change over the given domain is not 'excessively' large.

Last of all, a general comment on the magnitude of the average percent difference values in Table 13 is in order. Though these values may seem large in relation to the percent difference values in Table 11 and Table 12, it must be remembered that these are average values. In this particular situation, the values of the dependent variable over $0 < x < 0.2$ are quite small, i.e., on the order of 0.08 and less. Thus, an approximate solution at one node which is in absolute error of only 0.0023 from an exact solution value of 0.0100 yields a 23 percent error. Therefore, the larger average percent difference values stem from minor errors in the approximations at those nodes where the magnitude of the dependent variable is very small. The overall approximate solutions provided by these procedures is much better than the average percent difference values indicate as the actual percent difference values over a majority of the domain was less than 0.5 percent. The effect of each problem parameter on the performance of the various solution procedures which provided convergent solutions of equations (4.1) and (4.2) is now examined.

b. Boundary Conditions

The use of an E-E boundary condition combination provided convergence with less iterations than the E-N combination in all instances except for the solution of equation (4.1) over domain three. No explanation for this behavior could be determined and it remains an open question requiring further investigation. The accuracy of the approximations within a specific combination of iteration and interpolation strategies was not greatly affected by the boundary conditions, as the average percent difference values provided by the E-E and E-N combinations were similar.

c. Initial Iteration Strategy

In solving equations (4.1) and (4.2) over domain one, the initial iteration strategy is developed utilizing the prescribed essential boundary condition(s) as described in III.C.1. An initial iteration strategy based on the average value of the two essential boundary conditions consistently provided convergence with slightly less iterations than the other two strategies.

Over domains two and three, only convergent solutions of equation (4.1) were obtained; no valid approximate solutions were obtained of equation (4.2) over these domains. Initial iteration strategies defined by equations (4.9) and (4.10) were both utilized in the analysis over domain two to determine which was most effective. Equations (4.9) and (4.10) provided nearly the same numerical values in the approximation, but the use of equation (4.10) consistently enabled the solution procedure to converge with less iterations. This behavior requires further research and remains an open question, as it was felt that equation (4.9), which accounted for the dominance of the u'' term over the first part of the domain, should have provided better initial iteration values. In the solution of equation (4.2) over domains two and three, both initial iteration strategies led to convergent/nonconvergent nonsolutions of the differential equation.

d. Subsequent Iteration Strategy

Over domain one for both equations (4.1) and (4.2), the use of the previous value iteration strategy consistently provided convergence with slightly less iterations than the average value strategy. The one exception to this was when an E-N boundary condition combination was specified. In that case, both strategies yielded convergence with the same number of iterations. Over domains two and three, the average value strategy provided for convergence with significantly less iterations than the previous value method. This fact is especially evident in the solution of equation (4.1) over domain three where the previous value strategy could not converge within 200 iterations while the average value method provided convergence in 13 iterations or less. These two observations tend to indicate that the classical linearization strategy provides for monotonic convergence when the function order is one or less and oscillatory convergence when the function order is greater than one.

e. Interpolation Strategy

The linear interpolation strategy consistently provided more accurate approximations than the midpoint strategy as indicated by the lower average percent difference values in Table 11 through Table 13. The main reason for this relates to the

fact that the solutions of equations (4.1) and (4.2) are polynomial functions of x. The number of elements utilized in each of the solution procedures was sufficient enough to make the solution curve almost linear over each element. Thus, the linear approximation induced less error in the evaluation of the Galerkin linearization matrix integral than the midpoint method. For those cases where only a few elements were utilized, the midpoint method occasionally provided a more accurate approximation than the linear strategy. But, the overall accuracy of the approximation was poor due to the decrease in the number of system DOF.

f. Overall Performance

When the function order of the differential equation solution is one or less, a solution procedure which utilizes a previous value iteration and linear interpolation strategy tends to provide the most efficient solutions based on the average percent difference values in Table 11 and CPU* values in Table 13. The CPU* values in Table 11 should be evaluated with caution as the CPU times upon which they are based are at or below the accuracy level of the clock subroutine which is $\pm$ 0.03 seconds. When the function order is greater than one, an average value iteration and linear interpolation strategy combination provide the most efficient approximation as indicated by the CPU* values in Table 12.

3. Conclusions

The classical linearization strategy can generally provide approximations of second order nonlinear differential equations when the function order is two or less and the rate of change of the dependent variable is not extremely large. Provided these conditions are met, an average value iteration strategy combined with a linear interpolation strategy should provide an efficient, valid approximation of the differential equation. If the function order is later found to be one or less, the use of a previous value iteration strategy should result in similar numerical results and converge using slightly less iterations. In either case, both the E-E and E-N boundary condition combinations can be accommodated, although the use of an E-E combination is preferred.

D. QUASILINEARIZATION

1. Problem Formulation

Quasilinearization transforms the nonlinear u^2 term into a linear term using the relation given by equation (3.4.a), where $q(u^*) = (u^*)^2$ and $q'(u^*) = 2u^*$. Substitution of these functions into equation (3.4.a) yields

$$u^2 \approx \mathscr{L}^* u = 2u^* u + \left((u^*)^2 - u^*(2u^*)\right) = 2u^* u - (u^*)^2 \tag{4.22}$$

where u^* is determined as outlined in III.C. Substitution of equation (4.22) into equations (4.1) and (4.2) yields a linear differential equation of the form

$$u'' \pm 2u^* u = f(x) \pm (u^*)^2 \qquad x \in D \tag{4.23}$$

where both '−' signs are for equation (4.1), both '+' signs are for equation (4.2) and $f(x)$ is the respective excitation function in each equation. The Galerkin FEM formulation process transforms equation (4.23) into

$$\mathbf{G}(\mathbf{G}^T)'\mathbf{u}\Big|_a^b - \int_D \mathbf{G}'(\mathbf{G}^T)' dx\, \mathbf{u} \pm 2\int_D \mathbf{G}\mathbf{G}^T u^* dx\, \mathbf{u} = \int_D \mathbf{G} f(x) dx \pm \int_D \mathbf{G}(u^*)^2 dx \tag{4.24}$$

The first two terms on the left side of equation (4.24) yield $\mathbf{B} - \mathbf{A}\mathbf{u}$ where the $\mathbf{B}$ vector is present only when a natural boundary condition is prescribed. The third term on the left side of the equation is the linearization matrix integral which was evaluated in IV.C.1 and yields $\mathbf{L}^*$. Both integrals on the right side of equation (4.24), the excitation and linearization vector, were evaluated in IV.B.1 and yield $\mathbf{F}$ and $\mathbf{F}^*$, respectively.

Substitution of the matrix and vector equivalents for each term into equation (4.24) yields a system of equations given by

$$\mathbf{B} - \mathbf{A}\mathbf{u} + 2\mathbf{L}^*\mathbf{u} = \mathbf{F} + \mathbf{F}^* \tag{4.25.a}$$

where the $\pm$ signs in equation (4.24) are incorporated into $\mathbf{L}^*$ and $\mathbf{F}^*$. Combining $\mathbf{F}$ and $\mathbf{B}$ to yield $\mathbf{F}_m$, the system of linear algebraic equations takes the final form

$$\left(-\mathbf{A} + 2\mathbf{L}^*_{i-1}\right)\mathbf{u}_i = \mathbf{F}_m + \mathbf{F}^*_{i-1} \tag{4.25.b}$$

Equation (4.25.b) is solved iteratively for $\mathbf{u}$, with both $\mathbf{L}^*$ and $\mathbf{F}^*$ changing after each iteration, where i is the iteration counter.

2. Results

a. General

Forty eight different solution procedures were evaluated in solving equations (4.1) and (4.2) over domain one while there were twenty four procedures available for approximating each equation over both domains two and three. The

FORTRAN programs for quasilinearization are provided in Appendix G. The different strategies utilized in each solution procedure and the corresponding results in approximating the solution of equation (4.1) are provided in Table 15 on page 65 through Table 19 on page 69. For equation (4.2), this information is provided in Table 20 on page 70 through Table 25 on page 75. The number of elements utilized in each solution procedure is given in Table 5 on page 47.

Four general observations can be made about the quasilinearization strategy based on these results. The first is that this linearization technique provided valid approximations of both equations (4.1) and (4.2) over all domains, although some individual solution procedures were much more accurate than others. The difference in performance between the various solution procedures became noticeable as the function order approached three or more and the solution function gradient became large, as in equation (4.2) over domains two and three.

The second point is that this linearization technique provides for convergence with a minimum of iterations due to its quadratic rate of convergence [Ref. 1: pp. 38-40]. Some of the solution procedures utilized to solve equations (4.1) and (4.2) over domain three converged in just two iterations and thus did not yield as accurate solutions as was anticipated. In order to determine if these solution procedures could provide more accurate approximations, the convergence criterion was changed from .0001 to .0000001, to allow for slightly more iterations. The effect of changing the convergence criterion for those applicable solution procedures is noted in Table 18, Table 19, Table 24, and Table 25.

The third point, which also relates to convergence, is that the number of iterations required for convergence is not always a function of the overall iteration strategy as it was for the previous two linearization strategies. When the function order is one or slightly over two, as in equation (4.1) over domains one and two, and equation (4.2) over domain one, the number of iterations is dictated by the iteration strategies utilized. For those situations where the function order is almost three or more, the number of iterations required for convergence also appears to be affected by the specific combination of interpolation strategies used in the solution procedure.

Table 15. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAIN ONE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 1$	Prev. Value	Midpt. Midpt.	E	E	L	C	S	4	0.00	0.0000
			E	E	M	C	S	4	0.00	0.0000
			E	E	R	C	S	4	0.00	0.0000
			E	N	L	C	S	5	0.00	0.0000
$0 < x < 1$	Prev. Value	Midpt. 1/4-3/4	E	E	L	C	S	4	0.60	0.0100
			E	E	M	C	S	4	0.60	0.0080
			E	E	R	C	S	4	0.60	0.0100
			E	N	L	C	S	5	0.16	0.0023
$0 < x < 1$	Prev. Value	Midpt. Linear	E	E	L	C	S	4	0.32	0.0053
			E	E	M	C	S	4	0.32	0.0064
			E	E	R	C	S	4	0.32	0.0064
			E	E	R	C	S	5	0.24	0.0057
$0 < x < 1$	Prev. Value	Linear Midpt.	E	E	L	C	S	4	0.16	0.0021
			E	E	M	C	S	4	0.16	0.0021
			E	E	R	C	S	4	0.16	0.0032
			E	N	L	C	S	5	0.61	0.0122
$0 < x < 1$	Prev. Value	Linear 1/4-3/4	E	E	L	C	S	4	0.44	0.0059
			E	E	M	C	S	4	0.44	0.0059
			E	E	R	C	S	4	0.44	0.0044
			E	N	L	C	S	5	0.54	0.0108
$0 < x < 1$	Prev. Value	Linear Linear	E	E	L	C	S	4	0.16	0.0027
			E	E	M	C	S	4	0.16	0.0027
			E	E	R	C	S	4	0.16	0.0027
			E	N	L	C	S	5	0.02	0.0004

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

Table 16. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAIN ONE (CONT.)

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i')_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 1$	Avg. Value	Midpt. Midpt.	E	E	L	C	S	5	0.00	0.0000
			E	E	M	C	S	5	0.00	0.0000
			E	E	R	C	S	6	0.00	0.0000
			E	N	L	C	S	5	0.02	0.0004
$0 < x < 1$	Avg. Value	Midpt. 1/4-3/4	E	E	L	C	S	5	0.60	0.0120
			E	E	M	C	S	5	0.60	0.0080
			E	E	R	C	S	6	0.61	0.0141
			E	N	L	C	S	5	0.11	0.0015
$0 < x < 1$	Avg. Value	Midpt. Linear	E	E	L	C	S	5	0.32	0.0053
			E	E	M	C	S	5	0.32	0.0064
			E	E	R	C	S	6	0.32	0.0075
			E	N	L	C	S	5	0.27	0.0053
$0 < x < 1$	Avg. Value	Linear Midpt.	E	E	L	C	S	5	0.16	0.0027
			E	E	M	C	S	5	0.16	0.0032
			E	E	R	C	S	6	0.16	0.0038
			E	N	L	C	S	5	0.60	0.0099
$0 < x < 1$	Avg. Value	Linear 1/4-3/4	E	E	L	C	S	5	0.45	0.0089
			E	E	M	C	S	5	0.45	0.0089
			E	E	R	C	S	6	0.44	0.0118
			E	N	L	C	S	5	0.53	0.0070
$0 < x < 1$	Avg. Value	Linear Linear	E	E	L	C	S	5	0.16	0.0037
			E	E	M	C	S	5	0.16	0.0038
			E	E	R	C	S	6	0.16	0.0037
			E	N	L	C	S	5	0.36	0.0084

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

Table 17. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAIN TWO

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i')_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 2$	Prev. Value	Midpt. Midpt.	E	E	(1)	C	S	4	0.00	0.0001
			E	N	(1)	C	S	4	0.00	0.0001
$0 < x < 2$	Prev. Value	Midpt. 1/4-3/4	E	E	(1)	C	S	4	0.97	0.0547
			E	N	(1)	C	S	4	0.89	0.0475
$0 < x < 2$	Prev. Value	Midpt. Linear	E	E	(1)	C	S	4	0.52	0.0291
			E	N	(1)	C	S	5	0.47	0.0249
$0 < x < 2$	Prev. Value	Linear Midpt.	E	E	(1)	C	S	4	0.26	0.0130
			E	N	(1)	C	S	4	0.22	0.0119
$0 < x < 2$	Prev. Value	Linear 1/4-3/4	E	E	(1)	C	S	4	0.71	0.0424
			E	N	(1)	C	S	4	0.69	0.0411
$0 < x < 2$	Prev. Value	Linear Linear	E	E	(1)	C	S	4	0.26	0.0137
			E	N	(1)	C	S	4	0.27	0.0153
$0 < x < 2$	Avg. Value	Midpt. Midpt.	E	E	(1)	C	S	6	0.00	0.0001
			E	N	(1)	C	S	5	0.00	0.0002
$0 < x < 2$	Avg. Value	Midpt. 1/4-3/4	E	E	(1)	C	S	5	0.97	0.0610
			E	N	(1)	C	S	5	0.89	0.0594
$0 < x < 2$	Avg. Value	Midpt. Linear	E	E	(1)	C	S	5	0.51	0.0308
			E	N	(1)	C	S	5	0.47	0.0326
$0 < x < 2$	Avg. Value	Linear Midpt.	E	E	(1)	C	S	6	0.26	0.0198
			E	N	(1)	C	S	6	0.22	0.0179
$0 < x < 2$	Avg. Value	Linear 1/4-3/4	E	E	(1)	C	S	5	0.71	0.0448
			E	N	(1)	C	S	5	0.69	0.0503
$0 < x < 2$	Avg. Value	Linear Linear	E	E	(1)	C	S	5	0.26	0.0163
			E	N	(1)	C	S	5	0.27	0.0180

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: (1) - See equation (4.10)

Table 18. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAIN THREE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 5$	Prev. Value	Midpt. Midpt.	E	E	(1)	C[a]	S	5	0.00	0.0002
			E	N	(1)	C[a]	S	6	0.04	0.0046
$0 < x < 5$	Prev. Value	Midpt. 1/4-3/4	E	E	(1)	C	S	3	1.43	0.0948
			E	N	(1)	C	S	7	1.55	0.2216
$0 < x < 5$	Prev. Value	Midpt. Linear	E	E	(1)	C[b]	S	2	0.53	0.0230
			E	N	(1)	C	S	5	0.83	0.0911
$0 < x < 5$	Prev. Value	Linear Midpt.	E	E	(1)	C[a]	S	5	0.38	0.0352
			E	N	(1)	C	S	5	0.43	0.0408
$0 < x < 5$	Prev. Value	Linear 1/4-3/4	E	E	(1)	C[b]	S	2	0.80	0.0400
			E	N	(1)	C	S	3	1.08	0.0717
$0 < x < 5$	Prev. Value	Linear Linear	E	E	(1)	C[b]	S	2	0.15	0.0063
			E	N	(1)	C[b]	S	2	0.15	0.0068

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Convergence criterion changed to 0.0000001 in order to allow for additional iterations and a more efficient approximation

b - Convergence criterion kept at .0001 as use of tighter criterion in note *a* leads to a less efficient or nonconvergent approximation.

(1) - See equation (4.10)

Table 19. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.1) OVER DOMAIN THREE (CONT.)

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 5$	Avg. Value	Midpt. Midpt.	E	E	(1)	C[a]	S	5	0.00	0.0001
			E	N	(1)	C[a]	S	9	0.04	0.0069
$0 < x < 5$	Avg. Value	Midpt. 1/4-3/4	E	E	(1)	C[b]	S	2	2.95	0.1276
			E	N	(1)	C	S	9	1.55	0.2782
$0 < x < 5$	Avg. Value	Midpt. Linear	E	E	(1)	C[a]	S	7	0.76	0.1040
			E	N	(1)	C	S	6	0.83	0.1049
$0 < x < 5$	Avg. Value	Linear Midpt.	E	E	(1)	C[a]	S	7	0.38	0.0528
			E	N	(1)	C	S	5	0.42	0.0422
$0 < x < 5$	Avg. Value	Linear 1/4-3/4	E	E	(1)	C[a]	S	8	1.05	0.1644
			E	N	(1)	C	S	4	1.07	0.0818
$0 < x < 5$	Avg. Value	Linear Linear	E	E	(1)	C[c]	S	2	3.35	0.1559
			E	N	(1)	C[a]	S	6	0.42	0.0519

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Convergence criterion changed to 0.0000001 in order to allow for additional iterations and a more efficient approximation

b - Convergence criterion kept at .0001 as use of tighter criterion in note *a* leads to a less efficient or nonconvergent approximation.

c - Tightening of convergence criterion had no effect

(1) - See equation (4.10)

Table 20. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAIN ONE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i')_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 1$	Prev. Value	Midpt. Midpt.	E	E	L	C	S	5	1.05	0.0663
			E	E	M	C[a]	S	11	----	-----
			E	E	R	C[a]	S	5	----	-----
			E	N	L	C[a]	S	6	----	-----
$0 < x < 1$	Prev. Value	Midpt. 1/4-3/4	E	E	L	C	S	5	16.1	1.0700
			E	E	M	C[a]	S	12	----	-----
			E	E	R	C[a]	S	5	----	-----
			E	N	L	C[a]	S	6	----	-----
$0 < x < 1$	Prev. Value	Midpt. Linear	E	E	L	C	S	5	9.28	0.6487
			E	E	M	C[a]	S	12	----	-----
			E	E	R	C[a]	S	5	----	-----
			E	N	L	C[a]	S	6	----	-----
$0 < x < 1$	Prev. Value	Linear Midpt.	E	E	L	C	S	5	3.61	0.2163
			E	E	M	C[a]	S	11	----	-----
			E	E	R	C[a]	S	5	----	-----
			E	N	L	C[a]	S	6	----	-----
$0 < x < 1$	Prev. Value	Linear 1/4-3/4	E	E	L	C	S	5	11.4	0.6809
			E	E	M	C[a]	S	12	----	-----
			E	E	R	C[a]	S	5	----	-----
			E	N	L	C[a]	S	6	----	-----
$0 < x < 1$	Prev. Value	Linear Linear	E	E	L	C	S	5	4.64	0.3087
			E	E	M	C[a]	S	12	----	-----
			E	E	R	C[a]	S	5	----	-----
			E	N	L	C[a]	S	6	----	-----

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Converges to another solution of the differential equation

Table 21. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAIN ONE (CONT.)

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s		$(u_i')_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
			Lt	Rt						
$0 < x < 1$	Avg. Value	Midpt. Midpt.	E	E	L	C	S	6	1.07	0.0852
			E	E	M	C[a]	S	15	----	-----
			E	E	R	C[a]	S	7	----	-----
			E	N	L	C[a]	S	7	----	-----
$0 < x < 1$	Avg. Value	Midpt. 1/4-3/4	E	E	L	C	S	6	16.1	1.3399
			E	E	M	C[a]	S	16	----	-----
			E	E	R	C[a]	S	7	----	-----
			E	N	L	C[a]	S	7	----	-----
$0 < x < 1$	Avg. Value	Midpt. Linear	E	E	L	C	S	6	9.30	0.7422
			E	E	M	C[a]	S	15	----	-----
			E	E	R	C[a]	S	7	----	-----
			E	N	L	C[a]	S	7	----	-----
$0 < x < 1$	Avg. Value	Linear Midpt.	E	E	L	C	S	7	3.61	0.3486
			E	E	M	C[a]	S	15	----	-----
			E	E	R	C[a]	S	6	----	-----
			E	N	L	C[a]	S	7	----	-----
$0 < x < 1$	Avg. Value	Linear 1/4-3/4	E	E	L	C	S	7	11.4	0.9091
			E	E	M	C[a]	S	16	----	-----
			E	E	R	C[a]	S	7	----	-----
			E	N	L	C[a]	S	7	----	-----
$0 < x < 1$	Avg. Value	Linear Linear	E	E	L	C	S	7	4.60	0.4440
			E	E	M	C[a]	S	15	----	-----
			E	E	R	C[a]	S	7	----	-----
			E	N	L	C[a]	S	7	----	-----

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Converges to another solution of the differential equation

Table 22. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAIN TWO

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 2$	Prev. Value	Midpt. Midpt.	E	E	(1)	C	S	4	4.62[e]	1.1534
			E	N	(1)	C	S	3	0.85	0.1777
$0 < x < 2$	Prev. Value	Midpt. 1/4-3/4	E	E	(1)	C	CS[b]	4	56.6[c]	15.252
			E	N	(1)	----	CS[b]	200+	*d*	-----
$0 < x < 2$	Prev. Value	Midpt. Linear	E	E	(1)	C	S	4	32.0[e]	9.0524
			E	N	(1)	----	CS[b]	200+	*d*	-----
$0 < x < 2$	Prev. Value	Linear Midpt.	E	E	(1)	C	S	3	9.68[f]	2.0618
			E	N	(1)	C	S	5	273[d]	92.669
$0 < x < 2$	Prev. Value	Linear 1/4-3/4	E	E	(1)	C	CS[b]	4	38.9[e]	10.626
			E	N	(1)	C	S	5	188[d]	64.094
$0 < x < 2$	Prev. Value	Linear Linear	E	E	(1)	C	S	4	16.0[f]	4.3182
			E	N	(1)	C	S	3	3.28[e]	0.6880

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES:
a - Majority of error occurs over $0 < x < 0.2$
b - Divergence results when certain number of elements are utilized
c - Majority of error occurs over $0 < x < 0.6$
d - Provides a reasonable approximation over $1 < x < 2$
e - Majority of error occurs over $0 < x < 0.5$
f - Majority of error occurs over $0 < x < 0.3$
(1) - See equation (4.11)

Table 23. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAIN TWO (CONT.)

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 2$	Avg. Value	Midpt. Midpt.	E	E	(1)	C	S	4	4.70[a]	1.2206
			E	N	(1)	C	S	4	0.92	0.2580
$0 < x < 2$	Avg. Value	Midpt. 1/4-3/4	E	E	(1)	C	S	4	55.5[b]	14.952
			E	N	(1)	C	CS[c]	129	487[d]	4089.7
$0 < x < 2$	Avg. Value	Midpt. Linear	E	E	(1)	C	S	4	31.5[e]	9.1124
			E	N	(1)	C	S	21	593[d]	811.74
$0 < x < 2$	Avg. Value	Linear Midpt.	E	E	(1)	C	S	4	9.74[f]	2.4966
			E	N	(1)	C	S	5	275[d]	88.739
$0 < x < 2$	Avg. Value	Linear 1/4-3/4	E	E	(1)	C	S	4	38.8[e]	11.233
			E	N	(1)	C	S	5	192[d]	68.267
$0 < x < 2$	Avg. Value	Linear Linear	E	E	(1)	C	S	4	16.0[f]	4.0571
			E	N	(1)	C	S	4	3.12[a]	0.8615

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES:
a - Majority of error occurs over $0 < x < 0.2$
b - Majority of error occurs over $0 < x < 0.6$
c - Divergence results when certain number of elements are utilized
d - Provides a reasonable approximation over $1 < x < 2$
e - Majority of error occurs over $0 < x < 0.5$
f - Majority of error occurs over $0 < x < 0.3$
(1) - See equation (4.11)

Table 24. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAIN THREE

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i')_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 5$	Prev. Value	Midpt. Midpt.	E	E	(1)	C[a]	S	5	0.16	0.0875
			E	N	(1)	C[a]	S	11	0.38	0.4189
$0 < x < 5$	Prev. Value	Midpt. 1/4-3/4	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	Midpt. Linear	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	Linear Midpt.	E	E	(1)	C[b]	CS[c]	2	2692[d]	636.11
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	Linear 1/4-3/4	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Prev. Value	Linear Linear	E	E	(1)	C[a]	S	5	0.88	0.5378
			E	N	(1)	C[b]	S	3	7.48[e]	2.5872

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Convergence criterion changed to 0.0000001 in order to allow for additional iterations and a more efficient approximation

b - Convergence criterion kept at .0001 as use of tighter criterion in note *a* leads to a less efficient or nonconvergent approximation.

:.c - Divergence results when certain number of elements are used

d - Provided an adequate approximation over $2.5 < x < 5.0$

e - Majority of error occurs over $0 < x < 0.5$ where the values of $u < 2.0$

(1) - See equation (4.12)

Table 25. SOLUTION PROCEDURES AND RESULTS USING QUASI-LINEARIZATION TO SOLVE EQUATION (4.2) OVER DOMAIN THREE (CONT.)

Domain	Solution Procedure					Results				
	Iter. Strat.	Interp. Strat.	B. C.'s Lt	B. C.'s Rt	$(u_i^*)_0$	Conv.	Stab.	# of Iter.	% Dif	CPU* (sec)
$0 < x < 5$	Avg. Value	Midpt. Midpt.	E	E	(1)	C[a]	S	4	11.1[b]	5.2562
			E	N	(1)	C[a]	S	4	10.6[b]	4.8829
$0 < x < 5$	Avg. Value	Midpt. 1/4-3/4	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	Midpt. Linear	E	E	(1)	D	U	-----	----	-----
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	Linear Midpt.	E	E	(1)	C[c]	S	2	1143[d]	262.37
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	Linear 1/4-3/4	E	E	(1)	C[c]	S	2	672[e]	149.82
			E	N	(1)	D	U	-----	----	-----
$0 < x < 5$	Avg. Value	Linear Linear	E	E	(1)	C[a]	S	7	1.69	1.4075
			E	N	(1)	C[a]	S	31	1.46	5.2343

LEGEND: E = essential boundary condition; N = natural boundary condition; L = left essential boundary condition; R = right essential boundary condition; M = average of L and R; C = convergence; D = divergence S = unconditional stability; CS = conditional stability; U = unstable

NOTES: *a* - Convergence criterion changed to 0.0000001 in order to allow for additional iterations and a more efficient approximation

b - Majority of error occurs over $0 < x < 0.5$ where the values of $u < 2.0$

c - Convergence criterion kept at .0001 as use of tighter criterion in note *a* leads to a less efficient or nonconvergent approximation.

d - Provided an adequate approximation over $2.0 < x < 5.0$

e - Provided an adequate approximation over $1.5 < x < 5.0$

(1) - See equation (4.12)

Last of all, a general comment is required with respect to the average percent difference values in Table 22 through Table 25. As previously noted in IV.C.3, the value of the dependent variable for equation (4.2) is very small over the first part of the domain. Thus, errors in the approximation which are on the same order of magnitude as the value of the dependent variable result in large percent difference values. Therefore, average percent difference values of three or more are amplified with a superscript

which advises the reader as to the true accuracy of the approximation. The term 'a majority of error' in each of these accompanying notes means that the percent difference at each node is 10 percent or more over the indicated domain. The effect of each parameter on the overall performance of the various solution procedures is now evaluated.

b. Boundary Conditions

Except in a few isolated instances, the accuracy and number of iterations to convergence for a specific combination of iteration and interpolation strategies was unaffected by the boundary condition combination utilized. The one exception to this is shown in Table 20 and Table 21 where the use of a E-N combination caused the solution procedure to converge to a second solution of the differential equation. But, when the initial iteration strategy was changed from using the value of the left boundary condition to the strategy defined by equation (4.12), valid approximations of the solution $u = 10x^3$ were obtained by all procedures using an E-N combination. Thus, it appears that this linearization technique is quite favorable to both E-E and E-N type boundary value problems, provided that a valid initial iteration strategy is utilized.

c. Initial Iteration Strategy

Over domain one for equation (4.1), use of the three initial iteration strategies described in III.C.1 provided convergence with nearly the same number of iterations. Over the same domain for equation (4.2), only the use of the left essential boundary condition as values for the initial iteration strategy yielded a convergent approximation of the original exact solution. The other two strategies, when utilized with an E-E boundary condition combination, converged to a second solution of equation (4.2). The reason that the use of the left essential boundary condition for the initial iteration strategy outperformed the other two strategies is because it has a value of zero. With $(u_i^*)_0 = 0$, the first iteration solves the differential equation neglecting the effect of the nonlinear term, as $\mathbf{F}^*$ and $\mathbf{L}^*$ are zero. Thus, the values of $(u_i^*)_1$ utilized in the next iteration are very close to values of $(u_i^*)_0$ that would have been obtained by neglecting the u^2 term and integrating the differential equation twice with the imposition of boundary conditions. From Figure 17 on page 77, it can be seen that the u'' term dominates over a majority of the domain. Thus, if equation (4.2) had been analyzed as order two over this domain and the initial iteration strategy developed by neglecting the u^2 term, similar results using one less iteration would probably have been obtained.

For equation (4.1) over domain two, the initial iteration strategies defined by equations (4.9) and (4.10) were both used in the analysis to determine which was most effective. As in the classical linearization results, they both provided almost identical

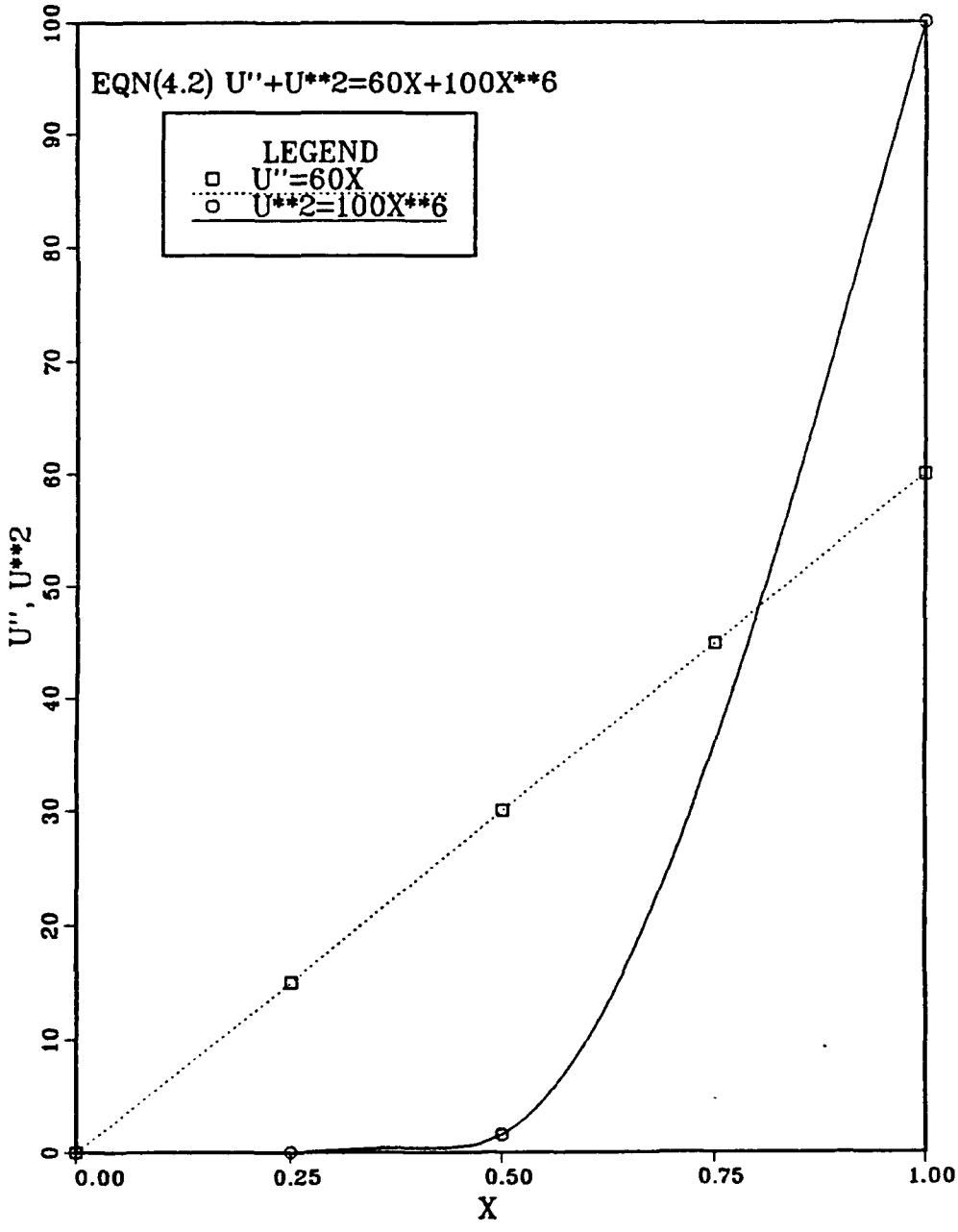


Figure 17. Dominance of Terms in Equation (4.2) Over Domain One

numerical approximations, and the use of equation (4.10) again resulted in convergence with less iterations. Likewise, equations (4.11) and (4.12) were both utilized as initial iteration strategies for equation (4.2) over domain two. They both provided the same accuracy in the approximation, but the use of equation (4.11), which divides the dominance equally over the domain, converged using less iterations as was originally expected. The use of equations (4.10) and (4.12) for the initial iteration strategies of equations (4.1) and (4.2), respectively, over domain three, lead to accurate approximations for those solution procedures which did converge.

d. Subsequent Iteration Strategy

The use of either the previous value or average value iteration strategy, in general, had no effect on the number of iterations required for convergence. This is most likely due to the quadratic rate of convergence guaranteed by the quasilinearization method, as previously mentioned in IV.D.2.a.

e. Interpolation Strategy

The quasilinearization method requires the use of two interpolation strategies; one for the linearization vector and one for the linearization matrix. In the interpolation strategy column of Table 15 through Table 25, the upper strategy is for the linearization matrix and the lower one is for the linearization vector. A general trend in the accuracy provided by the various combinations of interpolation strategies is evident. The different combinations are ranked from least to most accurate in the following list, where the first strategy indicated is for the linearization matrix and the second is for the linearization vector.

- Midpoint;1/4-3/4
- Linear;1/4-3/4
- Midpoint;Linear
- Linear;Midpoint
- Linear;Linear
- Midpoint;Midpoint

Two conclusions can be drawn from the above list. First is that the most accurate interpolation strategies utilize the same interpolation technique for both linearization integrals. Thus, if a 1/4-3/4 interpolation strategy for the linearization matrix had been developed, there is a good possibility that an overall 1/4-3/4;1/4-3/4 interpolation strategy would have provided accurate approximations. Second, the least

refined interpolation strategy, namely midpoint;midpoint, provides the most accurate approximations. This result is not very surprising, as in many situations, the simplest method provides the best results.

f. Overall Performance

In almost all cases, the solution procedure consisting of a previous iteration and a midpoint;midpoint interpolation strategy provided the most efficient approximations regardless of the function order of the equation or the boundary conditions imposed. The only case where this procedure faltered slightly was in approximating equation (4.2) over domain two. But, as shown in note *a* of Table 22 and Table 23, it only had a problem approximating the solution over that part of the domain where $u < 0.1$. The solution procedure utilizing a previous value iteration and a linear;linear interpolation strategy was not quite as efficient as the above solution procedure, but performed in an acceptable manner.

3. Conclusions

Quasilinearization provides a viable method of approximating nonlinear differential equations that contain the u^2 term, regardless of the function order of the equation and the nature of the boundary conditions imposed. The use of a previous value iteration and either a midpoint;midpoint or linear;linear interpolation strategy should provide an accurate approximation with a minimum number of iterations, provided that the initial iteration strategy is adeptly chosen. The actual shape of the solution curve and the discretization invoked, i.e., the number of elements, are the two factors most likely to determine which interpolation technique provides the more accurate approximation.

E. FINAL REMARKS

An overall solution procedure involving quasilinearization combined with a previous value iteration and either a midpoint;midpoint or linear;linear interpolation strategy provides excellent approximations of second order, nonlinear, one dimensional, differential equations which contain the u^2 nonlinear term. As the u^2 term has a more nonlinear nature than some of the other nonlinear terms encountered, i.e., $u''u$, $(u')^2$, etc.; it is felt that this solution procedure should provide viable approximations of many nonlinear, second order differential equations. It cannot be overemphasized that the success or failure of the above solution procedure depends greatly on the initial iteration strategy developed. Thus, utilization of this solution procedure requires that the user have an in-depth understanding of the physics involved in the system being analyzed.

This research has provided a fundamental baseline for the future investigation of techniques for solving nonlinear differential equations. The following steps provide a logical progression for determining the actual capabilities of a Galerkin FEM solution procedure utilizing quasilinearization.

- Conduct an analysis of second order, one dimensional, nonlinear, differential equations which contain nonlinear terms other than u^2. These nonlinear equations should be of an engineering nature for which experimental data exists to allow for a confirmation of the results developed by the mathematical model.
- Investigate the ability of this solution method to solve one dimensional, nonlinear, fourth order differential equations. This requires some modification of the interpolation strategies as the Galerkin FEM must utilize cubic shape functions for developing a fourth order differential equation approximation.
- Extrapolate the concepts and principals developed by this and future research to the solution of two dimensional, second and fourth order, nonlinear differential equations.

APPENDIX A. FORCING FUNCTION FORMULATION STRATEGIES

On an elemental level, $\int_0^2 G(6x)dx$. becomes $\int_0^{l_e} g(6(\alpha_i + \xi))d\xi$ where

- ξ is the local element coordinate, $0 \le \xi \le l_e$
- α_i is the sum of all element lengths prior to the element being evaluated. For equal length elements, $\alpha_i = (i-1)l_e$ where i is the element number.

Midpoint Lumped Approximation

This method evaluates $6(\alpha + \xi)$ at the midpoint of the element, $\xi = \frac{l_e}{2}$ and brings it outside the integral as a constant yielding

$$\mathbf{f} = 6\left(\alpha + \frac{l_e}{2}\right)\int_0^{l_e}\begin{bmatrix} 1 - \frac{\xi}{l_e} \\ \frac{\xi}{l_e} \end{bmatrix} d\xi$$
$$= 3l_e\begin{bmatrix} \alpha + \frac{l_e}{2} \\ \alpha + \frac{l_e}{2} \end{bmatrix} \qquad (A.1)$$

Quarter/Three Quarter Lumped Approximation

This method takes $6(\alpha + \xi)$ inside the shape function vector yielding

$$\mathbf{f} = \int_0^{l_e}\begin{bmatrix} (6(\alpha + \xi))\left(1 - \frac{\xi}{l_e}\right) \\ (6(\alpha + \xi))\left(\frac{\xi}{l_e}\right) \end{bmatrix} d\xi \qquad (A.2.a)$$

The first $6(\alpha + \xi)$ term is evaluated at $\xi = \frac{l_e}{4}$ and the second term at $\xi = \frac{3l_e}{4}$, yielding

$$\mathbf{f} = \int_0^{l_e} \begin{bmatrix} 6\left(\alpha + \frac{l_e}{4}\right)\left(1 - \frac{\xi}{l_e}\right) \\ 6\left(\alpha + \frac{3l_e}{4}\right)\left(\frac{\xi}{l_e}\right) \end{bmatrix} d\xi$$

$$= 3l_e \begin{bmatrix} \alpha + \frac{l_e}{4} \\ \alpha + \frac{3l_e}{4} \end{bmatrix} \qquad (A.2.b)$$

Consistent Evaluation

This method calculates the exact value of the Galerkin excitation integral yielding

$$\mathbf{f} = 6\int_0^{l_e} \begin{bmatrix} 1 - \frac{\xi}{l_e} \\ \frac{\xi}{l_e} \end{bmatrix} (\alpha + \xi) d\xi$$

$$= 6\int_0^{l_e} \begin{bmatrix} \alpha - \frac{\alpha\xi}{l_e} + \xi - \frac{\xi^2}{l_e} \\ \frac{\alpha\xi}{l_e} + \frac{\xi^2}{l_e} \end{bmatrix} d\xi$$

$$= 6\begin{bmatrix} \alpha\xi - \frac{\alpha\xi^2}{2l_e} + \frac{\xi^2}{2} - \frac{\xi^3}{3l_e} \\ \frac{\alpha\xi^2}{2l_e} + \frac{\xi^3}{3l_e} \end{bmatrix}_0^{l_e} \qquad (A.3)$$

$$= 6\begin{bmatrix} \frac{\alpha l_e}{2} + \frac{l_e^2}{6} \\ \frac{\alpha l_e}{2} + \frac{l_e^2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 3\alpha l_e + l_e^2 \\ 3\alpha l_e + 2l_e^2 \end{bmatrix}$$

APPENDIX B. PROGRAM LISTINGS AND RESULTS FOR THE LINEAR APPLICATION OF THE GALERKIN FEM

```
C     ************************************************************************
C     *                          PROGRAM LIN1                                *
C     *                                                                      *
C     * THIS PROGRAM SOLVES THE DIFFERENTIAL EQUATION U'' = 2., 0<X<2        *
C     * U(0)=0; U'(2)=4 WITH UEXACT=X**2.                                    *
C     ************************************************************************

110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP

C     INPUT NUMBER OF ELEMENTS AND TOTAL LENGTH OF DOMAIN

120   PRINT*,'INPUT NUMBER OF ELEMENTS DESIRED AND TOTAL LENGTH.'
130   READ(6,*) NEL,TLEN

C     CALCULATE NUMBER OF NODAL POINTS AND DEFINE LEFT BOUNDARY OF
C     DOMAIN

135   NSNP=NEL+1
150   COORD(1)=0.

C     DETERMINE ELEMENT LENGTH, LOCAL TO GLOBAL NODAL POINT
C     CORRESPONDENCE, AND X-COORDINATE OF EACH NODAL POINT

155   ELEN=TLEN/FLOAT(NEL)
160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
164     COORD(IEL+1)=COORD(IEL)+ELEN
169   CONTINUE

C     CALL SUBROUTINE SYM1A TO DETERMINE A MATRIX AND F VECTOR AND SOLVE
C     THE LINEAR SYSTEM OF EQUATIONS AU = F

170   CALL SYM1A

C     CALL SUBROUTINE UX2EXT TO DETERMINE EXACT SOLUTION

190   CALL UX2EXT

C     CALL SUBROUTINE OUTLIN TO OUTPUT RESULTS

200   CALL OUTLIN
210   END
```

```
C     ************************************************************************
C     *                        SUBROUTINE SYM1A                              *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND F VECTOR FOR MAIN          *
C     * PROGRAM LIN1 AND SOLVES THE LINEAR SET OF EQUATIONS AU = F.          *
C     ************************************************************************

100   SUBROUTINE SYM1A


110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP
120   DIMENSION AE(2,2), FS1E(2), FS2E(2), WKAREA(40600)

C     ZERO OUT A MATRIX AND F VECTOR

140   DO 210 IZ = 1,NSNP
150      FT(IZ) = 0.
160      DO 200 JZ = 1,NSNP
170           A(IZ,JZ) = 0.
200      CONTINUE
210   CONTINUE
213   ALPHA=0.

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

215   DO 375 IEL=1,NEL

C     DETERMINE ELEMENTAL A MATRIX AND ELEMENTAL F VECTOR

220     AE(1,1)=1./ELEN
230     AE(1,2)=(-1./ELEN)
240     AE(2,1)=AE(1,2)
250     AE(2,2)=AE(1,1)
260     FS1E(1)=ELEN
265     FS1E(2)=ELEN

C     DISTRIBUTE AE MATRICES AND FE VECTORS INTO SYSTEM A MATRIX AND F
C     VECTOR

300       DO 370 II=1,2
310         DO 350 JJ=1,2
320           IN=ICORR(IEL,II)
330           JN=ICORR(IEL,JJ)
340           A(IN,JN)=A(IN,JN) - AE(II,JJ)
350         CONTINUE
360         FT(IN)=FS1E(II) + FT(IN)
370       CONTINUE
372     ALPHA=ALPHA + ELEN
375   CONTINUE

C     IMPOSE KINEMATIC AND NATURAL BOUNDARY CONDITIONS

376   A(1,1)=1.
377   A(1,2)=0.
378   FT(1)=0.
```

```
379    FT(NSNP)=FT(NSNP)-4.
380    M=1
390    IDGT=3
400    IQ=100

C      CALL SUBROUTINE LEQT2F TO SOLVE AU = F

410    CALL LEQT2F(A,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
420    DO 440 NEW=1,NSNP
430      U(NEW)=FT(NEW)
440    CONTINUE
450    RETURN
460    END
```

```
C     ************************************************************************
C     *                          SUBROUTINE UX2EXT                           *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE VALUE OF U=X**2 AT THE SPECIFIED        *
C     * NODAL POINTS FOR MAIN PROGRAM LIN1.                                  *
C     ************************************************************************

100   SUBROUTINE UX2EXT
110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = COORD(NN)**2
150   CONTINUE
160   RETURN
170   END
```

SOLUTION OF U'' = 2. USING CONSISTENT FORCING FUNCTION

X-COORD	U EXACT	U FEM	% DIFF
0.000	0.0000	0.0000	0.0
0.200	0.0400	0.0400	0.0
0.400	0.1600	0.1600	0.0
0.600	0.3600	0.3600	0.0
0.800	0.6400	0.6400	0.0
1.000	1.0000	1.0000	0.0
1.200	1.4400	1.4400	0.0
1.400	1.9600	1.9600	0.0
1.600	2.5600	2.5600	0.0
1.800	3.2400	3.2400	0.0
2.000	4.0000	4.0000	0.0

```
C     ************************************************************************
C     *                          PROGRAM LIN2                                *
C     *                                                                      *
C     * THIS PROGRAM SOLVES THE DIFFERENTIAL EQUATION U'' = 6X, 0<X<2        *
C     * U(0)=0; U'(2)=12 WITH UEXACT=U**3.                                   *
C     ************************************************************************

110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP

C     INPUT NUMBER OF ELEMENTS AND TOTAL LENGTH OF DOMAIN

120   PRINT*,'INPUT NUMBER OF ELEMENTS DESIRED AND TOTAL LENGTH.'
130   READ(6,*) NEL,TLEN

C     CALCULATE NUMBER OF NODAL POINTS AND DEFINE LEFT BOUNDARY OF
C     DOMAIN

135   NSNP=NEL+1
150   COORD(1)=0.

C     DETERMINE ELEMENT LENGTH, LOCAL TO GLOBAL NODAL POINT
C     CORRESPONDENCE, AND X-COORDINATE OF EACH NODAL POINT

155   ELEN=TLEN/FLOAT(NEL)
160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
164     COORD(IEL+1)=COORD(IEL)+ELEN
169   CONTINUE

C     CALL SUBROUTINE SYM2A TO DETERMINE A MATRIX AND F VECTOR AND SOLVE
C     THE LINEAR SYSTEM OF EQUATIONS AU = F

170   CALL SYM2A

C     CALL SUBROUTINE UX3EXT TO DETERMINE EXACT SOLUTION

190   CALL UX3EXT

C     CALL SUBROUTINE OUTLIN TO OUTPUT RESULTS

200   CALL OUTLIN
210   END
```

```
C     ************************************************************************
C     *                        SUBROUTINE SYM2A                              *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND F VECTOR FOR MAIN          *
C     * PROGRAM LIN2 AND SOLVES THE LINEAR SET OF EQUATIONS AU = F.          *
C     ************************************************************************

100   SUBROUTINE SYM2A
110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP
120   DIMENSION AE(2,2), FS1E(2), FS2E(2), WKAREA(40600)

C     ZERO OUT A MATRIX AND F VECTOR

140   DO 210 IZ = 1,NSNP
150      FT(IZ) = 0.
160      DO 200 JZ = 1,NSNP
170           A(IZ,JZ) = 0.
200      CONTINUE
210   CONTINUE
213   ALPHA=0.
214   PRINT*,'WHAT TYPE OF FORCING FUNCTION IS TO BE USED?'
215   PRINT*,'MIDPOINT = 1; 1/4 -3/4 APPROX = 2; CONSISTENT = 3'
216   READ(6,*) NFF

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

217   DO 375 IEL=1,NEL

C     DETERMINE ELEMENTAL A MATRIX AND ELEMENTAL F VECTOR

220     AE(1,1)=1./ELEN
230     AE(1,2)=(-1./ELEN)
240     AE(2,1)=AE(1,2)
250     AE(2,2)=AE(1,1)
263     IF (NFF.EQ.1) THEN
264       FS1E(1)=3.*ELEN*(ALPHA+ELEN/2.)
265       FS1E(2)=FS1E(1)
266     ELSEIF (NFF.EQ.2) THEN
267       FS1E(1)=3.*ELEN*(ALPHA+ELEN/4.)
268       FS1E(2)=3.*ELEN*(ALPHA+3.*ELEN/4.)
269     ELSE
270       FS1E(1)=3.*ALPHA*ELEN+ELEN**2
271       FS1E(2)=3.*ALPHA*ELEN+2.*(ELEN**2)
272     ENDIF

C     DISTRIBUTE AE MATRICES AND FE VECTORS INTO SYSTEM A MATRIX AND F
C     VECTOR

300       DO 370 II=1,2
310         DO 350 JJ=1,2
320           IN=ICORR(IEL,II)
330           JN=ICORR(IEL,JJ)
340           A(IN,JN)=A(IN,JN) - AE(II,JJ)
350         CONTINUE
360         FT(IN)=FS1E(II) + FT(IN)
```

```
370         CONTINUE
372       ALPHA=ALPHA + ELEN
375     CONTINUE

C       IMPOSE KINEMATIC AND NATURAL BOUNDARY CONDITIONS

376     A(1,1)=1.
377     A(1,2)=0.
378     FT(1)=0.
379     FT(NSNP)=FT(NSNP)-12.
380     M=1
390     IDGT=3
400     IQ=100

C       CALL SUBROUTINE LEQT2F TO SOLVE AU = F

410     CALL LEQT2F(A,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
420     DO 440 NEW=1,NSNP
430       U(NEW)=FT(NEW)
440     CONTINUE
450     RETURN
460     END
```

```
C     ************************************************************************
C     *                        SUBROUTINE UX3EXT                             *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE VALUE OF U=X**3 AT THE SPECIFIED        *
C     * NODAL POINTS FOR MAIN PROGRAM LIN2.                                  *
C     ************************************************************************

100   SUBROUTINE UX3EXT
110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = COORD(NN)**3
150   CONTINUE
160   RETURN
170   END
```

SOLUTION OF U" = 6X USING LUMPED MIDPOINT FORCING FUNCTION

X-COORD	U EXACT	U FEM	% DIFF
0.000	0.0000	0.0000	0.0
0.200	0.0080	0.0120	50.0
0.400	0.0640	0.0720	12.5
0.600	0.2160	0.2280	5.6
0.800	0.5120	0.5280	3.1
1.000	1.0000	1.0200	2.0
1.200	1.7280	1.7520	1.4
1.400	2.7440	2.7720	1.0
1.600	4.0960	4.1280	0.8
1.800	5.8320	5.8680	0.6
2.000	8.0000	8.0400	0.5

SOLUTION OF U" = 6X USING 1/4 - 3/4 LUMPED FORCING FUNCTION

X-COORD	U EXACT	U FEM	% DIFF
0.000	0.0000	0.0000	0.0
0.200	0.0080	0.0060	-25.0
0.400	0.0640	0.0600	-6.3
0.600	0.2160	0.2100	-2.8
0.800	0.5120	0.5040	-1.6
1.000	1.0000	0.9900	-1.0
1.200	1.7280	1.7160	-0.7
1.400	2.7440	2.7300	-0.5
1.600	4.0960	4.0800	-0.4
1.800	5.8320	5.8140	-0.3
2.000	8.0000	7.9800	-0.2

SOLUTION OF U" = 6X USING CONSISTENT FORCING FUNCTION

X-COORD	U EXACT	U FEM	% DIFF
0.000	0.0000	0.0000	0.0
0.200	0.0080	0.0080	0.0
0.400	0.0640	0.0640	0.0
0.600	0.2160	0.2160	0.0
0.800	0.5120	0.5120	0.0
1.000	1.0000	1.0000	0.0
1.200	1.7280	1.7280	0.0
1.400	2.7440	2.7440	0.0
1.600	4.0960	4.0960	0.0
1.800	5.8320	5.8320	0.0
2.000	8.0000	8.0000	0.0

```
C     ************************************************************************
C     *                          PROGRAM LIN3                                *
C     *                                                                      *
C     * THIS PROGRAM SOLVES THE DIFFERENTIAL EQUATION U'' = 12X**2.          *
C     * 0 < X < 2  U(0)=0; U'(2)=32 WITH UEXACT=X**4                         *
C     ************************************************************************

110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP

C     INPUT NUMBER OF ELEMENTS AND TOTAL LENGTH OF DOMAIN

120   PRINT*,'INPUT NUMBER OF ELEMENTS DESIRED AND TOTAL LENGTH.'
130   READ(6,*) NEL,TLEN

C     CALCULATE NUMBER OF NODAL POINTS AND DEFINE LEFT BOUNDARY OF
C     DOMAIN

135   NSNP=NEL+1
150   COORD(1)=0.

C     DETERMINE ELEMENT LENGTH, LOCAL TO GLOBAL NODAL POINT
C     CORRESPONDENCE, AND X-COORDINATF OF EACH NODAL POINT

155   ELEN=TLEN/FLOAT(NEL)
160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
164     COORD(IEL+1)=COORD(IEL)+ELEN
169   CONTINUE

C     CALL SUBROUTINE SYM3A TO DETERMINE A MATRIX AND F VECTOR AND SOLVE
C     THE LINEAR SYSTEM OF EQUATIONS AU = F

170   CALL SYM3A

C     CALL SUBROUTINE UX4EXT TO DETERMINE EXACT SOLUTION

190   CALL UX4EXT

C     CALL SUBROUTINE OUTLIN TO OUTPUT RESULTS

200   CALL OUTLIN
210   END
```

```
C     ************************************************************************
C     *                          SUBROUTINE SYM3A                            *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND F VECTOR FOR MAIN          *
C     * PROGRAM LIN3 AND SOLVES THE LINEAR SET OF EQUATIONS AU = F.          *
C     ************************************************************************

100   SUBROUTINE SYM3A
110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP
120   DIMENSION AE(2,2), FS1E(2), FS2E(2), WKAREA(40600)

C     ZERO OUT A MATRIX AND F VECTOR

140   DO 210 IZ = 1,NSNP
150      FT(IZ) = 0.
160      DO 200 JZ = 1,NSNP
170           A(IZ,JZ) = 0.
200      CONTINUE
210   CONTINUE
213   ALPHA=0.
214   PRINT*,'WHAT TYPE OF FORCING FUNCTION IS TO BE USED?'
215   PRINT*,'MIDPOINT = 1; 1/4 -3/4 APPROX = 2; CONSISTENT = 3'
216   READ(6,*) NFF

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

217   DO 375 IEL=1,NEL

C     DETERMINE ELEMENTAL A MATRIX AND ELEMENTAL F VECTOR

220     AE(1,1)=1./ELEN
230     AE(1,2)=(-1./ELEN)
240     AE(2,1)=AE(1,2)
250     AE(2,2)=AE(1,1)
263     IF (NFF.EQ.1) THEN
264       FS1E(1)=6.*ELEN*(ALPHA+ELEN/2.)**2
265       FS1E(2)=FS1E(1)
266     ELSEIF (NFF.EQ.2) THEN
267       FS1E(1)=6.*ELEN*(ALPHA+ELEN/4.)**2
268       FS1E(2)=6.*ELEN*(ALPHA+3.*ELEN/4.)**2
269     ELSE
270       FS1E(1)=6.*(ALPHA**2)*ELEN+4.*ALPHA*(ELEN**2)+ELEN**3
271       FS1E(2)=6.*(ALPHA**2)*ELEN+8.*ALPHA*(ELEN**2)+3.*ELEN**3
272     ENDIF

C     DISTRIBUTE AE MATRICES AND FE VECTORS INTO SYSTEM A MATRIX AND F
C     VECTOR

300       DO 370 II=1,2
310         DO 350 JJ=1,2
320           IN=ICORR(IEL,II)
330           JN=ICORR(IEL,JJ)
340           A(IN,JN)=A(IN,JN) - AE(II,JJ)
350         CONTINUE
360         FT(IN)=FS1E(II) + FT(IN)
```

```
370         CONTINUE
372       ALPHA=ALPHA + ELEN
375     CONTINUE

C       IMPOSE KINEMATIC AND NATURAL BOUNDARY CONDITIONS

376     A(1,1)=1.
377     A(1,2)=0.
378     FT(1)=0.
379     FT(NSNP)=FT(NSNP)-32.
380     M=1
390     IDGT=3
400     IQ=100

C       CALL SUBROUTINE LEQT2F TO SOLVE AU = F

410     CALL LEQT2F(A,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
420     DO 440 NEW=1,NSNP
430       U(NEW)=FT(NEW)
440     CONTINUE
450     RETURN
460     END
```

```
C     ************************************************************************
C     *                          SUBROUTINE UX4EXT                           *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE VALUE OF U=X**4 AT THE SPECIFIED        *
C     * NODAL POINTS FOR MAIN PROGRAM LIN3.                                  *
C     ************************************************************************

100   SUBROUTINE UX4EXT
110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = COORD(NN)**4
150   CONTINUE
160   RETURN
170   END
```

SOLUTION OF U'' = 12X**2 USING MIDPOINT LUMPED FORCING FUNCTION

X-COORD	U EXACT	U FEM	% DIFF
0.000	0.0000	0.0000	0.0
0.200	0.0016	0.0184	1050.3
0.400	0.0256	0.0608	137.5
0.600	0.1296	0.1848	42.6
0.800	0.4096	0.4864	18.8
1.000	1.0000	1.1000	10.0
1.200	2.0736	2.1984	6.0
1.400	3.8416	3.9928	3.9
1.600	6.5536	6.7328	2.7
1.800	10.4976	10.7064	2.0
2.000	16.0000	16.2400	1.5

SOLUTION OF U'' = 12X**2 USING 1/4 - 3/4 LUMPED FORCING FUNCTION

X-COORD	U EXACT	U FEM	% DIFF
0.000	0.0000	0.0000	0.0
0.200	0.0016	0.0046	187.8
0.400	0.0256	0.0296	15.7
0.600	0.1296	0.1326	2.3
0.800	0.4096	0.4096	0.0
1.000	1.0000	0.9950	-0.5
1.200	2.0736	2.0616	-0.6
1.400	3.8416	3.8206	-0.5
1.600	6.5536	6.5216	-0.5
1.800	10.4976	10.4526	-0.4
2.000	16.0000	15.9400	-0.4

SOLUTION OF U" = 12X**2 USING CONSISTENT FORCING FUNCTION

X-COORD	U EXACT	U FEM	% DIFF
0.000	0.0000	0.0000	0.0
0.200	0.0016	0.0016	0.1
0.400	0.0256	0.0256	0.0
0.600	0.1296	0.1296	0.0
0.800	0.4096	0.4096	0.0
1.000	1.0000	1.0000	0.0
1.200	2.0736	2.0736	0.0
1.400	3.8416	3.8416	0.0
1.600	6.5536	6.5536	0.0
1.800	10.4976	10.4976	0.0
2.000	16.0000	16.0000	0.0

```
C     ************************************************************************
C     *                      SUBROUTINE OUTLIN                               *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE PERCENT ERROR BETWEEN THE EXACT         *
C     * AND FEM VALUES OF U AND OUTPUTS THE APPROPRIATE INFORMATION          *
C     * FOR MAIN PROGRAMS LIN|, LIN2, AND LIN3.                              *
C     ************************************************************************

100   SUBROUTINE OUTLIN
110   COMMON A(100,100),FT(100),U(100),UEXT(100),UDIF(100),COORD(100),
     :ELEN,ICORR(100,2),NEL,NSNP

C     COMPUTE PERCENT ERROR AT EACH NODAL POINT

120   DO 150 IK=2,NSNP
130     UDIF(IK)=100.*(U(IK)-UEXT(IK))/UEXT(IK)
150   CONTINUE
160   UDIF(1)=U(1)-UEXT(1)

C     OUTPUT RESULTS TO THE SCREEN AND TO A FILE

170   WRITE(6,180)
175   WRITE(30,180)
180   FORMAT(/,1X,'X-COORD',4X,'U EXACT',4X,'U FEM',6X,'% DIFF')
190   WRITE(6,200) (COORD(NP),UEXT(NP),U(NP),UDIF(NP), NP=1,NSNP)
195   WRITE(30,200) (COORD(NP),UEXT(NP),U(NP),UDIF(NP), NP=1,NSNP)
200   FORMAT(/,2X,F5.3,5X,F7.4,3X,F7.4,4X,F6.1)
230   RETURN
240   END
```

APPENDIX C. LINEARIZATION VECTORS FOR CONSTANT LINEARIZATION TECHNIQUE

For this analysis, $h(u^*)$ in equation (3.6) is replaced by $(u^*)^2$ and the element linearization vector becomes

$$\mathbf{f}^* = \int_0^{l_e} \mathbf{g}(u^*)^2 d\xi \tag{C.1}$$

The three approximations of this integral outlined in III.B.1 are now determined.

Midpoint Lumped Approximation

Replacing $h(u^*)$ in equation (3.7) with $(u^*)^2$ yields

$$\mathbf{f}^* = \left(\frac{(u_j^*)_i + (u_{j+1}^*)_i}{2} \right)^2 \begin{bmatrix} \frac{l_e}{2} \\ \frac{l_e}{2} \end{bmatrix} \tag{C.2}$$

1/4 - 3/4 Lumped Approximation

Replacing $h(u^*)$ in equation (3.8.d) with $(u^*)^2$ yields

$$\mathbf{f}^* = \frac{l_e}{2} \begin{bmatrix} \left(\frac{3}{4}(u_j^*)_i + \frac{1}{4}(u_{j+1}^*)_i \right)^2 \\ \left(\frac{1}{4}(u_j^*)_i + \frac{3}{4}(u_{j+1}^*)_i \right)^2 \end{bmatrix} \tag{C.3}$$

Linear Approximation

Replacing $h(u^*)$ with $(u^*)^2$ in equation (3.9.b) gives

$$\mathbf{f}^* = \int_0^{l_e} \begin{bmatrix} 1-\frac{\xi}{l_e} \\ \frac{\xi}{l_e} \end{bmatrix} \left((u_j^*)_i \left(1-\frac{\xi}{l_e}\right) + (u_{j+1}^*)_i \left(\frac{\xi}{l_e}\right) \right)^2 d\xi$$

$$= \int_0^{l_e} \begin{bmatrix} 1-\frac{\xi}{l_e} \\ \frac{\xi}{l_e} \end{bmatrix} \left((u_j^*)_i^2 \left(1-\frac{\xi}{l_e}\right)^2 + (u_j^*)_i (u_{j+1}^*)_i \left(\frac{\xi}{l_e} - \frac{\xi^2}{l_e^2}\right) + (u_{j+1}^*)_i^2 \left(\frac{\xi}{l_e}\right)^2 \right) d\xi \quad (C.4)$$

$$= l_e \begin{bmatrix} \frac{(u_j^*)_i^2}{4} + \frac{(u_j^*)_i (u_{j+1}^*)_i}{6} + \frac{(u_{j+1}^*)_i^2}{12} \\ \frac{(u_j^*)_i^2}{12} + \frac{(u_j^*)_i (u_{j+1}^*)_i}{6} + \frac{(u_{j+1}^*)_i^2}{4} \end{bmatrix}$$

APPENDIX D. LINEARIZATION MATRICES FOR THE CLASSICAL LINEARIZATION TECHNIQUE

For this analysis, $h(u^*)$ in equation (3.10) is replaced by u^* and the element linearization matrix integral becomes

$$\int_0^{l_e} \mathbf{g}\mathbf{g}^T u^* \, d\xi \tag{D.1}$$

The two approximations of this integral outlined in III.B.2 are now determined.

Midpoint Lumped Approximation

Replacing $h(u^*)$ in equation (3.11) with u^* yields

$$\mathbf{l}^* = \left(\frac{(u_j^*)_i + (u_{j+1}^*)_i}{2} \right) l_e \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \tag{D.2}$$

Linear Approximation

Replacing $h(u^*)$ with u^* in equation (3.12) gives

$$\begin{aligned} \mathbf{l}^* &= \int_0^{l_e} \begin{bmatrix} 1 - \frac{\xi}{l_e} \\ \frac{\xi}{l_e} \end{bmatrix} \begin{bmatrix} 1 - \frac{\xi}{l_e} & \frac{\xi}{l_e} \end{bmatrix} \left((u_j^*)_i \left(1 - \frac{\xi}{l_e} \right) + (u_{j+1}^*)_i \left(\frac{\xi}{l_e} \right) \right) d\xi \\ &= \int_0^{l_e} \begin{bmatrix} \left(1 - \frac{\xi}{l_e} \right)^2 & \frac{\xi}{l_e} \left(1 - \frac{\xi}{l_e} \right) \\ \frac{\xi}{l_e} \left(1 - \frac{\xi}{l_e} \right) & \left(\frac{\xi}{l_e} \right)^2 \end{bmatrix} \left((u_j^*)_i \left(1 - \frac{\xi}{l_e} \right) + (u_{j+1}^*)_i \left(\frac{\xi}{l_e} \right) \right) d\xi \\ &= \frac{l_e}{12} \begin{bmatrix} 3(u_j^*)_i + (u_{j+1}^*)_i & (u_j^*)_i + (u_{j+1}^*)_i \\ (u_j^*)_i + (u_{j+1}^*)_i & (u_j^*)_i + 3(u_{j+1}^*)_i \end{bmatrix} \end{aligned} \tag{D.3}$$

APPENDIX E. PROGRAM LISTINGS FOR CONSTANT LINEARIZATION

```
C     ************************************************************************
C     *                        PROGRAM NU2CAN                                *
C     *  THIS PROGRAM SOLVES THE NONLINEAR SECOND ORDER DIFFERENTIAL          *
C     *  EQUATION:                                                           *
C     *     U" - U**2 = 6 - 9X**4;  UEXACT=3X**2  WITH VARIABLE DOMAIN       *
C     *  TREATING THE U**2 TERM AS AN EXCITATION AND TAKING IT TO THE         *
C     *  RIGHT SIDE OF THE EQUATION.                                         *
C     *  THE USER SELECTS:                                                   *
C     *    1) NUMBER OF ELEMENTS                                             *
C     *    2) SIZE OF DOMAIN                                                 *
C     *    3) X AND U(X) AT THE LEFT BOUNDARY                                *
C     *    4) U(X) OR U'(X) AT THE RIGHT BOUNDARY                            *
C     *    5) ITERATION STRATEGY FOR DETERMINING U*                          *
C     *    6) APPROXIMATION TECHNIQUE FOR THE EXCITATION INTEGRAL            *
C     ************************************************************************

110   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE
115   CONV=.0001

C     READ IN PARAMETERS FROM DATA FILE

130   READ(29,*) NEL,TLEN,COORD(1),ULBC,ITYPE,URBC

C     CALCULATE NUMBER OF NODAL POINTS

135   NSNP=NEL+1

C     DETERMINE ELEMENT SIZE OF EQUAL LENGTHS

137   ELEN=TLEN/FLOAT(NEL)

C     ESTABLISH LOCAL TO GLOBAL CORRESPONDENCE AND X COORDINATE OF
C     EACH NODE

160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
165     COORD(IEL+1)=COORD(IEL) + ELEN
169   CONTINUE

C     CALL SUBROUTINE NU2CAM TO CREATE A MATRIX AND F VECTOR

170   CALL NU2CAM

C     CALL SUBROUTINE NU2CAI TO PERFORM SOLUTION ITERATION

180   CALL NU2CAI(IET)
```

```
C     CALL SUBROUTINE U2EXTA TO COMPUTE EXACT SOLUTION U=3X**2

190   CALL U2EXTA

C     CALL SUBROUTINE OUTPUT TO PRINT OUT DATA, COMPUTATIONAL EFFICIENCY

200   CALL OUTPUT(CPUSTAR,IET)
210   END
```

```
C     ************************************************************************
C     *                        SUBROUTINE NU2CAM                             *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND F VECTOR FOR MAIN          *
C     * PROGRAM NU2CAN.                                                      *
C     ************************************************************************

100   SUBROUTINE NU2CAM
110   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE
120   DIMENSION AE(2,2), FS1E(2), FS2E(2)
122   IF (TLEN.LE.1.0) THEN
123     PRINT*,'CHOOSE BOUNDARY FOR INITIAL GUESS:'
124     PRINT*,'1 = LEFT ESSENTIAL BOUNDARY CONDITION'
125     PRINT*,'2 = RIGHT ESSENTIAL BOUNDARY CONDITION'
126     PRINT*,'3 = AVERAGE OF THE TWO ESSENTIAL BOUNDARY CONDITIONS'
127     READ(6,*) INITGS
128   ELSE
129     CONTINUE
130   ENDIF
140   DO 210 IZ = 1,NSNP

C     ZERO OUT STEADY FORCE VECTOR

150      FS(IZ) = 0.

C     DETERMINE INITIAL VALUE OF USTAR TO BEGIN THE ITERATION PROCESS

157      IF (INITGS.EQ.1) THEN
158        U(IZ)=ULBC
159        UOLD(IZ)=ULBC
160      ELSEIF (INITGS.EQ.2) THEN
161        U(IZ)=URBC
162        UOLD(IZ)=URBC
163      ELSEIF (INITGS.EQ.3) THEN
164        U(IZ)=(ULBC+URBC)/2.
165        UOLD(IZ)=U(IZ)

166      ELSE
167        U(IZ)=SQRT(ABS(9.*COORD(IZ)**4 - 6.))
168        UOLD(IZ)=U(IZ)
169      ENDIF

C     ZERO OUT A MATRIX

170      DO 200 JZ = 1,NSNP
180           A(IZ,JZ) = 0.
200      CONTINUE
210   CONTINUE

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

213   ALPHA=0.
215   DO 375 IEL=1,NEL
220     AE(1,1)=1./ELEN
```

```
230      AE(1,2)=(-1./ELEN)
240      AE(2,1)=AE(1,2)
250      AE(2,2)=AE(1,1)
260      FS1E(1)=3.*ELEN
270      FS1E(2)=FS1E(1)
272      F1=(ALPHA**4)*ELEN/2.
274      F2=2.*(ALPHA**3)*(ELEN**2)/3.
276      F3=(ALPHA**2)*(ELEN**3)/2.
278      F4=ALPHA*(ELEN**4)/5.
280      F5=(ELEN**5)/30.
287      FS2E(1)=(-9.)*(F1 + F2 + F3 + F4 + F5)
290      FS2E(2)=(-9)*(F1 + 2.*F2 + 3.*F3 + 4.*F4 + 5.*F5)
300        DO 370 II=1,2
310          DO 350 JJ=1,2
320            IN=ICORR(IEL,II)
330            JN=ICORR(IEL,JJ)
340            A(IN,JN)=A(IN,JN) - AE(II,JJ)
350          CONTINUE
360          FS(IN)=FS1E(II) + FS2E(II) + FS(IN)
370        CONTINUE
372      ALPHA=ALPHA + ELEN
375    CONTINUE
420    RETURN
430    END
```

```
C     ************************************************************************
C     *                        SUBROUTINE NU2CAI                             *
C     *                                                                      *
C     * THIS SUBROUTINE PERFORMS THE ITERATIVE SOLUTION PROCESS FOR          *
C     * MAIN PROGRAM NU2CAN.                                                 *
C     ************************************************************************

100   SUBROUTINE NU2CAI(IET)
102   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE
104   DIMENSION WKAREA(40600), DIF(100), FUE(2), USTAR(100)

C     SELECT METHOD OF DETERMINING USTAR

105   PRINT*,'SELECT METHOD OF U* DETERMINATION.'
106   PRINT*,'1: U* = U'
107   PRINT*,'2: U* = AVERAGE OF LAST TWO COMPUTED VALUES OF U'
109   READ(6,*) METHU

C     SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR

116   PRINT*,'SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR.'
117   PRINT*,'1:  MIDPOINT APPROXIMATION'
118   PRINT*,'2:  1/4 - 3/4 APPROXIMATION'
119   PRINT*,'3:  LINEAR'
120   READ(6,*) METHFU

C     CALL SUBROUTINE SETIME TO BEGIN TIMING ITERATION PROCESS

121   CALL SETIME

C     BEGIN ITERATION PROCESS

122   DO 450 ITER=1,200

C     RESET VALUE OF UNSTEADY F VECTOR TO ZERO

123     DO 138 IU=1,NSNP
124       FU(IU)=0.

C     DETERMINE VALUE OF U* AT EACH NODE

125       IF (METHU.EQ.1) THEN
126         USTAR(IU)=U(IU)
127       ELSEIF (METHU.EQ.2) THEN
128         USTAR(IU)=(U(IU)+UOLD(IU))/2.
132       ENDIF
138     CONTINUE

C     DETERMINE UNSTEADY FORCE VECTOR

140     DO 210 IEL=1,NEL
145       IF (METHFU.EQ.1) THEN
146         FUE(1)=(ELEN/2.)*((USTAR(IEL)+USTAR(IEL+1))/2.)**2
147         FUE(2)=FUE(1)
```

```
149         ELSEIF (METHFU.EQ.2) THEN
151       FUE(1)=(ELEN/2.)*(3.*USTAR(IEL)/4. + USTAR(IEL+1)/4.)**2
152       FUE(2)=(ELEN/2.)*(USTAR(IEL)/4. + 3.*USTAR(IEL+1)/4.)**2
153         ELSE
154       FUE(1)=ELEN*(USTAR(IEL)**2/4. +USTAR(IEL)*USTAR(IEL+1)/6.
     :         + USTAR(IEL+1)**2/12.)
155       FUE(2)=ELEN*(USTAR(IEL)**2/12. + USTAR(IEL)*USTAR(IEL+1)
     :         /6. + USTAR(IEL+1)**2/4.)
156         ENDIF
170         DO 200 II=1,2
180           IN=ICORR(IEL,II)
190           FU(IN)=FUE(II) + FU(IN)
200         CONTINUE
210       CONTINUE

C       DETERMINE TOTAL FORCE VECTOR

220       DO 240 NP=1,NSNP
230         FT(NP)=FS(NP)+FU(NP)
235         UOLD(NP)=U(NP)
240       CONTINUE

C       IMPOSE BOUNDARY CONDITIONS

241       A(1,1)=1.
242       A(1,2)=0.
243       FT(1)=ULBC
244       IF (ITYPE.EQ.1) THEN
245         A(NSNP,NSNP-1)=0.
246         A(NSNP,NSNP)=1.
247         FT(NSNP)=URBC
248       ELSE
249         FT(NSNP)=FT(NSNP)-URBC*TLEN
250       ENDIF
255       M=1
260       IDGT=3
270       IQ=100

C       CALL SUBROUTINE LEQT2F TO SOLVE SET OF LINEAR ALGEBRAIC EQUATIONS

280       CALL LEQT2F(A,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
290       DO 310 NEW=1,NSNP
300         U(NEW)=FT(NEW)
C           WRITE(*,*) 'UNEW=',U(NEW)

C       TEST FOR CONVERGENCE

306         DIF(NEW)=ABS(U(NEW)-UOLD(NEW))
310       CONTINUE
320       DIFMAX=DIF(1)
325       NMAX=1
330       DO 390 IJ=1,NEL
340         IF (DIF(IJ+1).GE.DIF(IJ)) THEN
350           DIFMAX=DIF(IJ+1)
355           NMAX=IJ+1
360         ELSE
```

```
370         CONTINUE
380       ENDIF
390     CONTINUE
405     IF (ABS(DIFMAX/U(NMAX)).LT.CONV) THEN
410       GO TO 451
420     ELSE
430       CONTINUE
440     ENDIF
450   CONTINUE
451   CONTINUE

C     CALL SUBROUTINE GETIME TO OBTAIN CPU TIME FOR ITERATION PROCESS

452   CALL GETIME(IET)

C     OUTPUT HEADER INFORMATION

462   WRITE(6,464)
463   WRITE(30,464)
464   FORMAT(1X,'EQUATION:  U'' - U**2 = 6 - 9X**4')
465   IF (ITYPE.EQ.1) THEN
466     WRITE(6,468) COORD(1),ULBC,COORD(NSNP),URBC
467     WRITE(30,468) COORD(1),ULBC,COORD(NSNP),URBC
468     FORMAT(1X,'B.C.: U(',F2.0,')=',F2.0,'; U(',F3.0,')=',F4.0,/)
469   ELSE
470     WRITE(6,472) COORD(1),ULBC,COORD(NSNP),URBC
471     WRITE(30,472) COORD(1),ULBC,COORD(NSNP),URBC
472     FORMAT(1X,'B.C.: U(',F2.0,')=',F2.0,';DU/DX(',F3.0,')=',F4.0,/)
473   ENDIF
475   IF (METHU.EQ.1) THEN
476     WRITE(6,478)
477     WRITE(30,478)
478     FORMAT(1X,'ITERATION METHOD: U*=U',/)
479   ELSE
480     WRITE(6,482)
481     WRITE(30,482)
482     FORMAT(1X,'ITERATION METHOD: U*=(U+UOLD)/2',/)
483   ENDIF
488   IF (METHFU.EQ.1) THEN
489     WRITE(6,491)
490     WRITE(30,491)
491    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: MIDPOINT',/)
492   ELSEIF (METHFU.EQ.2) THEN
493     WRITE(6,495)
494     WRITE(30,495)
495    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: 1/4-3/4',/)
496   ELSE
497     WRITE(6,499)
498     WRITE(30,499)
499    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: LINEAR',/)
500   ENDIF
502   IF (ITER.GE.200) THEN
503     WRITE(6,505)
504     WRITE(30,505)
505     FORMAT(1X,'CONVERGENCE NOT OBTAINED AFTER 200 ITERATIONS.')
506   ELSEIF (ABS(U(NMAX)).GT.(10.**20).OR.ABS(U(NSNP-1)).GT.
```

```
     :(10.**20)) THEN
507     WRITE(6,509)
508     WRITE(30,509)
509     FORMAT(1X,'SOLUTION PROCESS DIVERGES.')
510   ELSE
511     WRITE(6,520) ITER,NEL
515     WRITE(30,520) ITER,NEL
520     FORMAT(1X,'CONVERGENCE OBTAINED AFTER ',I3,' ITERATIONS USING ',
     :I3,' ELEMENTS.',/)
525   ENDIF
530   RETURN
540   END
```

```
C     ************************************************************************
C     *                        SUBROUTINE U2EXTA                             *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE EXACT SOLUTION, U=3X**2, FOR            *
C     * MAIN PROGRAM NU2CA AT THE SPECIFIED NODAL POINTS.                    *
C     ************************************************************************

100   SUBROUTINE U2EXTA
110   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = 3.*COORD(NN)**2
150   CONTINUE
160   RETURN
170   END
```

```
C     ************************************************************************
C     *                      PROGRAM NU2CBN                                  *
C     *  THIS PROGRAM SOLVES THE NONLINEAR SECOND ORDER DIFFERENTIAL         *
C     *  EQUATION:                                                           *
C     *     U" + U**2 = 60X + 100X**6;  UEXACT=10X**3  WITH VARIABLE         *
C     *  DOMAIN, TREATING THE U**2 TERM AS AN EXCITATION AND TAKING IT       *
C     *  TO THE RIGHT SIDE OF THE EQUATION.  THE USER SELECTS:               *
C     *    1) NUMBER OF ELEMENTS                                             *
C     *    2) SIZE OF DOMAIN                                                 *
C     *    3) X AND U(X) AT THE LEFT BOUNDARY                                *
C     *    4) U(X) OR U'(X) AT THE RIGHT BOUNDARY                            *
C     *    5) ITERATION STRATEGY FOR DETERMINING U*                          *
C     *    6) APPROXIMATION TECHNIQUE FOR THE EXCITATION INTEGRAL            *
C     ************************************************************************

110   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE

115   CONV=.0001

C     READ IN PARAMETERS FROM DATA FILE

130   READ(29,*) NEL,TLEN,COORD(1),ULBC,ITYPE,URBC

C     CALCULATE NUMBER OF NODAL POINTS

135   NSNP=NEL+1

C     DETERMINE ELEMENT SIZE OF EQUAL LENGTHS

137   ELEN=TLEN/FLOAT(NEL)

C     ESTABLISH LOCAL TO GLOBAL CORRESPONDENCE AND X COORDINATE OF
C     EACH NODE

160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
164     COORD(IEL+1)=COORD(IEL)+ELEN
169   CONTINUE

C     CALL SUBROUTINE NU2CBM TO CREATE A MATRIX AND F VECTOR

170   CALL NU2CBM

C     CALL SUBROUTINE NU2CBI TO PERFORM SOLUTION ITERATION

180   CALL NU2CBI(IET)

C     CALL SUBROUTINE U2EXTB TO COMPUTE EXACT SOLUTION U=10X**3

190   CALL U2EXTB

C     CALL SUBROUTINE OUTPUT TO PRINT OUT DATA, COMPUTATIONAL EFFICIENCY
```

```
200     CALL OUTPUT(CPUSTAR,IET)
210     END
```

```
C     ************************************************************************
C     *                        SUBROUTINE NU2CBM                             *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND F VECTOR FOR MAIN          *
C     * PROGRAM NU2CBN.                                                      *
C     ************************************************************************

100   SUBROUTINE NU2CBM
110   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE
120   DIMENSION AE(2,2), FS1E(2), FS2E(2)
122   IF (TLEN.LT.1.0) THEN
123     PRINT*,'CHOOSE BOUNDARY FOR INITIAL GUESS:'
124     PRINT*,'1 = LEFT ESSENTIAL BOUNDARY CONDITION'
125     PRINT*,'2 = RIGHT ESSENTIAL BOUNDARY CONDITION'
126     PRINT*,'3 = AVERAGE OF THE TWO ESSENTIAL BOUNDARY CONDITIONS'
127     READ(6,*) INITGS
128   ELSE
129     CONTINUE
130   ENDIF
140   DO 210 IZ = 1,NSNP

C     ZERO OUT STEADY FORCE VECTOR

150      FS(IZ) = 0.

C     DETERMINE INITIAL VALUE OF USTAR TO BEGIN THE ITERATION PROCESS

157      IF (INITGS.EQ.1) THEN
158        U(IZ)=ULBC
159        UOLD(IZ)=ULBC
160      ELSEIF (INITGS.EQ.2) THEN
161        U(IZ)=URBC
162        UOLD(IZ)=URBC
163      ELSEIF (INITGS.EQ.3) THEN
164        U(IZ)=(ULBC+URBC)/2.
165        UOLD(IZ)=U(IZ)

166      ELSE
167        U(IZ)=SQRT(60.*COORD(IZ) + 100.*COORD(IZ)**6)
168        UOLD(IZ)=U(IZ)
169      ENDIF

C     ZERO OUT A MATRIX

170      DO 200 JZ = 1,NSNP
175           A(IZ,JZ) = 0.
200      CONTINUE
210   CONTINUE

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

213   ALPHA=0.
215   DO 375 IEL=1,NEL
220     AE(1,1)=1./ELEN
```

```
230      AE(1,2)=(-1./ELEN)
240      AE(2,1)=AE(1,2)
250      AE(2,2)=AE(1,1)
260      FS1E(1)=30.*ALPHA*ELEN + 10.*ELEN**2
270      FS1E(2)=30.*ALPHA*ELEN + 20.*ELEN**2
272      F1=50.*(ALPHA**6)*ELEN
274      F2=100.*(ALPHA**5)*(ELEN**2)
276      F3=125.*(ALPHA**4)*(ELEN**3)
278      F4=100.*(ALPHA**3)*(ELEN**4)
280      F5=50.*(ALPHA**2)*(ELEN**5)
281      F6=100.*ALPHA*(ELEN**6)/7.
282      F7=25.*(ELEN**7)/16.
287      FS2E(1)=F1 + F2 + F3 + F4 + F5 + F6 + F7
290      FS2E(2)=F1 + 2.*F2 + 3.*F3 + 4.*F4 + 5.*F5 + 6.*F6 + 7*F7
300        DO 370 II=1,2
310          DO 350 JJ=1,2
320            IN=ICORR(IEL,II)
330            JN=ICORR(IEL,JJ)
340            A(IN,JN)=A(IN,JN) - AE(II,JJ)
350          CONTINUE
360          FS(IN)=FS1E(II) + FS2E(II) + FS(IN)
370        CONTINUE
372      ALPHA=ALPHA + ELEN
375    CONTINUE
420    RETURN
430    END
```

```
C     ************************************************************************
C     *                      SUBROUTINE NU2CBI                               *
C     *                                                                      *
C     * THIS SUBROUTINE PERFORMS THE ITERATIVE SOLUTION PROCESS FOR          *
C     * MAIN PROGRAM NU2CBN.                                                 *
C     ************************************************************************

100   SUBROUTINE NU2CBI(IET)
102   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE
104   DIMENSION WKAREA(40600), DIF(100), FUE(2), USTAR(100)

C     SELECT METHOD OF DETERMINING USTAR

105   PRINT*,'SELECT METHOD OF U* DETERMINATION.'
106   PRINT*,'1: U* = U'
107   PRINT*,'2: U* = AVERAGE OF LAST TWO COMPUTED VALUES OF U'
109   READ(6,*) METHU

C     SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR

116   PRINT*,'SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR.'
117   PRINT*,'1:  MIDPOINT APPROXIMATION'
118   PRINT*,'2:  1/4 - 3/4 APPROXIMATION'
119   PRINT*,'3:  CONSISTENT'
120   READ(6,*) METHFU

C     CALL SUBROUTINE SETIME TO BEGIN TIMING ITERATION PROCESS

121   CALL SETIME

C     BEGIN ITERATION PROCESS

122   DO 450 ITER=1,200

C     RESET VALUE OF UNSTEADY F VECTOR TO ZERO

123     DO 138 IU=1,NSNP
124       FU(IU)=0.

C     DETERMINE VALUE OF U* AT EACH NODE

125       IF (METHU.EQ.1) THEN
126         USTAR(IU)=U(IU)
127       ELSE
128         USTAR(IU)=(U(IU)+UOLD(IU))/2.
132       ENDIF
138     CONTINUE

C     DETERMINE UNSTEADY FORCE VECTOR

140     DO 210 IEL=1,NEL
145       IF (METHFU.EQ.1) THEN
146         FUE(1)=(ELEN/2.)*((USTAR(IEL)+USTAR(IEL+1))/2.)**2
147         FUE(2)=FUE(1)
```

```
149         ELSEIF (METHFU.EQ.2) THEN
151           FUE(1)=(ELEN/2.)*(3.*USTAR(IEL)/4. + USTAR(IEL+1)/4.)**2
152           FUE(2)=(ELEN/2.)*(USTAR(IEL)/4. + 3.*USTAR(IEL+1)/4.)**2
153         ELSE
154           FUE(1)=ELEN*(USTAR(IEL)**2/4. +USTAR(IEL)*USTAR(IEL+1)/6.
     :       + USTAR(IEL+1)**2/12.)
155           FUE(2)=ELEN*(USTAR(IEL)**2/12. + USTAR(IEL)*USTAR(IEL+1)/6.
     :       + USTAR(IEL+1)**2/4.)
156         ENDIF
170         DO 200 II=1,2
180           IN=ICORR(IEL,II)
190           FU(IN)=FUE(II) + FU(IN)
200         CONTINUE
210       CONTINUE

C       DETERMINE TOTAL FORCE VECTOR

220       DO 240 NP=1,NSNP
230         FT(NP)=FS(NP)-FU(NP)
235         UOLD(NP)=U(NP)
240       CONTINUE

C       IMPOSE BOUNDARY CONDITIONS

241       A(1,1)=1.
242       A(1,2)=0.
243       FT(1)=ULBC
244       IF (ITYPE.EQ.1) THEN
245         A(NSNP,NSNP-1)=0.
246         A(NSNP,NSNP)=1.
247         FT(NSNP)=URBC
248       ELSE
249         FT(NSNP)=FT(NSNP)-URBC
250       ENDIF
255       M=1
260       IDGT=3
270       IQ=100

C       CALL SUBROUTINE LEQT2F TO SOLVE SET OF LINEAR ALGEBRAIC EQUATIONS

280       CALL LEQT2F(A,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
290       DO 310 NEW=1,NSNP
300         U(NEW)=FT(NEW)
C           WRITE(*,*) 'UNEW=',U(NEW)

C       TEST FOR CONVERGENCE

306         DIF(NEW)=ABS(U(NEW)-UOLD(NEW))
310       CONTINUE
320       DIFMAX=DIF(1)
325       NMAX=1
330       DO 390 IJ=1,NEL
340         IF (DIF(IJ+1).GE.DIF(IJ)) THEN
350           DIFMAX=DIF(IJ+1)
355           NMAX=IJ+1
360         ELSE
```

```
370         CONTINUE
380       ENDIF
390     CONTINUE
405     IF (ABS(DIFMAX/U(NMAX)).LT.CONV) THEN
410       GO TO 460
420     ELSE
430       CONTINUE
440     ENDIF
450   CONTINUE
460   CONTINUE

C     CALL SUBROUTINE GETIME TO OBTAIN CPU TIME FOR ITERATION PROCESS

461   CALL GETIME(IET)

C     OUTPUT HEADER INFORMATION

462   WRITE(6,464)
463   WRITE(30,464)
464   FORMAT(1X,'EQUATION:  U" + U**2 = 60X + 100X**6')
465   IF (ITYPE.EQ.1) THEN
466     WRITE(6,468) COORD(1),ULBC,COORD(NSNP),URBC
467     WRITE(30,468) COORD(1),ULBC,COORD(NSNP),URBC
468     FORMAT(1X,'B.C.: U(',F2.0,')=',F3.0,'; U(',F2.0,')=',F4.0,/)
469   ELSE
470     WRITE(6,472) COORD(1),ULBC,COORD(NSNP),URBC
471     WRITE(30,472) COORD(1),ULBC,COORD(NSNP),URBC
472     FORMAT(1X,'B.C.: U(',F2.0,')=',F3.0,';DU/DX(',F2.0,')=',F4.0,/)
473   ENDIF
475   IF (METHU.EQ.1) THEN
476     WRITE(6,478)
477     WRITE(30,478)
478     FORMAT(1X,'ITERATION METHOD: U*=U',/)
479   ELSE
480     WRITE(6,482)
481     WRITE(30,482)
482     FORMAT(1X,'ITERATION METHOD: U*=(U+UOLD)/2',/)
487   ENDIF
488   IF (METHFU.EQ.1) THEN
489     WRITE(6,491)
490     WRITE(30,491)
491    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: MIDPOINT',/)
492   ELSEIF (METHFU.EQ.2) THEN
493     WRITE(6,495)
494     WRITE(30,495)
495    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: 1/4-3/4',/)
496   ELSE
497     WRITE(6,499)
498     WRITE(30,499)
499    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: LINEAR',/)
500   ENDIF
502   IF (ITER.GE.200) THEN
503     WRITE(6,505)
504     WRITE(30,505)
505     FORMAT(1X,'CONVERGENCE NOT OBTAINED AFTER 200 ITERATIONS.')
506   ELSEIF (ABS(U(NMAX)).GT.(10.**20).OR.ABS(U(NSNP-1)).GT.
```

```
     :(10.**20)) THEN
507     WRITE(6,509)
508     WRITE(30,509)
509     FORMAT(1X,'SOLUTION PROCESS DIVERGES.')
510    ELSE
511     WRITE(6,520) ITER,NEL
515     WRITE(30,520) ITER,NEL
520     FORMAT(1X,'CONVERGENCE OBTAINED AFTER ',I3,' ITERATIONS USING ',
     :I3,' ELEMENTS.',/)
525    ENDIF
530    RETURN
540    END
```

```
C     ************************************************************************
C     *                        SUBROUTINE U2EXTB                             *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE EXACT SOLUTION, U=10X**3,               *
C     * FOR MAIN PROGRAM NU2CBN AT THE SPECIFIED NODAL POINTS.               *
C     ************************************************************************

100   SUBROUTINE U2EXTB
110   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = 10.*COORD(NN)**3
150   CONTINUE
160   RETURN
170   END
```

```
C     ***************************************************************
C     *                       SUBROUTINE OUTPUT                     *
C     * THIS SUBROUTINE COMPUTES THE PER CENT ERROR BETWEEN THE EXACT *
C     * AND FEM SOLUTIONS, CPU* FOR THE ITERATION PROCESS, AND PRINTS *
C     * OUT ALL DATA IN TABULAR FORM FOR PROGRAMS NU2CAN AND NU2CBN   *
C     ***************************************************************

100   SUBROUTINE OUTPUT(CPUSTAR,IET)
110   COMMON A(100,100),FS(100),FU(100),FT(100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),NEL,NSNP,ITYPE
115   SUMDIF=0.

C     CALCULATE PER CENT ERROR AT EACH NODE AND SUM THE ABSOLUTE VALUE
C     OF ALL THE ERRORS

120   DO 150 IK=2,NSNP
130     UDIF(IK)=100.*(U(IK)-UEXT(IK))/UEXT(IK)
140     SUMDIF=SUMDIF + ABS(UDIF(IK))
150   CONTINUE
160   UDIF(1)=U(1)-UEXT(1)

C     COMPUTE THE ELAPSED TIME OF THE ITERATION PROCESS

164   ELTIME=IET*.000026
165   WRITE(6,169) ELTIME
166   WRITE(30,169)ELTIME
169   FORMAT(1X,'ELAPSED TIME FOR THE ITERATION PROCESS IS ',F9.4,
     :' SECONDS.')

C     OUTPUT DATA IN TABULAR FORMAT

170   WRITE(6,180)
175   WRITE(30,180)
180   FORMAT(/,1X,'X-COORD',3X,'U EXACT',3X,'U FEM',4X,'% DIFF')
190   WRITE(6,200) (COORD(NP),UEXT(NP),U(NP),UDIF(NP), NP=1,NSNP)
195   WRITE(30,200) (COORD(NP),UEXT(NP),U(NP),UDIF(NP), NP=1,NSNP)
200   FORMAT(/,2X,F5.3,5X,F7.4,3X,F7.4,4X,F5.1)

C     CALCULATE CPU* FOR THE ITERATION PROCESS

205   CPUSTAR=ELTIME*SUMDIF/NSNP
210   WRITE(6,220) CPUSTAR
215   WRITE(30,220) CPUSTAR
220   FORMAT(/,1X,'CPU* FOR THE ITERATION PROCESS IS ',F9.4,' SECONDS.')
230   RETURN
240   END
```

APPENDIX F. PROGRAM LISTINGS FOR CLASSICAL LINEARIZATION

```
C     ************************************************************************
C     *                          PROGRAM NU2KA                               *
C     *  THIS PROGRAM SOLVES THE NONLINEAR SECOND ORDER DIFFERENTIAL         *
C     *  EQUATION:                                                           *
C     *     U'' - U**2 = 6 - 9X**4;  UEXACT=3X**2  WITH VARIABLE DOMAIN      *
C     *  BY LINEARIZING THE U**2 TERM AS USTAR*U AND KEEPING IT ON THE       *
C     *  LEFT SIDE OF THE EQUATION.  THE USER SELECTS:                       *
C     *    1) NUMBER OF ELEMENTS                                             *
C     *    2) SIZE OF DOMAIN                                                 *
C     *    3) X AND U(X) AT THE LEFT BOUNDARY                                *
C     *    4) U(X) OR U'(X) AT THE RIGHT BOUNDARY                            *
C     *    5) ITERATION STRATEGY FOR DETERMINING U*                          *
C     *    6) APPROXIMATION TECHNIQUE FOR THE EXCITATION INTEGRAL            *
C     ************************************************************************

110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
115   CONV=.0001

C     READ IN PARAMETERS FROM DATA FILE

130   READ(29,*) NEL,TLEN,COORD(1),ULBC,ITYPE,URBC

C     CALCULATE NUMBER OF NODAL POINTS

135   NSNP=NEL+1

C     DETERMINE ELEMENT SIZE OF EQUAL LENGTHS

137   ELEN=TLEN/FLOAT(NEL)

C     ESTABLISH LOCAL TO GLOBAL CORRESPONDENCE AND X COORDINATE OF
C     EACH NODE

160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
164     COORD(IEL+1)=COORD(IEL)+ELEN
169   CONTINUE

C     CALL SUBROUTINE NU2CAM TO CREATE A MATRIX AND F VECTOR

170   CALL NU2KAM

C     CALL SUBROUTINE NU2CAI TO PERFORM SOLUTION ITERATION

180   CALL NU2KAI(IET)
```

```
C     CALL SUBROUTINE U2EXTA TO COMPUTE EXACT SOLUTION U=3X**2
190   CALL CLEXTA
C     CALL SUBROUTINE OUTPUT TO PRINT OUT DATA, COMPUTATIONAL EFFICIENCY
200   CALL CLOTPT(CPUSTAR,IET)
210   END
```

```
C     ************************************************************************
C     *                        SUBROUTINE NU2KAM                             *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND F VECTOR FOR MAIN          *
C     * PROGRAM NU2KA.                                                       *
C     ************************************************************************

100   SUBROUTINE NU2KAM
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
120   DIMENSION AE(2,2), FS1E(2), FS2E(2)
122   IF (TLEN.LE.1.0) THEN
123     PRINT*,'CHOOSE BOUNDARY FOR INITIAL GUESS:'
124     PRINT*,'1 = LEFT ESSENTIAL BOUNDARY CONDITION'
125     PRINT*,'2 = RIGHT ESSENTIAL BOUNDARY CONDITION'
126     PRINT*,'3 = AVERAGE OF THE TWO ESSENTIAL BOUNDARY CONDITIONS'
127     READ(6,*) INITGS
128   ELSE
129     CONTINUE
130   ENDIF
140   DO 210 IZ = 1,NSNP

C     ZERO OUT STEADY FORCE VECTOR
150     FS(IZ) = 0.

C     DETERMINE INITIAL VALUE OF USTAR TO BEGIN THE ITERATION PROCESS

157     IF (INITGS.EQ.1) THEN
158       U(IZ)=ULBC
159       UOLD(IZ)=U(IZ)
160     ELSEIF (INITGS.EQ.2) THEN
161       U(IZ)=URBC
162       UOLD(IZ)=U(IZ)
163     ELSEIF (INITGS.EQ.3) THEN
164       U(IZ)=(ULBC+URBC)/2.
165       UOLD(IZ)=U(IZ)
166     ELSE
167       U(IZ) = SQRT(ABS(9.*COORD(IZ)**4 - 6.))
168       UOLD(IZ) = U(IZ)
169     ENDIF

C     ZERO OUT ALL MATRICES

170     DO 200 JZ = 1,NSNP
171           A(IZ,JZ) = 0.
175           B(IZ,JZ) = 0.
176           C(IZ,JZ) = 0.
200     CONTINUE
210   CONTINUE

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

213   ALPHA=0.
215   DO 375 IEL=1,NEL
220     AE(1,1)=1./ELEN
```

```
230      AE(1,2)=(-1./ELEN)
240      AE(2,1)=AE(1,2)
250      AE(2,2)=AE(1,1)
260      FS1E(1)=3.*ELEN
270      FS1E(2)=FS1E(1)
272      F1=(ALPHA**4)*ELEN/2.
274      F2=2.*(ALPHA**3)*(ELEN**2)/3.
276      F3=(ALPHA**2)*(ELEN**3)/2.
278      F4=ALPHA*(ELEN**4)/5.
280      F5=(ELEN**5)/30.
287      FS2E(1)=(-9.)*(F1 + F2 + F3 + F4 + F5)
290      FS2E(2)=(-9)*(F1 + 2.*F2 + 3.*F3 + 4.*F4 + 5.*F5)
300        DO 370 II=1,2
310          DO 350 JJ=1,2
320            IN=ICORR(IEL,II)
330            JN=ICORR(IEL,JJ)
340            A(IN,JN)=A(IN,JN) - AE(II,JJ)
350          CONTINUE
360          FS(IN)=FS1E(II) + FS2E(II) + FS(IN)
370        CONTINUE
372      ALPHA=ALPHA + ELEN
375    CONTINUE
420    RETURN
430    END
```

```
C     ************************************************************************
C     *                        SUBROUTINE NU2KAI                             *
C     *                                                                      *
C     * THIS SUBROUTINE PERFORMS THE ITERATIVE SOLUTION PROCESS FOR          *
C     * MAIN PROGRAM NU2KA.                                                  *
C     ************************************************************************

100   SUBROUTINE NU2KAI(IET)
102   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
104   DIMENSION WKAREA(40600), DIF(100), BE(2,2), USTAR(100), FT(100)

C     SELECT METHOD OF DETERMINING USTAR

105   PRINT*,'SELECT METHOD OF U* DETERMINATION.'
106   PRINT*,'1: U* = U'
107   PRINT*,'2: U* = AVERAGE OF LAST TWO COMPUTED VALUES OF U'
109   READ(6,*) METHU

C     SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR

116   PRINT*,'SELECT METHOD OF DETERMINING UNSTEADY B MATRIX.'
117   PRINT*,'1:  MIDPOINT APPROXIMATION FOR U OVER THE ELEMENT.'
119   PRINT*,'2:  U LINEARIZED OVER THE LENGTH OF THE ELEMENT.'
120   READ(6,*) METHBM

C     CALL SUBROUTINE SETIME TO BEGIN TIMING ITERATION PROCESS

121   CALL SETIME

C     BEGIN ITERATION PROCESS

122   DO 450 ITER=1,200

C     DETERMINE VALUE OF U* AT EACH NODE

130     DO 138 IU=1,NSNP
131       IF (METHU.EQ.1) THEN
132         USTAR(IU)=U(IU)
133       ELSE
134         USTAR(IU)=(U(IU)+UOLD(IU))/2.
137       ENDIF
138     CONTINUE

C     DETERMINE UNSTEADY ELEMENT B MATRIX

140     DO 210 IEL=1,NEL
145       IF (METHBM.EQ.1) THEN
146         BE(1,1)=(ELEN/6.)*(USTAR(IEL)+USTAR(IEL+1))
147         BE(1,2)=(ELEN/12.)*(USTAR(IEL)+USTAR(IEL+1))
148         BE(2,1)=BE(1,2)
149         BE(2,2)=BE(1,1)
150       ELSE
151         BE(1,1)=(ELEN/12.)*(3.*USTAR(IEL) + USTAR(IEL+1))
152         BE(1,2)=(ELEN/12.)*(USTAR(IEL) + USTAR(IEL+1))
```

```
153         BE(2,1)=BE(1,2)
154         BE(2,2)=(ELEN/12.)*(USTAR(IEL) + 3.*USTAR(IEL+1))
156       ENDIF
170       DO 200 II=1,2
175         DO 195 JJ=1,2
180           IN=ICORR(IEL,II)
185           JN=ICORR(IEL,JJ)
190           B(IN,JN)=BE(II,JJ) + B(IN,JN)
195         CONTINUE
200       CONTINUE
210     CONTINUE

C       DETERMINE TOTAL SYSTEM MATRIX

220     DO 240 IP=1,NSNP
221       DO 232 JP=1,NSNP
230         C(IP,JP)=A(IP,JP)-B(IP,JP)

C       RESET B MATRIX TO ZERO AND LET UOLD=U AND FT=FS

231         B(IP,JP)=0.
232       CONTINUE
235       UOLD(IP)=U(IP)
236       FT(IP)=FS(IP)
240     CONTINUE

C       IMPOSE BOUNDARY CONDITIONS

241     C(1,1)=1.
242     C(1,2)=0.
243     FT(1)=ULBC
244     IF (ITYPE.EQ.1) THEN
245       C(NSNP,NSNP-1)=0.
246       C(NSNP,NSNP)=1.
247       FT(NSNP)=URBC
248     ELSE
249       FT(NSNP)=FT(NSNP)-URBC
250     ENDIF
255     M=1
260     IDGT=3
270     IQ=100

C       CALL SUBROUTINE LEQT2F TO SOLVE SET OF LINEAR ALGEBRAIC EQUATIONS

280     CALL LEQT2F(C,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
290     DO 310 NEW=1,NSNP
300       U(NEW)=FT(NEW)
C         WRITE(*,*) 'UNEW=',U(NEW)

C       TEST FOR CONVERGENCE

306       DIF(NEW)=ABS(U(NEW)-UOLD(NEW))
310     CONTINUE
320     DIFMAX=DIF(1)
325     NMAX=1
330     DO 390 IJ=1,NEL
```

```
340         IF (DIF(IJ+1).GE.DIF(IJ)) THEN
350           DIFMAX=DIF(IJ+1)
355           NMAX=IJ+1
360         ELSE
370           CONTINUE
380         ENDIF
390       CONTINUE
405       IF (ABS(DIFMAX/U(NMAX)).LT.CONV) THEN
410         GO TO 460
420       ELSE
430         CONTINUE
440       ENDIF
450     CONTINUE
460     CONTINUE

C       CALL SUBROUTINE GETIME TO OBTAIN CPU TIME FOR ITERATION PROCESS

461     CALL GETIME(IET)

C       OUTPUT HEADER INFORMATION

462     WRITE(6,464)
463     WRITE(30,464)
464     FORMAT(1X,'EQUATION:  U" - U**2 = 6 - 9X**4')
465     IF (ITYPE.EQ.1) THEN
466       WRITE(6,468) COORD(1),ULBC,COORD(NSNP),URBC
467       WRITE(30,468) COORD(1),ULBC,COORD(NSNP),URBC
468       FORMAT(1X,'B.C.: U(',F2.0,')=',F2.0,'; U(',F2.0,')=',F4.0,/)
469     ELSE
470       WRITE(6,472) COORD(1),ULBC,COORD(NSNP),URBC
471       WRITE(30,472) COORD(1),ULBC,COORD(NSNP),URBC
472       FORMAT(1X,'B.C.: U(',F2.0,')=',F2.0,';DU/DX(',F2.0,')=',F4.0,/)
473     ENDIF
475     IF (METHU.EQ.1) THEN
476       WRITE(6,478)
477       WRITE(30,478)
478       FORMAT(1X,'ITERATION METHOD: U*=U',/)
479     ELSE
480       WRITE(6,482)
481       WRITE(30,482)
482       FORMAT(1X,'ITERATION METHOD: U*=(U+UOLD)/2',/)
487     ENDIF
488     IF (METHBM.EQ.1) THEN
489       WRITE(6,491)
490       WRITE(30,491)
491      FORMAT(1X,'METHOD OF B MATRIX INTEGRAL EVALUATION: MIDPOINT',/)
496     ELSE
497       WRITE(6,499)
498       WRITE(30,499)
499      FORMAT(1X,'METHOD OF B MATRIX INTEGRAL EVALUATION: LINEAR',/)
500     ENDIF
502     IF (ITER.GE.200) THEN
503       WRITE(6,505)
504       WRITE(30,505)
505       FORMAT(1X,'CONVERGENCE NOT OBTAINED AFTER 200 ITERATIONS.')
506     ELSEIF (ABS(U(NMAX)).GT.(10.**20).OR.ABS(U(NSNP-1)).GT.
```

```
     :(10.**20)) THEN
507     WRITE(6,509)
508     WRITE(30,509)
509     FORMAT(1X,'SOLUTION PROCESS DIVERGES.')
510    ELSE
511     WRITE(6,520) ITER,NEL
515     WRITE(30,520) ITER,NEL
520     FORMAT(1X,'CONVERGENCE OBTAINED AFTER ',I3,' ITERATIONS USING ',
     :I3,' ELEMENTS.',/)
525    ENDIF
530    RETURN
540    END
```

```
C     ***************************************************************
C     *                      SUBROUTINE CLEXTA                      *
C     *                                                             *
C     * THIS SUBROUTINE COMPUTES THE EXACT SOLUTION, U=3X**2, FOR   *
C     * MAIN PROGRAM NU2KA AT THE SPECIFIED NODAL POINTS.           *
C     ***************************************************************

100   SUBROUTINE CLEXTA
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = 3.*COORD(NN)**2
150   CONTINUE
160   RETURN
170   END
```

```
C     ************************************************************************
C     *                        PROGRAM NU2KB                                 *
C     *  THIS PROGRAM SOLVES THE NONLINEAR SECOND ORDER DIFFERENTIAL         *
C     *  EQUATION:                                                           *
C     *     U" + U**2 = 60X + 100X**6;  UEXACT=10X**3  WITH VARIABLE         *
C     *  DOMAIN BY LINEARIZING THE U**2 TERM AS USTAR*U AND KEEPING IT       *
C     *  ON THE LEFT SIDE OF THE EQUATION.  THE USER SELECTS:                *
C     *    1) NUMBER OF ELEMENTS                                             *
C     *    2) SIZE OF DOMAIN                                                 *
C     *    3) X AND U(X) AT THE LEFT BOUNDARY                                *
C     *    4) U(X) OR U'(X) AT THE RIGHT BOUNDARY                            *
C     *    5) ITERATION STRATEGY FOR DETERMINING U*                          *
C     *    6) APPROXIMATION TECHNIQUE FOR THE EXCITATION INTEGRAL            *
C     ************************************************************************

110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
115   CONV=.0001

C     READ IN PARAMETERS FROM DATA FILE

130   READ(29,*) NEL,TLEN,COORD(1),ULBC,ITYPE,URBC

C     CALCULATE NUMBER OF NODAL POINTS

135   NSNP=NEL+1

C     DETERMINE ELEMENT SIZE OF EQUAL LENGTHS

137   ELEN=TLEN/FLOAT(NEL)

C     ESTABLISH LOCAL TO GLOBAL CORRESPONDENCE AND X COORDINATE OF
C     EACH NODE

160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
164     COORD(IEL+1)=COORD(IEL)+ELEN
169   CONTINUE

C     CALL SUBROUTINE NU2KBM TO CREATE A MATRIX AND F VECTOR

170   CALL NU2KBM

C     CALL SUBROUTINE NU2KBI TO PERFORM SOLUTION ITERATION

180   CALL NU2KBI(IET)

C     CALL SUBROUTINE CLEXTB TO COMPUTE EXACT SOLUTION U=3X**2

190   CALL CLEXTB

C     CALL SUBROUTINE CLOTPT TO PRINT OUT DATA, COMPUTATIONAL EFFICIENCY
```

```
200    CALL CLOTPT(CPUSTAR,IET)
210    END
```

```
C     ***************************************************************
C     *                      SUBROUTINE NU2KBM                      *
C     *                                                             *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND F VECTOR FOR MAIN  *
C     * PROGRAM NU2KB.                                              *
C     ***************************************************************

100   SUBROUTINE NU2KBM
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
120   DIMENSION AE(2,2), FS1E(2), FS2E(2)

C     ZERO OUT A MATRIX AND ALL VECTORS

140   DO 210 IZ = 1,NSNP
150      FS(IZ) = 0.
154      U(IZ) =SQRT(60.*COORD(IZ) + 100.*COORD(IZ)**6)
156      UOLD(IZ)=0.
160      DO 200 JZ = 1,NSNP
170           A(IZ,JZ) = 0.
175           B(IZ,JZ) = 0.
176           C(IZ,JZ) = 0.
200      CONTINUE
210   CONTINUE

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

213   ALPHA=0.
215   DO 375 IEL=1,NEL
220     AE(1,1)=1./ELEN
230     AE(1,2)=(-1./ELEN)
240     AE(2,1)=AE(1,2)
250     AE(2,2)=AE(1,1)
260     FS1E(1)=30.*ALPHA*ELEN + 10.*ELEN**2
270     FS1E(2)=30.*ALPHA*ELEN + 20.*ELEN**2
272     F1=50.*(ALPHA**6)*ELEN
274     F2=100.*(ALPHA**5)*(ELEN**2)
276     F3=125.*(ALPHA**4)*(ELEN**3)
278     F4=100.*(ALPHA**3)*(ELEN**4)
280     F5=50.*(ALPHA**2)*(ELEN**5)
281     F6=100.*ALPHA*(ELEN**6)/7.
282     F7=25.*(ELEN**7)/16.
287     FS2E(1)=F1 + F2 + F3 + F4 + F5 + F6 + F7
290     FS2E(2)=F1 + 2.*F2 + 3.*F3 + 4.*F4 + 5.*F5 + 6.*F6 + 7*F7
300       DO 370 II=1,2
310         DO 350 JJ=1,2
320           IN=ICORR(IEL,II)
330           JN=ICORR(IEL,JJ)
340           A(IN,JN)=A(IN,JN) - AE(II,JJ)
350         CONTINUE
360         FS(IN)=FS1E(II) + FS2E(II) + FS(IN)
370       CONTINUE
372     ALPHA=ALPHA + ELEN
375   CONTINUE
```

```
420     RETURN
430     END
```

```
C     ************************************************************************
C     *                        SUBROUTINE NU2KBI                             *
C     *                                                                      *
C     * THIS SUBROUTINE PERFORMS THE ITERATIVE SOLUTION PROCESS FOR          *
C     * MAIN PROGRAM NU2KA.                                                  *
C     ************************************************************************

100   SUBROUTINE NU2KBI(IET)
102   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
104   DIMENSION WKAREA(40600), DIF(100), BE(2,2), USTAR(100), FT(100)

C     SELECT METHOD OF DETERMINING USTAR

105   PRINT*,'SELECT METHOD OF U* DETERMINATION.'
106   PRINT*,'1: U* = U'
107   PRINT*,'2: U* = AVERAGE OF LAST TWO COMPUTED VALUES OF U'
108   PRINT*,'3: U* = WEIGHTED AVERAGE OF LAST TWO COMPUTED VALUES OF U'
109   READ(6,*) METHU
110   IF (METHU.EQ.3) THEN
111     PRINT*,'CHOOSE WEIGHTING VALUES A AND B'
112     READ(6,*) AW,BW
113   ELSE
114     CONTINUE
115   ENDIF

C     SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR

116   PRINT*,'SELECT METHOD OF DETERMINING UNSTEADY B MATRIX.'
117   PRINT*,'1:  MIDPOINT APPROXIMATION FOR U OVER THE ELEMENT.'
119   PRINT*,'2:  U LINEARIZED OVER THE LENGTH OF THE ELEMENT.'
120   READ(6,*) METHBM

C     CALL SUBROUTINE SETIME TO BEGIN TIMING ITERATION PROCESS

121   CALL SETIME

C     BEGIN ITERATION PROCESS

122   DO 450 ITER=1,200

C     DETERMINE VALUE OF U* AT EACH NODE

130     DO 138 IU=1,NSNP
131       IF (METHU.EQ.1) THEN
132         USTAR(IU)=U(IU)
133       ELSEIF (METHU.EQ.2) THEN
134         USTAR(IU)=(U(IU)+UOLD(IU))/2.
135       ELSE
136         USTAR(IU)=(AW*U(IU)+BW*UOLD(IU))/(AW+BW)
137       ENDIF
138     CONTINUE

C     DETERMINE UNSTEADY ELEMENT B MATRIX
```

```
140     DO 210 IEL=1,NEL
145       IF (METHBM.EQ.1) THEN
146         BE(1,1)=(ELEN/6.)*(USTAR(IEL)+USTAR(IEL+1))
147         BE(1,2)=(ELEN/12.)*(USTAR(IEL)+USTAR(IEL+1))
148         BE(2,1)=BE(1,2)
149         BE(2,2)=BE(1,1)
150       ELSE
151         BE(1,1)=(ELEN/12.)*(3.*USTAR(IEL) + USTAR(IEL+1))
152         BE(1,2)=(ELEN/12.)*(USTAR(IEL) + USTAR(IEL+1))
153         BE(2,1)=BE(1,2)
154         BE(2,2)=(ELEN/12.)*(USTAR(IEL) + 3.*USTAR(IEL+1))
156       ENDIF
170       DO 200 II=1,2
175         DO 195 JJ=1,2
180           IN=ICORR(IEL,II)
185           JN=ICORR(IEL,JJ)
190           B(IN,JN)=BE(II,JJ) + B(IN,JN)
195         CONTINUE
200       CONTINUE
210     CONTINUE

C       DETERMINE TOTAL SYSTEM MATRIX

220     DO 240 IP=1,NSNP
221       DO 232 JP=1,NSNP
230         C(IP,JP)=A(IP,JP)+B(IP,JP)

C       RESET B MATRIX TO ZERO AND LET UOLD=U AND FT=FS

231         B(IP,JP)=0.
232       CONTINUE
235       UOLD(IP)=U(IP)
236       FT(IP)=FS(IP)
240     CONTINUE

C       IMPOSE BOUNDARY CONDITIONS

241     C(1,1)=1.
242     C(1,2)=0.
243     FT(1)=ULBC
244     IF (ITYPE.EQ.1) THEN
245       C(NSNP,NSNP-1)=0.
246       C(NSNP,NSNP)=1.
247       FT(NSNP)=URBC
248     ELSE
249       FT(NSNP)=FT(NSNP)-URBC
250     ENDIF
255     M=1
260     IDGT=3
270     IQ=100

C       CALL SUBROUTINE LEQT2F TO SOLVE SET OF LINEAR ALGEBRAIC EQUATIONS

280     CALL LEQT2F(C,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
290     DO 310 NEW=1,NSNP
```

```
300       U(NEW)=FT(NEW)
C         WRITE(*,*) 'UNEW=',U(NEW)

C     TEST FOR CONVERGENCE

306       DIF(NEW)=ABS(U(NEW)-UOLD(NEW))
310     CONTINUE
320     DIFMAX=DIF(1)
325     NMAX=1
330     DO 390 IJ=1,NEL
340       IF (DIF(IJ+1).GE.DIF(IJ)) THEN
350         DIFMAX=DIF(IJ+1)
355         NMAX=IJ+1
360       ELSE
370         CONTINUE
380       ENDIF
390     CONTINUE
405     IF (ABS(DIFMAX/U(NMAX)).LT.CONV) THEN
410       GO TO 460
420     ELSE
430       CONTINUE
440     ENDIF
450   CONTINUE
460   CONTINUE

C     CALL SUBROUTINE GETIME TO OBTAIN CPU TIME FOR ITERATION PROCESS

461   CALL GETIME(IET)

C     OUTPUT HEADER INFORMATION

462   WRITE(6,464)
463   WRITE(30,464)
464   FORMAT(1X,'EQUATION:  U" + U**2 = 60X + 100X**6')
465   IF (ITYPE.EQ.1) THEN
466     WRITE(6,468) COORD(1),ULBC,COORD(NSNP),URBC
467     WRITE(30,468) COORD(1),ULBC,COORD(NSNP),URBC
468     FORMAT(1X,'B.C.: U(',F2.0,')=',F3.0,'; U(',F2.0,')=',F5.0,/)
469   ELSE
470     WRITE(6,472) COORD(1),ULBC,COORD(NSNP),URBC
471     WRITE(30,472) COORD(1),ULBC,COORD(NSNP),URBC
472     FORMAT(1X,'B.C.: U(',F2.0,')=',F3.0,';DU/DX(',F2.0,')=',F5.0,/)
473   ENDIF
475   IF (METHU.EQ.1) THEN
476     WRITE(6,478)
477     WRITE(30,478)
478     FORMAT(1X,'ITERATION METHOD: U*=U',/)
479   ELSEIF (METHU.EQ.2) THEN
480     WRITE(6,482)
481     WRITE(30,482)
482     FORMAT(1X,'ITERATION METHOD: U*=(U+UOLD)/2',/)
483   ELSE
484     WRITE(6,486)AW,BW,AW,BW
485     WRITE(30,486)AW,BW,AW,BW
486     FORMAT(1X,'ITERATION METHOD: U*=(',F3.0,'*U +',F3.0,'*UOLD)/(',
```

```
     :F3.0,'+',F3.0,')',/)
487   ENDIF
488   IF (METHBM.EQ.1) THEN
489     WRITE(6,491)
490     WRITE(30,491)
491    FORMAT(1X,'METHOD OF B MATRIX INTEGRAL EVALUATION: MIDPOINT',/)
496   ELSE
497     WRITE(6,499)
498     WRITE(30,499)
499    FORMAT(1X,'METHOD OF B MATRIX INTEGRAL EVALUATION: LINEAR',/)
500   ENDIF
502   IF (ITER.GE.200) THEN
503     WRITE(6,505)
504     WRITE(30,505)
505     FORMAT(1X,'CONVERGENCE NOT OBTAINED AFTER 200 ITERATIONS.')
506   ELSEIF (ABS(U(NMAX)).GT.(10.**20).OR.ABS(U(NSNP-1)).GT.
     :(10.**20)) THEN
507     WRITE(6,509)
508     WRITE(30,509)
509     FORMAT(1X,'SOLUTION PROCESS DIVERGES.')
510   ELSE
511     WRITE(6,520) ITER,NEL
515     WRITE(30,520) ITER,NEL
520     FORMAT(1X,'CONVERGENCE OBTAINED AFTER ',I3,' ITERATIONS USING ',
     :I3,' ELEMENTS.',/)
525   ENDIF
530   RETURN
540   END
```

```
C     ***************************************************************************
C     *                        SUBROUTINE CLEXTB                                *
C     *                                                                         *
C     * THIS SUBROUTINE COMPUTES THE EXACT SOLUTION, U=10X**3, FOR              *
C     * MAIN PROGRAM NU2KB AT THE SPECIFIED NODAL POINTS.                       *
C     ***************************************************************************

100   SUBROUTINE CLEXTB
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = 10.*COORD(NN)**3
150   CONTINUE
160   RETURN
170   END
```

```
C     ************************************************************************
C     *                          SUBROUTINE CLOTPT                           *
C     * THIS SUBROUTINE COMPUTES THE PER CENT ERROR BETWEEN THE EXACT        *
C     * AND FEM SOLUTIONS, CPU* FOR THE ITERATION PROCESS, AND PRINTS        *
C     * OUT ALL DATA IN TABULAR FORM FOR MAIN PROGRAMS NU2KA & NU2KB.        *
C     ************************************************************************

100   SUBROUTINE CLOTPT(CPUSTAR,IET)
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :TLEN,ICORR(100,2),ITYPE,NEL,NSNP
115   SUMDIF=0.

C     CALCULATE PER CENT ERROR AT EACH NODE AND SUM THE ABSOLUTE VALUE
C     OF ALL THE ERRORS

120   DO 150 IK=2,NSNP
130     UDIF(IK)=100.*(U(IK)-UEXT(IK))/UEXT(IK)
140     SUMDIF=SUMDIF + ABS(UDIF(IK))
150   CONTINUE
160   UDIF(1)=U(1)-UEXT(1)

C     COMPUTE THE ELAPSED TIME OF THE ITERATION PROCESS

164   ELTIME=IET*.000026
165   WRITE(6,169) ELTIME
166   WRITE(30,169)ELTIME
169   FORMAT(1X,'ELAPSED TIME FOR THE ITERATION PROCESS IS ',F9.4,
     :' SECONDS.')

C     OUTPUT DATA IN TABULAR FORMAT

170   WRITE(6,180)
175   WRITE(30,180)
180   FORMAT(/,1X,'X-COORD',3X,'U EXACT',3X,'U FEM',7X,'% DIFF')
190   WRITE(6,200) (COORD(NP),UEXT(NP),U(NP),UDIF(NP), NP=1,NSNP)
195   WRITE(30,200) (COORD(NP),UEXT(NP),U(NP),UDIF(NP), NP=1,NSNP)
200   FORMAT(/,2X,F5.3,4X,F9.4,3X,F9.4,4X,F5.1)

C     CALCULATE CPU* FOR THE ITERATION PROCESS

205   CPUSTAR=ELTIME*SUMDIF/NSNP
210   WRITE(6,220) CPUSTAR
215   WRITE(30,220) CPUSTAR
220   FORMAT(/,1X,'CPU* FOR THE ITERATION PROCESS IS ',F9.4,' SECONDS.')
230   RETURN
240   END
```

APPENDIX G. PROGRAM LISTINGS FOR QUASILINEARIZATION

```
C     ************************************************************************
C     *                        PROGRAM NU2QA                                 *
C     *  THIS PROGRAM SOLVES THE NONLINEAR SECOND ORDER DIFFERENTIAL         *
C     *  EQUATION:                                                           *
C     *     U" - U**2 = 6 - 9X**4;  UEXACT=3X**2  WITH VARIABLE DOMAIN       *
C     *  BY THE PROCESS OF QUASILINEARIZATION.  THE USER SELECTS:            *
C     *    1) NUMBER OF ELEMENTS                                             *
C     *    2) SIZE OF DOMAIN                                                 *
C     *    3) X AND U(X) AT THE LEFT BOUNDARY                                *
C     *    4) U(X) OR U'(X) AT THE RIGHT BOUNDARY                            *
C     *    5) ITERATION STRATEGY FOR DETERMINING U*                          *
C     *    6) INTERPOLATION STRATEGY FOR THE B MATRIX INTEGRAL               *
C     *    7) INTERPOLATION STRATEGY FOR THE EXCITATION INTEGRAL             *
C     ************************************************************************

110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
115   CONV=.0001

C     READ IN PARAMETERS FROM DATA FILE

130   READ(29,*) NEL,TLEN,COORD(1),ULBC,ITYPE,URBC

C     CALCULATE NUMBER OF NODAL POINTS

135   NSNP=NEL+1

C     DETERMINE ELEMENT SIZE OF EQUAL LENGTHS

137   ELEN=TLEN/FLOAT(NEL)

C     ESTABLISH LOCAL TO GLOBAL CORRESPONDENCE AND X COORDINATE OF
C     EACH NODE

160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
164     COORD(IEL+1)=COORD(IEL)+ELEN
169   CONTINUE

C     CALL SUBROUTINE NU2QAM TO CREATE A MATRIX AND F VECTOR

170   CALL NU2QAM

C     CALL SUBROUTINE NU2QAI TO PERFORM SOLUTION ITERATION

180   CALL NU2QAI(IET)

C     CALL SUBROUTINE QLEXTA TO COMPUTE EXACT SOLUTION U=3X**2
```

```
190   CALL QLEXTA

C     CALL SUBROUTINE QLOTPT TO PRINT OUT DATA, COMPUTATIONAL EFFICIENCY

200   CALL QLOTPT(CPUSTAR,IET)
210   END
```

```
C     ************************************************************************
C     *                        SUBROUTINE NU2QAM                             *
C     *                                                                      *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND STEADY F VECTOR FOR        *
C     * MAIN PROGRAM NU2QA.                                                  *
C     ************************************************************************

100   SUBROUTINE NU2QAM
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
120   DIMENSION AE(2,2), FS1E(2), FS2E(2)

C     ZERO OUT A MATRIX AND ALL VECTORS

140   DO 210 IZ = 1,NSNP
150      FS(IZ) = 0.
154      U(IZ) =SQRT(ABS(9.*COORD(IZ)**4-6.))
156      UOLD(IZ)=0.
160      DO 200 JZ = 1,NSNP
170           A(IZ,JZ) = 0.
175           B(IZ,JZ) = 0.
176           C(IZ,JZ) = 0.
200      CONTINUE
210   CONTINUE

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

213   ALPHA=0.
215   DO 375 IEL=1,NEL
220     AE(1,1)=1./ELEN
230     AE(1,2)=(-1./ELEN)
240     AE(2,1)=AE(1,2)
250     AE(2,2)=AE(1,1)
260     FS1E(1)=3.*ELEN
270     FS1E(2)=FS1E(1)
272     F1=(ALPHA**4)*ELEN/2.
274     F2=2.*(ALPHA**3)*(ELEN**2)/3.
276     F3=(ALPHA**2)*(ELEN**3)/2.
278     F4=ALPHA*(ELEN**4)/5.
280     F5=(ELEN**5)/30.
287     FS2E(1)=(-9.)*(F1 + F2 + F3 + F4 + F5)
290     FS2E(2)=(-9)*(F1 + 2.*F2 + 3.*F3 + 4.*F4 + 5.*F5)
300       DO 370 II=1,2
310         DO 350 JJ=1,2
320           IN=ICORR(IEL,II)
330           JN=ICORR(IEL,JJ)
340           A(IN,JN)=A(IN,JN) - AE(II,JJ)
350         CONTINUE
360         FS(IN)=FS1E(II) + FS2E(II) + FS(IN)
370       CONTINUE
372     ALPHA=ALPHA + ELEN
375   CONTINUE
420   RETURN
430   END
```

```
C     ************************************************************************
C     *                      SUBROUTINE NU2QAI                               *
C     *                                                                      *
C     * THIS SUBROUTINE PERFORMS THE ITERATIVE SOLUTION PROCESS FOR          *
C     * MAIN PROGRAM NU2QA.                                                  *
C     ************************************************************************

100   SUBROUTINE NU2QAI(IET)
102   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
104   DIMENSION WKAREA(40600), DIF(100), BE(2,2), USTAR(100), FT(100),
     :FUE(2),FU(100)
C     SELECT METHOD OF DETERMINING USTAR

105   PRINT*,'SELECT METHOD OF U* DETERMINATION.'
106   PRINT*,'1: U* = U'
107   PRINT*,'2: U* = AVERAGE OF LAST TWO COMPUTED VALUES OF U'
108   PRINT*,'3: U* = WEIGHTED AVERAGE OF LAST TWO COMPUTED VALUES OF U'
109   READ(6,*) METHU
110   IF (METHU.EQ.3) THEN
111     PRINT*,'CHOOSE WEIGHTING VALUES A AND B'
112     READ(6,*) AW,BW
113   ELSE
114     CONTINUE
115   ENDIF

C     SELECT METHOD OF DETERMINING UNSTEADY B MATRIX

116   PRINT*,'SELECT METHOD OF DETERMINING UNSTEADY B MATRIX.'
117   PRINT*,'1:  MIDPOINT APPROXIMATION FOR U OVER THE ELEMENT.'
119   PRINT*,'2:  U LINEARIZED OVER THE LENGTH OF THE ELEMENT.'
120   READ(6,*) METHBM

C     SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR

121   PRINT*,'SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR.'
122   PRINT*,'1:  MIDPOINT APPROXIMATION'
123   PRINT*,'2:  1/4 - 3/4 APPROXIMATION'
124   PRINT*,'3:  LINEAR'
125   READ(6,*) METHFU

C     CALL SUBROUTINE SETIME TO BEGIN TIMING ITERATION PROCESS

127   CALL SETIME

C     BEGIN ITERATION PROCESS

128   DO 450 ITER=1,200

C     DETERMINE VALUE OF U* AT EACH NODE AND SET VALUE OF UNSTEADY
C     FORCE VECTOR TO ZERO

129     DO 138 IU=1,NSNP
130       FU(IU)=0.
131       IF (METHU.EQ.1) THEN
```

```
132         USTAR(IU)=U(IU)
133       ELSEIF (METHU.EQ.2) THEN
134         USTAR(IU)=(U(IU)+UOLD(IU))/2.
135       ELSE
136         USTAR(IU)=(AW*U(IU)+BW*UOLD(IU))/(AW+BW)
137       ENDIF
138     CONTINUE

C     DETERMINE UNSTEADY ELEMENT B MATRIX

140     DO 210 IEL=1,NEL
145       IF (METHBM.EQ.1) THEN
146         BE(1,1)=(ELEN/3.)*(USTAR(IEL)+USTAR(IEL+1))
147         BE(1,2)=(ELEN/6.)*(USTAR(IEL)+USTAR(IEL+1))
148         BE(2,1)=BE(1,2)
149         BE(2,2)=BE(1,1)
150       ELSE
151         BE(1,1)=(ELEN/6.)*(3.*USTAR(IEL) + USTAR(IEL+1))
152         BE(1,2)=(ELEN/6.)*(USTAR(IEL) + USTAR(IEL+1))
153         BE(2,1)=BE(1,2)
154         BE(2,2)=(ELEN/6.)*(USTAR(IEL) + 3.*USTAR(IEL+1))
156       ENDIF

C     DETERMINE SYSTEM B MATRIX BY DISTRIBUTING ELEMENT B MATRICES
C     ACCORDING TO THE LOCAL TO GLOBAL CORRESPONDENCE

157       DO 163 II=1,2
158         DO 162 JJ=1,2
159           IN=ICORR(IEL,II)
160           JN=ICORR(IEL,JJ)
161           B(IN,JN)=BE(II,JJ) + B(IN,JN)
162         CONTINUE
163       CONTINUE

C     DETERMINE UNSTEADY ELEMENT FORCE VECTOR

165       IF (METHFU.EQ.1) THEN
166         FUE(1)=(ELEN/2.)*((USTAR(IEL)+USTAR(IEL+1))/2.)**2
167         FUE(2)=FUE(1)
169       ELSEIF (METHFU.EQ.2) THEN
171         FUE(1)=(ELEN/2.)*(3.*USTAR(IEL)/4. + USTAR(IEL+1)/4.)**2
172         FUE(2)=(ELEN/2.)*(USTAR(IEL)/4. + 3.*USTAR(IEL+1)/4.)**2
173       ELSE
174         FUE(1)=ELEN*(USTAR(IEL)**2/4. +USTAR(IEL)*USTAR(IEL+1)/6.
    :       + USTAR(IEL+1)**2/12.)
175         FUE(2)=ELEN*(USTAR(IEL)**2/12. + USTAR(IEL)*USTAR(IEL+1)/6.
    :       + USTAR(IEL+1)**2/4.)
176       ENDIF

C     DETERMINE UNSTEADY SYSTEM FORCE VECTOR BY DISTRIBUTING ELEMENTAL
C     FORCE VECTORS ACCORDING TO THE LOCAL TO GLOBAL CORRESPONDENCE

177       DO 180 II=1,2
178         IN=ICORR(IEL,II)
179         FU(IN)=FUE(II) + FU(IN)
```

```
180       CONTINUE
210     CONTINUE

C     DETERMINE TOTAL SYSTEM MATRIX

220     DO 240 IP=1,NSNP
221       DO 232 JP=1,NSNP
230         C(IP,JP)=A(IP,JP)-B(IP,JP)

C     RESET B MATRIX TO ZERO

231         B(IP,JP)=0.
232       CONTINUE

C     UPDATE VALUE OF U AT THE PREVIOUS ITERATION

235       UOLD(IP)=U(IP)

C     DETERMINE TOTAL SYSTEM FORCE VECTOR

236       FT(IP)=FS(IP)-FU(IP)
240     CONTINUE

C     IMPOSE BOUNDARY CONDITIONS

241     C(1,1)=1.
242     C(1,2)=0.
243     FT(1)=ULBC
244     IF (ITYPE.EQ.1) THEN
245       C(NSNP,NSNP-1)=0.
246       C(NSNP,NSNP)=1.
247       FT(NSNP)=URBC
248     ELSE
249       FT(NSNP)=FT(NSNP)-URBC
250     ENDIF
255     M=1
260     IDGT=3
270     IQ=100

C     CALL SUBROUTINE LEQT2F TO SOLVE SET OF LINEAR ALGEBRAIC EQUATIONS

280     CALL LEQT2F(C,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
290     DO 310 NEW=1,NSNP
300       U(NEW)=FT(NEW)
C         WRITE(*,*) 'UNEW=',U(NEW)

C     TEST FOR CONVERGENCE

306       DIF(NEW)=ABS(U(NEW)-UOLD(NEW))
310     CONTINUE
320     DIFMAX=DIF(1)
325     NMAX=1
330     DO 390 IJ=1,NEL
340       IF (DIF(IJ+1).GE.DIF(IJ)) THEN
350         DIFMAX=DIF(IJ+1)
355         NMAX=IJ+1
```

```
360         ELSE
370           CONTINUE
380         ENDIF
390       CONTINUE
400       IF (U(NMAX).EQ.0.) GO TO 450
405       IF (ABS(DIFMAX/U(NMAX)).LT.CONV) THEN
410         GO TO 460
420       ELSE
430         CONTINUE
440       ENDIF
450     CONTINUE
460     CONTINUE

C       CALL SUBROUTINE GETIME TO OBTAIN CPU TIME FOR ITERATION PROCESS

461     CALL GETIME(IET)

C       OUTPUT HEADER INFORMATION

462     WRITE(6,464)
463     WRITE(30,464)
464     FORMAT(1X,'EQUATION:  U" - U**2 = 6 - 9X**4')
465     IF (ITYPE.EQ.1) THEN
466       WRITE(6,468) COORD(1),ULBC,COORD(NSNP),URBC
467       WRITE(30,468) COORD(1),ULBC,COORD(NSNP),URBC
468       FORMAT(1X,'B.C.: U(',F2.0,')=',F2.0,'; U(',F2.0,')=',F4.0,/)
469     ELSE
470       WRITE(6,472) COORD(1),ULBC,COORD(NSNP),URBC
471       WRITE(30,472) COORD(1),ULBC,COORD(NSNP),URBC
472       FORMAT(1X,'B.C.: U(',F2.0,')=',F2.0,';DU/DX(',F2.0,')=',F4.0,/)
473     ENDIF
475     IF (METHU.EQ.1) THEN
476       WRITE(6,478)
477       WRITE(30,478)
478       FORMAT(1X,'ITERATION METHOD: U*=U',/)
479     ELSEIF (METHU.EQ.2) THEN
480       WRITE(6,482)
481       WRITE(30,482)
482       FORMAT(1X,'ITERATION METHOD: U*=(U+UOLD)/2',/)
483     ELSE
484       WRITE(6,486)AW,BW,AW,BW
485       WRITE(30,486)AW,BW,AW,BW
486       FORMAT(1X,'ITERATION METHOD: U*=(',F4.1,'*U +',F4.1,'*UOLD)/(',
        :F4.1,'+',F4.1,')',/)
487     ENDIF
488     IF (METHBM.EQ.1) THEN
489       WRITE(6,491)
490       WRITE(30,491)
491      FORMAT(1X,'METHOD OF B MATRIX INTEGRAL EVALUATION: MIDPOINT',/)
496     ELSE
497       WRITE(6,499)
498       WRITE(30,499)
499      FORMAT(1X,'METHOD OF B MATRIX INTEGRAL EVALUATION: LINEAR',/)
500     ENDIF
501     IF (METHFU.EQ.1) THEN
502       WRITE(6,504)
```

```
503     WRITE(30,504)
504    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: MIDPOINT',/)
505   ELSEIF (METHFU.EQ.2) THEN
506     WRITE(6,508)
507     WRITE(30,508)
508    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: 1/4-3/4',/)
509   ELSE
510     WRITE(6,512)
511     WRITE(30,512)
512    FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: LINEAR',/)
513   ENDIF
515   IF (ITER.GE.200) THEN
516     WRITE(6,518)
517     WRITE(30,518)
518     FORMAT(1X,'CONVERGENCE NOT OBTAINED AFTER 200 ITERATIONS.')
519   ELSEIF (ABS(U(NMAX)).GT.(10.**20).OR.ABS(U(NSNP-1)).GT.
     :(10.**20)) THEN
520     WRITE(6,522)
521     WRITE(30,522)
522     FORMAT(1X,'SOLUTION PROCESS DIVERGES.')
523   ELSE
524     WRITE(6,526) ITER,NEL
525     WRITE(30,526) ITER,NEL
526     FORMAT(1X,'CONVERGENCE OBTAINED AFTER ',I3,' ITERATIONS USING ',
     :I3,' ELEMENTS.',/)
530   ENDIF
540   RETURN
550   END
```

```
C     ***************************************************************
C     *                      SUBROUTINE QLEXTA                      *
C     *                                                             *
C     * THIS SUBROUTINE COMPUTES THE EXACT SOLUTION, U=3X**2, FOR   *
C     * MAIN PROGRAM NU2QA AT THE SPECIFIED NODAL POINTS.           *
C     ***************************************************************

100   SUBROUTINE QLEXTA
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = 3.*COORD(NN)**2
150   CONTINUE
160   RETURN
170   END
```

```
C     ********************************************************************
C     *                          PROGRAM NU2QB                           *
C     *  THIS PROGRAM SOLVES THE NONLINEAR SECOND ORDER DIFFERENTIAL     *
C     *  EQUATION:                                                       *
C     *     U" + U**2 = 60X + 100X**6;  UEXACT=10X**3 WITH VARIABLE      *
C     *  DOMAIN BY THE PROCESS OF QUASILINEARIZATION. THE USER SELECTS:  *
C     *    1) NUMBER OF ELEMENTS                                         *
C     *    2) SIZE OF DOMAIN                                             *
C     *    3) X AND U(X) AT THE LEFT BOUNDARY                            *
C     *    4) U(X) OR U'(X) AT THE RIGHT BOUNDARY                        *
C     *    5) ITERATION STRATEGY FOR DETERMINING U*                      *
C     *    6) INTERPOLATION STRATEGY FOR THE B MATRIX INTEGRAL           *
C     *    7) INTERPOLATION STRATEGY FOR THE EXCITATION INTEGRAL         *
C     ********************************************************************

110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
115   CONV=.0000001

C     READ IN PARAMETERS FROM DATA FILE

130   READ(29,*) NEL,TLEN,COORD(1),ULBC,ITYPE,URBC

C     CALCULATE NUMBER OF NODAL POINTS

135   NSNP=NEL+1

C     DETERMINE ELEMENT SIZE OF EQUAL LENGTHS

137   ELEN=TLEN/FLOAT(NEL)

C     ESTABLISH LOCAL TO GLOBAL CORRESPONDENCE AND X COORDINATE OF
C     EACH NODE

160   DO 169 IEL=1,NEL
162     ICORR(IEL,1)=IEL
163     ICORR(IEL,2)=IEL+1
164     COORD(IEL+1)=COORD(IEL)+ELEN
169   CONTINUE

C     CALL SUBROUTINE NU2QBM TO CREATE A MATRIX AND F VECTOR

170   CALL NU2QBM

C     CALL SUBROUTINE NU2QBI TO PERFORM SOLUTION ITERATION

180   CALL NU2QBI(IET)

C     CALL SUBROUTINE QLEXTB TO COMPUTE EXACT SOLUTION U=3X**2

190   CALL QLEXTB

C     CALL SUBROUTINE QLOTPT TO PRINT OUT DATA, COMPUTATIONAL EFFICIENCY
```

```
200    CALL QLOTPT(CPUSTAR,IET)
210    END
```

```
C     ***************************************************************
C     *                      SUBROUTINE NU2QBM                      *
C     *                                                             *
C     * THIS SUBROUTINE COMPUTES THE A MATRIX AND STEADY F VECTOR FOR *
C     * MAIN PROGRAM NU2QB.                                         *
C     ***************************************************************

100   SUBROUTINE NU2QBM
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
120   DIMENSION AE(2,2), FS1E(2), FS2E(2), G1(100), G2(100), G3(100)

C     ZERO OUT A MATRIX AND ALL VECTORS

140   DO 210 IZ = 1,NSNP
145      FS(IZ) = 0.
C        IF (COORD(IZ).LE.1.) THEN
C          U(IZ)=0.
C        ELSE
C        G1(IZ) = 60.*COORD(IZ) + 100.*COORD(IZ)**6
C        G2(IZ) = 1500.*COORD(IZ)**4/SQRT(G1(IZ))
C        G3(IZ) = (60.+600.*COORD(IZ)**5)**2/(4.*G1(IZ)**1.5)
C        U(IZ)=SQRT(ABS(G1(IZ)-G2(IZ)+G3(IZ)))
C        ENDIF
157      U(IZ) = SQRT(60.*COORD(IZ) + 100.*COORD(IZ)**6)
158      UOLD(IZ)= U(IZ)
160      DO 200 JZ = 1,NSNP
170           A(IZ,JZ) = 0.
175           B(IZ,JZ) = 0.
176           C(IZ,JZ) = 0.
200      CONTINUE
210   CONTINUE

C     ELEMENTAL DO LOOP TO DETERMINE A MATRIX AND F VECTOR

214   ALPHA=0.
215   DO 375 IEL=1,NEL
220     AE(1,1)=1./ELEN
230     AE(1,2)=(-1./ELEN)
240     AE(2,1)=AE(1,2)
250     AE(2,2)=AE(1,1)
260     FS1E(1)=30.*ALPHA*ELEN + 10.*ELEN**2
270     FS1E(2)=30.*ALPHA*ELEN + 20.*ELEN**2
272     F1=50.*(ALPHA**6)*ELEN
274     F2=100.*(ALPHA**5)*(ELEN**2)
276     F3=125.*(ALPHA**4)*(ELEN**3)
278     F4=100.*(ALPHA**3)*(ELEN**4)
280     F5=50.*(ALPHA**2)*(ELEN**5)
281     F6=100.*ALPHA*(ELEN**6)/7.
282     F7=25.*(ELEN**7)/16.
287     FS2E(1)=F1 + F2 + F3 + F4 + F5 + F6 + F7
290     FS2E(2)=F1 + 2.*F2 + 3.*F3 + 4.*F4 + 5.*F5 + 6.*F6 + 7*F7
300       DO 370 II=1,2
310         DO 350 JJ=1.2
320           IN=ICORR(IEL,II)
```

```
330           JN=ICORR(IEL,JJ)
340           A(IN,JN)=A(IN,JN) - AE(II,JJ)
350         CONTINUE
360         FS(IN)=FS1E(II) + FS2E(II) + FS(IN)
370       CONTINUE
372     ALPHA=ALPHA + ELEN
375   CONTINUE
420   RETURN
430   END
```

```
C     ************************************************************************
C     *                        SUBROUTINE NU2QBI                             *
C     *                                                                      *
C     * THIS SUBROUTINE PERFORMS THE ITERATIVE SOLUTION PROCESS FOR          *
C     * MAIN PROGRAM NU2QA.                                                  *
C     ************************************************************************

100   SUBROUTINE NU2QBI(IET)
102   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
104   DIMENSION WKAREA(40600), DIF(100), BE(2,2), USTAR(100), FT(100),
     :FUE(2),FU(100)
C     SELECT METHOD OF DETERMINING USTAR

105   PRINT*,'SELECT METHOD OF U* DETERMINATION.'
106   PRINT*,'1: U* = U'
107   PRINT*,'2: U* = AVERAGE OF LAST TWO COMPUTED VALUES OF U'
108   PRINT*,'3: U* = WEIGHTED AVERAGE OF LAST TWO COMPUTED VALUES OF U'
109   READ(6,*) METHU
110   IF (METHU.EQ.3) THEN
111     PRINT*,'CHOOSE WEIGHTING VALUES A AND B'
112     READ(6,*) AW,BW
113   ELSE
114     CONTINUE
115   ENDIF

C     SELECT METHOD OF DETERMINING UNSTEADY B MATRIX

116   PRINT*,'SELECT METHOD OF DETERMINING UNSTEADY B MATRIX.'
117   PRINT*,'1:  MIDPOINT APPROXIMATION FOR U OVER THE ELEMENT.'
119   PRINT*,'2:  U LINEARIZED OVER THE LENGTH OF THE ELEMENT.'
120   READ(6,*) METHBM

C     SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR

121   PRINT*,'SELECT METHOD OF DETERMINING UNSTEADY FORCE VECTOR.'
122   PRINT*,'1:  MIDPOINT APPROXIMATION'
123   PRINT*,'2:  1/4 - 3/4 APPROXIMATION'
124   PRINT*,'3:  LINEAR'
125   READ(6,*) METHFU

C     CALL SUBROUTINE SETIME TO BEGIN TIMING ITERATION PROCESS

127   CALL SETIME

C     BEGIN ITERATION PROCESS

128   DO 450 ITER=1,200

C     DETERMINE VALUE OF U* AT EACH NODE AND SET VALUE OF UNSTEADY
C     FORCE VECTOR TO ZERO

129     DO 138 IU=1,NSNP
130       FU(IU)=0.
131       IF (METHU.EQ.1) THEN
```

```
132         USTAR(IU)=U(IU)
133       ELSEIF (METHU.EQ.2) THEN
134         USTAR(IU)=(U(IU)+UOLD(IU))/2.
135       ELSE
136         USTAR(IU)=(AW*U(IU)+BW*UOLD(IU))/(AW+BW)
137       ENDIF
138     CONTINUE

C     DETERMINE UNSTEADY ELEMENT B MATRIX

140     DO 210 IEL=1,NEL
145       IF (METHBM.EQ.1) THEN
146         BE(1,1)=(ELEN/3.)*(USTAR(IEL)+USTAR(IEL+1))
147         BE(1,2)=(ELEN/6.)*(USTAR(IEL)+USTAR(IEL+1))
148         BE(2,1)=BE(1,2)
149         BE(2,2)=BE(1,1)
150       ELSE
151         BE(1,1)=(ELEN/6.)*(3.*USTAR(IEL) + USTAR(IEL+1))
152         BE(1,2)=(ELEN/6.)*(USTAR(IEL) + USTAR(IEL+1))
153         BE(2,1)=BE(1,2)
154         BE(2,2)=(ELEN/6.)*(USTAR(IEL) + 3.*USTAR(IEL+1))
156       ENDIF

C     DETERMINE SYSTEM B MATRIX BY DISTRIBUTING ELEMENT B MATRICES
C     ACCORDING TO THE LOCAL TO GLOBAL CORRESPONDENCE

157       DO 163 II=1,2
158         DO 162 JJ=1,2
159           IN=ICORR(IEL,II)
160           JN=ICORR(IEL,JJ)
161           B(IN,JN)=BE(II,JJ) + B(IN,JN)
162         CONTINUE
163       CONTINUE

C     DETERMINE UNSTEADY ELEMENT FORCE VECTOR

165       IF (METHFU.EQ.1) THEN
166         FUE(1)=(ELEN/2.)*((USTAR(IEL)+USTAR(IEL+1))/2.)**2
167         FUE(2)=FUE(1)
169       ELSEIF (METHFU.EQ.2) THEN
171         FUE(1)=(ELEN/2.)*(3.*USTAR(IEL)/4. + USTAR(IEL+1)/4.)**2
172         FUE(2)=(ELEN/2.)*(USTAR(IEL)/4. + 3.*USTAR(IEL+1)/4.)**2
173       ELSE
174         FUE(1)=ELEN*(USTAR(IEL)**2/4. +USTAR(IEL)*USTAR(IEL+1)/6.
     :       + USTAR(IEL+1)**2/12.)
175         FUE(2)=ELEN*(USTAR(IEL)**2/12. + USTAR(IEL)*USTAR(IEL+1)/6.
     :       + USTAR(IEL+1)**2/4.)
176       ENDIF

C     DETERMINE UNSTEADY SYSTEM FORCE VECTOR BY DISTRIBUTING ELEMENTAL
C     FORCE VECTORS ACCORDING TO THE LOCAL TO GLOBAL CORRESPONDENCE

177       DO 180 II=1,2
178         IN=ICORR(IEL,II)
179         FU(IN)=FUE(II) + FU(IN)
```

```
180         CONTINUE
210       CONTINUE

C       DETERMINE TOTAL SYSTEM MATRIX

220       DO 240 IP=1,NSNP
221         DO 232 JP=1,NSNP
230           C(IP,JP)=A(IP,JP)+B(IP,JP)

C       RESET B MATRIX TO ZERO

231           B(IP,JP)=0.
232         CONTINUE

C       UPDATE VALUE OF U AT THE PREVIOUS ITERATION

235         UOLD(IP)=U(IP)

C       DETERMINE TOTAL SYSTEM FORCE VECTOR

236         FT(IP)=FS(IP)+FU(IP)
240       CONTINUE

C       IMPOSE BOUNDARY CONDITIONS

241       C(1,1)=1.
242       C(1,2)=0.
243       FT(1)=ULBC
244       IF (ITYPE.EQ.1) THEN
245         C(NSNP,NSNP-1)=0.
246         C(NSNP,NSNP)=1.
247         FT(NSNP)=URBC
248       ELSE
249         FT(NSNP)=FT(NSNP)-URBC
250       ENDIF
257       M=1
260       IDGT=3
270       IQ=100

C       CALL SUBROUTINE LEQT2F TO SOLVE SET OF LINEAR ALGEBRAIC EQUATIONS

280       CALL LEQT2F(C,M,NSNP,IQ,FT,IDGT,WKAREA,IER)
290       DO 310 NEW=1,NSNP
300         U(NEW)=FT(NEW)
C           WRITE(*,*) 'UNEW=',U(NEW)

C       TEST FOR CONVERGENCE

306         DIF(NEW)=ABS(U(NEW)-UOLD(NEW))
310       CONTINUE
320       DIFMAX=DIF(1)
325       NMAX=1
330       DO 390 IJ=1,NEL
340         IF (DIF(IJ+1).GE.DIF(IJ)) THEN
350           DIFMAX=DIF(IJ+1)
355           NMAX=IJ+1
```

```
360         ELSE
370           CONTINUE
380         ENDIF
390       CONTINUE
405       IF (ABS(DIFMAX/U(NMAX)).LT.CONV) THEN
410         GO TO 460
420       ELSE
430         CONTINUE
440       ENDIF
450     CONTINUE
460     CONTINUE

C       CALL SUBROUTINE GETIME TO OBTAIN CPU TIME FOR ITERATION PROCESS

461     CALL GETIME(IET)

C       OUTPUT HEADER INFORMATION

462     WRITE(6,464)
463     WRITE(30,464)
464     FORMAT(1X,'EQUATION:  U" + U**2 = 60X + 100X**6')
465     IF (ITYPE.EQ.1) THEN
466       WRITE(6,468) COORD(1),ULBC,COORD(NSNP),URBC
467       WRITE(30,468) COORD(1),ULBC,COORD(NSNP),URBC
468       FORMAT(1X,'B.C.: U(',F2.0,')=',F2.0,'; U(',F2.0,')=',F5.0,/)
469     ELSE
470       WRITE(6,472) COORD(1),ULBC,COORD(NSNP),URBC
471       WRITE(30,472) COORD(1),ULBC,COORD(NSNP),URBC
472       FORMAT(1X,'B.C.: U(',F2.0,')=',F2.0,';DU/DX(',F2.0,')=',F4.0,/)
473     ENDIF
475     IF (METHU.EQ.1) THEN
476       WRITE(6,478)
477       WRITE(30,478)
478       FORMAT(1X,'ITERATION METHOD: U*=U',/)
479     ELSEIF (METHU.EQ.2) THEN
480       WRITE(6,482)
481       WRITE(30,482)
482       FORMAT(1X,'ITERATION METHOD: U*=(U+UOLD)/2',/)
483     ELSE
484       WRITE(6,486)AW,BW,AW,BW
485       WRITE(30,486)AW,BW,AW,BW
486       FORMAT(1X,'ITERATION METHOD: U*=(',F4.1,'*U +',F4.1,'*UOLD)/(',
    :F4.1,'+',F4.1,')',/)
487     ENDIF
488     IF (METHBM.EQ.1) THEN
489       WRITE(6,491)
490       WRITE(30,491)
491      FORMAT(1X,'METHOD OF B MATRIX INTEGRAL EVALUATION: MIDPOINT',/)
496     ELSE
497       WRITE(6,499)
498       WRITE(30,499)
499      FORMAT(1X,'METHOD OF B MATRIX INTEGRAL EVALUATION: LINEAR',/)
500     ENDIF
501     IF (METHFU.EQ.1) THEN
502       WRITE(6,504)
503       WRITE(30,504)
```

```
504     FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: MIDPOINT',/)
505    ELSEIF (METHFU.EQ.2) THE
506      WRITE(6,508)
507      WRITE(30,508)
508     FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: 1/4-3/4',/)
509    ELSE
510      WRITE(6,512)
511      WRITE(30,512)
512     FORMAT(1X,'METHOD OF EXCITATION INTEGRAL EVALUATION: LINEAR',/)
513    ENDIF
515    IF (ITER.GE.200) THEN
516      WRITE(6,518)
517      WRITE(30,518)
518      FORMAT(1X,'CONVERGENCE NOT OBTAINED AFTER 200 ITERATIONS.')
519    ELSEIF (ABS(U(NMAX)).GT.(10.**20).OR.ABS(U(NSNP-1)).GT.
     :(10.**20)) THEN
520      WRITE(6,522)
521      WRITE(30,522)
522      FORMAT(1X,'SOLUTION PROCESS DIVERGES.')
523    ELSE
524      WRITE(6,526) ITER,NEL
525      WRITE(30,526) ITER,NEL
526      FORMAT(1X,'CONVERGENCE OBTAINED AFTER ',I3,' ITERATIONS USING ',
     :I3,' ELEMENTS.',/)
530    ENDIF
540    RETURN
550    END
```

```
C     ***************************************************************
C     *                      SUBROUTINE QLEXTB                      *
C     *                                                             *
C     * THIS SUBROUTINE COMPUTES THE EXACT SOLUTION, U=10X**3, FOR  *
C     * MAIN PROGRAM NU2QB AT THE SPECIFIED NODAL POINTS.           *
C     ***************************************************************

100   SUBROUTINE QLEXTB
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
130   DO 150 NN = 1,NSNP
140      UEXT(NN) = 10.*COORD(NN)**3
150   CONTINUE
160   RETURN
170   END
```

```
C     ************************************************************************
C     *                          SUBROUTINE QLOTPT                           *
C     * THIS SUBROUTINE COMPUTES THE PER CENT ERROR BETWEEN THE EXACT  *
C     * AND FEM SOLUTIONS, CPU* FOR THE ITERATION PROCESS, AND PRINTS  *
C     * OUT ALL DATA IN TABULAR FORM FOR MAIN PROGRAM NU2QA AND NU2QB. *
C     ************************************************************************

100   SUBROUTINE QLOTPT(CPUSTAR,IET)
110   COMMON A(100,100),FS(100),B(100,100),C(100,100),U(100),UOLD(100),
     :UEXT(100),UDIF(100),COORD(100),ELEN,CONV,ELTIME,ULBC,URBC,
     :ICORR(100,2),ITYPE,NEL,NSNP
115   SUMDIF=0.

C     CALCULATE PER CENT ERROR AT EACH NODE AND SUM THE ABSOLUTE VALUE
C     OF ALL THE ERRORS

120   DO 150 IK=2,NSNP
130     UDIF(IK)=100.*(U(IK)-UEXT(IK))/UEXT(IK)
140     SUMDIF=SUMDIF + ABS(UDIF(IK))
150   CONTINUE
160   UDIF(1)=U(1)-UEXT(1)

C     COMPUTE THE ELAPSED TIME OF THE ITERATION PROCESS

164   ELTIME=IET*.000026
165   WRITE(6,169) ELTIME
166   WRITE(30,169)ELTIME
169   FORMAT(1X,'ELAPSED TIME FOR THE ITERATION PROCESS IS ',F9.4,
     :' SECONDS.')

C     OUTPUT DATA IN TABULAR FORMAT

170   WRITE(6,180)
175   WRITE(30,180)
180   FORMAT(/,1X,'X-COORD',4X,'U EXACT',7X,'U FEM',4X,'% DIFF')
190   WRITE(6,200) (COORD(NP),UEXT(NP),U(NP),UDIF(NP), NP=1,NSNP)
195   WRITE(30,200) (COORD(NP),UEXT(NP),U(NP),UDIF(NP), NP=1,NSNP)
200   FORMAT(/,2X,F5.3,4X,F9.4,3X,F10.4,4X,F5.1)

C     CALCULATE CPU* FOR THE ITERATION PROCESS

205   CPUSTAR=ELTIME*SUMDIF/NSNP
210   WRITE(6,220) CPUSTAR
215   WRITE(30,220) CPUSTAR
220   FORMAT(/,1X,'CPU* FOR THE ITERATION PROCESS IS ',F9.4,' SECONDS.')
230   RETURN
240   END
```

LIST OF REFERENCES

1. United States Air Force Project Rand Report R-438-PR, *Quasilinearization and Nonlinear Boundary-value Problems*, by R. E. Bellman and R. E. Kalaba, June 1965.

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