523

NPSOR-92-009

NAVAL POSTGRADUATE SCHOOL

Monterey, California

EQUATIONS FOR APPROXIMATE LOWER CONFIDENCE LIMITS ON PROCESS CAPABILITY INDICES

W. Max Woods

December 1991

Prepared for:
Naval Postgraduate School
Monterey, CA

NAVAL POSTGRADUATE SCHOOL,
MONTEREY, CALIFORNIA

Rear Admiral R. W. West, Jr.
Superintendent

Harrison Shull
Provost

This report was prepared for and funded by the Naval Postgraduate School, Monterey, CA.

This report was prepared by:

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION
UNCLASSIFIED

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE

3. DISTRIBUTION /AVAILABILITY OF REPORT
Approved for public release; distribution is unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)
NPSOR-92-009

5. MONITORING ORGANIZATION REPORT NUMBER(S)

6a. NAME OF PERFORMING ORGANIZATION
Naval Postgraduate School

6b. OFFICE SYMBOL (If applicable)
OR

7a. NAME OF MONITORING ORGANIZATION
Naval Postgraduate School

6c. ADDRESS (City, State, and ZIP Code)
Monterey, CA 93943

7b. ADDRESS (City, State, and ZIP Code)
Monterey, CA 93943-5006

8a. NAME OF FUNDING/SPONSORING ORGANIZATION
Naval Postgraduate School

8b. OFFICE SYMBOL (If applicable)

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
OM&N Direct Funding

8c. ADDRESS (City, State, and ZIP Code)
Monterey, CA 93943

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
Equations for Approximate Lower Confidence Limits on Process Capability

12. PERSONAL AUTHOR(S)
W. Max Woods

13a. TYPE OF REPORT
Technical Report

13b. TIME COVERED
FROM_______TO_______

14. DATE OF REPORT (Year, month day)
December, 1991

15. PAGE COUNT
9

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
Quality control; process capability; confidence limit

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Equations that provide approximate but highly accurate lower confidence limits on process capability indices are provided. These equations yield the same value to the nearest hundredth or better than the values provided in tables in the literature. These equations can be used for a much more extensive set of data than those of the existing set of tables.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT
[x] UNCLASSIFIED/UNLIMITED [ ] SAME AS RPT. [ ] DTIC USERS

21. ABSTRACT SECURITY CLASSICIATION
UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL
W. Max Woods

22b. TELEPHONE (Include Area Code)
(408) 646-2743

2c. OFFICE SYMBOL
OR/Wo

DD Form 1473, JUN 86 Previous editions are obsolete.
S/N 0102-LF-014-6603

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

EQUATIONS FOR APPROXIMATE LOWER CONFIDENCE LIMITS ON PROCESS CAPABILITY INDICES

W. Max Woods
Naval Postgraduate School, Monterey, CA 93908

Closed form equations are given for approximate lower confidence limits for process capability CPU and CPL. The equations yield values that are accurate to one one-hundredth for any sample size greater than 9 at confidence levels 95% and any value of the statistics CPU or CPL in the interval (.333, 3). The procedures assume that the process is in control and the data observed are normally distributed.

INTRODUCTION

Three commonly used measures of process capability are C_p, CPU and CPL. Equations that define these expressions and the respective estimates are as follows:

a) Two specification limits, U and L, given:

$$C_p = \frac{U-L}{6\sigma} \quad \hat{C}_p = \frac{U-L}{6s} \tag{1}$$

b) One specification limit, L or U, given:

$$CPU = \frac{U-\mu}{3\sigma}, \quad \hat{CPU} = \frac{U-\overline{X}}{3s}$$
$$CPL = \frac{\mu-L}{3\sigma}, \quad \hat{CPL} = \frac{\overline{X}-L}{3s} \tag{2}$$

where $\overline{X}$ and s are the sample mean and sample standard deviation. Chow, Owen and Borrego [1], give the exact closed expressions for the lower 100γ% confidence limit, C_{pL} for C_p. Namely

$$C_{pL} = \hat{C}_p \left(x^2_{1-\gamma,n-1}/n-1)\right)^{1/2} \tag{3}$$

where $x^2_{1-\gamma,n-1}$ is the 100(1−γ) percentile point of the chi-square distribution with n−1 degrees of freedom. Some hand-held calculators can compute $x^2_{1-\gamma,n-1}$ values. For these calculators equation (1) can be programmed on the calculator and C_{pL} readily computed for all values of $\hat{C}_p$, sample size n and confidence level γ. If a lower confidence limit value c_0 is given, there is a unique minimum value, $\hat{C}_p(c_0)$, of $\hat{C}_p$ which will yield c_0 as the lower confidence limit for C_p. Chow, Owen and Borrego [1] also supply the equation for $\hat{C}_p(c_0)$; namely

$$\hat{C}_p(c_0) = c_0((n-1)/x^2_{1-\gamma,n-1})^{1/2}. \tag{4}$$

They have constructed two tables that display values of C_{pL} and $\hat{C}_p(c_0)$ using equations (1) and (2) for various n, $\hat{C}_p$, c_0 and confidence level 95%. They also present similar tables for the case when one specification limit is given. In the single specification case, however, much more effort is required to develop their tables using the exact classical method which they followed. In this paper, equations are given that can be used to compute their tabled values and many others for the single specification case. These equations can be programmed on some hand-held computers.

APPROXIMATE CONFIDENCE LIMITS FOR CPU AND CPL

Suppose only a lower specification limit L is given. Then if $\delta = (\mu - L)/\sigma$

$$CPL = \delta/3. \tag{5}$$

Consequently, if δ_L is a $100\gamma\%$ lower confidence limit for δ, then $\delta_L/3$ is a $100\gamma\%$ lower confidence limit for CPL, and

$$CPL_L = \delta_L/3. \tag{6}$$

An approximate but accurate expression for computing CPL_L at the 95% confidence level is

$$CPL_L = \hat{CPL} - \frac{1}{3}\left(\frac{1}{n} + \frac{9(\hat{CPL})^2}{2(n-1)}\right)^{1/2} t_{.95,n^*} \tag{7}$$

where n = sample size, $\hat{CPL}$ is defined in equation (2), $n^* = 60$ if $10 \le n \le 60$ and $n^* = n$ if $n > 60$ and $t_{.95,n}$ is the 95th percentile of the *t* distribution with *n* degrees of freedom. Applying equation (7) to all of the $\hat{CPL}$ and n values in Table 4 in Chin, Owen and Borrego [1] will yield the same lower confidence limit values given in their table. Equation (7) is accurate to the nearest one one-hundredth for all $\hat{CPL}$ values in (1/3, 3) and all sample sizes $n \ge 10$. Equation (7) can be used for confidence levels 90% and 80% if $n^* = n$ when $n \ge 10$. See Woods and Yang [3]. Equation (7) can also be used to find an approximate but accurate 95% lower confidence limit, $\Phi(\delta_L)$, for $P(X > L) = \Phi\left(\frac{\mu - L}{\sigma}\right) = \Phi(\delta)$ where $\Phi(z)$ is the standard normal cumulative function. Specifically,

$$\Phi(\delta_L) \equiv \Phi(3CPL_L) \tag{8}$$

is an approximate 95% lower confidence limit for $P(X > L)$. Owen and Hua [2] give tables of exact confidence limits for $P(X > L)$ which were used to choose

n^* in Equation (7) to make the equation accurate. Woods and Yang [3] have extended this result to obtain approximate lower confidence limits for P[X > Y] where X and Y are independent and have normal distributions with unknown means and variances with either equal variances or unequal variances.

If a lower confidence limit goal c_1 is given for CPL_L then Equation (7) can be used to solve for the corresponding minimum value $\hat{CPL}(c_1)$, of $\hat{CPL}$ for that sample size n. That is, if the process is capable when CPL $\geq c_1$, then $\hat{CPL}(c_1)$ is the minimum value of observed $\hat{CPL}$ for which we could say the process is capable with confidence 95%. The expression for the minimum value is

$$\hat{CPL}(c_1) = \frac{2c_1 + 2t\left(\frac{1}{9n} + \frac{c_1^2}{2(n-1)} - \frac{t^2}{18(n)(n-1)}\right)^{1/2}}{2\left(1 - \frac{t^2}{2(n-1)}\right)} \tag{9}$$

where $t = t_{.95,60}$ if $n \leq 60$ and $t = t_{.95,n}$ if $n > 60$.

Finally Equation (7) can be used to obtain lower 95% confidence limits for CPU by replacing $\hat{CPL}$ with $\hat{CPU}$. Equation (9) can be used to compute minimum value, $CPU(c_1)$, of $\hat{CPU}$ corresponding to a lower 9% confidence limit goal, c_1, for CPU.

APPENDIX A: EXPLANATION OF EQUATION (7)

It can be shown that $\frac{\bar{x} - L}{s} \equiv \hat{\delta}\,(\bar{x},s) \equiv \hat{\delta}$ is a consistent estimator for $\frac{\mu - L}{\sigma}$. Expand $\hat{\delta}\,(\bar{x},s)$ in a Taylor series using only first order terms to get

$$\hat{\delta} \doteq \frac{\mu - L}{\sigma} + \frac{(\bar{x} - \mu)}{\sigma} - \frac{(s-\sigma)(\mu - L)}{\sigma^2} \tag{10}$$

Taking the expected value and variance of Equation (10) gives

$$E(\hat{\delta}) \doteq \frac{\mu-L}{\sigma} = \delta$$

$$\mathrm{var}(\hat{\delta}) \doteq \frac{1}{n} + \frac{1}{2(n-1)}\left(\frac{\mu-L^2}{\sigma}\right)$$

and

$$\hat{\sigma}_{\hat{\delta}} = \frac{1}{n} + \frac{1}{2(n-1)}\left(\frac{\bar{x}-L}{s}\right)$$

Consequently an approximate lower 100γ% confidence limit for δ is

$$\delta_L = \hat{\delta} - \hat{\sigma}_{\hat{\delta}}\, t_{\gamma,n^*} \qquad (11)$$

assuming the distribution of $\hat{\delta}$ can be approximated by the student t distribution with some appropriate degrees of freedom, n*. Dividing both members of Equation (11) by three yields Equation (7).

REFERENCES

1. Chou, Y.; Owen, D. B.; and Borrego, S. A. (1990. "Lower Confidence Limits on Process Capability Indices." *Journal of Quality Technology*, 22, pp. 223-229.

2. Owen, D. B.; and Hua, A. (1977). "Tables of Confidence Limits on the Tail Area of the Normal Distribution." *Communications in Statistics*, B6(3), 285-311.

3. Woods, W. M.,; Yang, W. (1991). "Approximate Confidence Limits for P(X>y) and P(X>Y) under Normality Assumptions." (to be supplied in final draft)

INITIAL DISTRIBUTION LIST

1. Library (Code 52)..2
   Naval Postgraduate School
   Monterey, CA 93943-5000

2. Defense Technical Information Center..2
   Cameron Station
   Alexandria, VA 22314

3. Office of Research Administration (Code 81)..1
   Naval Postgraduate School
   Monterey, CA 93943-5000

4. Prof. Peter Purdue..1
   Code OR-Pd
   Naval Postgraduate School
   Monterey, CA 93943-5000

4. Technical Typist...1
   Department of Operations Research
   Naval Postgraduate School
   Monterey, CA 93943-5000

5. Prof. W. Max Woods...20
   Code OR-Wo
   Naval Postgraduate School
   Monterey, CA 93943-5000