

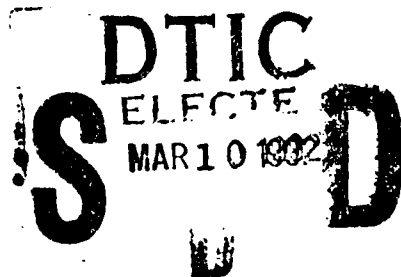


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Decomposition of Balanced Matrices.

Part I: Introduction

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Abstract

In this seven part paper, we prove the following theorem:

At least one of the following alternatives occurs for a bipartite graph G :

- The graph G has no cycle of length $4k + 2$.
- The graph G has a chordless cycle of length $4k + 2$.
- There exist two complete bipartite graphs K_1, K_2 in G having disjoint node sets, with the property that the removal of the edges in K_1, K_2 disconnects G .
- There exists a subset S of the nodes of G with the property that the removal of S disconnects G , where S can be partitioned into three disjoint sets T, A, N such that $T \neq \emptyset$, some node $x \in T$ is adjacent to every node in $A \cup N$ and, if $|T| \geq 2$, then $|A| \geq 2$ and every node of T is adjacent to every node of A .

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. Balanced matrices are important in integer programming and combinatorial optimization since the associated set packing and set covering polytopes have integral vertices.

To a 0,1 matrix A we associate a bipartite graph $G(V^r, V^c; E)$ as follows: The node sets V^r and V^c represent the row set and the column set of A and edge ij belongs to E if and only if $a_{ij} = 1$. Since a 0,1 matrix is balanced if and only if the associated bipartite graph does not contain a chordless cycle of length $4k + 2$, the above theorem provides a decomposition of balanced matrices into elementary matrices whose associated bipartite graphs have no cycle of length $4k + 2$. In Part VII of the paper, we show how to use this decomposition theorem to test in polynomial time whether a 0,1 matrix is balanced.



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1 Introduction

1.1 Balanced Matrices

This study concerns 0,1 matrices that do not contain a square submatrix of odd order with two ones per row and per column. Such matrices are called *balanced*. They were first introduced by Berge [2] and we summarize here their relevance in combinatorial optimization.

Given a 0,1 matrix A with m rows and n columns, we define the polytopes associated with the linear programming relaxation of the set packing and covering problems as follows:

$$P(A) = \{x \in R^n \mid x \geq 0; Ax \leq 1\},$$

$$Q(A) = \{y \in R^m \mid 1 \geq y \geq 0; yA \geq 1\}.$$

Berge [3] has shown that if A is balanced, the polytopes $P(A)$ and $Q(A)$ have only integral vertices. The polytopes $P(A)$ and $Q(A)$ have a fractional vertex when A is a square matrix of odd order with two ones per row and per column. It follows that A is balanced if and only if, for every submatrix A' obtained from A by removing some of its rows and columns, the polytopes $P(A')$ and $Q(A')$ have integral vertices.

Berge and Las Vergnas [8] have shown that a matrix A is balanced if and only if, for any submatrix A' of A , the maximum number of 1's in a 0,1 vector in $P(A')$ is equal to the minimum number of 1's in a 0,1 vector in $Q(A')$. This property is known as the *König property*.

Let b be an n -vector of nonnegative integers. If A is balanced, Fulkerson, Hoffman and Oppenheim [19] have shown that the following two linear programs:

$$\max \{1x \mid x \geq 0; Ax \leq b\}$$

$$\min \{yb \mid y \geq 0; yA \geq 1\}$$

have integral solutions with the same optimal objective function value. This property is called the *Menger property*. If we define the transversal matrix of A as the matrix $T(A)$ whose rows are all minimal 0,1 vectors in the polytope $Q(A)$, then Berge [6] has shown that if A is balanced, $T(A)$ has the Menger property. The above results show the importance of balanced matrices in the study of combinatorial packing and covering problems. Balanced

hypergraphs (or, equivalently, balanced matrices) can also be viewed as a natural generalization of bipartite graphs. This is the motivation that led Berge to introduce the notion of balancedness. For example, balanced hypergraphs can be characterized elegantly by a bicoloring theorem, see [2]. Further results on balanced matrices can be found in [5], [7].

It follows from the definition of balanced matrices that checking balancedness is in co-NP. An interesting question is whether checking balancedness is in NP, i.e. whether there exists a "good" characterization of balanced matrices. Also of interest is whether there exists a polynomial algorithm to test whether a matrix is balanced. This paper gives an affirmative answer to these questions. Note that such recognition problems have been solved for some classes of matrices, the most notable result being that of Seymour [23] for totally unimodular matrices.

An approach to the characterization of matrices in a given class is to exhibit a sequence of composition operations that produce all the matrices in the class and no other, starting from "elementary" matrices that can be easily recognized. Seymour's [23] characterization of totally unimodular matrices, the result of Yannakakis [25] for restricted unimodular matrices, the results of Anstee and Farber [1], Hoffman, Kolen and Sakarovitch [22], Golumbic and Goss [21] for totally balanced matrices and the results of Conforti and Rao for strongly balanced matrices [12] and linear balanced matrices [13] follow this line of argument and will be surveyed in Section 2.

1.2 Definitions and Notation

Bipartite representation of a 0,1 Matrix

Given a 0,1 matrix A , the *bipartite representation of A* is the bipartite graph $G(V^r, V^c; E)$ having a node in V^r for every row of A and a node in V^c for every column of A and an edge ij joining nodes $i \in V^r$ and $j \in V^c$ if and only if the entry a_{ij} of A equals 1. The sets V^r and V^c are the *sides* of the bipartition. Conversely, let $G(V^r, V^c; E)$ be a bipartite graph with no parallel edges. Up to permutations of rows and columns, there is a unique matrix A having G as bipartite representation. In this paper we consider properties of a 0,1 matrix that are invariant with respect to permutations of rows and columns, hence we study the corresponding bipartite representation. We say that G is balanced if A is. More generally, for any property of 0,1 matrices

that is invariant with respect to permutations of rows and columns, we say that G has the property if A has the property.

For $S \subset V^r \cup V^c$, the *induced subgraph* $G \setminus S$ is obtained by removing the nodes of S and all the edges that have at least one endnode in S . For $E' \subseteq E$, the *partial subgraph* $G \setminus E'$ of G is obtained by removing from G the edges in E' . Let G' be a partial induced subgraph of G . We denote by $E(G')$ and $V(G')$ its edge set and its node set. We define $N(u)$ to be the set of nodes adjacent to node u . For $S \subset V^r \cup V^c$, $N(S)$ denotes the set of nodes in $V^r \cup V^c \setminus S$ which are adjacent to at least one node in S .

Paths and Cycles

A path P is a sequence of distinct nodes $x_1, x_2, \dots, x_n$, $n \geq 1$ such that $x_i x_{i+1}$ is an edge of E for all $1 \leq i \leq n-1$. Nodes x_1 and x_n are the *endnodes* of P . The other nodes are *intermediate*. A path having x_1 and x_n as endnodes is an $x_1 x_n$ -path. A *cycle* C is a sequence of nodes $x_1, x_2, \dots, x_n, x_1$, $n \geq 3$, such that the nodes $x_1, x_2, \dots, x_n$ form a path and x_1, x_n are adjacent. An edge connecting two nonconsecutive nodes of a path or a cycle is a *chord*. A chordless cycle is a *hole*. Let x_i and x_l be two nodes of P , where $l \geq i$. The path $x_i, x_{i+1}, \dots, x_l$ is called the $x_i x_l$ -subpath of P and is denoted by $P_{x_i x_l}$. When $n \geq 3$, we denote by $\hat{P}$ the $x_2 x_{n-1}$ -subpath of P . We write $P = x_1, \dots, x_{i-1}, P_{x_i x_l}, x_{l+1}, \dots, x_n$ or $P = x_1, \dots, x_i, P_{x_i x_l}, x_l, \dots, x_n$.

For a path P (a cycle C), the edges connecting consecutive nodes of P (of C) are called the *edges of P* (*edges of C*) and this edge set is denoted by $E(P)$ ($E(C)$ respectively). The length of P or C is the cardinality of $E(P)$ or $E(C)$. Paths having endnodes in the same side of the bipartition have length congruent to 0 mod 4 or 2 mod 4. Paths having endnodes in opposite sides of the bipartition have length congruent to 1 mod 4 or 3 mod 4. For the sake of brevity, the word "congruent" will be omitted. A cycle is *quad* if its length is 0 mod 4 and is *unquad* if its length is 2 mod 4. Hence a bipartite graph G is balanced if and only if G does not contain an unquad hole.

Direct Connections

Let A, B, C be three disjoint node sets such that no node of A is adjacent to a node of B . A path $P = x_1, x_2, \dots, x_n$ connects A and B if x_1 is adjacent to at least one node in A and x_n is adjacent to at least one node in B . The path P , connecting A and B is a *direct connection between A and B* if, in

the subgraph induced by the nodes $V(P) \cup A \cup B$, no path connecting A and B is shorter than P . A direct connection between A and B *avoids* C if $V(P) \cap C = \emptyset$.

Bicliques and Wheels

An induced subgraph G' of G is a *biclique* if $V(G') \cap V^r \neq \emptyset$, $V(G') \cap V^c \neq \emptyset$ and any two nodes of $V(G')$ in opposite sides of the bipartition are adjacent. A *wheel* (H, x) is an induced subgraph comprising a hole H and a node x not belonging to H but having at least three neighbors in H . Wheels will be further discussed in Section 3.

Organization of the Paper and Figures

The seven parts of this study follow the definitions and notation contained in this part. To avoid repetitions, the references for all parts can be found at the end of this part (Part I). If, in Part II or later, we refer to Theorem 3.3 of this part, this will be referred to as Theorem 3.3(I). Some theorems describe a classification of subgraphs into distinct types: If, in Part II, we refer to a node v of Type 1, according to the classification introduced in Theorem 3.3 of this part, we will refer to v as being of Type 1[3.3(I)]. Figures are numbered consecutively in each part. We use the following convention: Bold lines represent edges and dotted lines represent paths that have length at least one. Nodes in V^c are represented by solid (black) dots and nodes in V^r are hollow (white). A node in V which could be in either V^c or V^r is checked.

2 Classes of Balanced Graphs and Decomposition Theorems

In the remainder, let G denote a connected bipartite graph. A set S of nodes and edges of G is a *cutset* if the partial subgraph of G , obtained by removing the nodes and the edges in S , is disconnected. In this paper, we consider cutsets containing only edges or only nodes (*edge* or *node cutsets*) and in this section we survey decomposition results that have been obtained for some classes of balanced bipartite graphs.

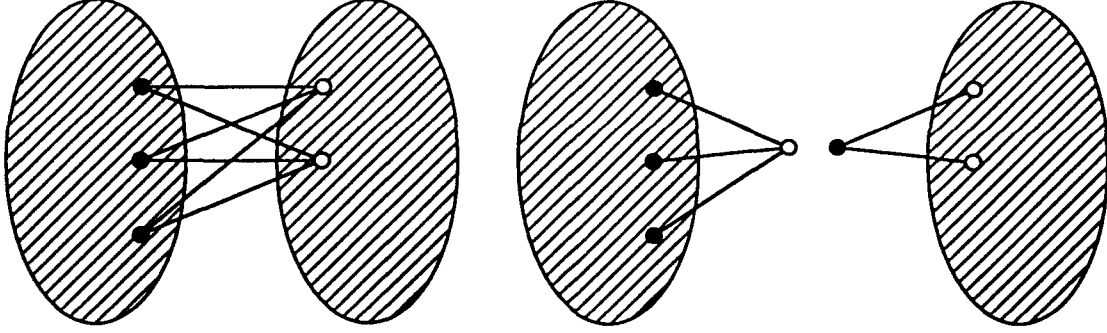


Figure 1: 1-join decomposition

2.1 Decompositions with Edge Cutsets

1-Joins

Let K_{BD} be a biclique with the property that its edge set $E(K_{BD})$ is a cutset and no connected component of $G \setminus E(K_{BD})$ contains both a node of B and a node of D . Note that, since G is connected, every connected component of $G \setminus E(K_{BD})$ contains either a node of B or a node of D . Let V_B be the set of nodes belonging to the components with at least one node in B . Similarly, let V_D be the set of nodes belonging to the components with at least one node in D . Let G'_B and G'_D be the subgraphs induced by V_B and V_D respectively. The *blocks* of $G \setminus E(K_{BD})$ are the graphs G_B, G_D obtained from G'_B, G'_D respectively by adding to G'_B a node d adjacent to all nodes in B and to G'_D a node b adjacent to all nodes in D , as in Figure 1.

The set $E(K_{BD})$ forms a *1-join* if neither of the blocks G_B and G_D coincides with G . This concept was introduced by Cunningham and Edmonds [18] and corresponds to a 2-sum in the 0, 1 matrix having G as its bipartite representation, see [23].

Theorem 2.1 *Let G_B, G_D be the blocks of the decomposition of G with a 1-join $E(K_{BD})$. Then G is balanced if and only if both G_B, G_D are balanced.*

Proof: Since G_B, G_D are induced subgraphs of G , the "only if" part is obvious. Furthermore, since the edge set of a 1-join belongs to a biclique, every hole of length greater than 4 in G belongs to G_B or to G_D . This proves the "if" part. $\square$

Definition 2.2 *A bipartite graph is strongly balanced if every unquad cycle has at least two chords.*

It is obvious from the definition that every strongly balanced bipartite graph is balanced. In fact, it can be shown that a strongly balanced bipartite graph is totally unimodular, see [12]. Conforti and Rao [12] prove the following decomposition theorem for strongly balanced bipartite graphs:

Theorem 2.3 *In a strongly balanced bipartite graph G , let uv and xy be two chords of a shortest unquad cycle C , where $u, x \in V^r$ and $v, y \in V^c$. Then x is adjacent to v and y is adjacent to u and the edge set of every maximal biclique containing nodes u, v, x, y is a 1-join of G .*

2-Joins and Strong 2-Joins

The concept of 2-join is related to the 3-sum operation [23] for a 0, 1 matrix and was introduced by Cornuéjols and Cunningham [17] for general graphs. Here we discuss 2-joins in a bipartite graph G . Let K_{BD} and K_{EF} be two bicliques of G where B, D, E, F are disjoint node sets and neither $E(K_{BD})$ nor $E(K_{EF})$ is an edge cutset of G . Further assume that no connected component of $G \setminus E(K_{BD}) \cup E(K_{EF})$ has a node in B and one in D , or a node in E and one in F . Then, we can assume w.l.o.g. that every component of $G \setminus E(K_{BD}) \cup E(K_{EF})$ contains either a node of B and a node of E or a node of D and a node of F . Let G'_1 be the union of the components of $G \setminus E(K_{BD}) \cup E(K_{EF})$ containing a node of B and a node of E . Similarly, let G'_2 be the union of the components of $G \setminus E(K_{BD}) \cup E(K_{EF})$ containing a node of D and a node of F . The block G_1 is constructed from G'_1 as follows:

- Add two nodes d and f , connected respectively to all nodes in B and to all nodes in E .
- If node set $B \cup E$ does not induce a biclique, let P_2 be the length of a shortest path in G_2 connecting a node in D to a node in F . If the length of P_2 is $0 \bmod 4$, nodes d and f are connected by a path of length 4 in G_1 . If the length of P_2 is $1 \bmod 4$, nodes d and f are adjacent in G_1 . If the length of P_2 is $2 \bmod 4$ ($3 \bmod 4$), nodes d and f are connected by a path of length 2 (3). Denote this path by P_{df} .
- If node set $B \cup E$ induces a biclique, no path connecting nodes d and f is added in G_1 .

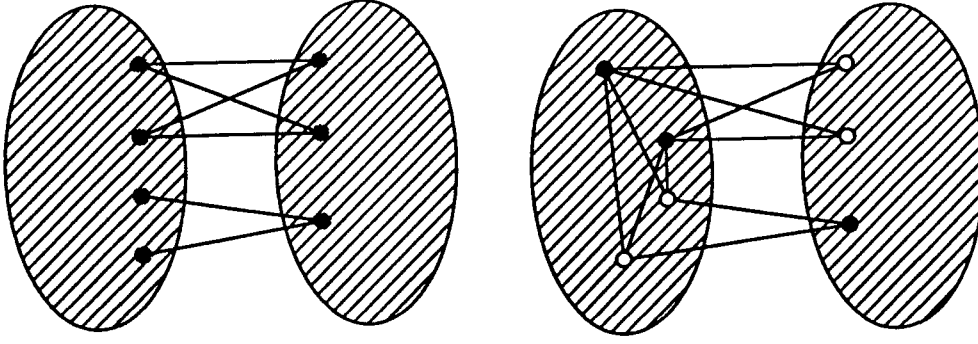


Figure 2: 2-join and strong 2-join

The block G_2 is defined analogously from G'_2 . The set $E(K_{BD}) \cup E(K_{EF})$ is a 2-join if G is distinct from both G_1, G_2 . If the set $B \cup E$ or $D \cup F$ induces a biclique, the 2-join is said to be *strong*. Figure 2 shows both types of 2-joins. Let G be a bipartite graph having a 1-join or a 2-join. An induced subgraph G^* of G is *separated* by the 1-join or the 2-join if neither block contains G^* .

Theorem 2.4 *Let G_1, G_2 be the blocks of the decomposition of the bipartite graph G by a 2-join $E(K_{BD}) \cup E(K_{EF})$. Then G is balanced if and only if both G_1 and G_2 are balanced.*

Proof: It follows from the definition of the blocks of a 2-join that if G is balanced, then G_1 and G_2 are balanced. To prove the other direction, assume that G is not balanced but G_1 and G_2 are balanced.

Let H be an unquad hole of G . If H contains no edge of G'_2 , there exists a hole in G_1 having length $|H|$. The same argument holds for G'_1 .

Let $H = b', d', Q_2, f', e', Q_1, b'$ where $b' \in B$, $d' \in D$, $e' \in E$, $f' \in F$ be an unquad hole of G , see Figure 3.

At least one of the sets $B \cup E$, $D \cup F$ does not induce a biclique, else H has length 4, a contradiction. If the node set $B \cup E$ induces a biclique and $D \cup F$ does not, a hole H' of the same length as H belongs to G_2 .

Hence the 2-join is not strong. Since G_1 contains no unquad hole, the length of P_{df} is not congruent to the length of Q_2 modulo 4. It follows that G_2 contains a chordless path P_2 connecting a node $d'' \in D$ to a node $f'' \in F$ whose length is not congruent to the length of Q_2 .

The holes $H_1 = d'', P_2, f'', e, P_{eb}, b, d''$ and $H_2 = d', Q_2, f', e, P_{eb}, b, d'$ have distinct lengths modulo 4. Hence one of them is unquad, contradicting the fact that G_2 is balanced. $\square$

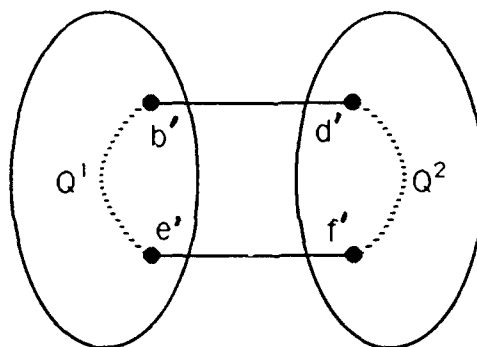


Figure 3:

Definition 2.5 *A bipartite graph is restricted balanced if it has no unquad cycle.*

It follows from the definition that every restricted balanced bipartite graph is strongly balanced. We define a bipartite graph to be *basic* if all the nodes in V^r or V^c have degree at most two. Testing whether a bipartite graph is basic amounts to testing whether a graph is bipartite, see [25]. Restricted balanced graphs have been studied by Commoner [9] and Yannakakis [25] has proven the following decomposition theorem:

Theorem 2.6 *Let G be a restricted balanced bipartite graph which is not basic. Then either G has a cutnode or G has a 2-join consisting of two edges bd and ef .*

An algorithm for testing whether a bipartite graph is restricted balanced follows from this theorem, see [25]. Conforti and Rao [12] give an algorithm to test whether a graph is restricted balanced that does not use any decomposition. In this study, we prove decomposition theorems for graphs that are balanced but have unquad cycles. Hence we consider restricted balanced bipartite graphs as building blocks of our decompositions.

Definition 2.7 *A bipartite graph is totally balanced if every hole has length 4.*

Totally balanced bipartite graphs arise in location theory and were the first balanced bipartite graphs to be the object of an extensive study. Several

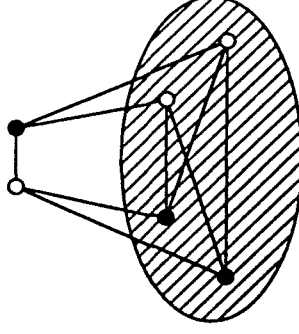


Figure 4: A bisimplicial edge

authors (Golumbic and Goss [20], Golumbic [21], Anstee and Farber [1] and Hoffman, Kolen and Sakarovitch [22]) have given properties of these graphs.

Definition 2.8 *An edge uv is bisimplicial if either u or v has degree 1 or the node set $N(u) \cup N(v)$ induces a biclique.*

Figure 4 shows a bisimplicial edge. Note that if uv is a bisimplicial edge and nodes u and v have degree at least 2, then G has a strong 2-join formed by the edges adjacent to exactly one node in the set $\{u, v\}$. The 2-join is strong since $N(u) \cup N(v) \setminus \{u, v\}$ induces a biclique.

The following theorem of Golumbic and Goss characterizes totally balanced bipartite graphs, see [20].

Theorem 2.9 *A totally balanced bipartite graph has a bisimplicial edge.*

2.2 Decompositions with Node Cutsets

Let $S \subset V(G)$ be a node cutset of G and $G'_1, G'_2, \dots, G'_n$ be the connected components of $G \setminus S$. The *block* G_i is the graph induced by $V(G'_i) \cup S$. We say that an induced subgraph G^* is *separated* in $G \setminus S$ if no block contains G^* .

Extended Stars

Definition 2.10 *An extended star $S = (x; T; A; N)$ is defined by disjoint subsets T, A, N of $V(G)$ and a node $x \in T$ such that $A \cup N \subseteq N(x)$ and the node set $T \cup A$ induces a biclique. If $|T| \geq 2$, then $|A| \geq 2$. An extended star cutset is one where $T \cup A \cup N$ is a node cutset.*

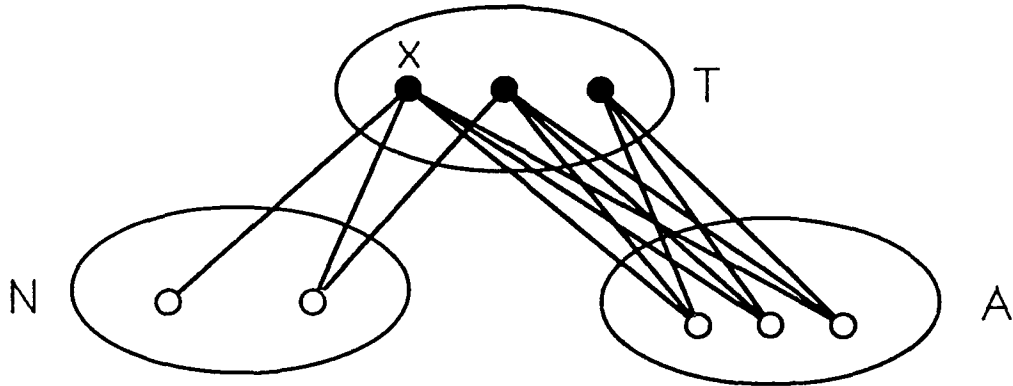


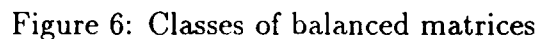
Figure 5: An extended star

Figure 5 shows an extended star. Since the nodes in $T \cup A$ induce a biclique, an extended star cutset with $N = \emptyset$ is called a *biclique cutset*. An extended star cutset having $T = \{x\}$ is called a *star cutset*, since it is composed by a node x and a subset of its neighbors. Note that a star cutset is a special case of a biclique cutset. The following theorem shows the relevance of biclique cutsets for a class of bipartite graphs.

Theorem 2.11 *Let G be a bipartite graph not containing any wheel and let K_{BD} be a biclique cutset of G . Then G is balanced if and only if all the blocks of $G \setminus (B \cup D)$ are balanced.*

Proof: The "only if" part is obvious, since the blocks are induced subgraphs of G .

To prove the "if" part, assume G is not balanced but all the blocks of $G \setminus (B \cup D)$ are. Let H be an unquad hole of G . At least two nonconsecutive nodes of H , say v_i and v_j , belong to $B \cup D$, else H is contained in some block. Furthermore nodes v_i and v_j belong to the same set, else H has a chord. Assume w.l.o.g. that $v_i, v_j \in B$. Let w be any node in D . If $w \in V(H)$, then wv_i and wv_j are edges of H and H contains no other node of $B \cup D$, else H has a chord. Now, it follows that H is contained in some block, a contradiction. So $w \notin V(H)$. Assume that w has no neighbor in $V(H)$ other than nodes v_i, v_j , and let P_1, P_2 be the two subpaths of H , connecting v_i and v_j . Then the holes $H_1 = v_i, w, v_j, P_1, v_i$ and $H_2 = v_i, w, v_j, P_2, v_i$ have opposite parity and each one belongs to a block, a contradiction to the assumption that all



Definition 2.12 *A bipartite graph is linear if it does not contain a cycle of length 4.*

Note that an extended star cutset $(x; T; A; N)$ in a linear bipartite graph is always a star cutset since $|T| = 1$. Conforti and Rao [13] prove the following decomposition theorem for balanced linear bipartite graphs:

This theorem has been used in [16] to give a polynomial algorithm to test whether a linear bipartite graph is balanced.

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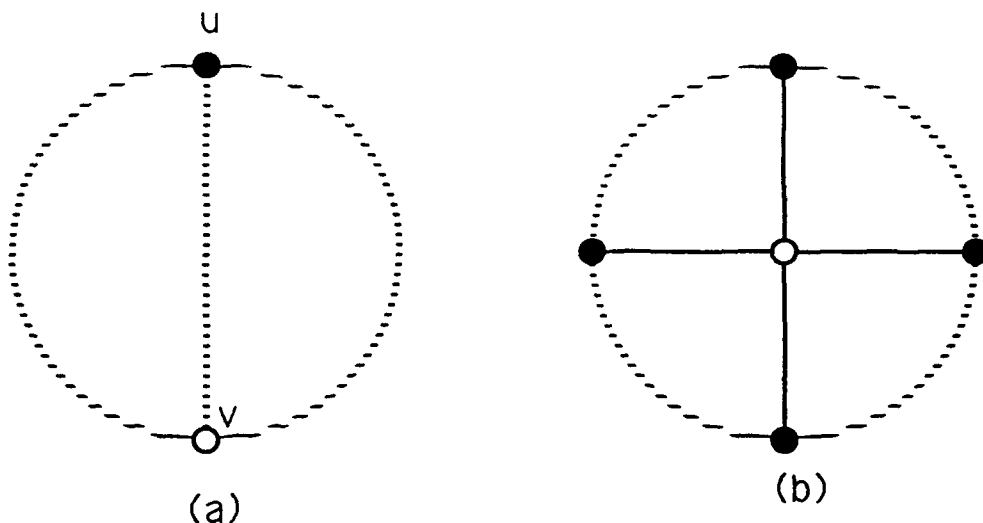


Figure 7: A 3-path configuration and a wheel

3 Configurations and a Theorem of Truemper

A *configuration* is an induced subgraph of G . Given a configuration Σ of G , we say that G *contains* Σ and we indicate with $E(\Sigma)$ and $V(\Sigma)$ its edge set and node set. A node $v \notin V(\Sigma)$ is *strongly adjacent* to Σ if $|N(v) \cap V(\Sigma)| \geq 2$. We say that a strongly adjacent node v is a *twin* of a node $x \in V(\Sigma)$ relative to Σ if $N(v) \cap V(\Sigma) = N(x) \cap V(\Sigma)$.

3-Path Configurations

Let u, v be two nonadjacent nodes in opposite sides of the bipartition. A *3-path configuration connecting u and v* , denoted by $3PC(u, v)$, is defined by three chordless paths P_1, P_2, P_3 connecting u and v , having no common intermediate nodes and such that the subgraph induced by the nodes of these three paths contains no other edge than those of the paths. Figure 7(a) shows a $3PC(u, v)$.

Wheels

A *wheel*, denoted by (H, x) is defined by a hole H and a node $x \notin V(H)$ having at least three neighbors in H . Node x is the *center* of the wheel, see Figure 7(b). Let $x_1, x_2, \dots, x_n$ be the nodes in $N(x) \cap V(H)$. An edge xx_i is a *spoke* of (H, x) . If n is even, (H, x) is an *even wheel*, otherwise (H, x) is an

odd wheel. Two nodes x_i, x_j are *consecutive* if one of the two $x_i x_j$ -subpaths of H contains no intermediate node in $N(x)$. A subpath with this property is a *sector* of (H, x) . Two sectors are *adjacent* if they have a common endnode. A *bicoloring* of (H, x) is an assignment of colors to the intermediate nodes of the sectors of (H, x) so that nodes in the same sector have the same color and nodes of adjacent sectors have opposite colors. Note that (H, x) is bicolorable if and only if (H, x) is an even wheel.

Signing Graphs and a Theorem of Truemper

The *signing* of a graph G is the assignment of weights $+1, -1$ (positive and negative weights) to the edges of G . A signed bipartite graph G is *balanced* if, for every hole H of G , the sum of the weights of the edges in H is $0 \pmod{4}$. A $0, +1, -1$ matrix is *balanced* if the corresponding signed bipartite graph is balanced.

A bipartite graph G is *signable to be balanced* if there exists a signing of its edges so that the resulting signed graph is balanced.

Remark 3.1 *Since cuts and cycles of a graph G have even intersection, it follows that if G is signed to be balanced, then the graph G' , obtained by switching signs on the edges of a cut is also signed to be balanced. This implies that all the edges of a given spanning tree of G can be assumed to be signed positive since for every edge uv of a spanning tree there is a cut containing uv and no other edge of the tree.*

Since paths P_1, P_2, P_3 of a $3PC(u, v)$ are of length $1 \pmod{4}$ or $3 \pmod{4}$ a $3PC(u, v)$ is not signable to be balanced. Consider a wheel (H, x) which is signed to be balanced. By Remark 3.1, all spokes of (H, x) can be assumed to be signed positive. This implies that the sum of the weights of the edges in each sector is $2 \pmod{4}$. Hence (H, x) must be an even wheel. The following important theorem of Truemper [24] characterizes the bipartite graphs that are signable to be balanced:

Theorem 3.2 *A bipartite graph G is signable to be balanced if and only if G does not contain an odd wheel or a 3-path configuration.*

Remark 3.1 implies that if G is signable to be balanced, one can sign it with the following signing algorithm:

Choose a spanning tree T , sign its edges positive and recursively choose an edge uv which closes a hole H of G with the previously chosen edges, and sign uv so that the sum of the weights of the edges in H is $0 \pmod 4$.

Hence, an algorithm to test whether a bipartite graph G is signable to be balanced can be used to test whether G is balanced, as follows:

Apply the signing algorithm to G and reject G as not being balanced if an edge is signed negative. Now test whether G is signable to be balanced. If it is, then G is balanced. If not, then G is not balanced.

Another way of testing balancedness of G using an algorithm to test whether a graph is signable to be balanced is to construct G' by adding to G a node u in V^r adjacent to all the nodes of G in V^c . Then G' contains an odd wheel (H, u) if and only if H is an unquad hole of G . No 3-path configuration of G' can contain node u . Therefore G' is signable to be balanced if and only if G is balanced.

Cycles with a Unique Chord

Let C be a cycle with unique chord uv , and let H_1, H_2 be the two holes in C , containing edge uv . The strongly adjacent nodes to C have been studied in [13]. We report here the main result and give a proof that holds for bipartite graphs that are signable to be balanced.

Theorem 3.3 *Let C be a cycle with a unique chord uv and let H_1 and H_2 be the two holes of the graph induced by $V(C)$. Let x be a node, strongly adjacent to C . Then x is of one of the following types:*

Type 1 *The set $N(x)$ is contained in $V(H_1)$ or in $V(H_2)$.*

Then $|N(x) \cap V(C)|$ is even.

Type 2 *The set $N(x)$ is not contained in $V(H_1)$ or in $V(H_2)$ and $N(x) \cap \{u, v\} \neq \emptyset$.*

Then $|N(x) \cap V(H_1)|$ and $|N(x) \cap V(H_2)|$ are even.

Type 3 *The set $N(x)$ is not contained in $V(H_1)$ or $V(H_2)$ and $N(x) \cap \{u, v\} = \emptyset$.*

Then either $|N(x) \cap V(H_1)|$ is even and $|N(x) \cap V(H_2)| = 1$ or $|N(x) \cap V(H_2)|$ is even and $|N(x) \cap V(H_1)| = 1$. Furthermore the unique neighbor of x in H_1 or H_2 is adjacent to u or v .

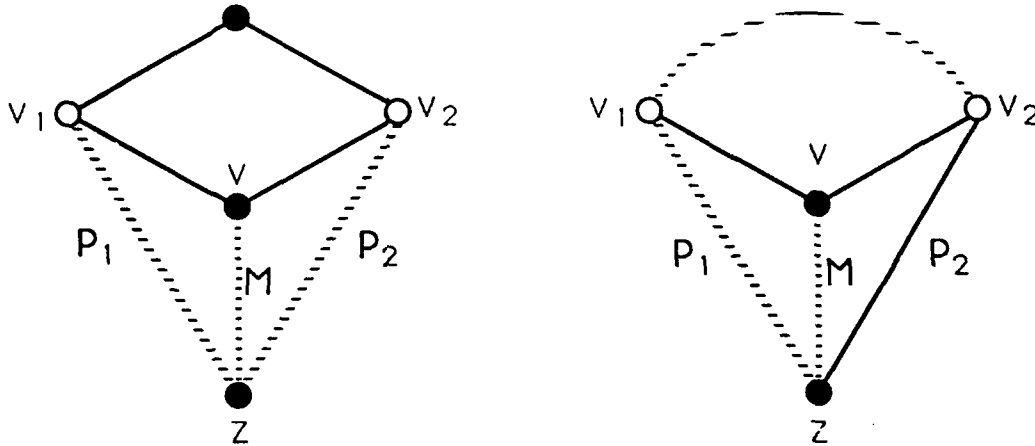


Figure 8: Parachutes

Proof: If, for $i = 1$ or 2 , $N(x)$ is contained in $V(H_i)$, then $|N(x) \cap V(C)|$ is even, else (H_i, x) is an odd wheel. This yields the nodes x of Type 1.

If $N(x)$ is not contained in $V(H_1)$ or $V(H_2)$ and x is adjacent to u or v , then $|N(x) \cap V(H_i)|$ is even for $i = 1, 2$, else (H_i, x) is an odd wheel. This yields the nodes x of Type 2.

If $N(x)$ is not contained in $V(H_1)$ or $V(H_2)$ and x is not adjacent to u or v , then assume w.l.o.g. that $u \in V^c$ and $x \in V^r$.

Case 1 $|N(x) \cap V(H_1)|$ or $|N(x) \cap V(H_2)|$ is even.

There is a $3PC(u, x)$ unless x has a unique neighbor adjacent to v in $V(H_1)$ or in $V(H_2)$. This yields nodes x of Type 3.

Case 2 $|N(x) \cap V(H_i)| = 1$ for $i = 1, 2$.

If both neighbors of x in $V(H)$ are adjacent to v , then there is a wheel with three spokes centered at v . If one neighbor of x in $V(C)$, say y , is not adjacent to v , then there is a $3PC(y, v)$. $\square$

Parachutes

A *parachute*, denoted by $Par(T, P_1, P_2, M)$, is defined by four chordless paths of positive lengths, $T = v_1, \dots, v_2$; $P_1 = v_1, \dots, z$; $P_2 = v_2, \dots, z$; $M = v, \dots, z$, where (v_1, v_2, v, z) are distinct nodes, and two edges vv_1 and vv_2 called the *spokes*. No other edge exists in $Par(T, P_1, P_2, M)$, except the ones mentioned above, see Figure 8. Furthermore $|E(P_1)| + |E(P_2)| \geq 3$. This implies that a parachute contains a cycle with a unique chord. Node v is the *center node*, nodes v_1, v_2 are the *side nodes* and node z is the *bottom node*.

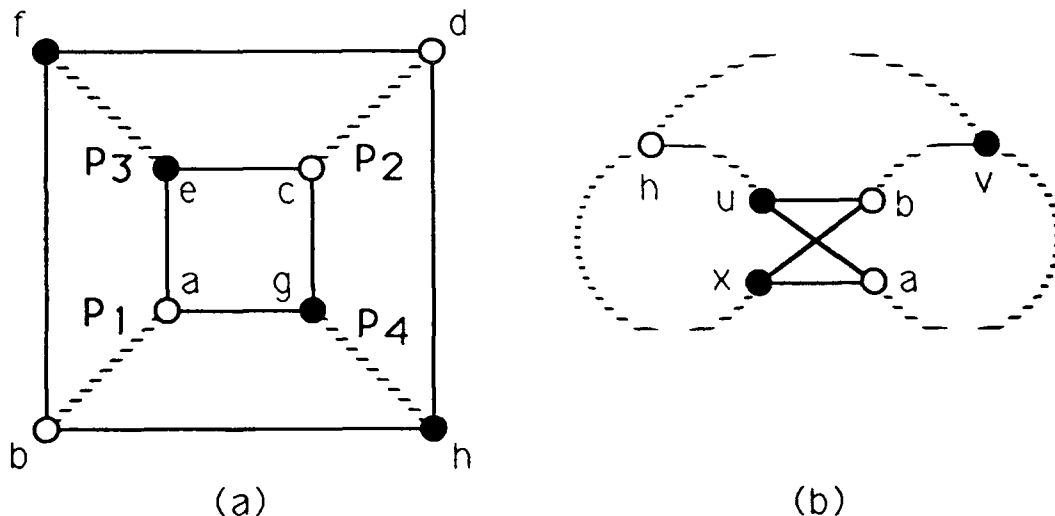


Figure 9: Connected squares and goggles

Note that if G is balanced then nodes v, z are on the same side of the bipartition, else the parachute contains a $3PC(v, z)$ or an odd wheel (H, v) with three spokes. A parachute has *long top* if $|E(T)| \geq 4$, *long sides* if $|E(P_1)| \geq 2$ and $|E(P_2)| \geq 2$, *one short side* if either $|E(P_1)| = 1$ or $|E(P_2)| = 1$ and *short middle* if $|E(M)| \leq 2$. Figure 8(a) shows a parachute with long sides and short top and Figure 8(b) shows a parachute with long top and short side.

Connected Squares and Goggles

Connected squares, denoted by $CS(P_1, P_2, P_3, P_4)$, are defined by four chordless paths of positive lengths $P_1 = a, \dots, b$; $P_2 = c, \dots, d$; $P_3 = e, \dots, f$; $P_4 = g, \dots, h$, where nodes a and c are adjacent to both e and g and b and d are adjacent to both f and h , as in Figure 9(a). No other adjacency exists in the connected squares. Note that node a has the same sign as b , else the connected squares contain a $3PC(a, b)$ or, if $|E(P_1)| = 1$, an odd wheel with center a . Therefore the nodes a, b, c, d are in one side of the bipartition and e, f, g, h are in the other. Note that connected squares that can be signed to be balanced contain a parachute with long sides and short top.

Goggles, denoted by $Go(P, Q, R, S, T)$, are defined by a cycle $C = h, P, x, a, Q, v, R, b, u, S, h$, with two chords ua and xb , and chordless paths P, Q, R, S of length greater than one, and a chordless path $T = h, \dots, v$ of length at least one, such that no intermediate node of T belongs to C . No other edge exists, connecting nodes of the goggles, see Figure 9(b).

4 Decomposition Theorems for Balanced Bipartite Graphs

In this section we introduce the main decomposition theorem obtained in this paper and we summarize the intermediate results that are used in its proof. We first introduce the following classes of graphs:

- Wheel-free balanced bipartite graphs are those which do not contain wheels.
- WP-free balanced bipartite graphs are those which do not contain wheels or parachutes.

In Part II, we study WP-free graphs and prove the following theorem:

Theorem 4.1 *Let G be a WP-free balanced bipartite graph. Then either G is strongly balanced or there exists a cycle C with unique chord uv , and a strong 2-join, $E(K_{BD}) \cup E(K_{EF})$, separating C , where the node set $B \cup E$ induces a biclique containing edge uv .*

In Part III, we study cutsets separating parachutes in wheel-free balanced bipartite graphs and we prove the following theorems:

Theorem 4.2 *Let G be a wheel-free balanced bipartite graph not containing parachutes with long sides. Then, either G is WP-free or G contains a parachute Π with spokes vv_1, vv_2 and one short side, such that Π is separated by an extended star cutset $(x; T; A; N)$, where $x = v$ and $v_1, v_2 \in A$.*

Theorem 4.3 *Let G be a wheel-free balanced bipartite graph containing at least one parachute with long sides and long top. Then G contains a parachute Π with spokes vv_1, vv_2 and long sides, such that Π is separated by an extended star cutset $(x; T; A; N)$, where $x = v$ and $v_1, v_2 \in A$.*

Theorem 4.4 *Let G be a wheel-free balanced bipartite graph containing at least one parachute with long sides, short top and short middle. Then G contains at least one of the following:*

- A parachute Π with spokes vv_1, vv_2 , long sides and short middle, such that Π is separated by an extended star cutset $(x; T; A; N)$, where $x = v$ and $v_1, v_2 \in A$.

- *Connected squares.*

Theorem 4.5 *If a wheel-free balanced bipartite graph G contains a parachute Π with long sides, short top, long middle and spokes vv_1, vv_2 , then G contains at least one of the following:*

- *An extended star cutset $(x; T; A; N)$, where $x = v$ and $v_1, v_2 \in A$, separating the parachute Π .*
- *A parachute with long sides and long top.*
- *A parachute with long sides, short top and short middle.*
- *Connected squares.*
- *Goggles.*

Part IV studies cutsets in wheel-free balanced bipartite graphs containing connected squares. The following is the main result:

Theorem 4.6 *Let G be a wheel-free balanced bipartite graph containing connected squares. If G has no bichique cutset, then G contains a 2-join separating $V(P_1) \cup V(P_2)$ from $V(P_3) \cup V(P_4)$ for some connected squares $CS(P_1, P_2, P_3, P_4)$.*

Part V studies wheel-free balanced bipartite graphs containing goggles:

Theorem 4.7 *Let G be a wheel-free balanced bipartite graph containing goggles. If G contains no parachute with long sides and long top, no parachute with long sides, short top and short middle, and no connected squares, then either G contains an extended star cutset or G contains a 2-join separating $V(P) \cup V(S) \setminus \{h\}$ from $V(Q) \cup V(R) \cup V(T) \setminus \{h\}$ in some goggles $Go(P, Q, R, S, T)$, where $\{h\} = V(P) \cap V(Q)$.*

Part VI contains the following decomposition result.

Theorem 4.8 *Let G be a balanced bipartite graph containing wheels. Then G has an extended star cutset $(x; T; A; N)$ and a wheel (H, v) such that the extended star cutset separates the sectors of (H, v) of opposite colors, where $x = v$ and $|N(v) \cap V(H) \cap A| \geq 2$.*

All these theorems imply the following:

Theorem 4.9 *Let G be a balanced bipartite graph which is not restricted balanced. Then G has a 1-join, or a 2-join, or an extended star cutset.*

In all proofs of the results contained in this paper, we only use the fact that a balanced bipartite graph contains no hole of length 6, no odd wheel and no 3-path configuration.

Definition 4.10 *A bipartite graph G is said to be weakly balanced if G contains no odd wheel, no 3-path configuration and no hole of length 6.*

Balanced bipartite graphs are weakly balanced and weakly balanced bipartite graphs are signable to be balanced.

Hence Theorems 4.1-4.9 hold even when substituting the word "balanced" with the word "weakly balanced", given appropriate extensions of the concepts of strongly balanced and restricted balanced bipartite graphs, as follows:

Definition 4.11 *A bipartite graph is said to be signable to be restricted balanced if it contains no 3-path configuration and no cycle with at least one chord.*

A bipartite graph is said to be signable to be strongly balanced if it contains no 3-path configuration and no cycle with a unique chord.

Since a 3-path configuration or a cycle with at least one chord implies the existence of an unquad cycle, a restricted balanced bipartite graph is also signable to be restricted balanced. It follows from Definitions 2.2 and 4.11 that every strongly balanced bipartite graph is also signable to be strongly balanced. Testing whether a bipartite graph belongs to either of these two classes can be done in polynomial time. We outline an algorithm for testing whether a graph is signable to be restricted balanced in the proposition below. Testing whether a bipartite graph is signable to be strongly balanced can be done with a similar algorithm.

Proposition 4.12 *Testing whether a bipartite graph G is signable to be restricted balanced can be done in polynomial time as follows: Apply the signing algorithm to G . Let G' be the resulting signed graph. Apply the algorithm in [12] to G' to test the existence of a cycle whose edge weights add up to 2 mod 4. If such a cycle exists, then G is not signable to be restricted balanced. Otherwise G is signable to be restricted balanced.*

Proof: First assume that G is signable to be restricted balanced. Since G is signable to be balanced, G' is a signed balanced graph. Since every cycle of G is a hole, G' does not contain a cycle of length $2 \bmod 4$.

If G is not signable to be restricted balanced, the existence of a 3-path configuration or of a cycle with at least one chord implies that every signing of G has a cycle of length $2 \bmod 4$. $\square$

The following theorem extends Theorem 4.9:

Theorem 4.13 *Let G be a weakly balanced bipartite graph which is not signable to be restricted balanced. Then G has a 1-join, or a 2-join, or an extended star cutset.*

5 Decomposition Theorems for Bipartite Graphs that are Signable to be Balanced

The class of bipartite graphs that are signable to be balanced properly contains the class of bipartite graphs that are signable to be totally unimodular (i.e. the nonzero entries of the corresponding 0,1 matrix can be signed so that the resulting matrix is totally unimodular, see [23]).

Our results in Parts II-V of this series are given in terms of bipartite graphs that are signable to be balanced. To state the results, we need to introduce the following configurations:

Let R_{2n} , $n \geq 3$, be a graph defined by a cycle $C = x_1, x_2, \dots, x_{2n}, x_1$ with chords $x_i x_{i+n}$, $1 \leq i \leq n$. Note that R_{2n} is bipartite when n is odd. Figure 10 shows R_{10} . The graph R_6 is a biclique, hence it is balanced. Any graph R_{2n} , $n \geq 7$, contains an odd wheel, hence it is not signable to be balanced.

However the graph R_{10} of Figure 10 does not contain an odd wheel, a 3-path configuration and each edge belongs to a 6-hole. Hence R_{10} is signable to be balanced but it is not balanced. Furthermore R_{10} has no 1-join, no 2-join, and no extended star cutset. This shows that the decomposition theorem 4.9 cannot be extended to the class of bipartite graphs that are signable to be balanced.

A connected 6-hole, see Figure 10, is defined by a hole $H = h_1, h_2, h_3, h_4, h_5, h_6$, paths P_i , $1 \leq i \leq 6$, where P_i connects t to h_i for i odd and b to h_i for i even. The two endnodes of each path P_i are in the same side of the

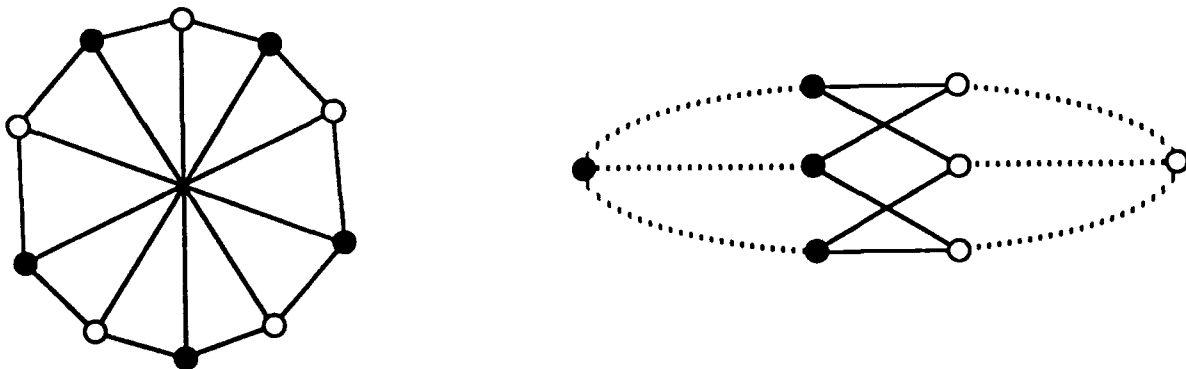


Figure 10: The R_{10} and connected 6-hole configurations

bipartition and there are no other edges in the connected 6-hole than those described above.

In Part II we prove the following theorem:

Theorem 5.1 *Let G be a WP-free bipartite graph which is signable to be balanced but not signable to be strongly balanced. Then G contains a cycle C with unique chord uv , and a strong 2-join, $E(K_{BD}) \cup E(K_{EF})$, separating C , where the node set $B \cup E$ induces a biclique containing edge uv .*

The results contained in Parts III-V imply the following theorem:

Theorem 5.2 *Let G be a wheel-free bipartite graph which is signable to be balanced but not signable to be strongly balanced. If G contains no R_{10} and no connected 6-hole, then G contains a 2-join or an extended star cutset.*

These theorems will be used in [11] to prove a decomposition result for all wheel-free bipartite graphs that are signable to be balanced. In part VI of this series, the wheel theorem 4.8 is stated in terms of balanced bipartite graphs. Figure 11 exhibits two configurations that are signable to be balanced but are not balanced, showing that Theorem 4.8 does not hold in the present form for bipartite graphs that are signable to be balanced. In [11] a wheel theorem for bipartite graphs that are signable to be balanced is proved.

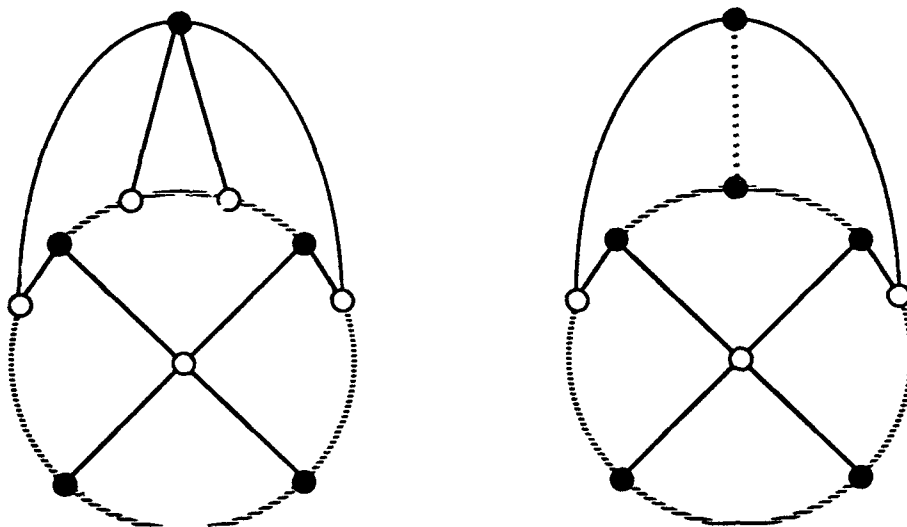


Figure 11:

6 Some Conjectures and Open Questions

The following conjecture has been formulated in [13]:

Conjecture 6.1 *Every balanced bipartite graph G has an edge uv with the property that the partial graph, obtained from G by removing the edge uv , remains balanced.*

This conjecture is obviously equivalent to the following:

Conjecture 6.2 *Every balanced bipartite graph contains an edge which is not the unique chord of a cycle.*

Note that every edge of the graph R_{10} is the unique chord of a cycle of length 8, hence the above conjecture cannot be extended to the class of bipartite graphs that are signable to be balanced. (Note that a cycle with a unique chord in a bipartite graph that is signable to be balanced can be quad, but every cycle with a unique chord in a balanced bipartite graph must be unquad). However, we believe that the above graph may be the only exception, and we propose the following conjecture:

Conjecture 6.3 *The graph R_{10} is the only bipartite graph which is signable to be balanced and has the property that every edge is the unique chord of a cycle but has no 1-join, no 2-join and no extended star cutset.*

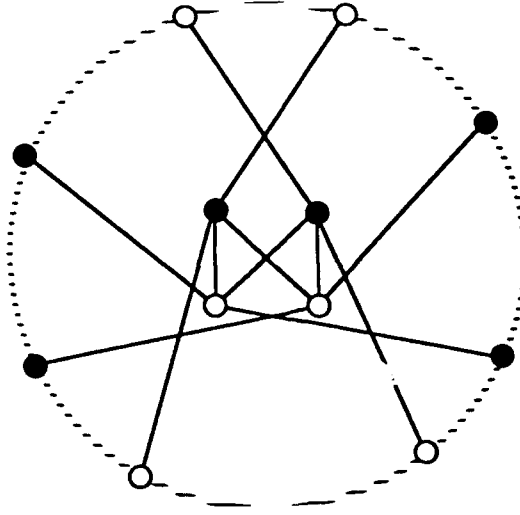


Figure 12: A W_{pq} configuration

As shown in Section 2.2, a biclique cutset is a special case of an extended star cutset, hence the question arises whether Theorem 4.9 can be strengthened, by showing that every bipartite graph that is balanced but not restricted balanced has a 1-join or a 2-join or a biclique cutset.

Conjecture 6.4 *Every wheel-free balanced bipartite graph which is not restricted balanced has a 1-join or a 2-join or a biclique cutset.*

Note that Theorem 4.6 proves the above conjecture if G contains connected squares. The graph in Figure 12 shows that this conjecture cannot be extended to all balanced bipartite graphs.

More generally, we define an infinite family of graphs as follows.

Let H be a hole where nodes $u_1, \dots, u_p, v_1, \dots, v_q, w_1, \dots, w_p, x_1, \dots, x_q$ appear in this order when traversing H , but are not necessarily adjacent. Let $Y = \{y_1, \dots, y_p\}$ and $Z = \{z_1, \dots, z_q\}$ be two node sets having empty intersection with $V(H)$ and inducing a biclique K_{YZ} . Node y_i is connected to u_i and w_i for $1 \leq i \leq p$. Node z_i is connected to v_i and x_i for $1 \leq i \leq q$. Any balanced bipartite graph of this form for $p, q \geq 2$ is denoted by W_{pq} . For all values of $p, q \geq 2$, the graphs W_{pq} have no 1-join, no 2-join and no biclique cutset.

Since the graphs W_{pq} contain a wheel, a stronger form of the above conjecture is the following.

Conjecture 6.5 *Every balanced bipartite graph which is not restricted balanced is either W_{pq} or has a 1-join, a 2-join or a biclique cutset.*

7 Efficient Algorithms

A node cutset $S \subset V$ is a *double star cutset* if there exist two adjacent nodes $u, v \in V$ such that $S = N(u) \cup N(v)$.

Part VII gives a polynomial algorithm to test whether a bipartite graph is balanced. This algorithm uses the following corollary of Theorem 4.9.

Corollary 7.1 *Let G be a balanced bipartite graph which is not restricted balanced. Then G has either a 1-join, or a 2-join, or a star cutset or a double star cutset.*

Neither this corollary nor Theorem 4.9 gives a sufficient condition for a bipartite graph G to be balanced when the blocks of the decomposition of G are balanced.

To overcome this problem, our recognition algorithm creates a polynomial number of induced subgraphs of G with specific properties and tests each one of them for balancedness. This family of subgraphs is such that G is balanced if and only if all blocks produced in the various decompositions are balanced. However, this process has a polynomial running time of high degree. It would be interesting to design better algorithms by using Theorems 4.1-4.8 directly instead of Corollary 7.1.

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In this seven part paper, we prove the following theorem:

At least one of the following alternatives occurs for a bipartite graph G :

- The graph G has no cycle of length $4k+2$.
- The graph G has a chordless cycle of length $4k+2$.

- There exist two complete bipartite graphs K_1, K_2 in G having disjoint node sets, with the property that the removal of the edges in K_1, K_2 disconnects G .
- There exists a subset S of the nodes of G with the property that the removal of S disconnects G , where S can be partitioned into three disjoint sets T, A, N such that $T \neq \emptyset$, some node $x \in T$ is adjacent to every node in $A \cup N$ and, if $|T| \geq 2$, then $|A| \geq 2$ and every node of T is adjacent to every node of A .

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. Balanced matrices are important in integer programming and combinatorial optimization since the associated set packing and set covering polytopes have integral vertices.

To a 0,1 matrix A we associate a bipartite graph $G(V^r, V^c; E)$ as follows: The node sets V^r and V^c represent the row set and the column set of A and edge ij belongs to E if and only if $a_{ij} = 1$. Since a 0,1 matrix is balanced if and only if the associated bipartite graph does not contain a chordless cycle of length $4k+2$, the above theorem provides a decomposition of balanced matrices into elementary matrices whose associated bipartite graphs have no cycle of length $4k+2$. In Part VII of the paper, we show how to use this decomposition theorem to test in polynomial time whether a 0,1 matrix is balanced.