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Decomposition of Balanced Matrices.
Part II:
Wheel-and-Parachute-Free Graphs

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1 Introduction

In this part, we consider bipartite graphs G containing neither a wheel nor a parachute. These bipartite graphs are said to be *WP-free.* We prove that, if G is a WP-free bipartite graph which is signable to be balanced and contains a cycle with a unique chord, then G contains a strong 2-join. This shows that if G is a WP-free balanced bipartite graph which is not strongly balanced, then G contains a strong 2-join.

Strongly and totally balanced bipartite graphs were introduced in Part I. We repeat the definitions here. A bipartite graph is *strongly balanced* if every unquad cycle has at least two chords. Theorem 2.3(I) states that the chords of every minimal unquad cycle belong to a 1-join. A bipartite graph is *totally balanced* if it has no hole of length greater than 4. Theorem 2.9(I) states that every totally balanced bipartite graph G has a bisimplicial edge, namely an edge uv such that every node of $N(u)$ is adjacent to every node of $N(v)$.

Remark 1.1 *The class of WP-free balanced bipartite graphs properly contains totally balanced bipartite graphs and strongly balanced bipartite graphs.*

Proof: The cycle H of a wheel (H, v) and the cycle induced by the paths T, P_1, P_2 in a parachute $Par(T, P_1, P_2, M)$ are holes of length strictly greater than 4. Hence totally balanced bipartite graphs are WP-free.

In a wheel (H, v), two consecutive sectors, together with node v, induce a cycle with a unique chord. In a parachute, assume w.l.o.g. that P_1 has length greater than 1. Then the graph obtained from the parachute by removing the intermediate nodes of P_1 is a cycle with a unique chord. Hence strongly balanced bipartite graphs are WP-free.

To see that the inclusion is proper, note that a cycle C with a unique chord is not strongly balanced, nor is it totally balanced when C has length 10 or more. Yet, when the two induced holes are quad, the cycle C is a WP-free balanced bipartite graph. □

In this part, we show that if a WP-free bipartite graph contains no 3-path configuration and no odd wheel but contains a cycle with a unique chord, then it has a strong 2-join, see Figure 1.

Our proof of the decomposition theorem is organized as follows. In Section 2, we show that every edge which is the unique chord of a cycle belongs to some biclique cutset. In Section 3, we show that G contains a strong 2-join.

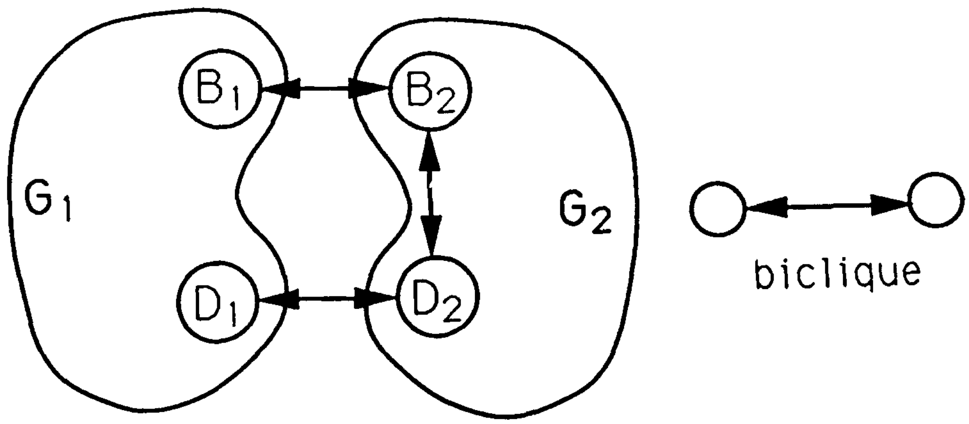

Figure 1: Strong 2-join

The strong 2-join decomposition generalizes the bisimplicial edge decomposition for totally balanced bipartite graphs since, in this case, the subgraph G_1 of Figure 1 is reduced to an edge. Strong 2-joins are used to decompose WP-free balanced bipartite graphs into strongly balanced bipartite blocks which in turn can be decomposed into restricted balanced bipartite components by 1-join decompositions, using Theorem 2.3(I).

As shown in Theorem 2.4(I), decomposition of a graph G by strong 2-join preserves balancedness, i.e. G is balanced if and only if each of the blocks in the decomposition is balanced. Furthermore it can be shown that G is WP-free if and only if each of the blocks in the decomposition is WP-free. Therefore an algorithm to find a strong 2-join decomposition of a graph can be used to test whether a graph is a balanced WP-free bipartite graph.

2 Biclique Cutsets

Let G be a WP-free bipartite graph which is signable to be balanced. In this section we show that, for every edge uv which is the unique chord of at least one cycle, the graph G has a biclique cutset K_{BD} with $u \in B$ and $v \in D$.

For a cycle C with unique chord uv, we use the notation of Figure 2. It will be convenient to write $C = (C_1, C_2)$, where C_1 and C_2 are the two holes induced by C and the chord uv. We assume w.l.o.g. that u is in V^r and that v is in V^c.

Lemma 2.1 *Every node x which is strongly adjacent to C is either of Type 1 [3.3(I)] and has two neighbors in C_1 or in C_2, or is a twin of u or v relative to C.*

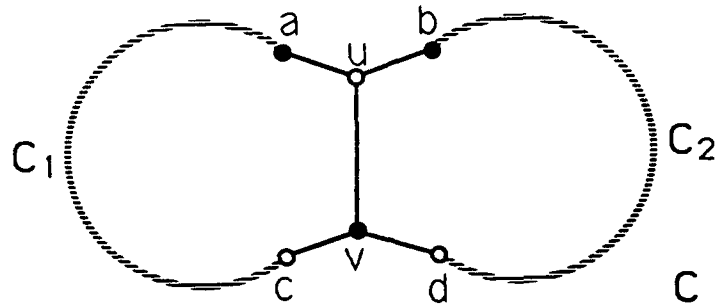

Figure 2: Cycle with a unique chord

Proof: Every strongly adjacent node x is of Type 1, 2 or 3 of Theorem 3.3(I) and has at most two neighbors in C_1 and in C_2, since G contains no wheel.

If x is of Type 2 [3.3(I)], assume w.l.o.g. that x is adjacent to u. Then x has exactly two other neighbors in C, one in C_1 and one in C_2, say x_1 and x_2 respectively. If x_1 is distinct from c (see Figure 2), then there is a parachute with side paths $P_1 = u, v$ and $P_2 = x_1, \ldots, c, v$, top path $T = u, a, \ldots, x_1$ and middle path $M = x, x_2, \ldots, d, v$. So $x_1 = c$. Similarly, it follows that $x_2 = d$. If x is of Type 3 [3.3(I)], assume w.l.o.g. that x is adjacent to b. Then x has exactly two neighbors in $V(C_1) \setminus \{u, v\}$, say x_1 and x_2. The nodes of $V(C_1) \cup \{b, x\}$ induce a parachute, a contradiction. □

Let $V^*(C)$ consist of nodes u, v and the twins of nodes u and v relative to C.

Lemma 2.2 *The nodes of $V^*(C)$ induce a biclique.*

Proof: Assume not. Then there exist twins u^* of u and v^* of v that are not adjacent. This implies the existence of an odd wheel with center v and hole induced by the nodes $(V(C) \setminus \{u, v\}) \cup \{u^*, v^*\}$. □

In the remainder we use the concept of direct connection, as defined in Part I. A direct connection $P = x_1, x_2, \ldots, x_n$ from $V(C_1) \setminus \{u, v\}$ to $V(C_2) \setminus \{u, v\}$ avoiding $V^*(C)$ is said to be *C-reducible* if all its nodes in V^r are adjacent to v, all its nodes in V^c are adjacent to u, node x_1 is adjacent to both a and v or to both c and u, and node x_n is adjacent to both b and v or to both d and u.

Lemma 2.3 *Every direct connection from $V(C_1) \setminus \{u,v\}$ to $V(C_2) \setminus \{u,v\}$ avoiding $V^*(C)$ is C-reducible.*

Proof: Let $P = x_1, x_2, \ldots, x_n$ be a direct connection as defined above. By Lemma 2.1, $n \geq 2$. We assume w.l.o.g. that $x_1 \in V^r$.

Claim Node x_1 is adjacent to both a and v.

Proof of Claim:

Case 1 Node x_1 is not strongly adjacent to C.

Let $x_0 \in V^c$ be the unique neighbor of x_1 in $V(C_1) \setminus \{u,v\}$.

Case 1.1 No node of P is adjacent to v.

Case 1.1.1 At least one node of P is adjacent to u.

If $x_0 \neq a$, there is a $3PC(x_0, u)$. If $x_0 = a$, there is a wheel with center u.

Case 1.1.2 No node of P is adjacent to u, node x_n is adjacent to d and to no other node of $V(C)$.

Then $V(C) \cup V(P)$ induces a parachute with center v, side nodes d, u and bottom node x_0.

Case 1.1.3 No node of P is adjacent to u and x_n is adjacent to at least one node of C_2 distinct from d.

If $x_0 \neq a$, there is a $3PC(x_0, u)$. If $x_0 = a$ and node x_n is adjacent to b and no other node of C, then there is an odd wheel with center u. Otherwise $V(C) \cup V(P)$ contains a parachute with center u and side nodes a, v.

So Case 1.1 cannot occur.

Case 1.2 At least one node of P is adjacent to v.

Let x_j be the node of P adjacent to v which has the lowest index. Note that $j > 1$, since x_1 is not a strongly adjacent node.

Case 1.2.1 No node of P is adjacent to u and $x_j = x_n$.

Then x_n is a strongly adjacent node of Type 1[3.3(I)], having neighbors v and z in C_2. By replacing the vz-subpath of C_2 not containing u by the path v, x_n, z, we are back to Case 1.1.2 when $n \geq 3$. Otherwise, when $n = 2$, we have a strongly adjacent node contradicting Lemma 2.1.

Case 1.2.2 No node of P is adjacent to u and $x_j \neq x_n$.

There is a wheel with center v or a parachute with center v, side nodes u, x_j and bottom node x_0.

Case 1.2.3 At least one node of P is adjacent to u.

So let x_i be the node of P with lowest index which is adjacent to u. If $i < j$, then there is a $3PC(x_0, u)$ or a wheel with center u depending on whether x_0 is adjacent to u or not. If $i > j$ and some node x_k is adjacent to v for

$j < k < i$, then there exists a wheel with center v. If no such node x_k exists there is a parachute with center v, side nodes u, x_j and bottom node x_0. So Case 1.2 cannot occur.

Case 2 Node x_1 is strongly adjacent to C.

If x_1 is not adjacent to v, then a parachute exists: it is induced by $V(C_1)$ and the x_1x_j-subpath of P, where x_j is the first node of P adjacent to u or v. If no such node x_j exists, the middle path of the parachute contains all nodes of P and a subpath of C_2.

So x_1 is adjacent to v and to another node y of C_1. Let P^* be a shortest path from x_1 to u using nodes of $V(P) \cup V(C_2) \setminus \{v\}$. Note that no intermediate node of P^* is adjacent to v, else there is a wheel with center v. Now $y = a$, otherwise the nodes of P^* and C_1 induce a parachute with center v, side nodes u, x_1 and bottom node y. This completes the proof of the claim.

To complete the proof of the lemma, we modify C_1 and P as follows: let $C_1 = u, a, x_1, v$ and redefine P by removing node x_1. Note that if the new path P contains only one node, we are done by Lemma 2.1. Otherwise, by repeating the above analysis with the new cycle C and the new path P, it follows that x_2 is adjacent to u. By induction, the nodes of P in V^r are adjacent to v and those of V^c are adjacent to u. □

Lemma 2.4 *In a C-reducible path P, the nodes in V^c (V^r resp.) are adjacent to all twins of u (v resp.).*

Proof: Assume some node of P is not adjacent to a twin u^* of node u. Let C^* be the cycle obtained from C by substituting u^* for u and let C_1^* and C_2^* be the two resulting holes. Then, since $V^*(C) = V^*(C^*)$, P is a direct connection which is not C^*-reducible, a contradiction to Lemma 2.3. □

Lemma 2.5 *Let $P = x_1, x_2, \ldots, x_p$ and $Q = y_1, y_2, \ldots, y_q$ be C-reducible paths such that x_1 is in V^r and y_1 is in V^c. Then x_1 and y_1 are adjacent.*

Proof: Assume that x_1 and y_1 are not adjacent. Then y_1 is not adjacent to x_3, else there is a wheel with center u (or v). By induction, y_1 is not adjacent to x_{2k+1}, for $3 \leq 2k+1 \leq p$, else there is a wheel. Similarly, y_2 is not adjacent to x_2, else there is a wheel. By induction, y_2 is not adjacent to x_{2k}, for $2 \leq 2k \leq p$. It follows by induction that the paths P and Q are node disjoint and that x_i is not adjacent to y_j for $1 \leq i \leq p$ and $1 \leq j \leq q$.

The nodes $V(P) \cup V(Q) \cup (V(C) \setminus \{u,v\})$ induce a hole. Therefore, there is a wheel with center u (or v), a contradiction. □

Given a cycle $C = (C_1, C_2)$ with unique chord uv, we define a *good biclique* K_{BD} relative to (C_1, C_2) as follows. The node set $B \cup D$ comprises $V^*(C)$ and all the nodes x_1 such that there exists a C-reducible direct connection $P = x_1, x_2, \ldots, x_n$. The fact that K_{BD} so defined is a biclique follows from Lemmas 2.2-2.5. Note that the above definition of a good biclique is not symmetrical with respect to C_1 and C_2, but once the pair (C_1, C_2) has been ordered, there is a unique good biclique.

Theorem 2.6 *Let G be a WP-free bipartite graph that is signable to be balanced. Let C be a cycle with unique chord uv and let C_1 and C_2 be the two induced holes. Then the good biclique relative to (C_1, C_2) is a cutset of G separating $V(C_1) \setminus \{u,v\}$ from $V(C_2) \setminus \{u,v\}$.*

Proof: Define K_{BD} to be the good biclique relative to (C_1, C_2). By Lemma 2.1, there is no node in $V \setminus (B \cup D)$ which is adjacent to both $V(C_1) \setminus \{u,v\}$ and $V(C_2) \setminus \{u,v\}$. So every direct connection P from C_1 to C_2 avoiding $B \cup D$ contains at least two nodes. By Lemma 2.3, P is C-reducible and, by Lemma 2.5 and our choice of K_{BD}, P contains at least one node in $B \cup D$, a contradiction. □

It follows from Theorem 2.6 and from the definition, that a good biclique is a node minimal cutset separating $V(C_1) \setminus \{u,v\}$ from $V(C_2) \setminus \{u,v\}$. Recall from Part I that the blocks in the decomposition of G by a biclique cutset K_{BD} are the graphs induced by the nodes in $B \cup D$ together with those in the connected components of $G \setminus B \cup D$.

A property that follows from the definition of a good biclique and that will be useful in the next section is stated below.

Remark 2.7 *Let K_{BD} be a good biclique relative to $C = (C_1, C_2)$ and let G_1 and G_2 be the blocks containing C_1 and C_2 respectively in the decomposition by K_{BD}. For every pair of nodes y, z in $B \cup D$, there is a path connecting y to z with intermediate nodes in $V(G_1) \setminus (B \cup D)$ as well as a path connecting y to z with intermediate nodes in $V(G_2) \setminus (B \cup D)$.*

3 Strong 2-Joins

Let G be a WP-free bipartite graph which is signable to be balanced and contains a cycle with a unique chord. In this section, we show that G has a strong 2-join. First, we need a technical lemma.

Lemma 3.1 *Among all cycles $C = (C_1, C_2)$ with a unique chord, choose C and the ordering (C_1, C_2) so that the block G_1 containing C_1 in the decomposition of G by the good biclique K_{BD} relative to (C_1, C_2) has the smallest possible number of nodes. Let $r \in V(G_1) \setminus (B \cup D)$ and let $y \in B$ be adjacent to r. Then there cannot exist a cycle $H = (H_1, H_2)$ with unique chord rx $(x \neq y)$ such that $V(H_1) \setminus \{x\} \subseteq V(G_1) \setminus (B \cup D)$ and $y \in V(H_2)$.*

Proof: Assume such a cycle H exists, contradicting the theorem. By Theorem 2.6, the good biclique K_{EF} relative to (H_1, H_2) is a cutset separating $V(H_1) \setminus \{x, r\}$ from $V(H_2) \setminus \{x, r\}$. Assume w.l.o.g. that B, E are contained in V^r and that D, F are contained in V^c. Note that $E \cup F$ is included in $V(G_1)$ since, by construction, every node of $E \cup F$ is adjacent to a node of $V(H_1) \setminus \{r, x\}$.

Let G^* be the block containing H_1 in the decomposition of G by K_{EF}. We will show that $V(G^*)$ is included in $V(G_1)$. Since $y \in V(G_1) \setminus V(G^*)$, the inclusion will be strict, contradicting the minimality of block G_1. Assume $V(G^*)$ contains a node of $V(G) \setminus V(G_1)$, say p. Then, there must be a direct connection P between p and $V(H_1) \setminus \{x, r\}$ avoiding $E \cup F$. Since $V(H_1) \setminus \{x, r\} \in V(G_1) \setminus (B \cup D)$, the path P must contain at least one node of $B \cup D$. Let z be such a node, which is closest to p in the path P.

If $z \in D \setminus F$, then by using the fact that z is adjacent to $y \in B$, it follows that there is a direct connection between p and y avoiding $E \cup F$. This implies that $y \in V(H_2) \setminus (E \cup F)$ belongs to $V(G^*)$, a contradiction.

If $z \in B \setminus E$, then by Remark 2.7, there exists a path with intermediate nodes in $V(G) \setminus V(G_1)$ connecting z to y. Since $E \cup F$ is included in $V(G_1)$, this path together with the path P implies the existence of a path from p to y avoiding $E \cup F$. Therefore, in both cases, the nodes of $V(G) \setminus V(G_1)$ cannot belong to $V(G^*)$, completing the proof. □

Theorem 3.2 *Let G be a WP-free bipartite graph that is signable to be balanced. If G contains a cycle with a unique chord, then G has a strong 2-join.*

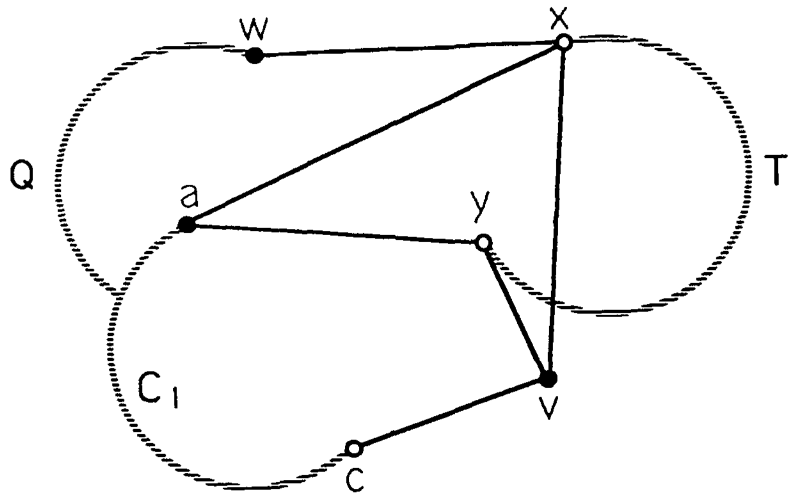

Figure 3:

Proof: Among all cycles $C = (C_1, C_2)$ with a unique chord, choose C and the ordering (C_1, C_2) so that the block G_1 containing C_1 in the decomposition of G by the good biclique relative to (C_1, C_2) has the smallest possible number of nodes. Denote this good biclique cutset by K_{BD}. Assume that the edges incident with $B \cup D$ in G_1 do not induce a 2-join. Then, there must be a node w of G_1 which is adjacent to $x \in B$ but not to $y \in B$. By the definition of a good biclique cutset, all the nodes of B are adjacent to node a in C_1, and therefore node w does not belong to $V(C_1)$. Let Q be a shortest path with nodes in $V(G_1) \setminus (B \cup D)$ connecting w to $V(C_1) \setminus (B \cup D)$. Such a path exists since, otherwise, w would be in a different block in the decomposition of G by K_{BD}. Finally, let T be a path of $V(G) \setminus V(G_1)$ connecting x to y. Such a path exists by Remark 2.7, see Figure 3.

Case 1 Some node of Q other than w is adjacent to x.
Let r be a node of Q which is adjacent to x. If r is not adjacent to y, then we can replace w by r, remove the portion of Q from w to r and repeat the argument with a shorter path Q. So, w.l.o.g., we can assume that the nodes of Q which are adjacent to x are also adjacent to y. Let r be the node of Q adjacent to x which is closest to w. Denote by R the subpath of Q connecting w to r. If y has two or more neighbors in R, in addition to r, then there is a wheel. If y has one neighbor q in R, other than node r, then there is a parachute induced by the nodes of R and T with center node r, bottom node q and side nodes x and y. If y has no neighbor in R, other than r, then there is a cycle (H_1, H_2) with a unique chord xr which satisfies the hypothesis of Lemma 3.1, namely H_1 induced by $V(R) \cup \{x\}$ and H_2 induced by $V(T) \cup \{r\}$. Now Lemma 3.1 contradicts our choice of G_1 with smallest number of nodes.

Case 2 No node of Q other than w is adjacent to x.
Let R be the unique chordless path connecting x to a using edges of Q and of the ac-subpath of C_1 in $G_1(V \setminus (B \cup D))$. Denote by H the hole formed by R together with edge ax. If y has two or more neighbors in H, other than a, then there is a wheel. If y has one neighbor in H, other than a, then $V(H) \cup V(T)$ induces a parachute. If y has no neighbor in H, other than a, then there is a cycle (H_1, H_2) with unique chord xa which satisfies the hypothesis of Lemma 3.1, namely $H_1 = H$ and H_2 induced by $V(T) \cup \{a\}$. But this contradicts the choice of G_1 with smallest number of nodes. □

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In this seven part paper, we prove the following theorem:

At least one of the following alternatives occurs for a bipartite graph G:

- The graph G has no cycle of length $4k+2$.
- The graph G has a chordless cycle of length $4k+2$.

- There exist two complete bipartite graphs K_1, K_2 in G having disjoint node sets, with the property that the removal of the edges in K_1, K_2 disconnects G.
- There exists a subset S of the nodes of G with the property that the removal of S disconnects G, where S can be partitioned into three disjoint sets T, A, N such that $T \neq \emptyset$, some node $x \in T$ is adjacent to every node in $A \cup N$ and, if $|T| \geq 2$, then $|A| \geq 2$ and every node of T is adjacent to every node of A.

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. Balanced matrices are important in integer programming and combinatorial optimization since the associated set packing and set covering polytopes have integral vertices.

To a 0,1 matrix A we associate a bipartite graph $G(V^r, V^c; E)$ as follows: The node nets V^r and V^c represent the row set and the column set of A and edge ij belongs to E if and only if $a_{ij}=1$. Since a 0,1 matrix is balanced if and only if the associated bipartite graph does not contain a chordless cycle of length $4k+2$, the above theorem provides a decomposition of balanced matrices into elementary matrices whose associated bipartite graphs have no cycle of length $4k+2$. In Part VII of the paper, we show how to use this decomposition theorem to test in polynomial time whether a 0,1 matrix is balanced.